



# On the comparison between jump processes and subordinated diffusions

Guanhua Liu and Mathav Murugan

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.  
*E-mail address:* [gliu@math.uni-bielefeld.de](mailto:gliu@math.uni-bielefeld.de)

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada.  
*E-mail address:* [mathav@math.ubc.ca](mailto:mathav@math.ubc.ca)  
*URL:* <https://personal.math.ubc.ca/~mathav/>

**Abstract.** Given a symmetric diffusion process and a jump process on the same underlying space, is there a subordinator such that the jump process and the subordinated diffusion process are comparable? We address this question when the diffusion satisfies a sub-Gaussian heat kernel estimate and the jump process satisfies a polynomial-type jump kernel bounds. Under these assumptions, we obtain necessary and sufficient conditions on the jump kernel estimate for such a subordinator to exist. As an application of our results and the recent stability results of Chen, Kumagai and Wang, we obtain parabolic Harnack inequality for a large family of jump processes. In particular, we show that any jump process with polynomial-type jump kernel bounds on such a space satisfy the parabolic Harnack inequality.

## 1. Introduction

Let  $(X(t))$  and  $(Y_\alpha(t))$  denote the Brownian motion and symmetric  $\alpha$ -stable process on  $\mathbb{R}^n$  respectively, where  $\alpha \in (0, 2)$ . These processes form the basic examples of symmetric diffusions and jump processes respectively. The jump kernel of the  $\alpha$ -stable process  $(Y_\alpha(t))$  on  $\mathbb{R}^n$  is given by

$$J(x, y) = \frac{c_{n,\alpha}}{d(x, y)^{n+\alpha}}, \quad \text{for all } x, y \in \mathbb{R}^n,$$

where  $d$  denotes the Euclidean distance. The processes  $(X(t))$  and  $(Y_\alpha(t))$  are related via a *subordinator*. A subordinator is a one-dimensional Lévy process with non-decreasing paths. The process  $(Y_\alpha(t))$  has the same law as  $(X(S(t)))$ , where  $(S(t))$  is a subordinator independent of  $(X(t))$  and defined by its Laplace transform  $\mathbb{E} \exp(-\lambda S(t)) = \exp(-t\lambda^{\alpha/2})$  for all  $t, \lambda \geq 0$ . Therefore one could study  $\alpha$ -stable processes using properties of Brownian motion and the subordinator. In particular,

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we have

$$\mathbb{P}^x(Y_\alpha(t) \in A) = \int_0^\infty \mathbb{P}^x(X(s) \in A) \eta_t(ds), \quad (1.1)$$

where  $\eta_t$  is the law of the subordinator  $S(t)$  described above.

A well-known important application of (1.1) is that one can obtain heat kernel bounds and parabolic Harnack inequality for the jump process  $(Y_\alpha)$  by transferring heat kernel bounds and parabolic Harnack inequality for the diffusion  $(X)$  along with heat kernel estimates on the subordinator (Chen et al. (2022, Section 5.2) and Bae et al. (2019a, Section 4.1)). In this work, we investigate the extent to which subordination can be used to analyze jump processes. In particular, our work addresses the following questions:

- (a) Let  $X$  be a  $\mu$ -symmetric diffusion that satisfies the parabolic Harnack inequality. Given a  $\mu$ -symmetric jump process  $Y$ , does there exist a subordinator  $S$  such that the subordinated process  $X \circ S$  has jump kernel comparable to that of  $Y$ ?
- (b) In the setting above, does the jump process  $Y$  also inherit the parabolic Harnack inequality from  $X$ ? If so, what is the space time scaling of the process  $Y$ ?

Under fairly mild conditions on the jump kernel, we obtain a positive answer to both these questions. Our answer to question (a) seems to be new even on  $\mathbb{R}^n$  (see Remark 2.4). The motivation for question (a) arises from the beautiful recent results of Chen, Kumagai, Wang concerning the stability of parabolic Harnack inequality for jump processes (Chen et al., 2020). Using their results a positive answer to question (b) follows from a positive answer to question (a).

## 2. Framework and results

2.1. *Dirichlet forms and symmetric Markov processes.* Throughout this work, we consider a complete, locally compact, separable, unbounded metric space  $(M, d)$  equipped with a Radon measure  $\mu$  with full support, i.e., a Borel measure  $\mu$  on  $M$  that is finite on any compact set and positive on any non-empty open set. Such a triple  $(M, d, \mu)$  is called a *metric measure space*.

Let  $(\mathcal{E}, \mathcal{F})$  be a *symmetric Dirichlet form* on  $L^2(M, \mu)$ . That is, the domain  $\mathcal{F}$  is a dense linear subspace of  $L^2(M, \mu)$ , such that  $\mathcal{E} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$  is a non-negative definite symmetric bilinear form which is *closed* ( $\mathcal{F}$  is a Hilbert space under the inner product  $\mathcal{E}_1(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + \langle \cdot, \cdot \rangle_{L^2(M, \mu)}$ ) and *Markovian* (the unit contraction operates on  $\mathcal{F}$ ;  $\widehat{u} := (u \vee 0) \wedge 1 \in \mathcal{F}$  and  $\mathcal{E}(\widehat{u}, \widehat{u}) \leq \mathcal{E}(u, u)$  for any  $u \in \mathcal{F}$ ). Recall that  $(\mathcal{E}, \mathcal{F})$  is called *regular* if  $\mathcal{F} \cap \mathcal{C}_c(M)$  is dense both in  $(\mathcal{F}, \mathcal{E}_1)$  and in  $(\mathcal{C}_c(M), \|\cdot\|_{\text{sup}})$ . Here  $\mathcal{C}_c(M)$  is the space of  $\mathbb{R}$ -valued continuous functions on  $M$  with compact support.

Given a Dirichlet form  $(\mathcal{E}, \mathcal{F})$ , there is an associated *Markov semigroup*  $(P_t)_{t \geq 0}$  on  $L^2(M, \mu)$  and a non-positive definite self-adjoint *generator*  $\mathcal{L}$  such that  $P_t = e^{t\mathcal{L}}$ . Furthermore, by Fukushima et al. (2011, Theorem 1.3.1 and Lemma 1.3.4) the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  is given in terms of the semigroup by

$$\mathcal{F} = \left\{ f \in L^2(M, \mu) : \lim_{t \downarrow 0} \frac{1}{t} \langle f - P_t f, f \rangle < \infty \right\}, \quad (2.1)$$

$$\mathcal{E}(f, f) = \lim_{t \downarrow 0} \frac{1}{t} \langle f - P_t f, f \rangle, \quad \text{for all } f \in \mathcal{F}, \quad (2.2)$$

where  $\langle \cdot, \cdot \rangle$  denotes the  $L^2(M, \mu)$  inner product. Recall that by the spectral representation,  $t \mapsto \frac{1}{t} \langle f - P_t f, f \rangle$  is non-increasing and

$$\limsup_{t \downarrow 0} \frac{1}{t} \langle f - P_t f, f \rangle = \lim_{t \downarrow 0} \frac{1}{t} \langle f - P_t f, f \rangle \quad \text{for any } f \in L^2(M, \mu). \quad (2.3)$$

It is known that the semigroup extends to a contraction on any  $L^p(M, \mu)$ , where  $p \in [1, \infty]$ . The Markov semigroup is said to be *conservative* if  $P_t 1 = 1$  for any  $t > 0$ .

Associated with a regular Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(M, \mu)$  is a  $\mu$ -symmetric *Hunt process*  $(X_t, t \geq 0, \mathbb{P}_x, x \in M \setminus \mathcal{N})$ , where  $\mathcal{N}$  is a properly exceptional set for  $(\mathcal{E}, \mathcal{F})$ . Recall that a Hunt process is a strong Markov process that is right continuous and quasi-left continuous on the one-point compactification  $M_\partial := M \cup \{\partial\}$  of  $M$ . The *heat kernel* associated with the Markov semigroup  $\{P_t\}$  (if it exists) is a family of measurable functions  $p(t, \cdot, \cdot) : M \times M \mapsto [0, \infty)$  for every  $t > 0$ , such that

$$P_t f(x) = \int p(t, x, y) f(y) \mu(dy) \quad \text{for all } f \in L^2(M, \mu), t > 0 \text{ and } x \in M, \tag{2.4}$$

$$p(t, x, y) = p(t, y, x) \quad \text{for all } x, y \in M \text{ and } t > 0, \tag{2.5}$$

$$p(t + s, x, y) = \int p(s, x, y) p(t, y, z) \mu(dy) \quad \text{for all } t, s > 0 \text{ and } x, y \in M. \tag{2.6}$$

We recall the notion of strongly local and pure jump type Dirichlet forms. For a Borel measurable function  $f : M \rightarrow \mathbb{R}$  or a  $\mu$ -equivalence class  $f$  of such functions,  $\text{supp}_\mu[f]$  denotes the support of the measure  $|f| d\mu$ , i.e., the smallest closed subset  $F$  of  $M$  with  $\int_{M \setminus F} |f| d\mu = 0$ , which exists since  $M$  is separable. A Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(M, \mu)$  is said to be *strongly local* if  $\mathcal{E}(f, g) = 0$  for all functions  $f, g \in \mathcal{F}$  with  $\text{supp}_\mu[f], \text{supp}_\mu[g]$  compact and  $\text{supp}_\mu[f - a\mathbb{1}_M] \cap \text{supp}_\mu[g] = \emptyset$  for some  $a \in \mathbb{R}$ . We say that a Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(M, \mu)$  is of pure jump type, if there exists a symmetric positive Radon measure  $\tilde{J}$  on  $M \times M \setminus \text{diag}$  such that

$$\mathcal{E}(f, f) = \int_{M \times M \setminus \text{diag}} (f(x) - f(y))^2 \tilde{J}(dx, dy) \quad \text{for all } f \in \mathcal{F},$$

where  $\text{diag} = \{(x, x) \mid x \in M\}$  denotes the diagonal. The Radon measure  $\tilde{J}$  is called the *jumping measure*; we refer to the Beurling-Deny decomposition for the reason behind this terminology Fukushima et al. (2011, Theorem 3.2.1 and Lemma 4.5.4). A symmetric Borel measurable function  $J : M \times M \setminus \text{diag} \rightarrow [0, \infty)$  is said to be a *jump kernel* of a Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(M, \mu)$  of pure jump type, if  $\tilde{J}(dx, dy) = J(x, y)\mu(dx)\mu(dy)$ , where  $\tilde{J}$  is the jumping measure.

Let  $\mathbb{R}_+ = [0, \infty)$ . We say that a homeomorphism  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a *scale function* if there exist  $C \geq 1, 0 < \beta_1 \leq \beta_2$  such that

$$C^{-1} \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\psi(R)}{\psi(r)} \leq C \left(\frac{R}{r}\right)^{\beta_2} \quad \text{for all } 0 < r \leq R. \tag{2.7}$$

Let  $\psi_j$  be a scale function and let  $\tilde{J}$  be a jumping measure. We say that the jumping measure  $\tilde{J}$  satisfies  $J(\psi_j)$  if there exists a density  $J$  such that  $\tilde{J}(dx, dy) = J(x, y)\mu(dx)\mu(dy)$  and there exists  $C > 0$  such that

$$\frac{C^{-1}}{\mu(B(x, d(x, y)))\psi_j(d(x, y))} \leq J(x, y) \leq \frac{C}{\mu(B(x, d(x, y)))\psi_j(d(x, y))}, \tag{J(\psi_j)}$$

for  $\mu$ -a.e.  $x, y \in M \times M \setminus \text{diag}$ . If the density  $J$  satisfies the upper or lower bounds on  $J$  in the above estimate, we say that the jumping measure satisfies  $J(\psi_j)_\leq$  or  $J(\psi_j)_\geq$  respectively. Jump processes satisfying  $J(\psi_j)$  have been widely studied in the context of heat kernel estimates and Harnack inequalities (see Chen et al. (2022); Bae et al. (2019a,b); Murugan and Saloff-Coste (2019)).

**2.2. Parabolic Harnack inequality.** We recall the definition of parabolic Harnack inequality and its relationship to heat kernel bounds. Let  $(\mathcal{E}, \mathcal{F})$  be a Dirichlet form on  $L^2(M, \mu)$  and let  $I$  be an open interval in  $\mathbb{R}$ . We say that a function  $u : I \rightarrow L^2(M, \mu)$  is weakly differentiable at  $t_0 \in I$  if the function  $t \mapsto \langle u(t), f \rangle$  is differentiable at  $t_0$  for all  $f \in L^2(M, \mu)$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $L^2(M, \mu)$ . By the uniform boundedness principle, there exists a (unique) function

$w \in L^2(M, \mu)$  such that

$$\lim_{t \rightarrow t_0} \left\langle \frac{u(t) - u(t_0)}{t - t_0}, f \right\rangle = \langle w, f \rangle, \quad \text{for all } f \in L^2(M, \mu).$$

In this case, we say that the function  $w$  above is the *weak derivative* of  $u$  at  $t_0$  and write  $w = u'(t_0)$ . Let  $\Omega$  be an open subset of  $M$ . A function  $u : I \rightarrow \mathcal{F}$  is said to be *caloric* in  $I \times \Omega$  if  $u$  is weakly differentiable in the space  $L^2(\Omega)$  at any  $t \in I$ , and for any  $f \in \mathcal{F} \cap C_c(\Omega)$ , and for any  $t \in I$ ,

$$\langle u', f \rangle + \mathcal{E}(u, f) = 0. \quad (2.8)$$

**Definition 2.1** (Parabolic Harnack inequality). Let  $(\mathcal{E}, \mathcal{F})$  be a Dirichlet form on  $L^2(M, \mu)$  and let  $\psi$  be a scale function. We say that a metric measure space  $(M, d, \mu)$  equipped with a Dirichlet form  $(\mathcal{E}, \mathcal{F})$  satisfies the *parabolic Harnack inequality* with scale function  $\psi$  (abbreviated as PHI( $\psi$ )), if there exist  $0 < C_1 < C_2 < C_3 < C_4 < \infty$ ,  $C_5 > 1$  and  $\delta \in (0, 1)$  such that for all  $x \in M$ ,  $r > 0$ , for any function  $u$  non-negative and bounded on  $(a, a + \psi(C_4r)) \times M$  and caloric on the space-time cylinder  $Q = (a, a + \psi(C_4r)) \times B(x, r)$ , we have

$$\operatorname{ess\,sup}_{Q_-} u \leq C_5 \operatorname{ess\,inf}_{Q_+} u, \quad \text{PHI}(\psi)$$

where  $Q_- = (a + \psi(C_1r), a + \psi(C_2r)) \times B(x, \delta r)$  and  $Q_+ = (a + \psi(C_3r), a + \psi(C_4r)) \times B(x, \delta r)$ .

We recall the following sub-Gaussian heat kernel estimate that is equivalent to the above parabolic Harnack inequality for strongly local, regular Dirichlet forms.

Let  $(\mathcal{E}, \mathcal{F})$  be a strongly local, regular, Dirichlet form on  $L^2(M, \mu)$  and let  $\psi$  be a scale function. We say that the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on  $L^2(M, \mu)$  satisfies HKE( $\psi$ ), if there exist  $C_1, c_1, c_2, c_3, \delta \in (0, \infty)$  and a heat kernel  $\{p_t\}_{t>0}$  such that for any  $t > 0$ ,

$$p_t(x, y) \leq \frac{C_1}{m(B(x, \psi^{-1}(t)))} \exp(-c_1 \Phi(c_2 d(x, y), t)) \quad \text{for } \mu\text{-a.e. } x, y \in M, \quad (2.9)$$

$$p_t(x, y) \geq \frac{c_3}{m(B(x, \psi^{-1}(t)))} \quad \text{for } \mu\text{-a.e. } x, y \in M \text{ with } d(x, y) \leq \delta \psi^{-1}(t), \quad (2.10)$$

where

$$\Phi(R, t) := \Phi_\psi(R, t) := \sup_{r>0} \left( \frac{R}{r} - \frac{t}{\psi(r)} \right), \quad \text{for all } R \geq 0, t > 0. \quad (2.11)$$

We recall volume doubling and reverse volume doubling properties of a metric measure space. We say that a metric measure space  $(M, d, \mu)$  satisfies the *volume doubling property* VD if there exists  $C_D > 1$  such that

$$\mu(B(x, 2r)) \leq C_D \mu(B(x, r)), \quad \text{for all } x \in M, r > 0. \quad \text{VD}$$

We say that a metric measure space  $(M, d, \mu)$  satisfies the *reverse volume doubling property* RVD, if there exists  $A, C > 1$  such that

$$\mu(B(x, Ar)) \geq C \mu(B(x, r)), \quad \text{for all } x \in M, r > 0.$$

We recall the following well-known equivalence between parabolic Harnack inequality and heat kernel estimates.

**Theorem 2.2.** (*Barlow et al., 2012, Theorem 3.1*) Let  $(M, d, \mu)$  be a metric measure space that satisfies the VD, RVD and let  $\psi$  be a scale function. Let  $(\mathcal{E}, \mathcal{F})$  be a strongly local, regular Dirichlet form on  $L^2(M, \mu)$ . Then for the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  on the metric measure space  $(M, d, \mu)$ , the parabolic Harnack inequality PHI( $\psi$ ) is equivalent to the heat kernel estimate HKE( $\psi$ ).

*Proof.* We first assume that the constant  $\beta_1$  in (2.7) satisfies  $\beta_1 > 1$  and we will later show that we could always take  $\beta_1 > 1$  under our assumptions. The equivalence mentioned above is essentially contained in Barlow et al. (2012, Theorem 3.1). Our formulation of PHI( $\psi$ ) is same as the weak

parabolic Harnack inequality in Barlow et al. (2012, §3.1) but since our version of  $\text{HKE}(\psi)$  is slightly different we provide the details. In order to use Barlow et al. (2012, Theorem 3.1), we first need to verify that all balls are precompact. This follows from the completeness of  $d$  and the doubling property of  $\mu$  as we explain below. Since  $\mu$  is a doubling measure, we obtain that  $d$  is a doubling metric by Heinonen (2001, Chapter 13) (that is, there exists  $N \in \mathbb{N}$  such that every ball of radius  $r$  can be covered by  $N$  balls of radius  $r/2$ ). This in turn implies that all metric balls are totally bounded by Heinonen (2001, Definition 10.15 and Exercise 10.17). Therefore all metric balls are totally bounded and hence precompact by the completeness of  $X$ . By Grigor'yan and Telcs (2012, Lemma 3.19) our version of  $\text{HKE}(\psi)$  implies w-HKE( $\psi$ ) in Barlow et al. (2012, p. 1102). Therefore, the fact that  $\text{HKE}(\psi)$  implies  $\text{PHI}(\psi)$  follows from Barlow et al. (2012, Theorem 3.1) along with Grigor'yan and Telcs (2012, Lemma 3.19).

For the converse implication we use Barlow et al. (2012, Theorem 3.1) to obtain that  $\text{PHI}(\psi)$  implies that the heat kernel exists and there exist  $c_1, C_1 \in (0, \infty)$  such that the heat kernel  $p_t(\cdot, \cdot)$  satisfies the following upper bound

$$p_t(x, y) \leq \frac{C_1}{\mu(B(x, \psi^{-1}(t)))} \exp\left(-c_1 \left(\frac{\psi(d(x, y))}{t}\right)^{1/(\beta_2-1)}\right) \quad \text{for } \mu\text{-a.e. } x, y \in M. \tag{2.12}$$

Furthermore by Barlow et al. (2012, Theorem 3.1), we have the following *local lower estimate* for the corresponding Dirichlet heat kernel  $p_t^{B(x_0, r)}$ : there exist  $c_2, \epsilon_0 \in (0, 1)$  such that for all  $x_0 \in M, r > 0$  we have

$$p_t^{B(x_0, r)}(x, y) \geq \frac{c_2}{\mu(B(x_0, \psi^{-1}(t)))} \quad \text{for all } 0 < t < \psi(\epsilon_0 r), \mu\text{-a.e. } x, y \in B(x_0, \epsilon_0 \psi^{-1}(t)). \tag{2.13}$$

In order to verify  $\text{HKE}(\psi)$ , by Grigor'yan et al. (2015, Theorem 1.2) it suffices to verify the elliptic Harnack inequality and exit time upper and lower bounds. Since the elliptic Harnack inequality follows from  $\text{PHI}(\psi)$ , it remains to show the following exit time bounds: there exist  $\epsilon \in (0, 1), C > 1$  such that for all  $x_0 \in M, r > 0$ ,

$$\text{ess sup}_{x \in B(x_0, r)} \mathbb{E}_x(\tau_{B(x_0, r)}) \leq C\psi(r), \quad \text{ess inf}_{x \in B(x_0, \epsilon r)} \mathbb{E}_x(\tau_{B(x_0, r)}) \geq C^{-1}\psi(r), \tag{2.14}$$

where  $\tau_{B(x_0, r)}$  denotes the exit time of  $B(x_0, r)$ . The lower bound for the exit time follows from (2.13) as

$$\begin{aligned} \text{ess inf}_{x \in B(x_0, \epsilon_0 r)} \mathbb{E}_x(\tau_{B(x_0, r)}) &\geq \psi(\epsilon_0 r)/2 \quad \text{ess inf}_{x \in B(x_0, \epsilon_0 r)} \mathbb{P}_x(\tau_{B(x_0, r)} > \epsilon_0 r)/2 \\ &\geq \psi(\epsilon_0 r)/2 \int_{B(x_0, r)} p_{\psi(\epsilon_0 r)/2}^{B(x_0, r)}(x, y) \mu(dy) \stackrel{(2.13)}{\gtrsim} \psi(r). \end{aligned}$$

For the upper bound, we use reverse volume doubling to choose  $A > 1$  such that  $\mu(B(x, Ar)) > 2\mu(B(x, r))$ . Then there exists  $c_2 \in (0, 1)$  such that for all  $x_0 \in M, r > 0$ , for  $\mu$ -a.e  $x \in B(x_0, r)$  we have

$$\begin{aligned} \mathbb{P}_x(\tau_{B(x_0, r)} \leq \psi(Ar)/2) &\geq \int_{B(x_0, Ar) \setminus B(x_0, r)} p_{\psi(Ar)/2}(x, y) \mu(dy) \\ &\geq \int_{B(x_0, Ar) \setminus B(x_0, r)} p_{\psi(Ar)/2}^{B(x_0, r)}(x, y) \mu(dy) \\ &\geq \int_{B(x_0, Ar) \setminus B(x_0, r)} p_t^{B(x_0, \epsilon_0^{-1} Ar)}(x, y) \mu(dy) \stackrel{(2.13)}{\geq} c_2. \end{aligned}$$

Therefore for all  $x_0 \in M, r > 0, n \in \mathbb{N}$ , we have

$$\text{ess sup}_{x \in B(x_0, \epsilon r)} \mathbb{P}_x(\tau_{B(x_0, r)} > n\psi(Ar)/2) \leq (1 - c_2)^n.$$

Hence there exists  $C_4 \geq 1, c_5 \in (0, 1)$  such that for all  $x_0 \in M, r > 0, t > 0$ , we have

$$\operatorname{ess\,sup}_{x \in B(x_0, \epsilon r)} \mathbb{P}_x(\tau_{B(x_0, r)} > t) \leq C_4 \exp(-c_5 t / \psi(r)).$$

This estimate along with  $\operatorname{ess\,sup}_{x \in B(x_0, \epsilon r)} \mathbb{E}_x(\tau_{B(x_0, r)}) = \operatorname{ess\,sup}_{x \in B(x_0, \epsilon r)} \int_0^\infty \mathbb{P}_x(\tau_{B(x_0, r)} > t) dt$  yields the desired upper bound in (2.14). We refer the reader to [Kajino and Murugan \(2023, Theorem 4.5 and Remark 4.6\)](#) for further discussion on related results.

Finally, we consider the case  $\beta_1 \leq 1$  in (2.7). Since replacing the metric  $d$  with  $d^\alpha$  where  $\alpha = \frac{3}{4}\beta_1$  leads to replacing  $\psi$  with  $r \mapsto \psi(r^{1/\alpha})$  for parabolic Harnack inequality and heat kernel estimates, by choosing  $\alpha < \beta_1$  we are in the case  $\beta_1 > 1$  (since  $\beta_1$  will be replaced with  $\beta_1/\alpha = \frac{4}{3}$ ). This follows immediately from the definitions of [HKE\( \$\psi\$ \)](#) and [PHI\( \$\psi\$ \)](#) along with  $\{y : d^\alpha(x, y) < r\} = \{y : d(x, y) < r^{1/\alpha}\}$ . By [Grigor'yan et al. \(2015, Theorem 1.2\)](#) and the previous case, we obtain the Poincaré inequality and capacity upper bounds under the new metric  $d^\alpha$ . By [Murugan \(2020, Corollary 1.10\)](#), we can choose a ‘new’  $\beta'_1 > 0$  for  $\psi$  so that  $\beta'_1 \geq 2\alpha = \frac{3}{2}\beta_1$ . We repeat this procedure finitely many times to conclude that  $\beta_1 > 1$ . In fact, we obtain that  $\beta_1$  can be chosen to be 2.  $\square$

**2.3. Subordinator and Lévy measure.** A subordinator  $(S_t)$  is a non-decreasing Lévy process with  $S_0 = 0$ ; that is,  $(S_t)$  has independent, stationary increments such that  $t \mapsto S_t$  is continuous in probability. A subordinator is characterized by its *Laplace exponent*  $\phi : [0, \infty) \rightarrow [0, \infty)$  such that

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad (2.15)$$

where  $\phi$  is a *Bernstein function* determined by its *drift*  $a \in [0, \infty)$  and *Lévy measure*  $\nu$  of the subordinator  $S_t$ , where  $\nu$  is a Borel measure on  $(0, \infty)$  such that

$$\int_{(0, \infty)} (1 \wedge s) \nu(ds) < \infty, \quad \text{and} \quad \phi(\lambda) = a\lambda + \int_0^\infty (1 - e^{-s\lambda}) \nu(ds). \quad (2.16)$$

Conversely, any drift  $a \in [0, \infty)$  and Borel measure  $\nu$  on  $(0, \infty)$  that satisfy (2.16) uniquely determine the subordinator  $(S_t)$ . We refer the reader to [Schilling et al. \(2012, Chapter 5\)](#) or [Sato \(1999, Chapter 6\)](#) for background on subordinators.

**Notation.** In the following, we will use the notation  $A \lesssim B$  for quantities  $A$  and  $B$  to indicate the existence of an implicit constant  $C \geq 1$  depending on some inessential parameters such that  $A \leq CB$ . We write  $A \asymp B$ , if  $A \lesssim B$  and  $B \lesssim A$ .

**2.4. Statement of the main results.** We now state the main results of this work. The following theorem characterizes all polynomial type jump kernels on a metric measure space that admits a diffusion satisfying parabolic Harnack inequality. Our theorem establishes a one-to-one correspondence between jump processes with polynomial type jump kernels and processes with jump kernels comparable to that of subordinated diffusion processes.

**Theorem 2.3** (Characterization of jump kernels). *Let  $(M, d, \mu)$  be an unbounded, complete, separable, locally compact metric measure space that satisfies [VD](#). Let  $(\mathcal{E}^c, \mathcal{F}^c)$  be a strongly local, regular Dirichlet form on  $L^2(M, \mu)$  that satisfies [PHI\( \$\psi\_c\$ \)](#), where  $\psi_c$  is a scale function. Let  $X$  be the  $\mu$ -symmetric Hunt process corresponding to  $(\mathcal{E}^c, \mathcal{F}^c)$  on  $L^2(M, \mu)$ . Given a scale function  $\psi_j$ , the following are equivalent.*

- (a) *There exists a regular Dirichlet form  $(\mathcal{E}^j, \mathcal{F}^j)$  on  $L^2(M, \mu)$  of pure jump type whose jump kernel satisfies [J\( \$\psi\_j\$ \)](#).*
- (b) *There exists a subordinator  $S$  such that the subordinated process  $X \circ S$  corresponds to a regular Dirichlet form  $(\mathcal{E}^j, \mathcal{F}^j)$  on  $L^2(M, \mu)$  of pure jump type and satisfies [J\( \$\psi\_j\$ \)](#).*

(c) The scale function  $\psi_j$  satisfies

$$\int_0^1 \frac{\psi_c(s)}{s\psi_j(s)} ds < \infty. \tag{2.17}$$

*Remark 2.4.* By [Murugan \(2020, Corollary 1.10\)](#), we have  $\psi_c(s) \lesssim s^2$  for all  $s \in [0, 1]$ . Therefore the condition

$$\int_0^1 \frac{s}{\psi_j(s)} ds < \infty \tag{2.18}$$

implies (2.17). The above sufficient condition (2.18) for (2.17) was assumed in the context of jump processes on  $d$ -regular subsets of the Euclidean space in [Chen and Kumagai \(2008, eq. \(1.3\)\)](#). Furthermore, the Brownian motion on Euclidean space satisfies  $\text{PHI}(\psi_c)$  with  $\psi_c(r) = r^2$  in which case (2.17) is same as (2.18). The integrability condition (2.17) was recently used to obtain heat kernel estimates in [Bae et al. \(2019a, eq. \(2.19\)\)](#). [Theorem 2.3](#) can be viewed as a justification for the assumptions in these previous works. Although these versions of (2.18) were used in earlier works to obtain heat kernel bounds and parabolic Harnack inequality, the necessity of (2.17) is new and is the key contribution of our work.

**Corollary 2.5** (Parabolic Harnack inequality via subordination). *Let  $(M, d, \mu)$  be an unbounded, complete, separable, locally compact metric measure space that satisfies  $\text{VD}$ . Let  $(\mathcal{E}^c, \mathcal{F}^c)$  be a regular, strongly local Dirichlet form on  $L^2(M, \mu)$  that satisfies  $\text{PHI}(\psi_c)$ , where  $\psi_c$  is a scale function. Let  $(\mathcal{E}^j, \mathcal{F}^j)$  be a regular Dirichlet form  $L^2(M, \mu)$  of pure jump type that satisfies  $\text{J}(\psi_j)$  for some scale function  $\psi_j$ .*

(a) The jump type Dirichlet form  $(\mathcal{E}^j, \mathcal{F}^j)$  satisfies  $\text{PHI}(\widehat{\psi}_j)$ , where  $\widehat{\psi}_j$  is a scale function satisfying the following estimate: there exists  $C \geq 1$  such that

$$C^{-1} \frac{\psi_c(r)}{\int_0^r \frac{\psi_c(s)}{s\psi_j(s)} ds} \leq \widehat{\psi}_j(r) \leq C \frac{\psi_c(r)}{\int_0^r \frac{\psi_c(s)}{s\psi_j(s)} ds} \quad \text{for all } r > 0.$$

(b) The scale functions  $\psi_c, \psi_j, \widehat{\psi}_j$  satisfy the following estimates:

$$\widehat{\psi}_j(r) \lesssim \psi_j(r) \quad \text{for all } r > 0, \tag{2.19}$$

$$\psi_c(r) \lesssim \psi_j(r) \quad \text{for all } r \leq 1, \tag{2.20}$$

$$\frac{\widehat{\psi}_j(R)}{\widehat{\psi}_j(r)} \lesssim \frac{\psi_c(R)}{\psi_c(r)} \quad \text{for all } 0 < r \leq R, \tag{2.21}$$

$$\widehat{\psi}_j(r) \lesssim \psi_c(r) \quad \text{for all } r \geq 1, \text{ and } \psi_c(r) \lesssim \widehat{\psi}_j(r) \quad \text{for all } r \leq 1. \tag{2.22}$$

We provide a probabilistic interpretation of the [Corollary 2.5\(b\)](#). Let  $(X_t)_{t \geq 0}, (Y_t)_{t \geq 0}$  denote the diffusion and jump processes corresponding to the Dirichlet forms  $(\mathcal{E}^c, \mathcal{F}^c)$  and  $(\mathcal{E}^j, \mathcal{F}^j)$  in [Corollary 2.5](#) respectively. Then by the results of [Grigor'yan et al. \(2015\)](#); [Grigor'yan and Telcs \(2012\)](#); [Chen et al. \(2020\)](#), the functions  $\psi_c$  and  $\widehat{\psi}_j$  govern the exit times from balls of the processes  $X$  and  $Y$  respectively. In particular, the following two sided bounds for exit times hold:

$$\mathbb{E}_x[\tau_{B(x,r)}^X] \asymp \psi_c(r), \quad \mathbb{E}_x[\tau_{B(x,r)}^Y] \asymp \widehat{\psi}_j(r)$$

for all  $x \in M, r > 0$ , where  $\mathbb{E}_x$  denote the expectation when the process starts at  $x$  and  $\tau_{B(x,r)}^X, \tau_{B(x,r)}^Y$  correspond to the exit time of  $B(x, r)$  for the process  $X$  and  $Y$  respectively. By the last estimate in [Corollary 2.5\(b\)](#), the diffusion process exits smaller balls (say balls of radii less than 1) faster than the jump process. On the other hand, the jump process exits larger balls faster than the diffusion process. A similar assumption can be found in [Chen et al. \(2022, \(1.13\)\)](#). This work grew from an attempt to understand and justify the above mentioned assumptions in [Chen et al. \(2022\)](#); [Bae et al. \(2019a\)](#); [Chen and Kumagai \(2008\)](#).

### 3. Subordinator with comparable jump kernel

We recall the following result from [Bae et al. \(2019a\)](#). We emphasize that the following result does not require  $\psi_c$  and  $\psi_j$  to satisfy condition (2.17). In the notation of [Bae et al. \(2019a\)](#), we only need that  $\mathcal{J}_\psi$  is defined by the equation that is four lines above the statement of [Bae et al. \(2019a\)](#). In [Bae et al. \(2019a\)](#), the assumption  $\beta_1 > 1$  was included but as explained in the proof of Theorem 2.2, this assumption is not needed.

**Lemma 3.1.** (*Bae et al., 2019a, Lemma 4.2*) *Let  $\psi_c, \psi_j$  be scale functions and let  $(t, x, y) \mapsto p_t^c(x, y)$  be a heat kernel that satisfies the estimate  $\text{PHI}(\psi_c)$ . Then*

$$\int_0^\infty \frac{p_t^c(x, y)}{t\psi_j \circ \psi_c^{-1}(t)} dt \asymp \frac{1}{\mu(B(x, d(x, y)))\psi_j(d(x, y))} \quad \text{for all } x, y \in M. \quad (3.1)$$

The following Poincaré inequality follows from the parabolic Harnack inequality and is a crucial ingredient in our proof.

**Lemma 3.2** (Poincaré inequality). (*Grigor'yan et al., 2015, Theorem 1.2*) *Let  $(M, d, \mu)$  be an unbounded, complete, locally compact, separable metric measure space. Let  $(\mathcal{E}^c, \mathcal{F}^c)$  be a regular, strongly local Dirichlet form on  $L^2(M, \mu)$  that satisfies  $\text{PHI}(\psi_c)$ , where  $\psi_c$  is a scale function. Then we have the following Poincaré inequality: there exist  $C_P, A > 1$  such that for any ball  $B(x, r)$  and for any function  $f \in \mathcal{F}^c$ ,*

$$\int_{B(x, r)} (f(y) - f_{B(x, r)})^2 \mu(dy) \leq C_P \int_{B(x, Ar)} \int_{B(x, Ar)} d\Gamma(f, f), \quad (\text{PI}(\psi_c))$$

where  $\Gamma(f, f)$  denotes the energy measure, and  $f_{B(x, r)}$  denotes the  $\mu$ -average of  $f$  in  $B(x, r)$  defined by  $f_{B(x, r)} = \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f d\mu$ .

*Proof.* This is an immediate consequence of [Grigor'yan et al. \(2015, Theorem 1.2\)](#) and Theorem 2.2.  $\square$

The following lemma is classical and is a special case of [Ôkura \(2002, Theorem 2.1\)](#)

**Lemma 3.3.** (*Ôkura, 2002, Theorem 2.1*) *Let  $X$  be a  $\mu$ -symmetric process with a conservative semigroup whose heat kernel is  $p_t^c(\cdot, \cdot)$ , and  $(S_t)$  be a subordinator with Lévy measure  $\nu$ . Then the Dirichlet form corresponding to the  $\mu$ -symmetric subordinated process  $Y_t = X_{S_t}$  is a pure jump process with jump kernel*

$$J(x, y) = \frac{1}{2} \int_0^\infty p_t^c(x, y) \nu(dt).$$

The following elementary estimate along with Lemma 3.3 provides the desired bounds on the jump kernel of subordinated process.

**Proposition 3.4.** *Let  $(M, d, \mu)$  be an unbounded, complete, locally compact, separable metric measure space. Let  $(\mathcal{E}^c, \mathcal{F}^c)$  be a regular, strongly local Dirichlet form on  $L^2(M, \mu)$  that satisfies  $\text{PHI}(\psi_c)$ , where  $\psi_c$  is a scale function. Let  $(\mathcal{E}^j, \mathcal{F}^j)$  be a pure jump type Dirichlet form such that the corresponding jumping measure satisfies  $\mathbf{J}(\psi_j)_\geq$ , where  $\psi_j$  is a scale function. Then*

$$\int_0^1 \frac{1}{\psi_j \circ \psi_c^{-1}(t)} dt < \infty.$$

*Proof.* Assume to the contrary that

$$\int_0^1 \frac{1}{\psi_j \circ \psi_c^{-1}(t)} dt = \infty. \quad (3.2)$$



Since  $t \mapsto \psi_j \circ \psi_c^{-1}(t)$  is a continuous positive function on  $(0, \infty)$ , we have

$$\int_0^{t_0} \frac{1}{\psi_j \circ \psi_c^{-1}(t)} dt = \infty, \tag{3.3}$$

for any  $t_0 > 0$ . Let  $(P_t^c)_{t>0}$  denote the Markov semigroup corresponding to the Dirichlet form  $(\mathcal{E}^c, \mathcal{F}^c)$  on  $L^2(M, \mu)$  and let  $p_t^c(\cdot, \cdot)$  denote the corresponding heat kernel (which exists by Theorem 2.2). Note that

$$\frac{1}{t} \langle f - P_t^c f, f \rangle = \frac{1}{2t} \int_M \int_M p_t^c(x, y) (f(x) - f(y))^2 \mu(dx) \mu(dy). \tag{3.4}$$

Therefore by (3.1) of Lemma 3.1 and (3.4), there exists  $C_1 > 0$  such that

$$C_1^{-1} \mathcal{E}^j(f, f) \leq \int_0^\infty \frac{1}{t \psi_j \circ \psi_c^{-1}(t)} \langle f - P_t^c f, f \rangle dt \leq C_1 \mathcal{E}^j(f, f) \quad \text{for all } f \in \mathcal{F}^j. \tag{3.5}$$

Let  $f \in \mathcal{F}^j$ . Choose  $N, \varepsilon \in (0, \infty)$ . We define  $\mathcal{E}^c(f, f)$  for any  $f \in L^2(M, \mu)$  by setting  $\mathcal{E}^c(f, f) = \infty$  whenever  $f \in L^2(M, \mu) \setminus \mathcal{F}^c$ . Let  $(P_t^c)_{t>0}$  denote the Markov semigroup corresponding to the Dirichlet form  $(\mathcal{E}^c, \mathcal{F}^c)$  on  $L^2(M, \mu)$ . By (2.3), there exists  $t_0 \in (0, 1)$  (depending on  $f, N, \varepsilon$ ) such that for any  $t \in (0, t_0)$ ,

$$\frac{1}{t} \langle f - P_t^c f, f \rangle \geq (N \wedge \mathcal{E}^c(f, f)) - \varepsilon. \tag{3.6}$$

Combining (3.5) and (3.6),

$$\infty > \mathcal{E}^j(f, f) \tag{3.7}$$

$$\geq C_1^{-1} \int_0^{t_0} \frac{1}{t \psi_j \circ \psi_c^{-1}(t)} \langle f - P_t^c f, f \rangle dt \tag{3.8}$$

$$\geq [(N \wedge \mathcal{E}^c(f, f)) - \varepsilon] \int_0^{t_0} \frac{1}{\psi_j \circ \psi_c^{-1}(t)} dt \tag{3.9}$$

Combining (3.3) and (3.9), we obtain that  $N \wedge \mathcal{E}^c(f, f) \leq \varepsilon$ . By letting  $\varepsilon \rightarrow 0$ , and using (2.1), (2.2), we obtain that

$$\mathcal{E}^c(f, f) = 0 \quad \text{for all } f \in \mathcal{F}^j. \tag{3.10}$$

By the Poincaré inequality (Lemma 3.2), this implies that any  $f \in \mathcal{F}^j$  is constant  $\mu$ -almost everywhere on every ball  $B(x, r)$ . In particular, every function in  $\mathcal{F}^j$  is constant  $\mu$ -almost everywhere. This implies that  $\mathcal{F}^j$  is not dense in  $L^2(M, \mu)$ , contradicting the assumption that  $(\mathcal{E}^j, \mathcal{F}^j)$  is a Dirichlet form.  $\square$

The following result is an elementary consequence of change of variables formula.

**Lemma 3.5.** *Let  $\psi_c, \psi_j$  be scale functions. Then  $\int_0^1 \frac{\psi_c(s)}{s \psi_j(s)} ds < \infty$  is equivalent to  $\int_0^1 \frac{1}{\psi_j \circ \psi_c^{-1}(t)} dt < \infty$ .*

*Proof.* It is easy to verify using (2.7) that  $\psi_c(t)$  is comparable to the function

$$t \mapsto \int_0^t \frac{\psi_c(r)}{r} dr.$$

Therefore, we assume without loss of generality that  $\psi_c$  is continuously differentiable and

$$\psi_c'(r) \asymp \frac{\psi_c(r)}{r} \tag{3.11}$$

for all  $r > 0$ . Using (3.11) and substituting  $t = \psi_c(s)$  in the integral  $\int_0^1 \frac{1}{\psi_j \circ \psi_c^{-1}(t)} dt$ , we obtain the desired equivalence.  $\square$

Next, we complete the proofs of the main results.

*Proof of Theorem 2.3.* The parabolic Harnack inequality implies that  $(M, d)$  is connected by Grigor'yan and Hu (2014, Proposition 5.6), Barlow et al. (2020, Theorem 5.4). By Heinonen (2001, Exercise 13.1),  $(M, d)$  satisfies RVD.

That (b) implies (a) is obvious.

Next, we show that (c) implies (b). By Lemma 3.5, the measure  $\nu(t) := \frac{1}{t\psi_j \circ \psi_c^{-1}(t)} dt$  is a Lévy measure of subordinator. Let  $S$  denote the subordinator corresponding to the Lévy measure  $\nu$ . By Grigor'yan et al. (2015, Theorems 1.2 and 1.3) and Theorem 2.2, the Markov semigroup corresponding to  $(\mathcal{E}^c, \mathcal{F}^c)$  is conservative. By Lemmas 3.3 and 3.1, the subordinated diffusion process  $X \circ S$  is a  $\mu$ -symmetric pure jump process that satisfies  $J(\psi_j)$ .

Finally, (a) implies (c), following from Proposition 3.4 and Lemma 3.5.  $\square$

*Proof of Corollary 2.5.*

(a) By Theorem 2.3, we obtain (2.17). Therefore, by Bae et al. (2019a, Theorem 2.19 and Lemma 4.5), Lemma 3.5, along with the characterization of parabolic Harnack inequality in Chen et al. (2020, Theorem 1.18), we obtain (a).

(b) The estimate (2.19) follows from

$$\int_0^r \frac{\psi_c(s)}{s\psi_j(s)} ds \geq \int_{r/2}^r \frac{\psi_c(s)}{s\psi_j(s)} ds \geq \frac{\psi_c(r/2)}{2\psi_j(r)} \gtrsim \frac{\psi_c(r)}{\psi_j(r)}.$$

Using the above estimate

$$\frac{\psi_c(r)}{\psi_j(r)} \lesssim \int_{r/2}^r \frac{\psi_c(s)}{s\psi_j(s)} ds \leq \int_0^1 \frac{\psi_c(s)}{s\psi_j(s)} ds$$

for any  $r \leq 1$ . This along with (2.17) in Theorem 2.3 yields (2.20). By part (a), for any  $0 < r \leq R$ , we obtain

$$\frac{\widehat{\psi}_j(R) \psi_c(r)}{\widehat{\psi}_j(r) \psi_c(R)} \lesssim \frac{\int_0^r \frac{\psi_c(s)}{s\psi_j(s)} ds}{\int_0^R \frac{\psi_c(s)}{s\psi_j(s)} ds} \lesssim 1.$$

This implies (2.21). The estimates in (2.22) follow from (2.21) by substituting  $r = 1$  and  $R = 1$  respectively.  $\square$

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