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Gaussian fluctuation for spatial average of the stochastic pseudo-partial differential equation with fractional noise

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Abstract. Consider a class of stochastic pseudo-partial differential equation on $\mathbb{R}$ of the form

$$\frac{\partial}{\partial t}u(t,x) = -\mathcal{L}u(t,x) + \sigma(u(t,x))\dot{W}(t,x),$$

where $\mathcal{L}$ is a nonlocal pseudo-differential operator and $\dot{W}$ denotes a Gaussian noise which is white in time and has a covariance function of fractional Brownian motion with Hurst index $H \in [1/2, 1)$ with respect to the spatial variable. If, in addition, $u(0,x) = 1, \forall x \in \mathbb{R}$, we prove that the normalized spatial average of the solution from $-R$ to R $(R > 0)$ converges in total variance distance to a standard normal distribution as R tends to infinity. We also establish a functional version of this central limit theorem. The Malliavin-Stein's method plays an important role.

1. Introduction

Consider the following stochastic pseudo-partial differential equation (SPDE for short) with Cauchy initial condition on $\mathbb{R}$,

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = -\mathcal{L}u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = 1, \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is a pseudo-differential operator with negative definite symbol of variable order which generates a stable-like process (see, for example, Jacob and Leopold (1993); Jacob et al. (2008, 2010) and etc). The noise $\dot{W}(t,x)$ is the Gaussian noise which is white in time and fractional in spatial variable with Hurst index $H \in [1/2, 1)$. Here $\sigma(\cdot)$ is assumed to be a Lipschitz continuous function with the property $\sigma(1) \neq 0$. We state the following assumption on the coefficient $\sigma(\cdot)$.

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Assumption A: There exists a constant $L_\sigma > 0$ such that for any $u, v \in \mathbb{R}$,

$$|\sigma(u) - \sigma(v)| \le L_\sigma |u - v|. \tag{1.2}$$

Following the references Dalang (1999); Walsh (1986), the definition of mild solution to the SPDE (1.1) is given as follows.

Definition 1.1. For $p \ge 1$, an $L^p(\Omega)$ valued $(\mathcal{F}_t)$-adapted process $u(t, x; \omega) : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a mild solution to SPDE (1.1) if

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}} G_{t-s}(x, y)\sigma(u(s, y))W(ds, dy), \tag{1.3}$$

with the Green function $\{G_{t-s}(x, y), t, s \in \mathbb{R}_+, x, y \in \mathbb{R}\}$ is the solution to Eq. (2.6) and the stochastic integral appearing in the right hand side of (1.3) is in the sense of Walsh (1986).

Theorem 1.2. *Suppose that the* **Assumption A** *holds. Then, with the assumptions* $\frac{1}{2} \le H < 1$ *and* (2.7)*, the SPDE* (1.1) *admits a unique mild solution given by* (1.3)*. Moreover, for any* $p \ge 2$*, we have*

$$\sup_{t \in \mathbb{R}_+} \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u(t, x)|^p\right] < \infty.$$

We may use Dalang-Walsh's theory (i.e. Dalang (1999); Walsh (1986)) to establish that the SPDE (1.1) has a unique solution which can be written in the mild form given by Definition 1.1. In fact, the proof of the existence can be done by using the Picard iteration scheme. The proof of the uniqueness can be done by the standard arguments (see, for example, Dalang (1999); Liu and Yan (2018); Walsh (1986)). Here we omit the details to the interested readers.

In this paper, we are interested in the Gaussian fluctuation of the following spatial average $F_R(t)$ of the solution u to SPDE (1.1) (see, for example, Assaad et al. (2022); Delgado-Vences et al. (2020); Huang et al. (2020a,b); Pu (2022)).

$$F_R(t) = \frac{1}{\sigma_R} \int_{-R}^{R} (u(t, x) - 1)\, dx, \tag{1.4}$$

with $R > 0$ and

$$\sigma_R^2 := \operatorname{Var}\left(\int_{-R}^{R} (u(t, x) - 1)\, dx\right). \tag{1.5}$$

The goal of this paper is to prove the following two central limit theorems. The first result is the following quantitative central limit theorem concerning the spatial average $F_R(t)$ of the solution $u(t, x)$ over $[-R, R], R > 0$ as $R \to +\infty$. It is stated as follows.

Theorem 1.3. *Let* $u = \{u(t, x), (t, x) \in (0, T] \times \mathbb{R}\}$ *be the mild solution to the SPDE* (1.1)*. For* $H \in [1/2, 1)$*, assume that the* **Assumption A** *and* (2.7) *hold. Denote by* d_{TV} *the total variation distance. Then there exists a constant* C*, depending only on* t*, such that*

$$d_{TV}\left(\frac{1}{\sigma_R}\int_{-R}^{R}(u(t, x) - 1)dx, N\right) \le CR^{H-1},$$

where N *is a standard normal random variable, and* σ_R^2 *is defined by* (1.5) *satisfying* $\sigma_R^2 \sim R^{2H}$ *as* $R \to +\infty$*.*

We also establish a functional version of Theorem 1.3.

Theorem 1.4. *Let* $u = \{u(t, x), (t, x) \in (0, T] \times \mathbb{R}\}$ *be the solution to the SPDE* (1.1)*. For* $H \in [1/2, 1)$*, assume that the* **Assumption A** *and* (2.7) *hold. Then for any* $T > 0$*,*

$$\left\{\frac{1}{R^H}\int_{-R}^{R}(u(t, x) - 1)dx\right\}_{t \in [0,T]} \to^{(W)} \left\{\int_0^t \rho(s)dB_s\right\}_{t \in [0,T]},$$

as $R \to +\infty$, where B is a standard Brownian motion, the notation "$\to^{(W)}$" denotes the weak convergence which takes place in the space of continuous functions $C([0,T])$ and $\rho(s)$ is defined by

$$\rho(s) = \begin{cases} \sqrt{2\xi(s)}, & H = \frac{1}{2}; \\ 2^H \eta(s), & H \in (1/2, 1), \end{cases} \tag{1.6}$$

with $\xi(s) = \mathbb{E}[\sigma^2(u(s,y))]$ and $\eta(s) = \mathbb{E}[\sigma(u(s,y))]$.

We will mainly rely on the methodology of Malliavin-Stein approach to prove the above results (Theorem 1.3 and Theorem 1.4). Such an approach was introduced by Nourdin and Peccati in Nourdin and Peccati (2012) to, among other things, quantify Nualart and Peccati's fourth moment theorem in Nourdin and Peccati (2009). Theorem 1.3 is proved using a combination of Stein's method and Malliavin calculus, following the ideas introduced in Assaad et al. (2022); Delgado-Vences et al. (2020); Huang et al. (2020b); Nourdin and Peccati (2012). An important aspect of our methodology is to use the representation of $F_R(t)$ defined by (1.4) as a divergence operator, taking into account that the Itô-Walsh integral is a particular case of the Skorohod integral (see, for example, Nourdin and Peccati (2012), Nualart (2006)).

This work continues the lines of researches initiated in Huang et al. (2020a,b), where a similar problem for the stochastic heat equation on $\mathbb{R}$ (or $\mathbb{R}^d$, respectively) driven by a space-time white noise (or spatial covariance given by the Riesz kernel, respectively) was considered. Later on, the results Huang et al. (2020a) and Huang et al. (2020b) have been extended to the stochastic fractional heat equation driven by a general Gaussian multiplicative noise in Assaad et al. (2022), in which the authors presented a quantitative central limit theorem for the stochastic fractional heat equation. The corresponding SPDE is driven by a general Gaussian multiplicative noise, including the cases of space-time white noise and the white-colored noise with spatial covariance given by the Riesz kernel or a bounded integrable function.

Our methods in this work are similar to those of the three references, Assaad et al. (2022), Huang et al. (2020a) and Huang et al. (2020b). However, we stress that we do not have fine properties of the Green function $\{G_{t-s}(x,y), t, s \in \mathbb{R}_+, x, y \in \mathbb{R}\}$ in our case, and hence one has to be more careful in the computations. In particular, our main contribution is the bound for the norm of the Malliavin derivative (cf. Lemma 3.5) that differs from the classical Laplacian and fractional Laplacian case (see, for example, Assaad et al. (2022), Huang et al. (2020a), Huang et al. (2020b)). Moreover, we follow a general approach proposed in Assaad et al. (2022), Chen et al. (2023), Delgado-Vences et al. (2020), Huang et al. (2020b) on how such bounds can be achieved, based on the boundedness properties of the convolution operator with the spatial covariance together with the semigroup property and some integrability of the Green kernel. One can consult the Section 2.2 for more details about the Green function $\{G_{t-s}(x,y), t, s \in \mathbb{R}_+, x, y \in \mathbb{R}\}$.

On some other related literatures, we also mention Chen et al. (2021) in which the authors studied the spatial ergodicity for a class of stochastic heat equation via Poincaré-type inequalities. While in Chen et al. (2022), the authors studied the central limit theorems for parabolic SPDEs driven by a Gaussian noise which is white in time and has a homogeneous spatial covariance function. They mainly used the Poincaré-type inequalities, Malliavin calculus, compactness arguments and Paul Lévy's characterization of Brownian motion. The authors in Chen et al. (2023) established quantitative central limit theorems for spatial averages of the form $N^{-d}\int_{[0,N]^d} g(u(t,x))dx$ as $N \to +\infty$, where g is a Lipschitz-continuous function or belongs to a class of locally-Lipschitz functions, using a combination of the Malliavin-Stein method for normal approximations. Furthermore, the authors in Delgado-Vences et al. (2020), Nualart and Zheng (2022) studied the asymptotic behavior of spatial averages of solutions to stochastic wave equation on $\mathbb{R}$ and $\mathbb{R}^d$. In this work, we consider that the driving noise in SPDE (1.1) was assumed to be the Gaussian multiplicative noise which is white in time and colored in space such that the correlation in the space variable is described by the fractional kernel. As such, our results complements the above mentioned works studying the stochastic heat and wave equation.

The rest of the paper is organized as follows. In Section 2 we recall some preliminaries on the fractional noise, the pseudo-differential operator $\mathcal{L}$, Malliavin calculus and Stein's method. Asymptotic behavior of the covariance with respect to the mild solution is proved in Section 3. Section 4 is devoted to the proofs of our main theorems (Theorem 1.3 and Theorem 1.4).

2. Preliminaries

This section contains some basic results on the fractional noise, the pseudo-differential operator $\mathcal{L}$ and Malliavin-Stein's method that will be needed in the following Section 3 and Section 4.

2.1. *Fractional noise.* We denote by $W = \{W(t,x), t \in \mathbb{R}_+, x \in \mathbb{R}\}$ a centered Gaussian family of random variables defined in some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with covariance function given by

$$\mathbb{E}[W(t,x)W(s,y)] = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right)(t \wedge s),$$

with $H \in [1/2, 1)$. Let $\mathcal{H}_0$ be the Hilbert space defined as the completion of the set of simple functions on $\mathbb{R}$ equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}_0} = \begin{cases} \displaystyle\int_{\mathbb{R}} \varphi(x)\psi(x)dx, \quad H = \frac{1}{2}, \\ \displaystyle\beta_H \int_{\mathbb{R}^2} \varphi(x)\psi(y)|x-y|^{2H-2}dxdy, \quad H \in (1/2, 1), \end{cases} \tag{2.1}$$

with $\varphi, \psi \in \mathcal{H}_0$ and $\beta_H := H(2H-1)$. Set $\mathcal{H} = L^2(\mathbb{R}; \mathcal{H}_0)$ and notice that

$$\mathbb{E}[W(t,x)W(s,y)] = \langle 1_{[0,t]\times[0,x]}, 1_{[0,s]\times[0,y]} \rangle_{\mathcal{H}}.$$

Therefore, the mapping $(t,x) \to W(t,x)$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian subspace of $L^2(\Omega)$ generated by W. We denote this isometry by $\varphi \to W(\varphi)$. When $H = 1/2$, the space $\mathcal{H}$ is simply $L^2(\mathbb{R}^+ \times \mathbb{R})$ and $W(\varphi)$ is the Wiener-Itô's integral of φ denoted by

$$W(\varphi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \varphi(s,x) W(ds, dx).$$

For $H \in (1/2, 1)$, the space $L^{1/H}(\mathbb{R})$ is known to be continuously embedded into $\mathcal{H}_0$.

For any $t \geq 0$, we denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s,x), 0 \leq s \leq t, x \in \mathbb{R}\}$. Then, for any adapted $\mathcal{H}_0$-valued stochastic process $\{X(s,\cdot), s \geq 0\}$ such that

$$\int_0^\infty \mathbb{E}\left[\|X(s,\cdot)\|^2_{\mathcal{H}_0}\right] ds < \infty, \tag{2.2}$$

then the following stochastic integral $\int_0^\infty \int_{\mathbb{R}} X(s,y)W(ds,dy)$ is well-defined and satisfies the isometry property

$$\mathbb{E}\left[\left(\int_0^\infty \int_{\mathbb{R}} X(s,y)W(ds,dy)\right)^2\right] = \mathbb{E}\left(\int_0^\infty \|X(s,\cdot)\|^2_{\mathcal{H}_0} ds\right). \tag{2.3}$$

2.2. *The pseudo-differential operator $\mathcal{L}$.* We maily recall the operator $\mathcal{L}$ in more details given in the references Bass (1988), Jacob and Leopold (1993), Kikuchi and Negoro (1997) and Kolokoltsov (2000). As shown in the above references, the operator $\mathcal{L}$, which is the Markov generator of a stable-like process, has the following representation

$$-\mathcal{L}\varphi(x) = \int_{\mathbb{R}\setminus\{0\}} \left[\varphi(x+z) - \varphi(x) - \frac{z\varphi'(x)1_{\{|z|<1\}}(z)}{1+|z|^2}\right] \frac{dz}{|z|^{1+\alpha(x)}}, \tag{2.4}$$

for some suitable functions $\varphi : \mathbb{R} \to \mathbb{R}$ (for example, φ could be a Schwartz test function on $\mathbb{R}$), where $\alpha(\cdot) : \mathbb{R} \to (0, 2)$ is a measurable function. The symbol of $\mathcal{L}$, denoted by q, is a complex-valued bivariate function $q : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. However, in our case, the symbol for the $\mathcal{L}$ is real-valued and $q(x, k) = q(x, -k), (x, k) \in \mathbb{R} \times \mathbb{R}$.

On the other hand, let us represent the operator $\mathcal{L}$ in polar coordinates as in the framework Kolokoltsov (2000). Following the notations of Kolokoltsov (2000), let $S_0 = \{-1, 1\}$ and $\mathcal{B} = \mathcal{P}(S_0) = \{\phi, \{-1\}, \{1\}, S_0\}$. Given any finite, centrally symmetric measure m on $(S_0, \mathcal{B})$, we define $s := \frac{z}{|z|}$ for $z \in \mathbb{R}\backslash\{0\}$. Then, (2.4) can be rewritten as

$$-\mathcal{L}\varphi(x) = \int_0^{+\infty} \int_{S_0} \left[\varphi(x+z) - \varphi(x) - \frac{z\varphi'(x) 1_{\{|z|<1\}}(z)}{1+|z|^2} \right] \frac{dz}{z^{1+\alpha(x)}} m(ds), \tag{2.5}$$

which is in the form of the generators for stable-like processes considered in Kolokoltsov (2000). Thus, by Theorem 5.1 of Kolokoltsov (2000), $\mathcal{L}$ generates a stable-like process with probability transition density $\{G_{t-s}(x, y), 0 < s < t < \infty, x, y \in \mathbb{R}\}$ which is nothing but the fundamental solution of the following parabolic equation

$$\begin{cases} \dfrac{\partial}{\partial t} G_{t-s}(x, y) = -\mathcal{L} G_{t-s}(x, y), \quad (t, x) \in (s, \infty) \times \mathbb{R}, \\ \lim\limits_{t \searrow s} G_{t-s}(x, y) = \delta_y(x), \quad x \in \mathbb{R}, \end{cases} \tag{2.6}$$

for $(s, y) \in [0, \infty) \times \mathbb{R}$.

The study of stable-like Markov generators with variable order can be traced back to the seminal paper Bass (1988) where the author studied pure jump Markov processes associated with such Lévy type Markov generators. Further works on sample path behaviors as well as transition densities related to stable-like processes can be found in Bass and Levin (2002), Jacob et al. (2010), Kolokoltsov (2000), Komatsu (1995) and etc. The topic of estimating the transition densities (or equivalently, the fundamental solutions) G associated with Lévy stable type Markov generators $\mathcal{L}$ (with fixed $\alpha \in (0, 2)$) was started in Komatsu (1988, 1995). There are further investigations for $\mathcal{L}$ with variable order in Bass and Levin (2002), Jacob et al. (2010), Kolokoltsov (2000), Komatsu (1995) and etc, where under certain further assumptions on $\alpha(x)$, interesting estimates for transition densities of the stable-like processes have been achieved.

Let

$$\alpha_m = \inf_{x \in \mathbb{R}} \alpha(x), \quad \alpha^M = \sup_{x \in \mathbb{R}} \alpha(x).$$

We collect the following proposition concerning with the Green function $G_t(x, y)$. We will write $G_t^{(\alpha)}(x - y)$ for $G_t(x, y)$ to emphasize that $\alpha(x)$ is a constant α.

Proposition 2.1. *If $0 < \alpha_m \le \alpha^M < 2$ and the derivative of $\alpha : \mathbb{R} \to [\alpha_m, \alpha^M]$ is uniformly continuous and bounded, assume the parameters α, β, λ satisfying the following*

$$\alpha \in \left[\alpha_m, \alpha^M\right], \quad \beta \in \left(0, \frac{1}{1+\alpha}\right), \quad \lambda \in (0, 1 - \beta(1+\alpha)). \tag{2.7}$$

then there exists a constant C only depending on α_m and α^M such that for any $0 \le s < t$ and $x, y \in \mathbb{R}$, the following estimates hold:

$$G_{t-s}(x, y) = G_{t-s}^{(\alpha)}(x - y) \left(1 + o((t-s)^\beta)\right) + \frac{o((t-s)^\lambda)}{1 + |x-y|^{1+\alpha}}, \tag{2.8}$$

$$C_1 H_{t-s}^{(\alpha)}(x - y) \le G_{t-s}^{(\alpha)}(x - y) \le C_2 H_{t-s}^{(\alpha)}(x - y), \tag{2.9}$$

$$H_{t-s}^{(\alpha)}(x - y) = (t-s)^{-\frac{1}{\alpha}} 1_{\left\{|x-y| \le (t-s)^{\frac{1}{\alpha}}\right\}} + \frac{t-s}{|x-y|^{1+\alpha}} 1_{\left\{|x-y| > (t-s)^{\frac{1}{\alpha}}\right\}}, \tag{2.10}$$

$$\begin{aligned}\left|\frac{\partial G_{t-s}(x,y)}{\partial y}\right| \le C &\left((t-s)^{-\frac{1}{\alpha}} \wedge |x-y|^{-1}\right)\\ &\cdot \left[(t-s)^{-\frac{1}{\alpha}} 1_{\left\{|x-y|\le (t-s)^{\frac{1}{\alpha}}\right\}} + \frac{t-s}{|x-y|^{1+\alpha}} 1_{\left\{|x-y|>(t-s)^{\frac{1}{\alpha}}\right\}}\right]\\ &\cdot (1+(t-s)^{\beta}) + \frac{C(t-s)^{\lambda}|x-y|^{\alpha}}{(1+|x-y|^{1+\alpha})^2},\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\left|\frac{\partial G_{t-s}(x,y)}{\partial t}\right| \le C &\left[(t-s)^{-1-\frac{1}{\alpha}} 1_{\left\{|x-y|\le (t-s)^{\frac{1}{\alpha}}\right\}} + \frac{1}{|x-y|^{1+\alpha}} 1_{\left\{|x-y|>(t-s)^{\frac{1}{\alpha}}\right\}}\right]\\ &\cdot \left(1+(t-s)^{\beta}\right) + \frac{C(t-s)^{\lambda-1}}{1+|x-y|^{1+\alpha}},\end{aligned} \tag{2.12}$$

where the standard notation $o(f)$ appearing in the above expression, for any positive function f , stands for a function that is less than $C \cdot f$ for some constant $C > 0$. Furthermore, we would like to clarify that the factors $o(t^{\beta})$ and $o(t^{\lambda})$ appeared above and in the sequel are indeed functions that do not depend on the space variables x, y and etc. Here the constant C is independent of α, β, and λ.

Remark 2.2. The inequality (2.8) and estimates (2.9), (2.10) and (2.11) can be found in Kolokoltsov (2000) and Jacob et al. (2010). Following the proof of Proposition 2.1 in Jacob et al. (2010), we can obtain the estimate (2.12). Here we omit the details. (One also can see Bass and Levin (2002), Jacob et al. (2010), Kolokoltsov (2000), Komatsu (1995) and references therein for more details)

2.3. *Malliavin-Stein's method.* Since $W = \{W(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ is Gaussian, one might develop the Malliavin calculus with respect to W (see, for example, Nualart (2006)). Let $\mathcal{S}$ be the class of smooth and cylindrical random variables of the form $F = f(W(\varphi_1), \ldots, W(\varphi_n))$, where $f \in C_b^{\infty}(\mathbb{R}^n)$ (i.e. the set of all functions with bounded derivatives of all orders) and $\varphi_i \in \mathcal{H}$ ($i = 1, \ldots, n$ and $n \in \mathbb{N}$). For each $F \in \mathcal{S}$, define the derivative DF by

$$DF := \sum_{i=1}^{n} \frac{\partial f}{\partial x}(W(\varphi_1), \ldots, W(\varphi_n))\varphi_i.$$

For any $p \ge 1$, let $\mathbb{D}^{1,p}$ be the completion of $\mathcal{S}$ under the norm

$$\|F\|_{1,p} = \left(\mathbb{E}\left[|F|^2 + \|DF\|_{\mathcal{H}}^2\right]\right)^{\frac{1}{p}}.$$

Then $\mathbb{D}^{1,2}$ is the domain of the closed operator D on $L^2(\Omega)$ with the domain $\mathbb{D}_h$ being the closure of $\mathcal{S}$ under the norm

$$\|F\|_h^2 = \mathbb{E}\left[|F|^2 + |D_h F|^2\right].$$

Let $\{h_n, n \ge 1\}$ be an orthonormal basis of $\mathcal{H}$. Then $F \in \mathbb{D}^{1,2}$ if and only if $F \in \mathbb{D}_{h_n}$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \mathbb{E}|D_{h_n} F|^2 < \infty$. In this case, $D_h F = \langle DF, h\rangle_{\mathcal{H}}$.

On the other hand, the divergence operator δ is the adjoint of the derivative operator D characterized by the following duality relationship

$$\mathbb{E}(\langle DF, u\rangle_{\mathcal{H}}) = \mathbb{E}(F\delta(u)), \quad \text{for any} \quad F \in \mathcal{S},$$

where $u \in \mathcal{H} = L^2(\Omega; \mathcal{H}_0)$. Then $Dom(\delta)$, the domain of δ, is the set of all functions $u \in L^2(\Omega; \mathcal{H}_0)$ such that

$$\mathbb{E}|\langle DF, u\rangle_{\mathcal{H}}| \le C(u)\|F\|_{L^2(\Omega)},$$

where $C(u)$ is some constant depending on u. The operator δ is also called the Skorohod integral. More generally, in the context of the (fractional) Gaussian noise W, and adapted random field X

which is jointly measurable and satisfying (2.2) belongs to the domain of δ and $\delta(X)$ coincides with the Dalang-Walsh integral

$$\delta(X)=\int_{\mathbb{R}^+}\int_{\mathbb{R}} X(s,x)W(ds,dx).$$

As a consequence, the mild equation (1.3) can also be rewritten as

$$u(t,x)=1+\delta\left(G_{t-\cdot}(x,*)\sigma(u(\cdot,*))\right). \tag{2.13}$$

Next let us provide a linear equation for the Malliavin derivative of the solution u to SPDE(1.1). The claim follows from (1.3), and the proof is rather standard (see, for example, Nualart (2006)). For this reason, we omit the details.

Lemma 2.3. *We have that the mild solution $u(t,x)$ to SPDE (1.1) belongs to $\mathbb{D}^{1,p}$ with $p\geq 2$ and the Malliavin derivative $Du(t,x)$ satisfies the following integral equation*

$$D_{s,y}u(t,x)=G_{t-s}(x,y)\sigma(u(s,y))+\int_s^t\int_{\mathbb{R}} G_{t-r}(x,z)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz), \tag{2.14}$$

where $\Sigma(r,z)$ is an adapted stochastic process (uniformly with respect to r and z) bounded by the Lipschitz constant L_σ of $\sigma(\cdot)$ that coincides with $\sigma'(u(r,z))$ whenever $\sigma(\cdot)$ is differentiable. Furthermore, for any $p\geq 2$

$$\sup_{x\in\mathbb{R}}\sup_{t\in[0,T]}\|Du(t,x)\|_p<+\infty.$$

Stein's method is a probabilistic method that allows one to measure the distance between a probability distribution and a target distribution, notably the normal distribution. Recall that the total variation distance between two real random variables F and G is defined by

$$d_{TV}(F,G):=\sup_{B\in\mathcal{B}(\mathbb{R})}|P(F\in B)-P(G\in B)|,$$

where $\mathcal{B}(\mathbb{R})$ is the collection of all Borel sets in $\mathbb{R}$. The following theorem provides the well-known Stein's bound in the total variation distance (see, Chapter 3 in Nourdin and Peccati (2012)).

Theorem 2.4. *For $N\sim N(0,1)$ and for any integrable random variable F,*

$$d_{TV}(F,N)\leq\sup_{f\in\mathcal{F}_{TV}}\left|\mathbb{E}[f'(F)]-\mathbb{E}[Ff(F)]\right|, \tag{2.15}$$

where $\mathcal{F}_{TV}$ is the class of continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ such that $\|f\|_\infty<\sqrt{\pi/2}$ and $\|f'\|_\infty\leq 2$.

For a proof of this theorem, one can see the Theorem 3.3.1 in Nourdin and Peccati (2012). Furthermore, Theorem 2.4 can be combined with Malliavin calculus to get a very useful estimate (see, Assaad et al. (2022), Delgado-Vences et al. (2020), Huang et al. (2020a), Huang et al. (2020b), Nourdin and Peccati (2009), Nualart and Zheng (2022)).

Proposition 2.5. *Let $F=\delta(v)$ for some $\mathcal{H}$-valued random variable $v\in\mathrm{Dom}(\delta)$. Assume $F\in\mathbb{D}^{1,2}$ and $\mathbb{E}[F^2]=1$ and let $N\sim N(0,1)$. Then we have*

$$d_{TV}(F,N)\leq 2\sqrt{\mathrm{Var}(\langle DF,v\rangle_{\mathcal{H}})}. \tag{2.16}$$

In the course of proving Theorem 1.4, we also need the following lemma, which is a generalization of Theorem 6.1.2 in Nourdin and Peccati (2012) and Proposition 2.3 in Huang et al. (2020a).

Proposition 2.6. *Let $F=(F^{(1)},\dots,F^{(m)})$ be a random vector such that $F^{(i)}=\delta(v^{(i)})$ for $v^{(i)}\in\mathrm{Dom}(\delta)$ and $F^{(i)}\in\mathbb{D}^{1,2}$, with $i=1,\dots,m$. Let $\mathbf{N}$ be an m-dimensional centered Gaussian vector with covariance $(C_{i,j})_{1\leq i,j\leq m}$. For any C^2-function $h:\mathbb{R}^m\to\mathbb{R}$ with bounded second* *partial derivatives, we have*

$$|\mathbb{E}[h(F)] - \mathbb{E}[h(z)]| \leq \frac{m}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^{m} \mathbb{E}\left[\left(C_{i,j} - \langle DF^{(i)}, v^{(j)}\rangle_{\mathcal{H}}\right)^2\right]},$$

where $\|h''\|_\infty := \sup\left\{\left|\frac{\partial^2}{\partial x_i \partial x_j} h(\mathbf{x})\right| : \mathbf{x} \in \mathbb{R}^m, i, j = 1, \ldots, m\right\}$.

3. Asymptotic behavior of the covariance

Our aim in this section is to prove the following proposition concerning with the asymptotic behavior of the covariance of the solution u to SPDE (1.1).

Proposition 3.1. *Denote by* $\xi(r) = \mathbb{E}\left[\sigma^2(u(r,x))\right]$, $\eta(r) = \mathbb{E}\left[\sigma(u(r,x))\right]$ *and set*

$$G_R(t) = \int_{-R}^{R} (u(t,x) - 1)dx. \tag{3.1}$$

Then for any $s, t \geq 0$, *one gets the following two limits*

$$\lim_{R\to\infty} \frac{1}{R}\mathrm{Cov}(G_R(t), G_R(s)) = 2\int_0^{s\wedge t} \xi(r)dr, \quad \text{if} \quad H = \frac{1}{2}, \tag{3.2}$$

and

$$\lim_{R\to\infty} \frac{1}{R^{2H}}\mathrm{Cov}(G_R(t), G_R(s)) = 2^{2H}\int_0^{s\wedge t} \eta(r)^2 dr, \quad \text{if} \quad H \in \left(\frac{1}{2}, 1\right). \tag{3.3}$$

From this proposition, we have found the result considered here that the square moment $\mathbb{E}\left[\sigma^2(u(r,x))\right]$ in the colored case (i.e., $H \in (1/2, 1)$) is replaced by the square of the first moment $(\mathbb{E}[\sigma(u(r,x))])^2$ in the white noise case (i.e. $H = 1/2$). Furthermore, the rate of convergence depends on the Hurst parameter H. Such a phenomenon also appeared in the case of the one-dimensional wave equation and $(d+1)$-stochastic fractional heat equation, see the recent paper Delgado-Vences et al. (2020) and Assaad et al. (2022), respectively, while the authors in Delgado-Vences et al. (2020) considered the Riesz kernel and more general spatial kernel in Assaad et al. (2022).

Before we can state the proof of Proposition 3.1, let us firstly give the following five useful lemmas [i.e., from Lemma 3.2 to Lemma 3.6].

Lemma 3.2. *Let* S *be a stable-like process whose transition probability is given by* $G_t(x,y)$ *with* $(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2$ *and* $H \in (1/2, 1)$. *Then the following inequality*

$$\sup_{t>0} \int_{\mathbb{R}} G_t(x,y)|y|^{2H-2}dy = \sup_{t>0} \mathbb{E}\left[|x+S|^{2H-2}\right] \leq C|x|^{2H-2}, \tag{3.4}$$

holds for some constant $C > 0$.

Proof: For the integral $\int_{\mathbb{R}} G_t(x,y)|y|^{2H-2}dy$, one can decompose it as follows

$$\int_{\mathbb{R}} G_t(x,y)|y|^{2H-2}dy = \int_{|y|<\frac{|x|}{2}} G_t(x,y)|y|^{2H-2}dy + \int_{|y|\geq\frac{|x|}{2}} G_t(x,y)|y|^{2H-2}dy. \tag{3.5}$$

The second term in (3.5) can be dealt with as follows.

$$\int_{|y|\geq\frac{|x|}{2}} G_t(x,y)|y|^{2H-2}dy \leq C|x|^{2H-2}\int_{|y|\geq\frac{|x|}{2}} G_t(x,y)dy \leq C|x|^{2H-2} \tag{3.6}$$

Now let us move the the first term in (3.5). According to (2.8), (2.9) and (2.10), one obtains that

$$\int_{|y|<\frac{|x|}{2}} G_t(x,y)|y|^{2H-2}dy \le C\int_{|y|<\frac{|x|}{2}} \left\{\left[t^{-\frac{1}{\alpha}}1_{\left\{|x-y|\le t^{\frac{1}{\alpha}}\right\}} + \frac{t}{|x-y|^{1+\alpha}}1_{\left\{|x-y|>t^{\frac{1}{\alpha}}\right\}}\right]\right. \\ \left. \cdot(1+t^{\beta}) + \frac{Ct^{\lambda}}{1+|x-y|^{1+\alpha}}\right\}|y|^{2H-2}dy. \tag{3.7}$$

Since for any $x, y \in \mathbb{R}$, it holds that

$$|x| - |y| \le |x-y| \le |x| + |y|.$$

The first term in the right hand of (3.7) can be estimated as follows

$$\begin{aligned}\int_{|y|<\frac{|x|}{2}} t^{-\frac{1}{\alpha}}1_{\left\{|x-y|\le t^{\frac{1}{\alpha}}\right\}}|y|^{2H-2}dy &\le t^{-\frac{1}{\alpha}}\int_{|y|<\frac{|x|}{2}} 1_{\left\{\frac{|x|}{2}\le t^{\frac{1}{\alpha}}\right\}}|y|^{2H-2}dy \\ &\le t^{-\frac{1}{\alpha}}\int_{|y|<t^{\frac{1}{\alpha}}} |y|^{2H-2}dy 1_{\left\{\frac{|x|}{2}\le t^{\frac{1}{\alpha}}\right\}} \\ &= Ct^{-\frac{2(1-H)}{\alpha}}1_{\left\{\frac{|x|}{2}\le t^{\frac{1}{\alpha}}\right\}} \le C|x|^{2H-2}.\end{aligned}$$

The second term in the right hand of (3.7) can be estimated as follows

$$\begin{aligned}\int_{|y|<\frac{|x|}{2}} \frac{t}{|x-y|^{1+\alpha}}1_{\left\{|x-y|>t^{\frac{1}{\alpha}}\right\}}|y|^{2H-2}dy &\le t^{-\frac{1}{\alpha}}\int_{|y|<\frac{|x|}{2}} 1_{\left\{|x-y|>t^{\frac{1}{\alpha}}\right\}}|y|^{2H-2}dy \\ &\le Ct^{-\frac{1}{\alpha}}|x|^{2H-1}1_{\left\{\frac{3}{2}|x|>t^{\frac{1}{\alpha}}\right\}} \\ &\le C|x|^{2H-2}.\end{aligned}$$

The third term in the right hand of (3.7) can be estimated as follows

$$\int_{|y|<\frac{|x|}{2}} \frac{t^{\lambda}}{1+|x-y|^{1+\alpha}}|y|^{2H-2}dy \le Ct^{\lambda-1}|x|^{2H-2}.$$

Combining these above three inequalities, one can conclude that, for all $t > 0$

$$\int_{|y|<\frac{|x|}{2}} G_t(x,y)|y|^{2H-2}dy \le C|x|^{2H-2}\left[1+t^{\lambda-1}\right] \le C|x|^{2H-2}.$$

Thus one can conclude the proof of this lemma by combining the estimates (3.5), (3.6) and (3.7). □

Lemma 3.3. *Suppose the parameters α, β, λ satisfying (2.7), then for any $x, y \in \mathbb{R}$ and $0 < s < t < +\infty$, there exists some constant $\kappa > \frac{1}{1+\alpha}$ such that*

$$\int_{\mathbb{R}} G_{t-s}(x,y)^{\kappa}dy \le C(t-s)^{\frac{1-\kappa}{\alpha}}, \tag{3.8}$$

for some constant $C > 0$.

Proof: Using the estimates (2.8), (2.9) and (2.10) for $G_t(x,y)$, with $\kappa > \frac{1}{1+\alpha}$, one can obtain that

$$\begin{aligned}
&\int_{\mathbb{R}} G_{t-s}(x,y)^{\kappa} dx \\
&\leq C \int_{\mathbb{R}} \Bigg[\left((t-s)^{-\frac{1}{\alpha}} 1_{\left\{|x-y| \leq (t-s)^{\frac{1}{\alpha}}\right\}} + \frac{t-s}{|x-y|^{1+\alpha}} 1_{\left\{|x-y| > (t-s)^{\frac{1}{\alpha}}\right\}} \right) \\
&\qquad\qquad \cdot \left(1+(t-s)^{\beta}\right) + \frac{(t-s)^{\lambda}}{1+|x-y|^{1+\alpha}} \Bigg]^{\kappa} dy \\
&\leq C(t-s)^{-\frac{\kappa}{\alpha}} \left(1+(t-s)^{\beta}\right)^{\kappa} \int_{\mathbb{R}} 1_{\left\{|x-y| \leq (t-s)^{\frac{1}{\alpha}}\right\}} dy \\
&\quad + (t-s)^{\kappa} \left(1+(t-s)^{\beta}\right)^{\kappa} \int_{\mathbb{R}} \frac{1_{\left\{|x-y| > (t-s)^{\frac{1}{\alpha}}\right\}}}{|x-y|^{\kappa(1+\alpha)}} dy + (t-s)^{\lambda\kappa} \int_{\mathbb{R}} \frac{1}{(1+|x|^{1+\alpha})^{\kappa}} dy \qquad (3.9) \\
&= C \Bigg[(t-s)^{\frac{1-\kappa}{\alpha}} \left(1+(t-s)^{\beta}\right)^{\kappa} \int_{\mathbb{R}} 1_{\{|u| \leq 1\}} du \\
&\quad + (t-s)^{\frac{1-\kappa}{\alpha}} \left(1+(t-s)^{\beta}\right)^{\kappa} \int_{\mathbb{R}} \frac{1}{|u|^{\kappa(1+\alpha)}} 1_{\{|u|>1\}} du + (t-s)^{\lambda\kappa} \int_{\mathbb{R}} \frac{1}{(1+|x|^{1+\alpha})^{\kappa}} dx \Bigg] \\
&\leq C \left[(t-s)^{\frac{1-\kappa}{\alpha}} \left(1+(t-s)^{\beta}\right)^{\kappa} + (t-s)^{\lambda\kappa} \right] \\
&= C(t-s)^{\frac{1-\kappa}{\alpha}} \left[\left(1+(t-s)^{\beta}\right)^{\kappa} + (t-s)^{\lambda\kappa - \frac{1-\kappa}{\alpha}} \right],
\end{aligned}$$

where in the above deviations we have used the facts that

$$\int_{|u| \leq 1} du = 2; \quad \int_{|u|>1} \frac{1}{|x|^{\kappa(1+\alpha)}} du = 2 \int_1^{\infty} x^{-\kappa(1+\alpha)} du < \infty, \quad \text{if} \quad \kappa > \frac{1}{1+\alpha};$$

and

$$\int_{\mathbb{R}} \frac{1}{(1+|u|^{1+\alpha})^{\kappa}} dx \leq 2 \int_0^1 dx + 2 \int_1^{\infty} x^{-\kappa(1+\alpha)} dx < \infty.$$

Thus one can conclude the proof of this lemma. □

The proof of the next lemma can be completed by using the similar arguments in the proof of Proposition 2.4 and (3.9) in Liu and Yan (2018). We omit the details here.

Lemma 3.4. *For any $t > 0, x, y \in \mathbb{R}$, there exists some positive constant C such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} G_t(x,z_1) G_t(x,z_2) |z_1 - z_2|^{2H-2} dz_1 dz_2 \leq C t^{\frac{2H-2}{\alpha}}. \qquad (3.10)$$

The following result provides two upper bounds for the p-th ($p \geq 2$) norm of the Malliavin derivative of the solution to SPDE (1.1) according to the value of H.

Lemma 3.5. *For every $0 < s < t < T$ and every $x, y \in \mathbb{R}$, with $p \geq 2$, there exist two positive constant C_1, C_2 such that*

(1) *If $H = 1/2$, then*

$$\|D_{s,y} u(t,x)\|_p \leq C_1 G_{t-s}(x,y). \qquad (3.11)$$

(2) *If $H \in (1/2, 1)$, then*

$$\|D_{s,y} u(t,x)\|_p \leq C_2 (t-s)^{-\frac{1-H}{\alpha}} G_{t-s}(x,y)^H. \qquad (3.12)$$

Proof: In a standard way we can show that, for every $t \in [0,T]$ and $x \in \mathbb{R}$, the random variable $u(t,x)$ belongs to the Sobolev space $\mathbb{D}^{1,p}$ for all $p \geq 2$ and its Malliavin derivative satisfies (2.14) (see, for example, Nualart (2006) and references therein). We will divide into two steps to prove this lemma.

Step 1. The case $H = 1/2$. In this case, from (2.14), one can write

$$\mathbb{E}\left[|D_{s,y}u(t,x)|^p\right] \leq CG_{t-s}(x,y)^p + C\left(\int_s^t \int_{\mathbb{R}} G_{t-s}(x,z)^2 (D_{s,y}u(r,z))^2 drdz\right)^{\frac{p}{2}}.$$

This implies that

$$\|D_{s,y}u(t,x)\|_p^2 \leq CG_{t-s}(x,y)^2 + C\int_s^t \int_{\mathbb{R}} G_{t-s}(x,z)^2 \|D_{s,y}u(r,z)\|_p^2 drdz.$$

By using Lemma A.1 in Huang et al. (2020a), one can conclude the proof of (3.11).

Step 2. The case $H \in (1/2, 1)$. Recall that the Dalang-Walsh integral satisfies the following version of the Burkholder-Davis-Gundy inequality (see, for example, Assaad et al. (2022)): for any $t \geq 0$ and $p \geq 2$, there exists some constant $c_p > 0$ such that

$$\begin{aligned} &\left\|\int_0^\infty \int_{\mathbb{R}} X(s,y)W(ds,dy)\right\|_p^2 \\ &\qquad \leq c_p \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \|X(s,y_1)X(s,y_2)\|_{\frac{p}{2}} |y_1 - y_2|^{2H-2} dy_1 dy_2 ds. \end{aligned} \tag{3.13}$$

We obtain that, for any $p \geq 2$, there exists some constant $C_p > 0$ such that

$$\begin{aligned} \|D_{s,y}u(t,x)\|_p^2 \leq C_p G_{t-s}(x,y)^2 + C_p \int_s^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_{t-r}(x,y_1) G_{t-r}(x,y_2) \\ \times \|D_{s,y}u(r,y_1)\|_p \|D_{s,y}u(r,y_2)\|_p |y_1 - y_2|^{2H-2} dy_1 dy_2 dr. \end{aligned}$$

To conclude the proof of (3.12), it suffices to apply the following Lemma 3.6 with $\theta = t-s, \eta = x-z$, and

$$g(\theta, \eta) = \|D_{s,y}u(\theta + s, \eta + z)\|_p.$$

□

In order to prove the above Lemma 3.5, we need the following lemma.

Lemma 3.6. *Assume that* $H \in (1/2, 1)$ *and* $g(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a non-negative function satisfying, for every* $t \in [0,T], x \in \mathbb{R}$,

$$g^2(t,x) \leq G_t(x) + \beta_H \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x,y) G_{t-s}(x,y') g(s,y) g(s,y') |y-y'|^{2H-2} dy' dy ds. \tag{3.14}$$

Then, for some constant $C > 0$, *one obtains that*

$$g(t,x) \leq Ct^{-\frac{1-H}{\alpha}} G_t(x,0)^H. \tag{3.15}$$

Proof: We follow the similar arguments in the proof of Lemma 5.3 in Assaad *et al* Assaad et al. (2022). As explained in Chen and Huang (2019) (see the proofs of Lemma 2.4 and Lemma 3.1), it suffices to prove the bound (3.15) in the case when (3.14) is an equality. Define iteratively as follows

$$g_0(t,x) = G_t(x,0),$$

and

$$\begin{aligned} g_{n+1}(t,x)^2 = G_t(x,0)^2 + \beta_H \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G_{t-s}(x,y) G_{t-s}(x,y') \\ \cdot g_n(s,y) g_n(s,y') |y-y'|^{2H-2} dy' dy ds. \end{aligned}$$

Denote by $\kappa = \frac{2(1-H)}{\alpha} < 1$, we prove by induction that for every $n \geq 0$

$$g_n(t,x)^2 \leq C \sum_{j=0}^n \frac{\Gamma^j(1-\kappa)}{\Gamma((j+1)(1-\kappa))} t^{j(1-\kappa)-\kappa} G_t(x,0)^{2H}. \tag{3.16}$$

By using the methods of induction, for $n=0$, one gets

$$g_0(t,x)=G_t(x,0)\le Ct^{-\frac{1-H}{\alpha}}G_t(x,0)^H. \tag{3.17}$$

In fact, one obtains

$$\frac{G_t(x,0)}{G_t(x,0)^H}=G_t(x,0)^{1-H}.$$

Based on the expressions (2.8), (2.10) and inequality (2.9), for $H\in[1/2,1)$, one can estimate $G_t(x)^{1-H}$ as follows.

$$\begin{aligned}
&G_t(x,0)^{1-H}\\
&=\left(G_t^{(\alpha)}(x)\left(1+o(t^\beta)\right)+\frac{o(t^\lambda)}{1+|x|^{1+\alpha}}\right)^{1-H}\\
&\le CG_t^{(\alpha)}(x)^{1-H}\left(1+o(t^\beta)\right)^{1-H}+\left(\frac{o(t^\lambda)}{1+|x|^{1+\alpha}}\right)^{1-H}\\
&\le C\left(t^{-\frac{1}{\alpha}}1_{\left\{|x|\le t^{\frac{1}{\alpha}}\right\}}+\frac{t}{|x|^{1+\alpha}}1_{\left\{|x|>t^{\frac{1}{\alpha}}\right\}}\right)^{1-H}\left(1+o(t^\beta)\right)^{1-H}+\left(\frac{o(t^\lambda)}{1+|x|^{1+\alpha}}\right)^{1-H}\\
&=Ct^{-\frac{1-H}{\alpha}}\left[1_{\left\{|x|\le t^{\frac{1}{\alpha}}\right\}}+\frac{t^{(1-H)\left(1+\frac{1}{\alpha}\right)}}{|x|^{(1+\alpha)(1-H)}}1_{\left\{|x|>t^{\frac{1}{\alpha}}\right\}}\right]\left(1+o(t^\beta)\right)^{1-H}+t^{-\frac{1-H}{\alpha}}\frac{t^{\lambda(1-H)+\frac{1}{\alpha}(1-H)}}{(1+|x|^{1+\alpha})^{1-H}}\\
&\le Ct^{-\frac{1-H}{\alpha}},
\end{aligned}$$

where the notations $o(t^\beta)$ and $o(t^\lambda)$ in the above expression are explained in Proposition 2.1.

Now suppose (3.16) holds for every $n\ge 1$. Denote by $C_j=\frac{\Gamma^j(1-\kappa)}{\Gamma((j+1)(1-\kappa))}$ and by induction hypothesis, one gets

$$\begin{aligned}
g_{n+1}^2(t,x)\le G_t(x,0)^2+\beta_H\int_0^t ds\int_{\mathbb{R}^2}dydy'G_{t-s}(x,y)G_{t-s}(x,y')\\
\times\sum_{j=0}^n C_js^{j(1-\kappa)-\kappa}G_s(y,0)^HG_s(y',0)^H|y-y'|^{2H-2}\\
:=G_t(x,0)^2+\beta_H\sum_{j=0}^n C_jI_j,
\end{aligned} \tag{3.18}$$

where we denote by

$$I_j:=\int_0^t s^{j(1-\kappa)-\kappa}\int_{\mathbb{R}^2}G_{t-s}(x,y)G_{t-s}(x,y')G_s(y,0)^HG_s(y',0)^H|y-y'|^{2H-2}dydy'ds.$$

By using the embedding inequality from $L^{1/H}(\mathbb{R})$ into $\mathcal{H}_0$, the inequality (3.17) and the semigroup property for $G_t(x,y)$, one obtains that

$$\begin{aligned}
I_j&\le C\int_0^t s^{j(1-\kappa)-\kappa}\left(\int_{\mathbb{R}}G_{t-s}(x,y)^{\frac{1}{H}}G_s(y,0)dy\right)^{2H}ds\\
&\le C\int_0^t s^{j(1-\kappa)-\kappa}\left(\int_{\mathbb{R}}(t-s)^{-\frac{1-H}{\alpha H}}G_{t-s}(x,y)G_s(y,0)dy\right)^{2H}ds\\
&\le CG_t(x,0)^{2H}\int_0^t s^{j(1-\kappa)-\kappa}(t-s)^{-\kappa}ds\\
&=CG_t(x,0)^{2H}t^{(j+1)(1-\kappa)-\kappa}\frac{\Gamma(1-\kappa)\Gamma((j+1)(1-\kappa))}{\Gamma((j+2)(1-\kappa))}.
\end{aligned} \tag{3.19}$$

Substituting both (3.17) and (3.19) into (3.18) yields

$$\begin{aligned} g_{n+1}^2(t,x) \leq & Ct^{-\kappa} G_t(x,0)^{2H} + C \sum_{j=0}^{n} C_j G_t(x,0)^{2H} t^{(j+1)(1-\kappa)-\kappa} \frac{\Gamma(1-\kappa)\Gamma((j+1)(1-\kappa))}{\Gamma((j+2)(1-\kappa))} \\ = & \, CG_t(x,0)^{2H} \sum_{j=0}^{n+1} t^{j(1-\kappa)-\kappa} \frac{\Gamma^j(1-\kappa)}{\Gamma((j+1)(1-\kappa))}. \end{aligned} \tag{3.20}$$

Finally, it follows from (3.16) that

$$\begin{aligned} g(t,x) &= \lim_{n\to+\infty} g_n(t,x) \\ &\leq C \left(G_t(x,0)^{2H} \sum_{j=0}^{\infty} t^{j(1-\kappa)-\kappa} \frac{\Gamma^j(1-\kappa)}{\Gamma((j+1)(1-\kappa))} \right)^{\frac{1}{2}} \\ &\leq Ct^{-\frac{\kappa}{2}} G_t(x,0)^H \\ &= Ct^{-\frac{1-H}{\alpha}} G_t(x,0)^H. \end{aligned}$$

This finishes the proof of (3.15).

□

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1: Recall the mild solution of SPDE (1.1) and the quantity $G_R(t)$ given by (3.1), one can rewrite $G_R(t)$ as follows

$$\begin{aligned} G_R(t) &= \int_{-R}^{R} \int_0^t \int_{\mathbb{R}} G_{t-s}(x,y)\sigma(u(s,y))W(ds,dy)dx \\ &= \int_0^t \int_{\mathbb{R}} \varphi_R(s,y)\sigma(u(s,y))W(ds,dy), \end{aligned} \tag{3.21}$$

where we denote by

$$\varphi_R(s,y) = \int_{-R}^{R} G_{t-s}(x,y)dx.$$

According to the value of H, we divide into two cases to prove this proposition.

Case 1. $H = 1/2$**.** Based on the Itô's isometry (2.3) for Dalang-Walsh stochastic integral (see, for example, Dalang (1999), Walsh (1986)), we have that, for any $s,t > 0$ and $x_1, x_2 \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}\left[u(t,x_1)u(s,x_2)\right] &= 1 + \int_0^{t\wedge s} \int_{\mathbb{R}} G_{t-r}(x_1,y)G_{s-r}(x_2,y)\mathbb{E}[\sigma(u(r,y))^2]dydr \\ &= 1 + \int_0^{t\wedge s} \xi(r)G_{t+s-2r}(x_1,x_2)dr, \end{aligned} \tag{3.22}$$

with $\xi(r) := \mathbb{E}[\sigma(u(r,y))^2]$. Recall that

$$\mathbb{E}\left[\int_{-R}^{R} u(t,x)dx\right] = 2R.$$

Since $G_R(t)$ is given by (3.1), then one has $\mathbb{E}[G_R(t)] = 0$. Thus we obtain that

$$\begin{aligned} \mathrm{Cov}(G_R(t), G_R(s)) &= \int_{-R}^{R}\int_{-R}^{R}\int_0^{t\wedge s} \xi(r)G_{t+s-2r}(x_1,x_2)drdx_1dx_2 \\ &= 2\int_0^{t\wedge s} \xi(r) \int_0^{2R} (2R-z)G_{t+s-2r}(z,0)dzdr. \end{aligned}$$

As a consequence,

$$\begin{aligned}\lim_{R\to\infty}\frac{1}{R}\mathrm{Cov}(G_R(t),G_R(s)) &= \lim_{R\to\infty} 2\int_0^{t\wedge s}\xi(r)\int_0^{2R}\left(2-\frac{z}{R}\right)G_{t+s-2r}(z,0)dzdr \\ &= 2\int_0^{t\wedge s}\xi(r)dr.\end{aligned} \tag{3.23}$$

Case 2. $H\in(1/2,1)$. Thanks to the Itô's isometry (2.3) and expression (3.21) for $G_R(t)$, we have

$$\begin{aligned}\mathrm{Cov}(G_R(t),G_R(s)) &= \mathbb{E}[G_R(t)G_R(r)] \\ &= \beta_H\int_0^{t\wedge r}\int_{\mathbb{R}^2}\varphi_R(s,y_1)\varphi_R(s,y_2) \\ &\quad\times\mathbb{E}\left[\sigma(u(s,y_1))\sigma(u(s,y_2))\right]|y_1-y_2|^{2H-2}dy_1dy_2ds.\end{aligned}$$

We remark that, for each fixed $t>0$, the process $\{u(t,x),x\in\mathbb{R}\}$ is strictly stationary[1], meaning that the finite-dimension distributions of the process $\{u(t,x+y),x\in\mathbb{R}\}$ do not depend on y. This fact yields that the process $\{\sigma(u(t,x)),x\in\mathbb{R}\}$ is stationary with respect to the space variable x, (see, for example, Lemma 18 in Dalang (1999)), we write

$$\mathbb{E}[\sigma(u(s,y_1))\sigma(u(s,y_2))] := \Psi(s,y_1-y_2).$$

Then

$$\mathbb{E}[G_R(t)G_R(r)] = \beta_H\int_0^{t\wedge r}\int_{\mathbb{R}}\varphi_R(s,\xi+z)\varphi_R(s,z)\Psi(s,\xi)|\xi|^{2H-2}d\xi dzds.$$

We claim that

$$\lim_{|\xi|\to+\infty}\sup_{0\le s\le t\wedge r}|\Psi(s,z)-\eta^2(s)|=0,$$

with $\eta(s):=\mathbb{E}[\sigma(u(s,x))]$. By using the two-parameter version of Clark-Ocone formula (see, for example, Nualart (2006)), we can write

$$\sigma(u(s,y)) = \mathbb{E}[\sigma(u(s,y))] + \int_0^s\int_{\mathbb{R}}\mathbb{E}\left[D_{r,\phi}(\sigma(u(s,y)))|\mathcal{F}_r\right]W(dr,d\phi),$$

and

$$\sigma(u(s,z)) = \mathbb{E}[\sigma(u(s,z))] + \int_0^s\int_{\mathbb{R}}\mathbb{E}\left[D_{r,\phi}(\sigma(u(s,z)))|\mathcal{F}_r\right]W(dr,d\phi).$$

As a consequence

$$\mathbb{E}\left[\sigma(u(s,y_1))\sigma(u(s,y_2))\right] = \eta(s)^2 + T(s;y_1,y_2),$$

with

$$\begin{aligned}&T(s;y_1,y_2) \\ &= \int_0^s\int_{\mathbb{R}^2}\mathbb{E}\left[\mathbb{E}\left(D_{r,z_1}(\sigma(u(s,y_1)))|\mathcal{F}_r\right)\mathbb{E}\left(D_{r,z_2}(\sigma(u(s,y_2)))|\mathcal{F}_r\right)\right]|z_1-z_2|^{2H-2}dz_1dz_2dr.\end{aligned}$$

By using the chain rule for the Malliavin derivative operator as follows (see, for example, Nualart (2006))

$$D_{r,z_1}(\sigma(u(s,y_1))) = \Sigma(s,y_1)D_{r,z_1}u(s,y_1),$$

and

$$D_{r,z_2}(\sigma(u(s,y_2))) = \Sigma(s,y_2)D_{r,z_2}u(s,y_2),$$

[1] To see the strict stationarity, we fix $y\in\mathbb{R}$ and put $v(t,x)=u(t,x+y)$. It is clear that v solves the SPDE (1.1) driven by the shifted noise $\{W(t,x+y),t\in\mathbb{R}_+,x\in\mathbb{R}\}$, which has stationary increments in the spatial variable.

where $\Sigma(s,y_1)$ and $\Sigma(s,y_2)$ are two adapted random fields uniformly bounded by the Lipschitz constant L_σ of $\sigma(\cdot)$. This implies

$$|\mathbb{E}\left[\mathbb{E}\left(D_{r,z_1}(\sigma(u(s,y_1)))|\mathcal{F}_r\right)\mathbb{E}\left(D_{r,z_2}(\sigma(u(s,y_2)))|\mathcal{F}_r\right)\right]| \le L_\sigma^2\|D_{r,z_1}u(s,y_1)\|_2\|D_{r,z_2}u(s,y_2)\|_2.$$

One obtains that

$$T(s;y_1,y_2) \le C\int_0^s\int_{\mathbb{R}^2}\|D_{r,z_1}u(s,y_1)\|_2\|D_{r,z_2}u(s,y_2)\|_2|z_1-z_2|^{2H-2}dz_1dz_2dr.$$

Hence by using inequality (3.12) in Lemma 3.5 with $\kappa\in\left(\frac{1}{1+\alpha},1\right)$, we obtain

$$\begin{aligned}|T(s;y_1,y_2)| &\le C\int_0^s(s-r)^{-\kappa}\int_{\mathbb{R}^2}G_{s-r}(y_1,z_1)^H G_{s-r}(y_2,z_2)^H|z_1-z_2|^{2H-2}dz_1dz_2dr\\ &:= T_1(s;y_1,y_2)\end{aligned}$$

We claim that

$$T_1(s;y_1,y_2)\to 0, \quad \text{as} \quad |y_1-y_2|\to+\infty.$$

We prove this claim by an argument based on the uniform integrability. Making the following change of variables

$$u=s-r, \quad \xi_1=y_1-z_1, \quad \xi_2=y_2-z_2.$$

Then we can write

$$T_1(s;y_1,y_2)=\int_0^s u^{-\kappa}\int_{\mathbb{R}^2}G_u(\xi_1,0)^H G_u(\xi_2,0)^H|y_1-y_2-\xi_1+\xi_2|^{2H-2}d\xi_1d\xi_2du.$$

For any fixed $\xi_1,\xi_2\in\mathbb{R}$, clearly $|y_1-y_2-\xi_1+\xi_2|^{2H-2}\to 0$ if $|y_1-y_2|$ tends to infinity. Taking into account that

$$\int_0^s u^{-\kappa}\int_{\mathbb{R}^2}G_u(\xi_1,0)^H G_u(\xi_2,0)^H d\xi_1d\xi_2du<\infty,$$

to show that

$$\lim_{|y_1-y_2|\to+\infty}T_1(s;y_1,y_2)=0. \tag{3.24}$$

It suffices to check that

$$\int_0^s u^{-\kappa}\int_{\mathbb{R}^2}G_u(\xi_1,0)^H G_u(\xi_2,0)^H|y_1-y_2-\xi_1+\xi_2|^{2H_0-2}d\xi_1d\xi_2du<\infty,$$

for some $1/2<H_0<H<1$. Making a change of variable, we can write

$$\begin{aligned}&\int_0^s u^{-\kappa}\int_{\mathbb{R}^2}G_u(\xi_1,0)^H G_u(\xi_2,0)^H|y_1-y_2-\xi_1+\xi_2|^{2H_0-2}d\xi_1d\xi_2du\\ &\le C\int_0^s u^{-\kappa}\left(\int_{\mathbb{R}}G_u(y,z)^{\frac{H}{H_0}}dz\right)^{2H_0}du.\end{aligned}$$

Next one can obtain the following estimate

$$\int_{\mathbb{R}}G_u(y,z)^{\frac{H}{H_0}}dz\le Cu^{\frac{H_0-H}{\alpha H_0}}.$$

Thus for $\kappa=\frac{2(1-H)}{\alpha}<1$, one gets

$$\begin{aligned}&\int_0^s u^{-\kappa}\int_{\mathbb{R}^2}G_u(\xi_1,0)^H G_u(\xi_2,0)^H|y_1-y_2-\xi_1+\xi_2|^{2H_0-2}d\xi_1d\xi_2du\\ &\le C\int_0^s u^{-\kappa}u^{\frac{2(H_0-H)}{\alpha}}du\\ &=C\int_0^s u^{\frac{2(H_0-1)}{\alpha}}du<\infty, \quad \text{if} \quad \alpha+2H_0-2>0.\end{aligned}$$

This concludes the proof of (3.24). That means

$$\Psi(s;y_1-y_2)=\mathbb{E}[\sigma(u(s,y_1))\sigma(u(s,y_2))]\to\eta^2(s)=\mathbb{E}[\sigma^2(u(s,x))],$$

as $|y_1-y_2|\to+\infty$.

Now for notational simplicity, we only consider the case $t=r$, while the general case $r,t\in[0,T]$ follows in a similar way. Recall the expression (3.21) for $G_R(t)$. Thanks to the Itô's isometry (2.3), we have

$$\mathbb{E}\left[G_R(t)^2\right]=\beta_H\int_0^t\int_{\mathbb{R}^2}\varphi_R(s,y_1)\varphi_R(s,y_2)\mathbb{E}[\sigma(u(s,y_1))\sigma(u(s,y_2))]|y_1-y_2|^{2H-2}dy_1dy_2ds.$$

Recall that we denote by

$$\Psi(s;y_1-y_2):=\mathbb{E}[\sigma(u(s,y_1))\sigma(u(s,y_2))]=\eta^2(s)+T(s;y_1,y_2).$$

Define the quantity T_R as

$$T_R:=\frac{1}{R^{2H}}\int_0^t\int_{\mathbb{R}^2}\varphi_R(s,\xi+z)\varphi_R(s,z)[\Psi(s,\xi)-\eta^2(s)]|\xi|^{2H-2}d\xi dzds.$$

By using Lemma 3.2 and dominated convergence theorem, one can use the similar arguments in the proof of (3.7) in Huang et al. (2020b) to prove that

$$T_R\to0,\quad\text{as}\quad R\to+\infty.$$

Here we omit the details.

Now, it suffices to show that,

$$\beta_HR^{-2H}\int_0^t\eta^2(s)\int_{\mathbb{R}^2}\varphi_R(s,\xi+z)\varphi_R(s,z)|\xi|^{2H-2}d\xi dzds\to2^{2H}\int_0^t\eta^2(s)ds,\tag{3.25}$$

as $R\to+\infty$. In fact, for some constant $c>0$, the left-hand of (3.25) is equal to

$$\begin{aligned}&\beta_HR^{-2H}\int_0^t\eta^2(s)\int_{\mathbb{R}^2}\varphi_R(s,\xi+z)\varphi_R(s,z)|\xi|^{2H-2}d\xi dzds\\&=\beta_HR^{-2H}\int_0^t\eta^2(s)\int_{-R}^R\int_{-R}^Rdxdx'\int_{\mathbb{R}^2}G_{t-s}(x,\xi+z)G_{t-s}(x',z)|\xi|^{2H-2}dzd\xi ds\\&=\beta_HR^{-2H}\int_0^t\eta^2(s)\int_{-R}^R\int_{-R}^Rdxdx'\int_{\mathbb{R}}G_{2(t-s)}(x,x'+\xi)|\xi|^{2H-2}dzds\\&=\beta_HR^{-2H}\int_0^t\eta^2(s)\int_{-R}^R\int_{-R}^R\mathbb{E}[|x-x'+cS|^{2H-2}]dxdx'ds\\&=\beta_H\int_0^t\eta^2(s)\int_{-1}^1\int_{-1}^1\mathbb{E}\left[|x-x'+cR^{-1}S|^{2H-2}\right]dxdx'ds.\end{aligned}$$

In view of Lemma 3.2 and dominated convergence theorem, we obtain the limit in (3.25) and hence the proof of this proposition is completed.

□

Remark 3.7. (1) It follows from Proposition 3.1 that, if $H\in(1/2,1)$, the random variable $G_R(t)$ is *not chaotic* in the linear case. More precisely, when $\sigma(x)=x$, the above proposition gives us that

$$\mathrm{Var}(G_R(t))\sim(2R)^{2H}t,\quad\text{as}\quad R\to+\infty.$$

On the other hand, when $\sigma(x)=x$, the random variable $G_R(t)$ defined by (3.21) has an explicit Wiener chaos expansion (see, for example, Pu (2022) and etc)

$$G_R(t)=\int_0^t\int_{\mathbb{R}}\varphi_R(s,y)W(ds,dy)+higher-order\quad chaoses.$$

The first chaotic component of $G_R(t)$ is centered Gaussian with variance equals to

$$\beta_H \int_0^t \int_{\mathbb{R}^2} \varphi_R(s,y)\varphi_R(s,z)|y-z|^{2H-2}dydz \sim (2R)^{2H}t, \quad \text{as} \quad R\to+\infty,$$

see (3.25). This shows that only the first chaos contributes to the limit, that is, there is a non-chaotic behavior of the spatial average of the linear SPDE (1.1), when $H\in(1/2,1)$.

(2) For $H=1/2$ and $\sigma(x)=x$, we obtain from Proposition 3.1 that

$$\mathrm{Var}(G_R(t)) \sim 2R\int_0^t \mathbb{E}[(u(s,x))^2]ds, \quad \text{as} \quad R\to+\infty,$$

whereas the variance of the projection on the first chaos is, using (3.23),

$$\int_0^t \int_{\mathbb{R}} \varphi_R(s,y)^2 dyds \sim 2Rt, \quad as \quad R\to+\infty.$$

Note that $\mathbb{E}[(u(s,x))^2] \geq [\mathbb{E}(u(s,x))]^2 = 1$ and the inequality is strict for all $s\in(0,t]$ (otherwise $u(s,x)$ would be a constant). This implies that the first chaos is not the only contributor to the limiting variance.

(3) We also would like to point out that, for the linear stochastic heat equation driven by space-time white noise as considered in Huang et al. (2020a), the central limit is chaotic, meaning that each projection on the Wiener chaos contributes to the Gaussian limit. In this case, the proof of asymptotic normality could be based on the chaotic central limit theorem (see, for example, Section 6.3 in Nourdin and Peccati (2012)). For the case of stochastic wave equation driven by fractional noise with Hurst index $H\in[1/2,1)$, one can see the Remark 1 in the work Delgado-Vences et al. (2020). For $H\in(1/2,1)$, the random variable G_R is not chaotic in the linear case (i.e. $\sigma(x)=x$). For $H=1/2$, the first chaos is not the only contributor to the limiting variance of G_R.

Before we give the proof of Theorem 1.3, by using the similar arguments as in the proof of Proposition 3.1, we can also obtain an asymptotic formula for $\mathbb{E}[G_R(t_i)G_R(t_j)]$ with $t_i,t_j\in\mathbb{R}^+$, which is useful in the proof of functional central limit theorem (i.e. Theorem 1.4).

Remark 3.8. By using the same arguments as in the proof of Proposition 3.1, we obtain an asymptotic formula for $\mathbb{E}[G_R(t_i)G_R(t_j)]$ with $t_i,t_j\in\mathbb{R}^+$, which is a useful ingredient for the proof of Theorem 1.4.

(1) Suppose $t_i,t_j\in\mathbb{R}^+$, for $H=1/2$, we write

$$\mathbb{E}[G_R(t_i)G_R(t_j)] = \int_0^{t_i\wedge t_j}\int_{\mathbb{R}} \varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,y)\xi(s)dyds,$$

with $\varphi_R^{(i)}(s,y)=\int_{-R}^R G_{t_i-s}(x,y)dx$, and we obtain

$$\begin{aligned}\lim_{R\to+\infty}\frac{\mathbb{E}[G_R(t_i)G_R(t_j)]}{R} &= \lim_{R\to+\infty} R^{-1}\int_0^{t_i\wedge t_j} ds\xi(s)\int_{\mathbb{R}}\varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,y)dy\\ &= 2\int_0^{t_i\wedge t_j}\xi(s)ds.\end{aligned}$$

(2) Suppose $t_i,t_j\in\mathbb{R}^+$, for $H\in(1/2,1)$, we write

$$\mathbb{E}[G_R(t_i)G_R(t_j)] = \int_0^{t_i\wedge t_j}\int_{\mathbb{R}^2} \varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,z)\Psi(s,y-z)|y-z|^{2H-2}dydzds,$$

with $\varphi_R^{(i)}(s,y) = \int_{-R}^R G_{t_i-s}(x,y)dx$, and we obtain

$$\lim_{R\to+\infty} \frac{\mathbb{E}\left[G_R(t_i)G_R(t_j)\right]}{R^{2H}} = \lim_{R\to+\infty} R^{-2H} \int_0^{t_i\wedge t_j} ds\eta^2(s) \int_{\mathbb{R}^2} \varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,z)|y-z|^{2H-2}dydz$$
$$= 2^{2H}\int_0^{t_i\wedge t_j} \eta^2(s)ds.$$

4. Proof of Theorem 1.3 and Theorem 1.4

In this section, we will apply Propositions 2.5 and Proposition 2.6 to prove Theorem 1.3 and Theorem 1.4.

4.1. *Proof of Theorem 1.3.* Now we can state the proof of Theorem 1.3.

Proof of Theorem 1.3: From (1.4), one can rewrite $F_R(t)$ as follows

$$F_R(t) = \frac{1}{\sigma_R}\int_0^t\int_{\mathbb{R}} \varphi_R(s,y)\sigma(u(s,y))W(ds,dy) := \delta(v_R),$$

with

$$v_R = 1_{[0,t]}(s)\frac{1}{\sigma_R}\varphi_R(s,y)\sigma(u(s,y)).$$

Moreover, from the expression (1.4), one can get

$$D_{s,y}F_R(t) = 1_{[0,t]}(s)\frac{1}{\sigma_R}\int_{-R}^R D_{s,y}u(t,x)dx.$$

From (2.14) and the stochastic Fubini's theorem, one obtains that

$$\int_{-R}^R D_{s,y}u(t,x)dx = \varphi_R(s,y)\sigma(u(s,y)) + \int_s^t\int_{\mathbb{R}} \varphi_R(s,y)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz).$$

Therefore, we have the following decomposition

$$\langle DF_R, v_R\rangle_{\mathcal{H}} := \mathcal{A}_1 + \mathcal{A}_2,$$

where the expressions of $\mathcal{A}_1, \mathcal{A}_2$ depend on the value of $H = 1/2$ and $H \in (1/2, 1)$.

Case $H = 1/2$. In this case, $\mathcal{A}_1, \mathcal{A}_2$ are given as

$$\mathcal{A}_1 = \frac{1}{\sigma_R^2}\int_0^t\int_{\mathbb{R}} \varphi_R^2(s,y)\sigma^2(u(s,y))dyds,$$
$$\mathcal{A}_2 = \frac{1}{\sigma_R^2}\int_0^t\int_{\mathbb{R}} \varphi_R(s,y)\sigma(u(s,y))\int_s^t\int_{\mathbb{R}} \varphi_R(r,z)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz)dyds.$$

Note that for any process $X = \{X_s, s\in[0,t]\}$ such that $\mathrm{Var}(X_s)$ is integrable on $[0,t]$, it holds that

$$\sqrt{\mathrm{Var}\left(\int_0^t X_s ds\right)} \le \int_0^t \sqrt{\mathrm{Var}(X_s)}ds. \tag{4.1}$$

So we can write

$$\sqrt{\mathrm{Var}(\langle DF_R, v_R\rangle_{\mathcal{H}})} \le \sqrt{2}\left(\sqrt{\mathrm{Var}(\mathcal{A}_1)} + \sqrt{\mathrm{Var}(\mathcal{A}_2)}\right) := \sqrt{2}(\mathcal{B}_1 + \mathcal{B}_2),$$

with

$$\mathcal{B}_1 = \frac{1}{\sigma_R^2}\int_0^t \left(\int_{\mathbb{R}^2} \varphi_R^2(s,y)\varphi_R^2(s,y')\mathrm{Cov}(\sigma^2(u(s,y)),\sigma^2(u(s,y')))dydy'\right)^{\frac{1}{2}} ds$$

$$\mathcal{B}_2 = \frac{1}{\sigma_R^2}\int_0^t \left(\int_{\mathbb{R}^3}\int_s^t \varphi_R^2(r,z)\varphi_R(s,y)\varphi_R(s,y')\right.$$
$$\left.\cdot\mathbb{E}\left[\Sigma^2(r,z)D_{s,y}u(r,z)D_{s,y'}u(r,z)\sigma(u(s,y))\sigma(u(s,y'))\right] dydy'dzdr\right)^{\frac{1}{2}} ds.$$

Let us firstly estimate the term $\mathcal{B}_2$. For any $p\geq 2$, as a consequence of stationarity, we write

$$K_p(t) = \sup_{0\leq s\leq t}\sup_{y\in\mathbb{R}} \|\sigma(u(s,y))\|_p = \sup_{0\leq s\leq t} \|\sigma(u(s,0))\|_p. \tag{4.2}$$

Thus one obtains that

$$\mathbb{E}\left[\Sigma^2(r,z)D_{s,y}u(r,z)D_{s,y'}u(r,z)\sigma(u(s,y))\sigma(u(s,y'))\right] \leq CK_4^2(t)L_\sigma^2 G_{r-s}(z,y)G_{r-s}(z,y'),$$

where the last inequality follows from Lemma 3.5. Together with Proposition 3.1, integrating $\widetilde{x},\widetilde{x}'$ over $\mathbb{R}$, then integrating y,y' on $\mathbb{R}$ and using the semigroup property for the Green function $G_t(x,y)$, we obtain that

$$\mathcal{B}_2 \leq C\frac{1}{R}\int_0^t \left(\int_{\mathbb{R}^2}\int_{[-R,R]^4}\int_s^t\int_{\mathbb{R}} G_{t-s}(x,y)G_{t-s}(x',y')G_{t-s}(\widetilde{x},z)G_{t-s}(\widetilde{x}',z)\right.$$
$$\left.\cdot\, G_{r-s}(z,y)G_{r-s}(z,y')dzdrdxdx'd\widetilde{x}d\widetilde{x}'dydy'\right)^{\frac{1}{2}} ds$$
$$\leq C\frac{1}{R}\int_0^t \left(\int_{-R}^R\int_{-R}^R\int_s^t\int_{\mathbb{R}} G_{t+r-2s}(x,z)G_{t+r-2s}(x',z)dzdrdxdx'\right)^{\frac{1}{2}} ds$$
$$= C\frac{1}{R}\int_0^t \left(\int_{-R}^R\int_{-R}^R\int_s^t G_{2t+2r-4s}(x,x')drdxdx'\right)^{\frac{1}{2}} ds.$$

Finally, integrating x over $\mathbb{R}$ and x' over $[-R,R]$, we get

$$\mathcal{B}_2 \leq C\frac{1}{\sqrt{R}}.$$

Next let us study the term $\mathcal{B}_1$. In order to bound $\mathcal{B}_1$, we need to find the bound for the covariance $\mathrm{Cov}(\sigma^2(u(s,y)),\sigma^2(u(s,y')))$. In fact, using the two-parameter version of Clark-Ocone formula (see, for example, Nualart (2006)), we can write

$$\sigma^2(u(s,y)) = \mathbb{E}[\sigma^2(u(s,y))] + \int_0^s\int_{\mathbb{R}} \mathbb{E}\left[D_{r,z}(\sigma^2(u(s,y)))|\mathcal{F}_r\right] W(dr,dz).$$

Then

$$\mathrm{Cov}(\sigma^2(u(s,y)),\sigma^2(u(s,y')))$$
$$= \int_0^s\int_{\mathbb{R}} \mathbb{E}\left[D_{r,z}(\sigma^2(u(s,y)))|\mathcal{F}_r\right]\mathbb{E}\left[D_{r,z}(\sigma^2(u(s,y')))|\mathcal{F}_r\right] W(dr,dz).$$

Applying the chain rule for the Lipschitz functions of Malliavin derivatives, we have

$$D_{r,z}(\sigma^2(u(s,y))) = 2\sigma(u(s,y))\Sigma(s,y)D_{r,z}u(s,y),$$

and

$$\left\|\mathbb{E}\left[D_{r,z}(\sigma^2(u(s,y)))|\mathcal{F}_r\right]\right\|_2 \leq 2K_4(t)\|D_{r,z}u(s,y)\|_4.$$

Then using the first point in Lemma 3.5, one can write

$$\begin{aligned}\mathrm{Cov}(\sigma^2(u(s,y)),\sigma^2(u(s,y'))) &\le 2K_4(t)\int_0^s\int_{\mathbb{R}}\|D_{r,z}u(s,y)\|_4\|D_{r,z}u(s,y')\|_4 dzdr\\ &\le C\int_0^s\int_{\mathbb{R}} G_{s-r}(y,z)G_{s-r}(y',z)dzdr\\ &= C\int_0^s G_{2s-2r}(y,y')dr.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{B}_1 &\le C\frac{1}{R}\int_0^t\left(\int_{\mathbb{R}^2}\left(\int_{-R}^R G_{t-s}(x,y)dx\right)^2\left(\int_{-R}^R G_{t-s}(x,y')dx'\right)^2\right.\\ &\qquad\left.\cdot\int_0^s G_{2s-2r}(y,y')drdydy'\right)^{\frac{1}{2}}ds\\ &\le C\frac{1}{R}\int_0^t\left(\int_0^s\int_{\mathbb{R}^2}\int_{[-R,R]^4} G_{t-s}(x,y)G_{t-s}(\widetilde{x},y)G_{t-s}(x',y')G_{t-s}(\widetilde{x}',y')\right.\\ &\qquad\left.\cdot\, G_{2s-2r}(y,y')dxd\widetilde{x}dx'd\widetilde{x}'dydy'dr\right)^{\frac{1}{2}}ds\end{aligned}$$

Again, integrate $\widetilde{x},\widetilde{x}'$ over $\mathbb{R}$, then integrate y,y' over $\mathbb{R}$ using the semigroup property, to obtain

$$\mathcal{B}_1 \le C\frac{1}{R}\int_0^t\left(\int_0^s\int_{-R}^R\int_{-R}^R G_{2t-2r}(x,x')dxdx'dr\right)^{\frac{1}{2}}ds.$$

Finally, integrating x over $\mathbb{R}$ and x' over $-R$ to R, we obtain

$$\mathcal{B}_1 \le C\frac{1}{R}.$$

This completes the proof of this theorem with $H=\frac{1}{2}$.

Case $H\in(1/2,1)$. In this case, the decomposition of $\langle DF_R, v_R\rangle_{\mathcal{H}}$ is given by

$$\langle DF_R, v_R\rangle_{\mathcal{H}} = \mathcal{C}_1+\mathcal{C}_2,$$

where

$$\begin{aligned}\mathcal{C}_1 &= \frac{\beta_H}{\sigma_R^2}\int_0^t\int_{\mathbb{R}^2}\varphi_R(s,y)\varphi_R(s,y')\sigma(u(s,y))\sigma(u(s,y'))|y-y'|^{2H-2}dydy'ds,\\ \mathcal{C}_2 &= \frac{\beta_H}{\sigma_R^2}\int_0^t\int_{\mathbb{R}^2}\left(\int_s^t\int_{\mathbb{R}}\varphi_R(r,z)\Sigma(r,z)D_{s,y}u(r,z)W(dr,dz)\right)\\ &\qquad\cdot\varphi_R(s,y')\sigma(u(s,y'))|y-y'|^{2H-2}dydy'ds.\end{aligned}$$

This decomposition implies that

$$\sqrt{\mathrm{Var}\langle DF_R, v_R\rangle_{\mathcal{H}}} \le \sqrt{2}(\mathcal{D}_1+\mathcal{D}_2),$$

with

$$\begin{aligned}\mathcal{D}_1 = \frac{\beta_H}{\sigma_R^2}\int_0^t&\left(\int_{\mathbb{R}^4}\varphi_R(s,y)\varphi_R(s,y')\varphi_R(s,\widetilde{y})\varphi_R(s,\widetilde{y}')|y-y'|^{2H-2}|\widetilde{y}-\widetilde{y}'|^{2H-2}\right.\\ &\left.\cdot\mathrm{Cov}(\sigma(u(s,y))\sigma(u(s,y')),\sigma(u(s,\widetilde{y}))\sigma(u(s,\widetilde{y}')))dydy'd\widetilde{y}d\widetilde{y}'\right)^{\frac{1}{2}}ds,\end{aligned}$$

and

$$\mathcal{D}_2 = \frac{\beta_H^{\frac{3}{2}}}{\sigma_R^2} \int_0^t \Big(\int_s^t \int_{\mathbb{R}^6} \varphi_R(r,z)\varphi_R(r,\widetilde{z})\varphi_R(s,y')\varphi_R(s,\widetilde{y}') \cdot \mathbb{E}\left[\Sigma(r,z) D_{s,y} u(r,z) \Sigma(r,\widetilde{z}) D_{s,\widetilde{y}} u(r,\widetilde{z}) \sigma(u(s,y'))\sigma(u(s,\widetilde{y}'))\right] \cdot |y-y'|^{2H-2} |\widetilde{y}-\widetilde{y}'|^{2H-2} |z-\widetilde{z}|^{2H-2} dy dy' d\widetilde{y} d\widetilde{y}' dz d\widetilde{z} dr \Big)^{\frac{1}{2}} ds.$$

Let us firstly prove the second term $\mathcal{D}_2$. As before, for any $p \geq 2$, recall the notation $K_p(t)$ defined by (4.2), we can write

$$\begin{aligned} &\mathbb{E}\left[\Sigma(r,z) D_{s,y} u(r,z) \Sigma(r,\widetilde{z}) D_{s,\widetilde{y}} u(r,\widetilde{z}) \sigma(u(s,y'))\sigma(u(s,\widetilde{y}'))\right] \\ &\quad \leq K_p(t)^2 L_\sigma^2 \|D_{s,y} u(r,z)\|_4 \|D_{s,\widetilde{y}} u(r,\widetilde{z})\|_4 \\ &\quad \leq C (r-s)^{-\frac{2(1-H)}{\alpha}} G_{r-s}(y,z)^H G_{r-s}(\widetilde{y},\widetilde{z})^H, \end{aligned}$$

where the last inequality follows from the second point in Lemma 3.5. Now by using Proposition 3.1, for any fixed $t > 0$, there exists a positive constant R_t that depends on t such that for any $R \geq R_t$,

$$\mathcal{D}_2 \leq C \frac{1}{R^{2H}} \int_0^t \Big(\int_s^t \int_{\mathbb{R}^6} \varphi_R(r,z)\varphi_R(r,\widetilde{z})\varphi_R(s,y')\varphi_R(s,\widetilde{y}') \cdot (r-s)^{-\frac{2(1-H)}{\alpha}} G_{r-s}(y,z)^H G_{r-s}(\widetilde{y},\widetilde{z})^H \cdot |y-y'|^{2H-2} |\widetilde{y}-\widetilde{y}'|^{2H-2} |z-\widetilde{z}|^{2H-2} dy dy' d\widetilde{y} d\widetilde{y}' dz d\widetilde{z} dr \Big)^{\frac{1}{2}} ds.$$

Following the similar arguments in the proof of Theorem 2.3 in Assaad et al. (2022), by using (3.8) in Lemma 3.3, and taking into account that

$$\sup_{z \in \mathbb{R}} \int_{-1}^1 |x+z|^{2H-2} dx < \infty, \tag{4.3}$$

we conclude that

$$\mathcal{D}_2 \leq C R^{H-1}.$$

We now estimate $\mathcal{D}_1$. We begin by estimating the following covariance

$$\operatorname{Cov}\left(\sigma(u(s,y))\sigma(u(s,y')), \sigma(u(s,\widetilde{y}))\sigma(u(s,\widetilde{y}'))\right). \tag{4.4}$$

Using a version of Clark-Ocone formula for some integrable functionals of the noise W (see, for example, Nualart (2006)), we can write, for any $s \geq 0$ and $y, y' \in \mathbb{R}$

$$\begin{aligned} \sigma(u(s,y))\sigma(u(s,y')) &= \mathbb{E}\left[\sigma(u(s,y))\sigma(u(s,y'))\right] \\ &\quad + \int_0^s \int_{\mathbb{R}} \mathbb{E}\left[D_{r,z}(\sigma(u(s,y))\sigma(u(s,y'))) | \mathcal{F}_r\right] W(dr,dz). \end{aligned}$$

Then, we represent the covariance (4.4) as

$$\int_0^s \int_{\mathbb{R}^2} \mathbb{E}\left\{ \mathbb{E}\left[D_{r,z}(\sigma(u(s,y))\sigma(u(s,y'))) | \mathcal{F}_r\right] \cdot \mathbb{E}\left[D_{r,z'}(\sigma(u(s,\widetilde{y}))\sigma(u(s,\widetilde{y}'))) | \mathcal{F}_r\right] \right\} |z-z'|^{2H-2} dz dz' dr.$$

By the chain rule,

$$D_{r,z}(\sigma(u(s,y))\sigma(u(s,y'))) = \Sigma(s,y) D_{r,z} u(s,y) \sigma(u(s,y')) + \sigma(u(s,y)) \Sigma(s,y') D_{r,z} u(s,y').$$

Therefore, the term $\mathbb{E}\left[D_{r,z}(\sigma(u(s,y))\sigma(u(s,y'))) | \mathcal{F}_r\right]$ can be bounded by

$$2 K_4(t) L_\sigma \left[\|D_{r,z} u(s,y)\|_4 + \|D_{r,z} u(s,y')\|_4 \right].$$

Using Lemma 3.5 again, we see that the covariance (4.4) is bounded by

$$
\begin{aligned}
&4L_\sigma^2 K_4^2(t) \int_0^s \int_{\mathbb{R}^2} \left[\|D_{r,z}u(s,y)\|_4 + \|D_{r,z}u(s,y')\|_4\right] \\
&\qquad\qquad \cdot \left[\|D_{r,z'}u(s,\widetilde{y})\|_4 + \|D_{r,z'}u(s,\widetilde{y}')\|_4\right] |z-z'|^{2H-2} dz dz' dr \\
&\leq C \int_0^s (s-r)^{-\frac{2(1-H)}{\alpha}} \int_{\mathbb{R}^2} \left(G_{s-r}(y,z)^H + G_{s-r}(y',z)^H\right) \\
&\qquad\qquad \cdot \left(G_{s-r}(\widetilde{y},z')^H + G_{s-r}(\widetilde{y}',z')^H\right) |z-z'|^{2H-2} dz dz' dr.
\end{aligned}
$$

Then, it follows from exactly the same arguments as in the estimation of $\mathcal{D}_2$ in the previous step and taking into account that the fact (4.3) again and by (3.8) in Lemma 3.3, we conclude that

$$\mathcal{D}_1 \leq C R^{H-1}.$$

Thus one can complete the proof of Theorem 1.3. □

4.2. *Proof of Theorem 1.4.* In order to prove the Theorem 1.4, we need to prove tightness and the convergence of the finite-dimensional distributions. Notice that tightness follows from the following Proposition 4.1 and the well-known criterion of Kolmogorov's continuity theorem.

Proposition 4.1. *Assume that $H \in [1/2, 1)$ and let $u(t,x)$ be the solution to SPDE (1.1). Then for any $0 \leq s < t \leq T$, $R > 0$ and any $p \geq 2$, there exists a constant $C_{p,T}$, depending on T and p, such that*

$$\mathbb{E}\left(\left|\int_{-R}^R u(t,x)dx - \int_{-R}^R u(s,x)dx\right|^p\right) \leq C_{p,T} R^{pH} (t-s)^{\frac{p(\alpha+2H-2)}{2\alpha}}. \tag{4.5}$$

Proof: For any $0 \leq s < t \leq T$, we can write

$$\int_{-R}^R (u(t,x) - u(s,x))dx = \int_0^T \int_{\mathbb{R}} \left(\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\right) \sigma(u(r,y)) W(dr,dy),$$

where we denote by

$$\varphi_{t,R}(r,y) := \int_{-R}^R G_{t-r}(x,y) 1_{\{r \leq t\}} dx.$$

The proof of this proposition consists two steps.

Step 1: Suppose that $H = 1/2$. Using Burkholder-Davis-Gundy inequality and Minkowski's inequality, we get, for some positive constant c_p,

$$
\begin{aligned}
&\mathbb{E}\left(\left|\int_{-R}^R u(t,x)dx - \int_{-R}^R u(s,x)dx\right|^p\right) \\
&\leq c_p \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}} \left(\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\right)^2 \sigma^2(u(r,y)) dr dy\right)^{\frac{p}{2}}\right] \\
&\leq c_p \left(\int_0^T \int_{\mathbb{R}} \left(\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\right)^2 \|\sigma(u(r,y))\|_p^2 dr dy\right)^{\frac{p}{2}} \\
&\leq c_p K_p(T)^p \left(\int_0^T \int_{\mathbb{R}} \left(\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)\right)^2 dr dy\right)^{\frac{p}{2}},
\end{aligned}
$$

where $K_p(T)$ has been defined in (4.2). Now we notice that

$$\begin{aligned}&|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)|\\&\le 1_{\{r\le s\}}\int_{-R}^{R}|G_{t-r}(x,y)-G_{s-r}(x,y)|dx+1_{\{s<r\le t\}}\int_{-R}^{R}|G_{t-r}(x,y)|dx.\end{aligned}$$

Thus we have,

$$\begin{aligned}&\mathbb{E}\left(\left|\int_{-R}^{R}u(t,x)dx-\int_{-R}^{R}u(s,x)dx\right|^p\right)\\&\le c_pK_p(T)^p\left(\int_0^T\int_{\mathbb{R}}\left(1_{\{r\le s\}}\int_{-R}^{R}|G_{t-r}(x,y)-G_{s-r}(x,y)|dx\right.\right.\\&\qquad\left.\left.+1_{\{s<r\le t\}}\int_{-R}^{R}|G_{t-r}(x,y)|dx\right)^2drdy\right)^{\frac{p}{2}}\\&\le c_pK_p(T)^p\left(\int_0^T1_{\{r\le s\}}\int_{-R}^{R}\int_{\mathbb{R}}|G_{t-r}(x,y)-G_{s-r}(x,y)|^2dydxdr\right.\\&\qquad\left.+\int_0^T1_{\{s<r\le t\}}\int_{-R}^{R}\int_{\mathbb{R}}|G_{t-r}(x,y)|^2dydxdr\right)^{\frac{p}{2}},\end{aligned}$$

where the last equality is derived by using Fubini's theorem. According to the estimate (2.12), following the similar arguments in the proof of (4.11) in Liu and Yan (2018), one gets that

$$\int_{\mathbb{R}}|G_{t-r}(x,y)-G_{s-r}(x,y)|^2dy\le C(t-s)^2(t-r)^{-2-\frac{1}{\alpha}}.$$

From inequality (3.9) in Liu and Yan (2018), one obtains that

$$\int_{\mathbb{R}}G_{t-r}(x,y)^2dy\le C(t-r)^{-\frac{1}{\alpha}}.$$

Thus we can get the following

$$\mathbb{E}\left(\left|\int_{-R}^{R}u(t,x)dx-\int_{-R}^{R}u(s,x)dx\right|^p\right)\le CR^{\frac{p}{2}}(t-s)^{\frac{p}{2}(1-\frac{1}{\alpha})}.$$

Step 2: Suppose that $H\in(1/2,1)$. Denote $\Theta_{t,s;x}(r,y)$ by

$$\Theta_{t,s;x}(r,y):=G_{t-r}(x,y)1_{\{r\le t\}}-G_{s-r}(x,y)1_{\{r\le s\}}.$$

In the same way, one can write

$$\begin{aligned}&\mathbb{E}\left(\left|\int_{-R}^{R}u(t,x)dx-\int_{-R}^{R}u(s,x)dx\right|^p\right)\\&\le c_p\mathbb{E}\left[\left(\int_0^T\|(\varphi_{t,R}(r,\cdot)-\varphi_{s,R}(r,\cdot))\sigma(u(r,\cdot))\|^2_{\mathcal{H}_0}dr\right)^{\frac{p}{2}}\right].\end{aligned}\tag{4.6}$$

As mentioned in Section 2, for $H\in(1/2,1)$, the space $L^{1/H}(\mathbb{R})$ is continuously embedded into $\mathcal{H}_0$. Consequently, there exists a constant $c_H>0$, depending on H, such that

$$\begin{aligned}&\|(\varphi_{t,R}(r,\cdot)-\varphi_{s,R}(r,\cdot))\sigma(u(r,\cdot))\|^2_{\mathcal{H}_0}\\&\qquad\le c_H\left(\int_{\mathbb{R}}|\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)|^{\frac{1}{H}}|\sigma(u(r,y))|^{\frac{1}{H}}dy\right)^{2H}.\end{aligned}\tag{4.7}$$

Substituting (4.7) into (4.6) and applying Hölder's and Minkowski's inequalities, one can write

$$\begin{aligned}
&\mathbb{E}\left(\left|\int_{-R}^{R} u(t,x)dx - \int_{-R}^{R} u(s,x)dx\right|^p\right) \\
&\le c_p c_H^{\frac{p}{2}} T^{\frac{p}{2}-1}\left[\int_0^T \mathbb{E}\left[\left(\int_{\mathbb{R}} |\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)|^{\frac{1}{H}} |\sigma(u(r,y))|^{\frac{1}{H}} dy\right)^{2H}\right] dr\right]^{\frac{p}{2}} \\
&\le c_p c_H^{\frac{p}{2}} T^{\frac{p}{2}-1}\left[\int_0^T \left(\int_{\mathbb{R}} |\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)|^{\frac{1}{H}} \|\sigma(u(r,y))\|_p^{\frac{1}{H}} dy\right)^{2H} dr\right]^{\frac{p}{2}} \\
&\le c_p c_H^{\frac{p}{2}} T^{\frac{p}{2}-1} K_p(T)^p \left[\int_0^T \left(\int_{\mathbb{R}} |\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)|^{\frac{1}{H}} dy\right)^{2H} dr\right]^{\frac{p}{2}}.
\end{aligned} \tag{4.8}$$

Now let us estimate the following integral $\int_{\mathbb{R}} |\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)|^{\frac{1}{H}} dy$. In fact

$$\begin{aligned}
&\int_{\mathbb{R}} |\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)|^{\frac{1}{H}} dy \\
&\le \int_{\mathbb{R}} \left| 1_{\{r\le s\}} \int_{-R}^{R} |G_{t-r}(x,y)-G_{s-r}(x,y)| dx + 1_{\{s<r\le t\}} \int_{-R}^{R} |G_{t-r}(x,y)| dx \right|^{\frac{1}{H}} dy \\
&\le C \int_{-R}^{R} 1_{\{r\le s\}} \int_{\mathbb{R}} |G_{t-r}(x,y)-G_{s-r}(x,y)|^{\frac{1}{H}} dy dx \\
&\qquad + \int_{-R}^{R} 1_{\{s<r\le t\}} \int_{\mathbb{R}} |G_{t-r}(x,y)|^{\frac{1}{H}} dy dx.
\end{aligned}$$

From the inequality (3.9) in Liu and Yan (2018), one obtains that

$$\int_{\mathbb{R}} |G_{t-r}(x,y)|^{\frac{1}{H}} dy \le C(t-r)^{-\frac{1-H}{\alpha H}}.$$

Moreover, according to the estimate (2.12), following the similar arguments in the proof of (4.11) in Liu and Yan (2018), one gets that

$$\int_{\mathbb{R}} |G_{t-r}(x,y)-G_{s-r}(x,y)|^{\frac{1}{H}} dy \le C(t-s)^{\frac{1}{H}} (t-r)^{-\frac{\alpha+1-H}{\alpha H}}.$$

These above two inequalities yield that

$$\begin{aligned}
&\int_{\mathbb{R}} |\varphi_{t,R}(r,y)-\varphi_{s,R}(r,y)|^{\frac{1}{H}} dy \\
&\qquad \le CR\left[1_{\{r\le s\}}(t-s)^{\frac{1}{H}}(t-r)^{-\frac{\alpha+1-H}{\alpha H}} + 1_{\{s<r\le t\}}(t-r)^{-\frac{1-H}{\alpha H}}\right].
\end{aligned} \tag{4.9}$$

Substitutes (4.9) into (4.8), one obtains that

$$\begin{aligned}
&\mathbb{E}\left(\left|\int_{-R}^{R} u(t,x)dx - \int_{-R}^{R} u(s,x)dx\right|^p\right) \\
&\le c_p c_H^{\frac{p}{2}} T^{\frac{p}{2}-1} K_p(T)^p R^{pH} \left[\int_0^T \left[1_{\{r\le s\}}(t-s)^{\frac{1}{H}}(t-r)^{-\frac{\alpha+1-H}{\alpha H}}\right.\right. \\
&\qquad \left.\left. + 1_{\{s<r\le t\}}(t-r)^{-\frac{1-H}{\alpha H}}\right]^{2H} dr\right]^{\frac{p}{2}} \\
&\le c_p c_H^{\frac{p}{2}} T^{\frac{p}{2}-1} K_p(T)^p R^{pH} (t-s)^{\frac{p(\alpha+2H-2)}{2\alpha}}.
\end{aligned}$$

Thus one can conclude the proof of this proposition. □

Now lets us give the proof of Theorem 1.4.

Proof of Theorem 1.4: Let us now show the convergence of the finite-dimensional distributions. We fix $0 \le t_1 < \cdots < t_m \le T$ and consider

$$F_R(t_i) := \frac{1}{R^H}\left(\int_{-R}^{R} u(t_i,x)dx - 2R\right) = \delta(v_R^{(i)}), \quad i = 1,2,\dots,m,$$

where

$$v_R^{(i)}(s,y) = 1_{[0,t_i]}(s)\frac{\sigma(u(s,y))}{R^H}\varphi_R^{(i)}(s,y),$$

with

$$\varphi_R^{(i)}(s,y) = \int_{-R}^{R} G_{t_i-s}(x,y)dx.$$

Set $\mathbf{F_R} = (F_R(t_1),\dots,F_R(t_m))$ and let $\mathbf{N}$ be a centered Gaussian vector on $\mathbb{R}^m$ with covariance $\mathbf{C} = (C_{i,j})_{1\le i,j\le m}$ given by

$$C_{i,j} := \begin{cases} 2\displaystyle\int_0^{t_i\wedge t_j} \xi(r)dr, & H = \dfrac{1}{2}; \\ 2^{2H}\displaystyle\int_0^{t_i\wedge t_j} \eta^2(r)dr, & H \in \left(\dfrac{1}{2},1\right). \end{cases} \tag{4.10}$$

Recall that $\xi(r) = \mathbb{E}[\sigma^2(u(r,y))]$ and $\eta(r) = \mathbb{E}[\sigma(u(r,y))]$. Then, we need to show $\mathbf{F_R}$ converges in distribution to $\mathbf{N}$. In view of Lemma 2.6, it suffices to show that for each i,j, the term $\langle DF_R(t_i), v_R^{(j)}\rangle_{\mathcal{H}}$ converges to $C_{i,j}$ defined by (4.10) in $L^2(\Omega)$, as $R\to+\infty$. The case $i=j$ has been tackled before and the other case can be dealt with by using arguments similar to those in the proof of Theorem 1.3. For the convenience of readers, we only sketch these arguments as follows.

We consider two cases: $H = 1/2$ and $H \in (1/2,1)$. In each case, we need to show, with $i,j = 1,2,\dots,m$

(1) $\mathbb{E}[F_R(t_i)F_R(t_j)] \to C_{i,j}$, as $R\to+\infty$;
(2) $\mathrm{Var}\left(\langle DF_R(t_i), v_R^{(j)}\rangle_{\mathcal{H}}\right) \to 0$, as $R\to+\infty$.

The above point 1 has been established in Remark 3.8. To see point 2, for the case $H = 1/2$, we begin with the decomposition

$$\langle DF_R(t_i), v_R^{(j)}\rangle_{\mathcal{H}} := \mathcal{E}_1(i,j) + \mathcal{E}_2(i,j),$$

with

$$\mathcal{E}_1(i,j) := \frac{1}{R}\int_0^{t_i\wedge t_j}\int_{\mathbb{R}} \varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,y)\sigma^2(u(s,y))dsdy,$$

and

$$\mathcal{E}_2(i,j) := \frac{1}{R}\int_0^{t_i\wedge t_j}\int_{\mathbb{R}} \varphi_R^{(j)}(s,y)\sigma(u(s,y)) \times \left(\int_s^{t_i}\int_{\mathbb{R}} \varphi_R^{(i)}(r,z)\Sigma(r,z)\sigma(u(r,z))W(dr,dz)\right)dsdy.$$

Then using (4.1) and going through the same lines as for the estimates of the two terms $\mathcal{B}_1, \mathcal{B}_2$ in the proof of Theorem 1.3, we can get

$$\begin{aligned}\sqrt{\operatorname{Var}(\mathcal{E}_2(i,j))} \leq & \frac{1}{R}\int_0^{t_i\wedge t_j}\left(\int_{\mathbb{R}^3}\int_s^{t_i}\varphi_R^{(i)}(r,z)^2\varphi_R^{(j)}(s,y)\varphi_R^{(j)}(s,y')\right.\\ & \left.\times\mathbb{E}\left[\Sigma^2(r,z)D_{s,y}u(r,z)D_{s,y'}u(r,z)\sigma(u(s,y))\sigma(u(s,y'))\right]dydy'drdz\right)^{\frac{1}{2}}ds\\ \leq & \frac{C}{\sqrt{R}}.\end{aligned}$$

That is, we have $\operatorname{Var}(\mathcal{E}_2(i,j)) \to 0$, as $R \to +\infty$. We can also get

$$\begin{aligned}\sqrt{\operatorname{Var}(\mathcal{E}_1(i,j))} \leq & \frac{1}{R}\int_0^{t_i\wedge t_j}\left(\int_{\mathbb{R}^2}\varphi_R^{(i)}(s,y)\varphi_R^{(j)}(s,y)\varphi_R^{(i)}(s,y')\varphi_R^{(j)}(s,y')\right.\\ & \left.\times\operatorname{Cov}\left(\sigma^2(u(s,y)),\sigma^2(u(s,y'))\right)dydy'\right)^{\frac{1}{2}}ds\\ \leq & \frac{C}{\sqrt{R}}.\end{aligned}$$

That is, we have $\operatorname{Var}(\mathcal{E}_1(i,j)) \to 0$, as $R \to +\infty$.

To see point (2) for the case $H \in (1/2, 1)$, one can begin with the same decomposition and then use (4.1) to arrive at similar estimates as those for $\mathcal{D}_1$ and $\mathcal{D}_2$. Therefore the same arguments ensure

$$\operatorname{Var}\left(\langle DF_R(t_i), v_R^{(j)}\rangle_{\mathcal{H}}\right) \leq CR^{2H-2}.$$

Now the proof of Theorem 1.4 is completed.

□

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