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Malliavin differentiability of solutions of hyperbolic stochastic partial differential equations with irregular drifts

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Abstract. We prove path-by-path uniqueness of solutions to hyperbolic stochastic partial differential equations when the drift coefficient is the difference of two componentwise monotone Borel measurable functions of spatial linear growth. The Yamada-Watanabe principle for SDEs driven by Brownian sheet then allows to derive strong uniqueness for such equation and thus extending the results in [Bogso, Dieye and Menoukeu Pamen, Elect. J. Probab., 27:1-26, 2022] and [Nualart and Tindel, Potential Anal., 7(3):661–680, 1997]. Assuming that the drift is globally bounded, we show that the unique strong solution is Malliavin differentiable. The case of a spatial linear growth drift coefficient is also studied.

1. Introduction

The existence, uniqueness and Malliavin differentiability of strong solutions of SDEs on the plane with smooth coefficients have been obtained in several settings of varying generality. However there are not many results when the coefficients of the such equation are singular. The purpose of the present paper is two-fold: first we obtain the existence and uniqueness of strong solution of the following integral form equation

$$X_{s,t} = \xi + \int_0^t \int_0^s b(s_1, t_1, X_{s_1,t_1}) \mathrm{d}s_1 \mathrm{d}t_1 + W_{s,t}, \text{ for } (s,t) \in \mathbb{R}_+^2, \tag{1.1}$$

when W is a d-dimensional Brownian sheet and the drift b is the difference of two componentwise monotone functions and of spatial linear growth. We address this problem by using the Yamada-Watanabe argument for SDEs driven by Brownian sheet derived in Nualart and Yeh (1989) (see also Yeh (1987), Tudor (1983, Remark 2)), that is, we combine weak existence and pathwise uniqueness

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to obtain the existence of a unique strong solution. More particularly, we replace the pathwise uniqueness by a stronger notion of uniqueness, namely, the path-by-path uniqueness introduced in Davie (2007) (see also Flandoli (2011)) in the case of SDEs driven by one-parameter Brownian motion. This notion was introduced in Bogso et al. (2022) for the two parameter process, as follows:

Definition 1.1. Let $\mathcal{V}$ (resp. $\partial\mathcal{V}$) be the space of $\mathbb{R}^d$-valued continuous functions on $[0,T]^2$ (resp. $\{0\}\times[0,T])\cup([0,T]\times\{0\})$ for some $T>0$. We say that the path-by-path uniqueness of solutions to (1.1) holds when there exists a full $\mathbb{P}$-measure set $\Omega_0\subset\Omega$ such that for all $\omega\in\Omega_0$ the following statement is true: there exists at most one function $y\in\mathcal{V}$ which satisfies

$$\int_0^T\int_0^T |b(s,t,y_{s,t})|\mathrm{d}s\mathrm{d}t<\infty,\ \partial y=x,\ \text{for some } x\in\partial\mathcal{V}$$

and

$$y_{s,t}=x+\int_0^s\int_0^t b(s_1,t_1,y_{s_1,t_1})\mathrm{d}s_1\mathrm{d}t_1+W_{s,t}(\omega),\ \forall\,(s,t)\in[0,T]^2. \tag{1.2}$$

The study of path-by-path uniqueness is motivated by the problem of regularisation by noise of random ordinary (or partial) differential equations (ODEs or PDEs). In the case of of SDEs driven by Brownian motion, path-by-path uniqueness of equation (1.1) was proved in Davie (2007) assuming that the drift is bounded and measurable, and the diffusion is constant. This result was extended to the non-constant diffusion in Davie (2011) using rough path analysis. There has now been several generalisation of this result. The authors in Beck et al. (2019) proved a Sobolev regularity of solutions to the linear stochastic transport and continuity equations with drift in critical L^p spaces. Such a result does not hold for the corresponding deterministic equations. In Butkovsky and Mytnik (2019), the authors analysed the regularisation by noise for a non-Lipschitz stochastic heat equation and proved path-by-path uniqueness for any initial condition in a certain class of a set of probability one. In Amine et al. (2023), the path-by-path uniqueness for transport equations driven by the fractional Brownian motion of Hurst index $H<1/2$ with bounded and integrable vector-fields is investigated. In Catellier and Gubinelli (2016); Galeati and Gubinelli (2022) the authors solved the regularisation by noise problem from the point of view of additive perturbations. In particular, the work Catellier and Gubinelli (2016) considered generic perturbations without any specific probabilistic setting whereas authors in Amine et al. (2017) construct a new Gaussian noise of fractional nature and proved that it has a strong regularising effect on a large class of ODEs. More recently, the regularisation by noise problem for ODEs with vector fields given by Schwartz distributions in the setting of non-linear Young type of integrals was studied in Harang and Perkowski (2021). It was also proved that if one perturbs such an equation by adding an infinitely regularising path, then it has a unique solution. Let us also mention the recent work Kremp and Perkowski (2022) in which the authors looked at multidimensional SDEs with distributional drift driven by symmetric α-stable Lévy processes for $\alpha\in(1,2]$. In all of the above mentioned works, the driving noise considered are one parameter processes.

Our method to prove path-by-path uniqueness follows as in Bogso et al. (2022). We show the path-by-path uniqueness on $\Gamma_0=[0,1]^2$. More precisely, we consider the integral equation

$$X_{s,t}=\xi+\int_0^t\int_0^s b(s_1,t_1,X_{s_1,t_1})\mathrm{d}s_1\mathrm{d}t_1+W_{s,t}\ \text{for } (s,t)\in\Gamma_0, \tag{1.3}$$

where the drift is of spatial linear growth. We denote by $\mathcal{V}_0^1$ the space of continuous $\mathbb{R}^d$-valued functions on Γ_0 which vanish on $\{0\}\times[0,1]\cup[0,1]\times\{0\}$. It is shown in Bogso et al. (2022, Section 1) (see also Davie (2007, Section 1)) that path-by-path uniqueness of solutions to (1.3) holds if and only if, with probability one, there is no nontrivial solution $u\in\mathcal{V}_0^1$ of

$$u(s,t)=\int_0^s\int_0^t\{b(s_1,t_1,W_{s_1,t_1}+u(s_1,t_1))-b(s_1,t_1,W_{s_1,t_1})\}\mathrm{d}s_1\mathrm{d}t_1,\ \text{for } (s,t)\in\Gamma_0. \tag{1.4}$$

This is the statement of Theorem 2.10 which is extended to unbounded drifts in Theorem 2.9. The proof of Theorem 2.10 relies on some estimates for an averaging operator along the sheet (see Lemma 2.3). This result plays a key role in the proof of a Gronwall type lemma (see Lemma 2.15) which enables us to prove path-by-path uniqueness of solutions to (1.3). The latter combined with the weak existence yield the existence of a unique strong solution. A crucial idea to obtain the Gronwall type inequality is to take advantage of the fact that the set of dyadic numbers is dense in $[-1,1]^d$. Note that Lemma 2.6 involves real numbers $x \in [-1,1]^d$ and not functions $u : [0,1]^2 \to [-1,1]^d$. In order to apply Lemma 2.6 in the proof of Lemma 2.15 for a drift that is the difference of two componentwise monotone functions, one needs to first rewrite (1.4) in each dyadic square and carefully replace the function u by the maximum of either its positive part or its negative part. Note that when the drift coefficient is componentwise nondecreasing, similar result can be found in Bogso et al. (2022); Nualart and Tindel (1997).

Secondly, in this paper, we prove Malliavin smoothness of the unique solution to the SDE (1.3). When the coefficients are smooth, the authors in Nualart and Sanz (1985, 1989) showed existence, uniqueness, Malliavin differentiability and smoothness of density of solutions to SDEs on the plane. Here, assuming that the drift is the difference of two componentwise nondecreasing functions, we show that the solution is Malliavin differentiable. In the one parameter case, the Malliavin differentiablity of solutions to SDEs with bounded and measeurable coefficients was studied in Meyer-Brandis and Proske (2010) under an additional commutativity assumption. The later assumption was removed in Menoukeu-Pamen et al. (2013). It is worth mentioning that in the above work, the Malliavin smoothness of the unique solutions to the SDEs with rough coefficients and driven by Brownian motion is obtained as a byproduct of the method used to study existence and uniqueness. This technique was introduced in Proske (2007) and has now been extensively utilised; see for example the work Haadem and Proske (2014) for the case of singular SDEs driven by Lévy noise, Menoukeu-Pamen and Tangpi (2019) for the case of random coefficients and Amine et al. (2023, 2017) for the case of singular SDEs driven by fractional noise. In order to prove the Malliavin differentiability of the solution to the SDE (1.3), we take advantage of Gaussian white noise theory and a local time-space integration formula provided in Bogso et al. (2023, Proposition 3.1) to show that the sequence of approximating sequence of solutions converges strongly in $L^2(\Omega,\mathbb{R}^d)$ to the solution of the SDE (compare with Menoukeu-Pamen et al. (2013)) and we use a compactness criteria given in Nualart (2006, Lemma 1.2.3) to conclude. An essential step in showing this is to obtain good enough estimates for the Malliavin derivative of the approximating sequence. This task is not trivial and requires the use of the Wendroff inequality (Theorem B.1) which plays a crucial role in the proof.

Equation (1.1) can also be written in a differential form as the following hyperbolic stochastic partial differential equation

$$\begin{cases} \dfrac{\partial^2 X_{s,t}}{\partial s \partial t} = b(s,t,X_{s,t}) + \dot{W}_{s,t}, \quad (s,t) \in \Gamma, \\ \\ \partial X = \xi, \end{cases} \tag{1.5}$$

where ∂X is the restriction of X to the boundary $\partial\Gamma = \{0\} \times \mathbb{R}_+ \cup \mathbb{R}_+ \times \{0\}$ of $\Gamma := \mathbb{R}_+^2$, $b : \Gamma \times \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable, $\dot{W} = (\dot{W}^{(1)}, \dots, \dot{W}^{(d)})$ is a d-dimensional white noise of the Brownian sheet on Γ given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(s,t) \longmapsto \xi_{s,t}(\omega)$ is continuous on $\partial\Gamma$ for all $\omega \in \Omega$. Recall that a d-dimensional white noise on Γ is a mean-zero Gaussian process $\dot{W} = (\dot{W}^{(1)}, \dots, \dot{W}^{(d)})$ indexed by the Borel field $\mathcal{B}(\Gamma)$ on Γ with covariance functions

$$\mathbb{E}\Big[\dot{W}^{(i)}(A)\dot{W}^{(j)}(B)\Big] = \delta_{i,j}|A \cap B|, \quad \forall\, A, B \in \mathcal{B}(\Gamma)$$

where $|\cdot|$ denotes the Lebesgue measure on Γ and $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. The process $W = \Big(W_{s,t} := \dot{W}([0,s]\times[0,t]), (s,t)\in\Gamma\Big)$ is mean-zero Gaussian process with covariance functions

$$\mathbb{E}\Big[W^{(i)}_{s,t}W^{(j)}_{s',t'}\Big] = \delta_{i,j}(s\wedge s')(t\wedge t'), \quad \forall\,(s,t),(s',t')\in\Gamma.$$

By the Kolmogorov continuity theorem, there exists a continuous version of W, still denoted by W, which is a d-dimensional Brownian sheet. We consider a nondecreasing and right-continuous family $\mathbb{F} = (\mathcal{F}_{s,t})$ of sub-σ-algebras of $\mathcal{F}$ each of which contains all negligible sets in $(\Omega,\mathcal{F},\mathbb{P})$ such that W and ξ are $\mathbb{F}$-adapted, that is $W_{s,t}$ (respectively, $\xi_{s,t}$) is $\mathcal{F}_{s,t}$-measurable for every $(s,t)\in\Gamma$ (respectively, $(s,t)\in\partial\Gamma$). We refer the reader to Khoshnevisan (2002) for a complete analysis on multi-parameter processes and their applications.

Equation (1.5) is a particular case of the quasilinear stochastic hyperbolic differential equation

$$\begin{cases} \dfrac{\partial^2 X_{s,t}}{\partial s\partial t} = b(s,t,X_{s,t}) + a(s,t,X_{s,t})\dot{W}_{s,t}, & (s,t)\in\Gamma \\ \\ \partial X = \xi, \end{cases} \tag{1.6}$$

where $a : \mathbb{R}^2_+\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d$ is a Borel measurable matrix function. A formal $\frac{\pi}{4}$ rotation transforms (1.6) into the following nonlinear stochastic wave equation

$$\frac{\partial^2 Y_{\rho,\theta}}{\partial\rho^2} - \frac{\partial^2 Y_{\rho,\theta}}{\partial\theta^2} = \tilde{b}(\rho,\theta,Y_{\rho,\theta}) + \tilde{a}(\rho,\theta,Y_{\rho,\theta})\dot{\tilde{W}}_{\rho,\theta}, \quad (\rho,\theta)\in\tilde{\Gamma}, \tag{1.7}$$

with the Goursat-Darboux type boundary condition $\partial Y = \tilde{\xi}$, where $\tilde{\Gamma} = \{(\rho,\theta) : \theta\geq 0 \text{ and } |\rho|\leq\theta\}$, $\dot{\tilde{W}}$ is a d-dimensional white noise of the Brownian sheet on $\tilde{\Gamma}$, $\tilde{b}(\rho,\theta,y) = b(\frac{\theta+\rho}{\sqrt{2}},\frac{\theta-\rho}{\sqrt{2}},y)$ (the same applies to $\tilde{a}$), $Y_{\rho,\theta} = X_{\frac{\theta+\rho}{\sqrt{2}},\frac{\theta-\rho}{\sqrt{2}}}$, $\tilde{\xi}_{\theta,\theta} = \xi_{\sqrt{2}\theta,0}$ and $\tilde{\xi}_{-\theta,\theta} = \xi_{0,\sqrt{2}\theta}$. The $\frac{\pi}{4}$ rotation has been used by Carmona and Nualart (1988) (see also Farré and Nualart (1993, Section 0) and Quer-Sardanyons and Tindel (2007, Section 1)) to prove existence and uniqueness of solution to (1.7) under a different boundary condition when $\tilde{a}$ and $\tilde{b}$ are time-homogeneous.

Equation (1.5) can also be seen as a noisy analog of the so-called Darboux problem given by

$$\frac{\partial^2 y}{\partial s\partial t} = b\Big(s,t,y,\frac{\partial y}{\partial s},\frac{\partial y}{\partial t}\Big) \quad \text{for } (s,t)\in[0,T]\times[0,T], \tag{1.8}$$

with the initial conditions

$$y(0,t) = \sigma(t) \text{ on } [0,T] \text{ and } y(s,0) = \tau(s) \text{ on } [0,T], \tag{1.9}$$

where σ and τ are absolutely continuous on $[0,T]$. Using Caratheodory's theory of differential equations, Deimling (1970) proved an existence theorem for the system (1.8)-(1.9) when b is Borel measurable in the first two variables and bounded and continuous in the last three variables. Hence the results obtained here can also be seen as a generalisation to the stochastic setting of the above mentioned one.

The remainder of the paper is organised as follows: In Section 2, we provide a path-by-path uniqueness result for (1.3) when the drift b is of linear growth. In Section 3, we study the Malliavin differentiability of the strong solution to (1.3). We show that this solution is Malliavin differentiable for uniformly bounded drifts and when the drift b is of linear growth, we obtain Malliavin differentiability of the solution only for sufficiently small time parameters.

2. Existence and uniqueness results

In this section, we show that the SDE (1.3) has a unique strong solution. Our approach is based on the Yamada-Watanabe principle introduced in Nualart and Yeh (1989). As pointed out earlier, instead of showing the weak existence and pathwise uniqueness, we show weak existence and path by path uniqueness (which implies pathwise uniqueness as shown in Catellier and Gubinelli (2016)). The following preliminary results that have been obtained by applying a local time-space integration formula for Brownian sheets (see Bogso et al. (2023) for more information) are needed to show path-by-path uniqueness.

2.1. *Preliminary results.* Let $f : [0,1]^2 \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that for any $(s,t) \in [0,1]^2$, $f(s,t,\cdot)$ is differentiable and for any $i \in \{1,\cdots,d\}$, the partial derivative $\partial_{x_i} f$ is continuous. We also know from Bogso et al. (2023, Proposition 3.1) that for a d-dimensional Brownian sheet $\left(W_{s,t} := (W^{(1)}_{s,t},\cdots,W^{(d)}_{s,t}); s \geq 0, t \geq 0\right)$ defined on a filtered probability space and for any $(s,t) \in [0,1]^2$ and any $i \in \{1,\cdots,d\}$, we have

$$\begin{aligned}
&\int_0^s \int_0^t \partial_{x_i} f(s_1,t_1,W_{s_1,t_1}) \mathrm{d}t_1 \mathrm{d}s_1 \\
&= -\int_0^s \int_0^t f(s_1,t_1,W_{s_1,t_1}) \frac{d_{t_1} W^{(i)}_{s_1,t_1}}{s_1} \mathrm{d}s_1 - \int_0^s \int_{1-t}^1 f(s_1,1-t_1,\widehat{W}_{s_1,t_1}) \frac{d_{t_1} B^{(i)}_{s_1,t_1}}{s_1} \mathrm{d}s_1 \\
&\quad + \int_0^s \int_{1-t}^1 f(s_1,1-t_1,\widehat{W}_{s_1,t_1}) \frac{\widehat{W}^{(i)}_{s_1,t_1}}{s_1(1-t_1)} \mathrm{d}t_1 \mathrm{d}s_1,
\end{aligned} \tag{2.1}$$

where $\widehat{W}^{(i)} = (\widehat{W}^{(i)}_{s_1,t_1} := W^{(i)}_{s_1,1-t_1}; 0 \leq s_1,t_1 \leq 1)$ and $B^{(i)} := (B^{(i)}_{s_1,t_1}; 0 \leq s_1,t_1 \leq 1)$ is a standard Brownian sheet with respect to the filtration of $\widehat{W}^{(i)}$, independent of $(W^{(i)}_{s,1}, s \geq 0)$. Here "$\mathrm{d}_{t_1} W^{(i)}_{s_1,t_1}$", resp. "$\mathrm{d}_{t_1} B^{(i)}_{s_1,t_1}$" denotes the stochastic line integral with respect to the Brownian motion $(W^{(i)}_{s_1,t_1}; 0 \leq t_1 \leq 1)$, resp. $(B^{(i)}_{s_1,t_1}; 0 \leq t_1 \leq 1)$ for s_1 fixed.

The following result will be extensively used and can be found in Bogso et al. (2022).

Proposition 2.1. *Let $W := \left(W^{(1)}_{s,t},\ldots,W^{(d)}_{s,t}; (s,t) \in [0,1]^2\right)$ be a $\mathbb{R}^d$-valued Brownian sheet defined on a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_{s,t}; s,t \in [0,1])$. Let $b \in \mathcal{C}\left([0,1]^2,\mathcal{C}^1(\mathbb{R}^d)\right)$, $\|b\|_\infty \leq 1$. Let $(a,a',\varepsilon,\varepsilon') \in [0,1]^4$. Then there exist positive constants α and C (independent of $\nabla_y b$, a, a', ε and ε') such that*

$$\mathbb{E}\Big[\exp\left(\alpha\varepsilon'\varepsilon\Big|\int_0^1\int_0^1 \nabla_y b\left(s,t,\widetilde{W}^{\varepsilon,\varepsilon'}_{s,t}\right)\mathrm{d}t\mathrm{d}s\Big|^2\right)\Big] \leq C. \tag{2.2}$$

Here $\nabla_y b$ denotes the gradient of b with respect to the third variable, $|\cdot|$ is the usual norm on $\mathbb{R}^d$ and the $\mathbb{R}^d$-valued two-parameter Gaussian process $\widetilde{W}^{\varepsilon,\varepsilon'} := \left(\widetilde{W}^{(\varepsilon,\varepsilon',1)}_{s,t},\ldots,\widetilde{W}^{(\varepsilon,\varepsilon',d)}_{s,t}; (s,t) \in [0,1]^2\right)$ is given by

$$\widetilde{W}^{(\varepsilon,\varepsilon',i)}_{s,t} = W^{(i)}_{a'+\varepsilon's,a+\varepsilon t} - W^{(i)}_{a',a+\varepsilon t} - W^{(i)}_{a'+\varepsilon's,a} + W^{(i)}_{a',a} \quad \textit{for all } i \in \{1,\ldots,d\}.$$

For every $0 \leq a < \gamma \leq 1$, $0 \leq a' < \gamma' \leq 1$ and for $(x,y) \in \mathbb{R}^d$ let us define the function ϱ by:

$$\varrho(x,y) = \int_{a'}^{\gamma'}\int_a^\gamma \Big\{b(s,t,W_{s,t}+x) - b(s,t,W_{s,t}+y)\Big\}\mathrm{d}t\mathrm{d}s.$$

Here is a direct consequence of the previous estimation.

Corollary 2.2. *Let $b : [0,1]^2 \times \mathbb{R}^d \to \mathbb{R}$ be a bounded and Borel measurable function such that $\|b\|_\infty \leq 1$. Let α, C and $\widetilde{W}^{\varepsilon,\varepsilon'}$ be defined as in Proposition 2.1. Then the following two bounds are valid:*

(1) *For every* $(x,y)\in\mathbb{R}^{2d}$, $x\neq y$ *and every* $(\varepsilon,\varepsilon')\in[0,1]^2$, *we have*

$$\mathbb{E}\Big[\exp\Big(\frac{\alpha\varepsilon'\varepsilon}{|x-y|^2}\Big|\int_0^1\int_0^1\Big\{b(s,t,\widetilde{W}^{\varepsilon',\varepsilon}_{s,t}+x)-b(s,t,\widetilde{W}^{\varepsilon',\varepsilon}_{s,t}+y)\Big\}\,\mathrm{d}t\mathrm{d}s\Big|^2\Big)\Big]\leq C. \tag{2.3}$$

(2) *For any* $(x,y)\in\mathbb{R}^2$ *and any* $\eta>0$, *we have*

$$\mathbb{P}\Big(|\varrho(x,y)|\geq\eta\sqrt{(\gamma-a)(\gamma'-a')}|x-y|\Big)\leq Ce^{-\alpha\eta^2}. \tag{2.4}$$

For any positive integer n, we divide $[0,1]$ into 2^n intervals $I_{nk}=]k2^{-n},(k+1)2^{-n}]$. We define the random real valued function $\varrho_{nkk'}$ on $[-1,1]^{2d}$ by

$$\varrho_{nkk'}(x,y):=\int_{I_{nk'}}\int_{I_{nk}}\{b(s,t,W_{s,t}+x)-b(s,t,W_{s,t}+y)\}\,\mathrm{d}t\mathrm{d}s.$$

The next two lemmas provide an estimate for $\varrho_{nkk'}(x,y)$ and $\varrho_{nkk'}(0,x)$ for every dyadic numbers $x,y\in[-1,1]^d$. Their proofs can be found in Bogso et al. (2022, Section 5).

Lemma 2.3. *Suppose* $b:\ [0,1]^2\times\mathbb{R}^d\to\mathbb{R}$ *is a Borel measurable function such that* $|b(s,t,x)|\leq 1$ *everywhere on* $[0,1]^2\times\mathbb{R}^d$. *Then there exists a subset* Ω_1 *of* Ω *with* $\mathbb{P}(\Omega_1)=1$ *such that for all* $\omega\in\Omega_1$,

$$|\varrho_{nkk'}(x,y)(\omega)|\leq C_1(\omega)2^{-n}\Big[\sqrt{n}+\Big(\log^+\frac{1}{|x-y|}\Big)^{1/2}\Big]|x-y|\ \ \textit{on }\Omega_1$$

for all dyadic numbers x, $y\in[-1,1]^d$ *and all choices of integers* n, k, k' *with* $n\geq 1$, $0\leq k,k'\leq 2^n-1$, *where* $\log^+z=\max\{0,\log z\}$ *for* $z\in(0,\infty)$ *and* $C_1(\omega)$ *is a positive random constant that does not depend on* x, y, n, k *and* k'.

Lemma 2.4. *Suppose* b *is as in Lemma 2.3. Then there exists a subset* Ω_2 *of* Ω *with* $\mathbb{P}(\Omega_2)=1$ *such that for all* $\omega\in\Omega_2$, *for any choice of* n, k, k', *and any choice of a dyadic number* $x\in[-1,1]^d$

$$|\varrho_{nkk'}(0,x)(\omega)|\leq C_2(\omega)\sqrt{n}2^{-n}\Big(|x|+2^{-4^n}\Big), \tag{2.5}$$

where $C_2(\omega)$ *is a positive random constant that does not depend on* x, n, k *and* k'.

Observe that the above two results require only the drift to be bounded and Borel measurable. Assuming in addition b is nondecreasing, the next two results state that Lemmas 2.3 and 2.4 can be extended to any $x,y\in[-1,1]^d$ (not only dyadic). The proof of Lemma 2.6 is omitted since it is similar to that of Lemma 2.5.

Lemma 2.5. *Suppose* b, Ω_1 *and* C_1 *are as in Lemma 2.3. Suppose in addition that* b *is componentwise nondecreasing. Then for all* $\omega\in\Omega_1$,

$$|\varrho_{nkk'}(x,y)(\omega)|\leq C_1(\omega)2^{-n}\Big[\sqrt{n}+\Big(\log^+\frac{1}{|x-y|}\Big)^{1/2}\Big]|x-y|\ \ \textit{on }\Omega_1$$

for all x, $y\in[-1,1]^d$ *and all choices of integers* n, k, k' *with* $n\geq 1$, $0\leq k,k'\leq 2^n-1$.

Proof: Fix $\omega\in\Omega_1$, $x,y\in[-1,1]^d$, $n\geq 1$ and $0\leq k,k'\leq 2^n-1$. Suppose without loss of generality that $\varrho_{nkk'}(x,y)(\omega)>0$. For every $i\in\{1,\dots,d\}$ and $\ell\in\mathbb{N}$, define $y^-_{i,\ell}=2^{-\ell}[2^\ell y_i]$, $x^+_{i,\ell}=1-2^{-\ell}[2^\ell(1-x_i)]$, $y^-_\ell=(y^-_{1,\ell},\dots,y^-_{d,\ell})$ and $x^+_\ell=(x^+_{1,\ell},\dots,x^+_{d,\ell})$. Observe that x^+_ℓ (respectively y^-_ℓ) is a componentwise non-increasing (respectively non-decreasing) sequence of dyadic vectors that converges to x (respectively y). Hence, as $b(s,t,W_{s,t}(\omega)+x) \le b(s,t,W_{s,t}(\omega)+x_\ell^+)$ and $b(s,t,W_{s,t}(\omega)+y_\ell^-) \le b(s,t,W_{s,t}(\omega)+y)$, it follows from Lemma 2.3 that

$$\begin{aligned}|\varrho_{nkk'}(x,y)(\omega)| = \varrho_{nkk'}(x,y)(\omega) &= \int_{I_{nk'}}\int_{I_{nk}} \{b(s,t,W_{s,t}(\omega)+x) - b(s,t,W_{s,t}(\omega)+y)\}\,\mathrm{d}t\mathrm{d}s \\ &\le \int_{I_{nk'}}\int_{I_{nk}} \{b(s,t,W_{s,t}(\omega)+x_\ell^+) - b(s,t,W_{s,t}(\omega)+y_\ell^-)\}\,\mathrm{d}t\mathrm{d}s \\ &\le C_1(\omega)2^{-n}\left[\sqrt{n} + \left(\log^+\frac{1}{|x_\ell^+ - y_\ell^-|}\right)^{1/2}\right]|x_\ell^+ - y_\ell^-|.\end{aligned}$$

Then, letting ℓ tends to ∞, we obtain the result. □

Lemma 2.6. *Suppose b, Ω_2 and C_2 are as in Lemma 2.4. Suppose in addition that b is componentwise nondecreasing. Then for all $\omega \in \Omega_2$*

$$|\varrho_{nkk'}(0,x)(\omega)| \le C_2(\omega)\sqrt{n}2^{-n}\left(|x| + 2^{-4^n}\right) \quad \text{on } \Omega_2$$

for all $x \in [-1,1]^d$ and all choices of integers n, k, k' with $n \ge 1$, $0 \le k, k' \le 2^n - 1$.

2.2. *Main results and proofs.* In this section, we prove the path-by-path uniqueness of the solution to (1.3). We use this result to derive the existence and uniqueness of a strong solution to (1.3). We assume the following conditions on the drift. We endow $\mathbb{R}^d$ with the partial order "$\preceq$" defined by

$$x \preceq y \text{ when } x_i \le y_i \text{ for all } i \in \{1,\dots,d\}.$$

Hypothesis 2.7.

(1) $b : [0,1]^2 \times \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable and admits the decomposition $b = \hat{b} - \check{b}$, where $\hat{b}(s,t,\cdot)$ and $\check{b}(s,t,\cdot)$ are componentwise nondecreasing functions, that is each component $\hat{b}_i$ and $\check{b}_i$, $1 \le i \le d$ is componentwise nondecreasing. Precisely, for every $x, y \in \mathbb{R}^d$,

$$x \preceq y \Rightarrow \hat{b}_i(s,t,x) \le \hat{b}_i(s,t,y) \text{ and } \check{b}_i(s,t,x) \le \check{b}_i(s,t,y).$$

(2) b is of linear growth uniformly on (s,t); precisely, there exists a positive constant M such that

$$|b(s,t,x)| \le M(1+|x|), \quad \forall\,(s,t,x) \in [0,1]^2 \times \mathbb{R}^d.$$

The main results of this section are the following :

Theorem 2.8. *Suppose b satisfies Hypothesis 2.7. Then the SDE (1.3) admits a unique strong solution.*

The above result constitutes an extension to those in Bogso et al. (2022); Nualart and Tindel (1997) by allowing the drift b to be the difference of two monotone functions. It is proved by using the Yamada-Watanabe principle. However, instead of showing the pathwise uniquess we show the following path-by-path uniqueness property.

Theorem 2.9. *Suppose b satisfies Hypothesis 2.7. Then for almost every Brownian sheet path W, there exists a unique continuous function $X : [0,1]^2 \to \mathbb{R}^d$ satisfying (1.3).*

The proof of Theorem 2.9 is omitted since it follows the same lines as that of Bogso et al. (2022, Theorem 3.2). It follows from both Gronwall inequality on the plane and the next result.

Theorem 2.10. *Suppose b is as in Theorem 2.9. Suppose in addition that b is uniformly bounded. Then for almost every Brownian sheet path W, there exists a unique continuous function X : $[0,1]^2 \to \mathbb{R}^d$ satisfying (1.3).*

Proof of Theorem 2.8: It follows from the conditions of the theorem that, (1.3) has a weak solution. In addition, since path-by-path uniqueness implies pathwise uniqueness (see e.g. Beck et al. (2019, Page 9, Section 1.8.4) where the result is provided in the one-parameter case. This may be extend easily to the two-parameter case.), the result follows from the Yamada-Watanabe type principle for SDEs driven by Brownian sheets (see e.g. Nualart and Yeh (1989)). □

Corollary 2.11. *Suppose that b is as in Theorem 2.9. Then for almost every Brownian sheet path W, there exists a unique continuous function $X : [0,1]^2 \to \mathbb{R}^d$ satisfying (1.5).*

Corollary 2.12. *Suppose that b is as in Theorem 2.9. Then for almost every Brownian sheet path W, there exists a unique continuous function $X : [0,1]^2 \to \mathbb{R}^d$ satisfying the stochastic wave equation (1.7) when a is the identity matrix.*

Below we provide a non trivial example of functions satisfying hypothesis of Theorem 2.9. This comes from the Jordan decomposition of real-valued functions of bounded variation on $\mathbb{R}$ (see e.g. Folland (1999)[Theorem 3.27, b.]).

Example 2.13. Let $g_1, \ldots, g_d$ be real-valued functions of bounded variation on $\mathbb{R}$ and let $h_1, \ldots, h_d$ be the functions defined on $\mathbb{R}$ by $h_i(z) = |z|g_i(z)$ for all $z \in \mathbb{R}$ and i. It follows from Jordan decomposition that $g_i = \hat{g}_i - \check{g}_i$, where $\hat{g}_i$, $\check{g}_i$ are two bounded nondecreasing functions on $\mathbb{R}$. It also holds that $h_i = \hat{h}_i - \check{h}_i$, where $\hat{h}_i$, $\check{h}_i$ are two nondecreasing functions of linear growth on $\mathbb{R}$. This follows from the fact that $z \longmapsto |z|$ is the difference of two non-decreasing functions and $z \longmapsto |z|(\hat{g}_i(z) - \hat{g}_i(0))$ (resp. $z \longmapsto |z|(\check{g}_i(z) - \check{g}_i(0))$) is non-decreasing on $\mathbb{R}$. Then

1. the function $b := (b_1, \ldots, b_d) : [0,1]^2 \times \mathbb{R}^d \to \mathbb{R}^d$ defined by
$$b_i(s,t,x_1,\ldots,x_d) = g_i\Big(\kappa_i(s,t) + \sum_{\ell=1}^{d} x_\ell\Big), \quad \text{for all } (s,t,x_1,\ldots,x_d) \text{ and } i$$
satisfies Hypothesis 2.7 for any Borel mesurable functions $\kappa_1, \ldots, \kappa_d : [0,1]^2 \to \mathbb{R}$,
2. the function $\tilde{b} := (\tilde{b}_1, \ldots, \tilde{b}_d) : [0,1]^2 \times \mathbb{R}^d \to \mathbb{R}^d$ defined by
$$\tilde{b}_i(s,t,x_1,\ldots,x_d) = h_i\Big(\zeta_i(s,t) + \sum_{\ell=1}^{d} x_\ell\Big), \quad \text{for all } (s,t,x_1,\ldots,x_d) \text{ and } i$$
satisfies Hypothesis 2.7 for any bounded Borel mesurable functions $\zeta_1, \ldots, \zeta_d : [0,1]^2 \to \mathbb{R}$.

2.3. *Proof of Theorem 2.10.* In this subsection, we prove Theorem 2.10. As already pointed out in the introduction, this is equivalent to showing that for almost all Brownian sheet path, the unique continuous solution u to (1.4) is zero. More precisely, Theorem 2.10 is equivalent to:

Theorem 2.14. *Let $W := \big(W_{s,t}, (s,t) \in [0,1]^2\big)$ be a d-dimensional Brownian sheet defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_{s,t}\}_{s,t\in[0,1]}$. Let $b : [0,1]^2 \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel measurable function such that for every $i \in \{1,\ldots,d\}$, $b_i(s,t,\cdot) = \hat{b}_i(s,t,\cdot) - \check{b}_i(s,t,\cdot)$, where $\hat{b}_i$, $\check{b}_i$ are bounded and componentwise nondecreasing in x for all (s,t). Then there exists $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that for any $\omega \in \Omega_1$, $u = 0$ is the unique continuous solution of the integral equation*

$$u(s,t) = \int_0^t \int_0^s \{b(s_1,t_1,W_{s_1,t_1}(\omega) + u(s_1,t_1)) - b(s_1,t_1,W_{s_1,t_1}(\omega))\}\,\mathrm{d}s_1\,\mathrm{d}t_1, \quad \forall\,(s,t) \in [0,1]^2. \tag{2.6}$$

The proof of Theorem 2.14 relies on the following Gronwall type result.

Lemma 2.15. *Suppose conditions of Theorem 2.14 are valid. Then there exists $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ and a positive random constant C_2 such that for any $\omega \in \Omega_1$, any sufficiently large positive integer n, any $(k, k') \in \{0, 1, 2, \cdots, 2^n\}^2$, any $\beta(n) \in \left[2^{-4^{3n/4}}, 2^{-4^{2n/3}}\right]$, and any solution u of the integral equation*

$$\begin{aligned} &u(s,t) - u(s,0) - u(0,t) + u(0,0) \\ &= \int_0^t \int_0^s \Big\{ b(s_1, t_1, W_{s_1,t_1}(\omega) + u(s_1,t_1)) - b(s_1, t_1, W_{s_1,t_1}(\omega)) \Big\} \mathrm{d}s_1 \, \mathrm{d}t_1, \quad \forall\, (s,t) \in [0,1]^2 \end{aligned} \tag{2.7}$$

satisfying

$$\max_{1 \le i \le d} \max\{|u_i|(s,0), |u_i|(0,t)\} \le \beta(n), \quad \forall\, (s,t) \in [0,1]^2, \tag{2.8}$$

we have

$$\max_{1 \le i \le d} \max\{\overline{u}_{n,i}(k,k'), \underline{u}_{n,i}(k,k')\} \le 3^{k+k'-1} \left(1 + 3C_2(\omega)\sqrt{dn}2^{-n}\right)^{k+k'} \beta(n)\,, \tag{2.9}$$

where $\overline{u}_n = (\overline{u}_{n,1}, \ldots, \overline{u}_{n,d})$, $\underline{u}_n = (\underline{u}_{n,1}, \ldots, \underline{u}_{n,d})$ and for every $i \in \{1, \ldots, d\}$,

$$\overline{u}_{n,i}(k,k') = \sup_{(s,t) \in I_{n,k-1} \times I_{n,k'-1}} \max\{0, u_i(s,t)\} \text{ and } \underline{u}_{n,i}(k,k') = \sup_{(s,t) \in I_{n,k-1} \times I_{n,k'-1}} \max\{0, -u_i(s,t)\}.$$

Proof: Suppose without loss of generality that $\|\hat{b}_i\|_\infty \le 1$ and $\|\check{b}_i\|_\infty \le 1$ for every $i \in \{1, \ldots, d\}$. By Lemma 2.6, there exists a subset $\Omega_2 \subset \Omega$ with $\mathbb{P}(\Omega_2) = 1$ such that for all $\omega \in \Omega_2$,

$$\max_{1 \le i \le d} \left| \hat{\varrho}^{(i)}_{nkk'}(0,x)(\omega) \right| \le C_2(\omega)\sqrt{n}2^{-n} \left(|x| + \beta(n)\right) \text{ on } \Omega_2 \tag{2.10}$$

and

$$\max_{1 \le i \le d} \left| \check{\varrho}^{(i)}_{nkk'}(0,x)(\omega) \right| \le C_2(\omega)\sqrt{n}2^{-n} \left(|x| + \beta(n)\right) \text{ on } \Omega_2 \tag{2.11}$$

for all integers n, k, k' with $n \ge 1$, $0 \le k, k' \le 2^n - 1$ and all $x \in [-1,1]^d$, where

$$\hat{\varrho}^{(i)}_{nkk'}(0,x)(\omega) = \int_{k2^{-n}}^{(k+1)2^{-n}} \int_{k'2^{-n}}^{(k'+1)2^{-n}} \Big\{ \hat{b}_i(s_1, t_1, W_{s_1,t_1}(\omega) + x) - \hat{b}_i(s_1, t_1, W_{s_1,t_1}(\omega)) \Big\} \mathrm{d}s_1 \, \mathrm{d}t_1$$

and

$$\check{\varrho}^{(i)}_{nkk'}(0,x)(\omega) = \int_{k2^{-n}}^{(k+1)2^{-n}} \int_{k'2^{-n}}^{(k'+1)2^{-n}} \Big\{ \check{b}_i(s_1, t_1, W_{s_1,t_1}(\omega) + x) - \check{b}_i(s_1, t_1, W_{s_1,t_1}(\omega)) \Big\} \mathrm{d}s_1 \, \mathrm{d}t_1.$$

For any $\omega \in \Omega_2$, we choose $n \in \mathbb{N}^*$ such that $C_2(\omega)\sqrt{dn}2^{-n} \le 1/6$ and split the set $[0,1] \times [0,1]$ onto 4^n squares $I_{nk} \times I_{nk'}$. We set $u = (u_1, \ldots, u_d)$, with $u^+ = (u_1^+, \ldots, u_d^+)$, $u^- = (u_1^-, \ldots, u_d^-)$, $u_i^+ = \max\{0, u_i\}$ and $u_i^- = \max\{0, -u_i\}$ for every $i \in \{1, \ldots, d\}$. Since $\hat{b}_i(s_1, t_1, \cdot)$ and $\check{b}_i(s_1, t_1, \cdot)$ are nondecreasing, we deduce from (2.7) that for all $i \in \{1, \ldots, d\}$ and all $(s,t) \in I_{nk} \times I_{nk'}$,

$$\begin{aligned} &u_i(s,t) - u_i(s, k'2^{-n}) - u_i(k2^{-n}, t) + u_i(2^{-n}(k,k')) \\ &= \int_{k2^{-n}}^{s} \int_{k'2^{-n}}^{t} \Big\{ \hat{b}_i(s_1, t_1, W_{s_1,t_1}(\omega) + u(s_1,t_1)) - \hat{b}_i(s_1, t_1, W_{s_1,t_1}(\omega)) \Big\} \mathrm{d}s_1 \, \mathrm{d}t_1 \\ &\quad - \int_{k2^{-n}}^{s} \int_{k'2^{-n}}^{t} \Big\{ \check{b}_i(s_1, t_1, W_{s_1,t_1}(\omega) + u(s_1,t_1)) - \check{b}_i(s_1, t_1, W_{s_1,t_1}(\omega)) \Big\} \mathrm{d}s_1 \, \mathrm{d}t_1 \\ &\le \int_{k2^{-n}}^{s} \int_{k'2^{-n}}^{t} \Big\{ \hat{b}_i(s_1, t_1, W_{s_1,t_1}(\omega) + u^+(s_1,t_1)) - \hat{b}_i(s_1, t_1, W_{s_1,t_1}(\omega)) \Big\} \mathrm{d}s_1 \, \mathrm{d}t_1 \\ &\quad - \int_{k2^{-n}}^{s} \int_{k'2^{-n}}^{t} \Big\{ \check{b}_i(s_1, t_1, W_{s_1,t_1}(\omega) - u^-(s_1,t_1)) - \check{b}_i(s_1, t_1, W_{s_1,t_1}(\omega)) \Big\} \mathrm{d}s_1 \, \mathrm{d}t_1. \end{aligned}$$

Then, using the fact that $\max\{0, x+y\} \leq \max\{0, x\} + \max\{0, y\}$, we have

$$\begin{aligned}
u_i^+(s,t) \leq &\max\{0, u_i(s, k'2^{-n}) + u_i(k2^{-n}, t) - u_i(2^{-n}(k,k'))\} \\
&+ \int_{k2^{-n}}^{s} \int_{k'2^{-n}}^{t} \Big\{\hat{b}_i(s_1, t_1, W_{s_1,t_1}(\omega) + u^+(s_1,t_1)) - \hat{b}_i(s_1, t_1, W_{s_1,t_1}(\omega))\Big\} \mathrm{d}s_1\, \mathrm{d}t_1 \\
&- \int_{k2^{-n}}^{s} \int_{k'2^{-n}}^{t} \Big\{\check{b}_i(s_1, t_1, W_{s_1,t_1}(\omega) - u^-(s_1,t_1)) - \check{b}_i(s_1, t_1, W_{s_1,t_1}(\omega))\Big\} \mathrm{d}s_1\, \mathrm{d}t_1.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
u_i^+(s,t) \leq u_i^+(s, k'2^{-n}) + u_i^+(k2^{-n}, t) + u_i^-(2^{-n}(k,k')) + \hat{\varrho}_{nkk'}^{(i)}\left(0, \overline{u}_n(k+1, k'+1)\right)(\omega) \\
- \check{\varrho}_{nkk'}^{(i)}\left(0, -\underline{u}_n(k+1, k'+1)\right)(\omega)
\end{aligned} \tag{2.12}$$

for all $(s,t) \in I_{nk} \times I_{nk'}$. Similarly, we can show that

$$\begin{aligned}
u_i^-(s,t) \leq u_i^-(s, k'2^{-n}) + u_i^-(k2^{-n}, t) + u_i^+(2^{-n}(k,k')) - \hat{\varrho}_{nkk'}^{(i)}\left(0, -\underline{u}_n(k+1, k'+1)\right)(\omega) \\
+ \check{\varrho}_{nkk'}^{(i)}\left(0, \overline{u}_n(k+1, k'+1)\right)(\omega)
\end{aligned} \tag{2.13}$$

for all $(s,t) \in I_{nk} \times I_{nk'}$. For any $k,\, k' \in \{1, 2, \cdots, 2^n\}$ and $i \in \{1, \ldots, d\}$, we define $\widehat{u}_{n,i}(k,k') = \max\{\overline{u}_{n,i}(k,k'), \underline{u}_{n,i}(k,k')\}$. We deduce from Inequalities (2.10)-(2.13) that

$$\begin{aligned}
\overline{u}_{n,i}(k+1, k'+1) \leq &\max_{1\leq j\leq d} \widehat{u}_{n,j}(k, k'+1) + \max_{1\leq j\leq d} \widehat{u}_{n,j}(k+1, k') + \max_{1\leq j\leq d} \widehat{u}_{n,j}(k, k') \\
&+ 2C_2(\omega)\sqrt{n}2^{-n}\left(\sqrt{d} \max_{1\leq j\leq d} \widehat{u}_{n,j}(k+1, k'+1) + \beta(n)\right)
\end{aligned}$$

and

$$\begin{aligned}
\underline{u}_{n,i}(k+1, k'+1) \leq &\max_{1\leq j\leq d} \widehat{u}_{n,j}(k, k'+1) + \max_{1\leq j\leq d} \widehat{u}_{n,j}(k+1, k') + \max_{1\leq j\leq d} \widehat{u}_{n,j}(k, k') \\
&+ 2C_2(\omega)\sqrt{n}2^{-n}\left(\sqrt{d} \max_{1\leq j\leq d} \widehat{u}_{n,j}(k+1, k'+1) + \beta(n)\right).
\end{aligned}$$

Then

$$\begin{aligned}
\max_{1\leq i\leq d} \widehat{u}_{n,i}(k+1, k'+1) \leq &\max_{1\leq i\leq d} \widehat{u}_{n,i}(k, k'+1) + \max_{1\leq i\leq d} \widehat{u}_{n,i}(k+1, k') + \max_{1\leq i\leq d} \widehat{u}_{n,i}(k, k') \\
&+ 2C_2(\omega)\sqrt{n}2^{-n}\left(\sqrt{d} \max_{1\leq i\leq d} \widehat{u}_{n,i}(k+1, k'+1) + \beta(n)\right).
\end{aligned}$$

Since $C_2(\omega)\sqrt{dn}2^{-n} \leq 1/6$, we have $(1 - 2C_2(\omega)\sqrt{dn}2^{-n})^{-1} \leq (1 + 3C_2(\omega)\sqrt{dn}2^{-n})$ and the above inequality implies

$$\begin{aligned}
\max_{1\leq i\leq d} \widehat{u}_{n,i}(k+1, k'+1) \leq &(1 + 3C_2(\omega)\sqrt{dn}2^{-n}) \cdot \Big(\max_{1\leq i\leq d} \widehat{u}_{n,i}(k, k'+1) + \max_{1\leq i\leq d} \widehat{u}_{n,i}(k+1, k') + \\
&+ \max_{1\leq i\leq d} \widehat{u}_{n,i}(k, k') + 2C_2(\omega)\sqrt{n}2^{-n}\beta(n)\Big).
\end{aligned}$$

The desired result then follows by induction on k and k' as in the proof of Lemma 3.9 in Bogso et al. (2022).

$\square$

We now turn to the proof of Theorem 2.14.

Proof of Theorem 2.14: Choose Ω_1, ω, n and $\beta(n)$ as in Lemma 2.15. Let u be a solution of (2.6). We have $\max\{|u|(s,0),|u|(0,t)\} = 0 \leq \beta(n)$ for all $(s,t) \in [0,1]^2$. Moreover, we deduce from (2.9) that

$$\sup_{k,k' \in \{0,1,2,\cdots,2^n\}} \max_{1\leq i\leq d} \max\{\overline{u}_{n,i}(k,k'), \underline{u}_{n,i}(k,k')\} \leq 2^{2^{n+2}}\beta(n) \tag{2.14}$$

for all n satisfying $C_2(\omega)\sqrt{dn}2^{-n} \leq 1/9$. Since the right hand side of (2.14) converges to 0 as n goes to ∞, then, for all (s,t), we have $u(s,t) = 0$ on Ω_1. □

3. Malliavin regularity

In this section we study the Malliavin regularity of the strong solution to (1.3).

3.1. *Basic facts on Malliavin calculus and compactness criterion on the plane.* We first recall some basic facts on Malliavin calculus for Wiener functionals on the plane which can be found in Nualart and Sanz (1985, Section 2) (see also Nualart and Sanz (1989, Section 1)). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the canonical space associated to the d-dimensional Brownian sheet, that is Ω is the space of all continuous functions $\omega : \Gamma \to \mathbb{R}^d$ which vanish on the axes, $\mathbb{P}$ is the Wiener measure and $\mathcal{F}$ is the completion of the Borel σ-algebra of Ω with respect to $\mathbb{P}$. Let $(\mathcal{F}_{s,t}, (s,t) \in \Gamma)$ denote the nondecreasing family of σ-algebras where $\mathcal{F}_{s,t}$ is generated by the functions $(s_1,t_1) \mapsto \omega(s_1 \wedge s, t_1 \wedge t)$, $(s_1,t_1) \in \Gamma$, $\omega \in \Omega$ and the null sets of $\mathcal{F}$. Consider the following subset H of Ω:

$$H = \left\{ \omega \in \Omega : \begin{array}{l} \text{there exists } \dot{\omega} \in L^2(\Gamma, \mathbb{R}^d) \text{ such that} \\ \omega(s,t) = \int_0^s \int_0^t \dot{\omega}(s_1,t_1)\,\mathrm{d}t_1\mathrm{d}s_1, \text{ for any } (s,t) \in \Gamma \end{array} \right\}.$$

Endowed with the inner product

$$\langle \omega_1, \omega_2 \rangle_H = \sum_{i=1}^d \int_\Gamma \dot{\omega}_1^{(i)}(s_1,t_1)\dot{\omega}_2^{(i)}(s_1,t_1)\mathrm{d}s_1\mathrm{d}t_1,$$

the set H is a Hilbert space. We call Wiener functional any measurable function defined on the Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$. A Wiener functional $F : \Omega \to \mathbb{R}$ is said to be smooth if there exists some integer $n \geq 1$ and an infinitely differentiable function f on $\mathbb{R}^n$ such that

(i) f and all its derivatives have at most polynomial growth order,
(ii) $F(\omega) = f(\omega(s_1,t_1), \ldots, \omega(s_n,t_n))$ for some $(s_1,t_1), \ldots, (s_n,t_n) \in \Gamma$.

Every smooth functional F is Fréchet-differentiable and the Fréchet-derivative of F along any vector $h \in H$ is given by

$$DF(h) = \sum_{j=1}^d \sum_{i=1}^n \frac{\partial f}{\partial x_i^{(j)}}(\omega(s_1,t_1), \ldots, \omega(s_n,t_n))h^{(j)}(s_i,t_i) = \sum_{j=1}^d \int_\Gamma \xi_j(q,r)\dot{h}^{(j)}(q,r)\mathrm{d}q\mathrm{d}r,$$

where

$$\xi_j(q,r) = \sum_{i=1}^n \frac{\partial f}{\partial x_i^{(j)}}(\omega(s_1,t_1), \ldots, \omega(s_n,t_n))\mathbf{1}_{[0,s_i]\times[0,t_i]}(q,r).$$

Let $\mathbb{D}_{2,1}$ denote the closed hull of the family of smooth functionals with respect to the norm

$$\|F\|_{2,1}^2 = \|F\|_{L^2(\Omega)}^2 + \|DF\|_{L^2(\Omega;H)}^2.$$

Now we present a useful characterization of relatively compact subsets in the space $L^2(\Omega, \mathbb{R}^d)$. Let us recall the following compactness criterion provided in Da Prato et al. (1992, Theorem 1).

Theorem 3.1. *Let A be a self-adjoint compact operator on H. Then, for any $c > 0$, the set*

$$\mathcal{G} = \left\{G \in \mathbb{D}_{2,1} : \|G\|_{L^2(\Omega)} + \|A^{-1}DG\|_{L^2(\Omega;H)} \leq c\right\}$$

is relatively compact in $L^2(\Omega, \mathbb{R}^d)$.

In order to apply the above result, we consider the fractional Sobolev space:

$$\mathbb{G}_{\beta}^{2,p}(U;\mathbb{R}) := \left\{g \in L^2(U,\mathbb{R}) : \int_U\int_U \frac{|g(u)-g(u')|^2}{|u-u'|^{p+2\beta}}\mathrm{d}u\mathrm{d}u' < \infty\right\},$$

where U is a domain of $\mathbb{R}^p$, $p \geq 1$ and the norm is given by

$$\|g\|_{\mathbb{G}_{\beta}^{2,p}(U;\mathbb{R})} := \|g\|_{L^2(U;\mathbb{R})} + \Big(\int_U\int_U \frac{|g(u)-g(u')|^2}{|u-u'|^{p+2\beta}}\mathrm{d}u\mathrm{d}u'\Big)^{1/2}.$$

We need the next compact embedding result from Palatucci et al. (2013, Lemma 10) (see also Di Nezza et al. (2012, Theorem 7.1)).

Lemma 3.2. *Let $p \geq 1$, $U \subset \mathbb{R}^p$ be a Lipschitz bounded open set and $\mathcal{J}$ be a bounded subset of $L^2(U;\mathbb{R})$. Suppose that*

$$\sup_{g\in\mathcal{J}}\int_U\int_U \frac{|g(u)-g(u')|^2}{|u-u'|^{p+2\beta}}\mathrm{d}u\mathrm{d}u' < \infty$$

for some $\beta \in (0,1)$. Then $\mathcal{J}$ is relatively compact in $L^2(U;\mathbb{R})$.

As a consequence of Theorem 3.1 and Lemma 3.2, we have the following compactness criterion for subsets in the space $L^2(\Omega, \mathbb{R}^d)$.

Corollary 3.3. *Denote by $\mathcal{F}_{\infty}^W$ the σ-algebra generated by the d-dimensional Brownian sheet $W = (W^{(1)},\ldots,W^{(d)})$. Let $(X^{(n)}, n \in \mathbb{N})$ be a sequence of $(\mathcal{F}_{\infty}^W, \mathcal{B}(\mathbb{R}^d))$-measurable random variables and let $D_{s,t}$ be the Malliavin derivative associated with the random vector $W_{s,t} = (W_{s,t}^{(1)},\ldots,W_{s,t}^{(d)})$. Suppose*

$$\sup_{n\in\mathbb{N}}\|X^{(n)}\|_{L^2(\Omega,\mathbb{R}^d)} < \infty \quad and \quad \sup_{n\in\mathbb{N}}\|D_{\cdot,\cdot}X^{(n)}\|_{L^2(\Omega\times[0,1]^2,\mathbb{R}^{d\times d})} < \infty \tag{3.1}$$

as well as

$$\sup_{n\in\mathbb{N}}\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\mathbb{E}\left[\|D_{s,t}X^{(n)} - D_{s',t'}X^{(n)}\|^2\right]}{(|s-s'|+|t-t'|)^{2+2\beta}}\mathrm{d}s\mathrm{d}s'\mathrm{d}t\mathrm{d}t' < \infty \tag{3.2}$$

for some $\beta \in (0,1)$. Then $(X^{(n)}, n \in \mathbb{N})$ is relatively compact in $L^2(\Omega, \mathbb{R}^d)$.

Proof: The proof is inspired from Haadem and Proske (2014, Section 5). We consider the symmetric form $\mathcal{L}$ on $L^2((0,1)^2, \mathbb{R}^d)$ defined as

$$\mathcal{L}(f,g) = \int_0^1\int_0^1 f(s,t)\cdot g(s,t)\mathrm{d}s\mathrm{d}t + \int_0^1\int_0^1\int_0^1\int_0^1 \frac{(f(s,t)-f(s',t'))\cdot(g(s,t)-g(s',t'))}{(|s-s'|+|t-t'|)^{2+2\beta}}\mathrm{d}s\mathrm{d}s'\mathrm{d}t\mathrm{d}t'$$

for functions f, g in the dense domain $\mathcal{D}(\mathcal{L}) \subset L^2((0,1)^2, \mathbb{R}^d)$ and a fixed $\beta \in (0,1)$, where

$$\mathcal{D}(\mathcal{L}) = \left\{g : \|g\|^2_{L^2((0,1)^2,\mathbb{R}^d)} + \int_0^1\int_0^1\int_0^1\int_0^1 \frac{|g(s,t)-g(s',t')|^2}{(|s-s'|+|t-t'|)^{2+2\beta}}\mathrm{d}s\mathrm{d}s'\mathrm{d}t\mathrm{d}t' < \infty\right\}.$$

Then $\mathcal{L}$ is a positive symmetric closed form and, by Kato's first representation theorem (see e.g. Kato (1995)), one can find a positive self-adjoint operator $T_{\mathcal{L}}$ such that

$$\mathcal{L}(f,g) = \langle f, T_{\mathcal{L}}g\rangle_{L^2((0,1)^2,\mathbb{R}^d)}$$

for all $g \in \mathcal{D}(T_{\mathcal{L}})$ and $f \in \mathcal{D}(\mathcal{L})$. Further, one may observe that the form $\mathcal{L}$ is bounded from below by a positive number. Indeed,

$$\mathcal{L}(g,g) \geq \|g\|_{L^2((0,1)^2,\mathbb{R}^d)} \tag{3.3}$$

for all $g \in \mathcal{D}(\mathcal{L})$. Hence, we have that $\mathcal{D}(\mathcal{L}) = \mathcal{D}(T_{\mathcal{L}}^{1/2})$.

Now, define the operator A as $A = T_{\mathcal{L}}^{1/2}$. It follows from Lemma 1 in Lewis (1982, Section 1) (see also Haadem and Proske (2014, Lemma 9)) and Lemma 3.2 applied to $p = 2$, $U = (0,1)^2$ that A has a discrete spectrum and a compact inverse A^{-1}. Then, using (3.1)-(3.3), the operator A and the sequence $(X^{(n)}, n \in \mathbb{N})$ satisfy the assumptions of Theorem 3.1. □

3.2. *Malliavin differentiability for bounded drifts.* In this subsection, we assume in addition the drift b is bounded. The main result of this subsection which significantly generalised those in Nualart and Sanz (1989) when the diffusion is constant is the following

Theorem 3.4. *The strong solution $\{X^{\xi}_{s,t},\ s,t \in [0,1]\}$ of the SDE (1.3) is Malliavin differentiable.*

The proof of this theorem is done is two steps:

Step 1: We use standard approximation procedure to approximate the drift coefficient $b = \hat{b} - \check{b}$ by a sequence of functions

$$b_n := \hat{b}_n - \check{b}_n, n \geq 1$$

such that $\hat{b}_n = (\hat{b}_{1,n}, \dots, \hat{b}_{d,n})$, $\check{b}_n = (\check{b}_{1,n}, \dots, \check{b}_{d,n})$, $(\hat{b}_{j,n})_{n\geq 1}$ and $(\check{b}_{j,n})_{n\geq 1}$ are smooth and componentwise non-decreasing functions with $\sup_n \|\hat{b}_{j,n}\|_\infty \leq \|\hat{b}_j\|_\infty < \infty$ and $\sup_n \|\check{b}_{j,n}\|_\infty \leq \|\check{b}_j\|_\infty < \infty$. In addition, $(\hat{b}_n)_{n\geq 1}$ (respctively, $(\check{b}_n)_{n\geq 1}$) converges to $\hat{b}$ (respctively, $\check{b}$) in $(s,t,x) \in [0,1]^2 \times \mathbb{R}^d$ $\mathrm{d}s \times \mathrm{d}t \times \mathrm{d}x$-a.e. We know that for such smooth drift coefficients, the corresponding SDEs have a unique strong solution denoted by $X^{\xi,n}$. We then show that for each $s,t \in [0,1]$, the sequence $(X^{\xi,n}_{s,t})_{n\geq 1}$ is relatively compact in $L^2(\Omega, \mathbb{P}; \mathbb{R}^d)$.

Step 2: We show that the the sequence of solutions $(X^{\xi,n})_{n\geq 1}$ converges strongly in $L^2(\Omega, \mathbb{P}; \mathbb{R}^d)$.

From **Step 1** and **Step 2**, the result will follow by application of a compactness criteria Nualart (2006, Lemma 1.2.3).

The next result corresponds to a $L^2(\Omega)$ compactness criteria. It is analogous to the result derived in Menoukeu-Pamen et al. (2013) for the case of Brownian motion.

Theorem 3.5. *For every $(s,t) \in \Gamma_0$, the sequence $(X^{\xi,n}_{s,t})_{n\geq 1}$, is relatively compact in $L^2(\Omega, \mathbb{P}; \mathbb{R}^d)$.*

The proof of the above theorem uses Corollary 3.3, which in our case is reduced to proving:

Lemma 3.6. *There exists $C_1 > 0$ such that for every $(s,t) \in \Gamma_0$, the sequence $(X^{\xi,n}_{s,t})_{n\geq 1}$ satisfies*

$$\sup_{n\geq 1} \|X^{\xi,n}_{s,t}\|^2_{L^2(\Omega,\mathbb{P};\mathbb{R}^d)} \leq C_1 \tag{3.4}$$

and

$$\sup_{n\geq 1} \sup_{\substack{0\leq r\leq s \\ 0\leq u\leq t}} \mathbb{E}\left[\|D_{r,u} X^{\xi,n}_{s,t}\|^2\right] \leq C_1. \tag{3.5}$$

Moreover for all $0 \leq r', r \leq s,\ 0 \leq u', u \leq t$,

$$\mathbb{E}\left[\|D_{r,u} X^{\xi,n}_{s,t} - D_{r',u'} X^{\xi,n}_{s,t}\|^2\right] \leq C_1(|r - r'| + |u - u'|), \tag{3.6}$$

where $\|\cdot\|$ denotes the max norm.

Remark 3.7. If (3.6) is satisfied, then for any $\beta \in (0, 1/2)$, (3.2) holds. Indeed, since, for any s, s', t, t', one has

$$|s - s'|^{1/2} |t - t'|^{1/2} \leq \frac{1}{2}(|s - s'| + |t - t'|),$$

then for any $n \in \mathbb{N}$,

$$\begin{aligned}&\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\mathbb{E}\left[\|D_{s,t}X^{(n)}-D_{s',t'}X^{(n)}\|^2\right]}{(|s-s'|+|t-t'|)^{2+2\beta}}\mathrm{d}s\mathrm{d}s'\mathrm{d}t\mathrm{d}t'\\ \leq & C_1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\mathrm{d}s\mathrm{d}s'\mathrm{d}t\mathrm{d}t'}{(|s-s'|+|t-t'|)^{1+2\beta}} \leq \frac{C_1}{2}\Big(\int_0^1\int_0^1 \frac{\mathrm{d}s\mathrm{d}s'}{|s-s'|^{1/2+\beta}}\Big)\Big(\int_0^1\int_0^1 \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|^{1/2+\beta}}\Big) < \infty.\end{aligned}$$

To prove the result above, we need some preliminary estimates.

Lemma 3.8. *There exists a non-decreasing function $\widetilde{C}_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $0 < r < s \leq 1$, any $0 < u < t \leq 1$, any $k \in \mathbb{R}_+$ and any $i, j \in \{1, \cdots, d\}$,*

$$\mathbb{E}\Big[\exp\Big(\frac{k}{\delta(r,s)\delta(u,t)}\int_r^s\int_u^t \partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big] \leq \widetilde{C}_1(k,\|\hat{b}_{j,n}\|_\infty) \tag{3.7}$$

and

$$\mathbb{E}\Big[\exp\Big(\frac{k}{\delta(r,s)\delta(u,t)}\int_r^s\int_u^t \partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big] \leq \widetilde{C}_1(k,\|\check{b}_{j,n}\|_\infty), \tag{3.8}$$

where $\delta(r,s) = \sqrt{s-r}$ and $\delta(u,t) = \sqrt{t-u}$.

Proof: We only prove (3.7) and the proof of (3.8) follows anagously. Using integration with respect to local time formula (see e.g. Bogso et al. (2023, Corollary 2.3)), we have

$$\begin{aligned}&\int_r^s\int_u^t \partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\\ =&-\int_r^s\int_u^t b_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{s_1}\mathrm{d}s_1+\int_r^s\int_{1-t}^{1-u} b_{j,n}(s_1,1-t_1,W_{s_1,1-t_1})\frac{\mathrm{d}_{t_1}B^{(i)}_{s_1,t_1}}{s_1}\mathrm{d}s_1\\ &-\int_r^s\int_{1-t}^{1-u} b_{j,n}(s_1,1-t_1,W_{s_1,1-t_1})\frac{W^{(i)}_{s_1,1-t_1}}{s_1(1-t_1)}\mathrm{d}t_1\mathrm{d}s_1,\end{aligned} \tag{3.9}$$

where $(B^{(i)}_{s,t}, (s,t) \in [0,1]^2)$ is the Brownian sheet given by the following representation provided by Dalang and Walsh (2002)

$$W^{(i)}_{s,1-t} = W^{(i)}_{s,1} + B^{(i)}_{s,t} - \int_0^t \frac{W^{(i)}_{s,1-u}}{1-u}\mathrm{d}u.$$

Since the function $x \longmapsto e^{3x}$ is convex,

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(\frac{k}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_u^t \partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\\ \leq&\frac{1}{3}\Big\{\mathbb{E}\Big[\exp\Big(\frac{3k}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_u^t \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{s_1}\mathrm{d}s_1\Big|\Big)\Big]\\ &+\mathbb{E}\Big[\exp\Big(\frac{3k}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_{1-t}^{1-u} \hat{b}_{j,n}(s_1,t_1,W_{s_1,1-t_1})\frac{\mathrm{d}_{t_1}B^{(i)}_{s_1,t_1}}{s_1}\mathrm{d}s_1\Big|\Big)\Big]\\ &+\mathbb{E}\Big[\exp\Big(\frac{3k}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_{1-t}^{1-u} \hat{b}_{j,n}(s_1,t_1,W_{s_1,1-t_1})\frac{W^{(i)}_{s_1,1-t_1}}{s_1(1-t_1)}\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\Big\} = \frac{1}{3}(I_1+I_2+I_3).\end{aligned} \tag{3.10}$$

By Jensen inequality

$$
\begin{aligned}
I_1 =&\mathbb{E}\Big[\exp\Big(\frac{3k}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_u^t \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{s_1}\mathrm{d}s_1\Big|\Big)\Big]\\
\le&\int_r^s\mathbb{E}\Big[\exp\Big(\frac{6k(\sqrt{s}-\sqrt{r})}{\delta(r,s)\delta(u,t)}\Big|\int_u^t \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{\sqrt{s_1}}\Big|\Big)\Big]\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}\\
\le&\int_r^s\mathbb{E}\Big[\exp\Big(\frac{6k}{\delta(u,t)}\Big|\int_u^t \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{\sqrt{s_1}}\Big|\Big)\Big]\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}.
\end{aligned}
$$

Since for every $s_1\in[r,s]$,

$$
\Big(Y_{s_1,t_1}:=\int_u^{t_1}\hat{b}_{j,n}(s_1,t_2,W_{s_1,t_2})\frac{\mathrm{d}_{t_2}W^{(i)}_{s_1,t_2}}{\sqrt{s_1}},u\le t_1\le t\Big)
$$

is a square integrable martingale, it follows from the Barlow-Yor inequality (see e.g. Barlow and Yor (1982, Proposition 4.2) and Carlen and Krée (1991, Theorem A in Appendix)) that there exists a positive constant c_1 such that for any positive integer m and any $s_1\in[r,s]$,

$$
\begin{aligned}
\mathbb{E}[|Y_{s_1,t}|^m]\le&\mathbb{E}\Big[\sup_{u\le t_1\le t}|Y_{s_1,t_1}|^m\Big]\le c_1^m\sqrt{m}^m\mathbb{E}[\langle Y_{s_1,\cdot}\rangle_t^{m/2}]\\
=&c_1^m\sqrt{m}^m\mathbb{E}\Big[\Big(\int_u^t\hat{b}^2_{j,n}(s_1,t_1,W_{s_1,t_1})\,\mathrm{d}t_1\Big)^{m/2}\Big]\\
\le&c_1^m\sqrt{m}^m\delta(u,t)^m\|\hat{b}_{j,n}\|_\infty^m.
\end{aligned}
$$

Hence, using the following exponential expansion formula,

$$
\exp\Big(\frac{6k}{\delta(u,t)}|Y_{s_1,t}|\Big)=\sum_{m=0}^{\infty}\frac{6^mk^m|Y_{s_1,t}|^m}{\delta(u,t)^m m!},
$$

we obtain

$$
\begin{aligned}
I_1\le&\int_r^s\mathbb{E}\Big[\exp\Big(\frac{6k}{\delta(u,t)}\Big|\int_u^t\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W_{s_1,t_1}}{\sqrt{s_1}}\Big|\Big)\Big]\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}\\
=&\int_r^s\mathbb{E}\Big[\exp\Big(\frac{6k}{\delta(u,t)}|Y_{s_1,t}|\Big)\Big]\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}=\int_r^s\sum_{m=0}^{\infty}\frac{6^mk^m\mathbb{E}[|Y_{s_1,t}|^m]}{\delta(u,t)^m m!}\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}\\
\le&\sum_{m=0}^{\infty}\frac{6^mk^mc_1^m\sqrt{m}^m\|\hat{b}_{j,n}\|_\infty^m}{m!}:=\widetilde{C}_{1,1}(k,\|\hat{b}_{j,n}\|_\infty),
\end{aligned}
$$

which is finite by ratio test. Similarly, we also have

$$
I_2\le\widetilde{C}_{1,1}(k,\|\hat{b}_{j,n}\|_\infty).
$$

To estimate I_3 we apply one more Jensen inequality and obtain

$$
\begin{aligned}
I_3 =&\mathbb{E}\Big[\exp\Big(\frac{3k}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_{1-t}^{1-u}\hat{b}_{j,n}(s_1,t_1,W_{s_1,1-t_1})\frac{W^{(i)}_{s_1,1-t_1}}{s_1(1-t_1)}\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\\
\le&\int_r^s\int_{1-t}^{1-u}\mathbb{E}\Big[\exp\Big(\frac{12k(\sqrt{s}-\sqrt{r})(\sqrt{1-u}-\sqrt{1-t})}{\delta(r,s)\delta(u,t)}|\hat{b}_{j,n}(s_1,t_1,W_{s_1,1-t_1})|\Big|\frac{W^{(i)}_{s_1,1-t_1}}{\sqrt{s_1}\sqrt{1-t_1}}\Big|\Big)\Big]\\
&\qquad\times\frac{\mathrm{d}t_1}{2\sqrt{1-t_1}(\sqrt{1-u}-\sqrt{1-t})}\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}\\
\le&\int_r^s\int_{1-t}^{1-u}\mathbb{E}\Big[\exp\Big(12k\|\hat{b}_{j,n}\|_\infty\Big|\frac{W^{(i)}_{s_1,1-t_1}}{\sqrt{s_1}\sqrt{1-t_1}}\Big|\Big)\Big]\frac{\mathrm{d}t_1}{2\sqrt{1-t_1}(\sqrt{1-u}-\sqrt{1-t})}\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}\\
\le&2\exp\Big(72k^2\|\hat{b}_{j,n}\|^2_\infty\Big):=\widetilde{C}_{1,2}(k,\|\hat{b}_{j,n}\|_\infty),
\end{aligned}
$$

since $\Big|\frac{W^{(i)}_{s_1,1-t_1}}{\sqrt{s_1}\sqrt{1-t_1}}\Big|$ is a mean-zero Gaussian random variable of variance 1.
The proof is completed by choosing $\widetilde{C}_1=2\widetilde{C}_{1,1}+\widetilde{C}_{1,2}$. □

Proof of Lemma 3.6: Without loss of generality, we suppose $\xi=0$. Further we set $X^n_{s,t}:=X^{0,n}_{s,t}$. We start with the proof of (3.5). Suppose for every $j\in\{1,\ldots,d\}$, $\hat{b}_{j,n}$ (respectively, $\check{b}_{j,n}$) is componentwise nondecreasing and continuously differentiable with bounded derivatives. Then for any $(r,u)\in[0,1]^2$ with $(0,0)\prec(r,u)\prec(s,t)$,

$$
D_{r,u}X^n_{s,t}=I_d+\int_r^s\int_u^t\nabla b_n(s_1,t_1,X^n_{s_1,t_1})\,D_{r,u}X^n_{s_1,t_1}\mathrm{d}t_1\mathrm{d}s_1, \tag{3.11}
$$

where I_d is the identity matrix, $\nabla b_n=(\partial_i b_{j,n})_{1\le i,j\le d}$, $\partial_i b_{j,n}$ being the partial derivative of $b_j(s,t,\cdot)$ with respect to x_i, and $D_{r,u}X^n_{s,t}=(D^i_{r,u}X^{n,j}_{s,t})_{1\le i,j\le d}$. Since $\partial_i\hat{b}_{j,n}$ and $\partial_i\check{b}_{j,n}$ are nonnegative, we have

$$
\|D_{r,u}X^n_{s,t}\|\le 1+\int_r^s\int_u^t\sum_{i,j=1}^d\Big\{\partial_i\hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1})+\partial_i\check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1})\Big\}\|D_{r,u}X^n_{s_1,t_1}\|\mathrm{d}t_1\mathrm{d}s_1.
$$

Therefore (see e.g. Theorem B.1),

$$
\|D_{r,u}X^n_{s,t}\|\le\exp\Big(\int_r^s\int_u^t\sum_{i,j=1}^d\Big\{\partial_i\hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1})+\partial_i\check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1})\Big\}\mathrm{d}t_1\mathrm{d}s_1\Big). \tag{3.12}
$$

Squaring both sides of the inequality, taking the expectation and using the Girsanov theorem (see e.g. Dalang and Mueller (2015, Theorem 3.5) and Nualart and Pardoux (1994, Proposition 1.6)) and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}[\|D_{r,u}X^n_{s,t}\|^2]\\
\leq&\mathbb{E}\Big[\exp\Big(2\int_r^s\int_u^t\sum_{i,j=1}^d\Big\{\partial_i\hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1})+\partial_i\check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1})\Big\}\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]\\
=&\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1\int_0^1 b_n(s_1,t_1,W_{s_1,t_1})\cdot\mathrm{d}W_{s_1,t_1}\Big)\cdot\\
&\qquad\exp\Big(2\int_r^s\int_u^t\sum_{i,j=1}^d\Big\{\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})+\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\Big\}\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]\\
\leq&C_0\mathbb{E}\Big[\exp\Big(4\int_r^s\int_u^t\sum_{i,j=1}^d\Big\{\partial_i\hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1})+\partial_i\check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1})\Big\}\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]^{\frac{1}{2}}\\
=&C_0\mathbb{E}\Big[\prod_{i,j=1}^d\exp\Big(4\int_r^s\int_u^t\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\exp\Big(4\int_r^s\int_u^t\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]^{\frac{1}{2}},
\end{aligned}
$$

where

$$
C_0:=\sup_{n\geq1}\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1\int_0^1 b_n(s_1,t_1,W_{s_1,t_1})\cdot\mathrm{d}W_{s_1,t_1}\Big)^2\Big]^{\frac{1}{2}}\tag{3.13}
$$

is finite (see Lemma A.1 and Remark A.2 in the Appendix), with

$$
\begin{aligned}
&\mathcal{E}\Big(\int_0^1\int_0^1 b_n(s_1,t_1,W_{s_1,t_1})\cdot\mathrm{d}W_{s_1,t_1}\Big)\\
&\quad=\exp\Big(\int_0^1\int_0^1 b_n(s_1,t_1,W_{s_1,t_1})\cdot\mathrm{d}W_{s_1,t_1}-\frac{1}{2}\int_0^1\int_0^1|b_n(s_1,t_1,W_{s_1,t_1})|^2\cdot\mathrm{d}s_1\mathrm{d}t_1\Big).
\end{aligned}
$$

Hence, by Hölder inequality, we have

$$
\begin{aligned}
\mathbb{E}[\|D_{r,u}X^n_{s,t}\|^2]\leq&C_0\prod_{i,j=1}^d\mathbb{E}\Big[\exp\Big(8d^2\int_r^s\int_u^t\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]^{\frac{1}{4d^2}}\\
&\cdot\mathbb{E}\Big[\exp\Big(8d^2\int_r^s\int_u^t\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]^{\frac{1}{4d^2}}.
\end{aligned}
$$

Then, by Lemma 3.8, we obtain

$$
\mathbb{E}[\|D_{r,u}X^n_{s,t}\|^2]\leq C_0\times\widetilde{C}_1\Big(8d^2,\max_{1\leq j\leq d}\{\|\hat{b}_{j,n}\|_\infty+\|\check{b}_{j,n}\|_\infty\}\Big)\leq C_0\times\widetilde{C}_1\Big(8d^2,\|\hat{b}\|_\infty+\|\check{b}\|_\infty\Big),
$$

which means that the Malliavin derivative of X^n is bounded in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$.

Next we prove (3.6). We deduce from (3.11) that for all $0\leq r'\leq r\leq s\leq1,\ 0\leq u'\leq u\leq t\leq1$,

$$
\begin{aligned}
&D_{r,u}X^n_{s,t}-D_{r',u'}X^n_{s,t}\\
=&\int_r^s\int_u^t\nabla b_n(s_1,t_1,X^n_{s_1,t_1})\,D_{r,u}X^n_{s_1,t_1}\,\mathrm{d}t_1\mathrm{d}s_1-\int_{r'}^s\int_{u'}^t\nabla b_n(s_1,t_1,X^n_{s_1,t_1})\,D_{r',u'}X^n_{s_1,t_1}\,\mathrm{d}t_1\mathrm{d}s_1\\
=&\int_r^s\int_u^t\nabla b_n(s_1,t_1,X^n_{s_1,t_1})\Big(D_{r,u}X^n_{s_1,t_1}-D_{r',u'}X^n_{s_1,t_1}\Big)\mathrm{d}t_1\mathrm{d}s_1\\
&-\int_{r'}^s\int_{u'}^u\nabla b_n s_1,t_1,X^n_{s_1,t_1})\,D_{r',u'}X^n_{s_1,t_1}\,\mathrm{d}t_1\mathrm{d}s_1-\int_{r'}^r\int_u^t\nabla b_n(s_1,t_1,X^n_{s_1,t_1})\,D_{r',u'}X^n_{s_1,t_1}\,\mathrm{d}t_1\mathrm{d}s_1.
\end{aligned}
$$

Taking the norm on both sides and using the fact that $b_n = \hat{b}_n - \check{b}_n$, with $\hat{b}_n, \check{b}_n$ nondecreasing, gives

$$\begin{aligned}
&\left\| D_{r,u}X^n_{s,t} - D_{r',u'}X^n_{s,t} \right\| \\
\leq &\int_r^s \int_u^t \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \left\| D_{r,u}X^n_{s_1,t_1} - D_{r',u'}X^n_{s_1,t_1} \right\| \mathrm{d}t_1 \mathrm{d}s_1 \\
&+ \int_{r'}^s \int_{u'}^u \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \left\| D_{r',u'}X^n_{s_1,t_1} \right\| \mathrm{d}t_1 \mathrm{d}s_1 \\
&+ \int_{r'}^r \int_u^t \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \left\| D_{r',u'}X^n_{s_1,t_1} \right\| \mathrm{d}t_1 \mathrm{d}s_1.
\end{aligned}$$

Applying Theorem B.1, we obtain

$$\begin{aligned}
&\left\| D_{r,u}X^n_{s,t} - D_{r',u'}X^n_{s,t} \right\| \\
\leq &\Big(\int_{r'}^s \int_{u'}^u \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \left\| D_{r',u'}X^n_{s_1,t_1} \right\| \mathrm{d}t_1 \mathrm{d}s_1 \\
&+ \int_{r'}^r \int_u^t \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \left\| D_{r',u'}X^n_{s_1,t_1} \right\| \mathrm{d}t_1 \mathrm{d}s_1 \Big) \\
&\times \exp\Big(\int_r^s \int_u^t \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}'_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \mathrm{d}t_1 \mathrm{d}s_1 \Big)
\end{aligned} \tag{3.14}$$

Since $\partial_i \hat{b}_{j,n}$ and $\partial_i \check{b}_{j,n}$ are nonnegative, it follows from (3.12) that

$$\| D_{r',u'}X^n_{s_1,t_1} \| \leq \exp\Big(\int_{r'}^s \int_{u'}^t \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_2,t_2,X^n_{s_2,t_2}) + \partial_i \check{b}_{j,n}(s_2,t_2,X^n_{s_2,t_2}) \right\} \mathrm{d}t_2 \mathrm{d}s_2 \Big).$$

Hence, we deduce from (3.14) that it holds:

$$\begin{aligned}
\left\| D_{r,u}X^n_{s,t} - D_{r',u'}X^n_{s,t} \right\| \leq &\Big(\int_{r'}^s \int_{u'}^u \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \mathrm{d}t_1 \mathrm{d}s_1 \\
&+ \int_{r'}^r \int_u^t \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \mathrm{d}t_1 \mathrm{d}s_1 \Big) \\
&\times \exp\Big(2 \int_{r'}^s \int_{u'}^t \sum_{i,j=1}^d \left\{ \partial_i \hat{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) + \partial_i \check{b}_{j,n}(s_1,t_1,X^n_{s_1,t_1}) \right\} \mathrm{d}t_1 \mathrm{d}s_1 \Big).
\end{aligned}$$

Squaring both sides of the inequality, taking the expectation and using Cauchy-Schwarz inequality and Girsanov theorem give

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|D_{r,u}X^n_{s,t}-D_{r',u'}X^n_{s,t}\Big\|^2\Big]\\
\leq&4^7C_0\Big(\sum_{i,j=1}^d\Big\{\mathbb{E}\Big[\Big(\int_{r'}^s\int_{u'}^u\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\,\mathrm{d}t_1\mathrm{d}s_1\Big)^8\Big]+\mathbb{E}\Big[\Big(\int_{r'}^s\int_{u'}^u\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\,\mathrm{d}t_1\mathrm{d}s_1\Big)^8\Big]\\
&\quad+\mathbb{E}\Big[\Big(\int_{r'}^r\int_{u}^t\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\,\mathrm{d}t_1\mathrm{d}s_1\Big)^8\Big]+\mathbb{E}\Big[\Big(\int_{r'}^r\int_{u}^t\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\,\mathrm{d}t_1\mathrm{d}s_1\Big)^8\Big]\Big\}\Big)^{\frac14}\\
&\times\mathbb{E}\Big[\exp\Big(16\int_{r'}^s\int_{u'}^t\sum_{i,j=1}^d\Big\{\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})+\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\Big\}\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]^{\frac14}\\
=&4^7C_0(J_1+J_2+J_3+J_4)^{1/4}\times J_5,
\end{aligned}$$

where C_0 is given by (3.13). It follows from Hölder inequality and the estimates (3.7)-(3.8) that

$$\begin{aligned}
J_5=&\mathbb{E}\Big[\exp\Big(16\int_{r'}^s\int_{u'}^t\sum_{i,j=1}^d\Big\{\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})+\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\Big\}\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]^{\frac14}\\
\leq&\prod_{i,j=1}^d\mathbb{E}\Big[\exp\Big(32d^2\int_r^s\int_u^t\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]^{\frac{1}{8d^2}}\\
&\qquad\cdot\mathbb{E}\Big[\exp\Big(32d^2\int_r^s\int_u^t\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big]^{\frac{1}{8d^2}}\\
\leq&\widetilde{C}_1(32d^2,\|\hat{b}\|_\infty+\|\check{b}\|_\infty)^{\frac14}.
\end{aligned}$$

Moreover, using the inequality $x^8\leq 8!e^x$ ($x\in\mathbb{R}$), we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{r'}^s\int_{u'}^u\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)^8\Big]+\mathbb{E}\Big[\Big(\int_{r'}^s\int_{u'}^u\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)^8\Big]\\
\leq&8!\delta(r',s)^8\delta(u',u)^8\mathbb{E}\Big[\exp\Big(\frac{1}{\delta(r',s)\delta(u',u)}\Big|\int_{r'}^s\int_{u'}^u\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\\
&\quad+8!\delta(r',s)^8\delta(u',u)^8\mathbb{E}\Big[\exp\Big(\frac{1}{\delta(r',s)\delta(u',u)}\Big|\int_{r'}^s\int_{u'}^u\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\\
\leq&2(8!)\widetilde{C}_1(1,\|\hat{b}\|_\infty+\|\check{b}\|_\infty)\delta(u',u)^4
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_{r'}^r\int_{u}^t\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)^8\Big]+\mathbb{E}\Big[\Big(\int_{r'}^r\int_{u}^t\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)^8\Big]\\
\leq&8!\delta(r',r)^8\delta(u,t)^8\mathbb{E}\Big[\exp\Big(\frac{1}{\delta(r',r)\delta(u,t)}\Big|\int_{r'}^r\int_{u}^t\partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\\
&\quad+8!\delta(r',r)^8\delta(u,t)^8\mathbb{E}\Big[\exp\Big(\frac{1}{\delta(r',r)\delta(u,t)}\Big|\int_{r'}^r\int_{u}^t\partial_i\check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\\
\leq&2(8!)\widetilde{C}_1(1,\|\hat{b}\|_\infty+\|\check{b}\|_\infty)\delta(r,r')^8.
\end{aligned}$$

As a consequence,

$$J_1+J_2+J_3+J_4\leq 4(8!)\widetilde{C}_1(1,\|\hat{b}\|_\infty+\|\check{b}\|_\infty)(|r-r'|^4+|u-u'|^4).$$

Therefore

$$\mathbb{E}\Big[\Big\|D_{r,u}X^n_{s,t}-D_{r',u'}X^n_{s,t}\Big\|^2\Big]\leq 4^8(8!)C_0\times\widetilde{C}_1(32d^2,\|\hat{b}\|_\infty+\|\check{b}\|_\infty)^{1/2}(|r-r'|+|u-u'|).$$

Finally, by the Girsanov theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sup_{n\geq 1}\|X^n_{s,t}\|^2_{L^2(\Omega,\mathbb{P};\mathbb{R}^d)}&=\sup_{n\geq 1}\mathbb{E}\left[|X^n_{s,t}|^2\right]=\sup_{n\geq 1}\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1\int_0^1 b_n(s_1,t_1,W_{s_1,t_1})\cdot \mathrm{d}W_{s_1,t_1}\Big)|W_{s,t}|^2\Big]\\&\leq \sup_{n\geq 1}\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1\int_0^1 b_n(s_1,t_1,W_{s_1,t_1})\cdot \mathrm{d}W_{s_1,t_1}\Big)^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[|W_{s,t}|^4\Big]^{\frac{1}{2}}\leq C_0\sqrt{3d},\end{aligned}$$

where C_0 is given by (3.13).

The proof is completed by taking $C_1=C_0\max\{\sqrt{3d},4^8(8!)\widetilde{C}_1(32d^2,\|\hat{b}\|_\infty+\|\check{b}\|_\infty)\}$. □

For $q\geq 1$, let us consider the following space $L^q(\mathbb{R}^d;\mathfrak{p}(x)\mathrm{d}x)$ defined by

$$L^q(\mathbb{R}^d;\mathfrak{p}(x)\mathrm{d}x)=\Big\{h:\mathbb{R}^d\to\mathbb{R}^d \text{ measurable and such that } \int_{\mathbb{R}^d}|h(x)|^q\mathfrak{p}(x)\mathrm{d}x<\infty\Big\}, \tag{3.15}$$

where the weight function $\mathfrak{p}(x)$ is defined by

$$\mathfrak{p}(x)=e^{\frac{-|x|^2}{2st}},\ x\in\mathbb{R}^d.$$

Theorem 3.9. *Let b_n be defined as before and let $(X^{\xi,n})_{n\geq 1}$ be the sequence of corresponding strong solutions to the SDE (1.3). Then for any fixed $s,t\in[0,1]$, $(X^{\xi,n}_{s,t})_{n\geq 1}$ converges strongly in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$ to $X^\xi_{s,t}$.*

In order to prove the above theorem we need the subsequent result.

Lemma 3.10. *Let $(X^{\xi,n})_{n\geq 1}$ be the sequence of corresponding strong solutions as given before. Then for every $s,t\in[0,1]$ and function $h\in L^4(\mathbb{R}^d;\mathfrak{p}(x)\mathrm{d}x)$, it holds that the sequence $(h(X^{\xi,n}_{s,t}))_{n\geq 1}$ converges weakly in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$ to $h(X^\xi_{s,t})$.*

Proof of Theorem 3.9: Using Theorem 3.5, we know that for each s,t, there exists a subsequence $(X^{\xi,n_k}_{s,t})_{k\geq 1}$ that converges strongly in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$. Set $h(x)=x$, $x\in\mathbb{R}^d$ and use Lemma 3.10 to obtain that $(X^{\xi,n_k}_{s,t})_{n\geq 1}$ converges weakly to $X^\xi_{s,t}$ in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$. Thanks to the uniqueness of the limit, there exists a subsequence n_k such that $(X^{\xi,n_k}_{s,t})_{n\geq 1}$ converges strongly to $X^\xi_{s,t}$ in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$. The convergence then holds for the entire sequence by uniqueness of the limit. To see this, suppose by contradiction that for some s,t, there exist $\epsilon>0$ and a subsequence $n_l, l\geq 0$ such that

$$\|X^{\xi,n_l}_{s,t}-X^\xi_{s,t}\|_{L^2(\Omega,\mathbb{P};\mathbb{R}^d)}\geq\epsilon.$$

We also know from the compactness criteria that there exists a further subsequence $n_m, m\geq 0$ of $n_l, l\geq 0$ such that

$$X^{\xi,n_{n_m}}_{s,t}\to\tilde{X}_{s,t} \text{ in } L^2(\Omega,\mathbb{P};\mathbb{R}^d) \text{ as } m\to\infty.$$

However, $(X^{\xi,n_k}_{s,t})_{n\geq 1}\to X^\xi_{s,t}$ as $k\to\infty$ weakly in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$, and hence by the uniqueness of the limit, we obtain

$$\tilde{X}_{s,t}=X^\xi_{s,t}.$$

Since

$$\|X^{\xi,n_{n_m}}_{s,t}-X^\xi_{s,t}\|_{L^2(\Omega,\mathbb{P};\mathbb{R}^d)}\geq\epsilon,$$

we obtain a contradiction. □

Proof of Theorem 3.4: We know from Theorem 3.9 that $(X^{\xi,n}_{s,t})_{n\geq 1}$ converges strongly in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$ to $X^{\xi}_{s,t}$ and from (3.5) in Lemma 3.6 that $(D_{r,u}X^{\xi,n}_{s,t})_{n\geq 1}$ is bounded in the $L^2([0,1]^2\times\Omega, \mathrm{d}s\times\mathrm{d}t\times\mathbb{P};\mathbb{R}^{d\times d})$-norm uniformly in n. Thefore, using Nualart (2006, Lemma 1.2.3), we also have that the limit $X^{\xi}_{s,t}$ is Malliavin differentiable. □

Proof of Lemma 3.10: Let us first noticing that the space

$$\Big\{\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big) : \varphi\in C_{2,b}([0,1]^2,\mathbb{R}^d)\Big\}$$

is a dense subspace of $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$. Here $C_{2,b}([0,1],\mathbb{R}^d)$ is the space of bounded vector functions φ such that each component φ^i has a second partial derivative $\frac{\partial^2\varphi^i_{s_1,t_1}}{\partial s_1\partial t_1}$ of bounded variation with values in $\mathbb{R}$. Hence, it suffices to show that for every i,

$$\mathbb{E}\Big[h_i(X^{\xi,n}_{s,t})\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big] \longrightarrow \mathbb{E}\Big[h_i(X^{\xi}_{s,t})\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big] \text{ as } n\to\infty.$$

Since Ω is a Wiener space, then, as in Kitagawa (1951, proof Lemma 2) or Yeh (1963, proof of Theorem 2), one can show a multidimensional analog of the Cameron-Martin translation theorem. Precisely for every $g:\mathbb{R}^d\to\mathbb{R}$ measurable, one has

$$\mathbb{E}\Big[g(X^{\xi}_{s,t})\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big] = \int_\Omega g(X^{\xi}_{s,t}(\omega+\varphi))\mathrm{d}\mathbb{P}(\omega). \tag{3.16}$$

Let $\varphi\in C_{2,b}([0,1]^2,\mathbb{R}^d)$. For every n, the process $\tilde{X}^{\xi,n}$ given by $\tilde{X}^{\xi,n}_{s,t}(\omega) := X^{\xi,n}_{s,t}(\omega+\varphi)$ solves the SDE

$$\mathrm{d}\tilde{X}^{\xi,n}_{s,t} = \Big(b_n(t,\tilde{X}^{\xi,n}_{s,t}) + \frac{\partial^2\varphi_{s,t}}{\partial s\partial t}\Big)\mathrm{d}s\mathrm{d}t + \mathrm{d}W_{s,t}. \tag{3.17}$$

Since X^ξ is also the solution to the SDE it holds that $\tilde{X}^\xi(\omega) := X^\xi(\omega+\varphi)$ satisfies

$$\mathrm{d}\tilde{X}^{\xi}_{s,t} = \Big(b(t,\tilde{X}^{\xi}_{s,t}) + \frac{\partial^2\varphi_{s,t}}{\partial s\partial t}\Big)\mathrm{d}s\mathrm{d}t + \mathrm{d}W_{s,t}, \quad \mathbb{P}\text{-a.s.} \tag{3.18}$$

Applying (3.16) and the Girsanov theorem, we have

$$\begin{aligned}
&\mathbb{E}\Big[h_i(X^{\xi,n}_{s,t})\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big) - h_i(X^{\xi}_{s,t})\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big]\\
=&\mathbb{E}\Big[\Big(h_i(X^{\xi,n}_{s,t}) - h_i(X^{\xi}_{s,t})\Big)\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big]\\
=&\mathbb{E}\Big[h_i(\xi+W_{s,t})\Big\{\mathcal{E}\Big(\int_0^1\int_0^1 \Big\{b_n(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big\}\cdot \mathrm{d}W_{s_1,t_1}\Big)\\
&\quad - \mathcal{E}\Big(\int_0^1\int_0^1 \Big\{b(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big\}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big\}\Big].
\end{aligned} \tag{3.19}$$

Using the fact that $|e^a - e^b| \leq |e^a + e^b||a-b|$, the Hölder inequality and Burkholder-Davis-Gundy inequality, we get

$$
\begin{aligned}
&\mathbb{E}\Big[h_i(X^{\xi,n}_{s,t})\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big) - h_i(X^{\xi}_{s,t})\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big]\\
\leq& C\mathbb{E}\Big[h_i(x+W_{s,t})^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big(\mathcal{E}\Big(\int_0^1\int_0^1 \Big\{b_n(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big\}\cdot \mathrm{d}W_{s_1,t_1}\Big)\\
&+\mathcal{E}\Big(\int_0^1\int_0^1 \Big\{b(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big\}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big)^4\Big]^{\frac{1}{4}}\\
&\times\Big\{\mathbb{E}\Big[\Big(\int_0^1\int_0^1 \Big(b_n(s_1,t_1,\xi+W_{s_1,t_1}) - b(s_1,t_1,\xi+W_{s_1,t_1})\Big)\cdot \mathrm{d}W_{s_1,t_1}\Big)^4\Big]\\
&+\mathbb{E}\Big[\Big(\frac{1}{2}\int_0^1\int_0^1\Big(b_n(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big)^2 - \Big(b(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big)^2\mathrm{d}s_1\mathrm{d}t_1\Big)^4\Big]\Big\}^{\frac{1}{4}}\\
\leq& C\mathbb{E}\Big[h_i(x+W_{s,t})^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big(\mathcal{E}\Big(\int_0^1\int_0^1 \Big\{b_n(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big\}\cdot \mathrm{d}W_{s_1,t_1}\Big)\\
&+\mathcal{E}\Big(\int_0^1\int_0^1 \Big\{b(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big\}\cdot \mathrm{d}W_{s_1,t_1}\Big)\Big)^4\Big]^{\frac{1}{4}}\\
&\times\Big\{\int_0^1\int_0^1 \mathbb{E}\Big[\Big|b_n(r,\xi+W_{s_1,t_1}) - b(s_1,t_1,\xi+W_{s_1,t_1})\Big|^4\Big]\mathrm{d}s_1\mathrm{d}t_1\\
&+\frac{1}{16}\int_0^1\int_0^1\mathbb{E}\Big[\Big|\Big(b_n(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big)^2 - \Big(b(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big)^2\Big|^4\Big]\mathrm{d}s_1\mathrm{d}t_1\Big\}^{\frac{1}{4}}\\
&= I_1\times I_{2,n}\times(I_{3,n}+I_{4,n}). \qquad (3.20)
\end{aligned}
$$

I_1 is finite since $h \in L^4(\mathbb{R};\mathfrak{p}(x)\mathrm{d}x)$. Next observe that

$$
\begin{aligned}
&\mathcal{E}\Big(\int_0^1\int_0^1 \Big\{b_n(s_1,t_1,\xi+W_{s_1,t_1}) + \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\Big\}\cdot \mathrm{d}W_{s_1,t_1}\Big)\\
=&\mathcal{E}\Big(\int_0^1\int_0^1 b_n(s_1,t_1,\xi+W_{s_1,t_1})\cdot \mathrm{d}W_{s_1,t_1}\Big)\mathcal{E}\Big(\int_0^1\int_0^1 \frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\cdot \mathrm{d}W_{s_1,t_1}\Big)\\
&\times\exp\Big(\int_0^1\int_0^1 b_n(s_1,t_1,\xi+W_{s_1,t_1})\frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}\mathrm{d}s_1\mathrm{d}t_1\Big).
\end{aligned}
$$

Using the boundedness of $\frac{\partial^2\varphi_{s_1,t_1}}{\partial s_1\partial t_1}$ and the uniform boundedness of b_n, it follows that $I_{2,n}$ is bounded. Using the dominated convergence theorem, we get that $I_{3,n}$ and $I_{4,n}$ converge to 0 as n goes to infinity. Let us for example consider the term $I_{3,n}$. Using the density of the Brownian sheet for every $q \geq 1$ it holds:

$$
\begin{aligned}
\mathbb{E}\Big[\Big|b_n(s,t,\xi+W_{s,t}) - b(s,t,\xi+W_{s,t})\Big|^q\Big] =& \frac{1}{\sqrt{2\pi st}}\int_{\mathbb{R}}|b_n(s,t,\xi+z) - b(s,t,\xi+z)|^q e^{\frac{-|z|^2}{2st}}\mathrm{d}z\\
=& \frac{1}{\sqrt{2\pi st}}\int_{\mathbb{R}}|b_n(s,t,z) - b(s,t,z)|^q e^{\frac{-|z-\xi|^2}{2st}}\mathrm{d}z\\
=& \frac{1}{\sqrt{2\pi st}}\int_{\mathbb{R}}|b_n(s,t,z) - b(s,t,z)|^q e^{\frac{-|z-2\xi|^2}{4st}}e^{\frac{-|z|^2}{4st}}e^{\frac{|\xi|^2}{4st}}\mathrm{d}z\\
\leq& \frac{e^{\frac{|\xi|^2}{4st}}}{\sqrt{2\pi st}}\int_{\mathbb{R}}|b_n(s,t,z) - b(s,t,z)|^q e^{\frac{-|z|^2}{4st}}\mathrm{d}z.
\end{aligned}
$$

Thus the result follows by the dominated convergence theorem. □

3.3. *Malliavin regularity under linear growth condition.* As in the previous section, we approximate the drift coefficient $b = \hat{b} - \check{b}$ by a sequence of functions $b_n := \hat{b}_n - \check{b}_n, n \geq 1$, where $\hat{b}_n = (\hat{b}_{1,n}, \ldots, \hat{b}_{d,n})$, $\check{b}_n = (\check{b}_{1,n}, \ldots, \check{b}_{d,n})$, $(\hat{b}_{j,n})_{n\geq 1}$ and $(\check{b}_{j,n})_{n\geq 1}$ are smooth, componentwise nondecreasing and bounded functions satisfying:

- There exists $\tilde{M} > 0$ such that $\|\hat{b}_{j,n}(s,t,x)\| \leq \tilde{M}(1+|x|)$ and $\|\hat{b}_{j,n}(s,t,x)\| \leq \tilde{M}(1+|x|)$ for all $n \geq 1$ and $(s,t,x) \in \Gamma_0 \times \mathbb{R}^d$,
- $(\hat{b}_n)_{n\geq 1}$ (respctively, $(\check{b}_n)_{n\geq 1}$) converges to $\hat{b}$ (respctively, $\check{b}$) in $(s,t,x) \in \Gamma_0 \times \mathbb{R}^d$ $\mathrm{d}s \times \mathrm{d}t \times \mathrm{d}x$-a.e.

One verifies that for such smooth drift coefficients, the corresponding SDEs have a unique strong solution denoted by $X^{\xi,n}$. We show that for s,t small enough, the sequence $(X^{\xi,n}_{s,t})_{n\geq 1}$ is relatively compact in $L^2(\Omega, \mathbb{P}; \mathbb{R}^d)$. The proofs of the next two results are similar to that of Lemmas 3.8 and 3.6 and are found in Appendix.

Lemma 3.11. *There exist $\widetilde{C}_2 > 0$ and $\zeta > 0$ such that, for any $i,j \in \{1, \cdots, d\}$, any $0 < r < s \leq 1$, any $0 < u < t \leq 1$ and any $k \in \mathbb{R}_+$,*

$$\mathbb{E}\Big[\exp\Big(\frac{\zeta}{\delta(r,s)\delta(u,t)} \int_r^s \int_u^t \partial_i \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big] \leq \widetilde{C}_2, \tag{3.21}$$

and

$$\mathbb{E}\Big[\exp\Big(\frac{\zeta}{\delta(r,s)\delta(u,t)} \int_r^s \int_u^t \partial_i \check{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big)\Big] \leq \widetilde{C}_2, \tag{3.22}$$

where $\delta(r,s) = \sqrt{s-r}$ and $\delta(u,t) = \sqrt{t-u}$.

Lemma 3.12. *There exist $C_2 > 0$ and $\tau \in (0,1)$ such that for every $(s,t) \in [0,\tau]$, the sequence $(X^{\xi,n}_{s,t})_{n\geq 1}$ satisfies*

$$\sup_{n\geq 1} \|X^n_{s,t}\|^2_{L^2(\Omega,\mathbb{P};\mathbb{R}^d)} \leq C_2 \tag{3.23}$$

and

$$\sup_{n\geq 1} \sup_{\substack{0\leq r\leq s \\ 0\leq u\leq t}} \mathbb{E}\left[\|D_{r,u}X^n_{s,t}\|^2\right] \leq C_2. \tag{3.24}$$

Moreover, for all $0 \leq r', r \leq s \leq \tau,\ 0 \leq u', u \leq t \leq \tau$,

$$\mathbb{E}\left[\|D_{r,u}X^n_{s,t} - D_{r',u'}X^n_{s,t}\|^2\right] \leq C_2(|r-r'| + |u-u'|). \tag{3.25}$$

Here is the main result of this section which is a consequence of Lemma 3.12 and the compactness criterion provided in Corollary 3.3.

Theorem 3.13. *There exist $\tau \in (0,1)$ such that for any ξ the strong solution $\{X^\xi_{s,t},\ s,t \in [0,\tau]\}$ of the SDE (1.3) is Malliavin differentiable.*

Proof: As in the proof of Lemma 3.10, we show that $(X^{\xi,n}_{s,t})_{n\geq 1}$ converges weackly in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$ to $X^\xi_{s,t}$ for every $(s,t) \in [0,\tau]^2$. Hence, using Lemma 3.12 and the compactness criterion provided in Corollary 3.3, we deduce that $(X^{\xi,n}_{s,t})_{n\geq 1}$ converges strongly in $L^2(\Omega,\mathbb{P};\mathbb{R}^d)$ to $X^\xi_{s,t}$. Then, since $(D_{r,u}X^{\xi,n}_{s,t})_{n\geq 1}$ is bounded in the $L^2([0,1]^2 \times \Omega, \mathrm{d}s \times \mathrm{d}t \times \mathbb{P}; \mathbb{R}^{d\times d})$-norm uniformly in n (see (3.24)), it follows from Nualart (2006, Lemma 1.2.3) that the limit $X^\xi_{s,t}$ is also Malliavin differentiable. $\square$

Appendix A. Proofs of Lemmas 3.11 and 3.12

In this section we provide the proofs of Lemmas 3.11 and 3.12. Let us start with a useful estimate.

Lemma A.1. *There exist $\tau_1 \in (0,1)$ such that*

$$\sup_{n\geq 1}\mathbb{E}\Big[\mathcal{E}\Big(\int_0^{\tau_1}\int_0^{\tau_1} b_n(s,t,W_{s,t})\cdot \mathrm{d}W_{s,t}\Big)^2\Big] < \infty. \tag{A.1}$$

Proof: By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{E}\Big(\int_0^{\tau_1}\int_0^{\tau_1} b_n(s,t,W_{s,t})\cdot \mathrm{d}W_{s,t}\Big)^2\Big]\\
=&\mathbb{E}\Big[\exp\Big(2\int_0^{\tau_1}\int_0^{\tau_1} b_n(s,t,W_{s,t})\cdot \mathrm{d}W_{s,t} - \int_0^{\tau_1}\int_0^{\tau_1}|b_n(s,t,W_{s,t})|^2\mathrm{d}s\mathrm{d}t\Big)\Big]\\
=&\mathbb{E}\Big[\exp\Big(2\int_0^{\tau_1}\int_0^{\tau_1} b_n(s,t,W_{s,t})\cdot \mathrm{d}W_{s,t} - 4\int_0^{\tau_1}\int_0^{\tau_1}|b_n(s,t,W_{s,t})|^2\mathrm{d}s\mathrm{d}t\\
&\qquad\qquad + 3\int_0^{\tau_1}\int_0^{\tau_1}|b_n(s,t,W_{s,t})|^2\mathrm{d}s\mathrm{d}t\Big)\Big]\\
\leq&\mathbb{E}\Big[\mathcal{E}\Big(4\int_0^{\tau_1}\int_0^{\tau_1} b_n(s,t,W_{s,t})\cdot \mathrm{d}W_{s,t}\Big)\Big]^{\frac{1}{2}}\mathbb{E}\Big[\exp\Big(6\int_0^{\tau_1}\int_0^{\tau_1}|b_n(s,t,W_{s,t})|^2\mathrm{d}s\mathrm{d}t\Big)\Big]^{\frac{1}{2}}.
\end{aligned}$$

Since b_n is bounded, we have (see e.g. Nualart and Pardoux (1994, Proposition 1.6))

$$\mathbb{E}\Big[\mathcal{E}\Big(4\int_0^{\tau_1}\int_0^{\tau_1} b_n(s,t,W_{s,t})\cdot \mathrm{d}W_{s,t}\Big)\Big] = 1, \quad \forall\, n\geq 1,\ \tau_1 > 0.$$

Moreover, by Jensen inequality,

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(6\int_0^{\tau_1}\int_0^{\tau_1}|b_n(s,t,W_{s,t})|^2\mathrm{d}s\mathrm{d}t\Big)\Big] \leq&\frac{1}{\tau_1^2}\int_0^{\tau_1}\int_0^{\tau_1}\mathbb{E}\Big[\exp\big(6\tau_1^2|b_n(s,t,W_{s,t})|^2\big)\Big]\mathrm{d}s\mathrm{d}t\\
\leq&\frac{1}{\tau_1^2}\int_0^{\tau_1}\int_0^{\tau_1}\mathbb{E}\Big[\exp\Big(24\tau_1^2\tilde{M}d(1+|W_{s,t}|^2)\Big)\Big]\mathrm{d}s\mathrm{d}t\\
=&\frac{\exp\big(24\tau_1^2\tilde{M}d\big)}{\tau_1^2}\int_0^{\tau_1}\int_0^{\tau_1}\mathbb{E}\Big[\exp(24\tau_1^2\tilde{M}d\,st|G|^2)\Big]\mathrm{d}s\mathrm{d}t\\
\leq&\exp\big(24\tau_1^2\tilde{M}d\big)\,\mathbb{E}\Big[\exp(24\tau_1^4\tilde{M}d|G|^2)\Big],
\end{aligned}$$

where $G = (G_1,\dots,G_d)$ is a mean-zero random vector with identity covariance matrix.

The proof is completed since $\mathbb{E}\Big[\exp(12\tau_1^4\tilde{M}d|G|^2)\Big] < \infty$ for τ_1 small enough. □

Remark A.2. When the drift b is bounded, the functions b_n, $n\geq 1$ are uniformly bounded and, as a consequence,

$$C_0 = \sup_{n\geq 1}\mathbb{E}\Big[\mathcal{E}\Big(\int_0^1\int_0^1 b_n(s,t,W_{s,t})\cdot \mathrm{d}W_{s,t}\Big)^2\Big]^{\frac{1}{2}} < \infty.$$

Proof of Lemma 3.11. We only prove (3.21) since the proof of (3.22) follows the same lines. We deduce from the local time-space integration formula (2.1) that

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\frac{\zeta}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_u^t \partial_i\hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\\
\leq&\frac{1}{3}\Big\{\mathbb{E}\Big[\exp\Big(\frac{3\zeta}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_u^t \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{s_1}\mathrm{d}s_1\Big|\Big)\Big]\\
&+\mathbb{E}\Big[\exp\Big(\frac{3\zeta}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_{1-t}^{1-u} \hat{b}_{j,n}(s_1,t_1,W_{s_1,1-t_1})\frac{\mathrm{d}_{t_1}B^{(i)}_{s_1,t_1}}{s_1}\mathrm{d}s_1\Big|\Big)\Big]\\
&+\mathbb{E}\Big[\exp\Big(\frac{3\zeta}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_{1-t}^{1-u} \hat{b}_{j,n}(s_1,t_1,W_{s_1,1-t_1})\frac{W^{(i)}_{s_1,1-t_1}}{s_1(1-t_1)}\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\Big\}=\frac{1}{3}(I_1+I_2+I_3).
\end{aligned}\tag{A.2}$$

By Jensen inequality,

$$\begin{aligned}
I_1=&\mathbb{E}\Big[\exp\Big(\frac{3\zeta}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_u^t \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{s_1}\mathrm{d}s_1\Big|\Big)\Big]\\
\leq&\int_r^s\mathbb{E}\Big[\exp\Big(\frac{6\zeta(\sqrt{s}-\sqrt{r})}{\delta(r,s)\delta(u,t)}\Big|\int_u^t \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{\sqrt{s_1}}\Big|\Big)\Big]\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}\\
\leq&\int_r^s\mathbb{E}\Big[\exp\Big(\frac{6\zeta}{\delta(u,t)}\Big|\int_u^t \hat{b}_{j,n}(s_1,t_1,W_{s_1,t_1})\frac{\mathrm{d}_{t_1}W^{(i)}_{s_1,t_1}}{\sqrt{s_1}}\Big|\Big)\Big]\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}
\end{aligned}$$

Since, for every $s_1\in[r,s]$,

$$\Big(Y_{s_1,t_1}:=\int_u^{t_1}\hat{b}_{j,n}(s_1,t_2,W_{s_1,t_2})\frac{\mathrm{d}_{t_2}W^{(i)}_{s_1,t_2}}{\sqrt{s_1}},u\leq t_1\leq t\Big)$$

is a square integrable martingale and similar reasoning as before gives

$$\mathbb{E}[|Y_{s_1,t}|^m]\leq 2^m c_1^{2m}\tilde{M}^m m^m\delta(u,t)^m.$$

From this and the exponential expansion formula, we get

$$I_1\leq\sum_{m=0}^{\infty}\frac{2^{4m}\zeta^m c_1^{2m}m^m\tilde{M}^m}{m!}:=\widetilde{C}_{1,1},$$

which is finite if $\zeta<e/2^4c_1^2\tilde{M}$ (by ratio test). Similarly, we also have

$$I_2\leq\widetilde{C}_{1,1}.$$

To estimate I_3 we apply Jensen inequality again and we obtain

$$\begin{aligned}
I_3=&\mathbb{E}\Big[\exp\Big(\frac{3\zeta}{\delta(r,s)\delta(u,t)}\Big|\int_r^s\int_{1-t}^{1-u} \hat{b}_{j,n}(s_1,t_1,W_{s_1,1-t_1})\frac{W^{(i)}_{s_1,1-t_1}}{s_1(1-t_1)}\mathrm{d}t_1\mathrm{d}s_1\Big|\Big)\Big]\\
\leq&\int_r^s\int_{1-t}^{1-u}\mathbb{E}\Big[\exp\Big(\frac{12\zeta(\sqrt{s}-\sqrt{r})(\sqrt{1-u}-\sqrt{1-t})}{\delta(r,s)\delta(u,t)}|\hat{b}_{j,n}(s_1,t_1,W_{s_1,1-t_1})|\Big|\frac{W^{(i)}_{s_1,1-t_1}}{\sqrt{s_1}\sqrt{1-t_1}}\Big|\Big)\Big]\\
&\times\frac{\mathrm{d}t_1}{2\sqrt{1-t_1}(\sqrt{1-u}-\sqrt{1-t})}\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}\\
\leq&\int_r^s\int_{1-t}^{1-u}\mathbb{E}\Big[\exp\Big\{24\tilde{M}\zeta\Big(1+\frac{|W^{(i)}_{s_1,1-t_1}|^2}{s_1(1-t_1)}\Big)\Big\}\Big]\frac{\mathrm{d}t_1}{2\sqrt{1-t_1}(\sqrt{1-u}-\sqrt{1-t})}\frac{\mathrm{d}s_1}{2\sqrt{s_1}(\sqrt{s}-\sqrt{r})}\\
:=&\widetilde{C}_{1,2},
\end{aligned}$$

with $\widetilde{C}_{1,2}$ finite provided that $24\tilde{M}\zeta < 1/2$.
The proof is completed by choosing $\zeta = 1/2^6 c_1^2 \tilde{M}$ and $\widetilde{C}_1 = (2\widetilde{C}_{1,1} + \widetilde{C}_{1,2})/3$. □

Proof of Lemma 3.12. Fix $\tau \in (0, \min\{\tau_1, \zeta/32d^2\})$, where τ_1 is the constant in Lemma A.1 and ζ is the constant in Lemma 3.11. We deduce from Theorem B.1 and the linear growth condition on the drift b_n that $\mathbb{E}[\|X^n_{s,t}\|^2] \le C_{2,1}$ for all $(s,t) \in [0,\tau]^2$ and $n \ge 1$, where $C_{2,1}$ does not depend of (s,t) and n. Let $0 \le r' \le r \le s \le \tau$ and $0 \le u' \le u \le t \le \tau$. Since

$$\tau \le \tau_1 \text{ and } \delta(r', s)\delta(u', t) \le \tau \le \frac{\zeta}{32d^2},$$

then, using similar computations as in the proof of Lemma 3.6, one can deduce from Lemma A.1, Girsanov theorem and Hölder inequality that

$$\sup_{n\ge1} \sup_{\substack{0\le r\le s\\ 0\le u\le t}} \mathbb{E}\Big[\|D_{r,u}X^n_{s,t}\|^2\Big] := C_{2,2} < \infty$$

and

$$\mathbb{E}\Big[\|D_{r,u}X^n_{s,t} - D_{r',u'}X^n_{s,t}\|^2\Big] \le C_{2,3}(|r - r'| + |u - u'|)$$

for some positive constant $\widetilde{C}_2$ independent of n.

The proof is completed by taking $C_2 = \max\{C_{2,1}, C_{2,2}, C_{2,3}\}$. □

Remark A.3. It is worth noting that if the drift is in addition the difference of two convex or concave functions, then the solution to the equation (1.3) is twice Malliavin differentiable. Indeed, $b = \hat{b} - \check{b}$ is Lipschitz with the second order weak derivatives of $\hat{b}, \check{b}$ positive or negative.

Appendix B. A Gronwall type inequality for functions of two variables

The next result which is originally due to Wendroff, extends Gronwall inequality to functions of two variables (see e.g. Qin (2016, Theorem 5.1.1)).

Theorem B.1. *Let $g(s,t)$, $a(s,t)$, $k(s,t)$ be non-negative continuous functions for all $s \ge s_0$, $t \ge t_0$, and let $a(s,t)$ be non-decreasing in each of the variables for all $s \ge s_0$, $t \ge t_0$. Suppose that for all $s \ge s_0$, $t \ge t_0$,*

$$g(s,t) \le a(s,t) + \int_{s_0}^{s}\int_{t_0}^{t} k(s_1,t_1) g(s_1,t_1)\, \mathrm{d}t_1 \mathrm{d}s_1.$$

Then for all $s \ge s_0$, $t \ge t_0$,

$$g(s,t) \le a(s,t) \exp\Big(\int_{s_0}^{s}\int_{t_0}^{t} k(s_1,t_1)\, \mathrm{d}t_1 \mathrm{d}s_1\Big).$$

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