ALEA, Lat. Am. J. Probab. Math. Stat. 20, 1359–1366 (2023)
DOI: 10.30757/ALEA.v20-50

Planar Brownian motion winds evenly along its trajectory

Isao Sauzedde

Department of Statistics, University of Warwick
E-mail address: isao.sauzedde@warwick.ac.uk
URL: http://perso.ens-lyon.fr/isao.sauzedde/index_en.html

Abstract. Let $\mathcal{D}_N$ be the set of points around which a planar Brownian motion winds at least N times. We prove that the random measure on the plane with density $2\pi N \mathbb{1}_{\mathcal{D}_N}$ with respect to the Lebesgue measure converges almost surely weakly, as N tends to infinity, towards the occupation measure of the Brownian motion.

1. Introduction

Let $X : [0,1] \to \mathbb{R}^2$ be a planar Brownian motion started from 0. Let $\bar{X}$ be the oriented loop obtained by concatenating X with the straight line segment joining X_1 to X_0.

For each point z in $\mathbb{R}^2$ outside the range of $\bar{X}$, let $\theta(z)$ be the number of times $\bar{X}$ winds around z. For z on the range of $\bar{X}$, we set $\theta(z) = 0$. Define

$$\mathcal{D}_N = \{z \in \mathbb{R}^2 : \theta(z) \geq N\}.$$

The Lebesgue measure $|\mathcal{D}_N|$ of this set is known to be of the order of $\frac{1}{2\pi N}$. More precisely, Werner proved in Werner (1994) that the following convergence holds:

$$2\pi N |\mathcal{D}_N| \xrightarrow[N\to\infty]{L^2} 1. \tag{1.1}$$

For all $N \geq 1$, we denote by μ_N the random measure on the plane with density $2\pi N \mathbb{1}_{\mathcal{D}_N}$ with respect to the Lebesgue measure:

$$\mathrm{d}\mu_N(z) = 2\pi N \mathbb{1}_{\mathcal{D}_N}(z)\, \mathrm{d}z.$$

Let ν be the occupation measure of X, defined as the push-forward of the Lebesgue measure on $[0,1]$ by X. In other words, ν is the random Borel probability measure on the plane characterised by the fact that for every continuous test function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\int_{\mathbb{R}^2} f \,\mathrm{d}\nu = \int_0^1 f(X_t)\, \mathrm{d}t.$$

Received by the editors November 26th, 2021; accepted September 21st, 2023.
2010 *Mathematics Subject Classification.* 60J65,60J55, 60F15.
Key words and phrases. Brownian motion, Brownian winding, local time.
This work has been supported by the École Normale Supérieure de Lyon and the Laboratoire de Probabilités, Statistique et Modélisation and Sorbonne Université during the writing process, and by the ERC Advanced Grant 740900 (LogCorRM) and the EPSRC grant EP/W006227/1 during the editorial process.

The main result of this paper is the following.

Theorem 1.1. *Almost surely,* $\mu_N \underset{N\to\infty}{\Longrightarrow} \nu$.

To be clear, we mean that almost surely, for all bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, the following convergence holds:

$$\lim_{N\to\infty} 2\pi N \int_{\mathbb{R}^2} f(z)\mathbb{1}_{[N,+\infty)}(\theta(z))\, \mathrm{d}z = \int_0^1 f(X_u)\, \mathrm{d}u.$$

The assumption that the test function is bounded is not essential, because almost surely, the supports of the measures μ_N, $N \geq 1$ and ν are contained in the convex hull of the range of X, which is compact.

In the course of the proof, we will obtain an estimation of the rate of convergence in terms of the modulus of continuity of the test function f (see Lemma 3.1).

The study of the windings of the planar Brownian motion has a long history. The first investigations were mostly concerned with the winding around a fixed point, the most prominent example being the celebrated Spitzer theorem Spitzer (1958). There followed among other works a computation by Yor of the exact law of the winding Mansuy and Yor (2008); Yor (1980), as well as many fine asymptotic results concerning related functionals (see for example Shi (1998) and references therein).

In Werner (1994, 1995), Werner shifted the attention from the winding around a point to the winding as a *function*, as well as to the set of points with a given winding number. He established, for instance, in Werner (1994), the convergence (1.1). His results suggest in particular that when N is large, the set $\mathcal{D}_N$, which is located near the trajectory X, is distributed very equally along the trajectory, with each part of the trajectory carrying a portion of $\mathcal{D}_n$ proportional to the length of the interval parameterising this part. Our main result gives a rigorous statement of this informal idea.

The proof uses some results that we obtained in our previous work Sauzedde (2022) on this subject, and which we recall briefly in the next section for the convenience of the reader.

2. Prior results

The Brownian motion X is defined under a probability that we denote by $\mathbb{P}$.

Let T be a positive integer. For all $i \in \{1, \dots, T\}$, let X^i be the restriction of X to the interval $[\frac{i-1}{T}, \frac{i}{T}]$. As we did for X, let us denote by $\bar{X}^i$ the concatenation of X^i with a straight line segment from $X_{\frac{i}{T}}$ to $X_{\frac{i-1}{T}}$, and by θ^i the winding function of the loop $\bar{X}^i$, taken to be 0 on the trajectory. We then set, for all $N \geq 1$,

$$\mathcal{D}^i_N = \{z \in \mathbb{R}^2 : \theta^i(z) \geq N\} \text{ and } \mathcal{D}^{i,j}_N = \{z \in \mathbb{R}^2 : |\theta^i(z)| \geq N, |\theta^j(z)| \geq N\},$$

with absolute values intended in the second definition.

Our proof of Theorem 1.1 relies on the following lemmas, which are mild reformulations of results that we proved in Sauzedde (2022) (Equation (28), Theorem 1.5 and Lemma 2.4 in Sauzedde (2022)).

Lemma 2.1. *Let μ be a Borel measure on $\mathbb{R}^2$, absolutely continuous with respect to the Lebesgue measure. For all positive integers N, T, M such that $T(M+1) < N$,*

$$\sum_{i=1}^{T} \mu\big(\mathcal{D}^i_{N+T+M(T-1)}\big) - \sum_{1\leq i<j\leq T} \mu\big(\mathcal{D}^{i,j}_M\big) \leq \mu(\mathcal{D}_N) \leq \sum_{i=1}^{T} \mu\big(\mathcal{D}^i_{N-T-M(T-1)}\big) + \sum_{1\leq i<j\leq T} \mu\big(\mathcal{D}^{i,j}_M\big).$$

Lemma 2.2. *For all $\delta < \frac{1}{2}$ and $p \geq 2$, there exists $C > 0$ such that for all $N \geq 1$ and all $R > 0$,*

$$\mathbb{P}\Big(N^\delta \big|2\pi N|\mathcal{D}_N| - 1\big| \geq R\Big) \leq CR^{-p}.$$

Lemma 2.3. *For all $\epsilon > 0$, there exists $C > 0$ such that for all positive integers T, M,*

$$\mathbb{E}\Big[\Big(\sum_{1\leq i<j\leq T} |\mathcal{D}_M^{i,j}|\Big)^2\Big] \leq CM^{-4+\epsilon}T^{1+\epsilon}.$$

3. Proof of the theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded continuous function. Let ω_f be the modulus of continuity of f: for all $t \geq 0$,

$$\omega_f(t) = \sup\{|f(z) - f(w)| : z, w \in \mathbb{R}^2, \|z - w\| \leq t\} \in [0, +\infty].$$

For all Borel subset E of $\mathbb{R}^2$, we also set $f(E) = \int_E f(z)\,\mathrm{d}z$.

For $\alpha \in (0, \frac{1}{2})$, let $\|X\|_{\mathcal{C}^\alpha}$ denote the α-Hölder norm of the Brownian motion:

$$\|X\|_{\mathcal{C}^\alpha} = \sup_{0\leq s<t\leq 1} \frac{\|X_t - X_s\|}{|t - s|^\alpha}.$$

We have the following quantitative estimation.

Lemma 3.1. *For all $t \in (0, \frac{2}{5})$ and $\alpha \in (0, \frac{1}{2})$, there exists $\eta > 0$ such that almost surely, there exists a constant C such that for all bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ and all $N \geq 1$,*

$$\Big|2\pi N f(\mathcal{D}_N) - \int_0^1 f(X_u)\,\mathrm{d}u\Big| \leq C\big(\omega_f(2\|X\|_{\mathcal{C}^\alpha} N^{-\alpha t}) + \|f\|_\infty N^{-\eta}\big).$$

Let us explain why this lemma directly implies Theorem 1.1.

Proof of Theorem 1.1 assuming Lemma 3.1: Thanks to the Portmanteau theorem, is suffices to show that $\mathbb{P}$-almost surely, for any bounded Lipschitz continuous function f,

$$\Big|2\pi N f(\mathcal{D}_N) - \int_0^1 f(X_u)\,\mathrm{d}u\Big| \underset{N\to+\infty}{\longrightarrow} 0.$$

For such a function f, one has $\omega_f(t) \leq \|f\|_{\mathrm{Lip}}\, t$ and the result follows from Lemma 3.1 applied for instance to $t = \frac{1}{5}$ and $\alpha = \frac{1}{4}$. □

In order to prove Lemma 3.1, we introduce the following subset of $\mathbb{N}$, which depends on a positive real parameter $\gamma > 1$:

$$\mathbb{N}^\gamma = \{\lfloor K^\gamma \rfloor : K \in \mathbb{N}\} \setminus \{0\}.$$

Let us fix two positive real parameters t and m with $m + t < 1$ and set, for all $N \geq 1$, $T = \lfloor N^t \rfloor$ and $M = \lfloor N^m \rfloor$. We advise the reader to think of m as being larger than $\frac{1}{2}$, and of t as a small number. Precise conditions can be found in the statement of Lemma 3.3.

We also set $N' = \max\{n \in \mathbb{N}^\gamma : n \leq N - T - M(T - 1)\}$, which is well defined when N is large enough. The difference between N and N' is $O(N^{1-1/\gamma} + N^{m+t})$.

We also define the following events, which depend on t and m, and also on other positive real parameters s, ζ, δ:

$$\begin{aligned}
E_N &= \big\{\forall i \in \{1, \dots, T\},\ N'^\delta \,\big|2\pi N'|\mathcal{D}_{N'}^i| - \tfrac{1}{T}\big| \leq T^{-\frac{1}{2}+\frac{s}{t}}\big\},\\
F_N &= \Big\{\sum_{1\leq i<j\leq T} |\mathcal{D}_M^{i,j}| \leq N^{-1-\zeta}\Big\},\\
G_N &= \big\{\forall i \in \{1, \dots, T\},\ 2\pi N|\mathcal{D}_{N'}^i| \leq \tfrac{2}{T}\big\}.
\end{aligned}$$

The proof goes in three steps. In the first (Lemma 3.2), we show that with an appropriate choice of γ, almost surely, the events E_N, F_N and G_N are realised for all $N \in \mathbb{N}^\gamma$ large enough. In a second step (Lemma 3.3), we show that on this almost sure event, for every bounded continuous function, and for all $N \in \mathbb{N}^\gamma$, the conclusion of Lemma 3.1 holds. In the third step, we show that the conclusion holds not only for $N \in \mathbb{N}^\gamma$, but for all $N \in \mathbb{N}$.

Let us collect in one place the assumptions that we make on the parameters that we introduced. These assumptions are organised in such a way that if enforced in the natural reading order, they are always satisfiable.

$$0 < \alpha < \tfrac{1}{2}\ ,\ 0 < t < \tfrac{2}{5}\ ,\ \begin{array}{c} \tfrac{1}{2} + \tfrac{t}{4} < m < 1 - t\ ,\ 0 < \zeta < 2m - 1 - \tfrac{t}{2}, \\ 0 < s < \tfrac{1}{2} - \tfrac{t}{2}\ ,\ \tfrac{t}{2} + s < \delta < \tfrac{1}{2}, \end{array} \quad \gamma > \max\left(\tfrac{1}{2s}, \tfrac{1}{4m-t-2-2\zeta}\right). \tag{A}$$

From now on, we always assume that these assumptions are satisfied.

Lemma 3.2. *The event* $\bigcup_{N_0 \geq 1} \bigcap_{\substack{N \in \mathbb{N}^\gamma \\ N \geq N_0}} (E_N \cap F_N \cap G_N)$ *has probability* 1.

Proof: The scaling properties of the Brownian motion imply that $|\mathcal{D}^i_{N'}|$ is equal in distribution to $T^{-1}|\mathcal{D}_{N'}|$. Thus,

$$1 - \mathbb{P}(E_N) \leq T\mathbb{P}({N'}^\delta \big| 2\pi N' |\mathcal{D}_{N'}| - 1 \big| \geq T^{\frac{1}{2} + \frac{s}{t}}).$$

Using Lemma 2.2 with $p = 2$ gives

$$1 - \mathbb{P}(E_N) \leq CT^{-\frac{2s}{t}},$$

and for N large enough, this quantity is smaller than $2CN^{-2s}$. In particular,

$$\sum_{N \in \mathbb{N}^\gamma} \big(1 - \mathbb{P}(E_N)\big) \leq 2C \sum_{K=1}^{+\infty} K^{-2s\gamma}.$$

Besides, by Markov inequality,

$$1 - \mathbb{P}(F_N) \leq N^{2+2\zeta}\, \mathbb{E}\Big[\Big(\sum_{1 \leq i < j \leq T} |\mathcal{D}^{i,j}_M|\Big)^2\Big].$$

By Lemma 2.3, for any $\epsilon > 0$, there exists C such that for all N,

$$1 - \mathbb{P}(F_N) \leq CN^{-4m+t+2+2\zeta+\epsilon}.$$

In particular,

$$\sum_{N \in \mathbb{N}^\gamma} \big(1 - \mathbb{P}(F_N)\big) \leq C \sum_{K=1}^{+\infty} K^{\gamma(-4m+t+2+2\zeta+\epsilon)}.$$

We assumed that $\gamma > \frac{1}{4m-t-2-2\zeta}$, so that there exists $\epsilon > 0$ such that $\gamma > \frac{1}{4m-t-2-2\zeta-\epsilon}$. Since we also assumed that $\gamma > \frac{1}{2s}$, the series

$$\sum_{K=1}^{+\infty} K^{-\gamma(4m-t-2-2\zeta-\epsilon)} \qquad \text{and} \qquad \sum_{K=1}^{+\infty} K^{-\gamma(2s)}$$

are both convergent.

Using Borel–Cantelli lemma, we conclude the proof, but for the presence of G_N. However, using the fact that N' is not larger than N and equivalent to N as N tends to infinity, and the inequality $T \leq N^t$, one verifies that if $t + 2s < 2\delta$, then for N large enough, the inclusion $E_N \subset G_N$ holds. Hence, the proof is complete. □

We now turn to the second step of the proof.

Lemma 3.3. *Almost surely, there exists a constant C such that for all $N \in \mathbb{N}^\gamma$ and all bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}$,*

$$\left|2\pi N f(\mathcal{D}_N) - \int_0^1 f(X_u)\,\mathrm{d}u\right| \\ \leq C\Big(\omega_f\big(\|X\|_{\mathcal{C}^\alpha} T^{-\alpha}\big) + \|f\|_\infty (N^{-1+m+t} + N^{-\frac{1}{\gamma}+1} + N^{-\delta+\frac{t}{2}+s} + N^{-\zeta})\Big).$$

Proof: We first assume that f is non-negative. Replacing C if necessary by a larger constant, it suffices to show the inequality for $N \geq N_0$, for a possibly random N_0 which does not depend on f. Using Lemma 3.2, we can thus assume that the event $E_N \cap F_N \cap G_N$ holds.

Using Lemma 2.1 to the measure $\mu = f\,\mathrm{d}z$ (which is not signed, since f is non-negative), and the fact that the sequence $(\mathcal{D}^i_N)_{N\geq 1}$ is non-increasing, we have

$$\begin{aligned} Nf(\mathcal{D}_N) &\leq \sum_{i=1}^T Nf(\mathcal{D}^i_{N-T-M(T-1)}) + \sum_{1\leq i<j\leq T} Nf(\mathcal{D}^{i,j}_M) \\ &\leq \sum_{i=1}^T Nf(\mathcal{D}^i_{N'}) + \sum_{1\leq i<j\leq T} Nf(\mathcal{D}^{i,j}_M). \end{aligned} \tag{3.1}$$

Besides, $\mathcal{D}^i_{N'}$ is contained in the convex hull of the trajectory of X between the times $\frac{i}{T}$ and $\frac{i+1}{T}$, hence in the ball of center $X_{\frac{i}{T}}$ and radius $\|X\|_{\mathcal{C}^\alpha} T^{-\alpha}$, so that

$$Nf(\mathcal{D}^i_{N'}) \leq N|\mathcal{D}^i_{N'}| f(X_{\frac{i}{T}}) + N|\mathcal{D}^i_{N'}|\omega_f(\|X\|_{\mathcal{C}^\alpha} T^{-\alpha}).$$

We replace in (3.1) and force the apparition of a Riemann sum by decomposing $N|\mathcal{D}^i_{N'}|$ into

$$\frac{1}{2\pi T} + \frac{N-N'}{2\pi T N'} + N\left(|\mathcal{D}^i_{N'}| - \tfrac{1}{2\pi T N'}\right).$$

We obtain

$$\begin{aligned} \sum_{i=1}^T Nf(\mathcal{D}^i_{N'}) \leq \sum_{i=1}^T \tfrac{1}{2\pi T} f(X_{\frac{i}{T}}) + \sum_{i=1}^T \tfrac{N-N'}{2\pi T N'} f(X_{\frac{i}{T}}) + N\sum_{i=1}^T \left(|\mathcal{D}^i_{N'}| - \tfrac{1}{2\pi T N'}\right) f(X_{\frac{i}{T}}) \\ + N\sum_{i=1}^T |\mathcal{D}^i_{N'}|\omega_f(\|X\|_{\mathcal{C}^\alpha} T^{-\alpha}). \end{aligned}$$

Comparing the Riemann sum with the integral and f to its upper bound, we turn this inequality into

$$\begin{aligned} 2\pi \sum_{i=1}^T Nf(\mathcal{D}^i_{N'}) \leq \int_0^1 f(X_u)\,\mathrm{d}u + \omega_f(\|X\|_{\mathcal{C}^\alpha} T^{-\alpha}) + \|f\|_\infty \tfrac{N-N'}{N'} \\ + \|f\|_\infty N \sum_{i=1}^T \left(2\pi|\mathcal{D}^i_{N'}| - \tfrac{1}{TN'}\right) + 2\pi\omega_f(\|X\|_{\mathcal{C}^\alpha} T^{-\alpha}) N \sum_{i=1}^T |\mathcal{D}^i_{N'}|. \end{aligned}$$

Our next goal is to bound the last three terms of the right-hand side. Let us discuss the first, then the third and finally the second.

For the first term, it follows from the definition of N' and by elementary arguments that for N large enough, indeed larger than a certain N_1 that does not depend on f,

$$\frac{N-N'}{N'} < 2(N^{m+t-1} + \gamma N^{-\frac{1}{\gamma}+1}).$$

For the third term, since the event G_N holds, we have

$$\sum_{i=1}^{T} |\mathcal{D}^i_{N'}| \le T \max_{i\in\{1,\dots,T\}} |\mathcal{D}^i_{N'}| \le \frac{1}{\pi N}.$$

Finally, since the event E_N holds, and for N large enough,

$$\sum_{i=1}^{T} \left(2\pi|\mathcal{D}^i_{N'}| - \tfrac{1}{TN'}\right) \le N'^{-1-\delta}T^{\frac{1}{2}+\frac{s}{t}} \le 2N^{-1-\delta+\frac{t}{2}+s}.$$

Here the second inequality holds for N larger than a certain N_2 which does not depend on f.

We end up with

$$\begin{aligned}2\pi \sum_{i=1}^{T} N f(\mathcal{D}^i_{N'}) - \int_0^1 f(X_u)\,\mathrm{d}u& \\ &\le 3\omega_f(\|X\|_{\mathcal{C}^\alpha}T^{-\alpha}) + 2\|f\|_\infty(N^{m+t-1}+\gamma N^{-\frac{1}{\gamma}+1}+N^{-\delta+\frac{t}{2}+s}).\end{aligned} \tag{3.2}$$

We now turn to the second term of the right-hand side of (3.1). Since F_N holds,

$$N \sum_{1\le i<j\le T} f(\mathcal{D}^{i,j}_M) \le N\|f\|_\infty \sum_{1\le i<j\le T} |\mathcal{D}^{i,j}_M| \le \|f\|_\infty N^{-\zeta}. \tag{3.3}$$

Using (3.1), (3.2) and (3.3), we get that almost surely, for $N \ge \max(N_0, N_1, N_2)$,

$$\begin{aligned}2\pi N f(\mathcal{D}_N) - \int_0^1 f(X_u)\,\mathrm{d}u& \\ &\le 3\omega_f(\|X\|_{\mathcal{C}^\alpha}T^{-\alpha}) + 2\|f\|_\infty(N^{m+t-1} + \gamma N^{-\frac{1}{\gamma}+1} + N^{-\delta+\frac{t}{2}+s} + N^{-\zeta}).\end{aligned} \tag{3.4}$$

To obtain this upper bound, we used the second inequality of Lemma 2.1, and the definition of N' which was suggested by the term $N - T - M(T-1)$ that appears in it. A repetition of the exact same arguments, with the difference that N' is now defined as the largest element of $\mathbb{N}^\gamma$ smaller than $N + T + M(T-1)$, and using the first inequality of Lemma 2.1 instead of the second, yields the corresponding lower bound, saying that the left-hand side of (3.4) is larger than the opposite of the right-hand side of (3.4).

This concludes the proof when f is non-negative. To remove this assumption, it suffices to decompose f into the sum of its positive and negative parts. □

We now extend Lemma 3.3 from $N \in \mathbb{N}^\gamma$ to $N \in \mathbb{N}^*$, in order to obtain Lemma 3.1.

Proof of Lemma 3.1: The reals t and α being given, choose positive real numbers $s, \zeta, m, \delta, \gamma$ which satisfy the assumptions (A). Set $\eta = \min(1-m-t, \frac{1}{\gamma} - 1, \delta - \frac{t}{2} - s, \zeta) > 0$.

Let us first assume f is non-negative. Set $\tilde{N} = \max\{n \in \mathbb{N}^\gamma : n \le N\}$, the largest integer smaller than N in $\mathbb{N}^\gamma$.

Since the sequence $(f(\mathcal{D}_N))_{N\ge 1}$ is non-increasing, we have

$$\begin{aligned}2\pi N f(\mathcal{D}_N) - \int_0^1 f(X_u)\,\mathrm{d}u &\le 2\pi N f(\mathcal{D}_{\tilde{N}}) - \int_0^1 f(X_u)\,\mathrm{d}u \\ &= \frac{N}{\tilde{N}}\Big(2\pi\tilde{N} f(\mathcal{D}_{\tilde{N}}) - \int_0^1 f(X_u)\,\mathrm{d}u\Big) + \Big(\frac{N}{\tilde{N}} - 1\Big)\int_0^1 f(X_u)\,\mathrm{d}u.\end{aligned}$$

The first term is taken care of by Lemma 3.3 and the fact that $N \le 2\tilde{N}$ for N large enough. The second term is bounded above, for N sufficiently large, by $2\gamma\|f\|_\infty N^{-\frac{1}{\gamma}+1}$. Altogether, we find the upper bound

$$2\pi N f(\mathcal{D}_N) - \int_0^1 f(X_u)\,\mathrm{d}u \le C\big(\omega_f(\|X\|_{\mathcal{C}^\alpha} T^{-\alpha}) + \|f\|_\infty N^{-\eta}\big)$$

for some constant C. The corresponding lower bound is obtained by the same argument with $\tilde{N}$ defined as $\min\{n \in \mathbb{N}^\gamma : n \ge N\}$. This concludes the proof when f is non-negative. For the general case, we simply decompose f into its positive and negative parts. This concludes the proof of Lemma 3.1, and also the proof of Theorem 1.1. □

4. Further perspectives

It is possible that a similar result also holds when we consider the joint windings of independent Brownian motions. To be more specific, for two independent planar Brownian motions X, X', we can define their intersection measure ℓ, which is carried by the plane (see Geman et al. (1984)).

One possible way to approximate the mass of this measure is to look at the Lebesgue measure of the intersection of Wiener sausages with small radius ϵ around X and X'. In Le Gall (1986) (and also in Le Gall (1992)), it is shown that $\ell(\mathbb{R}^2)$ can be obtained as the properly normalized limit of these measures as $\epsilon \to 0$.

For two independent planar Brownian motions X, X', define

$$\mathcal{D}_N^{(2)} = \{z \in \mathbb{R}^2 : \theta_X(z) \ge N, \theta_{X'}(z) \ge N\}.$$

Conjecture 4.1. *There exists a constant C which depends only $\|X_0 - X_0'\|$ and such that $CN^2|\mathcal{D}_N^{(2)}|$ converges, as $N \to \infty$, towards $\ell(\mathbb{R}^2)$. The converges holds both in L^p for any $p \in [1, +\infty)$ and almost surely.*

Besides, almost surely, the measure $CN^2 \mathbb{1}_{\mathcal{D}_N^{(2)}}\,\mathrm{d}z$ converges weakly towards ℓ.

For such a result to hold, it is necessary that the exponent of N is equal to 2. Nonetheless, we cannot exclude that some logarithmic corrections should be added.

Acknowledgements

I would like to thank Thierry Lévy for his outstanding help during the writing process, as well as the anonymous referee for their comments in a preliminary version.

References

Geman, D., Horowitz, J., and Rosen, J. A local time analysis of intersections of Brownian paths in the plane. *Ann. Probab.*, **12** (1), 86–107 (1984). MR723731.

Le Gall, J.-F. Sur la saucisse de Wiener et les points multiples du mouvement brownien. *Ann. Probab.*, **14** (4), 1219–1244 (1986). MR866344.

Le Gall, J.-F. Some properties of planar Brownian motion. In *École d'Été de Probabilités de Saint-Flour XX—1990*, volume 1527 of *Lecture Notes in Math.*, pp. 111–235. Springer, Berlin (1992). ISBN 3-540-56250-8. MR1229519.

Mansuy, R. and Yor, M. *Aspects of Brownian motion.* Universitext. Springer-Verlag, Berlin (2008). ISBN 978-3-540-22347-4. MR2454984.

Sauzedde, I. Lévy area without approximation. *Ann. Inst. Henri Poincaré Probab. Stat.*, **58** (4), 2165–2200 (2022). MR4492975.

Shi, Z. Windings of Brownian motion and random walks in the plane. *Ann. Probab.*, **26** (1), 112–131 (1998). MR1617043.

Spitzer, F. Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.*, **87**, 187–197 (1958). MR104296.

Werner, W. Sur les points autour desquels le mouvement brownien plan tourne beaucoup. *Probab. Theory Related Fields*, **99** (1), 111–144 (1994). MR1273744.

Werner, W. Formule de Green, lacet brownien plan et aire de Lévy. *Stochastic Process. Appl.*, **57** (2), 225–245 (1995). MR1334970.

Yor, M. Loi de l'indice du lacet brownien, et distribution de Hartman-Watson. *Z. Wahrsch. Verw. Gebiete*, **53** (1), 71–95 (1980). MR0576898.