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A functional limit theorem for lattice oscillating random walks

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Abstract. This paper is devoted to an invariance principle for Kemperman's model of oscillating random walk on $\mathbb{Z}$. This result appears as an extension of the invariance principal theorem for classical random walks on $\mathbb{Z}$ or reflected random walks on $\mathbb{N}_0$. Relying on some natural Markov sub-process which takes into account the oscillation of the random walks between $\mathbb{Z}^-$ and $\mathbb{Z}^+$, we first construct an aperiodic sequence of renewal operators acting on a suitable Banach space and then apply a powerful theorem proved by S. Gouëzel.

1. Model and setting

1.1. *Introduction.* Consider two independent sequences of i.i.d. discrete random variables $(\xi_n)_{n\geq 1}$ and $(\xi'_n)_{n\geq 1}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with respective distributions μ and μ'.

For any fixed $\alpha \in [0,1]$, the oscillating random walk $\mathcal{X}^{(\alpha)} = \left(X_n^{(\alpha)}\right)_{n\geq 0}$ is defined recursively by: $X_0^{(\alpha)} = x$, where $x \in \mathbb{Z}$ is fixed, and for $n \geq 0$,

$$X_{n+1}^{(\alpha)} = \begin{cases} X_n^{(\alpha)} + \xi_{n+1} & \text{if } X_n^{(\alpha)} \leq -1, \\ \eta_{n+1} & \text{if } X_n^{(\alpha)} = 0, \\ X_n^{(\alpha)} + \xi'_{n+1} & \text{if } X_n^{(\alpha)} \geq 1, \end{cases} \tag{1.1}$$

where $\eta_{n+1} := B_{n+1}\xi_{n+1} + (1 - B_{n+1})\xi'_{n+1}$ and $(B_n)_{n\geq 1}$ is a sequence of i.i.d. Bernoulli random variables (independent of $(\xi_n)_{n\geq 1}$ and $(\xi'_n)_{n\geq 1}$) with $\mathbb{P}[B_i = 1] = \alpha = 1 - \mathbb{P}[B_i = 0]$.

When we want to emphasize the dependence in μ and μ' of this oscillating process, we denote it by $\mathcal{X}^{(\alpha)}(\mu, \mu')$.

This spatially non-homogeneous random walk was first introduced by Kemperman (1974) to model discrete-time diffusions in one dimensional space with three different media $\mathbb{Z}^+$ and $\mathbb{Z}^-$ and a barrier $\{0\}$. Whenever the process $\mathcal{X}^{(\alpha)}(\mu, \mu')$ stays on the negative half line, its excursion is directed by the jumps ξ_n until it reaches the positive half line; then, it continues being directed by

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the jumps ξ'_n until returning in the negative half line and so on. After each visit of the origin, the increment is governed by the distribution of η_n, which is a convex combination of μ and μ'. Our considering system is referred to as a special case when the barrier is degenerated as a single point; in general context, it may be any determined interval $[a, b] \cap \mathbb{Z}$ which passes through the origin, see Kim and Lotov (2004) for instance. Another interesting variant has been studied by Madras and Tanny (1994) dealing with an oscillating random walk with a moving barrier at some constant speed. Although this basically leads to differences in its long-term behaviour compared to (1.1), we may be able to trace this model back to (1.1) by using some appropriate translations for its random increments.

In the present paper, we prove an invariant principle for $\mathcal{X}^{(\alpha)}(\mu, \mu')$ towards the skew Brownian motion $\left(B_t^\gamma\right)_{t>0}$ on $\mathbb{R}$ with parameter $\gamma \in [0,1]$. The diffusion $\left(B_t^\gamma\right)_{t>0}$ is obtained from the standard Brownian process by independently altering the signs of the excursions away from 0, each excursion being positive with probability γ and negative with probability $1-\gamma$. By Revuz and Yor (1999), its heat kernel is given by: for any $x, y \in \mathbb{R}$ and $t > 0$,

$$p_t^\gamma(x, y) := p_t(x, y) + (2\gamma - 1)\ \mathrm{sign}(y)\, p_t(0, |x| + |y|),$$

where $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$ is the transition density of the Brownian motion.

Throughout this paper, we suppose that the following general assumptions always hold:

H1 $\left(\xi_n\right)_{n\geq 1}$ and $\left(\xi'_n\right)_{n\geq 1}$ are independent sequences of i.i.d. $\mathbb{Z}$-valued random variables, with finite variances σ^2 and σ'^2, respectively.

H2 *Both distributions μ and μ' are centered (i.e. $\mathbb{E}[\xi_n] = \mathbb{E}[\xi'_n] = 0$).*

H3 *Both distributions μ and μ' are strongly aperiodic on $\mathbb{Z}$, i.e. their supports are not included in $b + a\mathbb{Z}$ for any $a > 1$ and $b \in \{0, \dots, a-1\}$.*

H4 *There exists $\delta > 0$ such that $\mathbb{E}[(\xi_n^+)^{3+\delta}] + \mathbb{E}[(\xi_n'^-)^{3+\delta}] < +\infty$, where $\xi_n^+ := \max\left\{0, \xi_n\right\}$ and $\xi_n'^- := \max\left\{0, -\xi'_n\right\}$.*

Let us emphasize that, under hypotheses **H1**, **H2** and **H3**, the oscillating random walk $\mathcal{X}^{(\alpha)}$ is irreducible and null recurrent on $\mathbb{Z}$; this property is not stated in Vo (2023) and we will detail the argument later (see Proposition 3.2).

We denote by $S = \left(S_n\right)_{n\geq 1}$ (resp. $S' = \left(S'_n\right)_{n\geq 1}$) the random walk defined by $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$ for $n \geq 1$ (resp. $S'_0 = 0$ and $S'_n = \xi'_1 + \ldots + \xi'_n$ for $n \geq 1$). Let $\left(\ell_i\right)_{i\geq 0}$ be the sequence of strictly ascending ladder epochs associated with S and defined recursively by $\ell_0 = 0$ and, for $i \geq 0$,

$$\ell_{i+1} := \inf\left\{k > \ell_i \mid S_k > S_{\ell_i}\right\}$$

(with the convention $\inf \emptyset = +\infty$). We also consider the sequence of descending ladder epochs $\left(\ell'_i\right)_{i\geq 0}$ of S', defined as follows:

$$\ell'_0 = 0, \quad \text{and} \quad \ell'_{i+1} := \inf\{k > \ell'_i \mid S'_k < S'_{\ell'_i}\}, \quad \text{for any } i \geq 0.$$

Under hypothesis **H2**, it holds $\mathbb{P}[\limsup_{n\to+\infty} S_n = +\infty] = \mathbb{P}[\liminf_{n\to+\infty} S'_n = -\infty] = 1$; hence, all the random variables ℓ_i and ℓ'_i are $\mathbb{P}$-a.s. finite. In addition, both sequences $\left(\ell_{i+1}-\ell_i\right)_{i\geq 0}$ and $\left(S_{\ell_{i+1}}-S_{\ell_i}\right)_{i\geq 0}$ contain i.i.d. random elements with distributions of ℓ_1 and S_{ℓ_1}, respectively; the same property holds for $\left(\ell'_{i+1}-\ell'_i\right)_{i\geq 0}$ and $\left(S'_{\ell'_{i+1}}-S'_{\ell'_i}\right)_{i\geq 0}$. Consequently, processes $\left(\ell_i\right)_{i\geq 0}$, $\left(S_{\ell_i}\right)_{i\geq 0}$, $\left(\ell'_i\right)_{i\geq 0}$ and $\left(S'_{\ell'_i}\right)_{i\geq 0}$ are all random walks with i.i.d. increments.

We denote μ_+ the distribution of S_{ℓ_1} and $\mathcal{U}_+$ its potential defined by $\mathcal{U}_+ := \sum_{n\geq 0}(\mu_+)^{*n}$. Similarly μ'_- denotes the distribution of $S'_{\ell'_1}$ and $\mathcal{U}'_- := \sum_{n\geq 0}(\mu'_-)^{*n}$.

In particular, the oscillating random walk $\mathcal{X}^{(\alpha)}$ visits $\mathbb{Z}^-$ and $\mathbb{Z}^+$ infinitely often; in order to control the excursions inside each of these these two half lines, it is natural to consider the following stopping times $\tau^S(-x)$, $\tau^{S'}(x)$ with $x \geq 1$, associated with S and S' respectively and defined by

$$\tau^S(-x) := \inf\{n \geq 1 \mid -x + S_n \geq 0\}, \quad \text{and} \quad \tau^{S'}(x) := \inf\{n \geq 1 \mid x + S'_n \leq 0\}.$$

In the sequel, we focus on the "ascending renewal function" h_a of S and the "descending renewal function" h'_d of S' defined by

$$h_a(x) := \begin{cases} \mathcal{U}_+[0,x] = \sum_{i\geq 0} \mathbb{P}[S_{\ell_i} \leq x] & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h'_d(x) := \begin{cases} \mathcal{U}'_-[-x,0] = \sum_{i\geq 0} \mathbb{P}[S'_{\ell'_i} \geq -x] & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\check{h}_a$ the function $x \mapsto h_a(-x)$, it appears in the definition of the parameter γ below.

Both functions h_a and h'_d are increasing and satisfy $h_a(x) = O(x)$ and $h'_d(x) = O(x)$. They appear crucially in the quantitative estimates of the fluctuations of S and S'; see Subsection 2.1 for precise statements.

Let us end this paragraph devoted to the presentation of quantities that play an important role in the rest of the paper.

- By classical results on 1-dimensional random walks (Feller, 1971), under hypotheses **H1** and **H2**, both constants $c = \dfrac{\mathbb{E}[S_{\ell_1}]}{\sigma\sqrt{2\pi}}$ and $c' = \dfrac{\mathbb{E}[-S_{\ell'_1}]}{\sigma'\sqrt{2\pi}}$ are finite.
- Under hypotheses **H1**, **H2** and **H3**, the "crossing sub-process" $\mathcal{X}^{(\alpha)}_{\mathbf{C}}$ which corresponds to the sign changes of the process $\mathcal{X}^{(\alpha)}$ is well defined (see Section 3.1) and it is positive recurrent on its unique irreducible class. We denote by ν its unique invariant probability measure on $\mathbb{Z}$.

1.2. *Main result.* **From now on, we fix** $\alpha \in [0,1]$ and consider the continuous and linearly interpolated version $(X^{(\alpha)}_{nt})$ of $\mathcal{X}^{(\alpha)}$, defined by: for any $n \geq 1$ and $t \in (0,1]$,

$$X^{(\alpha)}_{nt} = \sum_{i=1}^{n} \left(X^{(\alpha)}_{[nt]} + (nt - [nt]) \times J_{[nt]+1} \right) \mathbb{1}_{[\frac{i-1}{n}, \frac{i}{n}[}(t),$$

where

$$J_{[nt]+1} := \begin{cases} \xi_{[nt]+1} & if \quad X^{(\alpha)}_{[nt]} \leq -1 \\ \eta_{[nt]+1} & if \quad X^{(\alpha)}_{[nt]} = 0 \\ \xi'_{[nt]+1} & if \quad X^{(\alpha)}_{[nt]} \geq 1. \end{cases}$$

We also set

$$X^{(\alpha,n)}(t) := \begin{cases} \frac{X^{(\alpha)}_{nt}}{\sigma\sqrt{n}} & \text{if } X_{nt} \leq 0, \\ \frac{X^{(\alpha)}_{nt}}{\sigma'\sqrt{n}} & \text{if } X_{nt} \geq 0. \end{cases}$$

The main result of this paper is the following:

Theorem 1.1. *Assume that hypotheses* **H1**–**H4** *are satisfied. Then, as* $n \to +\infty$, *the normalized stochastic process* $\{X^{(\alpha,n)}(t), t \in [0,1]\}_{n\geq 1}$ *converges weakly in the space of continuous function* $C([0,1])$ *to the skew Brownian motion* $W_\gamma := \{W_\gamma(t), t \in [0,1]\}$ *with parameter* $\gamma = \dfrac{c'\nu(h'_d)}{c\nu(\check{h}_a) + c'\nu(h'_d)}$.

Let us clarify the value of the parameter γ in two peculiar cases of (1.1).

• When $\mu = \mu'$, the chain $\mathcal{X}^{(\alpha)}$ is an ordinary random walk on $\mathbb{Z}$ directed by the unique type of jumps $\big(\xi_n\big)_{n\geq 1}$ and the limit diffusion W_γ is the Brownian motion. In this case, the parameter γ equals $\frac{1}{2}$ since the sequences of ladder heights $\big(S_{\ell_i}\big)_{i\geq 1}$ and $\big(-S'_{\ell'_i}\big)_{i\geq 1}$ coincide.

• When $\mu(x) = \mu'(-x)$ for any $x \in \mathbb{Z}$, the random walk $\mathcal{X}^{(\alpha)}$ is the so-called "anti-symmetric random walk" (or "reflected random walk" as usual), which appears in several works, see for instance Essifi and Peigné (2015) and Peigné and Woess (2006). By setting $\xi_n = -\xi'_n$, the behaviour of the chain $\mathcal{X}^{(\alpha)}$ on positive and negative half lines, respectively, are mirror images of each other. Hence, we may "glue" them together to get an unifying Markov chain on $\mathbb{Z}^+ \cup \{0\}$ receiving $\{0\}$ as its reflecting boundary. Accordingly, $\gamma = 1$ in this case and it matches perfectly with the result in Ngo and Peigné (2021), which states that the normalized reflected random walk (constructed as above) converges weakly in $C([0,1])$ towards the absolute value of the standard Brownian motion.

• Notice that the limit process W_γ does not depend on $\alpha \in [0,1]$. Henceforth, we fix α and set $\mathcal{X}^{(\alpha)} = \mathcal{X}$ in order to simplify the notations.

1.3. *Notations.* We set $\mathbb{Z} := \mathbb{Z}^+ \cup \mathbb{Z}^- \cup \{0\}$ and $\overline{\mathbb{D}}$ the closed unit ball in $\mathbb{C}$. Given two positive real sequences $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ and $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$, we write as usual

- $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$,
- $a_n \approx b_n$ if $\lim_{n\to\infty} (a_n - b_n) = 0$,
- $a_n = O(b_n)$ if $\limsup_{n\to\infty} a_n/b_n < +\infty$ (we also write $\mathbf{a} \preceq \mathbf{b}$),
- $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$,
- $\mathbf{a} \asymp \mathbf{b}$ if $\mathbf{a} \preceq \mathbf{b} \preceq \mathbf{a}$, or equivalently $\dfrac{1}{c}\, b_n \leq a_n \leq c\, b_n$ for some constant $c \geq 1$.

The paper is organized as follows. In Section 2, we recall some important estimates in the theory of fluctuations of random walks; we introduce in particular the renewal functions associated with 1-dimensional random walks and relative conditional limit theorems. These helpful tools appear in Section 4 to compute the multi-dimensional distribution of the limit process. The center of gravity of the paper is Section 3 where we adapt the approach used in Ngo and Peigné (2021) in the case of the reflected random walk (with proper adjustments to derive Corollary 3.6 and to determine the parameter γ later on).

2. Auxiliary results for random walks

In this section, we present some classical results on fluctuations of random walk on $\mathbb{Z}$.

2.1. *Asymptotic estimates for fluctuations of a random walk.* The following statement summarizes classical results on fluctuations of random walks which are used below at various places (for instance, see Proposition 11 in Doney (2012), Theorem A in Kozlov (1977) et al). Recall that $c = \dfrac{\mathbb{E}[S_{\ell_1}]}{\sigma\sqrt{2\pi}}$ and $c' = \dfrac{\mathbb{E}[-S_{\ell'_1}]}{\sigma'\sqrt{2\pi}}$.

Lemma 2.1. *(Asymptotic property) Under assumptions* **H1**– **H3***, for any $x, y \geq 1$, it holds, as $n \to \infty$,*

a) $\mathbb{P}[\tau^S(-x) > n] \sim 2c\,\frac{h_a(x)}{\sqrt{n}}$, *and* $\mathbb{P}[\tau^{S'}(x) > n] \sim 2c'\,\frac{h'_d(x)}{\sqrt{n}}$;

b) $\mathbb{P}[\tau^S(-x) > n, -x + S_n = -y] \sim \frac{1}{\sigma\sqrt{2\pi}}\,\frac{h_a(x)\,h_d(y)}{n^{3/2}}$,

and $\mathbb{P}[\tau^{S'}(x) > n, x + S'_n = y] \sim \frac{1}{\sigma'\sqrt{2\pi}}\,\frac{h'_d(x)\,h'_a(y)}{n^{3/2}}$;

where h_d (resp. h'_a) is the descending (resp. ascending) renewal function associated with the random walk S (resp. S').

c) $\mathbb{P}[\tau^S(-x) = n] \sim c\, \frac{h_a(x)}{n^{3/2}}$, *and* $\mathbb{P}[\tau^{S'}(x) = n] \sim c'\, \frac{h'_d(x)}{n^{3/2}}$;

Lemma 2.2. *(Upper bound) For any $n \geq 1$, it holds*

a) $\mathbb{P}[\tau^S(-x) > n] \preceq \frac{1+x}{\sqrt{n}}$, *and* $\mathbb{P}[\tau^{S'}(x) > n] \preceq \frac{1+x}{\sqrt{n}}$;

b) $\mathbb{P}[\tau^S(-x) > n, -x + S_n = -y] \preceq \frac{(1+x)(1+y)}{n^{3/2}}$,

and $\mathbb{P}[\tau^{S'}(x) > n, x + S'_n = y] \preceq \frac{(1+x)(1+y)}{n^{3/2}}$;

c) $\mathbb{P}[\tau^S(-x) = n] \preceq \frac{1+x}{n^{3/2}}$, *and* $\mathbb{P}[\tau^{S'}(x) = n] \preceq \frac{1+x}{n^{3/2}}$.

As a direct consequence of b) in Lemmas 2.1 and 2.2, for any $x \geq 1$ and $w \geq 0$,

$$\mathbb{P}[\tau^S(-x) = n, -x + S_n = w] \preceq \frac{1+x}{n^{3/2}} \sum_{z \geq w+1} z\mu(z). \tag{2.1}$$

Indeed, for any $n \geq 1$,

$$\begin{aligned}
&\mathbb{P}[\tau^S(-x) = n, -x + S_n = w] \\
&\qquad = \sum_{y \geq 1} \mathbb{P}[\tau^S(-x) = n, -x + S_{n-1} = -y, -y + \xi_n = w] \\
&\qquad = \sum_{y \geq 1} \mathbb{P}[\tau^S(-x) > n-1, -x + S_{n-1} = -y, -y + \xi_n = w] \\
&\qquad = \sum_{y \geq 1} \mathbb{P}[\tau^S(-x) > n-1, -x + S_{n-1} = -y]\mu(y+w) \\
&\qquad \preceq \frac{1+x}{n^{3/2}} \underbrace{\sum_{y \geq 1} (1+y)\mu(y+w)}_{\preceq \sum_{z \geq w+1} z\mu(z) < +\infty} .
\end{aligned}$$

Notice also that, more precisely it holds

$$\mathbb{P}[\tau^S(-x) = n, -x + S_n = w] \sim \frac{h_a(x)}{\sigma\sqrt{2\pi} n^{3/2}} \sum_{y \geq 1} h_d(y)\mu(y+w).$$

2.2. *Conditional limit theorems.* It is worth remarking some necessary limit theorems which are very helpful for us to control the fluctuations of excursions between two consecutive crossing times and contribute significantly to reduce the complexity when dealing with multidimensional distribution of these excursions. Now, assume that $\mathbb{E}[\xi'_1] = 0$ and $\mathbb{E}[(\xi'_1)^2] < +\infty$ and let $\left(S'(t)\right)_{t \geq 0}$ be the continuous time process constructed from the sequence $\left(S'_n\right)_{n \geq 0}$ by using the linear interpolation between the values at integer points.

By Lemma 2.3 in Afanasyev et al. (2005), for $x \geq 1$, the rescaled process $\left(\frac{x + S'_{[nt]}}{\sigma'\sqrt{n}}, t \in [0,1]\right)$ conditioning on the event $[\tau^{S'}(x) > n]$ converges weakly on $C([0,1], \mathbb{R})$ towards the Brownian meander. In other words, for any bounded Lipschitz continuous function $\psi : \mathbb{R} \to \mathbb{R}$ and any $t \in (0,1]$ and $x \geq 1$,

$$\lim_{n \to +\infty} \mathbb{E}\left[\psi\left(\frac{x + S'_{[nt]}}{\sigma'\sqrt{n}}\right) \mid \tau^{S'}(x) > [nt]\right] = \frac{1}{t} \int_0^{+\infty} \psi(u) u \exp\left(-\frac{u^2}{2t}\right) du. \tag{2.2}$$

Let us also state the Caravena-Chaumont's result about random bridges conditioned to stay positive in the discrete case. Roughly speaking, as $n \to +\infty$, for any starting point $x \geq 1$ and any ending point $y \geq 1$, the random bridge of the random walk S, starting at x, ending at y at time n and conditioned to stay positive until time n, after a linear interpolation and a diffusive rescaling, converges in distribution on $C([0,1],\mathbb{R})$ towards the normalized Brownian excursion $\mathcal{E}^+$:

$$\left(\left(\frac{S'_{[nt]}}{\sigma'\sqrt{n}}\right)_{t\in[0,1]} \mid \tau^{S'}(x) > [nt], S'_n = y\right) \xrightarrow{\mathcal{L}} \mathcal{E}^+, \quad \text{as } n \to +\infty.$$

More precisely, for any $x, y \geq 1, 0 < s < t \leq 1$ and any bounded Lipschitz continuous function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to+\infty} \mathbb{E}\left[\psi\left(\frac{x + S'_{[ns]}}{\sigma'\sqrt{n}}\right) \mid \tau^{S'}(x) > [nt], x + S'_{[nt]} = y\right]$$
$$= \int_0^{+\infty} 2\psi(u\sqrt{t}) \exp\left(-\frac{u^2}{2\frac{s}{t}\frac{t-s}{t}}\right) \frac{u^2}{\sqrt{2\pi\frac{s^3}{t^3}\frac{(t-s)^3}{t^3}}} du. \tag{2.3}$$

3. Crossing times and renewal theory

In order to analyse the asymptotic behavior of the process $\mathcal{X}$, we decompose X_n as a sum of successive excursions in $\mathbb{Z}^-$ or $\mathbb{Z}^+$. It is therefore interesting to introduce the sequence $\mathbf{C} = \big(C_k\big)_{k\geq 0}$ of "crossing times", i.e. times at which the process $\mathcal{X}$ changes its sign: more precisely, $C_0 = 0$ and, for any $k \geq 0$,

$$C_{k+1} := \begin{cases} \inf\{n > C_k \mid X_{C_k} + (\xi_{C_k+1} + \cdots + \xi_n) \geq 0\} & \text{if } X_{C_k} \leq -1, \\ C_k + 1 & \text{if } X_{C_k} = 0, \\ \inf\{n > C_k \mid X_{C_k} + (\xi'_{C_k+1} + \cdots + \xi'_n) \leq -1\} & \text{if } X_{C_k} \geq 1. \end{cases} \tag{3.1}$$

Under hypothesis **H2**, the random times C_k are $\mathbb{P}$-a.s. finite and form a sequence of finite stopping times with respect to the canonical filtration $\big(\sigma(\xi_k, \xi'_k) \mid k \leq n\big)_{n\geq 1}$.

3.1. *On the crossing sub-process* $\mathcal{X}_{\mathbf{C}}$. We denote $\mathcal{X}_{\mathbf{C}} := \big(X_{C_k}\big)_{k\geq 0}$ the **crossing sub-process of** $\mathcal{X}$, which plays an important role in this paper.

Lemma 3.1. *The sub-process* $\mathcal{X}_{\mathbf{C}}$ *is a time-homogeneous Markov chain on* $\mathbb{Z}$ *with transition kernel* $\mathcal{C} = \big(\mathcal{C}(x,y)\big)_{x,y\in\mathbb{Z}}$ *given by*

$$\mathcal{C}(x,y) = \begin{cases} \displaystyle\sum_{t=0}^{-x-1} \mu_+(y-x-t)\,\mathcal{U}_+(t) & \text{if } x \leq -1 \text{ and } y \geq 0, \\ \alpha\mu(y) + (1-\alpha)\mu'(y) & \text{if } x = 0 \text{ and } y \in \mathbb{Z}, \\ \displaystyle\sum_{t=-x+1}^{0} \mu'_-(y-x-t)\ \mathcal{U}'_-(t) & \text{if } x \geq 1 \text{ and } y \leq 0. \end{cases} \tag{3.2}$$

Proof: The Markov property is obvious from the above definition.
Now, we compute $\mathcal{C}(x,y)$ for any $x \leq -1$ and $y \geq 0$ (other cases are similar) as follows. Noticing that the first crossing time C_1 belongs $\mathbb{P}$-a.s. to the set $\big\{\ell_k \mid k \geq 1\big\}$ and that the sequence $\big(S_{\ell_k}\big)_{k\geq 1}$

is increasing, we may write

$$\begin{aligned}\mathcal{C}(x,y) &= \sum_{k\geq 1} \mathbb{P}[x+S_{\ell_{k-1}} \leq -1, x+S_{\ell_k} \ = y] \\ &= \sum_{k\geq 1}\sum_{t=0}^{-x-1} \mathbb{P}[S_{\ell_{k-1}} = t]\,\mathbb{P}[S_{\ell_k} - S_{\ell_{k-1}} = y-x-t] \\ &= \sum_{t=0}^{-x-1} \mathbb{P}[S_{\ell_1} = y-x-t] \sum_{i\geq 0}\mathbb{P}[S_{\ell_i} = t] \\ &= \sum_{t=0}^{-x-1} \mu_+(y-x-t)\,\mathcal{U}_+(t).\end{aligned}$$

□

When **H2** holds, the crossing sub-process $\mathcal{X}_{\mathbf{C}}$ is well defined and it is irreducible, aperiodic and positive recurrent on its unique essential class $\mathcal{I}_{\mathbf{C}}(X_0)$. Notice that this essential class can be a proper subset of $\mathbb{Z}$; it occurs for instance when the support of μ is bounded from above or the one of μ' is bounded from below. Nevertheless, it admits a unique invariant probability measure ν supported by $\mathcal{I}_{\mathbf{C}}(X_0)$. Also in Vo (2023), the explicit expression of ν is only known when $\alpha \in \{0,1\}$ and the support of μ (resp. μ') is included in $\mathbb{Z}^+$ (resp. in $\mathbb{Z}^-$). However, the existence of ν is enough for our purpose regardless of its exact formula.

Furthermore, by Theorem 2.2 in Vo (2023), under hypothesis **H3**, the oscillating random walk $\mathcal{X}$ is irreducible on $\mathbb{Z}$. It is also important to notice the following property.

Proposition 3.2. *Under hypotheses* **H1**– **H3**, *the oscillating random walk is null recurrent. In other words, setting* $\mathbf{t}_0 := \big\{k \geq 1 \mid X_k = 0\big\}$ *then it holds*

$$\mathbb{P}_0[\mathbf{t}_0 < +\infty] = 1 \quad \textit{and} \quad \mathbb{E}_0[\mathbf{t}_0] = +\infty.$$

Proof: We may choose $x_0, y_0 \geq 1$ s.t. $\mu(-x_0) > 0, \mu'(y_0) > 0$ and write for any $n \geq 1$,

$$\begin{aligned}\mathbb{P}_0[\mathbf{t}_0 > n] &\geq \mathbb{P}_0[\mathbf{t}_0 > n, \xi_1 = -x_0] + \mathbb{P}_0[\mathbf{t}_0 > n, \xi'_1 = y_0] \\ &\geq \alpha\mu(-x_0)\mathbb{P}[\ell_1 > n-1] + (1-\alpha)\mu'(y_0)\mathbb{P}[\ell'_1 > n-1].\end{aligned}$$

Hence $\mathbb{E}_0[\mathbf{t}_0] \geq \alpha\mu(-x_0)(1+\mathbb{E}[\ell_1]) + (1-\alpha)\mu'(y_0)(1+\mathbb{E}[\ell'_1]) = +\infty$. □

Another point to insist on here is that any excursion between two consecutive crossing times is uniquely governed by S or S'; thus, all the results obtained in the previous section can be applied. The decomposition technique that exploits this fact is classical and extremely efficient in controlling the varying excursions over time of Markov processes; for example, we use it in the last section to estimate the convergence of finite dimensional distribution. As a direct application, we can prove that the strong law of large numbers still holds for the chain $\mathcal{X}$.

Lemma 3.3. *Assume that* $\mathbb{E}[|\xi_n|] + \mathbb{E}[|\xi'_n|] < +\infty$ *and* $\mathbb{E}[\xi_n] = \mathbb{E}[\xi'_n] = 0$. *Then, it holds*

$$\lim_{n\to+\infty} \frac{X_n}{n} = 0 \ \ \mathbb{P}\text{-}a.s.$$

Proof: We decompose X_n as $X_n = X_n \mathbb{1}_{\{X_n \geq 1\}} + X_n \mathbb{1}_{\{X_n \leq -1\}}$.

Let us estimate the first term. For any $n \geq 1$, there exists a random integer $k(n) \geq 0$ such that $C_{k(n)} \leq n < C_{k(n)+1}$; notice that the condition $X_n \geq 1$ yields $X_{C_{k(n)}} \geq 1$. Hence, we get

$$\begin{aligned} 0 \leq \frac{X_n \mathbb{1}_{\{X_n \geq 1\}}}{n} &= \frac{X_{C_{k(n)}} + S'_n - S'_{C_{k(n)}}}{n} \\ &\leq \frac{\max\{X_0, \xi_{C_{k(n)}}\}}{n} + \frac{S'_n}{n} - \frac{S'_{C_{k(n)}}}{C_{k(n)}} \frac{C_{k(n)}}{n} \\ &\leq \frac{\max\{X_0, \xi_{C_{k(n)}}\}}{n} + \frac{S'_n}{n} + \left| \frac{S'_{C_{k(n)}}}{C_{k(n)}} \right|. \end{aligned}$$

By the strong law of large numbers, the different terms on the right-hand side above converges $\mathbb{P}$-a.s. to 0; so does $\frac{X_n \mathbb{1}_{\{X_n \geq 1\}}}{n}$. The second term is treated in the same way. □

3.2. *On aperiodic renewal sequences of operators.* Let $\big(\mathbb{Z}^{\otimes \mathbb{N}}, (\mathcal{P}(\mathbb{Z}))^{\otimes \mathbb{N}}, \mathcal{X}, (\mathbb{P}_x)_{x \in \mathbb{Z}}, \theta\big)$ be the canonical space, i.e. the space of trajectories associated with the Markov chain $\mathcal{X}$. For any $x \in \mathbb{Z}$, we probability measure $\mathbb{P}_x$ is the conditional probability with respect to the event $[X_0 = x]$, we denote by $\mathbb{E}_x$ the corresponding conditional expectation. The operator θ is the classical shift transformation defined by $\theta\big((x_k)_{k \geq 0}\big) = \big(x_{k+1}\big)_{k \geq 0}$ for any $\big(x_k\big)_{k \geq 0} \in \mathbb{Z}^{\otimes \mathbb{N}}$.

In this section, we study the behavior as $n \to +\infty$ of the sequence

$$H_n(x, y) = \sum_{k=1}^{+\infty} \mathbb{P}_x[C_k = n, X_n = y],$$

for any $x, y \in \mathbb{Z}$. Since the position at time C_k may vary, so that the excursions of $\mathcal{X}$ between two successive crossing times are not independent, it thus motivates us to take into account the long-term behaviours of these quantities and express them in terms of operators related to the crossing sub-process $\mathcal{X}_{\mathbf{C}}$. For this purpose, we apply a general renewal theorem due to Gouëzel (2011). This theorem relies on the decomposition of the operator $\mathcal{C}$ using a sequence of operators $\big(\mathcal{C}_n\big)_{n \geq 1}$ acting on some Banach space and that are not so difficult to deal with.

It is natural in our context to deal with the operators $\mathcal{C}_n = \big(\mathcal{C}_n(x, y)\big)_{x, y \in \mathbb{Z}}, n \geq 1$, defined by: for any $x, y \in \mathbb{Z}$ and any $n \geq 1$,

$$\mathcal{C}_n(x, y) := \mathbb{P}_x[C_1 = n, X_n = y].$$

The relation $\mathcal{C}(x, y) = \sum_{n \geq 1} \mathcal{C}_n(x, y)$ is obvious since $\mathcal{C}(x, y) = \mathbb{P}_x[C_1 < +\infty, X_{C_1} = y]$. We also pay attention to the case $x = 0$, that is $\mathcal{C}_1(0, y) = \mathbb{P}_0[X_1 = y] = \alpha\mu(y) + (1 - \alpha)\mu'(y)$ and $\mathcal{C}_n(0, y) = 0$ if $n \geq 2$.

For a function $\varphi : \mathbb{Z} \to \mathbb{C}$, we formally set

$$\mathcal{C}_n \varphi(x) := \sum_{y \in \mathbb{Z}:\, xy \leq 0} \mathcal{C}_n(x, y)\varphi(y) = \mathbb{E}_x[\varphi(X_n), C_1 = n] \quad \text{if } x \in \mathbb{Z} \setminus \{0\},$$

and $\mathcal{C}_1 \varphi(0) = \sum_{y \in \mathbb{Z}} \mathcal{C}_1(0, y)\varphi(y) = \mathbb{E}_0[\varphi(X_1)]$ and $\mathcal{C}_n \varphi(0) = 0 \quad$ if $n \geq 2$. The quantity $\mathcal{C}_n \varphi(x)$ is well defined for instance when $\varphi \in L^\infty(\mathbb{Z})$. Other Banach spaces can be considered; under moment assumptions, we describe below the action of the $\mathcal{C}_n$ on a bigger Banach space $\mathcal{B}_\delta$, more suitable to the situation as explained a little further on.

Notice that $\mathcal{C}_n(x,y) = \mathcal{C}_n \mathbb{1}_{\{y\}}(x)$ for any $x, y \in \mathbb{Z}$, which yields, by induction,

$$\begin{aligned} H_n(x,y) &= \sum_{k=1}^{+\infty} \mathbb{P}_x[C_k = n, X_n = y] \\ &= \sum_{k=1}^{+\infty} \sum_{j_1+\ldots+j_k=n} \mathbb{P}_x[C_1 = j_1, C_2 - C_1 = j_2, \ldots, C_k - C_{k-1} = j_k, X_n = y] \\ &= \sum_{k=1}^{+\infty} \sum_{j_1+\ldots+j_k=n} \mathcal{C}_{j_1} \ldots \mathcal{C}_{j_k} 1_{\{y\}}(x). \end{aligned} \tag{3.3}$$

As announced above, we apply a result of S. Gouëzel, stated in a general framework Gouëzel (2011), *on aperiodic renewal sequence of operators*, i.e. the sequence $\big(\mathcal{C}_n\big)_{n\geq 1}$ of operators acting on a Banach space $\big(\mathcal{B}, |\cdot|_{\mathcal{B}}\big)$ and satisfying the following conditions:

- the operators $\mathcal{C}_n, n \geq 1$, act on $\mathcal{B}$ and $\sum_{n\geq 1} \|\mathcal{C}_n\|_{\mathcal{B}} < +\infty$ (where $\|\cdot\|_{\mathcal{B}}$ denotes the norm on the space $\mathcal{L}(\mathcal{B})$ of continuous operators on $\big(\mathcal{B}, |\cdot|_{\mathcal{B}}\big)$);
- the operator $\mathcal{C}(z) := \sum_{n\geq 1} z^n \mathcal{C}_n$, defined for any $z \in \overline{\mathbb{D}}$, satisfies

R1- $\mathcal{C}(1)$ has a simple eigenvalue at 1 (with corresponding eigenprojector Π) and the rest of its spectrum is contained in a disk of radius < 1;
R2- for any complex number $z \in \overline{\mathbb{D}} \setminus \{1\}$, the spectral radius of $\mathcal{C}(z)$ is < 1;
R3- for any $n \geq 1$, the real number r_n defined by $\Pi \mathcal{C}_n \Pi = r_n \Pi$ is ≥ 0.

Condition R2 implies that, for any $z \in \overline{\mathbb{D}} \setminus \{1\}$, the operator $I - \mathcal{C}(z)$ is invertible on $\mathcal{B}$ and

$$(I - \mathcal{C}(z))^{-1} = \sum_{k\geq 0} \mathcal{C}(z)^k = \sum_{k\geq 0} \left(\sum_{j\geq 1} \mathcal{C}_j z^j \right)^k = \sum_{n\geq 0} \mathcal{H}_n z^n$$

with $\mathcal{H}_0 = I$ and $\mathcal{H}_n = \sum_{k=1}^{+\infty} \sum_{j_1+\ldots+j_k=n} \mathcal{C}_{j_1} \ldots \mathcal{C}_{j_k}$. The above identity, called the *renewal equation*, is of fundamental importance to understand the asymptotic of H_n in the non-commutative setting; in particular, the equality (3.3) yields $H_n(x,y) = \mathcal{H}_n \mathbb{1}_{\{y\}}(x)$ so that the asymptotic behaviour of $\big(H_n(x,y)\big)_{n\geq 1}$ is related to that of $\big(\mathcal{H}_n\big)_{n\geq 1}$.

By Gouëzel (2011), if the sequence $\big(\mathcal{C}_n\big)_{n\geq 1}$ satisfies the following additional assumptions

R4(ℓ, β). $\quad \|\mathcal{C}_n\|_{\mathcal{B}} \leq C \dfrac{\ell(n)}{n^{1+\beta}}$,

R5(ℓ, β). $\quad \sum_{j>n} r_j \sim \dfrac{\ell(n)}{n^{\beta}}$,

where $C > 0$, $\beta \in (0,1)$ and ℓ is a slowly varying function, then the sequence $\big(n^{1-\beta}\ell(n)\mathcal{H}_n\big)_{n\geq 1}$ converges in $\big(\mathcal{L}(\mathcal{B}), \|\cdot\|_{\mathcal{B}}\big)$ to the operator $d_\beta \Pi$, with $d_\beta = \frac{1}{\pi} \sin \beta\pi$.

In the next subsection, we introduce some Banach space $\mathcal{B} = \mathcal{B}_\delta$ in order to be able to apply this general result.

3.3. *Spectral property of the transition matrix* $\mathcal{C} = \big(\mathcal{C}(x,y)\big)_{x,y\in\mathbb{Z}}$. The operator $\mathcal{C}$ acts on the space $L^\infty(\mathbb{Z})$ of bounded functions on $\mathbb{Z}$. By the following lemma, it satisfies some strong spectral property on this space.

Lemma 3.4. *Assume* **H1**– **H3** *hold. Then, the infinite matrix* $\mathcal{C}$ *satisfies the Doeblin condition and therefore, it is a quasi-compact operator on* $L^\infty(\mathbb{Z})$*, the space of bounded functions on* $\mathbb{Z}$*. Furthermore, the eigenvalue* 1 *is simple, with associated eigenvector* $\mathbf{1}$*, and the rest of the spectrum is included in a disk of radius* < 1*.*

Proof: Under the above assumptions, the positive random variable S_{ℓ_1} has finite first moment; hence, by the renewal theorem,

$$\lim_{t\to+\infty} \mathcal{U}_+(t) = \frac{1}{\mathbb{E}[S_{\ell_1}]} > 0.$$

The above convergence readily implies $\delta_+ := \inf_{z\in\mathbb{Z}^+} \mathcal{U}_+(z) > 0$.

Consequently, by (3.2), for any $x \le -1$ and $y \ge 0$,

$$\mathcal{C}(x, y) \ge \mu_+(y+1)\mathcal{U}_+(-x-1) \ge \delta_+\mu_+(y+1).$$

In the same vein, one gets $\mathcal{C}(x, y) \ge \delta'\mu'_-(y-1)$ for any $x \ge 1$ and $y \le 0$ with $\delta' := \inf_{z\in\mathbb{Z}^-} \mathcal{U'}_-(z) > 0$. Hence, it is easy to show that there exists a probability measure $\mathbf{m}$ and $\delta_0 > 0$ s.t. for any $x \in \mathbb{Z}$,

$$\mathcal{C}(x, .) \ge \delta_0\mathbf{m}(.),$$

which immediately implies the quasi-compactness of $\mathcal{C}$. The control of the peripheral spectrum readily follows. □

Thanks to this lemma, we could believe that hypothesis R1 is satisfied by the sequence $\left(\mathcal{C}_n\right)_{n\ge1}$ acting on $L^\infty(\mathbb{Z})$ since $\mathcal{C}(1) = \mathcal{C}$. Unfortunately, it holds $\sum_{n\ge1} \|\mathcal{C}_n\|_\infty = +\infty$. Indeed, it holds $\|\mathcal{C}_n\|_\infty = \sup_{x\in\mathbb{Z}} \mathbb{P}_x[C_1 = n]$; now, if we assume for instance $x \le 1$, it holds $\mathbb{P}_x[C_1 = n] = \mathbb{P}[\tau^S(x) = n]$ with

(*i*) $\mathbb{P}[\tau^S(x) = n] = O(1/n)$,

and

(*ii*) $\liminf_{n\to+\infty} n\mathbb{P}[\tau^S(x_n) = n] > 0$ when $x_n \asymp \sqrt{n}$.

(see Lemma 5 and Theorem (B) in Èppel' (1979)). Consequently $\|\mathcal{C}_n\|_\infty \asymp 1/n$.

Thus, we have to choose another Banach space $\mathcal{B}_\delta$. By (3.1), it is clear that $C_{k+1} = \tau_S(X_{C_k})$ when $X_{C_k} \le -1$ and $C_{k+1} = \tau_{S'}(X_{C_k})$ when $X_{C_k} \ge 1$. Consequently, the behaviour as $n \to +\infty$ of the k^{th}-term $\mathbb{P}_x[C_k = n, X_n = y]$ of the sum $\Sigma_n(x, y)$ is closely related to the distributions of τ_S and $\tau_{S'}$; in particular, by Lemma 2.1, its dependence on y is expressed in terms of $h_a(y)$ and $h'_d(y)$. This explains why we have to choose a Banach space on which the action of $\mathcal{C}$ has "nice" spectral properties - as compacity or quasi-compacity - and also does contain these functions h_a and h'_d. The fact that they are both sublinear leads us to examine the action of $\mathcal{C}$ on the space $\mathcal{B}_\delta$ of complex valued functions on $\mathbb{Z}$ defined by

$$\mathcal{B}_\delta := \left\{ f : \mathbb{Z} \to \mathbb{C} : |f|_{\mathcal{B}_\delta} := \sup_{x\in\mathbb{Z}} \frac{|f(x)|}{1 + |x|^{1+\delta}} < +\infty \right\}$$

with $\delta \ge 0$.

By Lemma 2.2 and the fact that $h_a(x) = O(x)$, $h'_d(x) = O(x)$, the functions $h_a, h'_d, \mathbf{h}_n : x \mapsto \sqrt{n}\mathbb{P}[\tau^S(-x) > n]$ and $\mathbf{h}'_n : x \mapsto \sqrt{n}\mathbb{P}[\tau^{S'}(x) > n]$ do belong to $\mathcal{B}_\delta$ for any $\delta \ge 0$; furthermore, applying Lemma 2.1, the sequence $\left(\mathbf{h}_n\right)_{n\ge0}$ (resp. $\left(\mathbf{h}'_n\right)_{n\ge0}$) converges to $2ch_a$ (resp. $2c'h'_d$) in $\mathcal{B}_\delta$ if $\delta > 0$. This last property is of interest in applying Gouëzel's renewal theorem and for this reason, we assume from now on $\delta > 0$.

Furthermore, the map $\mathcal{C}$ acts on $\mathcal{B}_\delta$ as a compact operator whose spectrum can be controlled as follows.

Proposition 3.5. *Assume that hypotheses* **H1**– **H4** *hold. Then,*

(i) *The map $\mathcal{C}$ acts on $\mathcal{B}_\delta$ and $\mathcal{C}(\mathcal{B}_\delta) \subset L^\infty(\mathbb{Z})$.*

(ii) *$\mathcal{C}$ is a compact operator on $\mathcal{B}_\delta$ with spectral radius $\rho_{\mathcal{B}_\delta} = 1$ and with the unique and simple dominant eigenvalue 1.*

(iii) *The rest of the spectrum of $\mathcal{C}$ on $\mathcal{B}_\delta$ is contained in a disk of radius < 1.*

Consequently, the operator $\mathcal{C}$ on $\mathcal{B}_\delta$ may be decomposed as

$$\mathcal{C} = \Pi + Q$$

where

- *Π is the eigenprojector from $\mathcal{B}_\delta$ to $\mathbb{C}\mathbf{1}$ corresponding to the eigenvalue 1 and $\Pi(\phi) = \nu(\phi)\mathbf{1}$, where ν is the unique $\mathcal{C}$-invariant probability measure on $\mathbb{Z}$;*
- *the spectral radius of Q on $\mathcal{B}_\delta$ is < 1;*
- *$\Pi Q = Q\Pi = 0$.*

Proof: i) Note that $\mathcal{U}'_-(t) = \sum_{n\geq 0} \mathbb{P}[S'_{\ell'_n} = t] = \mathbb{P}[\exists n \geq 0 : S'_{\ell'_n} = t] \leq 1$. For any $\varphi \in \mathcal{B}_\delta$ and $x \geq 1$, we have

$$\begin{aligned} |\mathcal{C}\varphi(x)| &\leq \sum_{y\leq 0} \sum_{t=-x+1}^{0} \mu'_-(y-x-t)\,|\varphi(y)| \\ &\leq |\varphi|_{\mathcal{B}_\delta} \sum_{y\leq 0} (1+|y|^{1+\delta})\,\mu'_-(-\infty, y-1) \\ &\leq |\varphi|_{\mathcal{B}_\delta} \Big(\mathbb{E}[|S'_{\ell'_1}|] + \mathbb{E}[|S'_{\ell'_1}|^{2+\delta}] \Big), \end{aligned}$$

which is finite if $\mathbb{E}[(\xi'^{-}_n)^{3+\delta}] < +\infty$ (see Chow and Lai (1979)). Other cases can be estimated in the same way and yield

$$|\mathcal{C}\varphi|_{\mathcal{B}_\delta} \leq |\mathcal{C}\varphi|_\infty \leq |\varphi|_{\mathcal{B}_\delta} \Big(\mathbb{E}[|S'_{\ell'_1}|] + \mathbb{E}[|S'_{\ell'_1}|^{2+\delta}] \Big) < +\infty. \tag{3.4}$$

ii) By (3.4), the operator $\mathcal{C}$ acts continuously from $\mathcal{B}_\delta$ into $L^\infty(\mathbb{Z})$; since the inclusion map $i : L^\infty(\mathbb{Z}) \hookrightarrow \mathcal{B}_\delta$ is compact, the operator $\mathcal{C}$ is also compact on $\mathcal{B}_\delta$.

Let us now compute the spectral radius $\rho_{\mathcal{B}_\delta}$ of $\mathcal{C}$. The fact that $\mathcal{C}$ is a stochastic matrix yields $\rho_{\mathcal{B}_\delta} \geq 1$. To prove $\rho_{\mathcal{B}_\delta} \leq 1$, it suffices to show that $\mathcal{C}$ has bounded powers on $\mathcal{B}_\delta$. For any $n \geq 1$ and $x \in \mathbb{Z}$,

$$|\mathcal{C}^n\varphi(x)| \leq \sum_{y\in\mathbb{Z}} \mathcal{C}^{n-1}(x,y)|\mathcal{C}\varphi(y)| \leq |\mathcal{C}\varphi|_\infty \sum_{y\in\mathbb{Z}} \mathcal{C}^{n-1}(x,y) = |\mathcal{C}\varphi|_\infty.$$

Together with (3.4), it implies

$$|\mathcal{C}^n\varphi|_{\mathcal{B}_\delta} \leq |\mathcal{C}^n\varphi|_\infty \leq |\mathcal{C}\varphi|_\infty \leq |\varphi|_{\mathcal{B}_\delta} \Big(\mathbb{E}[|S'_{\ell'_1}|] + \frac{1}{2}\mathbb{E}[|S'_{\ell'_1}|^{2+\delta}] \Big).$$

Hence $\|\mathcal{C}^n\|_{\mathcal{B}_\delta} \leq \mathbb{E}[|S'_{\ell'_1}|] + \mathbb{E}[|S'_{\ell'_1}|^{2+\delta}]$ for any $n \geq 1$ and $\rho_{\mathcal{B}_\delta} = \lim_{n\to+\infty} \|\mathcal{C}^n\|_{\mathcal{B}_\delta}^{1/n} \leq 1$.

Let us now control the peripheral spectrum of $\mathcal{C}$. Let $\theta \in \mathbb{R}$ and $\psi \in \mathcal{B}_\delta$ such that $\mathcal{C}\psi = e^{i\theta}\psi$. Obviously, the function ψ is bounded and $|\psi| \leq \mathcal{C}|\psi|$. Consequently, $|\psi|_\infty - |\psi|$ is non-negative and super-harmonic (i.e. $\mathcal{C}(|\psi|_\infty - |\psi|) \leq |\psi|_\infty - |\psi|$) on the unique irreducible class $\mathcal{I}_{\mathbf{C}}(X_0)$ of $\mathcal{X}$. According to the classical theory of denumerable Markov chains, it is thus constant on $\mathcal{I}_{\mathbf{C}}(X_0)$ which follows that $|\psi|$ is constant on $\mathcal{I}_{\mathbf{C}}(X_0)$.

Without loss of generality, we may assume $|\psi(x)| = 1$ for any $x \in \mathcal{I}_{\mathbf{C}}(X_0)$, i.e. $\psi(x) = e^{i\phi(x)}$ for some $\phi(x) \in \mathbb{R}$. We may rewrite the equality $\mathcal{C}\psi = e^{i\theta}\psi$ as

$$\forall x \in \mathcal{I}_{\mathbf{C}}(X_0) \qquad \sum_{y \in \mathcal{I}_{\mathbf{C}}(X_0)} \mathcal{C}(x,y) e^{i(\phi(y)-\phi(x))} = e^{i\theta}.$$

Note that $\mathcal{C}(x,y) > 0$ for all $x, y \in \mathcal{I}_{\mathbf{C}}(X_0)$; by convexity, one readily gets $e^{i\theta} = e^{i(\phi(y)-\phi(x))}$ for such points x, y. Taking $x = y \in \mathcal{I}_{\mathbf{C}}(X_0)$, we thus obtain $e^{i\theta} = 1$.

In particular, the function ψ is harmonic on $\mathcal{I}_{\mathbf{C}}(X_0)$, hence constant on this set, by *Liouville's theorem*. Furthermore, for any $x \in \mathbb{Z}$, it holds $\mathcal{C}(x,y) > 0 \iff y \in \mathcal{I}_{\mathbf{C}}(X_0)$; consequently, for any fixed $y_0 \in \mathcal{I}_{\mathbf{C}}(X_0)$ and any $x \in \mathbb{Z}$,

$$\psi(x) = \mathcal{C}\psi(x) = \sum_{y \in \mathcal{I}_{\mathbf{C}}(X_0)} \mathcal{C}(x,y)\psi(y) = \psi(y_0).$$

Therefore, the function ψ is constant on $\mathbb{Z}$.

iii) This is a direct consequence of (ii).

□

3.4. *A renewal limit theorem for the sequence of crossing times.* The main goal of this part is to prove the following statement.

Proposition 3.6. *The sequence* $\left(\sqrt{n}\mathcal{H}_n\right)_{n\geq 1}$ *converges in* $\left(\mathcal{L}(\mathcal{B}_\delta), \|\cdot\|_{\mathcal{B}_\delta}\right)$ *to the operator* $\mathbf{c}^{-1}\Pi$ *with* $\mathbf{c} = 2\pi\big(c\nu(\check{h}_a) + c'\nu(h'_d)\big)$. *In particular, for any* $x, y \in \mathbb{Z}$,

$$\lim_{n\to+\infty} \sqrt{n}H_n(x,y) = \frac{\nu(y)}{2\pi\big(c\nu(\check{h}_a) + c'\nu(h'_d)\big)}.$$

This is a consequence of the fact that $\left(\mathcal{C}_n\right)_{n\geq 1}$ is an aperiodic renewal sequence of operators on $\mathcal{B}_\delta$ satisfying R4 and R5 (with $\beta = 1/2$ and ℓ constant).

The fact that all the $\mathcal{C}_n, n \geq 1$, act on $\mathcal{B}_\delta$ and $\sum_{n\geq 1} \|\mathcal{C}_n\|_{\mathcal{B}_\delta} < +\infty$ is a consequence of the following lemma.

Lemma 3.7. *Under hypotheses* **H1**– **H4**, *for any* $n \geq 1$, *the operator* $\mathcal{C}_n$ *acts on* $\mathcal{B}_\delta$ *and*

$$\|\mathcal{C}_n\|_{\mathcal{B}_\delta} = O\left(\frac{1}{n^{3/2}}\right).$$

Proof: By (2.1), for any $x \geq 1$ and $\phi \in \mathcal{B}_\delta$,

$$\begin{aligned} |\mathcal{C}_n\phi(-x)| &\leq \sum_{w\geq 0} \mathbb{P}_{-x}[C_1 = n; X_n = w]\, |\phi(w)| \\ &= \sum_{w\geq 0} \mathbb{P}[\tau^S(-x) = n, -x + S_n = w]\, |\phi(w)| \\ &\preceq \frac{1+x}{n^{3/2}} \sum_{w\geq 0} \left(\sum_{z\geq w+1} z\mu(z)\right) |\phi(w)| \\ &\leq \frac{1+x}{n^{3/2}} |\phi|_{\mathcal{B}_\delta} \underbrace{\sum_{w\geq 0} (1 + w^{1+\delta}) \left(\sum_{z\geq w+1} z\mu(z)\right)}_{\preceq \sum_{z\geq 1} z^{3+\delta}\mu(z) = \mathbb{E}[(\xi_1^+)^{3+\delta}]}. \end{aligned}$$

Similarly,

$$|\mathcal{C}_n\phi(x)| \preceq \frac{1+x}{n^{3/2}}|\phi|_{\mathcal{B}_\delta}\mathbb{E}[(\xi_1'^-)^{3+\delta}].$$

Moreover, $|\mathcal{C}_1\phi(0)| \preceq |\phi|_{\mathcal{B}_\delta}$ and $\mathcal{C}_n\phi(0) = 0$ for all $n \geq 2$. This completes the proof. □

Condition R1 coincides with the statement of Proposition 3.5. Similarly, R2 and R3 correspond to assertions *i)* and *ii)* of the next proposition. Consequently, $\big(\mathcal{C}_n\big)_{n\geq 1}$ is an aperiodic renewal sequence of operators.

Proposition 3.8. *Suppose that* **H1**– **H4** *are satisfied. Then the sequence* $\big(\mathcal{C}_n\big)_{n\geq 1}$ *holds the following properties*

i) The spectral radius $\rho_{\mathcal{B}_\delta}(z)$ *of* $\mathcal{C}(z)$ *is strictly less than* 1 *for* $z \in \overline{\mathbb{D}} \setminus \{1\}$.

ii) For any $n \geq 1$, *it holds* $\Pi\mathcal{C}_n\Pi = r_n\Pi$ *with*

$$r_n := \nu(\mathcal{C}_n 1) = \sum_{x\in\mathbb{Z}} \nu(x)\mathbb{P}_x[C_1 = n] \geq 0.$$

iii) $\displaystyle\sum_{j>n} r_j \sim \frac{2\big(c\nu(\check{h}_a) + c'\nu(h_d')\big)}{\sqrt{n}}$ *as* $n \to +\infty$.

Proof: *i)* The argument is close to the one used to prove Proposition 3.5. For any $z \in \overline{\mathbb{D}} \setminus \{1\}$, the operator $\mathcal{C}(z)$ is compact on $\mathcal{B}_\delta$ with spectral radius $\rho_{\mathcal{B}_\delta}(z) \leq 1$. We now prove $\rho_{\mathcal{B}_\delta}(z) \neq 1$ by contraposition. Suppose $\rho_{\mathcal{B}_\delta}(z) = 1$; in other words, there exist $\theta \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\delta$ such that $\mathcal{C}(z)\varphi = e^{i\theta}\varphi$. Since $\mathcal{C}$ is bounded from $\mathcal{B}_\delta$ into $L^\infty(\mathbb{Z})$ and $0 \leq |\varphi| \leq \mathcal{C}|\varphi|$, the function $|\varphi|$ is $\mathcal{C}$- superharmonic, bounded and thus constant on its essential class $\mathcal{I}_{\mathbf{C}}(X_0)$.

Without loss of generality, we can suppose that $|\varphi(x)| = 1$ for any $x \in \mathcal{I}_{\mathbf{C}}(X_0)$; equivalently, $\varphi(x) = e^{i\phi(x)}$ for some function $\phi : \mathcal{I}_{\mathbf{C}}(X_0) \to \mathbb{R}$. For any $x \in \mathcal{I}_{\mathbf{C}}(X_0)$, we get

$$\mathcal{C}(z)\varphi(x) = e^{i\theta}\varphi(x) \Longleftrightarrow \sum_{n\geq 1}\sum_{y\in\mathcal{I}_{\mathbf{C}}(X_0)} z^n e^{i(\phi(y)-\phi(x))}\mathbb{P}_x[C_1 = n; X_n = y] = e^{i\theta},$$

with $\displaystyle\sum_{n\geq 1}\sum_{y\in\mathcal{I}_{\mathbf{C}}(X_0)} \mathbb{P}_x[C_1 = n; X_n = y] = 1$. By convexity, it readily holds $z^n e^{i(\phi(y)-\phi(x))} = e^{i\theta}$ for all $x, y \in \mathcal{I}_{\mathbf{C}}(X_0)$ and $n \geq 1$. By taking $x = y$, we obtain $z^n = e^{i\theta}$ for all $n \geq 1$; consequently $z = 1$, contradiction.

ii) For any $\phi \in \mathcal{B}_\delta$ and $n \geq 1$,

$$\Pi\mathcal{C}_n\Pi\phi = \nu(\phi)\Pi(\mathcal{C}_n\mathbf{1}) = \nu(\phi)\mathbf{1}\sum_{z\in\mathbb{Z}}\nu(z)\mathbb{P}_z[C_1 = n] = r_n\Pi(\phi)$$

with $r_n = \displaystyle\sum_{z\in\mathbb{Z}}\nu(z)\mathbb{P}_z[C_1 = n] \geq 0$.

iii) On the one hand, by Proposition 3.5, the eigenprojector Π acts on $\mathcal{B}_\delta$; thus, since $\check{h}_a \in \mathcal{B}_\delta$, it holds $\nu(\check{h}_a) < +\infty$. On the other hand, the support $\mathcal{I}_{\mathbf{C}}(X_0)$ of ν intersects $\mathbb{Z}^-$ and the support of $\check{h}_a$ equals $\mathbb{Z}^-$; hence $\nu(\check{h}_a) > 0$. Similarly $0 < \nu(h_d') < +\infty$.

Now, let us write

$$\begin{aligned}\sum_{j>n} r_j &= \sum_{j>n}\sum_{x\in\mathbb{Z}} \nu(x)\mathbb{P}_x[C_1 = j]\\ &= \sum_{x\in\mathbb{Z}} \nu(x)\mathbb{P}_x[C_1>n]\\ &= \sum_{x\le -1} \nu(x)\mathbb{P}[\tau^S(x)>n] + \sum_{x\ge 1}\nu(x)\mathbb{P}[\tau^{S'}(x)>n] \quad (\text{since } \mathbb{P}_0[C_1=1]=1)\\ &\sim \frac{2c}{\sqrt{n}}\sum_{x\le -1}\nu(x)h_a(-x) + \frac{2c'}{\sqrt{n}}\sum_{x\ge 1}\nu(x)h'_d(x) = 2\,\frac{c\,\nu(\check{h}_a)+c'\,\nu(h'_d)}{\sqrt{n}}.\end{aligned}$$

□

Finally, combining Lemma 3.7 and Proposition 3.8 *iii*), we see that conditions R4 and R5 are satisfied with $\ell = \text{const} = 2\big(c\,\nu(\check{h}_a)+c'\,\nu(h'_d)\big)$ and $\beta = 1/2$.

Consequently, by Gouëzel (2011), the sequence $\big(\sqrt{n}\mathcal{H}_n\big)_{n\ge 1}$ converges in $\big(\mathcal{L}(\mathcal{B}_\delta), \|\cdot\|_{\mathcal{B}_\delta}\big)$ to the operator $\mathbf{c}^{-1}\Pi$ with $\mathbf{c} = 2\pi\big(c\nu(\check{h}_a)+c'\nu(h'_d)\big)$. Formally, one may write

$$\left\|\sqrt{[ns]}\mathcal{H}_{[ns]} - \mathbf{c}^{-1}\Pi\right\|_{\mathcal{B}_\delta} \longrightarrow 0, \quad \text{as} \quad n\to+\infty.$$

4. Proof of Theorem 1.1

For $m\ge 1$, let $\big\{\varphi_i : \mathbb{R}\to\mathbb{R} \mid i = 1,\dots,m\big\}$ be a sequence of bounded and Lipschitz continuous functions with corresponding Lipschitz coefficients $Lip(\varphi_i)$. Assume that the time sequence $\big\{t_i\big\}_{1\le i\le m}$ is strictly increasing with values in $(0,1]$ and $t_0 = 0$. In this part, we prove that

$$\lim_{n\to+\infty}\mathbb{E}_x\left[\prod_{i=1}^m \varphi_i\Big(X^{(n)}(t_i)\Big)\right] = \int_{\mathbb{R}^m}\prod_{i=1}^m \varphi_i(u_i)p^{\gamma}_{t_i-t_{i-1}}(u_{i-1},u_i)\,du_1\dots du_m \tag{4.1}$$

with $u_0 = 0$.

Without loss of generality, we assume $\sigma=\sigma'$ and $x\ge 1$ to reduce unnecessary complexity associated with subcases.

4.1. *Convergence of the one dimensional distributions $m = 1$.*

We first notice that $\mathbb{E}_x[\varphi_1(X^{(n)}(t_1))] \approx \mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\right]$ since

$$\begin{aligned}\left|\mathbb{E}_x[\varphi_1(X^{(n)}(t_1))] - \mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\right]\right| &\le Lip(\varphi_1)\,\mathbb{E}_x\left[\left|X^{(n)}(t_1) - \frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right|\right]\\ &\le Lip(\varphi_1)\,\frac{\mathbb{E}[|\xi_{[nt_1]+1}|] + \mathbb{E}[|\eta_{[nt_1]+1}|] + \mathbb{E}[|\xi'_{[nt_1]+1}|]}{\sigma\sqrt{n}},\end{aligned}$$

which tends to 0 as $n\to+\infty$. Now, we can decompose $\mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\right]$ as

$$\underbrace{\mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right), X_{[nt_1]}=0\right]}_{A_0(n)} + \underbrace{\mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right), X_{[nt_1]}>0\right]}_{A^+(n)} + \underbrace{\mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right), X_{[nt_1]}<0\right]}_{A^-(n)}.$$

The term $A_0(n)$ tends to 0 as $n\to+\infty$ since $\big(X_n\big)_{n\ge 0}$ is null recurrent. It remains to control the two other terms.

• Estimate of $A^+(n)$

$$A^+(n) \approx \sum_{k_1=1}^{[nt_1]-1} \sum_{\ell\geq 1}\sum_{y\geq 1} \mathbb{E}_x\Big[\varphi_1\Big(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\Big), C_\ell = k_1, X_{k_1} = y, y+\xi'_{k_1+1} > 0, \ldots, y+\xi'_{k_1+1}+\ldots+\xi'_{[nt_1]} > 0\Big]$$

$$= \sum_{k_1=1}^{[nt_1]-1} \sum_{y\geq 1} \mathbb{E}\Big[\varphi_1\Big(\frac{y+\xi'_{k_1+1}+\ldots+\xi'_{[nt_1]}}{\sigma\sqrt{n}}\Big), \tau^{S'}(y) > [nt_1]-k_1\Big] \Big(\sum_{\ell\geq 1}\mathbb{P}_x[C_\ell = k_1, X_{k_1} = y]\Big)$$

$$= \sum_{k_1=1}^{[nt_1]-1} \sum_{y\geq 1} H_{k_1}(x,y)\mathbb{E}\Big[\varphi_1\Big(\frac{y+S'_{[nt_1]-k_1}}{\sigma\sqrt{n}}\Big), \tau^{S'}(y) > [nt_1]-k_1\Big].$$

For any $0 \leq s_1 \leq t_1$ and $n \geq 1$, let f_n be the function defined by

$$f_n(s_1) := n\sum_{y\geq 1} H_{[ns_1]}(x,y)\mathbb{E}\Big[\varphi_1\Big(\frac{y+S'_{[nt_1]-[ns_1]}}{\sigma\sqrt{n}}\Big), \tau^{S'}(y) > [nt_1]-[ns_1]\Big]$$

if $0 \leq s_1 < \frac{[nt_1]}{n}$ and $f_n(s_1) = 0$ if $\frac{[nt_1]}{n} \leq s_1 \leq t_1$. Hence,

$$A^+(n) = \int_0^{t_1} f_n(s_1)\, ds_1 + O\Big(\frac{1}{\sqrt{n}}\Big).$$

The convergence of the term $A^+(n)$ as $n \to +\infty$ is a consequence of the two following properties:

• for any $n \geq 1$,

$$|f_n(s_1)| \preceq \frac{1+|x|}{\sqrt{s_1(t_1-s_1)}} \in L^1([0,t_1]). \tag{4.2}$$

• for any $s_1 \in [0,t_1]$,

$$\lim_{n\to+\infty} f_n(s_1) = \frac{\gamma}{\pi\sqrt{s_1(t_1-s_1)}}\int_0^{+\infty} \varphi_1(z\sqrt{t_1-s_1})z\exp\Big(\frac{-z^2}{2}\Big)dz$$

$$= \frac{\gamma}{\pi}\int_0^{+\infty} \varphi_1(u)u\,\frac{\exp\Big(\frac{-u^2}{2(t_1-s_1)}\Big)}{\sqrt{s_1(t_1-s_1)^3}}du \qquad (\text{set } u = z\sqrt{t_1-s_1}). \tag{4.3}$$

Indeed, applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} A^+(n) = \frac{\gamma}{\pi}\int_0^{+\infty}\varphi_1(u)u\Big(\int_0^{t_1}\frac{1}{\sqrt{s_1(t_1-s_1)^3}}\exp\Big(\frac{-u^2}{2(t_1-s_1)}\Big)ds_1\Big)du$$

$$= \frac{\gamma}{\pi}\int_0^{+\infty}\varphi_1(u)u\Big[\frac{1}{t_1}\exp\Big(\frac{-u^2}{2t_1}\Big)\underbrace{\int_0^{+\infty}\frac{1}{\sqrt{s}}\exp\Big(\frac{-u^2}{2t_1}s\Big)ds}_{=\frac{\sqrt{2\pi t_1}}{u}}\Big]du \quad (\text{set } s := s_1(t_1-s_1)^{-1})$$

$$= \gamma\int_0^{+\infty}\varphi_1(u)\frac{2\exp\Big(-u^2/2t_1\Big)}{\sqrt{2\pi t_1}}du. \tag{4.4}$$

Similarly,

$$\lim_{n\to+\infty} A^-(n) = (1-\gamma)\int_{-\infty}^{0} \varphi_1(u)\frac{2\exp\Big(-u^2/2t_1\Big)}{\sqrt{2\pi t_1}}du. \tag{4.5}$$

Combining (4.4) and (4.5), we thus obtain

$$\lim_{n\to+\infty} \mathbb{E}_x[\varphi_1(X_{t_1}^{(n)})] = \int_{\mathbb{R}} \varphi_1(u)p_{t_1}^{\gamma}(0,u)du = \int_{\mathbb{R}} \tilde{\varphi}_1(u)\frac{2\exp(-u^2/2t_1)}{\sqrt{2\pi t_1}}du,$$

where $\tilde{\varphi}_1(u) = \gamma\varphi_1(u)\mathbb{1}_{(0,+\infty)}(u) + (1-\gamma)\varphi_1(u)\mathbb{1}_{(-\infty,0)}(u)$.

It thus remains to establish (4.2) and (4.3). The first natural idea is to set

$$\psi_n(y) := \sqrt{n}\mathbb{E}\left[\varphi_1\left(\frac{y+S'_{[nt_1]-[ns_1]}}{\sigma\sqrt{n}}\right), \tau^{S'}(y) > [nt_1]-[ns_1]\right]$$

and to remark that $f_n(s_1) = \sqrt{n}\mathcal{H}_{[ns_1]}(\psi_n)(x)$ with $\psi_n \in \mathcal{B}_\delta$. One can easily check that $\big(\psi_n\big)_{n\geq 0}$ converges point-wise to some function $\psi \in \mathcal{B}_\delta$ but it is much more complicated to prove that this convergence holds in $\mathcal{B}_\delta$. This can be done when $\delta \geq 1$ with a strong moment assumption (namely moments of order ≥ 4 for μ') by using a recent result in Grama et al. (2018); unfortunately, such a result does not exist for the Brownian meander, which is useful in the sequel for convergence of multidimensional distributions. This forces us to propose another strategy that we now present.

For this purpose, for any $n \geq 1$ and any fixed $0 < s_1 < t_1$, we decompose $f_n(s_1)$ as $f_n(s_1) = \sum_{y\geq 1} a_n(y)b_n(y)$, where

$$a_n(y) := nH_{[ns_1]}(x,y)\,\mathbb{P}[\tau^{S'}(y) > [nt_1]-[ns_1]],$$
$$b_n(y) := \mathbb{E}\left[\varphi_1\left(\frac{y+S'_{[nt_1]-[ns_1]}}{\sigma\sqrt{n}}\right) \mid \tau^{S'}(y) > [nt_1]-[ns_1]\right]$$

and apply the following classical lemma with $V = \mathbb{Z}^+$:

Lemma 4.1. *Let V be denumerable and $\big(a_n(v)\big)_{v\in V}$, $\big(b_n(v)\big)_{v\in V}$ be real sequences satisfying*

(i) $a_n(v) \geq 0$ for any $n \geq 1, v \in V$ and $\displaystyle\lim_{n\to+\infty}\sum_{v\in V} a_n(v) = A$,

(ii) for any $\epsilon > 0$, there exists a finite set $V_\epsilon \subset V$ s.t. $\displaystyle\sup_{n\geq 1}\sum_{v\notin V_\epsilon} a_n(v) < \epsilon$.

(iii) $\displaystyle\lim_{n\to+\infty} b_n(v) = b$ for any $v \in V$ and $\displaystyle\sup_{n\geq 1, v\in V} |b_n(v)| < +\infty$.

Then $\displaystyle\lim_{n\to+\infty}\sum_{v\in V} a_n(v)b_n(v) = Ab.$

Proof: Let us fix $\eta > 0$. We want to find an integer $n_\eta \geq 1$ such that for any $n > n_\eta$,

$$\left|\sum_{v\in V} a_n(v)b_n(v) - Ab\right| < \eta. \tag{4.6}$$

As a consequence of assumptions (i) and (iii), for any $\epsilon > 0$, there exists $n_\epsilon \geq 1$ such that for any $n > n_\epsilon$ and $v \in V$,

$$\left|\sum_{v\in V} a_n(v) - A\right| < \epsilon \quad \text{and} \quad |b_n(v) - b| < \epsilon.$$

Hence, together with the assumption (ii), we obtain

$$\begin{aligned}\Big|\sum_{v\in V} a_n(v)b_n(v) - Ab\Big| &\le \sum_{v\notin V_\epsilon} a_n(v)|b_n(v)-b| + \sum_{v\in V_\epsilon} a_n(v)|b_n(v)-b| + |b|\Big|\sum_{v\in V} a_n(v) - A\Big| \\ &\le \underbrace{\Big(\sup_{n\ge 1, v\in V}|b_n(v)|+|b|\Big)\sup_{n\ge 1}\sum_{v\notin V_\epsilon} a_n}_{<C_1\epsilon} + \underbrace{\Big(\sum_{v\in V} a_n(v)\Big)}_{<\ C_2\ \text{(for all } n\ge 1)}\epsilon + |b|\epsilon,\end{aligned}$$

for some positive constants C_1, C_2. This immediately yields (4.6) by taking $\epsilon = \dfrac{\eta}{C_1+C_2+|b|}$ and $n_\eta = n_\epsilon$.

□

Let us check that these conditions are satisfied by the families $\big(a_n(y)\big)_{y\ge 1}$, $\big(b_n(y)\big)_{y\ge 1}$ defined above.

Condition (i). The sum $\sum_{y\ge 1} a_n(y)$ may be written as

$$\sum_{y\ge 1} a_n(y) = \frac{1+o(n)}{\sqrt{s_1(t_1-s_1)}}\ \sqrt{[ns_1]}\mathcal{H}_{[ns_1]}(\mathbf{h}'_{[nt_1]-[ns_1]})(x). \tag{4.7}$$

On the one hand, the sequence $\big(\sqrt{[ns_1]}\mathcal{H}_{[ns_1]}\big)_{n\ge 1}$ converges in $\big(\mathcal{L}(\mathcal{B}_\delta), \|\cdot\|_{\mathcal{B}_\delta}\big)$ to the operator $\frac{1}{2\pi\left(c\nu(\check{h}_a)+c'\nu(h'_d)\right)}\Pi$; furthermore, the sequence $\big(\mathbf{h}'_{[nt_1]-[ns_1]}\big)_{n\ge 1}$ converges in $\mathcal{B}_\delta$ to $2c'h'_d$. Hence condition (i) holds with

$$A = \frac{1}{\sqrt{s_1(t_1-s_1)}}\frac{c'\nu(h'_d)}{\pi\big(c\nu(\check{h}_a)+c'\nu(h'_d)\big)} = \frac{\gamma}{\pi\sqrt{s_1(t_1-s_1)}}.$$

Condition (ii). Fix $\epsilon>0$. We want to find $y_\epsilon \ge 1$ s.t. $\sum_{y\ge y_\epsilon} a_n(y) \le \epsilon$ for any $n\ge 1$. By Lemma 2.2, there exists a constant $C_0>0$ s.t. $0\le \mathbf{h}'_k(y)\le C_0(1+y)$ for any $y,k\ge 1$; hence, for $y\ge y_\epsilon$,

$$0\le \mathbf{h}'_k(y) \le C_0\left(1+\frac{y^{1+\delta}}{y_\epsilon^\delta}\right) \le 2C_0\frac{1+y^{1+\delta}}{y_\epsilon^\delta}.$$

Consequently the function $\mathbf{h}'_k\mathbb{1}_{[y_\epsilon,+\infty[}$ belongs to $\mathcal{B}_\delta$ and $|\mathbf{h}'_k\mathbb{1}_{[y_\epsilon,+\infty[}|_{\mathcal{B}_\delta} \le \frac{2C_0}{y_\epsilon^\delta}$ for any $k\ge 1$. By (4.7), it follows

$$0\le \sum_{y\ge y_\epsilon} a_n(y) \preceq \underbrace{\sup_{n\ge 1}\sqrt{[ns_1]}\|\mathcal{H}_{[ns_1]}\|_{\mathcal{B}_\delta}}_{<+\infty}\ \underbrace{\sup_{n\ge 1}|\mathbf{h}'_{[nt_1]-[ns_1]}\mathbb{1}_{[y_\epsilon,+\infty[}|_{\mathcal{B}_\delta}}_{\preceq \frac{1}{y_\epsilon^\delta}}.$$

We conclude choosing y_ϵ large enough.

Condition (iii). By (4.3), it holds with $b = \displaystyle\int_0^{+\infty}\varphi_1(u)u\ \frac{\exp\left(\frac{-u^2}{2(t_1-s_1)}\right)}{t_1-s_1}du.$

4.2. *Convergence of the multidimensional distributions.* We first consider the case $m=2$.

We fix $0<t_1<t_2$ and for $n\ge 1$ given, let $\kappa=\kappa_{t_1}$ be the first crossing time after time $[nt_1]$ defined by

$$\kappa := \min\big\{k > [nt_1] : X_{[nt_1]}X_k \le 0\big\}.$$

As in the case $m = 1$, it holds

$$\mathbb{E}_x[\varphi_1(X^{(n)}(t_1))\varphi_2(X^{(n)}(t_2))] \approx \mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt_2]}}{\sigma\sqrt{n}}\right)\right],$$

and the right hand side term may be decomposed as $A_0(n) + A_1^{\pm}(n) + A_2^{\pm}(n)$, where

$$A_0(n) := \mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt_2]}}{\sigma\sqrt{n}}\right), X_{[nt_1]} = 0\right],$$

$$A_1^{\pm}(n) := \sum_{k_2=[nt_1]+1}^{[nt_2]} \mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt_2]}}{\sigma\sqrt{n}}\right)\mathbb{1}_{[\kappa=k_2]}\mathbb{1}_{[\pm X_{[nt_1]}>0]}\right],$$

and

$$A_2^{\pm}(n) := \mathbb{E}_x\left[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt_2]}}{\sigma\sqrt{n}}\right)\mathbb{1}_{[\kappa>[nt_2]]}\mathbb{1}_{[\pm X_{[nt_1]}>0]}\right].$$

As previously, the term $A_0(n)$ tends to 0 since $(X_n)_{n\geq 0}$ is null recurrent.

• Estimate of $A_1^{\pm}(n)$

$$\begin{aligned}
A_1^{+}(n) \approx & \sum_{k_1=1}^{[nt_1]-1}\sum_{k_2=[nt_1]+1}^{[nt_2]}\sum_{\ell\geq 1}\sum_{y\geq 1}\sum_{z\geq 1}\sum_{w\leq 0}\mathbb{E}_x\Bigg[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt_2]}}{\sigma\sqrt{n}}\right), C_\ell = k_1,\\
& X_{k_1} = y, y+\xi'_{k_1+1} > 0, \ldots, y+\xi'_{k_1+1}+\ldots+\xi'_{k_2-2} > 0,\\
& y+\xi'_{k_1+1}+\ldots+\xi'_{k_2-1} = z, y+\xi'_{k_1+1}+\ldots+\xi'_{k_2} = w\Bigg]\\
= & \sum_{k_1=1}^{[nt_1]-1}\sum_{k_2=[nt_1]+1}^{[nt_2]}\sum_{\ell\geq 1}\sum_{y\geq 1}\sum_{z\geq 1}\sum_{w\leq 0}\mathbb{E}_x\Bigg[\varphi_1\left(\frac{y+\xi'_{k_1+1}+\ldots+\xi'_{[nt_1]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt_2]}}{\sigma\sqrt{n}}\right),\\
& C_\ell = k_1, X_{k_1} = y, y+\xi'_{k_1+1} > 0, \ldots, y+\xi'_{k_1+1}+\ldots+\xi'_{k_2-2} > 0,\\
& y+\xi'_{k_1+1}+\ldots+\xi'_{k_2-1} = z\Bigg]\mathbb{P}[\xi'_{k_2} = w-z]\\
= & \sum_{k_1=1}^{[nt_1]-1}\sum_{y\geq 1}H_{k_1}(x,y)\sum_{k_2=[nt_1]+1}^{[nt_2]}\sum_{z\geq 1}\mathbb{E}\Bigg[\varphi_1\left(\frac{y+S'_{[nt_1]-k_1}}{\sigma\sqrt{n}}\right), \tau^{S'}(y) > k_2-k_1-1,\\
& y+S'_{k_2-k_1-1} = z\Bigg]\sum_{w\leq 0}\mathbb{E}_w\left[\varphi_2\left(\frac{X_{[nt_2]-k_2}}{\sigma\sqrt{n}}\right)\right]\mu'(w-z).
\end{aligned}$$

For any $(s_1, s_2) \in [0, t_1] \times [t_1, t_2]$ and $n \geq 1$, let g_n be the function defined by

$$\begin{aligned}
g_n(s_1,s_2) = n^2\sum_{y\geq 1}H_{[ns_1]}(x,y)\sum_{z\geq 1}\mathbb{E}\Bigg[\varphi_1\left(\frac{y+S'_{[nt_1]-[ns_1]}}{\sigma\sqrt{n}}\right), \tau^{S'}(y) > [ns_2]-[ns_1]-1,\\
y+S'_{[ns_2]-[ns_1]-1} = z\Bigg]\sum_{w\leq 0}\mathbb{E}_w\left[\varphi_2\left(\frac{X_{[nt_2]-[ns_2]}}{\sigma\sqrt{n}}\right)\right]\mu'(w-z)
\end{aligned}$$

if $0 \leq s_1 < \frac{[nt_1]}{n}$ and $\frac{[nt_1]+1}{n} \leq s_2 \leq \frac{[nt_2]}{n}$, and 0 otherwise. Hence,

$$A_1^{+}(n) = \int_0^{t_1}\int_{t_1}^{t_2} g_n(s_1,s_2)ds_1ds_2 + O\left(\frac{1}{\sqrt{n}}\right).$$

The convergence of the term $A_1^+(n)$ as $n \to +\infty$ is a consequence of the two following properties whose proofs are postponed at the end of the present subsection:

• for any $n \geq 1$,

$$|g_n(s_1, s_2)| \preceq \frac{1+|x|}{\sqrt{s_1(s_2-s_1)^3}} \in L^1([0,t_1]\times[t_1,t_2]); \tag{4.8}$$

• for any $(s_1, s_2) \in [0,t_1]\times[t_1,t_2]$,

$$\lim_{n\to+\infty} g_n(s_1,s_2) = \frac{\gamma}{\pi^2}\int_0^{+\infty}\int_{-\infty}^{+\infty} \varphi_1(u)\tilde{\varphi}_2(v)u^2 \frac{e^{\frac{-v^2}{2(t_2-s_2)}}\, e^{\frac{-u^2}{2\frac{(t_1-s_1)(s_2-t_1)}{s_2-s_1}}}}{\sqrt{s_1(t_1-s_1)^3(s_2-t_1)^3(t_2-s_2)}}dudv. \tag{4.9}$$

Indeed, applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{n\to+\infty} A_1^+(n)\\
&= \frac{\gamma}{\pi^2}\int_0^{+\infty}\int_{-\infty}^{+\infty}\varphi_1(u)\tilde{\varphi}_2(v)\Bigg(\int_0^{t_1}\int_{t_1}^{t_2}\frac{e^{\frac{-v^2}{2(t_2-s_2)}}u^2\exp\left(\frac{-u^2}{2\frac{(t_1-s_1)(s_2-t_1)}{s_2-s_1}}\right)}{\sqrt{s_1(t_1-s_1)^3(s_2-t_1)^3(t_2-s_2)}}ds_1ds_2\Bigg)dudv\\
&= \frac{\gamma}{\pi^2}\frac{\sqrt{2\pi t_1}}{t_1}\int_0^{+\infty}\int_{-\infty}^{+\infty}\varphi_1(u)\tilde{\varphi}_2(v)|u|\Bigg(\int_{t_1}^{t_2}\frac{e^{\frac{-u^2 s_2}{2t_1(s_2-t_1)}}\,e^{\frac{-v^2}{2(t_2-s_2)}}}{\sqrt{(t_2-s_2)(s_2-t_1)^3}}ds_2\Bigg)dudv\\
&= \frac{2\gamma}{\pi\sqrt{t_1(t_2-t_1)}}\int_0^{+\infty}\int_{-\infty}^{+\infty}\varphi_1(u)\tilde{\varphi}_2(v)e^{-\frac{u^2t_2+v^2t_1+2|uv|t_1}{t_1(t_2-t_1)}}dudv
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\lim_{n\to+\infty} A_1^+(n) &= \frac{2\gamma^2}{\pi\sqrt{t_1(t_2-t_1)}}\int_0^{+\infty}\int_0^{+\infty}\varphi_1(u)\varphi_2(v)e^{-\frac{u^2}{2t_1}}e^{-\frac{(u+v)^2}{2(t_2-t_1)}}dudv\\
&\quad+\frac{2\gamma(1-\gamma)}{\pi\sqrt{t_1(t_2-t_1)}}\int_0^{+\infty}\int_{-\infty}^{0}\varphi_1(u)\varphi_2(v)e^{-\frac{u^2}{2t_1}}e^{-\frac{(u-v)^2}{2(t_2-t_1)}}dudv,
\end{aligned}\tag{4.10}$$

by using the classical integral $\displaystyle\int_0^{+\infty}\frac{1}{\sqrt{x}}\exp\left(-\lambda_1 x-\frac{\lambda_2}{x}\right)dx = \sqrt{\frac{\pi}{\lambda_1}}e^{-2\sqrt{\lambda_1\lambda_2}}$ for any $\lambda_1>0$ and $\lambda_2\geq 0$.

The same argument holds for the term $A_1^-(n)$ and yields

$$\begin{aligned}
\lim_{n\to+\infty} A_1^-(n) &= \frac{2(1-\gamma)^2}{\pi\sqrt{t_1(t_2-t_1)}}\int_{-\infty}^0\int_{-\infty}^0\varphi_1(u)\varphi_2(v)e^{-\frac{u^2}{2t_1}}e^{-\frac{(u+v)^2}{2(t_2-t_1)}}dudv\\
&\quad+\frac{2\gamma(1-\gamma)}{\pi\sqrt{t_1(t_2-t_1)}}\int_{-\infty}^{0}\int_0^{+\infty}\varphi_1(u)\varphi_2(v)e^{-\frac{u^2}{2t_1}}e^{-\frac{(u-v)^2}{2(t_2-t_1)}}dudv.
\end{aligned}\tag{4.11}$$

• Estimate of $A_2^+(n)$

$$\begin{aligned}
A_2^+(n) &= \sum_{k=1}^{[nt_1]-1}\sum_{\ell\geq1}\sum_{y\geq1}\mathbb{E}_x\Bigg[\varphi_1\left(\frac{X_{[nt_1]}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{X_{[nt_2]}}{\sigma\sqrt{n}}\right), C_\ell=k, X_k=y,\\
&\qquad y+\xi'_{k+1}>0,\dots,y+\xi'_{k+1}+\ldots+\xi'_{[nt_1]}>0,\dots,y+\xi'_{k+1}+\ldots+\xi'_{[nt_2]}>0\Bigg]\\
&= \sum_{k=1}^{[nt_1]-1}\sum_{y\geq1}H_k(x,y)\mathbb{E}\Bigg[\varphi_1\left(\frac{y+S'_{[nt_1]-k}}{\sigma\sqrt{n}}\right)\varphi_2\left(\frac{y+S'_{[nt_2]-k}}{\sigma\sqrt{n}}\right),\tau^{S'}(y)>[nt_2]-k\Bigg].
\end{aligned}$$

For any $n \geq 1$, let $g_n : r \mapsto g_n(r)$ be the real function defined on $[0, t_1]$ by

$$g_n(r) := n \sum_{y\geq 1} H_{[nr]}(x,y)\mathbb{E}\Big[\varphi_1\Big(\frac{y+S'_{[nt_1]-[nr]}}{\sigma\sqrt{n}}\Big)\varphi_2\Big(\frac{y+S'_{[nt_2]-[nr]}}{\sigma\sqrt{n}}\Big), \tau^{S'}(y) > [nt_2]-[nr]\Big]$$

if $0 \leq r < \frac{[nt_1]}{n}$ and $g_n(r) = 0$ if $\frac{[nt_1]}{n} \leq r \leq t_1$.
In the same way as above, we set

$$a_n(y) := nH_{[nr]}(x,y)\mathbb{P}[\tau^{S'}(y) > [nt_2]-[nr]]$$

and

$$b_n(y) := \mathbb{E}\Big[\varphi_1\Big(\frac{y+S'_{[nt_1]-[nr]}}{\sigma\sqrt{n}}\Big)\varphi_2\Big(\frac{y+S'_{[nt_2]-[nr]}}{\sigma\sqrt{n}}\Big) \mid \tau^{S'}(y) > [nt_2]-[nr]\Big].$$

Sequences $\big(a_n(y)\big)_{y\geq 1}$ and $\big(b_n(y)\big)_{y\geq 1}$ satisfy assumptions of Lemma 4.1. Indeed, the limit of $\sum_{y\geq 1} a_n(y)$ is given by (4.7) and condition (ii) of this lemma has been checked previously. Furthermore, by Theorem 3.2 in Bolthausen (1976) and Theorems 2.23 and 3.4 in Iglehart (1974), it holds

$$\begin{aligned}\lim_{n\to+\infty} b_n(y) &= \lim_{n\to+\infty} \mathbb{E}\Big[\varphi_1\Big(\frac{y+S'_{[nt_1]-[nr]}}{\sigma\sqrt{[nt_2]-[nr]}}\frac{\sqrt{[nt_2]-[nr]}}{\sqrt{n}}\Big)\\ &\qquad \varphi_2\Big(\frac{y+S'_{[nt_2]-[nr]}}{\sigma\sqrt{n}}\frac{\sqrt{[nt_2]-[nr]}}{\sqrt{n}}\Big) \mid \tau^{S'}(y) > [nt_2]-[nr]\Big]\\ &= \frac{1}{\sqrt{2\pi(t_2-t_1)}}\int_0^{+\infty}\int_0^{+\infty}\varphi_1(u)\varphi_2(v)\frac{\sqrt{t_2-r}}{\sqrt{(t_1-r)^3}}ue^{\frac{-u^2}{2(t_1-r)}}\\ &\qquad\times\Big(e^{-\frac{(u-v)^2}{2(t_2-t_1)}} - e^{\frac{-(u+v)^2}{2(t_2-t_1)}}\Big)dudv.\end{aligned}$$

It immediately yields

$$\begin{aligned}\lim_{n\to+\infty} A_2^+(n) &= \frac{\gamma}{\pi\sqrt{2\pi(t_2-t_1)}}\int_0^{t_1}\Big(\frac{1}{\sqrt{2\pi(t_2-t_1)}}\int_0^{+\infty}\int_0^{+\infty}\varphi_1(u)\varphi_2(v)\\ &\qquad\times\frac{\sqrt{t_2-r}}{\sqrt{r(t_2-r)(t_1-r)^3}}ue^{\frac{-u^2}{2(t_1-r)}}\Big(e^{-\frac{(u-v)^2}{2(t_2-t_1)}} - e^{\frac{-(u+v)^2}{2(t_2-t_1)}}\Big)dudv\Big)dr\\ &= \frac{\gamma}{\pi\sqrt{t_1(t_2-t_1)}}\int_0^{+\infty}\int_0^{+\infty}\varphi_1(u)\varphi_2(v)e^{\frac{-u^2}{2t_1}}\Big(e^{-\frac{(u-v)^2}{2(t_2-t_1)}} - e^{\frac{-(u+v)^2}{2(t_2-t_1)}}\Big)dudv. \qquad (4.12)\end{aligned}$$

Analogously, one gets

$$\lim_{n\to+\infty} A_2^-(n) = \frac{1-\gamma}{\pi\sqrt{t_1(t_2-t_1)}}\int_{-\infty}^0\int_{-\infty}^0\varphi_1(u)\varphi_2(v)e^{\frac{-u^2}{2t_1}}\Big(e^{-\frac{(u-v)^2}{2(t_2-t_1)}} - e^{-\frac{(u+v)^2}{2(t_2-t_1)}}\Big)dudv \qquad (4.13)$$

Combining (4.10), (4.11), (4.12) and (4.13), we conclude

$$\lim_{n\to+\infty}\mathbb{E}_x\Big[\varphi_1(X^{(n)}(t_1))\varphi_2(X^{(n)}(t_2))\Big] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\varphi_1(u)\varphi_2(v)p^\gamma_{t_1}(0,u)p^\gamma_{t_2-t_1}(u,v)dudv.$$

Proof of properties (4.8) and (4.9)

Following the same strategy as in the one dimensional case, we decompose $g_n(s_1,s_2)$ as $g_n(s_1,s_2) = \sum_{y\geq 1}\sum_{z\geq 1}\sum_{w\leq 0} a_n(y,z,w)b_n(y,z,w)$, where

$$a_n(y,z,w) := n^2 H_{[ns_1]}(x,y)\mathbb{P}[\tau^{S'}(y) > [ns_2]-[ns_1]-1, y+S'_{[ns_2]-[ns_1]-1} = z]\ \mu'(w-z)$$

and

$$\begin{aligned} b_n(y,z,w) :=& \mathbb{E}\Big[\varphi_1\Big(\frac{y+S'_{[nt_1]-[ns_1]}}{\sigma\sqrt{n}}\Big) \mid \tau^{S'}(y) > [ns_2]-[ns_1]-1, y+S'_{[ns_2]-[ns_1]-1} = z\Big] \\ & \times \mathbb{E}_w\Big[\varphi_2\Big(\frac{X_{[nt_2]-[ns_2]}}{\sigma\sqrt{n}}\Big)\Big]. \end{aligned}$$

Properties (4.8) and (4.9) are direct consequences of Lemma 4.1, applied to the families $\big(a_n(y,z,w)\big)_{y,z\geq 1,w\leq 0}$ and $\big(b_n(y,z,w)\big)_{y,z\geq 1,w\leq 0}$; it thus suffices to check that conditions (i), (ii) and (iii) of this lemma are satisfied in this new situation.

Condition (i). The sum $\sum_{y\geq 1}\sum_{z\geq 1}\sum_{w\leq 0} a_n(y,z,w)$ may be written as

$$\sum_{y\geq 1}\sum_{z\geq 1}\sum_{w\leq 0} a_n(y,z,w) = \frac{1+o(n)}{\sqrt{s_1(s_2-s_1)^3}}\sqrt{[ns_1]}\mathcal{H}_{[ns_1]}(\mathbf{b}'_{[ns_2]-[ns_1]})(x), \tag{4.14}$$

where we set $\mathbf{b}'_k(y) = k^{3/2}\mathbb{P}[\tau^{S'}(y) = k]$.

As previously, the sequence $\big(\sqrt{[ns_1]}\mathcal{H}_{[ns_1]}\big)_{n\geq 1}$ converges in $\big(\mathcal{L}(\mathcal{B}_\delta), \|\cdot\|_{\mathcal{B}_\delta}\big)$ to the operator $\frac{1}{2\pi\big(c\nu(\check{h}_a)+c'\nu(h'_d)\big)}\Pi$; furthermore, the sequence $\big(\mathbf{b}'_{[ns_2]-[ns_1]}\big)_{n\geq 1}$ converges in $\mathcal{B}_\delta$ to the function $c'h'_d$. Hence condition (i) holds with

$$A = \frac{1}{\sqrt{s_1(s_2-s_1)^3}}\frac{c'\nu(h'_d)}{2\pi\big(c\nu(\check{h}_a)+c'\nu(h'_d)\big)} = \frac{\gamma}{2\pi\sqrt{s_1(s_2-s_1)^3}}.$$

Condition (ii). Fix $\epsilon > 0$ and $y_\epsilon \geq 1$. As above $\mathbf{h}'_k\mathbb{1}_{[y_\epsilon,+\infty[}$, the function $\mathbf{b}'_k\mathbb{1}_{[y_\epsilon,+\infty)}$ belongs to $\mathcal{B}_\delta$ and $|\mathbf{b}'_k\mathbb{1}_{[y_\epsilon,+\infty)}|_{\mathcal{B}_\delta} \leq \frac{C_1}{y_\epsilon^\delta}$ for some constant $C_1 > 0$. By (4.14), it follows

$$\begin{aligned} 0 \leq \sum_{y\geq y_\epsilon, z\geq 1, w\leq 0} a_n(y,z,w) &\preceq \sqrt{[ns_1]}\mathcal{H}_{[ns_1]}(\mathbf{b}'_{[ns_2]-[ns_1]}\mathbb{1}_{[y_\epsilon,+\infty)})(x) \\ &\preceq \underbrace{\sup_{n\geq 1}\|\sqrt{[ns_1]}\mathcal{H}_{[ns_1]}\|_{\mathcal{B}_\delta}}_{<+\infty}\ \underbrace{\sup_{n\geq 1}|\mathbf{b}'_{[ns_2]-[ns_1]}\mathbb{1}_{[y_\epsilon,+\infty)}|_{\mathcal{B}_\delta}}_{\preceq\frac{1}{y_\epsilon^\delta}}. \end{aligned}$$

This last right hand side term is $< \epsilon$ for sufficiently large y_ϵ.

Furthermore, $0 \leq a_n(y,z,w) \preceq (1+y)(1+z)\mu'(w-z)$ for any fixed y,z,w and any $n\geq 1$; hence, for any $0 \leq y < y_\epsilon$, it holds $\sum_{z+|w|>t} a_n(y,z,w) < \epsilon$ if t is large enough since $\sum_{z\geq 1} z\mu'(]-\infty,-z]) < +\infty$. This completes the argument.

Condition (iii). On the one hand, by (2.3), for any $y, z \geq 1$,

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{E}\left[\varphi_1\left(\frac{y+S'_{[nt_1]-[ns_1]}}{\sigma\sqrt{n}}\right) \mid \tau^{S'}(y) > [ns_2]-[ns_1]-1, y+S'_{[ns_2]-[ns_1]-1} = z\right] \\
= \int_0^{+\infty} 2\varphi_1(u'\sqrt{s_2-s_1}) \exp\left(\frac{-u'^2}{2\frac{t_1-s_1}{s_2-s_1}\frac{s_2-t_1}{s_2-s_1}}\right) \frac{u'^2}{\sqrt{2\pi\frac{(t_1-s_1)^3}{(s_2-s_1)^3}\frac{(s_2-t_1)^3}{(s_2-s_1)^3}}} du' \\
= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} \varphi_1(u) \exp\left(\frac{-u^2}{2\frac{(t_1-s_1)(s_2-t_1)}{s_2-s_1}}\right) \frac{u^2}{\sqrt{\frac{(t_1-s_1)^3(s_2-t_1)^3}{(s_2-s_1)^3}}} du.
\end{aligned}$$

On the other hand, the one dimensional case $m = 1$ studied above yields, for any $w \leq 0$,

$$\lim_{n\to+\infty} \mathbb{E}_w\left[\varphi_2\left(\frac{X_{[nt_2]-[ns_2]}}{\sigma\sqrt{n}}\right)\right] = \int_{\mathbb{R}} \tilde{\varphi}_2(v) \frac{2\exp\left(\frac{-v^2}{2(t_2-s_2)}\right)}{\sqrt{2\pi(t_2-s_2)}} dv$$

with $\tilde{\varphi}_2(v) = \gamma\varphi_2(v)\mathbb{1}_{(0,+\infty)}(v) + (1-\gamma)\varphi_2(v)\mathbb{1}_{(-\infty,0)}(v)$.

Case $m \geq 3$

The proof is done by induction on m. The strategy essentially stems from the case $m = 2$, reusing the random time $\kappa = \kappa_{t_1}$ defined above as the key point to control the fluctuations of the trajectory of the chain.

Suppose now that (4.1) is true up to the dimension $m \geq 2$, we then need to prove that it remains true for the dimension $m+1$. Indeed, we first have the following approximation and decomposition

$$\mathbb{E}_x\left[\prod_{i=1}^{m+1} \varphi_i(X^{(n)}(t_i))\right] \approx \mathbb{E}_x\left[\prod_{i=1}^{m+1} \varphi_i\left(\frac{X_{[nt_i]}}{\sigma\sqrt{n}}\right)\right] = B_0(n) + B_1^{\pm}(n) + B_2^{\pm}(n).$$

with

$$B_0(n) := \mathbb{E}_x\left[\prod_{i=1}^{m+1} \varphi_i\left(\frac{X_{[nt_i]}}{\sigma\sqrt{n}}\right), X_{[nt_1]} = 0\right],$$

$$B_1^{\pm}(n) := \sum_{j=1}^{m} \sum_{k_2=[nt_j]+1}^{[nt_{j+1}]} \mathbb{E}_x\left[\prod_{i=1}^{m+1} \varphi_i\left(\frac{X_{[nt_i]}}{\sigma\sqrt{n}}\right) \mathbb{1}_{[\kappa=k_2]} \mathbb{1}_{[\pm X_{[nt_1]}>0]}\right]$$

and

$$B_2^{\pm}(n) := \mathbb{E}_x\left[\prod_{i=1}^{m+1} \varphi_i\left(\frac{X_{[nt_i]}}{\sigma\sqrt{n}}\right) \mathbb{1}_{[\kappa>[nt_{m+1}]]} \mathbb{1}_{[\pm X_{[nt_1]}>0]}\right].$$

Again, the term $B_0(n)$ vanishes at infinity due to the null-recurrence of the chain. Let us now focus in particular on the two terms $B_1^+(n)$ and $B_2^+(n)$; the other terms follow exactly the same lines.

• Estimate of $B_1^+(n)$

$$
\begin{aligned}
B_1^+(n) \approx & \sum_{j=1}^{m} \sum_{k_2=[nt_j]+1}^{[nt_{j+1}]} \sum_{k_1=1}^{[nt_1]-1} \sum_{\ell\geq 1}\sum_{y\geq 1}\sum_{z\geq 1}\sum_{w\leq 0} \mathbb{E}_x\Big[\prod_{i=1}^{m+1}\varphi_i\Big(\frac{X_{[nt_i]}}{\sigma\sqrt{n}}\Big), C_\ell = k_1, \\
& X_{k_1} = y, y+\xi'_{k_1+1} > 0, \dots, y+\xi'_{k_1+1}+\dots+\xi'_{k_2-2} > 0, \\
& y+\xi'_{k_1+1}+\dots+\xi'_{k_2-1} = z, y+\xi'_{k_1+1}+\dots+\xi'_{k_2} = w\Big] \\
= & \sum_{j=1}^{m}\sum_{k_1=1}^{[nt_1]-1}\sum_{k_2=[nt_j]+1}^{[nt_{j+1}]} \sum_{\ell\geq 1}\sum_{y\geq 1}\sum_{z\geq 1}\sum_{w\leq 0} \mathbb{E}_x\Big[\prod_{i_1=1}^{j}\varphi_{i_1}\Big(\frac{y+S'_{[nt_{i_1}]-k_1}}{\sigma\sqrt{n}}\Big)\prod_{i_2=j+1}^{m+1}\varphi_{i_2}\Big(\frac{X_{[nt_{i_2}]}}{\sigma\sqrt{n}}\Big), \\
& C_\ell = k_1, X_{k_1} = y, y+\xi'_{k_1+1} > 0, \dots, y+\xi'_{k_1+1}+\dots+\xi'_{k_2-2} > 0, \\
& y+\xi'_{k_1+1}+\dots+\xi'_{k_2-1} = z\Big]\mathbb{P}[\xi'_{k_2} = w-z] \\
= & \sum_{j=1}^{m}\sum_{k_1=1}^{[nt_1]-1}\sum_{y\geq 1} H_{k_1}(x,y) \sum_{k_2=[nt_j]+1}^{[nt_{j+1}]}\sum_{z\geq 1} \mathbb{E}\Big[\prod_{i_1}^{j}\varphi_{i_1}\Big(\frac{y+S'_{[nt_{i_1}]-k_1}}{\sigma\sqrt{n}}\Big) \mid \tau^{S'}(y) > k_2-k_1-1, \\
& y+S'_{k_2-k_1-1} = z\Big] \times \sum_{w\leq 0}\mathbb{E}_w\Big[\prod_{i_2=j+1}^{m+1}\varphi_{i_2}\Big(\frac{X_{[nt_{i_2}]-k_2}}{\sigma\sqrt{n}}\Big)\Big] \\
& \times \mathbb{P}[\tau^{S'}(y) > k_2-k_1-1, y+S'_{k_2-k_1-1} = z]\ \mu'(w-z).
\end{aligned}
$$

The first expectation is treated by Corollary 2.5 in Caravenna and Chaumont (2013) while the second expectation is obviously derived from the induction hypothesis.

• Estimate of $B_2^+(n)$

$$
\begin{aligned}
B_2^+(n) = & \sum_{k=1}^{[nt_1]-1}\sum_{\ell\geq 1}\sum_{y\geq 1}\mathbb{E}_x\Big[\prod_{i=1}^{m+1}\varphi_i\Big(\frac{X_{[nt_i]}}{\sigma\sqrt{n}}\Big), C_\ell = k, X_k = y, y+\xi'_{k+1} > 0, \dots, \\
& y+\xi'_{k+1}+\dots+\xi'_{[nt_1]} > 0, \dots, y+\xi'_{k+1}+\dots+\xi'_{[nt_{m+1}]} > 0\Big] \\
= & \sum_{k=1}^{[nt_1]-1}\sum_{y\geq 1} H_k(x,y)\mathbb{E}\Big[\prod_{i=1}^{m+1}\varphi_i\Big(\frac{y+S'_{[nt_i]-k}}{\sigma\sqrt{n}}\Big) \mid \tau^{S'}(y) > [nt_{m+1}]-k\Big] \\
& \times \mathbb{P}[\tau^{S'}(y) > [nt_{m+1}]-k],
\end{aligned}
$$

which is again a direct application of Corollary 2.5 in Caravenna and Chaumont (2013).

4.3. *Tightness of the sequence* $\{X^{(n)}\}_{n\geq 1}$. Let us recall the modulus of continuity of a function $f : [0,T] \to \mathbb{R}$ is defined by

$$\omega_f(\delta) := \sup\big\{|f(t)-f(s)| : t,s \in [0,T]\ s.t.\ |t-s| \leq \delta\big\}.$$

By Theorems 7.1 and 7.3 in Billingsley (1999), we have to show that the following conditions hold:

(i) For every $\eta > 0$, there exist $a > 0$ and $n_\eta \geq 1$ such that

$$\mathbb{P}[|X^{(n)}(0)| \geq a] \leq \eta, \quad \forall n \geq n_\eta.$$

(ii) For every $\epsilon > 0$ and $\eta > 0$, there exist $\delta \in (0,1)$ and $n_{\epsilon,\eta} \geq 1$ such that

$$\mathbb{P}[\omega_{X^{(n)}}(\delta) \geq \epsilon] \leq \eta, \quad \forall n \geq n_{\epsilon,\eta}.$$

Proof: Condition (i) is obviously satisfied since $X_0 = x$.

Let us now check the condition (ii). Set $I_{n,\delta} := \big\{(i,j) \in \mathbb{N} \mid 1 \leq i < j \leq n \text{ and } |i-j| \leq n\delta\big\}$ and note that we have

$$\omega_{X^{(n)}}(\delta) \leq \frac{7}{(\sigma \wedge \sigma')\sqrt{n}} \Bigg(\sup_{(i,j)\in I_{n,\delta}} |S_i - S_j| + \sup_{(i,j)\in I_{n,\delta}} |S'_i - S'_j| \Bigg). \tag{4.15}$$

We suggest the following figure as a useful illustration of this bound.

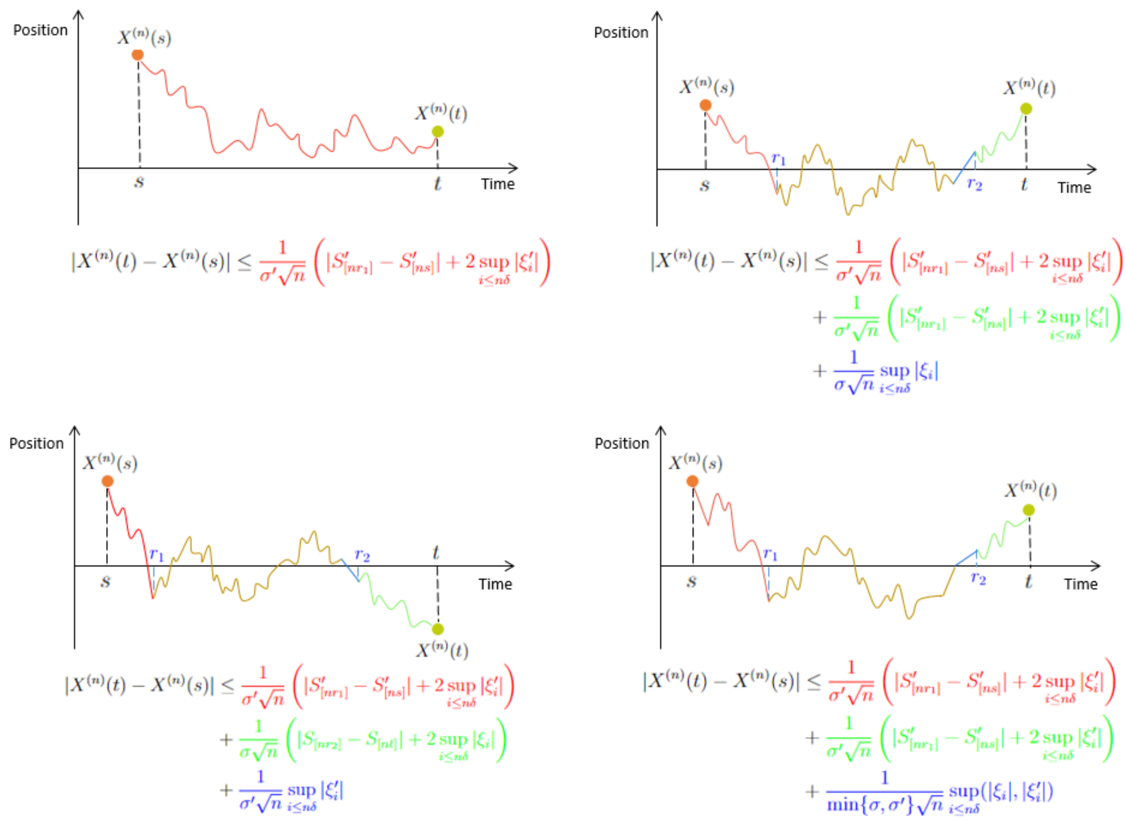

FIGURE 4.1. Fluctuation of the normalized oscillating random walk on the time interval $[s,t]$ with its first and last crossing times r_1 and r_2, respectively.

Moreover, by Billingsley (1999) (see Chapter 7) one also gets

$$\lim_{\delta\to 0}\lim_{n\to+\infty} \mathbb{P}\Bigg[\frac{1}{\sigma\sqrt{n}} \sup_{(i,j)\in I_{n,\delta}} |S_i - S_j| \geq \epsilon\Bigg] = \lim_{\delta\to 0}\lim_{n\to+\infty} \mathbb{P}\Bigg[\frac{1}{\sigma'\sqrt{n}} \sup_{(i,j)\in I_{n,\delta}} |S'_i - S'_j| \geq \epsilon\Bigg] = 0. \tag{4.16}$$

The condition (ii) immediately follows by (4.15) and (4.16). Hence, we conclude that the sequence $\{X^{(n)}\}_{n\geq 1}$ is tight. □

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