ALEA, Lat. Am. J. Probab. Math. Stat. 20, 1587–1614 (2023)
DOI: 10.30757/ALEA.v20-60

Uniformly and strongly consistent estimation for the random Hurst function of a multifractional process

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Abstract. Multifractional processes are extensions of Fractional Brownian Motion obtained by replacing its constant Hurst parameter by a deterministic or a random function $H(\cdot)$, called the Hurst function, which allows one to prescribe the local roughness of sample paths at each point. For that reason statistical estimation of $H(\cdot)$ is an important issue. Many articles have dealt with this issue in the case where $H(\cdot)$ is deterministic. However, statistical estimation of $H(\cdot)$ when it is random remains an open problem. The main goal of our present article is to propose, under a weak local Hölder regularity condition on $H(\cdot)$, a solution for this problem in the framework of Moving Average Multifractional Process with Random Exponent (MAMPRE), denoted by X. From the data consisting in a discrete realization of X on the interval $[0,1]$, we construct a continuous piecewise linear random function which almost surely converges to $H(\cdot)$ for the uniform norm, when the size of the discretization mesh goes to zero; we also provide an almost sure estimate of the uniform rate of convergence and we explain how it can be optimized. Such kind of strong consistency result in uniform norm is rather unusual in literature on statistical estimation of functions.

1. Introduction and background

All the stochastic processes considered in the present article are defined on the same probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. It is endowed with a complete filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$, and $\{B(s)\}_{s\in\mathbb{R}}$ is a standard Brownian Motion with respect to this filtration. Fractional Brownian Motion (FBM) of constant Hurst parameter $H \in (0,1)$, denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$, is a very classical centred self-similar Gaussian process with stationary increments, whose (sample) paths are, on each compact interval, Hölder continuous functions of any order strictly less than H. This process was first introduced by Kolmogorov (1940) related with his studies on turbulence, and was later made known to a large

Received by the editors March 25th, 2023; accepted November 13th, 2023.
2010 *Mathematics Subject Classification.* 60G22, 62G05, 60H07.
Key words and phrases. Statistical estimation of functions, Fractional Brownian Motion, varying Hurst parameter, quadratic variations, laws of large numbers.
This work has been partially supported by the Labex CEMPI (ANR-11-LABX-0007-01), the GDR 3475 (Analyse Multifractale et Autosimilarité) and the Australian Research Council's Discovery Projects funding scheme (project number DP220101680).

audience thanks to the seminal article Mandelbrot and Van Ness (1968). It can be defined, for all $t \in \mathbb{R}$, through the non-anticipative moving average Wiener integral:

$$B_H(t) := \int_{-\infty}^{t} \Big((t-s)^{H-1/2} - (-s)_+^{H-1/2} \Big)\, dB(s)\,, \tag{1.1}$$

with the usual convention that, for each $(y,b) \in \mathbb{R}^2$, one has

$$y_+^b := \begin{cases} y^b, \text{if } y > 0, \\ 0, \text{else.} \end{cases} \tag{1.2}$$

One refers to the two well-known books Embrechts and Maejima (2002); Samorodnitsky and Taqqu (1994) for a detailed presentation of FBM and many other related topics. FBM is a widespread model in signal processing (see e.g. Doukhan et al. (2003)). Unfortunately, in many situations, it does not fit very well to modeling of erratic real-life signals since it lacks flexibility. An important drawback of the FBM model comes from the fact that local roughness of its paths is not allowed to change from point to point. More precisely, for a generic stochastic process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ with continuous and nowhere differentiable paths, their roughness in a neighborhood of any arbitrary fixed point $\tau \in \mathbb{R}$ is usually measured through $\alpha_Y(\tau)$ and $\widetilde{\alpha}_Y(\tau)$, the pointwise Hölder exponent and local Hölder exponent of Y at τ, defined, for all $\omega \in \Omega$, as:

$$\alpha_Y(\tau,\omega) := \sup\left\{ a \in [0,1] : \limsup_{t\to\tau} \frac{|Y(t,\omega) - Y(\tau,\omega)|}{|t-\tau|^a} < +\infty \right\} \tag{1.3}$$

and

$$\widetilde{\alpha}_Y(\tau,\omega) := \sup\left\{ \widetilde{a} \in [0,1] : \limsup_{(t',t'')\to(\tau,\tau)} \frac{|Y(t',\omega) - Y(t'',\omega)|}{|t'-t''|^{\widetilde{a}}} < +\infty \right\}. \tag{1.4}$$

For a given $\omega \in \Omega$, the closer to zero are $\alpha_Y(\tau,\omega)$ and $\widetilde{\alpha}_Y(\tau,\omega)$, the rougher is the path $t \mapsto Y(t,\omega)$ in the vicinity of τ. In the case of FBM $\{B_H(t)\}_{t\in\mathbb{R}}$, path roughness remains the same everywhere because one has (see for instance Xiao (1997))

$$\mathbb{P}\big(\forall\, \tau \in \mathbb{R}\,,\, \alpha_{B_H}(\tau) = \widetilde{\alpha}_{B_H}(\tau) = H\big) = 1\,. \tag{1.5}$$

In order to overcome the latter limitation of FBM, multifractional processes have started to be constructed and studied since the mid 1990s. One refers to the recent book Ayache (2019) for a detailed presentation of such processes and their connections to wavelet methods. The paradigmatic example of them is the Classical Multifractional Brownian Motion (CMBM) which was introduced independently in the two pioneering articles Benassi et al. (1997); Peltier and Lévy Véhel (1995). CMBM is a Gaussian process with non-stationary increments and continuous paths. It is obtained by replacing the constant Hurst parameter H in a stochastic integral representation of FBM (as for instance the moving average representation (1.1)) by a deterministic continuous function $t \mapsto H(t)$, with values in an arbitrary compact interval $[\underline{H}, \overline{H}] \subset (0,1)$, which depends on the time variable t, that is the index of the process. The latter function is called the Hurst function. Under a local Hölder condition on it, the articles Benassi et al. (1997); Peltier and Lévy Véhel (1995) have shown that this deterministic function can be used for prescribing local path roughness of CMBM paths which is thus allowed to change from point to point in a deterministic way. Namely, for any point $\tau \in \mathbb{R}$ at which the local Hölder exponent $\widetilde{\alpha}_H(\tau)$ of the function $t \mapsto H(t)$ satisfies the inequality

$$H(\tau) < \widetilde{\alpha}_H(\tau)\,, \tag{1.6}$$

one has, almost surely,

$$\alpha_{\mathrm{CMBM}}(\tau) = \widetilde{\alpha}_{\mathrm{CMBM}}(\tau) = H(\tau)\,, \tag{1.7}$$

where $\alpha_{\mathrm{CMBM}}(\tau)$ and $\widetilde{\alpha}_{\mathrm{CMBM}}(\tau)$ are the pointwise Hölder exponent and the local Hölder exponent of the CMBM at τ. Even if the Gaussian CMBM is a more flexible model than FBM, it still has some limitations; a major one of them is that the two exponents $\alpha_{\mathrm{CMBM}}(\tau)$ and $\widetilde{\alpha}_{\mathrm{CMBM}}(\tau)$ are deterministic quantities since the Hurst function $t \mapsto H(t)$ itself is deterministic. The difficulty for overcoming the latter limitation of the CMBM comes from the fact that one can not replace the Hurst parameter H in (1.1), or in another stochastic integral representation of FBM, by a random variable $H(t)$ without imposing to it to be (stochastically) independent of the Brownian Motion $\{B(s)\}_{s\in\mathbb{R}}$ generating the stochastic integral. Indeed, when this very restrictive independence condition is dropped, then the stochastic integral, in which H is substituted by $H(t)$, is no longer well-defined. Therefore, the two articles Ayache and Taqqu (2005); Ayache et al. (2007) have proposed to use a random wavelet series representation of FBM, instead of a stochastic integral representation of it, in order to be allowed to make this substitution. The multifractional process with random Hurst function obtained in this way is called, in the present article, the Wavelet Multifractional Process with Random Exponent (WMPRE). It is a non-Gaussian process which generally has non-stationary increments and continuous paths. Thanks to wavelet methods, the paper Ayache and Taqqu (2005) has shown that, under the condition (1.6), the two fundamental equalities (1.7), relating $H(\cdot)$ to local path roughness, can be extended to the WMPRE. Moreover, the latter result has been significantly strengthened in the paper Ayache et al. (2007) in which it has been established that these two fundamental equalities are even valid on a universal event of probability 1 not depending on τ, and for a much more general class of multifractional processes.

It is worth mentioning that several articles (see for instance Bianchi et al. (2012, 2013); Bianchi and Pianese (2014)) have pointed out that multifractional processes with random Hurst functions are good candidates for modeling of financial time series. Indeed, they allow to replicate main stylized facts (non-Gaussianity, volatility clustering and so on) of such time series. Moreover, analysis of evolution over time of their random pointwise and local Hölder exponents can provide explanations for trading mechanisms over financial markets. For instance, at a given time one or the other of these two exponents can be viewed as a weight that investors assign to past prices when taking their trading decisions.

As we have already mentioned, there are significant difficulties in constructing and studying multifractional processes with random Hurst functions. Even if the WMPRE, constructed a long time ago in Ayache and Taqqu (2005), is a first breakthrough in this area, it is not at all clear how this process can be represented via Itô integral and how Itô calculus can be applied in its framework. In the last few years, another type of non-Gaussian multifractional process with random Hurst function having a natural representation via Itô intergral was introduced in Ayache et al. (2018). It generally has non-stationary increments and continuous paths. It is called the Moving Average Multifractional Process with Random Exponent (MAMPRE) in the present article. In contrast with the WMPRE for which the random Hurst function depends on the time variable t, in the case of the MAMPRE this function depends on the integration variable s. Indeed, the MAMPRE, denoted by $\{X(t)\}_{t\in\mathbb{R}}$, is obtained by substituting to the constant Hurst parameter H in (1.1) a stochastic process $\{H(s)\}_{s\in\mathbb{R}}$ with continuous paths, indexed by the integration variable s, which is adapted to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}}$ and satisfies

$$0 < \underline{H} \le H(s) \le \overline{H} < 1\,, \quad \text{for all } s \in \mathbb{R}, \tag{1.8}$$

for some deterministic constants $\underline{H}$ and $\overline{H}$ belonging to the open interval $(0,1)$. More formally, the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$ is defined, for all $t \in \mathbb{R}$, as the Itô integral:

$$X(t) := \int_{-\infty}^{t} \Big((t-s)^{H(s)-1/2} - (-s)_+^{H(s)-1/2} \Big)\, dB(s)\,. \tag{1.9}$$

Recently, in the article Loboda et al. (2021), under a very weak global regularity condition on paths of the process $\{H(s)\}_{s\in\mathbb{R}}$, for all $\tau \in \mathbb{R}$, the two fundamental equalities (1.7), relating $H(\cdot)$ to local path roughness, have been extended to MAMPRE. A short time later, the article Ayache and Bouly (2022) has shown that they are even valid on a universal event of probability 1 not depending on τ, as soon as paths of $\{H(s)\}_{s\in\mathbb{R}}$ are, on each compact interval, Hölder continuous functions of any arbitrarily small deterministic order $\gamma > 0$. Notice that in contrast with the condition (1.6) there is no need to impose to γ to be greater than $H(\tau)$.

As we have already emphasized, local roughness of paths of a multifractional process is governed by its deterministic or random Hurst function $H(\cdot)$. Therefore, statistical estimation of values of $H(\cdot)$ is an important issue both from a practical point of view and from a theoretical one. Many articles have dealt with this issue in the case where $H(\cdot)$ is deterministic (see e.g. Shen and Hsing (2020); Dang (2020); Lebovits and Podolskij (2017); Ayache and Hamonier (2017, 2015); Bardet and Surgailis (2013); Lopes et al. (2011); Le Guével (2013); Coeurjolly (2005, 2006); Lacaux (2004); Ayache and Lévy Véhel (2004); Benassi et al. (2000, 1998)). However, statistical estimation of $H(\cdot)$ when it is random remains an open problem. A major new difficulty in it, in comparison to the frameworks with $H(\cdot)$ being deterministic, is that little information is available on finite-dimensional distributions of multifractional process with random exponent. Another serious new difficulty in this problem comes from the fact that the randomness of the Hurst function $H(\cdot)$ makes the dependence structure of the associated multifractional process to be significantly more complex to study than those with deterministic Hurst function. The main goal of the present article is to propose a solution for this problem in the framework of the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$, defined through (1.9), under a weak local Hölder regularity condition on paths of the stochastic process $\{H(s)\}_{s\in\mathbb{R}}$.

Let us now describe in a more precise way the main contribution of the present article. Similarly to the previous literature on statistical estimation of $H(\cdot)$, we assume that, on the interval $[0,1]$, the discrete realization: $\big\{X(k/N) : k \in \{0,\ldots,N\}\big\}$ of the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$ is available for all integer N large enough; notice that our main result can be extended without great difficulty to the general case where the interval $[0,1]$ is replaced by any other compact interval with non-empty interior. Also, we suppose that, for some deterministic constants $\gamma \in (0,1]$ and $\rho \in (0,+\infty)$ paths of $\{H(s)\}_{s\in\mathbb{R}}$ satisfy, on the interval $[-1,1]$, the Hölder regularity condition of order γ:

$$\big|H(s') - H(s'')\big| \le \rho\,|s'-s''|^{\gamma}, \quad \text{for all } (s',s'') \in [-1,1]^2. \tag{1.10}$$

Then, we construct from generalized quadratic variations associated with $\big\{X(k/N)\colon k \in \{0,\ldots,N\}\big\}$ and $\big\{X(k/(QN)) : k \in \{0,\ldots,QN\}\big\}$, the integer $Q \ge 2$ being arbitrary and fixed, a continuous piecewise linear random function on $[0,1]$, denoted by $\widetilde{H}_N(\cdot)$, which provides a uniformly and strongly consistent estimator of the whole random Hurst function $H(\cdot)$ on $[0,1]$. More precisely we show that, when N goes to $+\infty$, the uniform norm $\sup_{s\in[0,1]}\big|H(s)-\widetilde{H}_N(s)\big|$ converges almost surely to zero at the same rate as a power function. It is noteworthy that such kind of strong consistency result in uniform norm is rather unusual in literature on statistical estimation of functions. Let us also emphasize that, when γ the deterministic order of Hölder regularity on $[-1,1]$ of the random Hurst function $H(\cdot)$ is assumed to be known, the uniform rate of convergence of its estimator $\widetilde{H}_N(\cdot)$ can be optimized so that it almost becomes $N^{-\frac{\gamma}{2\gamma+1}}$, which is rather reminiscent of the minimax asymptotic lower bound for the risk of estimating a deterministic smooth Hurst function in the Gaussian framework of Classical Multifractional Brownian Motion (CMBM); the latter lower bound is provided by Theorem 3.1 of the recent article Shen and Hsing (2020).

Remark 1.1. It might seem restrictive to impose that the positive constant ρ, in the Hölder regularity condition (1.10), be deterministic. In fact, thanks to a localization procedure via stopping times (see for instance Section 4.4.1 in Jacod and Protter (2012)) which is explained in the setting of MAMPRE in Section 3 of Loboda et al. (2021), our main result (Theorem 2.2) remains valid when ρ is an almost surely finite random variable. Moreover, a careful inspection of the proof of this same theorem shows that it also remains valid when the interval $[-1,1]$ in (1.10) is replaced by any other compact interval of the form $[-b,1]$, where b is a fixed arbitrarily small positive real number.

The remaining of our article is organized as follows. In Section 2, the construction via generalized quadratic variations of X of the estimator $\widetilde{H}_N(\cdot)$ is precisely explained, our main result is stated, and $\widetilde{H}_N(\cdot)$ is tested on some simulated data. In Section 3, it is shown that generalized quadratic variations of X can be simplified since some parts of them are negligible for our purpose. The goal of Section 4 is to precisely determine their asymptotic behavior when N goes to $+\infty$. At last, Section 5 is devoted to complete the proof of our main result.

2. Statement of the main result and simulations

The main lines of this section are the following. First, we define the estimator $\widetilde{H}_N(\cdot)$ of the random Hurst function $H(\cdot)$ of the MAMPRE $\{X(t)\}_{t\in\mathbb{R}}$ introduced in (1.9). Then, we state our main result and comment on it. Namely, under the hypothesis that γ is known, we explain how the almost sure uniform rate of convergence for the estimator $\widetilde{H}_N(\cdot)$, provided by our main result, can be optimized so that it almost becomes $N^{-\frac{\gamma}{2\gamma+1}}$; recall that each realization of the random Hurst function $H(\cdot)$ is assumed to satisfy on the interval $[-1,1]$ a Hölder regularity condition of deterministic order $\gamma\in(0,1]$ (see (1.10)), also recall that the rate $N^{-\frac{\gamma}{2\gamma+1}}$ is rather reminiscent of the minimax asymptotic lower bound for the risk of estimating a deterministic smooth Hurst function in the Gaussian framework of Classical Multifractional Brownian Motion (CMBM) (see Theorem 3.1 in the recent article Shen and Hsing (2020)). Finally, at the end of the section we provide some simulations for demonstrating the performance of the estimator $\widetilde{H}_N(\cdot)$.

The estimator $\widetilde{H}_N(\cdot)$ of the random Hurst function of the MAMPRE, that we will soon construct, will be defined in a rather similar way as the estimator of the deterministic Hurst function of the Linear Multifractional Stable Motion (LMSM) studied in Ayache and Hamonier (2017). Yet, since there are big differences between MAMPRE and LMSM, the strategy we will employ for proving the main result of the present article will to a large extent be different from the one previously used in Ayache and Hamonier (2017). We mention in passing that the big differences between MAMPRE and LMSM stem, among others, from the fact that MAMPRE is defined via Itô integral and has a random Hurst function depending on the integration variable s, while LMSM is defined via stable stochastic integral and has a deterministic Hurst function depending on the time variable t (that is the same index as the LMSM itself). Thus, little information is available on finite-dimensional distributions of MAMPRE, while it is known that those of LMSM are classical stable distributions whose characteristic functions have "nice" explicit forms.

In order to precisely define the estimator $\widetilde{H}_N(\cdot)$, first we need to introduce several notations. From now till the end of our article, the integer $L\geq 2$ is arbitrary and fixed. The deterministic coefficients $a_0,a_1,\ldots,a_L$ are defined, for every $l\in\{0,\ldots,L\}$, as:

$$a_l := (-1)^{L-l}\binom{L}{l} := (-1)^{L-l}\frac{L!}{l!\,(L-l)!}\,. \tag{2.1}$$

Observe that one can derive from (2.1) that the finite sequence of real numbers $(a_l)_{0\leq l\leq L}$ has exactly L vanishing first moments; that is, for all $q\in\{0,\ldots,L-1\}$, one has

$$\sum_{l=0}^{L} l^q a_l = 0 \quad \text{(with the convention } 0^0=1\text{), while} \quad \sum_{l=0}^{L} l^L a_l \neq 0. \tag{2.2}$$

For each integer N large enough, the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]}$ for paths of the stochastic process $\{H(s)\}_{s\in[0,1]}$ is built from generalized quadratic variations of the MAMPRE X (see (1.9)) associated with its generalized increments $d_{N,k}$, $0\leq k\leq N-L$, defined, for all $k\in\{0,\ldots,N-L\}$, as:

$$d_{N,k} := \sum_{l=0}^{L} a_l X\big((k+l)/N\big). \tag{2.3}$$

For any compact interval, with non-empty interior, $I \subseteq [0,1]$, the generalized quadratic variation of X on I is denoted by $V_N(I)$ and defined as the empirical mean:

$$V_N(I) := |\nu_N(I)|^{-1} \sum_{k\in\nu_N(I)} |d_{N,k}|^2\,, \tag{2.4}$$

where the finite set of indices

$$\nu_N(I) := \big\{k \in \{0,\dots,N-L\} : k/N \in I\big\}, \tag{2.5}$$

and $|\nu_N(I)|$ is the cardinality of $\nu_N(I)$. Observe that $|\nu_N(I)|$ does not really depend on the position of I but mainly on $\lambda(I)$, the Lebesgue measure of this interval. Indeed, it can easily be seen that one has

$$N\lambda(I) - L - 1 < |\nu_N(I)| \le N\lambda(I) + 1; \tag{2.6}$$

thus, as soon as $N \ge 2(L+1)\lambda(I)^{-1}$, one gets that

$$N\lambda(I)/2 < |\nu_N(I)| \le 7N\lambda(I)/6. \tag{2.7}$$

We mention in passing that a long time ago (see for instance the two seminal papers Taqqu (1975); Dobrushin and Major (1979)), it was shown that standardized quadratic variations of Fractional Brownian Motion (FBM) with Hurst parameter larger than 3/4 fail to have asymptotic normal distributions. Later, in order to recover asymptotic normality of standardized quadratic variations of such FBM and related Gaussian processes with stationary increments, the article Istas and Lang (1997) proposed to replace the usual increments in them by generalized increments of the type (2.3). In our present framework, defining the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]}$ via generalized increments, instead of usual increments, will allow us to derive its strong consistency for the uniform norm, and to significantly improve its almost sure rate of convergence for this norm.

Before giving a formal definition of the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]}$, let us explain, in a few sentences, its way of construction. Let $(\theta_N)_N$ be an arbitrary sequence of positive real numbers in the interval $(0,1/2]$ which converges to zero at a convenient rate (see (2.10) and (2.11)), when N goes to $+\infty$. For all N, the integer $\lfloor\theta_N^{-1}\rfloor \ge 2$ denotes the integer part of θ_N^{-1}. We split the interval $[0,1]$ into a finite sequence $(\mathcal{I}_{N,n})_{0\le n<\lfloor\theta_N^{-1}\rfloor}$ of $\lfloor\theta_N^{-1}\rfloor$ adjacent compact subintervals with the same length θ_N, except the last one $\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1} := \big[(\lfloor\theta_N^{-1}\rfloor - 1)\theta_N, 1\big]$ having a length lying between θ_N and $2\theta_N$. Then, for any fixed integer $Q\ge 2$, the estimator $\{\widetilde{H}_N(s)\}_{s\in[0,1]} = \{\widetilde{H}^Q_{N,\theta_N}(s))\}_{s\in[0,1]}$ is obtained as the linear interpolation between the $\lfloor\theta_N^{-1}\rfloor + 1$ random points whose coordinates are the following:

$$\big(0, \widehat{H}^Q_N(\mathcal{I}_{N,0})\big),\dots,\big((\lfloor\theta_N^{-1}\rfloor - 1)\theta_N, \widehat{H}^Q_N(\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1})\big),\big(1, \widehat{H}^Q_N(\mathcal{I}_{N,\lfloor\theta_N^{-1}\rfloor-1})\big), \tag{2.8}$$

where, for all $n \in \{0,\dots,\lfloor\theta_N^{-1}\rfloor - 1\}$,

$$\widehat{H}^Q_N(\mathcal{I}_{N,n}) := \min\left\{\max\left\{\log_{Q^2}\left(\frac{V_N(\mathcal{I}_{N,n})}{V_{QN}(\mathcal{I}_{N,n})}\right), 0\right\}, 1\right\}. \tag{2.9}$$

Notice that, for every $x\in(0,+\infty)$, $\log_{Q^2}(x) := \log(x)/\log(Q^2)$, with the convention that log is the Napierian logarithm. Also notice that it might be possible to define $\widehat{H}^Q_N(\mathcal{I}_{N,n})$ as $\widehat{H}^Q_N(\mathcal{I}_{N,n}) := \log_{Q^2}\Big(\frac{V_N(\mathcal{I}_{N,n})}{V_{QN}(\mathcal{I}_{N,n})}\Big)$, yet the definition (2.9) offers the advantage that $\widehat{H}^Q_N(\mathcal{I}_{N,n})$ always belongs to the interval $[0,1]$. At last, notice that the ordinate of the last point in (2.8) is assumed to be the same as that of the point which is just before it. This weak assumption comes from the fact that the set of the indices t of MAMPRE has been restricted to the interval $[0,1]$; it does not significantly alter the results on the estimation of $H(\cdot)$ on this interval.

Let us now define the estimator $\widetilde{H}_N(\cdot) = \widetilde{H}^Q_{N,\theta_N}(\cdot)$ in a formal and very precise way.

Definition 2.1. The integer $L \geq 2$ is the same as in (2.2). The positive integer N_0 is fixed and large enough. Let $(\theta_N)_{N \geq N_0}$ be an arbitrary sequence of positive real numbers belonging to the interval $(0, 1/2]$ and satisfying, for all integer $N \geq N_0$,

$$\theta_N \leq \kappa\, N^{-\mu} \tag{2.10}$$

and

$$\theta_N \geq \kappa' N^{-\mu'} \geq 2(L+1)\, N^{-1}\,, \tag{2.11}$$

where $\kappa > 0$, $\mu \in (0,1)$, $\kappa' > 0$ and $\mu' \in [\mu, 1)$ are four constants not depending on N. For each $n \in \{0, \ldots, \lfloor \theta_N^{-1} \rfloor - 1\}$, we denote by $\mathcal{I}_{N,n}$ the compact subinterval of $[0,1]$ defined as:

$$\mathcal{I}_{N,n} := \big[n\theta_N, (n+1)\theta_N\big] \text{ when } n < \lfloor \theta_N^{-1} \rfloor - 1, \text{ and } \mathcal{I}_{N,\lfloor \theta_N^{-1} \rfloor - 1} := \big[(\lfloor \theta_N^{-1} \rfloor - 1)\theta_N, 1\big]. \tag{2.12}$$

Observe that it follows from (2.7) and (2.12) that, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \ldots \lfloor \theta_N^{-1} \rfloor - 1\}$, the cardinality $|\nu_{QN}(\mathcal{I}_{N,n})|$ of the set of indices $\nu_{QN}(\mathcal{I}_{N,n})$ (see (2.5) and (2.4)) satisfies

$$QN\theta_N/2 < |\nu_{QN}(\mathcal{I}_{N,n})| \leq 7QN\theta_N/3\,, \tag{2.13}$$

which in particular implies that $\nu_{QN}(\mathcal{I}_{N,n})$ is non-empty. At last, for all fixed integer $Q \geq 2$ and for every integer $N \geq N_0$, we denote by $\{\widetilde{H}^Q_{N,\theta_N}(s)\}_{s \in [0,1]}$ the stochastic process with continuous piecewise linear paths, defined as:

$$\widetilde{H}^Q_{N,\theta_N}(s) := \widehat{H}^Q_N\big(\mathcal{I}_{N,\lfloor \theta_N^{-1} \rfloor - 1}\big), \text{ for all } s \in \mathcal{I}_{N,[\theta_N^{-1}]-1}, \tag{2.14}$$

and, for every $n \in \{0, \ldots, [\theta_N^{-1}] - 2\}$ and $s \in \mathcal{I}_{N,n}$, as:

$$\widetilde{H}^Q_{N,\theta_N}(s) := \big(1 - \theta_N^{-1}(s - n\theta_N)\big)\widehat{H}^Q_N\big(\mathcal{I}_{N,n}\big) + \theta_N^{-1}(s - n\theta_N)\widehat{H}^Q_N\big(\mathcal{I}_{N,n+1}\big)\,, \tag{2.15}$$

where $\widehat{H}^Q_N\big(\mathcal{I}_{N,n}\big)$ is defined through (2.9) for all $n \in \{0, \ldots, [\theta_N^{-1}] - 1\}$.

Let us now state the main result of our article.

Theorem 2.2. *Assume that the conditions (1.8), (1.10), (2.10) and (2.11) hold, and that the exponent $\mu \in (0,1)$ in (2.10) satisfies the further condition:*

$$\frac{\overline{H} - \underline{H}}{L - \overline{H}} < 1 - \mu\,, \tag{2.16}$$

where the arbitrary fixed integer $L \geq 2$ is as in (2.2). Let β be an arbitrary positive real number such that

$$\beta < \min\Big\{\gamma\mu,\, 2^{-1}(1 - \mu'),\, (1-\mu)(L - \overline{H}) + \underline{H} - \overline{H}\Big\}\,. \tag{2.17}$$

Then, one has almost surely, for all fixed integer $Q \geq 2$,

$$\lim_{N \to +\infty} \Big\{ N^\beta \sup_{s \in [0,1]} \big| H(s) - \widetilde{H}^Q_{N,\theta_N}(s) \big| \Big\} = 0\,. \tag{2.18}$$

We point out that a major ingredient of the proof of Theorem 2.2 is the important Burkholder-Davis-Gundy inequality (see for instance Mao (2008); Protter (2005)) as formulated in the following proposition:

Proposition 2.3. *Let $p \in [1, +\infty[$ be arbitrary and fixed. There is a universal deterministic finite constant $a(p)$ for which the following result holds: for any $(\mathcal{F}_s)_{s \in \mathbb{R}}$-adapted stochastic process $f = \{f(s)\}_{s \in \mathbb{R}}$ satisfying almost surely $\int_{-\infty}^{+\infty} |f(s)|^2\, ds < +\infty$, one has*

$$\mathbb{E}\bigg(\Big| \int_{-\infty}^{+\infty} f(s)\, dB(s) \Big|^p\bigg) \leq a(p)\mathbb{E}\bigg(\Big(\int_{-\infty}^{+\infty} |f(s)|^2\, ds\Big)^{p/2}\bigg)\,, \tag{2.19}$$

where $\int_{-\infty}^{+\infty} f(s)\, dB(s)$ denotes the Itô integral of f on $\mathbb{R}$.

Let us now explain how the almost sure uniform rate of convergence for the estimator $\widetilde{H}^{Q}_{N,\theta_N}(\cdot)$, provided by Theorem 2.2, can be optimized.

Remark 2.4. Observe that, in view of (2.10) and (2.11), a natural choice for the sequence $(\theta_N)_{N\ge N_0}$ in Definition 2.1 is

$$\theta_N = 2^{-1}N^{-\mu}\,, \quad \text{for all } N \ge N_0, \tag{2.20}$$

where the exponent $\mu \in (0,1)$ is arbitrary and satisfies the condition

$$\frac{\overline{H}-\underline{H}}{L-\overline{H}-2^{-1}} < 1-\mu\,, \tag{2.21}$$

which is a bit stronger than the condition (2.16). Notice that, for the choice (2.20) for the sequence $(\theta_N)_{N\ge N_0}$, one can take in (2.11) $\mu'=\mu$. Then using the latter equality and (2.21), it turns out that the expression in the right-hand side of the inequality (2.17) reduces to

$$\min\Big\{\gamma\mu,\, 2^{-1}(1-\mu'),\, (1-\mu)(L-\overline{H})+\underline{H}-\overline{H}\Big\} = \min\Big\{\gamma\mu,\, 2^{-1}(1-\mu)\Big\}\,. \tag{2.22}$$

Moreover, when $\gamma\in(0,1]$, the deterministic order of Hölder regularity on the interval $[-1,1]$ of the random Hurst function $H(\cdot)$ (see (1.10)), is known, then the quantity in the right-hand side of the (2.22) can be maximized by taking μ such that $\gamma\mu = 2^{-1}(1-\mu)$, that is

$$\mu = \frac{1}{2\gamma+1}\,. \tag{2.23}$$

Thus, we can derive from (2.22) and (2.23) that

$$\min\Big\{\gamma\mu,\, 2^{-1}(1-\mu'),\, (1-\mu)(L-\overline{H})+\underline{H}-\overline{H}\Big\} = \frac{\gamma}{2\gamma+1}\,. \tag{2.24}$$

Notice that, when μ is chosen as in (2.23), then the condition (2.21) is fulfilled when the integer L, associated with the generalized increments $d_{N,k}$ (see (2.3)), is such that

$$L \ge \frac{5\gamma+1}{2\gamma}\,. \tag{2.25}$$

In conclusion, under the hypotheses that γ is known and L satisfies (2.25), the sequence $(\theta_N)_{N\ge N_0}$ in Definition 2.1 can be chosen as in (2.20) with μ given by (2.23). Then, in view of (2.17) and (2.24), it follows from (2.18) that, for all fixed integer $Q\ge 2$, the almost sure uniform rate of convergence of the estimator $\widetilde{H}^{Q}_{N,\theta_N}(\cdot)$ is arbitrarily close to $N^{-\frac{\gamma}{2\gamma+1}}$, which is rather reminiscent of the minimax asymptotic lower bound for the risk of estimating a deterministic smooth Hurst function in the Gaussian framework of Classical Multifractional Brownian Motion (CMBM); the latter lower bound is provided by Theorem 3.1 of the recent article Shen and Hsing (2020).

Before ending the present section, let us give some simulations for illustrating Theorem 2.2 and Remark 2.4. We mention in passing that a full numerical study of the estimator $\widetilde{H}^{Q}_{N,\theta_N}(\cdot)$ requires in itself another article and will be done elsewhere. We denote by ψ the infinitely differentiable strictly increasing deterministic function from $\mathbb{R}$ into the interval $(0.1,0.9)\subset[0.1,0.9]\subset(0,1)$, defined, for all $x\in\mathbb{R}$, as $\psi(x) := 0.8(\pi^{-1}\arctan(x)) + 0.5) + 0.1$. We denote by $\{B_{0.25}(s)\}_s$, $\{B_{0.55}(s)\}_s$ and $\{B_{0.80}(s)\}_s$ three FBMs (see (1.1)) which are adapted to the filtration $(\mathcal{F}_s)_s$ and whose Hurst parameters are respectively equal to 0.25, 0.55 and 0.80. The three random Hurst functions $H^1(s) := \psi(B_{0.25}(s))$, $H^2(s) := \psi(B_{0.55}(s))$ and $H^3(s) := \psi(B_{0.80}(s))$, for which the orders of Hölder regularity on the interval $[-1,1]$ can be considered to be $\gamma_1 = 0.24$, $\gamma_2 = 0.54$ and $\gamma_3 = 0.79$, have been successively simulated on the interval $[0,1]$ in the first column of Figure 2.1 given below. The corresponding three MAMPREs have been simulated on the same interval in the second column, by using a simulation method, relying on the Haar wavelet basis, which is rather similar to that introduced in Ayache et al. (2018); we mention in passing that the three FBMs generating the random Hurst functions H^i, $i\in\{1,2,3\}$, have been simulated by the same method.

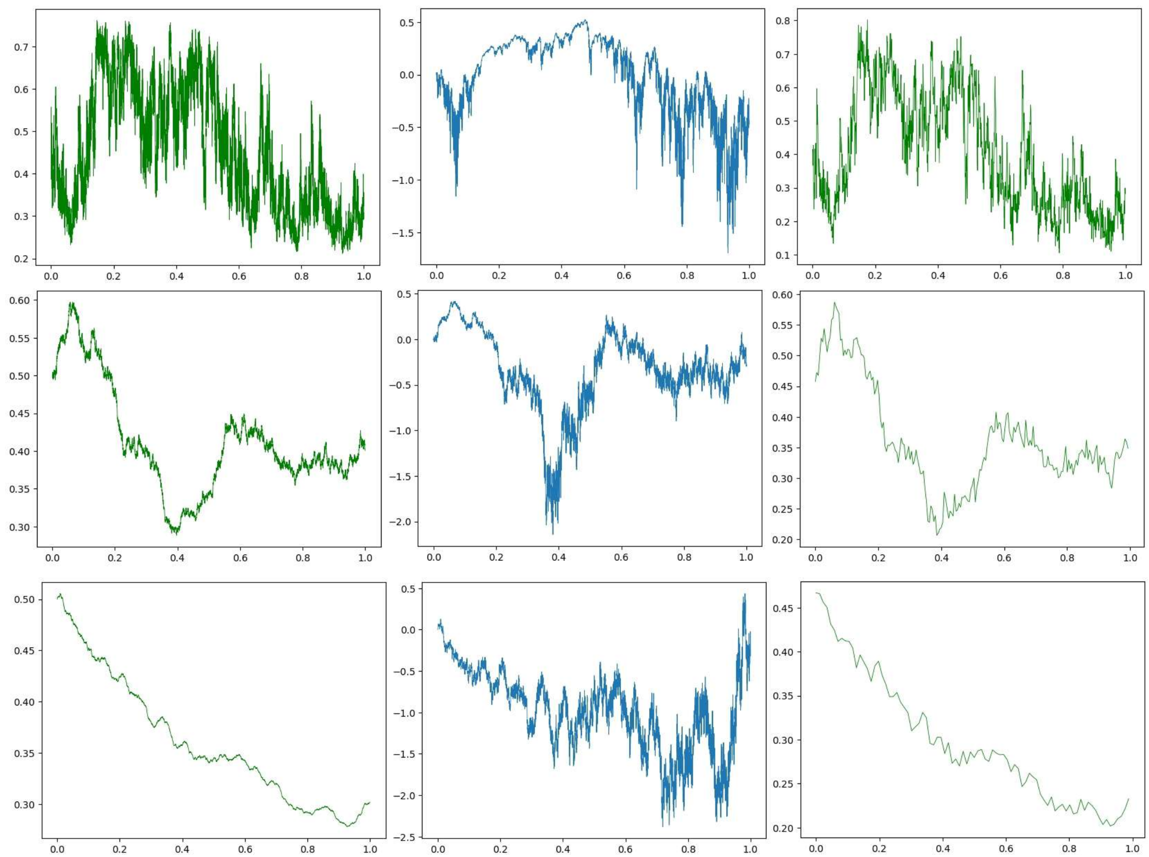

FIGURE 2.1. Some simulations

The estimated versions $\widetilde{H}^i$, $i \in \{1,2,3\}$, of these three random Hurst functions, via the statistical estimator introduced in Definition 2.1 with $N = 2^{14}$, have been simulated on the interval $[0,1]$ in the third column. In view of Remark 2.4, for obtaining each $\widetilde{H}^i$, we have taken $\theta_N = 2^{-1} N^{-\frac{1}{2\gamma_i+1}}$ and L_i the lowest integer, greater than 2, such that (2.25) holds with $\gamma = \gamma_i$.

In view of the simulations the statistical estimator of random Hurst functions, introduced in Definition 2.1, seems to work fairly well. Indeed, the simulations show that it allows to reconstruct random Hurst functions in a rather precise way, even when they are very erratic, as for instance the random Hurst function $H^1(s)$.

3. Negligible parts of generalized quadratic variations of X

Remark 3.1. From now till the end of the article we always assume that the four conditions (1.8), (1.10), (2.10) and (2.11) hold, without always mentioning it explicitly in the statements of the intermediate results which will allow us to prove Theorem 2.2.

In view of (2.9), (2.14) and (2.15), for proving Theorem 2.2 it is useful to study, for any fixed positive integer Q, asymptotic behavior of the generalized quadratic variations $V_{QN}(\mathcal{I}_{N,n})$, $n \in \{0,\ldots,\lfloor\theta_N^{-1}\rfloor - 1\}$, when N goes to $+\infty$. A first difficulty in this matter is that the domains of integration of the Itô integrals representing the generalized increments $d_{N,k}$ are unbounded intervals. Indeed, one can derive from (1.9), (2.3), (2.2), (1.2) and easy computations, that, for all $k \in$

$\{0,\dots,N-L\}$,

$$d_{N,k}=\int_{-\infty}^{N^{-1}(k+L)} N^{-(H(s)-1/2)}\,\Phi\big(Ns-k,H(s)\big)\,dB(s)\,, \tag{3.1}$$

where Φ is the real-valued deterministic function defined, for all $(u,v)\in\mathbb{R}\times(0,1)$, as:

$$\Phi(u,v):=\sum_{l=0}^{L} a_l(l-u)_+^{v-1/2}. \tag{3.2}$$

Roughly speaking, our first goal will be to show that $d_{N,k}$ can be expressed as the sum of an Itô integral over a well-chosen bounded interval and another term which is negligible in some sense. From now till the end of our article, we set

$$\delta:=1-\mu\,, \tag{3.3}$$

where $\mu\in(0,1)$ is as in (2.10) and satisfies the condition (2.16). For every integer $N\ge N_0$, where N_0 is as in Definition 2.1, the positive integer $e_N=e_N(\delta)$ is defined as:

$$e_N:=\lfloor N^\delta\rfloor. \tag{3.4}$$

Then we can derive from (3.1) that, for all $k\in\{0,\dots,N-L\}$,

$$d_{N,k}=\widetilde{d}^{\,\delta}_{N,k}+\breve{d}^{\,\delta}_{N,k}\,, \tag{3.5}$$

where

$$\widetilde{d}^{\,\delta}_{N,k}:=\int_{N^{-1}(k-e_N+L)}^{N^{-1}(k+L)} N^{-(H(s)-1/2)}\,\Phi\big(Ns-k,H(s)\big)\,dB(s) \tag{3.6}$$

and

$$\breve{d}^{\,\delta}_{N,k}:=\int_{-\infty}^{N^{-1}(k-e_N+L)} N^{-(H(s)-1/2)}\,\Phi\big(Ns-k,H(s)\big)\,dB(s)\,. \tag{3.7}$$

Definition 3.2. For any integers $Q\ge1$, $N\ge N_0$ and $n\in\big\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\big\}$, the generalized quadratic variations $\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ and $\breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ are defined as:

$$\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n})=|\nu_{QN}(\mathcal{I}_{N,n})|^{-1}\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}|\widetilde{d}^{\,\delta}_{QN,k}|^2 \tag{3.8}$$

and

$$\breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n})=|\nu_{QN}(\mathcal{I}_{N,n})|^{-1}\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}|\breve{d}^{\,\delta}_{QN,k}|^2. \tag{3.9}$$

Basically, the following lemma shows that the generalized quadratic variations $\breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$, $n\in\big\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\big\}$, are negligible when N goes to $+\infty$. In other words, when N goes to $+\infty$, the asymptotic behavior of the generalized quadratic variations $V_{QN}(\mathcal{I}_{N,n})$ is similar to that of the "less complicated" generalized quadratic variations $\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$.

Lemma 3.3. *Let $\underline{H}$, $\overline{H}$, L and δ be as in (1.8), (2.2) and (3.3). One has almost surely*

$$\limsup_{N\to+\infty}\Big\{N^\beta\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\breve{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\Big\}=0\,,\quad \textit{for all } Q\in\mathbb{N} \textit{ and } \beta<2\delta(L-\overline{H})+2\underline{H}. \tag{3.10}$$

For proving Lemma 3.3, one needs the following lemma whose proof will be given in the sequel.

Lemma 3.4. *Let c be the same constant as in (3.16). For all real number $p\in[1,+\infty)$ and integers $Q\ge1$ and $N\ge(3L)^{1/\delta}+N_0$, the following inequality is satisfied:*

$$\max_{0\le k\le N-L}\mathbb{E}\big(|\breve{d}^{\,\delta}_{QN,k}|^{2p}\big)\le c^{2p}\,a(2p)N^{-2p(\delta(L-\overline{H})+\underline{H})}\,, \tag{3.11}$$

where $a(2p)$ is the same constant only depending on p as in Proposition 2.3.

Proof of Lemma 3.3: Let $Q \in \mathbb{N}$ and b a fixed real number such that

$$\beta < b < 2\delta(L - \overline{H}) + 2\underline{H}\,, \tag{3.12}$$

where β is as in (3.10). Let $p \in [1, +\infty)$ be fixed and such that

$$p(2\delta(L - \overline{H}) + 2\underline{H} - b) > 2\,. \tag{3.13}$$

Using (3.9), Markov inequality, the fact that $z \mapsto |z|^p$ is a convex function on $\mathbb{R}$, (3.11) and (2.11), one gets that

$$\begin{aligned}
&\mathbb{P}\Big(N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} \breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) > 1\Big) \le \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} \mathbb{P}\Big(N^b\, \breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) > 1\Big)\\
&\le N^{pb} \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} \mathbb{E}\Big(\big|\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|^p\Big) \le N^{pb} \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\big(|\breve{d}_{QN,k}^{\delta}|^{2p}\big)\\
&\le c_1\lfloor\theta_N^{-1}\rfloor N^{-p(2\delta(L-\overline{H})+2\underline{H}-b)} \le c_1 N^{1-p(2\delta(L-\overline{H})+2\underline{H}-b)}\,,
\end{aligned} \tag{3.14}$$

where $c_1 > 0$ is a constant not depending on N and Q. Next, combining (3.13) and (3.14), one obtains that

$$\sum_{N=N_0}^{+\infty} \mathbb{P}\Big(N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} \breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) > 1\Big) < +\infty\,.$$

Thus, it results from the Borel-Cantelli Lemma that one has, almost surely,

$$\sup_{N\ge N_0}\Big\{N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} \breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\Big\} < +\infty\,. \tag{3.15}$$

Finally, (3.12) and (3.15) imply that (3.10) holds. □

Let us now focus on the proof of Lemma 3.4. It mainly relies on Proposition 2.3 and the following proposition.

Proposition 3.5. *One has*

$$c := \sup\Big\{\big(1 + L + |u|\big)^{L+1/2-\overline{H}}\,\big|\Phi(u,v)\big| \,:\, (u,v) \in (-\infty, -2L] \times [\underline{H}, \overline{H}]\Big\} < +\infty \tag{3.16}$$

and

$$c' := \sup\left\{\frac{\big(1 + L + |u|\big)^{L+1/2-\overline{H}}}{\log\big(1 + L + |u|\big)}\,\big|(\partial_v\Phi)(u,v)\big| \,:\, (u,v) \in (-\infty, -2L] \times [\underline{H}, \overline{H}]\right\} < +\infty. \tag{3.17}$$

The short proof of Proposition 3.5 is rather similar to that of Proposition 3.1 in Ayache and Hamonier (2017); we give it for the sake of completeness.

Proof of Proposition 3.5: Combining (3.2) and (1.2) one gets, for all $(u,v) \in (-\infty, -L) \times (0,1)$, that

$$\Phi(u,v) = |u|^{v-1/2} \sum_{l=0}^{L} a_l f(lu^{-1}, v) \tag{3.18}$$

and

$$(\partial_v\Phi)(u,v) = |u|^{v-1/2}\log(|u|) \sum_{l=0}^{L} a_l f(lu^{-1}, v) + |u|^{v-1/2} \sum_{l=0}^{L} a_l (\partial_v f)(lu^{-1}, v), \tag{3.19}$$

where f is the C^∞ function on $(-1,1) \times (-2,2)$ defined, for all $(y,v) \in (-1,1) \times (-2,2)$, as $f(y,v) = (1-y)^{v-1/2}$. Then noticing that when u belongs to $(-\infty, -2L]$ one equivalently has that

$z = u^{-1}$ belongs to $\big[-2^{-1}L^{-1},0\big) \subset \big(-L^{-1},L^{-1}\big)$, one can easily derive from (3.18), (3.19) and Lemma 3.6 below that (3.16) and (3.17) are satisfied. □

Lemma 3.6. *Assume that y_0 and v_0 are two arbitrary and fixed positive real numbers. Let φ be an arbitrary real-valued C^∞ function on $(-y_0,y_0)\times(-v_0,v_0)$ and let g be the C^∞ function on $\big(-L^{-1}y_0,L^{-1}y_0\big)\times(-v_0,v_0)$ defined, for all $(z,v)\in\big(-L^{-1}y_0,L^{-1}y_0\big)\times(-v_0,v_0)$, as:*

$$g(z,v) := \sum_{l=0}^{L} a_l \varphi(lz,v).$$

Then, one has, for every $(z,v)\in\big[-2^{-1}L^{-1}y_0,2^{-1}L^{-1}y_0\big]\times\big[-2^{-1}v_0,2^{-1}v_0\big]$,

$$\big|g(z,v)\big| \le c|z|^L,$$

where c is the finite constant defined as

$$c := (L!)^{-1}\sup\big\{|(\partial_z^L g)(z,v)| : (z,v)\in\big[-2^{-1}L^{-1}y_0,2^{-1}L^{-1}y_0\big]\times\big[-2^{-1}v_0,2^{-1}v_0\big]\big\}.$$

Proof of Lemma 3.6: Assume that $v\in(-v_0,v_0)$ is arbitrary and fixed. Applying Taylor formula to the function $z\mapsto g(z,v)$ it follows, for all $z\in\big[-2^{-1}L^{-1}y_0,2^{-1}L^{-1}y_0\big]$, that

$$g(z,v) = \Big(\sum_{q=0}^{L-1}\frac{(\partial_z^q g)(0,v)}{q!}z^q\Big) + \frac{(\partial_z^L g)(\theta,v)}{L!}z^L\,, \tag{3.20}$$

where $\theta\in\big(-2^{-1}L^{-1}y_0,2^{-1}L^{-1}y_0\big)$. Next, observe that, for each $z\in\big(-L^{-1}y_0,L^{-1}y_0\big)$ and $q\in\mathbb{N}$, one has

$$(\partial_z^q g)(z,v) = \sum_{l=0}^{L} l^q a_l (\partial_y^q\varphi)(lz,v)\,.$$

Therefore, one gets that

$$(\partial_z^q g)(0,v) = (\partial_y^q\varphi)(0,v)\big(\sum_{l=0}^{L} l^q a_l\big).$$

Then, in view of (2.2), it turns out that $(\partial_z^q g)(0,v)=0$, for all $q\in\{0,\dots,L-1\}$. Finally, combining the latter equality with (3.20), one obtains the lemma. □

We are now ready to prove Lemma 3.4.

Proof of Lemma 3.4: Using (3.7), (2.19), (1.8), (3.16) and (3.4) one gets, for all integers $Q\ge1$, $N\ge(3L)^{1/\delta}+N_0$ and $k\in\{0,\dots,N-L\}$, that

$$\begin{aligned}
&\mathbb{E}\big(|\breve{d}^{\delta}_{QN,k}|^{2p}\big) \le a(2p)\mathbb{E}\bigg(\Big(\int_{-\infty}^{(QN)^{-1}(k-e_{QN}+L)}(QN)^{-2H(s)+1}\big|\Phi\big(QNs-k,H(s)\big)\big|^2\,ds\Big)^p\bigg)\\
&\le c^{2p}\,a(2p)N^{-2p\underline{H}}\Big(QN\int_{-\infty}^{(QN)^{-1}(k-e_{QN}+L)}\big(1+L+k-QNs\big)^{2\overline{H}-2L-1}\,ds\Big)^p\\
&\le c^{2p}\,a(2p)N^{-2p\underline{H}}\,(1+e_{QN})^{2p(\overline{H}-L)} \le c^{2p}\,a(2p)N^{2p(\delta(\overline{H}-L)-\underline{H})}\,,
\end{aligned}$$

which proves that (3.11) holds. □

Roughly speaking, so far we have shown that, when N goes to $+\infty$, the asymptotic behavior of $V_{QN}(\mathcal{I}_{N,n})$ is similar to that of $\widetilde{V}^{\delta}_{QN}(\mathcal{I}_{N,n})$ defined in (3.8). There is still a difficulty in the study of the latter behavior. Basically, it comes from the $H(s)$ which figures in (3.6). It is convenient to replace $H(s)$ by a well-chosen random variable not depending on s. This is the main motivation behind the following definition.

Definition 3.7. Let $\delta \in (0,1)$ be as (3.3). For every integers $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, one sets

$$\zeta_{N,n} := n\theta_N - N^{-(1-\delta)} = n\theta_N - N^{-\mu}\,; \tag{3.21}$$

moreover, for each $Q \in \mathbb{N}$, the generalized quadratic variations $\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$ and $\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$ are defined as:

$$\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\widehat{d}_{QN,k}^{\delta,n}|^2 \tag{3.22}$$

and

$$\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) = |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k \in \nu_{QN}(\mathcal{I}_{N,n})} |\overline{d}_{QN,k}^{\delta,n}|^2\,, \tag{3.23}$$

where, for all $k \in \nu_{QN}(\mathcal{I}_{N,n})$,

$$\widehat{d}_{QN,k}^{\delta,n} := \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} (QN)^{-(H(\zeta_{N,n})-1/2)}\, \Phi\big(QNs-k, H(\zeta_{N,n})\big)\, dB(s) \tag{3.24}$$

and

$$\begin{aligned}\overline{d}_{QN,k}^{\delta,n} &:= \widetilde{d}_{QN,k}^{\delta} - \widehat{d}_{QN,k}^{\delta,n} \\ &= \int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)} \Big((QN)^{-(H(s)-1/2)}\, \Phi\big(QNs-k, H(s)\big) \\ &\qquad - (QN)^{-(H(\zeta_{N,n})-1/2)}\, \Phi\big(QNs-k, H(\zeta_{N,n})\big)\Big)\, dB(s)\,.\end{aligned} \tag{3.25}$$

Basically, the following lemma shows that the generalized quadratic variations $\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$, $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, are negligible when N goes to $+\infty$. In other words, when N goes to $+\infty$, the asymptotic behavior of $\widetilde{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$ (and consequently that of $V_{QN}(\mathcal{I}_{N,n})$) is similar to that of the "less complicated" generalized quadratic variation $\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$.

Lemma 3.8. *Let γ, μ and δ be as in (1.10), (2.10) and (3.3). One has almost surely*

$$\limsup_{N \to +\infty} \Big\{ N^{\beta} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} N^{2H(\zeta_{N,n})}\, \overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \Big\} = 0\,, \quad \text{for all } Q \in \mathbb{N} \text{ and } \beta < 2\gamma\mu. \tag{3.26}$$

In order to show that Lemma 3.8 holds, one needs the following lemma.

Lemma 3.9. *For any fixed $Q \in \mathbb{N}$ and $p \in [1, +\infty)$, there exists a finite constant $\overline{c}(Q,p)$ such that, for all integer $N \geq N_0$, one has*

$$\max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \max_{k \in \nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\Big(\big| N^{H(\zeta_{N,n})}\, \overline{d}_{QN,k}^{\delta,n} \big|^{2p} \Big) \leq \overline{c}(Q,p) (\log N)^{2p} N^{-2p\gamma\mu}\,. \tag{3.27}$$

Proof of Lemma 3.9: The integers $Q \in \mathbb{N}$, $N \geq N_0$, $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$ and $k \in \nu_{QN}(\mathcal{I}_{N,n})$ are arbitrary and fixed. It follows from (3.25), (2.19), the inequality $(QN)^{-2H(\zeta_{N,n})} \leq 1$, the inequality

$$|x+y|^2 \leq 2\big(|x|^2 + |y|^2\big), \quad \text{for all } (x,y) \in \mathbb{R}^2, \tag{3.28}$$

and the convexity on $\mathbb{R}_+$ of the function $z \mapsto z^p$ that

$$\begin{aligned} &\mathbb{E}\Big(\big|N^{H(\zeta_{N,n})}\,\overline{d}^{\delta,n}_{QN,k}\big|^{2p}\Big)\\ &\le a(2p)\mathbb{E}\bigg(\bigg(QN\int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)}\Big|(QN)^{(H(\zeta_{N,n})-H(s))}\,\Phi\big(QNs-k,H(s)\big)\\ &\qquad\qquad -\Phi\big(QNs-k,H(\zeta_{N,n})\big)\Big|^2\,ds\bigg)^p\bigg)\\ &\le 2^{2p-1}a(2p)\Big(\mathbb{E}\big((A^{\delta,n}_{QN,k})^p\big)+\mathbb{E}\big((B^{\delta,n}_{QN,k})^p\big)\Big)\,, \end{aligned} \tag{3.29}$$

where

$$A^{\delta,n}_{QN,k} := QN\int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)}\big|(QN)^{(H(\zeta_{N,n})-H(s))}-1\big|^2\,\big|\Phi\big(QNs-k,H(s)\big)\big|^2\,ds \tag{3.30}$$

and

$$B^{\delta,n}_{QN,k} := QN\int_{(QN)^{-1}(k-e_{QN}+L)}^{(QN)^{-1}(k+L)}\Big|\Phi\big(QNs-k,H(s)\big)-\Phi\big(QNs-k,H(\zeta_{N,n})\big)\Big|^2\,ds\,. \tag{3.31}$$

Next, using the mean value Theorem, (1.10), (3.21), the fact that $(QN)^{-1}k \in \mathcal{I}_{N,n}$ (see (2.5) and (2.12)), (3.4) and (2.10), for all $s \in \big[(QN)^{-1}(k-e_{QN}+L),(QN)^{-1}(k+L)\big]$, one gets that

$$\begin{aligned} \big|(QN)^{(H(\zeta_{N,n})-H(s))}-1\big| &\le c_1\rho\exp\big(c_1\rho(\log QN)N^{-\gamma\mu}\big)(\log QN)N^{-\gamma\mu}\\ &\le \exp\big(c_2(Q)\rho\big)(\log N)N^{-\gamma\mu}\,, \end{aligned} \tag{3.32}$$

where the deterministic finite constants c_1 and $c_2(Q)$ are defined as: $c_1 := (2\kappa+L+2)^\gamma$ and $c_2(Q) := c_1\big(2\log(3+Q)+\sup_{N\ge N_0}(\log QN)N^{-\gamma\mu}\big)$. Next, putting together (3.30), (3.32), the change of variable $u = QNs-k$, (3.16) and (3.2), one obtains that

$$\begin{aligned} A^{\delta,n}_{N,k} &\le \exp\big(2c_2(Q)\rho\big)(\log N)^2N^{-2\gamma\mu}\int_{-\infty}^{L}\big|\Phi\big(u,H((QN)^{-1}(u+k))\big)\big|^2\,du\\ &\le c_3\exp\big(2c_2(Q)\rho\big)(\log N)^2N^{-2\gamma\mu}, \end{aligned} \tag{3.33}$$

where the deterministic finite constant

$$c_3 := c_4^2\int_{-\infty}^{-2L}\big(1+L+|u|\big)^{2\overline{H}-2L-1}\,du+\int_{-2L}^{L}\Big(\sum_{l=0}^{L}|a_l|\big((l-u)_+^{\underline{H}-1/2}+(l-u)_+^{\overline{H}-1/2}\big)\Big)^2\,du, \tag{3.34}$$

c_4 being the constant c in (3.16). Next, notice that in view of (3.4) one can assume without any restriction that N is big enough so that $L-e_{QN}\le L-e_N < -2L$. Then, it follows from (3.31), the change of variable $u = QNs-k$ and (3.2) that

$$\begin{aligned} B^{\delta,n}_{QN,k} &= \int_{L-e_{QN}}^{L}\Big|\Phi\big(u,H((QN)^{-1}(u+k))\big)-\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du\\ &= \int_{L-e_{QN}}^{-2L}\Big|\Phi\big(u,H((QN)^{-1}(u+k))\big)-\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du\\ &\quad+\int_{-2L}^{0}\Big(\sum_{l=0}^{L}a_l(l-u)^{H((QN)^{-1}(u+k))-1/2}-\sum_{l=0}^{L}a_l(l-u)^{H(\zeta_{N,n})-1/2}\Big)^2\,du\\ &\quad+\sum_{p=0}^{L-1}\int_{p}^{p+1}\Big(\sum_{l=p+1}^{L}a_l(l-u)^{H((QN)^{-1}(u+k))-1/2}-\sum_{l=p+1}^{L}a_l(l-u)^{H(\zeta_{N,n})-1/2}\Big)^2\,du\,. \end{aligned}$$

Thus, one can derive from the mean value Theorem, (1.10), (3.21), the fact that $(QN)^{-1}k \in \mathcal{I}_{N,n}$, (3.4), (2.10), (3.17) and (1.8) that

$$B^{\delta,n}_{QN,k} \le c_5 \rho^2 N^{-2\gamma\mu}\,, \tag{3.35}$$

where the deterministic finite constant

$$\begin{aligned} c_5 \;\; := \;\; & c_7^2\, c_6^2 \int_{-\infty}^{-2L} \big(1+L+|u|\big)^{2\overline{H}-2L-1} \log^2\big(1+L+|u|\big)\, du \\ & + c_6^2 \int_{-2L}^{0} \Big(\sum_{l=0}^{L} |a_l| \big((l-u)^{\underline{H}-1/2} + (l-u)^{\overline{H}-1/2}\big) \big|\log(l-u)\big| \Big)^2 du \\ & + c_6^2 \sum_{p=0}^{L-1} \int_{p}^{p+1} \Big(\sum_{l=p+1}^{L} |a_l| \big((l-u)^{\underline{H}-1/2} + (l-u)^{\overline{H}-1/2}\big) \big|\log(l-u)\big| \Big)^2 du\,, \end{aligned}$$

$c_6 := (2\kappa+2)^\gamma$ and c_7 being the constant c' in (3.17). Finally, putting together (3.29), (3.33) and (3.35), one obtains (3.27). □

We are now ready to prove Lemma 3.8.

Proof of Lemma 3.8: Let $Q \in \mathbb{N}$ and b be a fixed real number such that

$$\beta < b < 2\gamma\mu\,, \tag{3.36}$$

where β is as in (3.26). Let $p \in [1,+\infty)$ be fixed and such that

$$p(2\gamma\mu - b) > 2\,. \tag{3.37}$$

Using (3.23), Markov inequality, the fact that $z \mapsto |z|^p$ is a convex function on $\mathbb{R}$, (3.27) and (2.11), one gets that

$$\begin{aligned} & \mathbb{P}\Big(N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\, \overline{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) > 1\Big) \le \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} \mathbb{P}\Big(N^{b+2H(\zeta_{N,n})}\, \overline{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) > 1\Big) \\ & \le N^{pb} \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} \mathbb{E}\Big(\big|N^{2H(\zeta_{N,n})}\, \overline{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|^p\Big) \\ & \le N^{pb} \sum_{n=0}^{\lfloor\theta_N^{-1}\rfloor-1} |\nu_{QN}(\mathcal{I}_{N,n})|^{-1} \sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\Big(\big|N^{H(\zeta_{N,n})}\, \overline{d}^{\delta,n}_{QN,k}\big|^{2p}\Big) \\ & \le c_1 \lfloor\theta_N^{-1}\rfloor (\log N)^{2p} N^{-p(2\gamma\mu-b)} \le c_1 (\log N)^{2p} N^{1-p(2\gamma\mu-b)}\,, \end{aligned} \tag{3.38}$$

where $c_1 > 0$ is a constant not depending on N. Next, combining (3.37) and (3.38), one obtains that

$$\sum_{N=N_0}^{+\infty} \mathbb{P}\Big(N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\, \overline{V}^{\delta}_{N}(\mathcal{I}_{N,n}) > 1\Big) < +\infty\,.$$

Thus, it results from the Borel-Cantelli Lemma that one has, almost surely,

$$\sup_{N\ge N_0} \Big\{ N^b \max_{0\le n<\lfloor\theta_N^{-1}\rfloor} N^{2H(\zeta_{N,n})}\, \overline{V}^{\delta}_{N}(\mathcal{I}_{N,n}) \Big\} < +\infty\,. \tag{3.39}$$

Finally, (3.36) and (3.39) imply that (3.26) holds. □

4. Asymptotic behavior of generalized quadratic variation of X

The main goal of the present section is to prove the following lemma.

Lemma 4.1. *Let $\underline{H}$, $\overline{H}$, γ, L, μ, μ', δ and β be as in (1.8), (1.10), (2.2), (2.10), (2.16), (2.11), (3.3) and (2.17). Then, one has almost surely, for all $Q\in\mathbb{N}$,*

$$\limsup_{N\to+\infty}\left\{N^{\beta}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|\right\}=0\,, \tag{4.1}$$

where $V_{QN}(\mathcal{I}_{N,n})$, $\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})$ and $\zeta_{N,n}$ are defined through (2.4), (3.22) and (3.21). Notice that $\mathbb{E}\big(\cdot\,|\mathcal{F}_{\zeta_{N,n}}\big)$ is the conditional expectation operator with respect to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$.

The proof of Lemma 4.1, which will be given at the end of the present section, relies on Lemma 3.3, Lemma 3.8, and the following crucial lemma.

Lemma 4.2. *Let μ' and δ be as in (2.11) and (3.3). One has almost surely*

$$\limsup_{N\to+\infty}\left\{N^{\beta(1-\mu')}\max_{0\le n<\lfloor\theta_N^{-1}\rfloor}\left|\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}-1\right|\right\}=0\,,\quad \textit{for all } Q\in\mathbb{N} \textit{ and } \beta<1/2. \tag{4.2}$$

In order to show that Lemma 4.2 holds, one needs some preliminary results.

Lemma 4.3. *For all integers $Q\in\mathbb{N}$, $N\ge N_0$ and $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$, and for each finite sequence $(z_k)_{k\in\nu_{QN}(\mathcal{I}_{N,n})}$ of real numbers, one has, almost surely,*

$$\begin{aligned}&\mathbb{E}\bigg(\exp\Big(i\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}z_k\,\widehat{d}_{QN,k}^{\delta,n}\Big)\Big|\mathcal{F}_{\zeta_{N,n}}\bigg)\\&\quad=\exp\bigg(-2^{-1}\int_{\zeta_{N,n}}^{1}\Big|\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}z_k\mathbb{1}_{\mathcal{D}_{QN,k}}(s)(QN)^{-(H(\zeta_{N,n})-1/2)}\,\Phi\big(QNs-k,H(\zeta_{N,n})\big)\Big|^2ds\bigg),\end{aligned} \tag{4.3}$$

where $\zeta_{N,n}$ is an in (3.21), and

$$\mathcal{D}_{QN,k}:=\big[(QN)^{-1}(k-e_{QN}+L),(QN)^{-1}(k+L)\big]\,. \tag{4.4}$$

Notice that (4.3) means that, for each $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$, conditionally to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$, the random vector $(\widehat{d}_{QN,k}^{\delta,n})_{k\in\nu_{QN}(\mathcal{I}_{N,n})}$ has a centred Gaussian distribution with covariance matrix $\big(\mathbb{E}(\widehat{d}_{QN,k'}^{\delta,n}\,\widehat{d}_{QN,k''}^{\delta,n}|\mathcal{F}_{\zeta_{N,n}})\big)_{k',k''\in\nu_{QN}(\mathcal{I}_{N,n})}$ such that, for every $k',k''\in\nu_{QN}(\mathcal{I}_{N,n})$,

$$\begin{aligned}&\mathbb{E}\big(\widehat{d}_{QN,k'}^{\delta,n}\,\widehat{d}_{QN,k''}^{\delta,n}|\mathcal{F}_{\zeta_{N,n}}\big)\\&=(QN)^{1-2H(\zeta_{N,n})}\\&\quad\times\int_{\zeta_{N,n}}^{1}\mathbb{1}_{\mathcal{D}_{QN,k'}}(s)\mathbb{1}_{\mathcal{D}_{QN,k''}}(s)\Phi\big(QNs-k',H(\zeta_{N,n})\big)\Phi\big(QNs-k'',H(\zeta_{N,n})\big)\,ds\,.\end{aligned} \tag{4.5}$$

Proof of Lemma 4.3: First observe that one can derive from (3.24), (2.5), (2.12), (3.21) and (4.4) that, for all integers $Q\in\mathbb{N}$, $N\ge N_0$ and $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$, and for each finite sequence $(z_k)_{k\in\nu_{QN}(\mathcal{I}_{N,n})}$ of real numbers, one has

$$\begin{aligned}&\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}z_k\,\widehat{d}_{QN,k}^{\delta,n}\\&=\int_{\zeta_{N,n}}^{1}\Big(\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}z_k\mathbb{1}_{\mathcal{D}_{QN,k}}(s)(QN)^{-(H(\zeta_{N,n})-1/2)}\,\Phi\big(QNs-k,H(\zeta_{N,n})\big)\Big)\,dB(s)\,.\end{aligned} \tag{4.6}$$

The main idea of the proof of this lemma consists in the observation that the Brownian motion B in (4.6) can be replaced by the Brownian motion $W_{N,n} = \{W_{N,n}(x)\}_{x\in\mathbb{R}_+} := \{B(x+\zeta_{N,n}) - B(\zeta_{N,n})\}_{x\in\mathbb{R}_+}$ which is independent of the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$. Therefore $W_{N,n}$ is independent of the integrand in (4.6), denoted by $K_{N,n}$, which is $\mathcal{F}_{\zeta_{N,n}}$-measurable. Having made this observation the proof becomes classical: it can be done in a standard way by approximating the integrand $K_{N,n} = \{K_{N,n}(s)\}_{s\in[\zeta_{N,n},1]}$ by a sequence $\big(K^j_{N,n}\big)_{j\in\mathbb{N}} = \big(\{K^j_{N,n}(s)\}_{s\in[\zeta_{N,n},1]}\big)_{j\in\mathbb{N}}$ of elementary processes of the form:

$$K^j_{N,n}(s) = \sum_{p=0}^{q-1} A_p \mathbb{1}_{[t_p,t_{p+1})}(s)\,,$$

where the random variables A_p, $0\le p<q$, are $\mathcal{F}_{\zeta_{N,n}}$-measurable, and the finite sequence $(t_p)_{0\le p\le q}$ is a subdivision of the interval $[\zeta_{N,n},1]$. □

Roughly speaking, the following lemma shows that $\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)$ behaves in the same way as $(QN)^{-2H(\zeta_{N,n})}$

Lemma 4.4. *One has almost surely, for all integers $Q\in\mathbb{N}$, $N\ge N_0$ and $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$, that*

$$\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) = (QN)^{-2H(\zeta_{N,n})}\int_{L-e_{QN}}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du \tag{4.7}$$

and consequently that

$$c'(QN)^{-2H(\zeta_{N,n})} \le \mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) \le c''(QN)^{-2H(\zeta_{N,n})}\,, \tag{4.8}$$

where c' and c'' are two finite, deterministic and strictly positive constants not depending on δ, Q, N and n.

Proof of Lemma 4.4: One can derive from (4.5), (4.4) and the change of variable $u=(QN)s-k$ that one has almost surely, for integers $Q\in\mathbb{N}$, $N\ge N_0$, $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$ and $k\in\nu_{QN}(\mathcal{I}_{N,n})$, that

$$\mathbb{E}\big(|\widehat{d}^{\,\delta,n}_{QN,k}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big) = (QN)^{-2H(\zeta_{N,n})}\int_{L-e_{QN}}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du\,. \tag{4.9}$$

Thus combining (3.22) and (4.9) one obtains (4.7). Then notice that (3.2), (1.2), (2.1), (3.4), (3.16) and (1.8) entail that

$$\int_{L-e_{QN}}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du \ge \int_{L-1}^{L}(L-u)^{2H(\zeta_{N,n})-1}\,du \ge c' \tag{4.10}$$

and

$$\int_{L-e_{QN}}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du \le \int_{-\infty}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du \le c''\,, \tag{4.11}$$

where the strictly positive constant $c' := (2\overline{H})^{-1}$ and the constant c'' is equal to the constant c_3 defined in (3.34). Finally, putting together (4.7), (4.10) and (4.11) one gets (4.8). □

Remark 4.5. The integers $Q\in\mathbb{N}$, $N\ge N_0$ and $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$ are arbitrary and fixed. One denotes by $\mathcal{G}$ a Gaussian Hilbert space on $\mathbb{R}$ spanned by a centred real-valued Gaussian vector $(G_k)_{k\in\nu_{QN}(\mathcal{I}_{N,n})}$ whose distribution is equal to the conditional distribution of the random vector $(\widehat{d}^{\,\delta,n}_{QN,k})_{k\in\nu_{QN}(\mathcal{I}_{N,n})}$ with respect to the sigma-algebra $\mathcal{F}_{\zeta_{N,n}}$ (see Lemma 4.3) for some given arbitrary value of the random variable $H(\zeta_{N,n})$. Then, for the same given value of $H(\zeta_{N,n})$, the conditional distribution with respect to $\mathcal{F}_{\zeta_{N,n}}$ of the random variable $\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)$ (see (3.22)) is equal to the distribution of the random variable $|\nu_{QN}(\mathcal{I}_{N,n})|^{-1}\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}\big(|G_k|^2 - \mathbb{E}(|G_k|^2)\big)$. Since the latter random variable belongs to $\overline{\mathcal{P}}_2(\mathcal{G})$ the second order chaos associated to

$\mathcal{G}$ (see Definition 2.1 on page 17 in Janson (1997)), one knows from Theorem 5.10 on page 62 in Janson (1997) that, for any fixed $q \in \mathbb{N}$, there exists a universal deterministic finite constant $\widehat{c}(q)$, only depending on q, such that

$$\mathbb{E}\bigg(\Big||\nu_{QN}(\mathcal{I}_{N,n})|^{-1}\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}\big(|G_k|^2-\mathbb{E}(|G_k|^2)\big)\Big|^{2q}\bigg)$$
$$\leq \widehat{c}(q)\Bigg(\mathbb{E}\bigg(\Big||\nu_{QN}(\mathcal{I}_{N,n})|^{-1}\sum_{k\in\nu_{QN}(\mathcal{I}_{N,n})}\big(|G_k|^2-\mathbb{E}(|G_k|^2)\big)\Big|^{2}\bigg)\Bigg)^q.$$

Therefore, one has

$$\mathbb{E}\bigg(\Big|\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})-\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big|^{2q}\Big|\mathcal{F}_{\zeta_{N,n}}\bigg) \tag{4.12}$$
$$\leq \widehat{c}(q)\Bigg(\mathbb{E}\bigg(\Big|\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})-\mathbb{E}\big(\widehat{V}^{\delta}_{QN}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big|^{2}\Big|\mathcal{F}_{\zeta_{N,n}}\bigg)\Bigg)^q.$$

Also, notice that one can derive from Theorem 3.9 on page 26 in Janson (1997) that

$$\mathbb{E}\Big(\big(|G_{k'}|^2-\mathbb{E}(|G_{k'}|^2)\big)\big(|G_{k''}|^2-\mathbb{E}(|G_{k''}|^2)\big)\Big)=2\big(\mathbb{E}(G_{k'}G_{k''})\big)^2,\quad \text{for all } k',k''\in\nu_{QN}(\mathcal{I}_{N,n}),$$

which implies that

$$\mathbb{E}\bigg(\Big(|\widehat{d}^{\delta,n}_{QN,k'}|^2-\mathbb{E}\big(|\widehat{d}^{\delta,n}_{QN,k'}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big)\Big(|\widehat{d}^{\delta,n}_{QN,k''}|^2-\mathbb{E}\big(|\widehat{d}^{\delta,n}_{QN,k''}|^2\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big)\Big|\mathcal{F}_{\zeta_{N,n}}\bigg)$$
$$=2\Big(\mathbb{E}\big(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}\big|\mathcal{F}_{\zeta_{N,n}}\big)\Big)^2,\quad \text{for all } k',k''\in\nu_{QN}(\mathcal{I}_{N,n}). \tag{4.13}$$

Lemma 4.6. *There exists a finite deterministic constant c such that, for all integers $Q\in\mathbb{N}$, $N\geq N_0$, $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$ and $k',k''\in\nu_{QN}(\mathcal{I}_{N,n})$, one has*

$$\big|\mathbb{E}(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}|\mathcal{F}_{\zeta_{N,n}})\big|\leq c\,(QN)^{-2H(\zeta_{N,n})}\big(1+|k'-k''|\big)^{-(L-\overline{H})}. \tag{4.14}$$

Proof of Lemma 4.6: The integers $Q\in\mathbb{N}$, $N\geq N_0$, $n\in\{0,\dots,\lfloor\theta_N^{-1}\rfloor-1\}$ and $k',k''\in\nu_N(\mathcal{I}_{N,n})$ are arbitrary; moreover one can assume without any restriction that $k''\geq k'$. One can derive from (4.5) and the change of variable $QNs-k'$ that

$$\big|\mathbb{E}(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}|\mathcal{F}_{\zeta_{N,n}})\big| \tag{4.15}$$
$$\leq (QN)^{-2H(\zeta_{N,n})}\int_{\mathbb{R}}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|\,du\,.$$

One denotes by c_1 the finite deterministic constant c_3 defined in (3.34) which does not depend on Q, N, n, $H(\zeta_{N,n})$, k' and k''. Using (4.15), the Cauchy-Scharwz inequality, (3.2) and (1.2), one gets that

$$\big|\mathbb{E}(\widehat{d}^{\delta,n}_{N,k'}\,\widehat{d}^{\delta,n}_{N,k''}|\mathcal{F}_{\zeta_{N,n}})\big|\leq (QN)^{-2H(\zeta_{N,n})}\int_{-\infty}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2du\leq c_1(QN)^{-2H(\zeta_{N,n})} \tag{4.16}$$

and

$$\begin{aligned}
&\big|\mathbb{E}(\widehat{d}^{\delta,n}_{QN,k'}\,\widehat{d}^{\delta,n}_{QN,k''}|\mathcal{F}_{\zeta_{N,n}})\big| \\
&\le (QN)^{-2H(\zeta_{N,n})}\int_{-\infty}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|\,du \\
&= (QN)^{-2H(\zeta_{N,n})}\bigg(\int_{2^{-1}(k'-k'')}^{L}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|\,du \\
&\qquad\qquad +\int_{-\infty}^{2^{-1}(k'-k'')}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|\,du\bigg) \\
&\le \sqrt{c_1}(QN)^{-2H(\zeta_{N,n})}\Bigg(\sqrt{\int_{2^{-1}(k'-k'')}^{L}\Big|\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|^2\,du} \qquad (4.17)\\
&\qquad\qquad +\sqrt{\int_{-\infty}^{2^{-1}(k'-k'')}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du}\Bigg).
\end{aligned}$$

Next observe that, under the condition that

$$k'-k''\le -4L\,, \qquad (4.18)$$

one clearly has $2^{-1}(k'-k'')\le -2L$, and thus one can derive from (3.16) that

$$\begin{aligned}
&\int_{-\infty}^{2^{-1}(k'-k'')}\Big|\Phi\big(u,H(\zeta_{N,n})\big)\Big|^2\,du \qquad (4.19)\\
&\le c_2^2\int_{-\infty}^{2^{-1}(k'-k'')}\big(1+L-u\big)^{2\overline{H}-2L-1}\,du \le c_3\big(1+L+k''-k'\big)^{-2(L-\overline{H})}\,,
\end{aligned}$$

where c_2 is the finite deterministic constant c in (3.16) and $c_3 := 2^{2(L-\overline{H})-1}(L-\overline{H})^{-1}c_2^2$. Also, observe that under the condition (4.18), for all $u\in\big[2^{-1}(k'-k''),L\big]$, one has $u+k'-k''\le -3L$, and thus one can derive from (3.16) that

$$\begin{aligned}
&\int_{2^{-1}(k'-k'')}^{L}\Big|\Phi\big(u+k'-k'',H(\zeta_{N,n})\big)\Big|^2\,du \le c_2^2\int_{-\infty}^{L}\big(1+L-u+k''-k'\big)^{2\overline{H}-2L-1}\,du \\
&\le \frac{c_2^2}{2(L-\overline{H})}\big(1+k''-k'\big)^{-2(L-\overline{H})}\le c_3\big(1+k''-k'\big)^{-2(L-\overline{H})}\,. \qquad (4.20)
\end{aligned}$$

Finally setting $c := c_1(4L)^{L-\overline{H}}+2\sqrt{c_1c_3}$, it follows from (4.16), (4.17), (4.19) and (4.20) that (4.14) is satisfied. □

We are now ready to prove Lemma 4.2.

Proof of Lemma 4.2: Let b be a fixed real number such that

$$\beta<b<1/2\,, \qquad (4.21)$$

where β is as in (4.2). Let $q\in\mathbb{N}$ be fixed and big enough so that

$$q(1-\mu')(1-2b)>\mu'+1\,, \qquad (4.22)$$

where $\mu' \in (0,1)$ is as in (2.11). Using Markov inequality one obtains, for all integers $Q \in \mathbb{N}$ and $N \geq N_0$, that

$$\begin{aligned}
&\mathbb{P}\bigg(N^{b(1-\mu')} \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor} \bigg| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \bigg| > 1 \bigg) \\
&\leq \sum_{n=0}^{\lfloor \theta_N^{-1}\rfloor -1} \mathbb{P}\bigg(N^{b(1-\mu')} \bigg| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \bigg| > 1 \bigg) \\
&\leq N^{2qb(1-\mu')} \sum_{n=0}^{\lfloor \theta_N^{-1}\rfloor -1} \mathbb{E}\bigg(\bigg| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \bigg|^{2q} \bigg). \qquad (4.23)
\end{aligned}$$

Moreover, the expectations in (4.23) can be expressed, for all $n \in \{0, \dots, \lfloor \theta_N^{-1}\rfloor - 1\}$, as:

$$\begin{aligned}
&\mathbb{E}\bigg(\bigg| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \bigg|^{2q} \bigg) \\
&= \mathbb{E}\bigg(\Big(\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)\Big)^{-2q} \Big| \widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) \Big|^{2q} \bigg) \qquad (4.24) \\
&= \mathbb{E}\bigg(\Big(\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)\Big)^{-2q} \mathbb{E}\Big(\Big| \widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) \Big|^{2q} \Big| \mathcal{F}_{\zeta_{N,n}} \Big) \bigg).
\end{aligned}$$

On the other hand, it follows from (3.22), (4.13), (4.14) and the inequality $L - \overline{H} > 1$ that, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1}\rfloor - 1\}$, one has

$$\begin{aligned}
&\mathbb{E}\Big(\Big| \widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) - \mathbb{E}\big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\big) \Big|^{2} \Big| \mathcal{F}_{\zeta_{N,n}} \Big) \\
&= \big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2} \sum_{k',k''\in \nu_{QN}(\mathcal{I}_{N,n})} \mathbb{E}\bigg(\Big(|\widehat{d}_{QN,k'}^{\delta,n}|^2 - \mathbb{E}\big(|\widehat{d}_{QN,k'}^{\delta,n}|^2 \big| \mathcal{F}_{\zeta_{N,n}}\big)\Big) \\
&\qquad\qquad \times \Big(|\widehat{d}_{QN,k''}^{\delta,n}|^2 - \mathbb{E}\big(|\widehat{d}_{QN,k''}^{\delta,n}|^2 \big| \mathcal{F}_{\zeta_{N,n}}\big)\Big) \Big| \mathcal{F}_{\zeta_{N,n}} \bigg) \\
&= 2\big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2} \sum_{k',k''\in \nu_{QN}(\mathcal{I}_{N,n})} \Big(\mathbb{E}\big(\widehat{d}_{QN,k'}^{\delta,n}\, \widehat{d}_{QN,k''}^{\delta,n} \big| \mathcal{F}_{\zeta_{N,n}}\big)\Big)^2 \\
&\leq 2c_1^2 \big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-2} (QN)^{-4H(\zeta_{N,n})} \sum_{k',k''\in \nu_{QN}(\mathcal{I}_{N,n})} \big(1 + |k'-k''|\big)^{-2(L-\overline{H})} \\
&\leq c_2 \big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-1} (QN)^{-4H(\zeta_{N,n})}, \qquad (4.25)
\end{aligned}$$

where c_1 denotes the constant c in (4.14) and $c_2 := 4c_1^2 \sum_{j=1}^{+\infty} j^{-2(L-\overline{H})} < +\infty$. Next, putting together (4.24), the first inequality in (4.8), (4.12), (4.25), the first inequality in (2.13) and (2.11), one gets, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1}\rfloor - 1\}$, that

$$\mathbb{E}\bigg(\bigg| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)} - 1 \bigg|^{2q} \bigg) \leq c_3 \big|\nu_{QN}(\mathcal{I}_{N,n})\big|^{-q} \leq c_4 N^{-q(1-\mu')}, \qquad (4.26)$$

where c_3 and c_4 are two deterministic finite constants not depending on Q, N and n. Then, one can derive from (4.23), (4.26) and (2.11) that, for all integers $Q \in \mathbb{N}$ and $N \geq N_0$,

$$\mathbb{P}\left(N^{b(1-\mu')} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)} - 1 \right| > 1 \right)$$
$$\leq c_4 N^{2qb(1-\mu')} \theta_N^{-1} N^{-q(1-\mu')} \leq \frac{c_4}{\kappa'} N^{\mu' - q(1-\mu')(1-2b)} . \tag{4.27}$$

Thus, it follows from (4.22) and (4.27) that

$$\sum_{N=N_0}^{+\infty} \mathbb{P}\left(N^{b(1-\mu')} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)} - 1 \right| > 1 \right) < +\infty .$$

Then the Borel-Cantelli Lemma entails that one has almost surely

$$\sup_{N \geq N_0} \left\{ N^{b(1-\mu')} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)} - 1 \right| \right\} < +\infty . \tag{4.28}$$

Finally, combining (4.21) and (4.28) one gets (4.2). □

We are now ready to prove Lemma 4.1.

Proof of Lemma 4.1: First observe that, for all integers $Q \in \mathbb{N}$ and $N \geq N_0$, one has

$$\max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)} - 1 \right|$$
$$= \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)}} - 1 \right| \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)}} + 1 \right|$$
$$\leq \left(\max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)}} - 1 \right| \right)^2$$
$$+2 \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)}} - 1 \right| .$$

Thus, in order to prove that (4.1) holds, it is enough to show that, almost surely,

$$\limsup_{N \to +\infty} \left\{ N^{\beta} \max_{0 \leq n < \lfloor \theta_N^{-1} \rfloor} \left| \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\left(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n}) \big| \mathcal{F}_{\zeta_{N,n}}\right)}} - 1 \right| \right\} = 0 . \tag{4.29}$$

Let us point out that throughout this proof β denotes an arbitrary fixed positive real number satisfying (2.17). Next observe that (2.4), (3.5), (3.25), (3.9), (3.22), (3.23) and the triangle inequality imply, for all integers $Q \in \mathbb{N}$, $N \geq N_0$ and $n \in \{0, \dots, \lfloor \theta_N^{-1} \rfloor - 1\}$, that

$$\begin{aligned}
&\sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} - \sqrt{\frac{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} - \sqrt{\frac{\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}\\
&\leq \sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}\\
&\leq \sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} + \sqrt{\frac{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} + \sqrt{\frac{\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}
\end{aligned}$$

and consequently that

$$\begin{aligned}
&\max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\left|\sqrt{\frac{V_{QN}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} - 1\right|\\
&\leq \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} - 1\right| + \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\sqrt{\frac{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}\\
&\qquad + \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\sqrt{\frac{\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}}\\
&\leq \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} - 1\right| + c_1 N^{\overline{H}} \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\sqrt{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}\\
&\qquad + c_1 \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\sqrt{N^{2H(\zeta_{N,n})}\,\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}\,, \qquad (4.30)
\end{aligned}$$

where c_1 is a deterministic finite constant not depending on N. Notice that the last inequality in (4.30) results from (4.8) and (1.8). It clearly follows from (2.17), (2.16), Lemma 3.3, (3.3) and Lemma 3.8 that one has almost surely

$$\limsup_{N\to+\infty}\left\{N^{\beta+\overline{H}} \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\sqrt{\breve{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}\right\} = 0$$

and

$$\limsup_{N\to+\infty}\left\{N^{\beta} \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\sqrt{N^{2H(\zeta_{N,n})}\,\overline{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}\right\} = 0\,.$$

Thus, in view of (4.30), in order to show that (4.29) holds, it is enough to prove that

$$\limsup_{N\to+\infty}\left\{N^{\beta} \max_{0\leq n<\lfloor \theta_N^{-1}\rfloor}\left|\sqrt{\frac{\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n})\big|\mathcal{F}_{\zeta_{N,n}}\Big)}} - 1\right|\right\} = 0\,. \qquad (4.31)$$

Combining (2.17) with Lemma 4.2 and the inequality $|\sqrt{z} - 1| \leq |z - 1|$, for every $z \in \mathbb{R}_+$, one gets (4.31). □

5. Final steps of the proof of Theorem 2.2

Lemma 5.1. *Let $\overline{H}$, γ, L, μ and δ be as in (1.8), (1.10), (2.2), (2.10) and (3.3). Let β be an arbitrary real number satisfying*

$$0 < \beta < \min\left\{\gamma\mu,\, 2(1-\mu)(L-\overline{H})\right\}. \tag{5.1}$$

Then, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,

$$\limsup_{N\to+\infty}\left\{N^{\beta}\sup_{s\in[0,1]}\left|\frac{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)} - Q^{2H(s)}\right|\right\} = 0\,, \tag{5.2}$$

where, for every $s \in [0,1]$,

$$n_0(N,s) := \begin{cases} \lfloor \theta_N^{-1} s\rfloor & \text{if } s \in \big[0, (\lfloor\theta_N^{-1}\rfloor - 1)\theta_N\big), \\ \lfloor\theta_N^{-1}\rfloor - 1 & \text{if } s \in \big[(\lfloor\theta_N^{-1}\rfloor - 1)\theta_N, 1\big], \end{cases} \tag{5.3}$$

and

$$n_1(N,s) := \begin{cases} \lfloor \theta_N^{-1} s\rfloor + 1 & \text{if } s \in \big[0, (\lfloor\theta_N^{-1}\rfloor - 1)\theta_N\big), \\ \lfloor\theta_N^{-1}\rfloor - 1 & \text{if } s \in \big[(\lfloor\theta_N^{-1}\rfloor - 1)\theta_N, 1\big]. \end{cases} \tag{5.4}$$

Proof of Lemma 5.1: One can derive from (4.7) and (1.8), that one has almost surely, for each real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$, that

$$\begin{aligned}
&\left|\frac{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)} - Q^{2H(s)}\right| \\
&= \left|Q^{2H(\zeta_{N,n_i(N,s)})}\frac{\int_{L-e_N}^{L}\big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du}{\int_{L-e_{QN}}^{L}\big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du} - Q^{2H(s)}\right| \\
&\leq U_{i,N}(s) + V_{i,N}(s)\,,
\end{aligned} \tag{5.5}$$

where

$$U_{i,N}(s) := Q^{2\overline{H}}\left|Q^{2(H(\zeta_{N,n_i(N,s)})-H(s))} - 1\right|\frac{\int_{L-e_N}^{L}\big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du}{\int_{L-e_{QN}}^{L}\big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du} \tag{5.6}$$

and

$$\begin{aligned}
V_{i,N}(s) &:= Q^{2\overline{H}}\left|\frac{\int_{L-e_N}^{L}\big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du}{\int_{L-e_{QN}}^{L}\big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du} - 1\right| \\
&= Q^{2\overline{H}}\frac{\int_{L-e_{QN}}^{L-e_N}\big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du}{\int_{L-e_{QN}}^{L}\big|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\big|^2\,du}\,.
\end{aligned} \tag{5.7}$$

Next observe that, one can derive from the mean value Theorem, (1.8), (1.10), (3.21), (5.3), (5.4) and (2.10) that

$$\begin{aligned}
&\left|Q^{2(H(\zeta_{N,n_i(N,s)})-H(s))} - 1\right| \\
&\leq 2(\log Q)\exp\Big(2(\log Q)\overline{H}\Big)\left|H(\zeta_{N,n_i(N,s)}) - H(s)\right| \\
&\leq \rho(2\kappa+1)^{\gamma}\log(Q^2)Q^{2\overline{H}}N^{-\gamma\mu}\,.
\end{aligned} \tag{5.8}$$

Moreover, it easily follows (3.4) that

$$\frac{\int_{L-e_N}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du}{\int_{L-e_{QN}}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du} \leq 1 \,. \tag{5.9}$$

Thus, combining (5.6), (5.8) and (5.9), one gets, for all $N \geq N_0$, that

$$\sup_{s\in[0,1]} U_{i,N}(s) \leq \rho(2\kappa+1)^{\gamma} \log(Q^2) Q^{4\overline{H}} N^{-\gamma\mu} \,. \tag{5.10}$$

Next observe that, similarly to (4.10), it can be shown that, for all real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$, one has

$$\int_{L-e_{QN}}^{L} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du \geq (2\overline{H})^{-1} \,.$$

Thus, one can derive from (5.7) that, for all real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$,

$$V_{i,N}(s) \leq 2\overline{H} Q^{2\overline{H}} \int_{-\infty}^{L-e_N} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du \,. \tag{5.11}$$

Notice that there is no restriction to assume that $N \geq (3L)^{1/\delta} + N_0$ which implies that $L - e_N \leq -2L$. Then, using (3.16), one gets that

$$\begin{aligned} &\int_{-\infty}^{L-e_N} \left|\Phi\big(u, H(\zeta_{N,n_i(N,s)})\big)\right|^2 du \\ &\leq c_1^2 \int_{-\infty}^{L-e_N} \big(1 + L - u\big)^{2\overline{H}-2L-1} du \leq \frac{c_1^2}{2(L-\overline{H})} N^{-2\delta(L-\overline{H})} \,, \end{aligned} \tag{5.12}$$

where c_1 denotes the finite deterministic constant c in (3.16) which does not depend on N and $\zeta_{N,n_i(N,s)}$. Then combining (5.11) and (5.12), one obtains, for all integers $i \in \{0,1\}$ and $N \geq (3L)^{1/\delta} + N_0$, that

$$\sup_{s\in[0,1]} V_{i,N}(s) \leq 2\overline{H} Q^{2\overline{H}} \frac{c_1^2}{2(L-\overline{H})} N^{-2\delta(L-\overline{H})} \,. \tag{5.13}$$

Finally, putting together (5.1), (5.5), (5.10), (5.13) and (3.3), it follows that (5.2) holds. □

Lemma 5.2. *Let β be an arbitrary real number satisfying the condition (2.17). Then, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,*

$$\limsup_{N\to+\infty} \left\{ N^{\beta} \sup_{s\in[0,1]} \left| \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} - Q^{2H(s)} \right| \right\} = 0 \,, \tag{5.14}$$

where $n_0(N,s)$ and $n_1(N,s)$ are as in (5.3) and (5.4).

Proof of Lemma 5.2: First observe that, for each real number $s \in [0,1]$ and integers $i \in \{0,1\}$, $Q \in \mathbb{N}$ and $N \geq N_0$, one has

$$\begin{aligned}
&\left|\frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} - Q^{2H(s)}\right| \\
&= \left|R_N^i(s) S_{Q,N}^i(s) \big(Z_{Q,N}^i(s)\big)^{-1} - Q^{2H(s)}\right| \\
&\leq Q^{2H(s)} \left|R_N^i(s)\big(Z_{Q,N}^i(s)\big)^{-1} - 1\right| + \frac{R_N^i(s)}{Z_{Q,N}^i(s)} \left|S_{Q,N}^i(s) - Q^{2H(s)}\right| \\
&\leq Q^{2\overline{H}} \left|R_N^i(s) - 1\right| + Q^{2\overline{H}} \frac{R_N^i(s)}{Z_{Q,N}^i(t)} \left|Z_{Q,N}^i(s) - 1\right| + \frac{R_N^i(s)}{Z_{Q,N}^i(s)} \left|S_{Q,N}^i(s) - Q^{2H(s)}\right|,
\end{aligned} \tag{5.15}$$

where

$$R_N^i(s) := \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}, \tag{5.16}$$

$$S_{Q,N}^i(s) := \frac{\mathbb{E}\Big(\widehat{V}_N^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)} \tag{5.17}$$

and

$$Z_{Q,N}^i(s) := \frac{V_{QN}(\mathcal{I}_{N,n_i(N,s)})}{\mathbb{E}\Big(\widehat{V}_{QN}^{\delta}(\mathcal{I}_{N,n_i(N,s)})\big|\mathcal{F}_{\zeta_{N,n_i(N,s)}}\Big)}. \tag{5.18}$$

Recall that $\delta := 1-\mu$ (see (3.3)) and that μ satisfies (2.16). Next notice that one knows from (2.17), (5.16), (5.18) and Lemma 4.1 that, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,

$$\sup_{s\in[0,1]} \left|R_N^i(s) - 1\right| = o\big(N^{-\beta}\big) \tag{5.19}$$

and

$$\sup_{s\in[0,1]} \left|Z_N^i(s) - 1\right| = o\big(N^{-\beta}\big). \tag{5.20}$$

Moreover, it results from (5.19) that, almost surely,

$$\sup_{N\geq N_0} \sup_{s\in[0,1]} R_N^i(s) < +\infty, \tag{5.21}$$

and it follows from (5.20) that, almost surely,

$$\inf_{s\in[0,1]} Z_N^i(s) \geq 1/2, \quad \text{for all } N \text{ big enough.} \tag{5.22}$$

On the other hand, one knows from (2.17), (5.17) and Lemma 5.1 that, one has almost surely, for all $i \in \{0,1\}$ and $Q \in \mathbb{N}$,

$$\sup_{s\in[0,1]} \left|S_N^i(s) - Q^{2H(s)}\right| = o\big(N^{-\beta}\big). \tag{5.23}$$

Finally, putting together (5.15) and (5.19) to (5.23) one obtains (5.14). □

We are now in position to complete the proof of Theorem 2.2.

End of the proof of Theorem 2.2: One can derive from (2.14), (2.15), (2.9), (5.3), (5.4) and (1.8) that, for all integers $Q \geq 2$ and $N \geq N_0$, one has

$$\begin{aligned} & \sup_{s\in[0,1]} \left|H(s) - \widetilde{H}^{Q}_{N,\theta_N}(s)\right| \\ & \leq \sum_{i=0}^{1} \sup_{s\in[0,1]} \left|\log_{Q^2}\left(Q^{2H(s)}\right) - \log_{Q^2}\left(\frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})}\right)\right| . \end{aligned} \tag{5.24}$$

Next observe that one knows from (5.14) and (1.8) that, one has almost surely, for all N large enough,

$$\inf_{s\in[0,1]} \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})} \geq 2^{-1} Q^{2\underline{H}} .$$

Thus, one can derive (5.24) and the mean value Theorem that one has almost surely, for all N large enough,

$$\begin{aligned} & \sup_{s\in[0,1]} \left|H(s) - \widetilde{H}^{Q}_{N,\theta_N}(s)\right| \\ & \leq \frac{2Q^{-2\underline{H}}}{\log(Q^2)} \sum_{i=0}^{1} \sup_{s\in[0,1]} \left|Q^{2H(s)} - \frac{V_N(\mathcal{I}_{N,n_i(N,s)})}{V_{QN}(\mathcal{I}_{N,n_i(N,s)})}\right| . \end{aligned} \tag{5.25}$$

Then, (5.25) and Lemma 5.2 imply that (2.18) holds. □

Acknowledgements

The authors are grateful to the anonymous referee for her/his valuable comments which have led to improvements of the manuscript.

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