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U-statistics of local sample moments under weak dependence

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Abstract. In this paper, we study the asymptotic distribution of some U-statistics whose entries are functions of empirical moments computed from non-overlapping consecutive blocks of an underlying weakly dependent process. The length of these blocks converges to infinity, and thus we consider U-statistics of triangular arrays. We establish asymptotic normality of such U-statistics. The results can be used to construct tests for changes of higher order moments.

1. Introduction

Given some real-valued data $Y_1, \ldots, Y_n$ and a symmetric measurable function $h : \mathbb{R}^m \to \mathbb{R}$, we define the U-statistic with kernel h as

$$U_n := U_n(h) = \frac{1}{\binom{n}{m}} \sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} h(Y_{i_1}, \ldots, Y_{i_m}).$$

U-statistics play an important role in nonparametric statistics, as many sample statistics can be expressed in this way, at least asymptotically. Well-known examples include the sample variance, Gini's mean difference, the Cramér-von Mises test statistic and the χ^2-test statistic for goodness of fit. For details and further examples see e.g. Serfling (1980) and Dehling (2006). U-statistics have been introduced independently by Halmos (1946) and Hoeffding (1948). Halmos (1946) showed that for i.i.d. data, $U_n(h)$ is an unbiased estimator of the parameter $\theta = \mathbb{E}\left[h\left(Y_1, \ldots, Y_m\right)\right]$, and that it is

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minimum variance unbiased in nonparametric models. Hoeffding (1948) proved that, again for i.i.d. data and for square integrable kernels, the U-statistic is asymptotically normal. More precisely,

$$\sqrt{n}\left(U_n(h)-\mathbb{E}\left[h\left(Y_1,\ldots,Y_m\right)\right]\right)\longrightarrow N(0,m^2\,\gamma_h^2)$$

in distribution, where $\gamma_h^2 := \operatorname{Var}\left(\mathbb{E}\left[h(Y_1,\ldots,Y_m)|Y_2,\ldots,Y_m\right]\right)$. In the so-called degenerate case $\gamma_h^2=0$, a different normalization is required to get a non-trivial limit, which will be a non-normal distribution; see Serfling (1980) for further details. Most of the results for i.i.d. data can be extended to weakly dependent stationary processes $(Y_i)_{i\geq 1}$; see e.g. Dehling (2006) for a survey.

In this paper, we study the asymptotic distribution of certain U-statistics whose entries are local summary statistics of an underlying weakly dependent process $(X_t)_{t\in\mathbb{Z}}$. More precisely, we consider local statistics $g\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t,\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^2,\ldots,\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^m\Big)$, $1\leq j\leq b_n$, which can be expressed as a function of the first m empirical moments of the consecutive non-overlapping blocks

$$B_{n,j} := \{(j-1)\ell_n+1,\ldots,j\ell_n\} \tag{1.1}$$

for $1\leq j\leq b_n$. We assume the block length ℓ_n to converge to infinity. Given an appropriate scaling factor $\sqrt{\ell_n}$ and certain regularity assumptions on $g\colon\mathbb{R}^m\to\mathbb{R}$, we will show that the statistics

$$W_{n,j} := \sqrt{\ell_n}g\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t,\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^2,\ldots,\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^m\Big) \tag{1.2}$$

are each asymptotically normal. We are then interested in U-statistics of the type

$$U_n := \frac{1}{b_n\left(b_n-1\right)}\sum_{1\leq j\neq k\leq b_n}h\left(W_{n,j},W_{n,k}\right). \tag{1.3}$$

Such U-statistics arise naturally in nonparametric tests for the constancy of parameters of the underlying process $(X_t)_{t\in\mathbb{Z}}$. Schmidt, Wornowizki, Fried, and Dehling (2021) test for the constancy of the variance by analysing Gini's mean difference of the logarithmic local sample variances, i.e. they choose $h(x,y)=|x-y|$ and $W_{n,j}=\sqrt{\ell_n}\log(\frac{1}{\ell_n}\sum_{t=(j-1)\ell_n+1}^{j\ell_n}(X_t-\frac{1}{\ell_n}\sum_{r=(j-1)\ell_n+1}^{jl\ell_n}X_r)^2)$. Schmidt (2023+) tests for changes in the mean by considering Gini's mean difference of the local sample means $W_{n,j}=\frac{1}{\sqrt{\ell_n}}\sum_{t\in B_{n,j}}X_t$. In both works, the behaviour of the test statistic under the hypothesis is determined by deriving a central limit theorem for U_n. The setup considered in the present paper allows for testing for constancy of higher order characteristics of the distribution of X_t, such as the skewness or kurtosis; see Example 2.2 below.

Note that the entries of U_n from (1.3) stem from a triangular array $(W_{n,j})_{1\leq j\leq b_n,n\geq 1}$ and that under certain regularity assumptions made in this paper, they each converge to a normal law as $\ell_n\to\infty$. Moreover, assuming the number b_n of blocks to converge to infinity as well, there additionally holds (under appropriate assumptions) a central limit theorem for the U-statistic itself. The limit distribution of U_n is hence determined by the double asymptotics of the U-statistic and its entries. It is the goal of this paper to investigate more systematically such structures and to find minimal conditions that guarantee asymptotic normality of the resulting U-statistics of type (1.3).

2. Main results

We are interested in U-statistics of triangular arrays of the form (1.3), where $h\colon\mathbb{R}^2\to\mathbb{R}$ denotes a symmetric kernel function. We will henceforth always assume that the kernel fulfils

$$\left|h\left(x,y\right)\right|\leq C\left(1+|x|+|y|\right) \tag{2.1}$$

for all $x,y\in\mathbb{R}$ and some constant C. For the results under dependence later on, we will require the stronger assumption of Lipschitz-continuity.

Our first result is a central limit theorem for U_n, given the triangular array $(W_{n,j})_{1\leq j\leq b_n,n\geq 1}$ is row-wise i.i.d. with a very mild assumption on the distribution of the random variables $W_{n,j}$.

Theorem 2.1. *Let $(W_{n,j})_{1\leq j\leq b_n, n\geq 1}$ be a row-wise i.i.d. triangular array such that $b_n \to \infty$ as $n \to \infty$. Assume that (2.1) holds, that $\gamma_n^2 := \operatorname{Cov}\left(h\left(W_{n,1}, W_{n,2}\right), h\left(W_{n,2}, W_{n,3}\right)\right) > 0$, and that the sequence $\left(W_{n,1}^2/\gamma_n^2\right)_{n\geq 1}$ is uniformly integrable. Then the following convergence in distribution holds*

$$\frac{\sqrt{b_n}}{\gamma_n}\left(\frac{1}{b_n\left(b_n-1\right)}\sum_{1\leq j\neq k\leq b_n} h\left(W_{n,j}, W_{n,k}\right) - \mathbb{E}\left[h\left(W_{n,1}, W_{n,2}\right)\right]\right) \to N\left(0,4\right). \tag{2.2}$$

A related result was obtained in Löwe and Terveer (2021) for incomplete U-statistics of independent data.

The above theorem lays the groundwork for the more specific problems we investigate in this paper. As opposed to Theorem 1, the triangular array $(W_{n,j})_{1\leq j\leq b_n, n\geq 1}$ we consider from now on is in general not row-wise independent as the underlying process $(X_t)_{t\in\mathbb{Z}}$ is weakly dependent. More specifically, we assume the stationary sequence $(X_t)_{t\in\mathbb{Z}}$ to be expressible as a functional of an i.i.d. process. Thus, we can write $X_t := f\left((\varepsilon_{t-u})_{u\in\mathbb{Z}}\right)$, where $f\colon \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ is measurable and $(\varepsilon_u)_{u\in\mathbb{Z}}$ is i.i.d. In order to quantify the dependence, let $(\varepsilon'_u)_{u\in\mathbb{Z}}$ be an independent copy of $(\varepsilon_u)_{u\in\mathbb{Z}}$ and define

$$\delta_i\left((X_t)_{t\in\mathbb{Z}}\right) := \left\|X_0 - X_0^{*,i}\right\|_2, \tag{2.3}$$

where $X_0^{*,i} = f\left(\left(\varepsilon_{-u}^{*,i}\right)_{u\in\mathbb{Z}}\right)$ and $\varepsilon_v^{*,i} = \varepsilon'_i$ if $v = i$ and $\varepsilon_v^{*,i} = \varepsilon_v$ otherwise. We thus measure the contribution of ε_i to X_0 by looking at the difference between X_0 and a coupled version $X_0^{*,i}$ for which ε_i is replaced by an independent copy. This weak dependence concept was introduced by Wu (2005) under the term *physical dependence measure* and is now frequently used in statistical applications (see, e.g., El Machkouri (2014), Liu, Xiao, and Wu (2013), Wu (2008) and Wu (2011)).

In the following, the triangular array $(W_{n,j})_{1\leq j\leq b_n, n\geq 1}$ is assumed to be of the form (1.2). Example 2.2 presents some problems that are covered by this structure.

Example 2.2.

(1) Schmidt et al. (2021) propose a test for constancy of the variance based on the test statistic

$$U_n = \frac{1}{b_n(b_n-1)}\sum_{1\leq j,k\leq b_n}\sqrt{\ell_n}\left|\log s_{n,j}^2 - \log s_{n,k}^2\right|,$$

where $s_{n,j}^2 := \sum_{t\in B_{n,j}}(X_t - \frac{1}{\ell_n}\sum_{r\in B_{n,j}} X_r)^2$. In our setting, this corresponds to $m = 2$, $g(x_1, x_2) = \log(x_2 - x_1^2)$ and $h\left(x, y\right) = |x - y|$.

(2) Considering higher moments, one can construct a test for the constancy of the skewness or the kurtosis in a similar fashion to (1) by considering Gini's mean difference (that is, $h\left(x, y\right) = |x - y|$) of the blockwise estimates $\hat{\gamma}_{n,j}$, $j = 1, \ldots, b_n$, or $\hat{\kappa}_{n,j}$, $j = 1, \ldots, b_n$, respectively. Note that an empirical version of the skewness is given by

$$\begin{aligned}\hat{\gamma}_{n,j} &= \frac{\frac{1}{\ell_n}\sum_{t\in B_{n,j}}\left(X_t - \frac{1}{\ell_n}\sum_{r\in B_{n,j}} X_r\right)^3}{\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}\left(X_t - \frac{1}{\ell_n}\sum_{r\in B_{n,j}} X_r\right)^2\right)^{3/2}} \\ &= \frac{\frac{1}{\ell_n}\sum_{t\in B_{n,j}} X_t^3 - 3\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}} X_t^2\right)\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}} X_t\right) + 2\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}} X_t\right)^3}{\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}} X_t^2 - \left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}} X_t\right)^2\right)^{3/2}},\end{aligned}$$

which is covered in our setting via the function

$$g(x_1, x_2, x_3) = \frac{x_3 - 3x_1x_2 + 2x_1^3}{\left(x_2 - x_1^2\right)^{3/2}}.$$

An empirical version of the kurtosis is given by

$$\begin{aligned}\hat{\kappa}_{n,j} =& \frac{\frac{1}{\ell_n}\sum_{t\in B_{n,j}}\left(X_t-\frac{1}{\ell_n}\sum_{r\in B_{n,j}}X_r\right)^4}{\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}\left(X_t-\frac{1}{\ell_n}\sum_{r\in B_{n,j}}X_r\right)^2\right)^2}\\ =&\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^2-\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t\Big)^2\Big)^{-2}\cdot\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^4-4\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t\Big)\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^3\Big)\\ &+6\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t\Big)^2\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^2\Big)-3\Big(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t\Big)^4\Big),\end{aligned}$$

which corresponds to the function

$$g(x_1,x_2,x_3,x_4)=\frac{x_4-4x_1x_3+6x_1^2x_2-3x_1^4}{(x_2-x_1^2)^2}.$$

We now state a central limit theorem for U-statistics of this more concrete type of triangular array (1.2). The question of the central limit theorem for U-statistics whose entries are Bernoulli shifts has been addressed in Hsing and Wu (2004) and Giraudo (2021), but these results do not treat the case of arrays.

Theorem 2.3. *Assume that the following conditions are satisfied.*

(1) *The function h is Lipschitz-continuous.*
(2) $\mathbb{E}\left[X_1^{2m}\right]<\infty$ *and*

$$\sum_{i\in\mathbb{Z}}\sum_{k=1}^{m}i^2\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)<\infty. \tag{2.4}$$

(3) *The function g satisfies $g(v_0)=0$, where $v_0=\left(\mathbb{E}\left[X_1^k\right]\right)_{k=1}^m\in\mathbb{R}^m$. There exists an $a>0$ such that g is differentiable at each point of $\prod_{k=1}^m\left(\mathbb{E}\left[X_1^k\right]-2a,\mathbb{E}\left[X_1^k\right]+2a\right)$ and the gradient of g is bounded on $\prod_{k=1}^m\left(\mathbb{E}\left[X_1^k\right]-2a,\mathbb{E}\left[X_1^k\right]+2a\right)$.*
(4) *The sequences $(b_n)_{n\geq1}$ and $(\ell_n)_{n\geq1}$ go to infinity as n goes to infinity. Moreover, the sequence (b_n/ℓ_n) goes to 0 as n goes to infinity.*

Let $\eta\colon\mathbb{R}^m\to[0,1]$ be a smooth function with $\eta(x)=1$ if $\|x-v_0\|_2\leq a$ and $\eta(x)=0$ if $\|x-v_0\|_2>2a$, and define

$$W_{n,j}^{(\eta)}:=\sqrt{\ell_n}\,(g\cdot\eta)\left(\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}X_t^k\right)_{k=1}^m\right).$$

If

$$\sigma^2:=\sum_{t\in\mathbb{Z}}\operatorname{Cov}\left(\sum_{k=1}^m\frac{\partial g}{\partial x_k}(v_0)\,X_0^k,\sum_{k=1}^m\frac{\partial g}{\partial x_k}(v_0)\,X_t^k\right)>0, \tag{2.5}$$

the following convergence in distribution holds

$$\sqrt{b_n}\left(U_n-\frac{1}{b_n(b_n-1)}\sum_{1\leq j\neq k\leq b_n}\mathbb{E}\left[h\left(W_{n,j}^{(\eta)},W_{n,k}^{(\eta)}\right)\right]\right)\to N\left(0,4\gamma^2\right), \tag{2.6}$$

where

$$\gamma^2:=\operatorname{Cov}\left(h\left(\sigma N,\sigma N'\right),h\left(\sigma N',\sigma N''\right)\right)$$

for independent standard normally distributed random variables N, N' and N''.

Remark 2.4. Note that the partial derivatives in (2.5) arise as a consequence of the delta method.

In the examples below, we provide some classes of processes $(X_t)_{t\in\mathbb{Z}}$ that fulfil the above condition (2.4). For more details, we refer to Section 3.4.

Example 2.5. Let X_t be a Hölder continuous function of a linear process as considered, for instance, in Ho and Hsing (1997) and Wu (2006). More precisely, define X_t as

$$X_t := \varphi\left(\sum_{j\in\mathbb{Z}} a_j \varepsilon_{t-j}\right),$$

where $\varphi\colon \mathbb{R}\to\mathbb{R}$ is γ-Hölder continuous for some $\gamma\in(0,1]$, $(a_j)_{j\in\mathbb{Z}}$ is a sequence of real numbers such that $\sum_{j\in\mathbb{Z}} j^2 |a_j|^\gamma < \infty$ and $(\varepsilon_u)_{u\in\mathbb{Z}}$ is an i.i.d. sequence such that $\mathbb{E}\left[|\varepsilon_0|^{2m\gamma}\right] < \infty$. Then $(X_t)_{t\in\mathbb{Z}}$ satisfies condition (2.4).

Example 2.6. Assume X_t can be written as a function of a Gaussian linear process: Let $X_t = \varphi(Y_t)$ with

$$Y_t := \sum_{j\in\mathbb{Z}} a_j \varepsilon_{t-j},$$

where $\varphi : \mathbb{R}\to\mathbb{R}$, $(\varepsilon_u)_{u\in\mathbb{Z}}$ is an i.i.d. sequence, ε_0 has a standard normal distribution and $a_j\in\mathbb{R}$ for all $j\in\mathbb{Z}$ with $\sum_{j\in\mathbb{Z}}|a_j| < \infty$ as well as $\sum_{j\in\mathbb{Z}} a_j^2 = 1$. Such processes were considered, e.g., in Nualart (2009). Given that $\varphi(Y_0)\in\mathbb{L}^2$ and $\mathbb{E}[\varphi(Y_0)] = 0$, the following expansion holds:

$$\varphi(Y_t) = \sum_{q=1}^{\infty} c_q(\varphi) H_q(Y_t),$$

where the q-th Hermite polynomial is defined by

$$H_q(x) := (-1)^q \exp\left(\frac{x^2}{2}\right)\frac{d^q}{dx^q}\exp\left(-\frac{x^2}{2}\right)$$

and

$$c_q(\varphi) := \frac{1}{q!}\mathbb{E}\left[\varphi(Y_0) H_q(Y_0)\right].$$

Since $\varphi(Y_0)\in\mathbb{L}^2$, the series $\sum_{q=1}^{\infty} q!\,(c_q(\varphi))^2$ converges. Then condition (2.4) is met if $\sum_{j\in\mathbb{Z}} j^2|a_j| < \infty$ and

$$\sum_{k=1}^{m}\sum_{q=1}^{\infty}\sqrt{q\cdot q!}\left|c_q\left(\varphi^k - \mathbb{E}\left[\varphi^k(Y_0)\right]\right)\right| < \infty.$$

Example 2.7. Let $(X_t)_{t\in\mathbb{Z}}$ be a Volterra process, i.e. let

$$X_t := \sum_{j,j'\in\mathbb{Z}, j\neq j'} a_{j,j'}\varepsilon_{t-j}\varepsilon_{t-j'},$$

where $(\varepsilon_u)_{u\in\mathbb{Z}}$ is i.i.d. centred, $\mathbb{E}\left[\varepsilon_0^2\right] < \infty$ and $\sum_{j,j'\in\mathbb{Z},j\neq j'} a_{j,j'}^2 < \infty$. Such processes and extensions to Volterra series are considered in Rugh (1981) and Priestley (1988). If $\mathbb{E}\left[\varepsilon_0^{2m}\right] < \infty$ as well as

$$\sum_{j\in\mathbb{Z}} j^2 \sqrt{\sum_{j'\in\mathbb{Z}, j'\neq j}\left(a_{j,j'}^2 + a_{j',j}^2\right)} < \infty,$$

then $(X_t)_{t\in\mathbb{Z}}$ satisfies condition (2.4).

It would be more natural to centre U_n in (2.6) by $\sum_{1\leq j\neq k\leq b_n}\mathbb{E}\left[h\left(W_{n,j},W_{n,k}\right)\right]$ rather than by the truncated version $\sum_{1\leq j\neq k\leq b_n}\mathbb{E}\left[h\left(W_{n,j}^{(\eta)},W_{n,k}^{(\eta)}\right)\right]$. However, the conditions of Theorem 2.3 do not guarantee that $\mathbb{E}\left[\left|h\left(W_{n,j},W_{n,k}\right)\right|\right]$ exists, as the following example shows.

Example 2.8. Consider the case $m = 2$, $g(x_1, x_2) = \log x_2 \mathbf{1}_{x_2>0}$ and $h(x, y) = |x - y|$ with i.i.d. observations $(X_t)_{t\in\mathbb{Z}}$. Since $W_{n,1}$ and $W_{n,2}$ are consequently likewise independent and identically distributed, finiteness of $\mathbb{E}[|W_{n,1} - W_{n,2}|]$ is equivalent to the finiteness of $\mathbb{E}[|W_{n,1}|]$. Now, it suffices to find an i.i.d. sequence $(X_t)_{t\in\mathbb{Z}}$ such that $\mathbb{E}\left[X_1^4\right] < \infty$ and

$$\mathbb{E}\left[\left|\log\left(\frac{1}{\ell_n}\sum_{t=1}^{\ell_n} X_t^2\right)\right|\right] = \infty.$$

By choosing the distribution of X_1 as

$$\mathbb{P}\left(X_1^2 = \exp\left(-\exp\left(\exp\left(k\right)\right)\right)\right) = 2^{-k}$$

for $k \geq 1$, it follows

$$\begin{aligned}\left|\log\left(\frac{1}{\ell_n}\sum_{t=1}^{\ell_n} X_t^2\right)\right| &\geq \sum_{k\geq 1}\left|\log\left(\frac{1}{\ell_n}\sum_{t=1}^{\ell_n}\exp\left(-\exp\left(\exp\left(k\right)\right)\right)\right)\right|\mathbf{1}_{\bigcap_{t=1}^{\ell_n}\left\{X_t^2=\exp(-\exp(\exp(k)))\right\}}\\ &= \sum_{k\geq 1}\exp\left(\exp k\right)\mathbf{1}_{\bigcap_{t=1}^{\ell_n}\left\{X_t^2=\exp(-\exp(\exp(k)))\right\}}.\end{aligned}$$

Taking expectations and using independence leads to

$$\mathbb{E}\left[\left|\log\left(\frac{1}{\ell_n}\sum_{t=1}^{\ell_n} X_t^2\right)\right|\right] \geq \sum_{k\geq 1}\exp\left(\exp(k)\right)2^{-k\ell_n} = \infty.$$

By imposing additional assumptions on the function g, the dependence coefficients as well as the sequences $(b_n)_{n\geq 1}$ and $(\ell_n)_{n\geq 1}$, we are able to replace the centring term in (2.6) computed from the $W_{n,j}^{(\eta)}$'s by an expression that does not require truncation.

Proposition 2.9. *Assume that the assumptions of Theorem 2.3 hold. If additionally all the second order partial derivatives of g at v_0 exist and if there exists a $\kappa \in (0, 1)$ such that $b_n/\ell_n^{1-\kappa} \to 0$ and $\sum_{i\in\mathbb{Z}}\sum_{k=1}^{m}|i|^{\frac{1}{2}+\frac{1}{2\kappa}}\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right) < \infty$, then the following convergence in distribution holds*

$$\sqrt{b_n}\left(U_n - \mathbb{E}\left[h\left(Z_n, Z_n'\right)\right]\right) \to N\left(0, 4\gamma^2\right),$$

where

$$Z_n := \frac{1}{\sqrt{\ell_n}}\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)\sum_{t=1}^{\ell_n}\left(X_t^k - \mathbb{E}\left[X_1^k\right]\right)$$

and Z_n' is an independent copy of Z_n.

Remark 2.10. The above proposition introduces a centring term which is easier to handle than the original one in Theorem 2.3. However, if we want to use U_n as a test statistic for testing for a constant value of the parameter $g\left(\mathbb{E}\left[X_t\right], \mathbb{E}\left[X_t^2\right], \ldots, \mathbb{E}\left[X_t^m\right]\right)$, we need to be able to explicitly calculate the centring term. This is achieved by the following corollary, where we show that, for the important example when $h(x, y) = |x - y|$, we can replace $\mathbb{E}\left[|Z_n - Z_n'|\right]$ by $\sigma\,\mathbb{E}\left[|N - N'|\right] = 2\,\sigma/\sqrt{\pi}$. The remaining parameter σ can be estimated by standard procedures for estimating long-run variances.

Corollary 2.11. *Suppose that the time series $(X_t)_{t\in\mathbb{Z}}$ can be written as a one-sided Bernoulli shift, that is, $X_t = f\left(\left(\varepsilon_{t-u}\right)_{u\geq 0}\right)$. We assume moreover that there exists a $2 < p \leq 3$ such that*

(1) $\mathbb{E}\left[|X_t|^{p\cdot m}\right] < \infty$ *and*

$$\sum_{i\geq 0}\sum_{k=1}^{m}\left(i^2\delta_{i,p}\left((X_t^k)_{t\in\mathbb{Z}}\right) + i^{5/2}\delta_{i,2}\left((X_t^k)_{t\in\mathbb{Z}}\right)\right) < \infty, \tag{2.7}$$

where $\delta_{i,p}((X_t)_{t\in\mathbb{Z}}) := \left\| X_0 - X_0^{*,i} \right\|_p$, *and*

(2) $\sqrt{b_n}\ell_n^{1-p/2}(\log \ell_n)^{p/2} + \sqrt{b_n/\ell_n} \to 0.$

Denote

$$U_n := \frac{1}{b_n(b_n-1)} \sum_{1\le j\ne k\le b_n} |W_{n,j} - W_{n,k}|,$$

where $W_{n,j}$ *is defined as in* (1.2). *Then the following convergence in distribution holds*

$$\sqrt{b_n}\left(U_n - \frac{2\sigma}{\sqrt{\pi}}\right) \longrightarrow N(0, 4\gamma^2),$$

where σ *and* γ^2 *are defined as in Theorem 2.3.*

3. Proofs

3.1. *Proof of Theorem 2.1.* We use the Hoeffding decomposition of the kernel function h and define

$$\theta_n := \mathbb{E}\left[h\left(W_{n,1}, W_{n,2}\right)\right],$$
$$h_{1,n}(x) := \mathbb{E}\left[h(x, W_{n,2})\right] - \theta_n,$$
$$h_{2,n}(x,y) := h(x,y) - h_{1,n}(x) - h_{1,n}(y) - \theta_n.$$

At the level of the U-statistic, we then obtain

$$\begin{aligned} U(n) - \mathbb{E}\left[U(n)\right] &= \frac{1}{b_n(b_n-1)} \sum_{1\le j\ne k\le b_n} h\left(W_{n,j}, W_{n,k}\right) - \theta_n \\ &= \frac{2}{b_n}\sum_{j=1}^{b_n} h_{1,n}\left(W_{n,j}\right) + \frac{1}{b_n(b_n-1)} \sum_{1\le j\ne k\le b_n} h_{2,n}\left(W_{n,j}, W_{n,k}\right). \end{aligned}$$

In the following, we will prove the convergence in distribution

$$\frac{2}{\sqrt{b_n}\gamma_n}\sum_{j=1}^{b_n} h_{1,n}\left(W_{n,j}\right) \to N(0,4), \tag{3.1}$$

and the convergence in probability

$$\frac{1}{b_n^{3/2}\gamma_n} \sum_{1\le j\ne k\le b_n} h_{2,n}\left(W_{n,j}, W_{n,k}\right) \to 0, \tag{3.2}$$

where $\gamma_n^2 = \mathrm{Cov}\left(h\left(W_{n,1}, W_{n,2}\right), h\left(W_{n,2}, W_{n,3}\right)\right)$. The assertion then follows by an application of Slutzky's Lemma.

Starting with (3.1), we will apply Lindeberg's central limit theorem to the triangular array $Y_{n,j} := h_{1,n}\left(W_{n,j}\right) / \left(\gamma_n\sqrt{b_n}\right)$. Note that by construction, the $Y_{n,j}$'s are identically distributed with $\mathbb{E}\left[Y_{n,1}\right] = 0$. Moreover, it holds $\mathrm{Var}\left(Y_{n,1}\right) = 1/b_n$ since by independence

$$\begin{aligned} \mathrm{Var}\left(h_{1,n}\left(W_{n,1}\right)\right) &= \mathrm{Var}\left(\mathbb{E}\left[h\left(W_{n,1}, W_{n,2}\right) \mid W_{n,1}\right]\right) \\ &= \mathbb{E}\left[\mathbb{E}\left[h\left(W_{n,1}, W_{n,2}\right) \mid W_{n,1}\right]^2\right] - \left(\mathbb{E}\left[\mathbb{E}\left[h\left(W_{n,1}, W_{n,2}\right) \mid W_{n,1}\right]\right]\right)^2 \\ &= \int_{\mathbb{R}} \mathbb{E}\left[h\left(x, W_{n,2}\right) h\left(x, W_{n,3}\right)\right] d\mathbb{P}_{W_{n,1}}(x) - \left(\mathbb{E}\left[h\left(W_{n,1}, W_{n,2}\right)\right]\right)^2 \\ &= \mathbb{E}\left[h\left(W_{n,1}, W_{n,2}\right) \cdot h\left(W_{n,1}, W_{n,3}\right)\right] - \left(\mathbb{E}\left[h\left(W_{n,1}, W_{n,2}\right)\right]\right)^2 \\ &= \mathrm{Cov}\left(h\left(W_{n,1}, W_{n,2}\right), h\left(W_{n,2}, W_{n,3}\right)\right) = \gamma_n^2. \end{aligned}$$

It thus remains to verify the Lindeberg condition, that is, to show that for all $\varepsilon > 0$,

$$\sum_{j=1}^{b_n} \mathbb{E}\left[Y_{n,j}^2 \mathbf{1}_{\{|Y_{n,j}|>\varepsilon\}}\right] \to 0.$$

Since the random variables $Y_{n,j}, 1 \leq j \leq b_n$, are identically distributed, Lindeberg's condition reduces to

$$\sum_{j=1}^{b_n} \mathbb{E}\left[Y_{n,j}^2 \mathbf{1}_{\{|Y_{n,j}|>\varepsilon\}}\right] = b_n \mathbb{E}\left[Y_{n,1}^2 \mathbf{1}_{\{|Y_{n,1}|>\varepsilon\}}\right] = \mathbb{E}\left[\frac{h_{1,n}^2\left(W_{n,1}\right)}{\gamma_n^2} \mathbf{1}_{\left\{\frac{\left|h_{1,n}\left(W_{n,1}\right)\right|}{\gamma_n} > \varepsilon\sqrt{b_n}\right\}}\right] \to 0.$$

Observe that by property (2.1) of the kernel function h, we have $|h_{1,n}(x)| \leq 3C\mathbb{E}\left[|W_{n,1}|\right] + C|x|$ and hence $h_{1,n}^2(x) \leq 18C^2\mathbb{E}\left[W_{n,1}^2\right] + 2C^2x^2$, from which it follows that

$$\frac{h_{1,n}^2\left(W_{n,1}\right)}{\gamma_n^2} \leq 18C^2 + 2C^2\frac{W_{n,1}^2}{\gamma_n^2}.$$

Consequently, the uniform integrability of the sequence $\left(h_{1,n}^2\left(W_{n,1}\right)/\gamma_n^2\right)_{n\geq 1}$ follows from that of the sequence $\left(W_{n,1}^2/\gamma_n^2\right)_{n\geq 1}$, and thus Lindeberg's condition is met.

It remains to verify (3.2). Since $\mathbb{E}\left[h_{2,n}\left(W_{n,j}, W_{n,k}\right) h_{2,n}\left(W_{n,j'}, W_{n,k'}\right)\right] = 0$ if $\{j,k\} \neq \{j',k'\}$, we obtain

$$\mathbb{E}\left[\left(\frac{1}{b_n^{3/2}\gamma_n}\sum_{1\leq j\neq k\leq b_n} h_{2,n}\left(W_{n,j}, W_{n,k}\right)\right)^2\right] = \frac{2\left(b_n-1\right)}{\gamma_n^2 b_n^2}\mathbb{E}\left[h_{2,n}^2\left(W_{n,1}, W_{n,2}\right)\right]. \tag{3.3}$$

By the properties of the kernel function h and the definition of $h_{2,n}$, there exists a constant C independent of n, x and y such that $|h_{2,n}(x,y)| \leq C\left(1 + |x| + |y| + \mathbb{E}\left[|W_{n,1}|\right]\right)$ and hence $\mathbb{E}\left[h_{2,n}^2\left(W_{n,1}, W_{n,2}\right)\right] \leq C'\mathbb{E}\left[W_{n,1}^2\right]$ for a constant C' independent of n. Combining this with (3.3), we get that

$$\mathbb{E}\left[\left(\frac{1}{b_n^{3/2}\gamma_n}\sum_{1\leq j\neq k\leq b_n} h_{2,n}\left(W_{n,j}, W_{n,k}\right)\right)^2\right] \leq \frac{2C}{b_n}\sup_{n\geq 1}\mathbb{E}\left[\frac{W_{n,1}^2}{\gamma_n^2}\right]$$

and the uniform integrability of $\left(W_{n,1}^2/\gamma_n^2\right)_{n\geq 1}$ guarantees the finiteness of the above supremum. This concludes the proof of (3.2) and that of Theorem 2.1.

3.2. *Sketch of proof for Theorem 2.3.* This section outlines the proof ideas for Theorem 2.3, while all details can be found in the next section. First, we reduce the problem to the case where in U_n, the term $W_{n,j}$ is replaced by $W_{n,j}^{(\eta)} = \sqrt{\ell_n}\,(g\cdot\eta)\left(\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}} X_t^k\right)_{k=1}^m\right)$. We thus define

$$U_n^{(\eta)} := \frac{1}{b_n\left(b_n-1\right)}\sum_{1\leq j\neq k\leq b_n} h\left(W_{n,j}^{(\eta)}, W_{n,k}^{(\eta)}\right).$$

The next lemma shows that we can replace U_n in the central limit theorem by $U_n^{(\eta)}$.

Lemma 3.1. *Let the assumptions of Theorem 2.3 hold. Then* $\mathbb{P}\left(U_n \neq U_n^{(\eta)}\right) \to 0$. *In particular,* $\left(\sqrt{b_n}\left(U_n - U_n^{(\eta)}\right)\right)_{n\geq 1}$ *converges in probability to zero.*

It thus suffices to prove the convergence in distribution

$$\sqrt{b_n}\left(U_n^{(\eta)} - \frac{1}{b_n\left(b_n-1\right)}\sum_{1\leq j\neq k\leq b_n}\mathbb{E}\left[h\left(W_{n,j}^{(\eta)}, W_{n,k}^{(\eta)}\right)\right]\right) \to N\left(0, 4\gamma^2\right). \tag{3.4}$$

To do so, we use a second approximation step and replace the $W_{n,j}^{(\eta)}$'s by

$$W_{n,j}^{(M)} := \sqrt{\ell_n}\,(g\cdot\eta)\left(\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}}\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{k=1}^{m}\right),$$

where $\mathcal{F}_M^N := \sigma\left(\varepsilon_u, M\le u\le N\right)$ for $M,N\in\mathbb{Z}$ with $M\le N$. The random variables X_t are thus replaced by random variables depending only on those ε_{t-u} with $|t-u|\le M$. Note that this way, the entries of the U-statistic become almost independent (up to some small overlap in consecutive blocks). We define

$$U_n^{(M)} := \frac{1}{b_n\left(b_n-1\right)}\sum_{1\le j\ne k\le b_n} h\left(W_{n,j}^{(M)}, W_{n,k}^{(M)}\right).$$

We can now decompose the expression on the left hand side of (3.4) for each fixed $M\ge 1$ via

$$\sqrt{b_n}(U_n^{(M)} - \mathbb{E}\left[U_n^{(M)}\right]) + R_{n,M}, \tag{3.5}$$

where the remainder term is given by the telescoping sum

$$R_{n,M} := \frac{1}{\sqrt{b_n}\left(b_n-1\right)}\sum_{N\ge M}\sum_{1\le j\ne k\le b_n}\Big(h\left(W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)}\right) - \mathbb{E}\left[h\left(W_{n,j}^{(N+1)}, W_{n,k}^{(N+1)}\right)\right]$$
$$- \left(h\left(W_{n,j}^{(N)}, W_{n,k}^{(N)}\right) - \mathbb{E}\left[h\left(W_{n,j}^{(N)}, W_{n,k}^{(N)}\right)\right]\right)\Big).$$

The following three lemmas show that the first term in (3.5) converges to the desired normal distribution, while the continuity of $g\cdot\eta$ will guarantee that the remainder term $R_{n,M}$ becomes asymptotically negligible.

Lemma 3.2. *Let the assumptions of Theorem 2.3 hold. Then, there exists an $M_0\in\mathbb{Z}_+$ such that for all fixed $M\ge M_0$, the sequence $\left(\sqrt{b_n}(U_n^{(M)}-\mathbb{E}[U_n^{(M)}])\right)_{n\ge 1}$ converges in distribution to a centred normally distributed random variable with variance*

$$4\gamma_M^2 := 4\,\mathrm{Cov}\left(h\left(\sigma_M N, \sigma_M N'\right), h\left(\sigma_M N, \sigma_M N''\right)\right),$$

where N, N' and N'' are three independent standard normal random variables and

$$\sigma_M^2 := \sum_{t=-M-1}^{M+1}\mathrm{Cov}\left(\mathbb{E}\left[\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)X_0^k\middle|\mathcal{F}_{-M}^{M}\right], \mathbb{E}\left[\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)X_t^k\middle|\mathcal{F}_{t-M}^{t+M}\right]\right).$$

The central ingredient in the proof of Lemma 3.2 is Theorem 2.1. To meet its conditions, we approximate $U_n^{(M)}$ by yet another U-statistic, which has independent entries (details are given in the next section).

Lemma 3.3. *Let the assumptions of Theorem 2.3 hold. Then the sequence $\left(4\gamma_M^2\right)_{M\ge 1}$ converges to $4\gamma^2 = 4\,\mathrm{Cov}\left(h\left(\sigma N,\sigma N'\right), h\left(\sigma N',\sigma N''\right)\right)$.*

Lemma 3.4. *Let the assumptions of Theorem 2.3 hold. Then, for each $\varepsilon>0$, it holds*

$$\lim_{M\to\infty}\limsup_{n\to\infty}\mathbb{P}(|R_{n,M}|>\varepsilon) = 0.$$

In the final step, we apply Theorem 4.2 of Billingsley (1968). This theorem states that for stochastic processes $(Y_{m,n})_{m,n\ge 1}$, $(Y_n')_{n\ge 1}$, $(Z_m)_{m\ge 1}$, and a random variable Z, satisfying

$$Y_{m,n} \longrightarrow Z_m \text{ in distribution, as } n\to\infty, \text{ for each } m,$$
$$Z_m \longrightarrow Z \text{ in distribution, as } m\to\infty,$$
$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\left|Y_n' - Y_{m,n}\right| > \varepsilon\right) = 0, \text{ for all } \varepsilon>0,$$

we may conclude that $Y_n' \to Z$ in distribution. A combination of Lemmas 3.2, 3.3 and 3.4 thus yields the desired convergence in (3.4) and thus finishes the proof.

3.3. *Proof details.*

Proof of Lemma 3.1: We start by noting the following inclusions

$$\left\{U_n \neq U_n^{(\eta)}\right\} \subset \bigcup_{j=1}^{b_n} \left\{W_{n,j} \neq W_{n,j}^{(\eta)}\right\} \subset \bigcup_{j=1}^{b_n} \bigcup_{k=1}^{m} \left\{ \left| \left(\frac{1}{\ell_n} \sum_{t \in B_{n,j}} X_t^k \right) - \mathbb{E}\left[X_1^k\right] \right| > 2a/m \right\}.$$

Hence, using the fact that for each k, the random variables $\left(\frac{1}{\ell_n}\sum_{t\in B_{n,j}} X_t^k - \mathbb{E}\left[X_1^k\right]\right)$, $j = 1, \ldots, b_n$, have the same distribution, and by an application of Chebychev's inequality, we obtain

$$\mathbb{P}(U_n \neq U_n^{(\eta)}) \leq m^2 \frac{b_n}{4a^2 \ell_n} \sum_{k=1}^{m} \mathrm{Var}\left(\frac{1}{\sqrt{\ell_n}} \sum_{t=1}^{\ell_n} X_t^k \right).$$

By Lemma A.1, it holds $\mathrm{Var}\left(\frac{1}{\sqrt{\ell_n}} \sum_{t=1}^{\ell_n} X_t^k\right) \leq \left(\sum_{i\in\mathbb{Z}} \delta_i((X_t^k)_{t\in\mathbb{Z}})\right)^2$ and thus, we have $\mathbb{P}(U_n \neq U_n^{(\eta)}) \leq C b_n/\ell_n$, which converges to zero by assumption. □

Proof of Lemma 3.2: Let $M \geq 1$ be fixed. We intend to approximate $U_n^{(M)}$ once more to obtain a U-statistic $\widetilde{U}_n^{(M)}$ with independent entries. Thus, we define

$$\widetilde{W}_{n,j}^{(M)} := \sqrt{\ell_n}\,(g\cdot\eta)\left(\left(\frac{1}{\ell_n} \sum_{t=(j-1)\ell_n+M+1}^{j\ell_n - M - 1} \mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right] \right)_{k=1}^{m} \right)$$

for $j = 1, \ldots, b_n$, assuming that n is large enough such that $2M \leq \ell_n$. Due to the shortening of the blocks $B_{n,j}$ by M observations on each side, the $\widetilde{W}_{n,j}^{(M)}$'s are independent. Next, we define

$$\widetilde{U}_n^{(M)} := \frac{1}{b_n\,(b_n-1)} \sum_{1\leq j\neq k\leq b_n} h\left(\widetilde{W}_{n,j}^{(M)}, \widetilde{W}_{n,k}^{(M)}\right).$$

Since the function $g\cdot\eta$ is Lipschitz-continuous, it holds

$$\left|W_{n,j}^{(M)} - \widetilde{W}_{n,j}^{(M)}\right| \leq C \sum_{k=1}^{m} \frac{1}{\sqrt{\ell_n}} \sum_{t=(j-1)\ell_n+1}^{(j-1)\ell_n+M} \mathbb{E}\left[\left|X_t^k\right| \middle| \mathcal{F}_{t-M}^{t+M}\right] + C \sum_{k=1}^{m} \frac{1}{\sqrt{\ell_n}} \sum_{t=j\ell_n-M}^{j\ell_n} \mathbb{E}\left[\left|X_t^k\right| \middle| \mathcal{F}_{t-M}^{t+M}\right].$$

Taking expectations, we obtain

$$\mathbb{E}\left[\left|W_{n,j}^{(M)} - \widetilde{W}_{n,j}^{(M)}\right|\right] \leq C\frac{M}{\sqrt{\ell_n}}$$

since $\mathbb{E}\left[X_1^m\right] < \infty$ by assumption. Thus, for the U-statistic, we have

$$\begin{aligned}
&\sqrt{b_n}\mathbb{E}\left[\left|\left(U_n^{(M)} - \mathbb{E}\left[U_n^{(M)}\right]\right) - \left(\widetilde{U}_n^{(M)} - \mathbb{E}\left[\widetilde{U}_n^{(M)}\right]\right)\right|\right] \\
\leq& 2\frac{\sqrt{b_n}}{b_n\,(b_n-1)} \sum_{1\leq j\neq k\leq b_n} \mathbb{E}\left[\left|h\left(W_{n,j}^{(M)}, W_{n,k}^{(M)}\right) - h\left(\widetilde{W}_{n,j}^{(M)}, \widetilde{W}_{n,k}^{(M)}\right)\right|\right] \\
\leq& C\sqrt{b_n}\mathbb{E}\left[\left|W_{n,1}^{(M)} - \widetilde{W}_{n,1}^{(M)}\right|\right] \leq CM\frac{\sqrt{b_n}}{\sqrt{\ell_n}},
\end{aligned}$$

where the last but one inequality follows by the Lipschitz-continuity of the kernel h and the stationarity of the $W_{n,j}^{(M)}$'s and of the $\widetilde{W}_{n,j}^{(M)}$'s.

In the following, it hence suffices to prove that $\sqrt{b_n}\left(\widetilde{U}_n^{(M)} - \mathbb{E}\left[\widetilde{U}_n^{(M)}\right]\right)$ converges in distribution to a centred normal random variable N_M with variance $4\gamma_M^2$. We first show that

$$\gamma_{M,n}^2 := \operatorname{Cov}\left(h\left(\widetilde{W}_{n,1}^{(M)}, \widetilde{W}_{n,2}^{(M)}\right), h\left(\widetilde{W}_{n,1}^{(M)}, \widetilde{W}_{n,3}^{(M)}\right)\right) \to \gamma_M^2$$

and afterwards the convergence in distribution

$$\frac{\sqrt{b_n}}{\gamma_{M,n}}\left(\widetilde{U}_n^{(M)} - \mathbb{E}\left[\widetilde{U}_n^{(M)}\right]\right) \to N(0,4),$$

which combined yield the assertion.

We start by verifying $\gamma_{M,n}^2 \to \gamma_M^2$ by means of Lemma A.6. To apply Lemma A.6, we have to check that the sequence $\left(\left(\widetilde{W}_{n,1}^{(M)}\right)^2\right)_{n\geq 1}$ is uniformly integrable. By the assumptions on g and η, the gradient of $g \cdot \eta$ is uniformly bounded over $\mathbb{R}^m$, such that

$$\begin{aligned}\left(\widetilde{W}_{n,1}^{(M)}\right)^2 =& \ell_n \left(g^2 \cdot \eta^2\right)\left(v_0 + \left(\frac{1}{\ell_n}\sum_{t=M+1}^{\ell_n - M-1} \mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{k=1}^m - v_0\right)\\ \leq& \ell_n C^2 \sum_{k=1}^m \left(\frac{1}{\ell_n}\sum_{t=M+1}^{\ell_n-M-1} \mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right] - \mathbb{E}\left[X_1^k\right]\right)^2\end{aligned}$$

and uniform integrability follows from Lemma A.2. We additionally have to show that $\widetilde{W}_{n,1}^{(M)} \to N\left(0, \sigma_M^2\right)$ in distribution to apply Lemma A.6. To do so, we will use the differentiability of $g \cdot \eta$ at v_0 and the fact that $(g\cdot\eta)(v_0) = 0$ in order to write

$$(g\cdot\eta)(v_0 + z) = \sum_{k=1}^m z_k \frac{\partial g}{\partial x_k}(v_0) + \varepsilon(z),$$

where $\varepsilon(z)/\|z\|_2 \to 0$ as $\|z\|_2 \to 0$. Setting $z = \left(\frac{1}{\ell_n}\sum_{t=M+1}^{\ell_n-M-1}\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right] - \mathbb{E}\left[X_1^k\right]\right)\right)_{k=1}^m$, we thus obtain

$$\begin{aligned}\widetilde{W}_{n,1}^{(M)} =& \frac{1}{\sqrt{\ell_n}}\sum_{t=M+1}^{\ell_n-M-1}\sum_{k=1}^m \frac{\partial g}{\partial x_k}(v_0)\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right] - \mathbb{E}\left[X_1^k\right]\right)\\ &+ \sqrt{\ell_n}\varepsilon\left(\left(\frac{1}{\ell_n}\sum_{t=M+1}^{\ell_n-M-1}\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right] - \mathbb{E}\left[X_1^k\right]\right)\right)_{k=1}^m\right).\end{aligned}$$

By the central limit theorem (see Lemma A.3 in the appendix), the first of the above terms converges to a centred normal distribution with variance σ_M^2. Moreover, due to the properties of the function ε and the strong law of large numbers, the second of the above terms converges in probability to 0.

Indeed,

$$\sqrt{\ell_n}\varepsilon\left(\left(\frac{1}{\ell_n}\sum_{t=M+1}^{\ell_n-M-1}\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right]-\mathbb{E}\left[X_1^k\right]\right)\right)_{k=1}^{m}\right)$$
$$=\frac{\varepsilon\left(\left(\frac{1}{\ell_n}\sum_{t=M+1}^{\ell_n-M-1}\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right]-\mathbb{E}\left[X_1^k\right]\right)\right)_{k=1}^{m}\right)}{\sqrt{\sum_{k=1}^{m}\frac{1}{\ell_n}\left(\sum_{t=M+1}^{\ell_n-M-1}\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right]-\mathbb{E}\left[X_1^k\right]\right)\right)^2}}$$
$$\cdot\sqrt{\sum_{k=1}^{m}\frac{1}{\ell_n}\left(\sum_{t=M+1}^{\ell_n-M-1}\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right]-\mathbb{E}\left[X_1^k\right]\right)\right)^2},$$

which is a product of the form A_nB_n, where $(A_n)_{n\geq 1}$ converges in probability to 0 and $(B_n)_{n\geq 1}$ is tight. We hence have $\widetilde{W}_{n,1}^{(M)} \to N\left(0,\sigma_M^2\right)$ in distribution and Lemma A.6 implies $\gamma_{M,n}^2 \to \gamma_M^2$.

Turning towards showing $\sqrt{b_n}\big(\widetilde{U}_n^{(M)}-\mathbb{E}[\widetilde{U}_n^{(M)}]\big)/\gamma_{M,n} \to N(0,4)$ in distribution, we have to check the assumptions of Theorem 2.1 and therefore have to prove the uniform integrability of $((\widetilde{W}_{n,1}^{(M)}/\gamma_{M,n})^2)_{n\geq 1}$. The uniform integrability of $\big((\widetilde{W}_{n,1}^{(M)})^2\big)_{n\geq 1}$ has already been shown above. Moreover, since we will show in the proof of Lemma 3.3 below that $\gamma_M^2 \to \gamma^2 > 0$, there exists an M_0 such that for all $M \geq M_0$, it holds $\gamma_M^2 > 0$. Since for fixed $M \geq M_0$, we have shown above that $\gamma_{M,n}^2 \to \gamma_M^2$, there exists an $n_{0,M}$ such that $\gamma_{M,n}^2 \geq \gamma_M^2/2 > 0$ for all $n \geq n_{0,M}$. Hence, the sequence $(\gamma_{M,n}^{-2})_{n\geq n_{0,M}}$ is bounded and $((\widetilde{W}_{n,1}^{(M)}/\gamma_{M,n})^2)_{n\geq n_{0,M}}$ is uniformly integrable, such that Theorem 2.1 yields the desired convergence. □

Proof of Lemma 3.3: It suffices to prove that $\gamma_M^2 \to \gamma^2$. Since h is Lipschitz-continuous, the mapping $x \mapsto \operatorname{Cov}\left(h\left(xN, xN'\right), h\left(xN, xN''\right)\right)$ is continuous, where N, N' and N'' denote three independent standard normal random variables. Therefore, we only need to check that $\sigma_M^2 \to \sigma^2$, where

$$\sigma_M^2=\sum_{t=-2M-1}^{2M+1}\operatorname{Cov}\left(\mathbb{E}\left[\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)X_0^k\middle|\mathcal{F}_{-M}^{M}\right],\mathbb{E}\left[\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)X_t^k\middle|\mathcal{F}_{t-M}^{t+M}\right]\right)$$

and

$$\sigma^2=\sum_{t\in\mathbb{Z}}\operatorname{Cov}\left(\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)X_0^k,\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)X_t^k\right).$$

By construction, $\sigma_M^2=\sum_{t\in\mathbb{Z}}\operatorname{Cov}\left(\mathbb{E}\left[\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)X_0^k\middle|\mathcal{F}_{-M}^{M}\right],\mathbb{E}\left[\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)X_t^k\middle|\mathcal{F}_{t-M}^{t+M}\right]\right)$ and a small calculation shows

$$\left|\sigma_M^2-\sigma^2\right|$$
$$=\left|\sum_{t\in\mathbb{Z}}\operatorname{Cov}\left(\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)\left(X_0^k-\mathbb{E}\left[X_0^k \mid \mathcal{F}_{-M}^{M}\right]\right),\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)\left(X_t^k-\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-M}^{t+M}\right]\right)\right)\right|.$$

Note that

$$\sum_{i\in\mathbb{Z}}\delta_i\left(\left(\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)\left(X_t^k-\mathbb{E}\left[X_t^k\middle|\mathcal{F}_{t-M}^{t+M}\right]\right)\right)_{t\in\mathbb{Z}}\right)$$
$$\leq\sum_{i\in\mathbb{Z}}\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)\delta_i\left(\left(X_t^k-\mathbb{E}\left[X_t^k\middle|\mathcal{F}_{t-M}^{t+M}\right]\right)_{t\in\mathbb{Z}}\right)\leq 2\sum_{i\in\mathbb{Z}}\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right),$$

which is finite by assumption (2.4). We can thus apply Lemma A.2 from the appendix to the above sum of covariances and obtain

$$\left|\sigma_M^2-\sigma^2\right| = \lim_{n\to\infty}\frac{1}{n}\cdot\mathbb{E}\left[\left(\sum_{t=1}^n\sum_{k=1}^m \frac{\partial g}{\partial x_k}(v_0)\left(X_t^k-\mathbb{E}\left[X_t^k\middle|\mathcal{F}_{t-M}^{t+M}\right]\right)\right)^2\right]$$
$$\le\left(\sum_{i\in\mathbb{Z}}\delta_i\left(\left(\sum_{k=1}^m \frac{\partial g}{\partial x_k}(v_0)\left(X_t^k-\mathbb{E}\left[X_t^k\middle|\mathcal{F}_{t-M}^{t+M}\right]\right)\right)_{t\in\mathbb{Z}}\right)\right)^2,$$

where the last inequality follows by Lemma A.1. We have already shown above that the last expression is finite. Moreover, for each fixed i, it holds by the martingale convergence theorem that

$$\lim_{M\to\infty}\delta_i\left(\left(\sum_{k=1}^m \frac{\partial g}{\partial x_k}(v_0)\left(X_t^k-\mathbb{E}\left[X_t^k\middle|\mathcal{F}_{t-M}^{t+M}\right]\right)\right)_{t\in\mathbb{Z}}\right)=0$$

and by dominated convergence, it thus follows $\left|\sigma_M^2-\sigma^2\right|\to 0$. □

Proof of Lemma 3.4: Due to Chebychev's inequality, it suffices to prove

$$\lim_{M\to\infty}\limsup_{n\to\infty}\left\|R_{n,M}\right\|_2=0.$$

By the definition of $R_{n,M}$, the above equality holds if the sum $\sum_{N\ge1}a_N$ converges, where

$$a_N:=\sup_{n\ge1}\frac{1}{b_n^{3/2}}\left\|\sum_{1\le j\ne k\le b_n}\left(h\left(W_{n,j}^{(N+1)},W_{n,k}^{(N+1)}\right)-\mathbb{E}\left[h\left(W_{n,j}^{(N+1)},W_{n,k}^{(N+1)}\right)\right]\right.\right.$$
$$\left.\left.-\left(h\left(W_{n,j}^{(N)},W_{n,k}^{(N)}\right)-\mathbb{E}\left[h\left(W_{n,j}^{(N)},W_{n,k}^{(N)}\right)\right]\right)\right)\right\|_2.$$

Splitting the supremum up into those cases of n for which $\ell_n\ge2(N+1)$ and those for which $\ell_n<2(N+1)$, we have to prove the convergence of $\sum_{N\ge1}a_{N,1}$ and $\sum_{N\ge1}a_{N,2}$, where

$$a_{N,1}:=\sup_{n\ge1,2(N+1)\le\ell_n}\frac{1}{b_n^{3/2}}\left\|\sum_{1\le j\ne k\le b_n}\left(h\left(W_{n,j}^{(N+1)},W_{n,k}^{(N+1)}\right)-\mathbb{E}\left[h\left(W_{n,j}^{(N+1)},W_{n,k}^{(N+1)}\right)\right]\right.\right.$$
$$\left.\left.-\left(h\left(W_{n,j}^{(N)},W_{n,k}^{(N)}\right)-\mathbb{E}\left[h\left(W_{n,j}^{(N)},W_{n,k}^{(N)}\right)\right]\right)\right)\right\|_2,$$

$$a_{N,2}:=\sup_{n\ge1,2(N+1)>\ell_n}\frac{1}{b_n^{3/2}}\left\|\sum_{1\le j\ne k\le b_n}\left(h\left(W_{n,j}^{(N+1)},W_{n,k}^{(N+1)}\right)-\mathbb{E}\left[h\left(W_{n,j}^{(N+1)},W_{n,k}^{(N+1)}\right)\right]\right.\right.$$
$$\left.\left.-\left(h\left(W_{n,j}^{(N)},W_{n,k}^{(N)}\right)-\mathbb{E}\left[h\left(W_{n,j}^{(N)},W_{n,k}^{(N)}\right)\right]\right)\right)\right\|_2.$$

We bound $a_{N,1}$ by an application of Lemma A.5 from the appendix to obtain

$$a_{N,1}\le C\sup_{n\ge1,2(N+1)\le\ell_n}\left\|W_{n,1}^{(N+1)}-W_{n,1}^{(N)}\right\|_2.$$

In order to bound $a_{N,2}$, we notice that by assumption, $b_n<\ell_n\le2(N+1)\le CN$ for n large enough. Consequently, by the Lipschitz-continuity of h,

$$a_{N,2}\le2\sup_{n\ge1,2(N+1)>\ell_n}\sqrt{b_n}\left\|W_{n,1}^{(N+1)}-W_{n,1}^{(N)}\right\|_2\le2\sup_{n\ge1,2(N+1)>\ell_n}\sqrt{N}\left\|W_{n,1}^{(N+1)}-W_{n,1}^{(N)}\right\|_2.$$

Using the Lipschitz-continuity of $g \cdot \eta$ and Lemma A.1, we derive that

$$\left\|W_{n,1}^{(N+1)} - W_{n,1}^{(N)}\right\|_2 \leq C \sum_{k=1}^{m} \sum_{i \in \mathbb{Z}} \delta_i \left(\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-N-1}^{t+N+1}\right] - \mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-N}^{t+N}\right]\right)_{t \in \mathbb{Z}} \right).$$

The above summands are 0 for $|i| \geq N+2$, while for $|i| < N+2$, we get by similar arguments as in the proof of Lemma A.4 that

$$\delta_i \left(\left(\mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-N-1}^{t+N+1}\right] - \mathbb{E}\left[X_t^k \mid \mathcal{F}_{t-N}^{t+N}\right]\right)_{t \in \mathbb{Z}} \right) \leq \delta_{N+1}\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right) + \delta_{-N-1}\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right).$$

We thus obtain the bounds

$$a_{N,1} \leq CN \sum_{k=1}^{m} \left(\delta_{N+1}\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right) + \delta_{-N-1}\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)\right)$$

and

$$a_{N,2} \leq CN^{3/2} \sum_{k=1}^{m} \left(\delta_{N+1}\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right) + \delta_{-N-1}\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)\right).$$

The convergence of $\sum_{N\geq 1} a_{N,1}$ and $\sum_{N\geq 1} a_{N,2}$ then follows from assumption (2.4). □

Proof of Proposition 2.9: The proof is divided into two parts. First, we show that

$$\lim_{n\to\infty} \sqrt{b_n} \left(\frac{1}{b_n(b_n-1)} \sum_{1\leq j\neq k\leq b_n} \left(\mathbb{E}\left[h\left(W_{n,j}^{(\eta)}, W_{n,k}^{(\eta)}\right)\right] - \mathbb{E}\left[h\left(Z_{n,j}, Z_{n,k}\right)\right]\right)\right) = 0,$$

where

$$Z_{n,j} = \frac{1}{\sqrt{\ell_n}} \sum_{k=1}^{m} \frac{\partial g}{\partial x_k}(v_0) \sum_{t\in B_{n,j}} \left(X_t^k - \mathbb{E}\left[X_1^k\right]\right),$$

and afterwards that

$$\lim_{n\to\infty} \sqrt{b_n} \left(\frac{1}{b_n(b_n-1)} \sum_{1\leq j\neq k\leq b_n} \mathbb{E}\left[h\left(Z_{n,j}, Z_{n,k}\right)\right] - \mathbb{E}\left[h\left(Z_n, Z_n'\right)\right]\right) = 0.$$

The assertion then follows from Theorem 2.3.

Starting with the first part, due to the Lipschitz-continuity of h and stationarity, it suffices to show that

$$\lim_{n\to\infty} \sqrt{b_n}\mathbb{E}\left[\left|W_{n,1}^{(\eta)} - Z_{n,1}\right|\right] = 0.$$

By the assumptions imposed on g and η, there exists a constant C such that for each $z = (z_k)_{k=1}^m \in \mathbb{R}^m$, it holds

$$\left|(g\cdot\eta)(z) - \sum_{k=1}^{m} \frac{\partial g}{\partial x_k}(v_0)\left(z_k - \mathbb{E}\left[X_1^k\right]\right)\right| \leq C\left\|z - v_0\right\|_2.$$

With the choice $z = \left(\frac{1}{\ell_n}\sum_{t\in B_{n,1}} X_t^k\right)_{k=1}^m$, this yields

$$\sqrt{b_n}\mathbb{E}\left[\left|W_{n,j}^{(\eta)} - Z_{n,1}\right|\right] \leq C\sqrt{b_n\ell_n} \sum_{k=1}^{m} \mathbb{E}\left[\left(\frac{1}{\ell_n}\sum_{t\in B_{n,1}}\left(X_t^k - \mathbb{E}\left[X_1^k\right]\right)\right)^2\right].$$

Lemma A.1 from the appendix now implies that $\sqrt{b_n}\mathbb{E}\left[\left|W_{n,j}^{(\eta)} - Z_{n,1}\right|\right] \leq C\sqrt{b_n/\ell_n}$, which converges to 0 by assumption.

Turning towards the second part, we define

$$A_{n,j} := \frac{1}{\sqrt{\ell_n}} \sum_{k=1}^{m} \frac{\partial g}{\partial x_k}(v_0) \sum_{t \in B_{n,j}} \left(\mathbb{E}\left[X_t^k \mid \sigma(\varepsilon_u, |u-t| \leq \ell_n^\kappa)\right] - \mathbb{E}\left[X_1^k\right] \right)$$

and

$$A'_{n,j} := \frac{1}{\sqrt{\ell_n}} \sum_{k=1}^{m} \frac{\partial g}{\partial x_k}(v_0) \sum_{t \in B'_{n,j}} \left(\mathbb{E}\left[X_t^k \mid \sigma(\varepsilon_u, |u-t| \leq \ell_n^\kappa)\right] - \mathbb{E}\left[X_1^k\right] \right),$$

where

$$B'_{n,j} := \{k \in \mathbb{Z}_+, (j-1)\ell_n + 1 + \ell_n^\kappa \leq k \leq j\ell_n - \ell_n^\kappa\}.$$

Note that by construction, the sequence of random variables $\left(A'_{n,j}\right)_{j \geq 1}$ is independent. Moreover, the following decomposition holds

$$\begin{aligned}
&\sqrt{b_n} \left| \frac{1}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} \mathbb{E}\left[h(Z_{n,j}, Z_{n,k})\right] - \mathbb{E}\left[h\left(Z_n, Z'_n\right)\right] \right| \\
\leq & \sqrt{b_n} \frac{1}{b_n(b_n-1)} \sum_{1 \leq j \neq k \leq b_n} \mathbb{E}\left[\left|h(Z_{n,j}, Z_{n,k}) - h\left(A'_{n,j}, A'_{n,k}\right)\right|\right] \\
+ & \sqrt{b_n} \frac{1}{b_n(b_n-1)} \left| \sum_{1 \leq j \neq k \leq b_n} \left(\mathbb{E}\left[h\left(A'_{n,j}, A'_{n,k}\right)\right] - \mathbb{E}\left[h\left(Z_n, Z'_n\right)\right] \right) \right|.
\end{aligned}$$

By the Lipschitz-continuity of h combined with stationarity, we can bound the first of the above terms by

$$C\sqrt{b_n}\mathbb{E}\left[\left|Z_{n,1} - A'_{n,1}\right|\right] \leq C\sqrt{b_n}\mathbb{E}\left[\left|Z_{n,1} - A_{n,1}\right|\right] + C\sqrt{b_n}\mathbb{E}\left[\left|A_{n,1} - A'_{n,1}\right|\right].$$

An application of Lemma A.4 from the appendix yields

$$\begin{aligned}
\sqrt{b_n}\mathbb{E}\left[\left|Z_{n,1} - A_{n,1}\right|\right] \leq & C\sqrt{b_n} \sum_{k=1}^{m} \ell_n^\kappa \sum_{i:|i|>\ell_n^\kappa} \delta_i\left(\left(X_t^k\right)_{t \in \mathbb{Z}}\right) \\
\leq & C\sqrt{b_n/\ell_n}\,\ell_n^{\kappa/2} \sum_{k=1}^{m} \sum_{i:|i|>\ell_n^\kappa} \ell_n^{\kappa/2+1/2} \delta_i\left(\left(X_t^k\right)_{t \in \mathbb{Z}}\right) \\
\leq & C\sqrt{b_n/\ell_n}\,\ell_n^{\kappa/2} \sum_{k=1}^{m} \sum_{i:|i|>\ell_n^\kappa} |i|^{1/2+1/(2\kappa)} \delta_i\left(\left(X_t^k\right)_{t \in \mathbb{Z}}\right),
\end{aligned}$$

which tends to zero by the assumptions on the dependence coefficients and since $b_n = o(\ell_n^{1-\kappa})$. By Lemma A.1,

$$\begin{aligned}
\sqrt{b_n}\mathbb{E}\left[\left|A_{n,1} - A'_{n,1}\right|\right] \leq & C\frac{\sqrt{b_n}}{\sqrt{\ell_n}} \sum_{k=1}^{m} \left\| \sum_{t=1}^{\ell_n^\kappa} \mathbb{E}\left[X_t^k \mid \sigma(\varepsilon_u, |u-t| \leq \ell_n^\kappa)\right] \right\|_2 \\
\leq & C\frac{\sqrt{b_n}}{\sqrt{\ell_n}} \sum_{k=1}^{m} \ell_n^{\kappa/2} \sum_{i \in \mathbb{Z}} \delta_i((\mathbb{E}\left[X_t^k \mid \sigma(\varepsilon_u, |u-t| \leq \ell_n^\kappa)\right])_{t \in \mathbb{Z}}) \\
\leq & C\frac{\sqrt{b_n}}{\sqrt{\ell_n}} \sum_{k=1}^{m} \ell_n^{\kappa/2} \sum_{i:|i| \leq \ell_n^\kappa} \delta_i((X_t^k)_{t \in \mathbb{Z}}),
\end{aligned}$$

which likewise tends to zero.

For the second of the above terms, we once more use the Lipschitz-continuity of h and stationarity in order to obtain the upper bound $\sqrt{b_n}\mathbb{E}\left[\left|A'_{n,1} - Z_n\right|\right]$, whose convergence has already been shown above since $Z_n = Z_{n,1}$. □

Proof of Corollary 2.11: It holds

$$\sqrt{b_n}\left(\mathbb{E}\left[\left|Z_n - Z'_n\right|\right] - \sigma\mathbb{E}\left[\left|N - N'\right|\right]\right) = \sqrt{2}\sigma\sqrt{b_n}\left(\frac{\mathbb{E}\left[\left|Z_n - Z'_n\right|\right]}{\sqrt{2}\sigma} - \mathbb{E}\left[|N|\right]\right)$$

$$= \sqrt{b_n}\mathbb{E}\left[\left|Z_n - Z'_n\right|\right]\frac{\sqrt{\operatorname{Var}(Z_n)} - \sigma}{\sqrt{\operatorname{Var}(Z_n)}} + \sqrt{2}\sigma\sqrt{b_n}\left(\frac{\mathbb{E}\left[\left|Z_n - Z'_n\right|\right]}{\sqrt{2\operatorname{Var}(Z_n)}} - \mathbb{E}\left[|N|\right]\right). \tag{3.6}$$

For the first of these terms, it suffices to show that $\sqrt{b_n}\left|\sqrt{\operatorname{Var}(Z_n)} - \sigma\right| \to 0$ since $\mathbb{E}\left[\left|Z_n - Z'_n\right|\right]$ can be bounded by a constant independent of n due to Lemma A.1 and since for $\operatorname{Var}(Z_n) \to \sigma^2 > 0$, it holds $\operatorname{Var}(Z_n) > \sigma^2/2$ for all n large enough. Proceeding as in the proof of Proposition 2 in El Machkouri, Volný, and Wu (2013), we derive that for a centered time series $(Y_t)_{t\in\mathbb{Z}}$ such that $Y_t = f\left((\varepsilon_{t-u})_{u\geq 0}\right)$, one has

$$\left|\operatorname{Var}\left(\frac{1}{\sqrt{N}}\sum_{t=1}^{N} Y_t\right) - \sum_{t\in\mathbb{Z}}\operatorname{Cov}\left(Y_0, Y_t\right)\right|$$

$$\leq \sum_{j:|j|>N}\sum_{i\in\mathbb{Z}}\delta_i\left((Y_t)_{t\in\mathbb{Z}}\right)\delta_{i-j}\left((Y_t)_{t\in\mathbb{Z}}\right) + 2\sum_{j=1}^{N}\frac{j}{N}\sum_{i\in\mathbb{Z}}\delta_i\left((Y_t)_{t\in\mathbb{Z}}\right)\delta_{i-j}\left((Y_t)_{t\in\mathbb{Z}}\right). \tag{3.7}$$

This follows from the following arguments. First, we expand $\operatorname{Var}\left(\sum_{t=1}^{N} Y_t/\sqrt{N}\right)$ in terms of $\operatorname{Cov}\left(Y_0, Y_t\right)$ via a use of stationarity. Then we write

$$\operatorname{Cov}\left(Y_0, Y_t\right) = \sum_{i\in\mathbb{Z}}\mathbb{E}\left[P_i\left(Y_0\right)P_i\left(Y_t\right)\right], \tag{3.8}$$

where $P_i\left(Y\right) := \mathbb{E}\left[Y \mid \sigma\left(\varepsilon_u, u \leq i\right)\right] - \mathbb{E}\left[Y \mid \sigma\left(\varepsilon_u, u \leq i-1\right)\right]$. Finally, we use the fact that

$$\left|\mathbb{E}\left[P_i\left(Y_0\right)P_i\left(Y_t\right)\right]\right| \leq \left\|P_i\left(Y_0\right)\right\|_2\left\|P_i\left(Y_t\right)\right\|_2$$

due to the Cauchy-Schwarz inequality and Theorem 1 in Wu (2005). Letting $N = \ell_n$, $Y_t = \sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)\left(X_t^k - \mathbb{E}\left[X_1^k\right]\right)$ and $a_i := \max_{1\leq k\leq m}\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)$, we infer that for some constant C independent of n,

$$\sqrt{b_n}\left|\operatorname{Var}(Z_n) - \sigma^2\right| \leq C\sqrt{b_n}\sum_{k=1}^{m}\sum_{i\geq 0}\sum_{j=1}^{\ell_n}\frac{j}{\ell_n}\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)\delta_{i-j}\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)\mathbf{1}_{i\geq j}$$

$$+ C\sqrt{b_n}\sum_{k=1}^{m}\sum_{i\geq 0}\sum_{j=\ell_n+1}^{\infty}\frac{j}{\ell_n}\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)\delta_{i-j}\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)\mathbf{1}_{i\geq j}.$$

Denote $A := \sum_{i\geq 0} a_i$. Then elementary computations give

$$\sum_{i\geq 0}\sum_{j=1}^{\ell_n}\frac{j}{\ell_n}a_i a_{i-j}\mathbf{1}_{i\geq j} = \sum_{j=1}^{\ell_n}\frac{j}{\ell_n}\sum_{i\geq j}a_i a_{i-j} = \sum_{j=1}^{\ell_n}\frac{j}{\ell_n}\sum_{k\geq 0}a_{k+j}a_k$$

$$\leq \sum_{j=1}^{\ell_n}\frac{j}{\ell_n}\sum_{k\geq 0}a_{k+j}A \leq A\sum_{j=1}^{\ell_n}\frac{j}{\ell_n}j^{-5/2}\sum_{k\geq 0}\left(k+j\right)^{5/2}a_{j+k} \leq C/\sqrt{\ell_n}$$

and

$$\sum_{i\geq 0}\sum_{j=\ell_n+1}^{\infty}\frac{j}{\ell_n}a_i a_{i-j}\mathbf{1}_{i\geq j}=\sum_{i\geq \ell_n+1}\sum_{j=\ell_n+1}^{i}\frac{j}{\ell_n}a_i a_{i-j}\leq \sum_{i\geq \ell_n+1}\frac{i}{\ell_n}\sum_{j=\ell_n+1}^{i}a_i a_{i-j}$$
$$=\sum_{i\geq \ell_n+1}a_i\frac{i}{\ell_n}\sum_{k=0}^{i-\ell_n-1}a_k\leq A\sum_{i\geq \ell_n+1}a_i\frac{i}{\ell_n}\leq C\ell_n^{-5/2}.$$

Consequently,

$$\sqrt{b_n}\left|\mathrm{Var}(Z_n)-\sigma^2\right|\leq C\sqrt{b_n/\ell_n},$$

and the convergence of $\sqrt{b_n}\left|\sqrt{\mathrm{Var}\left(Z_n\right)}-\sigma\right|\to 0$ follows from

$$\sqrt{b_n}\left|\sqrt{\mathrm{Var}\left(Z_n\right)}-\sigma\right|=\sqrt{b_n}\left|\mathrm{Var}(Z_n)-\sigma^2\right|\frac{1}{\sqrt{\mathrm{Var}\left(Z_n\right)}+\sigma}.$$

For the second term of (3.6), we can apply Corollary 2.6 in Jirak (2016) to obtain the bound

$$\sqrt{b_n}\left(\frac{\mathbb{E}\left[\left|Z_n-Z_n'\right|\right]}{\sqrt{2\mathrm{Var}(Z_n)}}-\mathbb{E}\left[|N|\right]\right)\leq C\sqrt{b_n}\ell_n^{1-p/2}\left(\log \ell_n\right)^{p/2},$$

provided the assumptions stated in Corollary 2.11 hold. Indeed, Z_n-Z_n' can be expressed as

$$Z_n-Z_n'=\frac{1}{\sqrt{\ell_n}}\sum_{t=1}^{\ell_n}\phi\left(\left(\varepsilon_{t-u},\varepsilon_{t-u}'\right)_{u\in\mathbb{Z}_+}\right),$$

where $\left(\varepsilon_u'\right)_{u\in\mathbb{Z}}$ is an independent copy of $\left(\varepsilon_u\right)_{u\in\mathbb{Z}}$ and

$$\phi\left(\left(x_u,y_u\right)_{u\in\mathbb{Z}_+}\right):=\sum_{k=1}^{m}\frac{\partial g}{\partial x_k}\left(v_0\right)\left(f^k\left(\left(x_u\right)_{u\in\mathbb{Z}_+}\right)-f^k\left(\left(y_u\right)_{u\in\mathbb{Z}_+}\right)\right).$$

□

3.4. *Verification of Examples.*

Details of Example 2.5: Since φ is γ-Hölder continuous, there exists a constant C such that for each $x,y\in\mathbb{R}$, $\left|\varphi\left(x\right)-\varphi\left(y\right)\right|\leq C\left|x-y\right|^{\gamma}$. In particular, $\left|\varphi\left(x\right)\right|\leq C\left|x\right|^{\gamma}+\left|\varphi\left(0\right)\right|$ and it follows that

$$\left|\varphi^k\left(x\right)-\varphi^k\left(y\right)\right|=\left|\varphi\left(x\right)-\varphi\left(y\right)\right|\left|\sum_{j=0}^{k-1}\varphi\left(x\right)^j\varphi\left(y\right)^{k-j-1}\right|$$
$$\leq\max\left\{C,\left|\varphi\left(0\right)\right|\right\}^{k-1}\left|\varphi\left(x\right)-\varphi\left(y\right)\right|\sum_{j=0}^{k-1}\left(1+\left|x\right|^{\gamma}\right)^j\left(1+\left|y\right|^{\gamma}\right)^{k-j-1}$$
$$\leq C'\left|x-y\right|^{\gamma}\left(1+\left|x\right|^{\gamma}+\left|y\right|^{\gamma}\right)^{k-1}.$$

For a fixed i, we thus have

$$\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)=\left\|\varphi\Big(\sum_{j\in\mathbb{Z}}a_j\varepsilon_{-j}\Big)^k-\varphi\Big(\sum_{j\in\mathbb{Z}}a_j\varepsilon_{-j}^{*,i}\Big)^k\right\|_2$$

$$\leq C'\left\|\left|a_{-i}\right|^{\gamma}\left|\varepsilon_i-\varepsilon_i'\right|^{\gamma}\left(1+\left|\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{-j}+a_{-i}\varepsilon_i\right|^{\gamma}+\left|\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{-j}+a_{-i}\varepsilon_i'\right|^{\gamma}\right)^{k-1}\right\|_2$$

$$\leq C'\left\|\left|a_{-i}\right|^{\gamma}\left|\varepsilon_i-\varepsilon_i'\right|^{\gamma}\left(1+\left|a_{-i}\varepsilon_i\right|^{\gamma}+\left|a_{-i}\varepsilon_i'\right|^{\gamma}\right)^{k-1}\right\|_2$$

$$+C'\left\|\left|a_{-i}\right|^{\gamma}\left|\varepsilon_i-\varepsilon_i'\right|^{\gamma}\left(1+\left|\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{-j}\right|^{\gamma}+\left|\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{-j}\right|^{\gamma}\right)^{k-1}\right\|_2$$

$$\leq C'\Big(\left\|\left|\varepsilon_0\right|^{k\gamma}\right\|_2+\left\|\left|\varepsilon_0\right|^{(k-1)\gamma}\right\|_2\cdot\left\|\left|\varepsilon_0\right|^{\gamma}\right\|_2\Big)\left|a_{-i}\right|^{k\gamma}+C'\left|a_{-i}\right|^{\gamma}\left\|\left|\varepsilon_0\right|^{\gamma}\right\|_2$$

$$+C'\left\|\left|\varepsilon_0\right|^{\gamma}\right\|_2\left|a_{-i}\right|^{\gamma}\left\|\left|\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{-j}\right|^{(k-1)\gamma}\right\|_2,$$

where the second inequality follows by a combination of the triangle inequality for $\gamma\in(0,1]$ and the c_r-inequality, and the third one is due to the independence of $\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{-j}$ and $(\varepsilon_i,\varepsilon_i')$.

Recall that we have to check that $\sum_{i\in\mathbb{Z}}i^2\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right)<\infty$ for each $k=1,\dots,m$. By our assumptions, we have $\mathbb{E}\left[\left|\varepsilon_0\right|^{2m\gamma}\right]<\infty$ and $\sum_{i\in\mathbb{Z}}i^2\left|a_i\right|^{\gamma}<\infty$, such that it remains to show $\mathbb{E}\left[\left|\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{-j}\right|^{2\gamma(k-1)}\right]<\infty$. In case $2\gamma(k-1)\leq 1$, we can simply employ

$$\mathbb{E}\left[\left|\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{-j}\right|^{2(k-1)\gamma}\right]\leq\sum_{j\in\mathbb{Z},j\neq 0}\left|a_j\right|^{2(k-1)\gamma}\mathbb{E}\left[\left|\varepsilon_0\right|^{2(k-1)\gamma}\right],$$

which is finite by assumption. If $2\gamma(k-1)\in(1,2]$, then the Von Bahr-Esseen inequality gives (up to a constant) the same upper bound. If $2\gamma(k-1)>2$, Rosenthal's inequality yields

$$\mathbb{E}\left[\left|\sum_{j\in\mathbb{Z},j\neq -i}a_j\varepsilon_{i-j}\right|^{2(k-1)\gamma}\right]\leq C\sum_{j\in\mathbb{Z},j\neq 0}\left|a_j\right|^{2(k-1)\gamma}\mathbb{E}\left[\left|\varepsilon_0\right|^{2(k-1)\gamma}\right]+C\left(\sum_{j\in\mathbb{Z},j\neq 0}a_j^2\mathbb{E}\left[\varepsilon_0^2\right]\right)^{(k-1)\gamma}.$$

□

Details of Example 2.6: This is a consequence of the estimation of the physical dependence measure in Example 3 on pages 5967-5968 of Bierme and Durieu (2014) applied separately to each function φ^k for $1\leq k\leq m$. □

Details of Example 2.7: In order to give a bound on $\delta_i((X_t^k)_{t\in\mathbb{Z}})$ for a fixed i and a $k\in\{1,\dots,m\}$, we decompose X_0 as follows: Set $X_0=\varepsilon_iY_i+Z_i$, where

$$Y_i:=\sum_{j'\in\mathbb{Z},j'\neq -i}a_{-i,j'}\varepsilon_{-j'}+\sum_{j\in\mathbb{Z},j\neq -i}a_{j,-i}\varepsilon_{-j}$$

$$Z_i:=\sum_{j,j'\in\mathbb{Z},j\neq j',j\neq -i,j'\neq -i}a_{j,j'}\varepsilon_{-j}\varepsilon_{-j'}.$$

Thus,

$$X_0^k - \left(X_0^{*,i}\right)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\varepsilon_i^\ell Y_i^\ell Z_i^{k-\ell} - \left(\varepsilon_i'\right)^\ell Y_i^\ell Z_i^{k-\ell}\right)$$

and since the term with index 0 vanishes, we derive that

$$\begin{aligned}\delta_i((X_t^k)_{t\in\mathbb{Z}}) = \left\|X_0^k - \left(X_0^{*,i}\right)^k\right\|_2 &\le \sum_{\ell=1}^{k} \binom{k}{\ell} \left\|\left(\varepsilon_i^\ell - \left(\varepsilon_i'\right)^\ell\right) Y_i^\ell Z_i^{k-\ell}\right\|_2 \\ &\le 2\sum_{\ell=1}^{k} \binom{k}{\ell} \left\|\varepsilon_0^\ell\right\|_2 \left\|Y_i^\ell Z_i^{k-\ell}\right\|_2 \le 2\sum_{\ell=1}^{k} \binom{k}{\ell} \|\varepsilon_0\|_{2\ell}^\ell \|Y_i\|_{2k}^\ell \|Z_i\|_{2k}^{k-\ell},\end{aligned}$$

where the second inequality is due to the independence of $(\varepsilon_i, \varepsilon_i')$ and (Y_i, Z_i), and the third inequality follows from an application of Hölder's inequality with conjugate exponents k/ℓ and $k/(k-\ell)$ for $\ell \le k-1$. Following the arguments given on pages 2376-2377 in Zhang, Reding, and Peligrad (2020), we obtain

$$\|Z_i\|_{2k} \le C\sqrt{\sum_{j,j'\in\mathbb{Z}\setminus\{-i\},j\neq j'} a_{j,j'}^2}\,\|\varepsilon_0\|_{2k} \le C\sqrt{\sum_{j,j'\in\mathbb{Z},j\neq j'} a_{j,j'}^2}\,\|\varepsilon_0\|_{2k},$$

such that $\|Z_i\|_{2k}$ can be bounded independently of i. Moreover, an application of Rosenthal's inequality yields

$$\|Y_i\|_{2k} \le C\sqrt{\sum_{j,j'\in\mathbb{Z},j\neq j'} a_{j,j'}^2} \cdot \sqrt{\sum_{j\in\mathbb{Z},j\neq -i} \left(a_{-i,j}^2 + a_{j,-i}^2\right)}\,\|\varepsilon_0\|_{2k}.$$

Thus, $\delta_i\left(\left(X_t^k\right)_{t\in\mathbb{Z}}\right) \le C\sqrt{\sum_{j\in\mathbb{Z},j\neq -i}\left(a_{-i,j}^2 + a_{j,-i}^2\right)}$ and the result follows. □

A. Appendix

A.1. *Auxiliary results for functionals of i.i.d. sequences.* This appendix collects some auxiliary results for functionals of i.i.d. sequences, which we require for our proofs. We consider sequences $(X_t)_{t\in\mathbb{Z}}$ of the form $X_t := f\left((\varepsilon_{t-u})_{u\in\mathbb{Z}}\right)$, where $f\colon \mathbb{R}^\mathbb{Z} \to \mathbb{R}$ is measurable, $(\varepsilon_u)_{u\in\mathbb{Z}}$ is an i.i.d. sequence and $\mathbb{E}[X_t] = 0$. Denote $\mathcal{F}_M^N := \sigma(\varepsilon_u, M \le u \le N)$. To quantify the dependence, let $(\varepsilon_u')_{u\in\mathbb{Z}}$ denote an independent copy of $(\varepsilon_u)_{u\in\mathbb{Z}}$ and define

$$\delta_i\left((X_t)_{t\in\mathbb{Z}}\right) := \left\|X_0 - X_0^{*,i}\right\|_2,$$

where $X_0^{*,i} = f\left(\left(\varepsilon_{-u}^{*,i}\right)_{u\in\mathbb{Z}}\right)$ and $\varepsilon_v^{*,i} = \varepsilon_i'$ if $v = i$ and $\varepsilon_v^{*,i} = \varepsilon_v$ otherwise.

We start by presenting a bound on the partial sum of $(X_t)_{t\in\mathbb{Z}}$, which is a special case of Proposition 1 in El Machkouri et al. (2013).

Lemma A.1. *The following inequality holds for all $N \in \mathbb{Z}_+$:*

$$\left\|\sum_{t=1}^{N} X_t\right\|_2 \le \sqrt{N}\sum_{i\in\mathbb{Z}} \delta_i\left((X_t)_{t\in\mathbb{Z}}\right).$$

Lemma A.2. *Suppose that $\sum_{i\in\mathbb{Z}} \delta_i\big((X_t)_{t\in\mathbb{Z}}\big) < \infty$. Then the sequence $\left(\frac{1}{N}\left(\sum_{t=1}^{N} X_t\right)^2\right)_{N\ge1}$ is uniformly integrable. Moreover, the series $\sum_{t\in\mathbb{Z}} |\mathrm{Cov}(X_0, X_t)|$ converges and*

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\left(\sum_{t=1}^{N} X_t\right)^2\right] = \sum_{t\in\mathbb{Z}} \mathrm{Cov}(X_0, X_t).$$

Proof: The convergence of the series is established in Proposition 2 of El Machkouri et al. (2013). It remains to check the uniform integrability of $\left(\frac{1}{N}\left(\sum_{t=1}^{N} X_t\right)^2\right)_{N\geq 1}$. Since the sequence is bounded in $\mathbb{L}^1$, we only have to check that

$$\lim_{\delta\to 0} \sup_{A:\mathbb{P}(A)<\delta} \mathbb{E}\left[\frac{1}{N}\left(\sum_{t=1}^{N} X_t\right)^2 \mathbf{1}_A\right] = 0.$$

To do so, let $X_t^{(M)} := \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]$. Then

$$\begin{aligned} &\sup_{A:\mathbb{P}(A)<\delta} \mathbb{E}\left[\frac{1}{N}\left(\sum_{t=1}^{N} X_t\right)^2 \mathbf{1}_A\right] \\ \leq& 2\sup_{A:\mathbb{P}(A)<\delta} \mathbb{E}\left[\frac{1}{N}\left(\sum_{t=1}^{N} X_t^{(M)}\right)^2 \mathbf{1}_A\right] + 2\mathbb{E}\left[\frac{1}{N}\left(\sum_{t=1}^{N}\left(X_t - X_t^{(M)}\right)\right)^2\right]\end{aligned}$$

and using Lemma A.1, we obtain

$$\begin{aligned}\sup_{A:\mathbb{P}(A)<\delta} \mathbb{E}\left[\frac{1}{N}\left(\sum_{t=1}^{N} X_t\right)^2 \mathbf{1}_A\right] \leq& 2\sup_{A:\mathbb{P}(A)<\delta} \mathbb{E}\left[\frac{1}{N}\left(\sum_{t=1}^{N} X_t^{(M)}\right)^2 \mathbf{1}_A\right] \\ &+2\left(\sum_{i\in\mathbb{Z}} \delta_i\left(\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{t\in\mathbb{Z}}\right)\right)^2.\end{aligned}$$

Employing the uniform integrability of $\left(\frac{1}{N}\left(\sum_{t=1}^{N} X_t^{(M)}\right)^2\right)_{N\geq 1}$ for any fixed M, we derive that

$$\limsup_{\delta\to 0} \sup_{A:\mathbb{P}(A)<\delta} \mathbb{E}\left[\frac{1}{N}\left(\sum_{t=1}^{N} X_t\right)^2 \mathbf{1}_A\right] \leq 2\left(\sum_{i\in\mathbb{Z}} \delta_i\left(\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{t\in\mathbb{Z}}\right)\right)^2.$$

Since $\delta_i\left(\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{t\in\mathbb{Z}}\right) \leq 2\delta_i\left((X_t)_{t\in\mathbb{Z}}\right)$, we conclude by an application of the dominated convergence theorem. □

There moreover holds a central limit theorem for $(X_t)_{t\in\mathbb{Z}}$ (see Theorem 1 in El Machkouri et al. (2013)):

Lemma A.3. *Suppose that $\sum_{i\in\mathbb{Z}} \delta_i\left((X_t)_{t\in\mathbb{Z}}\right) < \infty$. Then the following convergence in distribution holds*

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_t \to N\left(0,\sigma^2\right),$$

where

$$\sigma^2 = \sum_{t\in\mathbb{Z}} \operatorname{Cov}\left(X_0, X_t\right).$$

We also require an estimate on the $\mathbb{L}^2$-norm of partial sums of $\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{t\geq 1}$ for some $M\in\mathbb{Z}_+$.

Lemma A.4. *It holds*

$$\left\|\sum_{t=1}^{N}\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)\right\|_2 \leq (4M+3)\sqrt{N}\sum_{i:|i|\geq M} \delta_i\left((X_t)_{t\in\mathbb{Z}}\right).$$

Proof: Lemma A.1 applied to $X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]$ yields

$$\left\|\sum_{t=1}^{N}\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)\right\|_2 \leq \sqrt{N}\sum_{i\in\mathbb{Z}}\delta_i\left(\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{t\in\mathbb{Z}}\right).$$

For $|i| \geq M+1$, we use $\delta_i\left(\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{t\in\mathbb{Z}}\right) \leq \delta_i\left((X_t)_{t\in\mathbb{Z}}\right)$, whereas for $|i| \leq M$, we employ $\delta_i\left(\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)_{t\in\mathbb{Z}}\right) \leq 2\left\|X_0 - \mathbb{E}\left[X_0 \mid \mathcal{F}_{-M}^{M}\right]\right\|_2$, thus obtaining

$$\begin{aligned}&\left\|\sum_{t=1}^{N}\left(X_t - \mathbb{E}\left[X_t \mid \mathcal{F}_{t-M}^{t+M}\right]\right)\right\|_2 \\ \leq &2(2M+1)\sqrt{N}\left\|X_0 - \mathbb{E}\left[X_0 \mid \mathcal{F}_{-M}^{M}\right]\right\|_2 + \sqrt{N}\sum_{i\in\mathbb{Z}:|i|\geq M+1}\delta_i\left((X_t)_{t\in\mathbb{Z}}\right).\end{aligned} \tag{A.1}$$

In the following, we derive an upper bound for the first of the above terms. By the martingale convergence theorem, it holds

$$\left\|X_0 - \mathbb{E}\left[X_0 \mid \mathcal{F}_{-M}^{M}\right]\right\|_2^2 \leq \sum_{i\geq M+1}\left\|\mathbb{E}\left[X_0 \mid \mathcal{F}_{-i}^{i}\right] - \mathbb{E}\left[X_0 \mid \mathcal{F}_{-(i-1)}^{i-1}\right]\right\|_2^2.$$

Moreover,

$$\begin{aligned}&\left\|\mathbb{E}\left[X_0 \mid \mathcal{F}_{-i}^{i}\right] - \mathbb{E}\left[X_0 \mid \mathcal{F}_{-(i-1)}^{i-1}\right]\right\|_2 \\ \leq &\left\|\mathbb{E}\left[X_0 \mid \mathcal{F}_{-i}^{i}\right] - \mathbb{E}\left[X_0^{*,i} \mid \mathcal{F}_{-i}^{i}\right]\right\|_2 + \left\|\mathbb{E}\left[X_0^{*,i} \mid \mathcal{F}_{-i}^{i}\right] - \mathbb{E}\left[X_0 \mid \mathcal{F}_{-(i-1)}^{i-1}\right]\right\|_2 \\ \leq &\delta_i\left((X_t)_{t\in\mathbb{Z}}\right) + \delta_{-i}\left((X_t)_{t\in\mathbb{Z}}\right),\end{aligned}$$

where the second inequality follows from

$$\mathbb{E}\left[X_0^{*,i} \mid \mathcal{F}_{-i}^{i}\right] = \mathbb{E}\left[X_0^{*,i} \mid \mathcal{F}_{-i}^{i-1}\right] = \mathbb{E}\left[X_0 \mid \mathcal{F}_{-i}^{i-1}\right]$$

combined with

$$\mathbb{E}\left[X_0 \mid \mathcal{F}_{-(i-1)}^{i-1}\right] = \mathbb{E}\left[X_0^{*,i} \mid \mathcal{F}_{-(i-1)}^{i-1}\right] = \mathbb{E}\left[X_0^{*,-i} \mid \mathcal{F}_{-i}^{i-1}\right].$$

By the comparison of the ℓ^1- and ℓ^2-norm, we thus have

$$\left\|X_0 - \mathbb{E}\left[X_0 \mid \mathcal{F}_{-M}^{M}\right]\right\|_2 \leq \sum_{i:|i|\geq M+1}\delta_i\left((X_t)_{t\in\mathbb{Z}}\right).$$

Inserting the above bound into (A.1) concludes the proof. □

A.2. *A moment inequality for U-statistics.*

Lemma A.5. *Let $(\varepsilon_u)_{u\in\mathbb{Z}}$ be an i.i.d. sequence. Let $M \geq 0$ and $\ell > 2M$ be integers. Define the random vectors V_j by $V_j := (\varepsilon_u)_{u=j\ell+1-M}^{(j+1)\ell+M}$. Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz-continuous function, let $f_1, f_2\colon \mathbb{R}^{\ell+2M} \to \mathbb{R}$ be measurable functions and let U_N be defined by*

$$U_N := \sum_{1\leq j<k\leq N}\left(h\left(f_1(V_j), f_1(V_k)\right) - h\left(f_2(V_j), f_2(V_k)\right)\right).$$

Then the following inequality holds

$$N^{-3/2}\left\|U_N - \mathbb{E}\left[U_N\right]\right\|_2 \leq C\left\|f_1(V_0) - f_2(V_0)\right\|_2,$$

where C is a constant depending only on h.

Proof: The difficulty here lies in the fact that the vectors $V_j, j \geq 1$, are not independent. Denote

$$H_{j,k} := h\left(f_1\left(V_j\right), f_1\left(V_k\right)\right) - h\left(f_2\left(V_j\right), f_2\left(V_k\right)\right).$$

We will prove the inequality

$$\left\| U_N - \mathbb{E}\left[U_N\right] \right\|_2 \leq C N^{3/2} \sup_{k\geq 1} \left\| H_{0,k} \right\|_2,$$

from which the assertion then follows by the Lipschitz-continuity of h. To verify the above inequality, we will distinguish between the cases where N is even and those where N is odd. Let us first consider even values of N, in which case we can write $2N$ instead of N. Denote by $\mathcal{F}_k$ the σ-algebra generated by the random variables $V_{k'}$ for $k' \leq k$. Then it holds

$$\begin{aligned} \left\| U_{2N} - \mathbb{E}\left[U_{2N}\right] \right\|_2 &= \left\| \sum_{k=2}^{2N} \sum_{j=1}^{k-1} \left(H_{j,k} - \mathbb{E}\left[H_{j,k} \mid \mathcal{F}_{k-2}\right]\right) + \sum_{k=2}^{2N} \sum_{j=1}^{k-1} \left(\mathbb{E}\left[H_{j,k} \mid \mathcal{F}_{k-2}\right] - \mathbb{E}\left[H_{j,k}\right]\right) \right\|_2 \\ &\leq \left\| \sum_{i=1}^{N} \sum_{j=1}^{2i-1} \left(H_{j,2i} - \mathbb{E}\left[H_{j,2i} \mid \mathcal{F}_{2i-2}\right]\right) \right\|_2 + \left\| \sum_{i=1}^{N-1} \sum_{j=1}^{2i} \left(H_{j,2i+1} - \mathbb{E}\left[H_{j,2i+1} \mid \mathcal{F}_{2i+1-2}\right]\right) \right\|_2 \\ &\quad + \left\| \sum_{k=2}^{2N} \sum_{j=1}^{k-2} \left(\mathbb{E}\left[H_{j,k} \mid \mathcal{F}_{k-2}\right] - \mathbb{E}\left[H_{j,k}\right]\right) \right\|_2 + \left\| \sum_{k=2}^{2N} \left(\mathbb{E}\left[H_{k-1,k} \mid \mathcal{F}_{k-2}\right] - \mathbb{E}\left[H_{k-1,k}\right]\right) \right\|_2. \end{aligned}$$

For the first two terms, we additionally define $d_i := \sum_{j=1}^{2i-1} \left(H_{j,2i} - \mathbb{E}\left[H_{j,2i} \mid \mathcal{F}_{2i-2}\right]\right)$ and $d'_i := \sum_{j=1}^{2i} \left(H_{j,2i+1} - \mathbb{E}\left[H_{j,2i+1} \mid \mathcal{F}_{2i+1-2}\right]\right)$, such that the sequences $\left(d_i, \mathcal{F}_{2i}\right)_{i\geq 1}$ and $\left(d'_i, \mathcal{F}_{2i+1}\right)_{i\geq 1}$ are martingale differences. For the third term, we use the independence between V_k and $\mathcal{F}_{k-2}$ to get that $\mathbb{E}\left[H_{j,k} \mid \mathcal{F}_{k-2}\right] = \mathbb{E}\left[H_{j,-1} \mid V_j\right]$, and we have to bound the moments of a two-dependent identically distributed centred sequence. For the fourth term, we simply use the triangle inequality. By orthogonality of $\left(d_i\right)_{i\geq 1}$ and orthogonality of $\left(d'_i\right)_{i\geq 1}$, it follows

$$\begin{aligned} \left\| U_{2N} - \mathbb{E}\left[U_{2N}\right] \right\|_2 &\leq \left(\sum_{i=1}^{N} \left\| \sum_{j=1}^{2i-1} \left(H_{j,2i} - \mathbb{E}\left[H_{j,2i} \mid \mathcal{F}_{2i-2}\right]\right) \right\|_2^2 \right)^{1/2} \\ &\quad + \left(\sum_{i=1}^{N-1} \left\| \sum_{j=1}^{2i} \left(H_{j,2i+1} - \mathbb{E}\left[H_{j,2i+1} \mid \mathcal{F}_{2i+1-2}\right]\right) \right\|_2^2 \right)^{1/2} \\ &\quad + \left\| \sum_{k=2}^{2N} \sum_{j=1}^{k-2} \left(\mathbb{E}\left[H_{j,-1} \mid V_j\right] - \mathbb{E}\left[H_{j,-1}\right]\right) \right\|_2 + 4N \sup_{k\geq 1} \left\| H_{0,k} \right\|_2. \end{aligned}$$

The first of the above terms can be further bounded via

$$\begin{aligned} &\left(\sum_{i=1}^{N} \left\| \sum_{j=1}^{2i-1} \left(H_{j,2i} - \mathbb{E}\left[H_{j,2i} \mid \mathcal{F}_{2i-2}\right]\right) \right\|_2^2 \right)^{1/2} \leq \left(\sum_{i=1}^{N} \left(\sum_{j=1}^{2i-1} \left\| \left(H_{j,2i} - \mathbb{E}\left[H_{j,2i} \mid \mathcal{F}_{2i-2}\right]\right) \right\|_2 \right)^2 \right)^{1/2} \\ &\leq \left(\sum_{i=1}^{N} 4 \left(\sum_{j=1}^{2i-1} \left\| H_{j,2i} \right\|_2 \right)^2 \right)^{1/2} \leq \left(\sum_{i=1}^{N} 16 i^2 \sup_{k\geq 1} \left\| H_{0,k} \right\|_2^2 \right)^{1/2} \leq 4N^{3/2} \sup_{k\geq 1} \left\| H_{0,k} \right\|_2. \end{aligned}$$

The second term can be treated analogously. In order to bound the third term, we switch the sums over j and k to obtain

$$\begin{aligned}
&\left\|\sum_{k=2}^{2N}\sum_{j=1}^{k-2}\left(\mathbb{E}\left[H_{j,-1}\mid V_j\right]-\mathbb{E}\left[H_{j,-1}\right]\right)\right\|_2\\
=&\left\|\sum_{j=1}^{2N-2}\left(2N-j+1\right)\left(\mathbb{E}\left[H_{j,-1}\mid V_j\right]-\mathbb{E}\left[H_{j,-1}\right]\right)\right\|_2\\
\leq&\left\|\sum_{i=1}^{N-1}\left(2N-2i+1\right)\left(\mathbb{E}\left[H_{2i,-1}\mid V_{2i}\right]-\mathbb{E}\left[H_{2i,-1}\right]\right)\right\|_2\\
&+\left\|\sum_{i=1}^{N-1}\left(2N-(2i-1)+1\right)\left(\mathbb{E}\left[H_{2i-1,-1}\mid V_{2i-1}\right]-\mathbb{E}\left[H_{2i-1,-1}\right]\right)\right\|_2\\
=&\sqrt{\sum_{i=1}^{N-1}\left(2N-2i+1\right)^2\left\|Y\left(\mathbb{E}\left[H_{0,-1}\mid V_0\right]-\mathbb{E}\left[H_{0,-1}\right]\right)\right\|_2^2}\\
&+\sqrt{\sum_{i=1}^{N-1}\left(2N-(2i-1)+1\right)^2\left\|\left(\mathbb{E}\left[H_{0,-1}\mid V_0\right]-\mathbb{E}\left[H_{0,-1}\right]\right)\right\|_2^2}\\
\leq&CN^{3/2}\sup_{k\geq1}\left\|H_{0,k}\right\|_2.
\end{aligned}$$

This proves $\|U_{2N}-\mathbb{E}\left[U_{2N}\right]\|_2\leq CN^{3/2}\sup_{k\geq1}\|H_{0,k}\|_2$. In order to show the corresponding inequality for the index $2N+1$ instead of $2N$, we note that $U_{2N+1}-\mathbb{E}\left[U_{2N+1}\right]$ differs from $U_{2N}-\mathbb{E}\left[U_{2N}\right]$ only by the term $\sum_{j=1}^{2N}\left(H_{j,2N+1}-\mathbb{E}\left[H_{j,2N+1}\right]\right)$, whose $\mathbb{L}^2$-norm is smaller than $4N\sup_{k\geq1}\|H_{0,k}\|_2$. This ends the proof of Lemma A.5. □

A.3. *Tools for the proof of Lemma 3.2.*

Lemma A.6. *Let $(Y_n)_{n\geq1}$ be a sequence of random variables such that $\left(Y_n^2\right)_{n\geq1}$ is uniformly integrable and $Y_n\to N\left(0,\sigma^2\right)$ in distribution with $\sigma>0$. Let Y_n' and Y_n'' be independent copies of Y_n and let $h\colon\mathbb{R}^2\to\mathbb{R}$ be a Lipschitz-continuous function. Then*

$$\lim_{n\to\infty}\operatorname{Cov}\left(h\left(Y_n,Y_n'\right),h\left(Y_n,Y_n''\right)\right)=\operatorname{Cov}\left(h\left(N,N'\right),h\left(N,N''\right)\right),$$

where N, N', N'' are independent $N(0,\sigma^2)$-distributed random variables.

Proof: By independence, the sequence of random vectors (Y_n,Y_n',Y_n'') converges in distribution to (N,N',N''). By Skorohod's representation theorem, there exist a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, sequences of random variables $(Z_n)_{n\geq1}$, $(Z_n')_{n\geq1}$ and $(Z_n'')_{n\geq1}$ and random variables Z, Z' and Z'', each defined on $\tilde\Omega$, such that for all $n\geq1$, the vectors (Y_n,Y_n',Y_n'') and (Z_n,Z_n',Z_n'') have the same distribution, (Z,Z',Z'') has the same distribution as (N,N',N''), and the sequence $(Z_n)_{n\geq1}$ (respectively $(Z_n')_{n\geq1}$ and $(Z_n'')_{n\geq1}$) converges to Z (respectively Z' and Z'') almost surely. Note that for each fixed n, it holds

$$\operatorname{Cov}\left(h\left(Y_n,Y_n'\right),h\left(Y_n,Y_n''\right)\right)=\operatorname{Cov}\left(h\left(Z_n,Z_n'\right),h\left(Z_n,Z_n''\right)\right)$$

as well as

$$\operatorname{Cov}\left(h\left(N,N'\right),h\left(N,N''\right)\right)=\operatorname{Cov}\left(h\left(Z,Z'\right),h\left(Z,Z''\right)\right).$$

Due to the elementary fact that $\operatorname{Cov}(U_n, V_n) \to \operatorname{Cov}(U, V)$ if $U_n \to U$ and $V_n \to V$ in $\mathbb{L}^2$, it hence suffices to show

$$\left\| h\left(Z_n, Z_n'\right) - h\left(Z, Z'\right)\right\|_2 \to 0.$$

Since h is Lipschitz-continuous and the sequence $\left(Z_n^2 + \left(Z_n'\right)^2 + Z^2 + \left(Z'\right)^2\right)_{n\geq 1}$ is uniformly integrable, the sequence $\left(\left(h\left(Z_n, Z_n'\right) - h\left(Z, Z'\right)\right)^2\right)_{n\geq 1}$ is uniformly integrable as well. By the continuity of h, the sequence $\left(\left(h\left(Z_n, Z_n'\right) - h\left(Z, Z'\right)\right)^2\right)_{n\geq 1}$ converges to 0 almost surely. Combined, this yields the desired $\mathbb{L}^2$-convergence and finishes the proof. $\square$

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