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Scaling limit of the collision measures of multiple random walks

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Abstract. For an integer $k \geq 2$, let $S^{(1)}, S^{(2)}, \ldots, S^{(k)}$ be k independent simple symmetric random walks on $\mathbb{Z}$. A pair (n, z) is called a collision event if there are at least two distinct random walks, namely, $S^{(i)}, S^{(j)}$ satisfying $S^{(i)}_n = S^{(j)}_n = z$. We show that under the same scaling as in Donsker's theorem, the sequence of random measures representing these collision events converges to a non-trivial random measure on $[0, 1] \times \mathbb{R}$. Moreover, the limit random measure can be characterized using Wiener chaos. The proof is inspired by methods from statistical mechanics, especially, by a partition function that has been developed for the study of directed polymers in random environments.

1. Introduction

For an integer $k \geq 2$, let $S^{(1)}, S^{(2)}, \ldots, S^{(k)}$ be k independent simple symmetric random walks (SSRWs) on $\mathbb{Z}$, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ (see Révész, 2005, p.3). A pair $(n, z) \in \mathbb{N} \times \mathbb{Z}$ is called *a collision event* if there are at least two random walks that collide (occupy the same position at the same time) at the time n and the location z (see Figure 1.1), and n is then called a *collision time*.

First mentioned in Pólya's note (Pólya, 1984), the collision of random walks has since then been a classic topic in probability theory. Recently, this topic has gained more attention from researchers working on the random walks on graphs Barlow et al. (2012); Hutchcroft and Peres (2015) and random environments Avena et al. (2018); Chen (2016); Halberstam and Hutchcroft (2022).

When consider only two random walks, collision problems are strongly related to Brownian local time Knight (1963); Révész (1981); Szabados and Székely (2005). The convergence of collision times can be achieved by coupling a new SSRW, the difference between two given random walks, with a Brownian motion using Skorokhod's embedding Révész (2005, p.52), Durrett (2010, Theorem 8.6.1). However, these methods cannot be easily adapted to give convergence results for the collisions of

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more than two SSRWs because the couplings rely heavily on the choices of stopping times which are proper for each SSRW. To the best of our knowledge, scaling limit results for collisions of $k > 2$ random walks are still limited. Moreover, the limiting behaviour of collision positions are rarely researched.

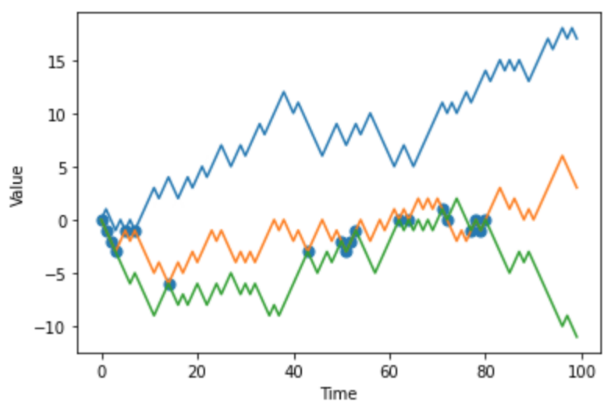
The main contribution of this paper is to consider the position of collision events, contrary to previous studies that focus only on their numbers (e.g. Bertini and Cancrini (1995), or Lygkonis and Zygouras (2023, 2022)).

We investigate a relatively uncommon aspect of random walk collisions, concerning their duality with the partition function of a directed polymer model in statistical mechanics Carmona and Hu (2002); Comets (2017). By following the ideas developped in Carmona and Hu (2002), and Alberts et al. (2014), we obtain new results on when and where the collisions of these random walks occur after long observation, or more precisely, on the scaling limit of the empirical measures of the collision events.

It is worth mentioning that independently with our research, Lygkonis and Zygouras have used similar ideas to extend the well-known Erdös-Taylor theorem on the number of collisions of 2-dimensional random walks, Lygkonis and Zygouras (2023, 2022).

Of related interest, in the continuous setting, it has been proven that the distribution of coincidences of Brownian motions has a close relation to the KPZ equation (Krajenbrink et al., 2019).

FIGURE 1.1. An example of collision events when $k = 3, N = 100$. The collision events are represented by blue dots.



Our study objects are as follows:

Definition 1.1. For each $N \in \mathbb{N}$, we define the collision measures of k random walks $S^{(1)}, S^{(2)}, \dots,$ $S^{(k)}$ until time N to be:

$$\Pi_N := \sum_{n=1}^{N} \sum_{z \in \mathbb{Z}} \sum_{\substack{1 \le i < j \le k: \\ S_n^{(i)} = S_n^{(j)} = z}} \delta_{\left(\frac{n}{N}, \frac{z}{\sqrt{N}}\right)}, \text{ and}$$

$$\Pi'_N := \sum_{n=1}^{N} \sum_{z \in \mathbb{Z}} \delta_{\left(\frac{n}{N}, \frac{z}{\sqrt{N}}\right)} \mathbb{1}_{\{(n,z) \text{ is a collision event}\}},$$

where δ is the Dirac measure.

For each $N \in \mathbb{N}$, the main difference between Π'_N and Π_N is that Π_N takes into account the multiplicity of collision events. For example, if the number of considered random walks is $k = 3$

and it happens that $S_n^{(1)} = S_n^{(2)} = S_n^{(3)} = z$ for some (n, z), then the Dirac measure $\delta_{(\frac{n}{N}, \frac{z}{\sqrt{N}})}$ will appear 3 times in the summation in Π_N while the number of its appearance in Π'_N is still one.

Concerning the scaling choice, one can observe that this is the same scaling as in Donker's theorem, which also suggests that the distribution of Π_N is closely related to Brownian local time. Indeed, when $k = 2$, the total measure of Π_N is equal to

$$\#\{n \in [\![1, N]\!] : S_n^{(1)} - S_n^{(2)} = 0\} \overset{(d)}{=} \#\{n \in [\![1, N]\!] : S_{2n}^{(1)} = 0\} = L^0_{S^{(1)}}(2N)\,, \tag{1.1}$$

where $[\![1, N]\!] = \{1, 2, ..., N\}$, $\overset{(d)}{=}$ is the equality in law, and $L^0_S(t)$ denotes the local time at position 0 during the period $(0, t]$ of some walk S. So by the convergence of local time of simple random walks, the equation (1.1) implies $\frac{1}{\sqrt{N}} \|\Pi_N\| = O_{\mathbf{P}}(1)$, where $\|\Pi_N\|$ denotes the total measure of Π_N.

In this work, we not only bound the sequence $(\frac{1}{\sqrt{N}}\Pi_N; N \in \mathbb{N})$, but also prove that this sequence of random measures converges to a non-trivial random measure $\mathcal{N}$ on $[0, 1] \times \mathbb{R}$. Before giving our main result, let us recall the convergence of random measures.

Definition 1.2. (Convergence of random measures) Suppose $\xi, \xi_1, \xi_2, \ldots$ are random finite measures on $([0, 1] \times \mathbb{R}, \mathcal{B}([0, 1] \times \mathbb{R}))$, we say that $\xi_n \xrightarrow[n\to\infty]{wd} \xi$ if the sequence of real random variables $(\xi_n(f), n \in \mathbb{N})$ converges in distribution to $\xi(f)$ when n goes to infinity for all bounded continuous function $f \in \mathcal{C}_b([0, 1] \times \mathbb{R})$. Here, $\mu(f)$ denotes the integral $\int f d\mu$ for any (random) measure μ and bounded measurable function f.

Here are our main results:

Theorem 1.3. *(Convergence of collision measures and characterization of the limit random measure)*

- *There is a random finite positive measure $\mathcal{N}$ on the measurable space $([0, 1] \times \mathbb{R}, \mathcal{B}([0, 1] \times \mathbb{R}))$ such that:*
$$\frac{1}{\sqrt{N}}\Pi_N \xrightarrow[N\to+\infty]{wd} \mathcal{N} \qquad \text{and} \qquad \frac{1}{\sqrt{N}}\Pi'_N \xrightarrow[N\to+\infty]{wd} \mathcal{N}.$$
- *Furthermore, for all nonnegative bounded continuous function $f \in \mathcal{C}_{b,+}([0, 1] \times \mathbb{R})$, the exponential moment of $\mathcal{N}$ with respect to f is equal to k-th moment of a positive random variable $\mathcal{Z}_{\sqrt{2f}}$:*
$$\mathbf{E}\left[e^{\mathcal{N}(f)}\right] = \mathbf{E}\left[\left(\mathcal{Z}_{\sqrt{2f}}\right)^k\right]. \tag{1.2}$$
For each $a \in \mathcal{C}_{b,+}([0, 1] \times \mathbb{R})$, the random variable $\mathcal{Z}_a$ is identified as the sum of multiple stochastic integrals given by:
$$\mathcal{Z}_a := 1 + \sum_{n=1}^{\infty} \int_{\Delta_n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \Big[a(\mathbf{t}_i, \mathbf{x}_i)\varrho(\mathbf{t}_i - \mathbf{t}_{i-1}, \mathbf{x}_i - \mathbf{x}_{i-1})W(d\mathbf{t}_i d\mathbf{x}_i)\Big], \tag{1.3}$$
where W is the white noise based on the Lebesque measure on $[0, 1] \times \mathbb{R}$, $\mathbf{x}_0 = 0$, ϱ is the standard Gaussian heat kernel
$$\varrho(t, x) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}},$$
*and Δ_n is the n-**dimensional simplex***
$$\Delta_n := \{\mathbf{t} \in [0, 1]^n : 0 = \mathbf{t}_0 < \mathbf{t}_1 < \mathbf{t}_2 < \cdots < \mathbf{t}_n \leq 1\}. \tag{1.4}$$

We refer to Walsh (1986, Chapter 1) for an introduction on white noise (see also Section 3.2 of this article). Regarding $\mathcal{Z}_a$, it is a convergent series in $L^2(\Omega)$, where Ω is the probability space on which the white noise is defined. In the relation with KPZ equation (Caravenna et al., 2020; Quastel and Spohn, 2015), $\mathcal{Z}_a$ can be recognized as $\mathcal{Z}(1,0)$ where $\mathcal{Z}(t,x)$ satisfies the stochastic PDE

$$\partial_t \mathcal{Z} = \frac{1}{2}\partial_{xx}\mathcal{Z} + a\mathcal{Z}W,$$

where the initial condition is $\mathcal{Z}(0,x) = 1$.

As mentioned earlier, we will prove this theorem by investigating the connection between the collision measures of random walks with a model in statistical mechanics. Hence, we will introduce many auxiliary notions in our paper, such as *random environment* ω, *partition function* $\mathfrak{Z}_N$ and *U-statistics* $\mathcal{S}_n^N(\cdot)$. We refer to Section 2.2 for a brief discussion of the connection between our collision measures and random polymer models, or more precisely, between Π_N and $\mathfrak{Z}_N$.

The general idea is that by associating each point (n,z) on the grid $\mathbb{N}\times\mathbb{Z}$ with a random variable, we can change the underlying framework from studying a deterministic grid to studying a collection of random variables indexed by $\mathbb{N}\times\mathbb{Z}$. For a such collection, the range of possible tools from statistical mechanics is large. Indeed, the partition function we will use is developped to study a directed polymer model Berger and Lacoin (2021); Caravenna et al. (2017, 2020), and Alberts et al. (2014) , which served as the primary inspiration for our work. More precisely, to meet our needs, we have upgraded many tools presented in Alberts et al. (2014) to incorporate inhomogeneous noise variance.

The organization of this paper is as follows: Section 2 introduces some basic notions that we will use in the sequel to explain our main ideas, especially, the relation between the concept of partition functions and the collision measures Π_N. Section 3 gives a brief review on U-statistics and Wiener chaos. At the end of this section, we prove some auxiliary results on the convergence of U-statistics, on which our asymptotic result on partition functions (Theorem 2.3) is based. Section 4 presents a short study on the random variable $\mathcal{Z}_a$ defined in (1.3), and our proof of Theorem 2.3. Finally, Section 5 combines all proved results to show the *weak tightness* of $(\frac{1}{\sqrt{N}}\Pi_N, N\in\mathbb{N})$, and prove Theorem 1.3.

Some auxiliary results are presented in the Appendix at the end of this article.

2. Partition functions and main ideas of proof

2.1. *Partition functions.* We introduce a collection $\omega := (\omega(i,z) : i\in\mathbb{N}, z\in\mathbb{Z})$ of independent Rademacher variables indexed by $\mathbb{N}\times\mathbb{Z}$, i.e., for all $(n,z)\in\mathbb{N}\times\mathbb{Z}$,

$$\mathbf{P}(\omega(n,z) = -1) = \mathbf{P}(\omega(n,z) = 1) = \frac{1}{2}.$$

These random variables are created by extending our existing probability space $(\Omega,\mathcal{A},\mathbf{P})$ so that $\omega, S^{(1)}, S^{(2)}, \ldots, S^{(k)}$ are independent.

In the sequel, for a real number β and a real function A on $\mathbb{N}\times\mathbb{Z}$, $\beta\omega$ and $A\omega$ are defined as:

$$A\omega := (A(i,z)\omega(i,z) : i\in\mathbb{N}, z\in\mathbb{Z}).$$
$$\beta\omega := (\beta\omega(i,z) : i\in\mathbb{N}, z\in\mathbb{Z}).$$

As briefly explained in Section 1, the role of ω is to add new degrees of freedom to the existing model, by which we have more flexibility to create more objects. The partition function $\mathfrak{Z}$ is one of such objects:

Definition 2.1. For any positive integer N and any real function A on $\mathbb{N}\times\mathbb{Z}$, the **partition function** $\mathfrak{Z}_N(A)$ is defined as the conditional expectation:

$$\mathfrak{Z}_N(A) := \mathbf{E}\left[\prod_{n=1}^{N}(1 + A(n,S_n^{(1)})\omega(n,S_n^{(1)}))\;\middle|\;\omega\right].$$

Note that $\mathfrak{Z}_N(A)$ is a random variable depending only on the value of ω.

2.2. *Main ideas.* The starting point of our paper and the proof of our main results is a heuristic relation between the partition functions $\mathfrak{Z}_N$ and the random measure Π_N:
Given a nonnegative bounded function A on $\mathbb{N}\times\mathbb{Z}$, since $S^{(1)},\cdots,S^{(k)}$ are i.i.d., we have:

$$\begin{aligned}
\mathbf{E}\left[\mathfrak{Z}_N\left(\frac{1}{N^{1/4}}A\right)^k\right] &= \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^k\prod_{n=1}^N(1+\frac{1}{N^{1/4}}A(n,S_n^{(i)})\omega(n,S_n^{(i)}))\middle|\omega\right]\right]\\
=&\mathbf{E}\left[\mathbf{E}\left[\prod_{n=1}^N\prod_{i=1}^k(1+\frac{1}{N^{1/4}}A(n,S_n^{(i)})\omega(n,S_n^{(i)}))\middle|S^{(1)},\cdots,S^{(k)}\right]\right]\\
=&\mathbf{E}\left[\prod_{n=1}^N\mathbf{E}\left[\prod_{i=1}^k(1+\frac{1}{N^{1/4}}A(n,S_n^{(i)})\omega(n,S_n^{(i)}))\middle|S^{(1)},\cdots,S^{(k)}\right]\right] \qquad (2.1)\\
=&\mathbf{E}\Bigg[\prod_{n=1}^N\Bigg[1+\frac{1}{N^{1/2}}\Bigg(\sum_{\substack{1\le i<j\le k:\\ S_n^{(i)}=S_n^{(j)}=z}}A^2(n,z)\Bigg)+\frac{1}{N^{3/4}}(\dots)+\dots\Bigg]\Bigg]
\end{aligned}$$

Then since $1+x\approx e^x$, heuristically, we deduce:

$$\begin{aligned}
\mathbf{E}\left[\mathfrak{Z}_N\left(\frac{1}{N^{1/4}}A\right)^k\right]&\approx\mathbf{E}\Bigg[\prod_{n=1}^N\Bigg[\exp\Big(\frac{1}{N^{1/2}}\sum_{\substack{1\le i<j\le k:\\ S_n^{(i)}=S_n^{(j)}=z}}A^2(n,z)\Big)\Bigg]\Bigg]\\
=&\mathbf{E}\Bigg[\exp\Bigg(\frac{1}{N^{1/2}}\sum_{n=1}^N\sum_{\substack{1\le i<j\le k:\\ S_n^{(i)}=S_n^{(j)}=z}}A^2(n,z)\Bigg)\Bigg]=\mathbf{E}\left[\exp\left(\frac{1}{N^{1/2}}\Pi_N(f_N)\right)\right],
\end{aligned}$$

where f_N is a measurable function such that $f_N\left(\frac{n}{N},\frac{z}{\sqrt{N}}\right)=A^2(n,z)$ for all $n\in\mathbb{N},z\in\mathbb{Z}$. In short, by abuse of notation, the above observation suggests that:

$$\mathbf{E}\left[e^{N^{-1/2}\Pi_N}\right]\approx\mathbf{E}\left[(\mathfrak{Z}_N)^k\right]. \qquad (2.2)$$

In other words, if we have a good understanding of $\mathfrak{Z}$, we will have good information on Π_N,Π'_N. Besides, we can observe that in the limit, Equation 2.2 essentially gives the connection 1.2 between the limiting random measure $\mathcal{N}$ and $\mathcal{Z}_{\sqrt{2f}}$ that we have in Theorem 1.3.

Then to study the partition function $\mathfrak{Z}_N$, we base our study on the paper Alberts et al. (2014), in which Alberts et al. studied the scaling limit of $\mathfrak{Z}$ when the function A is constant. In our study, we generalize their results for a sequence of functions $(A_N,N\in\mathbb{N})$ satisfying certain conditions. An expansion of Wiener chaoses emerges naturally in our limit objects because, as we will see, each term in the algebraic expansion of $\mathfrak{Z}_N$ (cf. Proposition 4.5) converges to a Wiener chaos. To this aim and following Alberts et al. (2014); Janson and Nowicki (1991), we will have to introduce some U-statistics and study their asymptotic behavior in Section 3.

2.3. *Results on partition functions.* We terminate this section by presenting our results on the asymptotic behavior of $\mathfrak{Z}_N$. The proofs will be presented later in Section 4.

Notation 2.2. For $(t,x)\in[0,1]\times\mathbb{R}$, $[t,x]_N$ denotes the unique pair of integer (i,z) such that :

- $(t,x)\in\left(\frac{i-1}{N},\frac{i}{N}\right]\times\left(\frac{z-1}{\sqrt{N}},\frac{z+1}{\sqrt{N}}\right]$,
- i and z have same parity.

Theorem 2.3. *Let $(A_n, n \in \mathbb{N})$ be a sequence of real functions whose domain is $\mathbb{N} \times \mathbb{Z}$ such that:*

i. $\sup_N \|A_N\|_\infty < +\infty$,
ii. *there is a measurable function $a \in L^\infty([0,1] \times \mathbb{R})$ such that:*

$$\lim_{N \to +\infty} A_N([t,x]_N) = a(t,x) \quad a.e.$$

Then as N converges to infinity, we have:

$$\mathfrak{Z}_N(N^{-1/4} A_N) \xrightarrow{(d)} \mathcal{Z}_{\sqrt{2}a}$$

This theorem is a mild generalization of Proposition 5.3 in Alberts et al. (2014) where the sequence $(A_N; N \in \mathbb{N})$ is replaced by a fixed constant $\beta \geq 0$. For a comprehensive approach to demonstrating the convergence of polynomial chaos to Wiener chaos, we refer interested readers to Caravenna et al. (2017).

We will also prove that under some conditions, the partition functions are uniformly bounded in L^k:

Theorem 2.4. *For a sequence of real functions $(A_n, n \in \mathbb{N})$ on $\mathbb{N} \times \mathbb{Z}$ such that $\sup_N \|A_N\|_\infty < +\infty$, we have:*

$$\limsup_N \mathbf{E}\left[\left(\mathfrak{Z}_N\left(N^{-1/4} A_N\right)\right)^k\right] < +\infty.$$

Notice that even though k is fixed in our study, the definition of $\mathfrak{Z}$ does not depend on k. So, the above sequence $\left(\mathfrak{Z}_N\left(N^{-1/4} A_N\right), N \in \mathbb{N}\right)$ is also uniformly bounded in L^p for all $p \in \mathbb{N}$, which implies directly the following corollary:

Corollary 2.5. *For a sequence of real functions $(A_n, n \in \mathbb{N})$ on $\mathbb{N} \times \mathbb{Z}$ such that $\sup_N \|A_N\|_\infty < +\infty$, the sequence of random variables $\left(\mathfrak{Z}_N\left(N^{-1/4} A_N\right), N \in \mathbb{N}\right)$ is uniformly L^k-integrable for any $k \in \mathbb{N}$..*

Remark 2.6. In Caravenna et al. (2020), authors utilized hypercontractivity to bound the moments of partition functions in (2+1) directed polymer model. This approach should also be applicable to our framework.

3. U-Statistics: related notions and limit theorems

Let $E_n^N := \{\mathbf{i} \in [\![1, N]\!]^n : \mathbf{i}_j \neq \mathbf{i}_l \text{ for } j \neq l\}$. In this paper, inspired by Alberts et al. (2014), we are interested in sums of the form:

$$\sum_{\mathbf{i} \in E_n^N} \sum_{\substack{\mathbf{z} \in \mathbb{Z}^n \\ \mathbf{i} \leftrightarrow \mathbf{z}}} \overline{g_N}(\mathbf{i}, \mathbf{z}) A_N(\mathbf{i}, \mathbf{z}) \omega(\mathbf{i}, \mathbf{z}), \tag{3.1}$$

for some weight functions $\overline{g_N}$ specified later.

Notation 3.1. $\mathbf{i} \leftrightarrow \mathbf{z}$ means that for all $j \in [\![1, n]\!]$, the corresponding j-th coordinates of $\mathbf{i}$ and $\mathbf{z}$, namely $\mathbf{i}_j$ and $\mathbf{z}_j$, have the same parity.

We will see that sums of this type appear naturally when we expand the partition functions $\mathfrak{Z}_N$ (see (4.4)). In Alberts et al. (2014), these forms appear without the extra terms $A_N(\mathbf{i}, \mathbf{z})$, so we have to extend the results.

The organization of this section is as follows: Sections 3.1 and 3.2 introduce the framework of Theorem 3.10 which is the interested result on the convergence of sums of the form (3.1). Section 3.3 presents the proof of Theorem 3.10.

The approach we used in this section is standard in the theory of U-statistics. Interested readers can consult the book Koroljuk and Borovskich (1994) of Korolyuk and Borovskikh for a more rigourous introduction of this theory.

3.1. *Introduction of U-statistics $\mathcal{S}_n^N$.* We first make precise the definition of the weight functions $(\overline{g_N}, N \in \mathbb{N})$ in the above sum.

Let g be a function in $L^2([0,1]^n \times \mathbb{R}^n)$. For each N, the weight functions $\overline{g_N}$ associated to g is defined by the following procedure:
First, we partition the space $(0,1]^n \times \mathbb{R}^n$ in rectangles of the form:

$$\mathcal{R}_n^N := \left\{ \left(\frac{\mathbf{i}-\mathbf{1}}{N}, \frac{\mathbf{i}}{N} \right] \times \left(\frac{\mathbf{z}-\mathbf{1}}{\sqrt{N}}, \frac{\mathbf{z}+\mathbf{1}}{\sqrt{N}} \right] : \mathbf{i} \in D_n^N, \mathbf{z} \in \mathbb{Z}^n, \mathbf{i} \leftrightarrow \mathbf{z} \right\},$$

with $\mathbf{1}$ being the vector of ones and D_n^N being the *integer simplex*:

$$D_n^N := \{\mathbf{i} \in [\![1, N]\!]^n : 0 =: \mathbf{i}_0 < \mathbf{i}_1 < \mathbf{i}_2 \cdots < \mathbf{i}_n \leq N\}. \tag{3.2}$$

Visually, $\mathcal{R}_n^N$ is a collection of nonoverlapping translations of the base rectangle:

$$\left(\frac{0}{N}, \frac{1}{N} \right]^n \times \left(\frac{-1}{\sqrt{N}}, \frac{1}{\sqrt{N}} \right]^n.$$

Then, the function $\overline{g_N}$ is defined as the average of g on each rectangles above. More precisely, for any $(\mathbf{t}, \mathbf{x}) \in (0,1]^n \times \mathbb{R}^n$, $\overline{g_N}(\mathbf{t}, \mathbf{x})$ is defined as the mean:

$$\overline{g_N}(\mathbf{t}, \mathbf{x}) := \frac{1}{|R|} \int_R g(\mathbf{s}, \mathbf{y}) d\mathbf{s}\, d\mathbf{y},$$

where R is the unique rectangle in $\mathcal{R}_n^N$ that contains $(\mathbf{t}, \mathbf{x})$, and $|R|$ denotes the Lebesque measure of R. In probabilistic terms, $\overline{g_N}$ is simply the conditional expectation of g onto the rectangles of $\mathcal{R}_n^N$. We note that $|R| = 2^n N^{-3n/2}$. This term will appear recurrently in most of our computations. Suppose $(A_N, N \in \mathbb{N})$ is a sequence of real-valued functions on $\mathbb{N} \times \mathbb{Z}$.

Notation 3.2. For any n-tuple $\mathbf{i} \in E_n^N$ and n-tuple $\mathbf{z} \in \mathbb{Z}^n$, $A_N(\mathbf{i}, \mathbf{z})$ and $\omega(\mathbf{i}, \mathbf{z})$ denote

$$A_N(\mathbf{i}, \mathbf{z}) := A_N(\mathbf{i}_1, \mathbf{z}_1) A_N(\mathbf{i}_2, \mathbf{z}_2) ... A_N(\mathbf{i}_n, \mathbf{z}_n),$$
$$\omega(\mathbf{i}, \mathbf{z}) := \omega(\mathbf{i}_1, \mathbf{z}_1) \omega(\mathbf{i}_2, \mathbf{z}_2) ... \omega(\mathbf{i}_n, \mathbf{z}_n),$$

with $\mathbf{i}_j$ being the j-th coordinate of $\mathbf{i}$ as defined previously.

Now, we define the weighted U-statistics $\mathcal{S}_n^N$.

Definition 3.3. Suppose $(A_n, n \in \mathbb{N})$ is a sequence of bounded real-valued functions on $\mathbb{N} \times \mathbb{Z}$. For any function $g \in L^2([0,1]^n \times \mathbb{R}^n)$, the U- statistics $\mathcal{S}_n^N$ is defined as:

$$\mathcal{S}_n^N(g) := 2^{n/2} \sum_{\mathbf{i} \in E_n^N} \sum_{\substack{\mathbf{z} \in \mathbb{Z}^n: \\ \mathbf{i} \leftrightarrow \mathbf{z}}} \overline{g_N} \left(\frac{\mathbf{i}}{N}, \frac{\mathbf{z}}{\sqrt{N}} \right) A_N(\mathbf{i}, \mathbf{z}) \omega(\mathbf{i}, \mathbf{z}). \tag{3.3}$$

We here give some basic properties of the U-statistics $\mathcal{S}_n^N$, extending Lemma 4.1 in Alberts et al. (2014), proved in Appendix C.

Proposition 3.4. *Suppose $(A_N, N \in \mathbb{N})$ is a sequence of bounded real-valued functions on $\mathbb{N} \times \mathbb{Z}$. For all positive integers n and N, we have:*

i. *(Well-posedness) $\mathcal{S}_n^N(g)$ is well-defined and has zero mean for all $g \in L^2([0,1]^n \times \mathbb{R}^n)$.*
ii. *(Linearity) For all $f, g \in L^2([0,1]^n \times \mathbb{R}^n)$, $\alpha, \beta \in \mathbb{R}$*

$$\mathcal{S}_n^N(\alpha f + \beta g) = \alpha \mathcal{S}_n^N(f) + \beta \mathcal{S}_n^N(g).$$

iii. *(L^2-boundedness) If $c > 0$ is a number such that such that $\|A_N\|_\infty \leq c$, then for all $g \in L^2([0,1]^n \times \mathbb{R}^n)$:*

$$\mathbf{E}[\mathcal{S}_n^N(g)^2] \leq c^{2n} N^{3n/2} \|g\|_2^2.$$

iv. *(Uncorrelatedness of U-statistics of different orders) If n_1, n_2 are two different positive integers, then* $\forall g_i \in L^2([0,1]^{n_i} \times \mathbb{R}^{n_i}) \quad i = 1, 2,$

$$\mathbf{E}[\mathcal{S}_{n_1}^N(g_1)\mathcal{S}_{n_2}^N(g_2)] = 0.$$

Now, to characterize rigorously the limit of the U-statistics $(\mathcal{S}_n^N, N \geq 1)$, we need to introduce the Wiener chaos.

3.2. *Wiener chaos.*

3.2.1. *White noise and stochastic integration on* $[0,1]\times\mathbb{R}$. This section recalls the elementary theory of white noise and stochastic integration on the measure space $([0,1]\times\mathbb{R}, \mathcal{B}, \mathrm{d}t\otimes\mathrm{d}x)$. Here $\mathcal{B}$ is the Borel σ-algebra, and $\mathrm{d}t\otimes\mathrm{d}x$ denotes Lebesque measure on $[0,1]\times\mathbb{R}$. For more details on Wiener chaos, we invite readers to read Nualart (2006, Chapter 1) or Kallenberg (1997, Chapter 11).

Let $\mathcal{B}_f$ be the collection of all Borel sets of $[0,1]\times\mathbb{R}$ with finite Lebesgue measure. Observe that $\mathcal{B} = \sigma(\mathcal{B}_f)$.

Definition 3.5. A **white noise** on $[0,1]\times\mathbb{R}$ is a collection of mean zero Gaussian random variables indexed by $\mathcal{B}_f$

$$W = \{W(A) : A \in \mathcal{B}_f\}$$

such that for any $h \in \mathbb{N}$ and every finite collection $(A_1, A_2, \ldots, A_h)$ of elements of $\mathcal{B}_f$, the tuple $(W(A_1), \ldots, W(A_h))$ is a h-dimensional Gaussian vector, with mean zero and covariance structure:

$$\mathbf{E}[W(A)W(B)] = |A\cap B|.$$

So in particular, if A and B are disjoint then $W(A)$ and $W(B)$ are independent. For any $g \in L^2([0,1]\times\mathbb{R}, \mathcal{B}, dt\,dx)$, the stochastic integral

$$I_1(g) := \int_0^1 \int_{\mathbb{R}} g(t,x)W(dt\,dx)$$

is constructed by first defining I_1 on simple functions then extending I_1 via density arguments Kallenberg (1997, p.210). In the end, for each $g \in L^2([0,1]\times\mathbb{R})$, we have that $I_1(g) \sim N(0, \|g\|_2^2)$, so in particular, I_1 preserves the Hilbert space structure of $L^2([0,1]\times\mathbb{R})$,

$$\mathbf{E}(I_1(g)I_1(h)) = \int_0^1 \int_{\mathbb{R}} g(t,x)h(t,x)dt\,dx.$$

This construction idea can be extended to higher dimensions (see Nualart (2006, p. 9,10)) to give a sense of the following notation of multiple stochastic integrals for any $n > 1$ and function $g \in L^2([0,1]^n \times \mathbb{R}^n)$:

$$I_n(g) := \int_{[0,1]^n} \int_{\mathbb{R}^n} g(\mathbf{t},\mathbf{x})W^{\otimes n}(d\mathbf{t}d\mathbf{x}),$$

where $W^{\otimes n}(d\mathbf{t}d\mathbf{x}) := W(d\mathbf{t}_1 d\mathbf{x}_1)W(d\mathbf{t}_2 d\mathbf{x}_2)\cdots W(d\mathbf{t}_n d\mathbf{x}_n)$.

Definition 3.6. A function $g \in L^2([0,1]^n \times \mathbb{R}^n)$ is said to be symmetric if $g(\mathbf{t},\mathbf{x}) = g(\pi\mathbf{t}, \pi\mathbf{x})$ for all $(\mathbf{t},\mathbf{x}) \in [0,1]^n \times \mathbb{R}^n$, permutation π on $\{1,\ldots,n\}$, where $\pi\mathbf{t} := \mathbf{t}_{\pi(1)},\ldots,\mathbf{t}_{\pi(n)}, \pi\mathbf{x} := \mathbf{x}_{\pi(1)},\ldots,\mathbf{x}_{\pi(n)}$. The set $L^2_{\mathrm{sym}}([0,1]^n\times\mathbb{R}^n)$ is then defined as the subspace of all symmetric functions of $L^2([0,1]^n\times\mathbb{R}^n)$.

The following theorem is a standard result in the theory of stochastic integration (cf. Nualart (2006, p.8,9)), which summarizes the above discussion:

Theorem 3.7 (Nualart (2006)). *There exists a continuous linear mapping $I_n : L^2([0,1]^n \times \mathbb{R}^n) \to L^2(\mathbf{P})$ such that for any n-tuple of disjoint finite measurable sets $A_1, A_2, \cdots A_n$ in $\mathcal{B}([0,1] \times \mathbb{R})$:*

$$I_n(\mathbb{1}_{A_1 \times A_2 \cdots \times A_n}) = W(A_1)W(A_2)\cdots W(A_n).$$

Furthermore, for all $g \in L^2([0,1]^n \times \mathbb{R}^n)$,

$$\mathbf{E}\left[I_n(g)^2\right] \le \|g\|_2^2, \tag{3.4}$$

and the equality occurs if and only if g is symmetric.

3.2.2. *Wiener chaos on* $[0,1] \times \mathbb{R}$. This section provides a short introduction to the Wiener chaos's theory. Wiener chaos may be regarded as a way of representing random variables as infinite sums of multiple stochastic integrals.

For a white noise W, we denote by $\mathcal{F}_W$ the complete σ-algebra generated by random variables $(W(A), A \in \mathcal{B}_f)$. The Wiener chaos decomposition theorem states (see Nualart (2006, Theorem 1.1.2)):

Proposition 3.8. *(Wiener chaos decomposition) For every random variable $X \in L^2(\Omega, \mathcal{F}_W, \mathbf{P})$, there is a unique sequence of symmetric functions $g_n \in L^2_{Sym}([0,1]^n \times \mathbb{R}^n), n \ge 1$, such that:*

$$X = \sum_{n=0}^{\infty} I_n(g_n).$$

Here g_0 is simply a constant and I_0 is the identity mapping on the constants.

In fact, for $n \ge 1$, the terms of the chaos series are all mean zero, so g_0 must be the mean of X. Moreover, by the orthogonality of $I_{n_1}(g_1)$ and $I_{n_2}(g_2)$ for $n_1 \neq n_2$(see Nualart (2006, p.9)), we have the relation $\mathbf{E}[X^2] = \sum_{n=0}^{\infty} \|g_n\|_2^2$. Now, we define two important spaces of collections of functions:

Definition 3.9. The Fock space over $L^2([0,1] \times \mathbb{R})$ is defined to be the Hilbert space:

$$F := \left\{ \mathbf{g} = (g_0, g_1, \dots) \in \bigoplus_{n=0}^{\infty} L^2([0,1]^n \times \mathbb{R}^n) : \sum_{n=0}^{\infty} \|g_n\|_2^2 < \infty \right\} \tag{3.5}$$

equipped with the inner product $\langle \mathbf{g}, \mathbf{f} \rangle_F = \sum_{n=0}^{\infty} \langle \mathbf{g}_n, \mathbf{f}_n \rangle_{L^2([0,1]^n \times \mathbb{R}^n)}$. Then, the symmetric Fock space F_{sym} is defined as the Hilbert subspace of F that contains only collections of symmetric functions, i.e.,

$$F_{\text{sym}} := F \bigcap \left(\bigoplus_{n=0}^{\infty} L^2_{\text{sym}}([0,1]^n \times \mathbb{R}^n) \right).$$

The result in Proposition 3.8 works also in reverse, that is, the mapping

$$\begin{aligned} \mathbf{I} \colon F_{\text{sym}} &\longrightarrow L^2(\Omega, \mathcal{F}_W, \mathbf{P}) \\ (g_0, g_1, \dots) &\longmapsto \sum_{n \ge 0} I_n(g_n) \end{aligned}$$

is an isometry. This fact will be useful for the justification for the well-posedness of $\mathcal{Z}_a$ in Section 4.

3.3. *A limit theorem for U-statistics.* In this section, we prove Theorem 3.10 for our U-statistics $\mathcal{S}_n^N$ defined by (3.3). This theorem extends Lemma 4.4 in Alberts et al. (2014) with non-constant A_n.

Theorem 3.10. *Suppose the functions $(A_n, n \in \mathbb{N})$ in the definition 3.3 of the U-statistics satisfy the following conditions:*

i. $\sup_N \|A_N\|_\infty < c$ *for some $c > 0$.*

ii. *There is a measurable function* $a \in L^\infty([0,1]\times\mathbb{R})$ *such that:*

$$\lim_{N\to+\infty} A_N([t,x]_N) = a(t,x) \quad a.e.$$

Then if $(g_n, n\in\mathbb{N}_0)$ *is a sequence of functions such that* $(c^n g_n, n\in\mathbb{N}_0)$ *belongs to the Fock space* F*, we have:*

$$\sum_{n=0}^{\infty} N^{-3n/4}\mathcal{S}_n^N(g_n) \xrightarrow[N\to\infty]{(d)} \sum_{n=0}^{\infty}\int_{[0,1]^n\times\mathbb{R}^n} g_n(\mathbf{t},\mathbf{x})a^{\otimes n}(\mathbf{t},\mathbf{x})W^{\otimes n}(d\mathbf{t}d\mathbf{x}).$$

Notation 3.11. For the sake of simplicity, for each $n\geq 1$ and $(\mathbf{t},\mathbf{x})\in[0,1]^n\times\mathbb{R}^n$, we define:

$$\tilde{I}_n(g) := \int_{[0,1]^n}\int_{\mathbb{R}^n} g(\mathbf{t},\mathbf{x})a^{\otimes n}(\mathbf{t},\mathbf{x})W^{\otimes n}(d\mathbf{t}d\mathbf{x}), \tag{3.6}$$

and

$$a^{\otimes n}(\mathbf{t},\mathbf{x}) := a(\mathbf{t}_1,\mathbf{x}_1)\cdots a(\mathbf{t}_n,\mathbf{x}_n). \tag{3.7}$$

Definition 3.12. For each $n,N\in\mathbb{N}$, and each function $g\in L^n([0,1]^n\times\mathbb{R}^n)$, we define $\hat{\mathcal{S}}_n^N$ as

$$\hat{\mathcal{S}}_n^N(g) := 2^{n/2}\sum_{\mathbf{i}\in E_n^N}\sum_{\substack{\mathbf{z}\in\mathbb{Z}^n:\\ \mathbf{i}\leftrightarrow\mathbf{z}}} \overline{g_N}\left(\frac{\mathbf{i}}{N},\frac{\mathbf{z}}{\sqrt{N}}\right)\omega(\mathbf{i},\mathbf{z}). \tag{3.8}$$

In other words, $\hat{\mathcal{S}}_n^N$ are just $\mathcal{S}_n^N$ with A_N being replaced by 1. The convergence of these statistics $\hat{\mathcal{S}}_n^N$ has been studied extensively in Alberts et al. (2014).

Proof of Theorem 3.10: Without loss of generality, we assume $c=1$.

This proof relies on the Lemma 4.1 and 4.4 in Alberts et al. (2014), and the following relation between $\hat{\mathcal{S}}_n^N$ and $\mathcal{S}_n^N$:

$$\mathcal{S}_n^N(g) = \hat{\mathcal{S}}_n^N(g a_N^{\otimes n}), \tag{3.9}$$

where $a_N(t,x) := A_N([t,x]_N)$ for all $t,x\in[0,1]^n\times\mathbb{R}^n$.

By Lemma 4.1 in Alberts et al. (2014), and using the orthogonality of U-statistics of different orders n, we have for each N:

$$\left\|\sum_{n=0}^{\infty}N^{-3n/4}\hat{\mathcal{S}}_n^N(g_n a^{\otimes n}) - \sum_{n=0}^{\infty}N^{-3n/4}\hat{\mathcal{S}}_n^N(g_n a_N^{\otimes n})\right\|_2^2 \leq \sum_{n=1}^{\infty}\left\|g_n a^{\otimes n} - g_n a_N^{\otimes n}\right\|_2^2$$

Hence, by noticing that $\lim_{N\to\infty}\|g_n a^{\otimes n} - g_n a_N^{\otimes n}\|_2 = 0$ and that $\left\|g_n a^{\otimes n} - g_n a_N^{\otimes n}\right\|_2 \leq 2\|g_n\|_2$, using the dominated convergence theorem, we imply that:

$$\sum_{n=0}^{\infty}N^{-3n/4}\hat{\mathcal{S}}_n^N(g_n a_N^{\otimes n}) - \sum_{n=0}^{\infty}N^{-3n/4}\hat{\mathcal{S}}_n^N(g_n a^{\otimes n}) \xrightarrow[N\to\infty]{L^2} 0,$$

Besides, by Lemma 4.4 in Alberts et al. (2014), we have:

$$\sum_{n=0}^{\infty}N^{-3n/4}\hat{\mathcal{S}}_n^N(g_n a^{\otimes n}) \xrightarrow[N\to\infty]{(d)} \sum_{n=0}^{\infty}\tilde{I}_n(g_n).$$

Therefore, by Equation (3.9) and by Slutsky's lemma van der Vaart (1998, Lemma 2.8), we conclude that:

$$\sum_{n=0}^{\infty}N^{-3n/4}\mathcal{S}_n^N(g_n) \xrightarrow[N\to\infty]{(d)} \sum_{n=0}^{\infty}\tilde{I}_n(g_n).$$

□

4. Limit theorems for partition functions

In this section, we study the convergence of partition functions $\mathfrak{Z}_N$ (Definition 2.1). First, we verify the well-posedness of the limit value $\mathcal{Z}_a$ given in Theorem 1.3, for all $a \in L^\infty([0,1]\times\mathbb{R})$ by

$$\mathcal{Z}_a := 1 + \sum_{n=1}^{\infty} \int_{\Delta_n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} \left[a(\mathbf{t}_i, \mathbf{x}_i) \varrho(\mathbf{t}_i - \mathbf{t}_{i-1}, \mathbf{x}_i - \mathbf{x}_{i-1}) W(d\mathbf{t}_i, d\mathbf{x}_i) \right].$$

where ϱ is the Gaussian kernel, and W is the white noise based on the Lebesgue measure on $[0,1]\times\mathbb{R}$.

4.1. *Study of $\mathcal{Z}_a$.*

4.1.1. *Brownian motion and simple random walk.* Let $(S_n, n \in \mathbb{N}_0)$ denote a simple random walk on $\mathbb{Z}$ and $(B_t, t \in \mathbb{R}_{\geq 0})$ denote a Brownian motion on $\mathbb{R}$ (Bingham et al., 1989). For $i \in \mathbb{N}$, $t \geq 0$ and $x \in \mathbb{R}$, we define:

$$p(i,x) := \mathbf{P}(S_i = x) \qquad \varrho(t,x) := \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \tag{4.1}$$

We will make heavy use of the finite dimensional distributions of both simple random walk and Brownian motion. For notations, we introduce for $n \in \mathbb{N}$, $\mathbf{i} \in D_n^N$ (D_n^N being the integer simplex (3.2)), $\mathbf{z} \in \mathbb{Z}^n$, $\mathbf{t} \in \Delta_n$ (Δ_n being the real simplex (1.4)), $\mathbf{x} \in \mathbb{R}^n$:

$$p_n(\mathbf{i}, \mathbf{z}) := \prod_{j=1}^{n} p(\mathbf{i}_j - \mathbf{i}_{j-1}, \mathbf{z}_j - \mathbf{z}_{j-1}) = \mathbf{P}(S_{\mathbf{i}_1} = \mathbf{z}_1, \ldots, S_{\mathbf{i}_n} = \mathbf{z}_n), \tag{4.2}$$

and

$$\varrho_n(\mathbf{t}, \mathbf{x}) := \prod_{j=1}^{n} \varrho(\mathbf{t}_j - \mathbf{t}_{j-1}, \mathbf{x}_j - \mathbf{x}_{j-1}). \tag{4.3}$$

For convenience, we respectively extend the domains of p_n and ϱ to $[\![1,N]\!]^n \times \mathbb{Z}^n$ and to $[0,1]^n \times \mathbb{R}^n$ by letting p_n and ϱ_n to be zero outside $D_n^N \times \mathbb{Z}^n$ and $\Delta_n \times \mathbb{R}^n$.

4.1.2. *Wiener chaos for Brownian transition probabilities.* The Brownian transition probabilites can generate many elements in the Fock space F (see Definition 3.9). Let us recall here Notation (3.7) of $a^{\otimes}$.

Proposition 4.1. *For every measurable bounded function $a \in L^\infty([0,1]\times\mathbb{R})$, let*

$$\boldsymbol{\varrho}(a) := (1, a\varrho_1, a^{\otimes 2}\varrho_2, a^{\otimes 3}\varrho_3, \ldots)$$

be a weighted ordered collection (indexed by $\mathbb{N}_0$) of Brownian transition probabilites ϱ_n that depends on a. Then, $\boldsymbol{\varrho}(a)$ is an element in the Fock space F,i.e., $\sum_{n\geq 0} \|a^{\otimes n}\varrho_n\|_2^2 < \infty$.

This proposition is proved in Appendix C. In particular, if a is a constant function, i.e., a is equal to some constant β then $\boldsymbol{\varrho}(\beta) = (1, \beta\varrho_1, \beta^2\varrho_2, \beta^3\varrho_3, \ldots)$.

So naturally, we have the following corollary on the well-posedness of $\mathcal{Z}_a$.

Proposition 4.2. *For all measurable bounded function a on $[0,1]\times\mathbb{R}$, the Wiener chaos $\mathcal{Z}_a$ is well-defined and has the representation $\mathcal{Z}_a = \mathbf{I}(\boldsymbol{\varrho}(a))$.*

4.2. *Relation between $\mathfrak{Z}$ and U-statistics.* We begin with establishing the relation between partition functions $\mathfrak{Z}$ and U-statistics, then we will prove Theorem 2.3.

For convenience, we extend Notation 2.2 $[t,x]_N$ for a pair $(t,x) \in [0,1] \times \mathbb{R}$ to higher dimensions:

Notation 4.3. For any pair $(\mathbf{t},\mathbf{x}) \in (0,1]^n \times \mathbb{R}^n$, we let $[\mathbf{t},\mathbf{x}]_N$ denote the unique pair $(\mathbf{i},\mathbf{z}) \in [\![1,N]\!]^n \times \mathbb{Z}^n$ such that:

i. $(\mathbf{t},\mathbf{x}) \in \left(\frac{\mathbf{i-1}}{N}, \frac{\mathbf{i}}{N}\right] \times \left(\frac{\mathbf{z-1}}{\sqrt{N}}, \frac{\mathbf{z+1}}{\sqrt{N}}\right]$,
ii. $\mathbf{i}$ and $\mathbf{z}$ have the same parity.

Definition 4.4. For $n, N \geq 1$, define $p_n^N : [0,1]^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$p_n^N(\mathbf{t},\mathbf{x}) = 2^{-n} p_n([\mathbf{t},\mathbf{x}]_N) \mathbb{1}_{\lceil N\mathbf{t}\rceil \in D_n^N},$$

where $\lceil N\mathbf{t}\rceil$ is the usual ceiling function, that is, for all $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{Z}^n$, $\lceil \mathbf{x} \rceil = \mathbf{z}$ if and only if for all i, $\mathbf{z}_i$ is the smallest integer bigger than or equal to $\mathbf{x}_i$.

We observe that the condition $\lceil N\mathbf{t}\rceil \in D_n^N$ implies that p_n^N is identically zero if $n > N$. Besides, we also see that p_n^N is constant on each rectangle in $\mathcal{R}_n^N$, so the average $\overline{p_n^N} = p_n^N$ and in particular, for $\mathbf{i} \in E_n^N, \mathbf{z} \in \mathbb{Z}^n$ such that $\mathbf{i} \leftrightarrow \mathbf{z}$, we have:

$$p_n^N\left(\frac{\mathbf{i}}{N}, \frac{\mathbf{z}}{\sqrt{N}}\right) \mathbf{1}_{\mathbf{i}\in D_n^N} = 2^{-n} p_n(\mathbf{i},\mathbf{z}) \mathbf{1}_{\mathbf{i}\in D_n^N}.$$

Thus, by definition of $\mathcal{S}_n^N$(see Definition 3.3) ,

$$\mathcal{S}_n^N(p_n^N) = 2^{-n/2} \sum_{\mathbf{i}\in D_n^N} \sum_{\mathbf{z}\in\mathbb{Z}^n} p_n(\mathbf{i},\mathbf{z}) \omega(\mathbf{i},\mathbf{z}) A_N(\mathbf{i},\mathbf{z}).$$

Note that the condition $\mathbf{i} \leftrightarrow \mathbf{z}$ is already handled by p_n. This leads to the following relation:

Proposition 4.5. *For all real number $\beta \in \mathbb{R}$ and positive integer $N \in \mathbb{N}$, the partition functions $\mathfrak{Z}_N$ can rewritten as:*

$$\mathfrak{Z}_N(\beta A_N) = \sum_{n=0}^{N} 2^{n/2} \beta^n \mathcal{S}_n^N(p_n^N).$$

Remark 4.6. So in particular,

$$\mathfrak{Z}_N(N^{-1/4} A_N) = \sum_{n=0}^{N} 2^{n/2} N^{-3n/4} \mathcal{S}_n^N(N^{n/2} p_n^N).$$

This equality is useful for our proof of Theorem 2.3.

Proof of Proposition 4.5: By definition,

$$\begin{aligned}
\mathfrak{Z}_N(\beta A_N) &= \mathbf{E}\left[\prod_{n=1}^{N}\left(1+\beta A_N(n,S_n)\omega(n,S_n)\right)\Big|\omega\right]\\
&= \mathbf{E}\left[1+\sum_{n=1}^{N}\sum_{\mathbf{i}\in D_n^N}\beta^n\left(\prod_{j=1}^{n}A_N(\mathbf{i}_j,S_{\mathbf{i}_j})\right)\left(\prod_{j=1}^{n}\omega(\mathbf{i}_j,S_{\mathbf{i}_j})\right)\Big|\omega\right]\\
&= \mathbf{E}\left[1+\sum_{n=1}^{N}\sum_{\mathbf{i}\in D_n^N}\sum_{\mathbf{z}\in\mathbb{Z}^n}\beta^n\left(\prod_{j=1}^{n}\mathbb{1}_{S_{\mathbf{i}}=\mathbf{z}_j}\right)\left(\prod_{j=1}^{n}A_N(\mathbf{i}_j,\mathbf{z}_j)\right)\left(\prod_{j=1}^{n}\omega(\mathbf{i}_j,\mathbf{z}_j)\right)\Big|\omega\right]\\
&= 1+\sum_{n=1}^{N}\sum_{\mathbf{i}\in D_n^N}\sum_{\mathbf{z}\in\mathbb{Z}^n}\beta^n\mathbf{E}\left[\prod_{j=1}^{n}\mathbb{1}_{S_{\mathbf{i}}=\mathbf{z}_j}\right]A_N(\mathbf{i},\mathbf{z})\omega(\mathbf{i},\mathbf{z})\\
&= 1+\sum_{n=1}^{N}\sum_{\mathbf{i}\in D_n^N}\sum_{\mathbf{z}\in\mathbb{Z}^n}\beta^n p_n(\mathbf{i},\mathbf{z})A_N(\mathbf{i},\mathbf{z})\omega(\mathbf{i},\mathbf{z}) = 1+\sum_{n=1}^{N}\beta^n 2^{n/2}\mathcal{S}_n^N(p_n^N).
\end{aligned}\tag{4.4}$$

Thus, $\mathfrak{Z}_N(\beta A_N) = 1+\sum_{n=1}^{N}2^{n/2}\beta^n\mathcal{S}_n^N(p_n^N)$. $\square$

Lemma 4.7. *For all n, we have the L^2-convergence:*

$$\lim_{N\to+\infty}\|\varrho_n - N^{n/2}p_n^N\|_2 = 0,$$

and moreover, there exists a constant C such that for all $n\in\mathbb{N}$,

$$\sup_N \|N^{n/2}p_n^N\|_2 \le C^n\|\varrho_n\|_2.$$

The proof for this lemma is presented at the end of this section. Now, we are ready to give a proof of Theorem 2.3.

Proof of Theorem 2.3: First observe that Theorem 3.10 and Proposition 4.1 imply that :

$$\sum_{n=0}^{\infty}N^{-3n/4}\mathcal{S}_n^N(2^{n/2}\varrho_n)\xrightarrow{(d)}\sum_{n=0}^{\infty}\tilde{I}_n(\varrho_n 2^{n/2}) = \mathbf{I}(\boldsymbol{\varrho}(\sqrt{2}a)) = \mathcal{Z}_{\sqrt{2}a}.$$

as N converges to infinity. Now we show that the difference between this term and $\mathfrak{Z}_N(N^{-1/4}A_N))$ goes to zero as N converges to infinity. Observe that:

$$\begin{aligned}
&\sum_{n=0}^{\infty}N^{-3n/4}\mathcal{S}_n^N(2^{n/2}\varrho_n) - \mathfrak{Z}_N(N^{-1/4}A_N\omega)\\
&\qquad = \sum_{n=0}^{N}2^{n/2}N^{-3n/4}\mathcal{S}_n^N(\varrho_n - N^{n/2}p_n^N) + \sum_{n=N+1}^{\infty}N^{-3n/4}\mathcal{S}_n^N(2^{n/2}\varrho_n).
\end{aligned}$$

By Proposition 3.4, the second term is bounded in L^2 by the square root of

$$\sum_{n=N+1}^{\infty}2^n c^{2n}\|\varrho_n\|_2^2.$$

which goes to zero as $N\to\infty$ by Proposition 4.1.

For the first term, using again Proposition 3.4, we note that its L^2-norm is bounded above by the square root of

$$\sum_{n=0}^{N}2^n c^{2n}\|\varrho_n - N^{n/2}p_n^N\|_2^2.$$

From Lemma 4.7 above, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $\sup_N \|\varrho_n - N^{n/2} p_n^N\|_2 \leq (1 + C^n)\|\varrho_n\|_2$. Besides, again by Proposition 4.1, the sequence $\sum_n 2^{n+1} c^{2n}(1 + C^n)\|\varrho_n\|_2^2 < \infty$. Hence, using the dominated convergence theorem, we can interchange limit and sum the following argument, then use the first part of Lemma 4.7 to conclude:

$$\lim_{N\to+\infty} \sum_{n=0}^{N} 2^n c^{2n} \|\varrho_n - N^{n/2} p_n^N\|_2^2 = \sum_{n=0}^{\infty} \lim_{N\to+\infty} 2^n c^{2n} \|\varrho_n - N^{n/2} p_n^N\|_2^2 = 0.$$

Theorem 2.3 is therefore proved. □

Proof of Lemma 4.7: From Gnedenko's local limit theorem Bingham et al. (1989, Theorem 8.4.1), we deduce that for any fixed $n \in \mathbb{N}$, $N^{n/2} p_n^N$ converges almost surely to ϱ_n as N goes to infinity. So by the general Lebesgue dominated convergence theorem Royden and Fitzpatrick (2010, Theorem 19), to prove our L^2 convergence, it suffices to find a function $g \in L^2([0,1]^n \times \mathbb{R}^n)$ and a sequence $(g_N, N \in \mathbb{N})$ of functions in $L^2([0,1]^n \times \mathbb{R}^n)$ such that:

i. $\left(N^{n/2} p_n^N\right)^2 \leq g_N$ for all N.
ii. g_N converges pointwise to g when N converges to infinity.
iii. $\lim_{N\to\infty} \int_{[0,1]^n \times \mathbb{R}^n} g_N = \int_{[0,1]^n \times \mathbb{R}^n} g < \infty$.

By Definition (4.1) of p and Stirling's formula (see Abramowitz and Stegun (1964, Stirling's formula, 6.1.37)), we observe that there exists a constant C such that $\sqrt{i} p(i, x) \leq C$ for all i and x , therefore:

$$\sup_{\mathbf{z} \in \mathbb{Z}^n} p_n(\mathbf{i}, \mathbf{z}) \leq C^n \prod_{j=1}^{n} \frac{1}{\sqrt{\mathbf{i}_j - \mathbf{i}_{j-1}}}.$$

From this and by Definition 4.4 of p_n^N, we have:

$$\left(N^{n/2} p_n^N(\mathbf{t}, \mathbf{x})\right)^2 \leq (C/2)^n h\left(\frac{\lceil N\mathbf{t} \rceil}{N}\right) N^{n/2} p_n^N(\mathbf{t}, \mathbf{x}).$$

where $h(\mathbf{t}) = \prod_{j=1}^{n} \frac{1}{\sqrt{\mathbf{t}_j - \mathbf{t}_{j-1}}} \mathbb{1}_{\{\mathbf{t} \in \Delta_n\}}$.
Let us choose for all N the function

$$g_N(\mathbf{t}, \mathbf{x}) := (C/2)^n h\left(\frac{\lceil N\mathbf{t} \rceil}{N}\right) N^{n/2} p_n^N(\mathbf{t}, \mathbf{x}),$$

and let

$$g(\mathbf{t}, \mathbf{x}) := (C/2)^n h(\mathbf{t}) \varrho_n(\mathbf{t}, \mathbf{x}).$$

Clearly, the conditions i. and ii. for the generalized dominated convergence Theorem are satisfied. For the last condition, we first notice that:

$$\int_{[0,1]^n \times \mathbb{R}^n} g(\mathbf{t}, \mathbf{x}) d\mathbf{t} d\mathbf{x} = (C/2)^n \int_{[0,1]^n} h(\mathbf{t}) d\mathbf{t}.$$

Then by definition of p_n^N, we have the following equalities:

$$\begin{aligned}
&\int_{[0,1]^n\times\mathbb{R}^n} g_N(\mathbf{t},\mathbf{x})d\mathbf{t}d\mathbf{x}\\
&= \sum_{\substack{\mathbf{i}\in[\![1,N]\!]^n,\mathbf{z}\in\mathbb{Z}^n:\\ \mathbf{i}\text{ and }\mathbf{z}\text{ have the same parity}}} \int_{\left(\frac{\mathbf{i}-\mathbf{1}}{N},\frac{\mathbf{i}}{N}\right]\times\left(\frac{\mathbf{z}-\mathbf{1}}{\sqrt{N}},\frac{\mathbf{z}+\mathbf{1}}{\sqrt{N}}\right]} g_N(\mathbf{t},\mathbf{x})\,d\mathbf{t}d\mathbf{x}\\
&= \sum_{\substack{\mathbf{i}\in[\![1,N]\!]^n,\mathbf{z}\in\mathbb{Z}^n:\\ \mathbf{i}\text{ and }\mathbf{z}\text{ have the same parity}}} \left(N^{-3n/2}2^n\right)\left[(C/2)^n h\left(\frac{\mathbf{i}}{N}\right)2^{-n}N^{n/2}p_n(\mathbf{i},\mathbf{z})\mathbb{1}_{\mathbf{i}\in D_n^N}\right]\\
&= (C/2)^n N^{-n}\sum_{\mathbf{i}\in D_n^N}\sum_{\mathbf{z}\in\mathbf{Z}^n} h\left(\frac{\mathbf{i}}{N}\right)p_n(\mathbf{i},\mathbf{z})\mathbb{1}_{\{\mathbf{i}\text{ and }\mathbf{z}\text{ have the same parity}\}}\\
&= (C/2)^n N^{-n}\sum_{\mathbf{i}\in D_n^N} h\left(\frac{\mathbf{i}}{N}\right) = (C/2)^n\int_{[0,1]^n} h\left(\frac{\lceil N\mathbf{t}\rceil}{N}\right)d\mathbf{t}.
\end{aligned}$$

So, what is left to do is prove that

$$\lim_{N\to\infty}\int_{[0,1]^n} h\left(\frac{\lceil N\mathbf{t}\rceil}{N}\right)d\mathbf{t} = \int_{[0,1]^n} h(\mathbf{t})\,d\mathbf{t}\quad\text{and}\quad\int_{[0,1]^n} h(\mathbf{t})\,d\mathbf{t}<\infty.$$

which is true because $h\left(\frac{\lceil N\mathbf{t}\rceil}{N}\right)$ converges pointwise to $h(\mathbf{t})$ for all $\mathbf{t}$ and they form a uniformly integrable sequence of functions in $L^2([0,1]^n)$. Indeed, the uniform integrability is due to the fact that:

$$\begin{aligned}
&\int_{\Delta_n}\left[h\left(\frac{\lceil N\mathbf{t}\rceil}{N}\right)\right]^{3/2}d\mathbf{t} = \sum_{\mathbf{i}\in D_n^N}\int_{\{\mathbf{t}\in\mathbb{R}^n:\lceil N\mathbf{t}\rceil=\mathbf{i}\}}\prod_{j=1}^n\left(\frac{\mathbf{i}_j-\mathbf{i}_{j-1}}{N}\right)^{-3/4}\mathrm{d}\mathbf{t}\\
&\le \sum_{\mathbf{i}\in D_n^N}\int_{\{\mathbf{t}\in\mathbb{R}^n:\lceil N\mathbf{t}\rceil=\mathbf{i}\}}\prod_{j=1}^n\left(\frac{\mathbf{t}_j-\mathbf{t}_{j-1}}{2}\right)^{-3/4}d\mathbf{t} \qquad (4.5)\\
&= 2^{3n/4}\int_{\{\mathbf{t}\in\mathbb{R}^n:\lceil N\mathbf{t}\rceil\in D_n^N\}}\prod_{j=1}^n(\mathbf{t}_j-\mathbf{t}_{j-1})^{-3/4}d\mathbf{t}\\
&\le 2^{3n/4}\int_{\Delta_n}\prod_{j=1}^n(\mathbf{t}_j-\mathbf{t}_{j-1})^{-3/4}d\mathbf{t}<\infty.
\end{aligned}$$

Note that in (4.5), we used the inequality: $\lceil a\rceil-\lceil b\rceil\ge\frac{a-b}{2}$ if $\lceil a\rceil-\lceil b\rceil\ge 1$. For the inequality in the latter part of our lemma, by what we have proved so far, we observe that:

$$\begin{aligned}
&\|N^{n/2}p_n^N\|_2^2\le (C/2)^n\int_{\Delta_n} h\left(\frac{\lceil N\mathbf{t}\rceil}{N}\right)d\mathbf{t}\\
&\le (C/2)^n 2^{n/2}\int_{\Delta_n}\prod_{j=1}^n(\mathbf{t}_j-\mathbf{t}_{j-1})^{-1/2}d\mathbf{t}\\
&= C^n 2^{-n/2}\int_{\Delta_n}\int_{\mathbb{R}^n}(4\pi)^{n/2}[\varrho_n(\mathbf{t},\mathbf{x})]^2d\mathbf{t}d\mathbf{x} = C^n(2\pi)^{n/2}\|\varrho_n\|_2^2,
\end{aligned}$$

where the second inequality is obtained similarly as (4.5). Hence, we have our desired conclusion.

□

5. Asymptotics of collision measures

5.1. *Convergence of exponential moments.* We first prove Theorem 2.4 on the uniform boundedness of moments of the partition functions. Then we will study the convergence of the exponential moments of $(\frac{1}{\sqrt{N}}\Pi_N; N \in \mathbb{N})$.

Proof of Theorem 2.4: Let c be a positive number such that $c \geq \sup_N \|A_N\|_\infty$. Without loss of generality, assume N is sufficiently large (i.e. $N > c^4$) such that the partition function $\mathfrak{Z}_N\left(\frac{1}{N^{1/4}}A_N\right)$ is a positive random variable.

Recall that in Equation (2.1), we have shown that:

$$\mathbf{E}\left[\mathfrak{Z}_N\left(\frac{1}{N^{1/4}}A_N\right)^k\right] =$$
$$\mathbf{E}\left[\prod_{n=1}^N \mathbf{E}\left[\prod_{i=1}^k \left(1+\frac{1}{N^{1/4}}A_N(n,S_n^{(i)})\omega(n,S_n^{(i)})\right)\middle| S^{(1)},S^{(2)},\ldots,S^{(k)}\right]\right].$$

Now, define for $n \geq 1$:

$$X_{N,n} := \mathbf{E}\Bigg[\prod_{i=1}^k \left(1+\frac{1}{N^{1/4}}A_N(n,S_n^{(i)})\omega(n,S_n^{(i)})\right)\Bigg| S^{(1)},S^{(2)},\ldots,S^{(k)}\Bigg] - 1, \tag{5.1}$$

and

$$T_N := \sum_{n=1}^N X_{N,n}. \tag{5.2}$$

Because ω is a collection of independent Rademacher random variables, we easily notice that $X_{N,n} \geq 0$ $\mathbf{P}-a.s$, since $X_{N,n}$ can be written as:

$$\sum_{l=2}^k \sum_{1\leq i_1<i_2<\ldots<i_l\leq k} N^{-l/4}\prod_{h=1}^l A_N(n,S_n^{(i_h)})\underbrace{\mathbf{E}\Bigg[\prod_{h=1}^l \omega(n,S_n^{(i_h)})\big|S^{(1)},\ldots,S^{(k)}\Bigg]}_{\text{is either 0 or 1}}. \tag{5.3}$$

Consequently,

$$\mathbf{E}\left[\mathfrak{Z}_N\left(\frac{1}{N^{1/4}}A_N\right)^k\right] = \mathbf{E}\left[\prod_{n=1}^N (1+X_{N,n})\right] \leq \mathbf{E}\left[e^{T_N}\right]. \tag{5.4}$$

where we have used the classical inequality that $\forall x \in \mathbb{R} : 1+x \leq e^x$ and $X_{N,n} \geq 0$.

Then, for each n, let us introduce the number $U^{(n)}$ of pairs (i,j) such that $S_n^{(i)} = S_n^{(j)}$, i.e.,

$$U^{(n)} := \sum_{1\leq i<j\leq k} \mathbf{1}_{S_n^{(i)}=S_n^{(j)}}.$$

We observe that on the event $\{U^{(n)} = 0\}$, $X_{N,n}$ is equal to zero, and on the event $\{U^{(n)} \geq 1\}$,

$$\sum_{1\leq i_1<i_2<\ldots<i_l\leq k} \mathbf{E}\Big[\omega(n,S_n^{(i_1)})\ldots\omega(n,S_n^{(i_l)})\big|S^{(1)},\ldots,S^{(k)}\Big] \leq \binom{k}{l} \leq \binom{k}{l}U^{(n)}. \tag{5.5}$$

Thus,

$$X_{N,n} \leq \sum_{l=2}^k N^{-l/4}c^l\binom{k}{l}U^{(n)} \leq (c+1)^k N^{-1/2}U^{(n)}. \tag{5.6}$$

So by combining the inequalities (5.4) and (5.6), one sees that:

$$\begin{aligned}
\mathbf{E}\left[\mathfrak{Z}_N\left(\frac{1}{N^{1/4}}A_N\omega\right)^k\right] &\leq \mathbf{E}\left[e^{T_N}\right] \\
\leq&\mathbf{E}\left[\exp\left((c+1)^k N^{-1/2}\sum_{1\leq i<j\leq k}\sum_{n=1}^{N}\mathbf{1}_{S_n^{(i)}=S_n^{(j)}}\right)\right] \\
=&\mathbf{E}\left[\prod_{(i,j):1\leq i<j\leq k}\exp\left((c+1)^k N^{-1/2}\sum_{n=1}^{N}\mathbf{1}_{S_n^{(i)}=S_n^{(j)}}\right)\right] \\
\leq&\mathbf{E}\left[\exp\left(\frac{k(k-1)}{2}(c+1)^k N^{-1/2}\sum_{n=1}^{N}\mathbf{1}_{S_n^{(1)}=S_n^{(2)}}\right)\right],
\end{aligned}$$

by Hölder's inequality. Besides, using Theorem B.2 in Appendices, we can prove that for all $\beta \geq 0$:

$$\sup_N \mathbf{E}\left[\exp\left(\beta N^{-1/2}\sum_{n=1}^{N}\mathbf{1}_{S_n^{(1)}=S_n^{(2)}}\right)\right] < +\infty.$$

Thus, this implies the desired conclusion. □

Remark 5.1. Using the same argument as in the above proof, one can see that:

$$\sup_N \mathbf{E}\left(e^{\beta T_N}\right) < \infty \qquad \forall \beta \geq 0.$$

Hence, in particular, $\left(e^{\beta T_N}, N \in \mathbb{N}\right)$ is uniformly integrable. If we do not care about $U^{(n)}$, we can just have $X_{N,n} \leq (c+1)^k N^{-1/2}$. This remark will be useful in our proof for Theorem 5.2.

We now give result on the converence of the exponential moments of $(\frac{1}{\sqrt{N}}\Pi_N, N \in \mathbb{N})$.

Theorem 5.2. *For any bounded positive continous function $f \in \mathcal{C}_{b,+}([0,1]\times\mathbb{R})$, we have:*

$$\mathbf{E}\left[\exp\left(\frac{1}{\sqrt{N}}\Pi_N(f)\right)\right] \xrightarrow[N\to+\infty]{} \mathbf{E}\left[(\mathcal{Z}_{\sqrt{2f}})^k\right].$$

Proof of Theorem 5.2: For any bounded nonnegative continous function $f \in \mathcal{C}_{b,+}([0,1]\times\mathbb{R})$, let

- $(A_n, n \in \mathbb{N})$ be a sequence of real functions defined on $\mathbb{N}\times\mathbb{Z}$ such that:

$$A_N(n,z) := \sqrt{f}\left(\frac{n}{N}, \frac{z}{\sqrt{N}}\right) \quad \forall n \in \mathbb{N}, z \in \mathbb{Z}.$$

- $a := \sqrt{f}$ and $c := \|a\|_\infty$.

Notice that due to the continuity of f, $\lim_{N\to\infty} A([t,x]_N) = a(t,x)$ for all $(t,x) \in [0,1]\times\mathbb{R}$. Thus, $(A_N, N \in \mathbb{N})$ satisfies the conditions of Theorem 2.3 and therefore:

$$\mathfrak{Z}_N(N^{-1/4}A_N) \xrightarrow[N\to\infty]{(d)} \mathcal{Z}_{\sqrt{2f}}.$$

Hence, from the uniform integrability in Corollary 2.5, we deduce that:

$$\mathbf{E}\left[\left(\mathfrak{Z}_N(N^{-1/4}A_N)\right)^k\right] \xrightarrow[N\to+\infty]{} \mathbf{E}\left[(\mathcal{Z}_{\sqrt{2f}})^k\right]. \tag{5.7}$$

Using again the quantity $X_{N,n}$ defined by (5.1), we have shown in (5.4) that:

$$\mathbf{E}\left[\left(\mathfrak{Z}_N(N^{-1/4}A_N)\right)^k\right] = \mathbf{E}\left[\prod_{n=1}^{N}(1+X_{N,n})\right]$$

So the convergence (5.7) can be rewritten as:

$$\mathbf{E}\left[\prod_{n=1}^{N}(1+X_{N,n})\right] \xrightarrow[N\to+\infty]{} \mathbf{E}\left[(\mathcal{Z}_{\sqrt{2f}})^k\right]. \tag{5.8}$$

From Remark 5.1, we know that the sequence $(T_N, N\in\mathbb{N})$ with $T_N = \sum_{n=1}^{N} X_{N,n}$ satisfies that $\big(e^{\beta T_N}, N\in\mathbb{N})\big)$ is uniformly integrable and that:

$$0 \le X_{N,n} \le (c+1)^k N^{-1/2}.$$

Hence, using Theorem A.2 in Appendix A and the convergence (5.8), we deduce that:

$$\mathbf{E}\left[e^{T_N}\right] \xrightarrow[N\to+\infty]{} \mathbf{E}\left[(\mathcal{Z}_{\sqrt{2f}})^k\right]. \tag{5.9}$$

We now investigate the relation between T_N and $\frac{1}{\sqrt{N}}\Pi_N(f)$. Observe that:

$$0 \le \frac{1}{\sqrt{N}}\Pi_N(f) \le T_N.$$

Indeed, from the expansion (5.3), we have:

$$\begin{aligned} &T_N - \frac{1}{\sqrt{N}}\Pi_N(f) = \\ &= \sum_{n=1}^{N}\sum_{l=3}^{k}\sum_{1\le i_1<i_2<\ldots<i_l\le k} N^{-l/4}\prod_{h=1}^{l} A_N(n,S_n^{(i_h)})\mathbf{E}\Big[\prod_{h=1}^{l}\omega(n,S_n^{(i_l)})\Big] \ge 0. \end{aligned}$$

Then following the same arguments used to bound T_N in (5.5) and (5.6), one can show that:

$$0 \le T_N - \frac{1}{\sqrt{N}}\Pi_N(f) \le (c+1)^k N^{-3/4}\sum_{1\le i<j\le k}\sum_{n=1}^{N}\mathbb{1}_{S_n^{(i)}=S_n^{(j)}}.$$

Besides, $\mathbf{E}\left(N^{-3/4}\sum_{1\le i<j\le k}\sum_{n=1}^{N}\mathbb{1}_{S_n^{(i)}=S_n^{(j)}}\right) = \frac{k(k-1)}{2N^{3/4}}\sum_{n=1}^{N}\frac{1}{2^{2n}}\binom{2n}{n} \to 0$ as $N \longrightarrow \infty$, which implies $T_N - \frac{1}{\sqrt{N}}\Pi_N(f) \xrightarrow[N\to\infty]{(d)} 0$.

Thus, by applying Lemma C.1 in Appendices to two sequences $\left(e^{\frac{1}{\sqrt{N}}\Pi_N(f)}, N\in\mathbb{N}\right)$ and $(e^{T_N}, N\in\mathbb{N})$, we conclude that:

$$\mathbf{E}\left[\exp\left(\frac{1}{\sqrt{N}}\Pi_N(f)\right)\right] \xrightarrow[N\to+\infty]{} \mathbf{E}\left[(\mathcal{Z}_{\sqrt{2f}})^k\right] < \infty.$$

□

5.2. *Convergence of collision measures.* We begin by proving the weak tightness of $\big(\frac{1}{\sqrt{N}}\Pi_N, N\in\mathbb{N}\big)$, then giving the proof of Theorem 1.3. We refer to Kallenberg (2017, p.118,119) for the weak tightness. The weak tightness is crucial as it allows us to take convergent subsequences of $(\frac{1}{\sqrt{N}}\Pi_N; N\in\mathbb{N})$.

Theorem 5.3. *The sequence of random measures* $\big(\frac{1}{\sqrt{N}}\Pi_N, N\in\mathbb{N}\big)$ *is weakly tight.*

Proof: Let $\mathcal{M}$ denote the set of all finite positive measures on the Polish space $[0,1]\times\mathbb{R}$, and

$$\begin{aligned} L_m &:= \{\mu\in\mathcal{M} : \|\mu\| \le m\},\\ M_m &:= \{\mu\in\mathcal{M} : \mathrm{supp}\mu \subset [0,1]\times[-m,m]\},\\ K_m &:= L_m \cap M_m. \end{aligned}$$

So K_m is a collection of some measures that are uniformly bounded and contained within the same compact set. Thus, by Lemma 4.4 in Kallenberg (2017), K_m is a weakly relatively compact subset of $\mathcal{M}$. So, by the definition of tightness, it suffices to prove that

$$\lim_{m\to+\infty}\sup_N \mathbf{P}(N^{-1/2}\Pi_N \notin K_m) = 0,$$

which is true because

$$\lim_{m\to+\infty}\sup_N \mathbf{P}(N^{-1/2}\Pi_N \notin M_m) = 0 \text{ and } \lim_{m\to+\infty}\sup_N \mathbf{P}(N^{-1/2}\Pi_N \notin L_m) = 0.$$

Indeed, for M_m, we observe that:

$$\begin{aligned}&\mathbf{P}(N^{-1/2}\Pi_N \notin M_m)\\ &\quad\le \mathbf{P}\left(\sup_{\substack{1\le n\le N\\1\le i\le k}} |S_n^{(i)}| > m\sqrt{N}\right) \le k\mathbf{P}\left(\frac{\sup_{1\le n\le N}|S_n|}{\sqrt{N}} > m\right).\end{aligned}$$

Since by Donsker's theorem the sequence $\left(\frac{1}{\sqrt{N}}\sup_{1\le n\le N}|S_n|, N\in\mathbb{N}\right)$ converges in distribution to a real random variable, then it is tight by Prokhorov's theorem Billingsley (1999, Theorem 5.2, p. 60). Thus,

$$\lim_{m\to+\infty}\sup_N \mathbf{P}(N^{-1/2}\Pi_N \notin M_m) \le k \lim_{m\to+\infty}\sup_N \mathbf{P}\left(\frac{\sup_{1\le n\le N}|S_n|}{\sqrt{N}} > m\right) = 0.$$

For L_m, we have:

$$\begin{aligned}\mathbf{P}(N^{-1/2}\Pi_N \notin L_m) &\le \mathbf{P}\left(\sum_{n=1}^N \sum_{1\le i<j\le k} \mathbb{1}_{S_n^{(i)}=S_n^{(j)}} > m\sqrt{N}\right)\\ &\le \frac{k(k-1)}{2}\mathbf{P}\left(\frac{1}{\sqrt{N}}\sum_{n=1}^N \mathbb{1}_{S_n^{(1)}=S_n^{(2)}} > \frac{2m}{k(k-1)}\right)\\ &= \frac{k(k-1)}{2}\mathbf{P}\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{2N} \mathbb{1}_{S_n=0} > \frac{2m}{k(k-1)}\right).\end{aligned}$$

Similarly, because $\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{2N}\mathbb{1}_{S_n=0}, N\in\mathbb{N}\right)$ also converges in distribution Révész (2005, Theorem 10.1), we have

$$\lim_{m\to+\infty}\sup_N \mathbf{P}(N^{-1/2}\Pi_N \notin L_m) = 0.$$

Hence the conclusion. □

Now, by combining all results we have shown so far, we can give the proof of Theorem 1.3.

Proof of Theorem 1.3: By Theorem 5.3 and Prokhorov's theorem Billingsley (1999, Theorem 5.1), there exists a random finite positive measure $\mathcal{N}'$ on $[0,1]\times\mathbb{R}$ such that there is a subsequence of $(\frac{1}{\sqrt{N}}\Pi_N, N\in\mathbb{N})$ that converges in distribution to $\mathcal{N}'$. For convenience, assume that $\mathcal{N}'$ is defined on the existing probability space $(\Omega,\mathcal{A},\mathbf{P})$.

Besides, for any $f\in C_{b,+}([0,1]\times\mathbb{R})$, by the proof of Theorem 5.2, it is known that: $(e^{\frac{1}{\sqrt{N}}\Pi_N(f)}, N\in\mathbb{N})$ is uniformly integrable. Thus, $\mathbf{E}\left[e^{\mathcal{N}'(f)}\right]$ is finite and equal to $\mathbf{E}\left[\left(\mathcal{Z}_{\sqrt{2f}}\right)^k\right]$.

We see that to show $\frac{1}{\sqrt{N}}\Pi_N \xrightarrow[N\to\infty]{wd} \mathcal{N}'$, it suffices to prove that $\mathcal{N}'$ is uniquely defined in distribution. Indeed, let $\mathcal{N}''$ be another random bounded measure on $[0,1]\times\mathbb{R}$ such that there is a subsequence of $(\frac{1}{\sqrt{N}}\Pi_N, N \in \mathbb{N})$ that converges in distribution to it. Assume $\mathcal{N}''$ is also defined on $(\Omega, \mathcal{A}, \mathbf{P})$.

In the following, we will prove that $\mathcal{N}'(h) \overset{(d)}{=} \mathcal{N}''(h)$ for all $h \in C_b([0,1]\times\mathbb{R})$, then the uniqueness of $\mathcal{N}'$ follows immediately from Lemma 4.7 in Kallenberg (2017). Let f, g be two continous nonnegative bounded functions on $[0,1]\times\mathbb{R}$. For any two nonnegative numbers a and b, $af+bg$ is also a continous bounded nonnegative function. Hence,

$$\mathbf{E}\left[e^{\mathcal{N}'(af+bg)}\right] = \mathbf{E}\left[\left(\mathcal{Z}_{\sqrt{2(af+bg)}}\right)^k\right] = \mathbf{E}\left[e^{\mathcal{N}''(af+bg)}\right].$$

Equivalently, for all $a, b \geq 0$, $\mathbf{E}\left[e^{a\mathcal{N}'(f)+b\mathcal{N}'(g)}\right] = \mathbf{E}\left[e^{a\mathcal{N}''(f)+b\mathcal{N}''(g)}\right]$. Thus,

$$a\mathcal{N}'(f) + b\mathcal{N}'(g) \overset{(d)}{=} a\mathcal{N}'' + b\mathcal{N}''(g) \qquad \forall a, b \geq 0.$$

So by Cramer-Wold theorem Kallenberg (1997, Corollary 4.5), we have:

$$(\mathcal{N}'(f), \mathcal{N}'(g)) \overset{(d)}{=} (\mathcal{N}''(f), \mathcal{N}''(g)).$$

Then using Cramer-Wold Theorem again, we deduce that $\mathcal{N}'(f-g) \overset{(d)}{=} \mathcal{N}''(f-g)$. So, $\mathcal{N}'(h) \overset{(d)}{=} \mathcal{N}''(h)$ for all $h \in \mathcal{C}_b([0,1]\times\mathbb{R})$ because any bounded continous function h can be written as the difference of two continuous bounded nonnegative functions.

Thus, we proved that $\frac{1}{\sqrt{N}}\Pi_N \xrightarrow[N\to\infty]{wd} \mathcal{N}$, where $\mathcal{N}$ is a positive random measure $[0,1]\times\mathbb{N}$ that is uniquely defined in distribution by the following equation for all $f \in \mathcal{C}_{b,+}([0,1]\times\mathbb{R})$:

$$\mathbf{E}(e^{\mathcal{N}(f)}) = \mathbf{E}\left[\left(\mathcal{Z}_{\sqrt{2f}}\right)^k\right].$$

Finally, the convergence of $(\frac{1}{\sqrt{N}}\Pi'_N, N \in \mathbb{N})$ follows directly from the convergence of $(\frac{1}{\sqrt{N}}\Pi_N, N \in \mathbb{N})$ and Lemma C.1 by noticing that $\Pi_N(f) \geq \Pi'_N(f) \geq 0$ for all $f \in \mathcal{C}_{b,+}([0,1]\times\mathbb{R})$, and

$$\begin{aligned}
&\mathbf{E}\left(\frac{1}{\sqrt{N}}\|\Pi_N - \Pi'_N\|\right) \\
&\leq \frac{1}{\sqrt{N}}\mathbf{E}\left[\sum_{n=1}^{N}\sum_{z\in\mathbb{Z}}\binom{k}{2}\sum_{1\leq i_1\leq i_2\leq i_3\leq k}\mathbf{1}_{\{S_n^{(i_1)}=S_n^{(i_2)}=S_n^{(i_3)}=z\}}\right] \\
&\leq \frac{k^5}{\sqrt{N}}\sum_{n=1}^{N}\mathbf{P}(S_n^{(1)}=S_n^{(2)}=S_n^{(3)}) \leq \frac{k^5}{\sqrt{N}}\sum_{n=1}^{N}\max_{z\in\mathbb{Z}}(\mathbf{P}(S_n^{(3)}=z))\mathbf{P}(S_n^{(1)}=S_n^{(2)}) \\
&= \frac{k^5}{\sqrt{N}}\sum_{n=1}^{N}\frac{1}{2^n}\binom{n}{\lceil n/2\rceil}\frac{1}{2^{2n}}\binom{2n}{n} \leq \frac{k^5}{\sqrt{N}}C^2\sum_{n=1}^{N}\frac{1}{n}\xrightarrow[N\to\infty]{} 0,
\end{aligned}$$

for some constant C such that $\frac{1}{2^n}\binom{n}{\lceil n/2\rceil} \leq C\frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$. Note that such C exists thanks to Stirling's formula. Hence, our theorem is proved. □

Appendix A. On the asymptotic relation between products and sums of independent random variables

We consider a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. For any N, let $X_N = (X_{N,n}, n \in \mathbb{N})$ be a sequence of nonnegative random variables such that the sum $S_N = \sum_{n\geq 1} X_{N,n}$ is almost surely finite.

Assumption A1. Suppose that there exists a sequence of numbers $(c_N, N \in \mathbb{N})$ converging to 0 such that for all N, $c_N \geq \sup_n |X_{N,n}|$.

Let $P_N := \prod_{n\geq 1}(1+X_{N,n})$. In this Apprendix, we establish two relations between the sum S_N and the product P_N when N converges to infinity. Note that we do not assume $(X_{N,n}; n\in\mathbb{N}, N\in\mathbb{N})$ to be independent nor identically distributed.

Theorem A.1. *(First relation) Assume A1, then for any real random variable Y, the following two assertions are equivalent:*

$$1)\quad S_N \xrightarrow[N\to+\infty]{(d)} Y \qquad\qquad 2)\quad P_N \xrightarrow[N\to+\infty]{(d)} e^Y.$$

Theorem A.2. *(Second relation) Assume A1 and that the sequence $(\exp(S_N), N\in\mathbb{N})$ is uniformly integrable. Then for any real constant C, the following two assertions are equivalent:*

$$1)\quad \mathbf{E}\left[e^{S_N}\right] \xrightarrow[N\to+\infty]{} C \qquad\qquad 2)\quad \mathbf{E}\left[P_N\right] \xrightarrow[N\to+\infty]{} C.$$

Proof of Theorem A.1: Let us first prove that 1) $\Rightarrow$ 2). The inequality $x-\frac{x^2}{2}\leq \ln(1+x)\leq x$ and the assumption imply that:

$$0\leq S_N-\ln(P_N)\leq \frac{1}{2}\sum_{n\geq 1}X_{N,n}^2\leq c_N S_N \xrightarrow[N\to+\infty]{(d)} 0.$$

Hence, by Slutsky's lemma van der Vaart (1998, Lemma 2.8), $\ln(P_N)\xrightarrow[N\to+\infty]{(d)} Y$.
Let us now prove that 2) $\Rightarrow$ 1), we see that for all $x>0, 0\leq x-\ln(1+x)\leq x\ln(1+x)$. We deduce

$$0\leq S_N-\ln(P_N)\leq \sum_{n\geq 1}X_{N,n}\ln(1+X_{N,n})\leq c_N\ln(P_N)\xrightarrow[N\to+\infty]{(d)} 0.$$

Thus, $S_N\xrightarrow[N\to+\infty]{(d)} Y$. The equivalence is proved. □

Proof of Theorem A.2: For the 1) $\Rightarrow$ 2) direction:
The sequence $(\exp(S_n), n\in\mathbb{N})$ being uniformly integrable, thus there is a subsequence $(n_k, k\in\mathbb{N})$ of $\mathbb{N}$ and a random variable $Z\in L^1$ such that:

$$\exp S_{n_k}\xrightarrow[n\to\infty]{(\mathrm{d})} Z \qquad \text{and} \qquad \mathbf{E}\left[\exp(S_{n_k})\right]\xrightarrow[k\to\infty]{}\mathbf{E}[Z].$$

We deduce that $\mathbf{E}[Z]=C$ and by Theorem A.1, we have $P_{n_k}\xrightarrow[n\to\infty]{(\mathrm{d})} Z$. Besides, the uniform integrability of $(\exp(S_n), n\in\mathbb{N})$ implies the uniform integrability of $(P_N, N\in\mathbb{N})$, since $0\leq P_N\leq e^{S_N}$. So,

$$\mathbf{E}[P_{n_k}]\xrightarrow[k\to\infty]{}\mathbf{E}[Z]=C.$$

Notice that the uniform integrability and the convergence $\mathbf{E}(S_N)\xrightarrow{N\to\infty} C$ are still valid if we take any subsequence $(S_{m_i}, i\in\mathbb{N})$ of $(S_N, N\in\mathbb{N})$. Thus, the result so far implies that for every subsequence $(m_i, i\in\mathbb{N})$ of $\mathbb{N}$, there is a subsequence $(m_{i_k}, k\in\mathbb{N})$ of (m_i) such that:

$$\mathbf{E}[P_{m_{i_k}}]\xrightarrow[k\to\infty]{} C.$$

The first implication is proved. The reciprocal is similar. □

Remark A.3. Note that uniform integrability implies tightness.

Appendix B. Some auxiliary results on random walks

Let $(S_n, n \in \mathbb{N}_0)$ be a simple symmetric random walks on $\mathbb{Z}$ and :

i. $(X_k, k \in \mathbb{N}_0)$ be a sequence of random variables such that $X_0 = 0$ and $X_k := \inf\{N > X_{k-1} : S_N = 0\}$ for all positive integer k,

ii. $\mathcal{F} := (\mathcal{F}_k, k \in \mathbb{N}_0)$ be the canonical filtration of the process $(X_k, k \in \mathbb{N}_0)$,

iii. $T_k := X_k - X_{k-1}$ for all positive integer k,

iv. $\tau_N := \inf\{k \geq 0 : X_k \geq N\}$.

Clearly, by definition, for each N, τ_N is a stopping time with respect to the filtration $\mathcal{F}$ and by Markov's property of S, $(T_k, k \in \mathbb{N})$ is a sequence of indepedent identically distributed random variables.

Notice that T_1 is the first time after 0 at which the random walk S returns to the position 0. Clearly, this stopping time is well-known. One of its properties is that

Lemma B.1. *There is a positive constant C such that for all $k \in \mathbb{N}$,*

$$\mathbf{P}(T_1 = 2k) = 2^{-2k+1}\frac{1}{k}\binom{2k-2}{k-1} \geq \frac{C}{k^{3/2}}.$$

Indeed, this lemma is just a combination of Theorem 9.2 in Révész (2005) and Stirling's formula. Concerning τ_N, by its definition, we have the following equality which will be useful for our later analysis:

$$\tau_N - 1 = \sup\{k \geq 0 : X_k \leq N-1\} = \sum_{n=1}^{N-1} \mathbb{1}_{S_n=0}.$$

In the following, we present the main theorem of this Section.

Theorem B.2. *(Boundedness of exponential moments of local times)*
Let S be a simple random walk on $\mathbb{Z}$ starting from 0, then for any constant $\beta \geq 0$, we have:

$$\sup_N \mathbf{E}\left[\exp\left(\beta N^{-1/2}\sum_{n=1}^{N}\mathbb{1}_{S_n=0}\right)\right] < +\infty.$$

This is a corollary of Sohier (2009, Lemma 4.2). Here, we give an alternative proof using Lemma B.1.

Proof: The main idea to prove this theorem is to construct many appropriate martingales to estimate the underlying exponential moment. The construction is as follows, for each $N \in \mathbb{N}$, define:

i. $X_n^N := \sum_{i=1}^n \min(T_i, N)$.

ii. $\gamma_N := \inf\{n \geq 1 : X_n^N \geq N\}$.

iii. $\lambda_N(\beta) := -\log \mathbf{E}(e^{-\beta\min(T_1,N)}) > 0 \quad \forall N \in \mathbb{N}, \beta > 0$.

iii. $M_n^N := \exp(-\beta X_n^N + n\lambda_N(\beta))$.

Then by noticing that the random variables $T_1, T_2, \dots$ are i.i.d, we see that for each N, $(M_n^N, n \in \mathbb{N})$ is a martingale with respect to the filtration $\mathcal{F}$. In addition, because $\forall n, N : X_n^N \geq n$, $\forall N : \tau_N \leq N$. Hence by the optional sampling theorem, $\forall N \in \mathbb{N}, \beta > 0$,

$$\mathbf{E}\left[\exp(-\beta X_{\gamma_N}^N + \gamma_N\lambda_N(\beta))\right] = 1.$$

Besides, by definition of γ_N and X^N, we have:

$$X_{\gamma_N}^N = X_{\gamma_N-1}^N + \min(T_{\gamma_N}, N) \leq N + N = 2N.$$

Thus, $e^{2\beta} \geq \mathbf{E}(e^{\gamma_N\lambda_N(\beta/N)})$, and from Lemma B.3 it follows that for all $\beta > 0$,

$$\sup_N \mathbf{E}\left[\exp(\frac{1}{2}c(\beta)\gamma_N/\sqrt{N})\right] < \infty,$$

where $c(\beta) := C\int_0^{1/2} \frac{1}{t^{3/2}}(1-e^{-2t\beta})dt$ and C is the constant defined in Lemma B.1. By noticing that $\lim_{\beta\to\infty} c(\beta) = \infty$ and $\forall N : \tau_N = \gamma_N$, we conclude that for all $\beta > 0$:

$$\sup_N \mathbf{E}\left[\exp(\beta\tau_N/\sqrt{N})\right] < \infty,$$

which is essentially our desired conclusion because $\tau_N - 1 = \sum_{n=1}^{N-1} \mathbb{1}_{S_n=0}$. ☐

Lemma B.3. *The sequence of functions* $(\lambda_N, N \in \mathbb{N})$ *given in the proof of Theorem B.2 sasitifies the following inequality:*

$$\liminf_{N\to\infty} \sqrt{N}\lambda_N(\beta/N) \geq c(\beta),$$

with $c(\beta) := C\int_0^{1/2} \frac{1}{t^{3/2}}(1-e^{-2t\beta})dt$*, where* C *is the constant defined in the Lemma B.1.*

Proof: For any $\beta > 0$ and $N \geq 2$, we have:

$$\begin{aligned}
&1 - \mathbf{E}\left[e^{-\beta\min(T_1,N)/N}\right] \\
&= \sum_{k=1}^{\lfloor N/2\rfloor} \mathbf{P}(T_1 = 2k)(1-e^{-2k\beta/N}) + \mathbf{P}(T_1 \geq 2\lfloor N/2\rfloor + 2)(1-e^{-\beta}) \\
&\geq \sum_{k=1}^{\lfloor N/2\rfloor} \frac{C}{k^{3/2}}(1-e^{-2k\beta/N}).
\end{aligned}$$

Thus,

$$\liminf_{N\to\infty} \sqrt{N}\left(1 - \mathbf{E}\left[e^{-\beta\min(T_1,N)/N}\right]\right) \geq C\int_0^{1/2} \frac{1}{t^{3/2}}(1-e^{-2t\beta})dt = c(\beta) > 0.$$

From which, we conclude $\liminf_{N\to\infty} \sqrt{N}\lambda_N(\beta/N) \geq c(\beta)$. ☐

Appendix C. Auxiliary proofs

Lemma C.1. *Let* $(U_n), (V_n)$ *be two sequences of positive random variables such that* $0 \leq U_n \leq V_n$ *for all* n*, and* $V_1, V_2, \ldots$ *are uniformly integrable. Then if* $\frac{V_n}{U_n} \xrightarrow[n\to+\infty]{(d)} 1$ *and* $\lim_{n\to\infty} \mathbf{E}(V_n) = C$*, then* $\lim_{n\to\infty} \mathbf{E}(U_n) = C$*.*

Proof: The uniform integrability of (V_n) implies the uniform integrability of (U_n). The uniform integrability of (U_n) implies that for every subsequence (n_k) of $\mathbb{N}$, there exists a subsequence (n_{k_l}) of (n_k) such that $(U_{n_{k_l}}, l \in \mathbb{N})$ converges in distribution to a random variable Z. The convergence of $(\frac{V_n}{U_n}, n \in \mathbb{N})$ implies that $(V_{n_{k_l}}, l \in \mathbb{N})$ also converges in distribution to Z. Then, the uniform integrability implies that $\lim_l \mathbf{E}(U_{n_{k_l}}) = C = \lim_l \mathbf{E}(V_{n_{k_l}})$. Hence the conclusion. ☐

Proof of Proposition 3.4: Assume that f and g have compact supports, then the sums in $S_n^N(f)$ and $S_n^N(g)$ have a finite number of terms; thus, point i is trivial. Point ii is also trivial by recalling that ω is a collection of centered random variables. Now, for point iii, observe that for any $\mathbf{i}, \mathbf{i}' \in E_n^N, \mathbf{x}, \mathbf{x}' \in \mathbb{Z}^n$:

$$\mathbf{E}\left[\prod_{l=1}^n \omega(\mathbf{i}_l, \mathbf{x}_l) \prod_{l=1}^n \omega(\mathbf{i}'_l, \mathbf{x}'_l)\right] = \mathbb{1}_{\{\mathbf{i}=\mathbf{i}', \mathbf{x}=\mathbf{x}'\}}.$$

Hence

$$\begin{aligned}\mathbf{E}\left[\mathcal{S}_n^N(g)^2\right] &= 2^n \sum_{\mathbf{i}\in E_n^N} \sum_{\substack{\mathbf{z}\in\mathbb{Z}^n \\ \mathbf{i}\leftrightarrow\mathbf{z}}} A_N(\mathbf{i},\mathbf{z})^2 \overline{g_N}\left(\frac{\mathbf{i}}{N}, \frac{\mathbf{z}}{\sqrt{N}}\right)^2 \\ &\leq 2^n \sum_{\mathbf{i}\in[\![1,N]\!]^n} \sum_{\substack{\mathbf{z}\in\mathbb{Z}^n \\ \mathbf{i}\leftrightarrow\mathbf{z}}} c^{2n} \frac{1}{|\mathcal{R}|} \int_{\mathcal{R}} g(\mathbf{t},\mathbf{y})^2 d\mathbf{t}d\mathbf{y} \\ &= N^{3n/2} c^{2n} \int_{[0,1]^n} \int_{\mathbb{R}^n} g(\mathbf{t},\mathbf{y})^2 d\mathbf{t}\mathbf{y}.\end{aligned}$$

The last inequality is simply an application of the Cauchy-Schwarz lemma. So the properties i, ii, and iii are valid for compactly supported functions. In other words, $g \mapsto S_n^N(g)$ is a linear Lipschitz continuous mapping that maps the space $L^2_{\text{compact}}([0,1]^n \times \mathbb{R}^n)$ into $L^2(\mathbf{P})$. Hence, all the properties i, ii, iii can be extended naturally to all $L^2([0,1]^n \times \mathbb{R}^n)$ by the density of $L^2_{\text{compact}}([0,1]^n \times \mathbb{R}^n)$ in $L^2([0,1]^n \times \mathbb{R}^n)$.

For the covariance relation in point iv, one can observe that if $\mathbf{i} \in E_{n_1}^N, \mathbf{x} \in \mathbb{Z}^{n_1}, \mathbf{i}' \in E_{n_2}^N, \mathbf{x}' \in \mathbb{Z}^{n_2}$, then

$$\mathbf{E}\left[\prod_{l=1}^{n_1} \omega(\mathbf{i}_l, \mathbf{x}_l) \prod_{l=1}^{n_2} \omega(\mathbf{i}'_l, \mathbf{x}'_l)\right] = 0.$$

because there is necessarily one ω term that is distinct from the others, and its independence from the rest implies zero expectation. Hence, iv is clearly true if g_1, g_2 have compact supports. The extension to non-compactly-supported functions can also be obtained by a density argument as above. □

Proof of Proposition 4.1: Recall that a is a bounded function, then there is a positive number β such that $\|a\|_\infty \leq \beta$. Hence, $\sum_{n\geq 0} \|a^{\otimes n}\varrho_n\|_2^2 \leq \sum_{n\geq 0} \beta^{2n}\|\varrho_n\|_2^2$. Thus, it suffices to prove that $\boldsymbol{\varrho}(\beta)$ belongs to the Fock space, which is proven in Alberts et al. (2014, Subsection 3.4). □

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