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Fluctuations of random Motzkin paths II

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Abstract. We compute limit fluctuations of random Motzkin paths with arbitrary end-points as the length of the path tends to infinity.

1. Introduction

1.1. *Model and main result.* A Motzkin path of length L is a sequence of steps on the integer lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ that starts at point $(0, n_0)$ with the initial altitude n_0 and ends at point (L, n_L) at the final altitude n_L for some non-negative integers n_0, n_L, L. The steps can be up, down, or horizontal, along the vectors $(1, 1)$, $(1, -1)$ and $(1, 0)$ respectively, and the path cannot fall below the horizontal axis, see Flajolet and Sedgewick (2009, Definition V.4, page 319) or Viennot (1985). We represent a Motzkin path of length $L \geq 1$ as a sequence of integers $(\gamma_0, \dots, \gamma_L) \in \mathbb{Z}_{\geq 0}^{L+1}$ such that $|\gamma_k - \gamma_{k-1}| \leq 1, k = 1, \dots, L$ subject to the non-negativity condition $\sum_{j=0}^{k} \gamma_j \geq 0$ for $k = 0, 1 \dots, L$. We say that the k-th step of the path is up, down, and horizontal respectively, if $\gamma_k - \gamma_{k-1} = 1, -1, 0$ respectively. By $\mathcal{M}_{i,j}^{(L)}$ we denote the family of all Motzkin paths of length L with the initial altitude $\gamma_0 = i$ and the final altitude $\gamma_L = j$. Our goal is to study statistical properties of random Motzkin paths, selected at random from the discrete set

$$\mathcal{M}^{(L)} = \bigcup_{i,j \geq 0} \mathcal{M}_{i,j}^{(L)}$$

in the limit as $L \to \infty$. Our setup generalizes our previous work Bryc and Wang (2019a), where we studied statistical properties of the three counting processes that count the up steps, the horizontal steps, and the down steps for a Motzkin path γ selected at random uniformly from the set $\mathcal{M}_{0,0}^{(L)}$.

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To define these counting processes, we first introduce the indicators of these steps:

$$\varepsilon_k^+ \equiv \varepsilon_k^+(\gamma) := \mathbf{1}_{\{\gamma_k > \gamma_{k-1}\}}, \quad \varepsilon_k^- \equiv \varepsilon_k^-(\gamma) := \mathbf{1}_{\{\gamma_k < \gamma_{k-1}\}}, \quad \varepsilon_k^0 \equiv \varepsilon_k^0(\gamma) := \mathbf{1}_{\{\gamma_k = \gamma_{k-1}\}}, \gamma \in \mathcal{M}^{(L)}, \tag{1.1}$$

and $k = 1, \dots, L$. For the sake of simplicity we drop the dependence on γ of ε's most of the time. Then given a path γ of length L, the counts of the up steps, down steps and horizontal steps up to position $\lfloor xL \rfloor$, where $x \in [0,1]$ are then

$$U_L(x) := \sum_{k=1}^{\lfloor Lx \rfloor} \varepsilon_k^+, \quad D_L(x) := \sum_{k=1}^{\lfloor Lx \rfloor} \varepsilon_k^-, \quad H_L(x) := \sum_{k=1}^{\lfloor Lx \rfloor} \varepsilon_k^0, \quad x \in [0,1]. \tag{1.2}$$

We introduce a probability measure on $\mathcal{M}^{(L)}$ as follows. For each path $\gamma \in \mathcal{M}_{i,j}^{(L)}$, we define its weight

$$w_\sigma(\gamma) := \sigma^{\sum_{k=1}^L \varepsilon_k^0}, \quad \gamma \in \mathcal{M}_{i,j}^{(L)}, L \in \mathbb{N}.$$

Note that with $\sigma = 1$ this gives each path the same weight. Since $\mathcal{M}_{i,j}^{(L)}$ is a finite set,

$$\mathfrak{W}_{i,j}^{(L)} = \sum_{\gamma \in \mathcal{M}_{i,j}^{(L)}} w(\gamma), i, j \geq 0,$$

are well defined. In addition to the weights of the edges, we also weight the initial and the final altitudes of each path with geometric weights

$$\alpha_{L,n} := (\rho_{L,0})^n, \quad \beta_{L,n} := (\rho_{L,1})^n, \quad n \geq 0, \tag{1.3}$$

with

$$\rho_{L,0} = 1 - \frac{\mathsf{c}}{\sqrt{L}} \quad \text{and} \quad \rho_{L,1} = 1 - \frac{\mathsf{a}}{\sqrt{L}} \quad \text{for some } \mathsf{a}, \mathsf{c} \in \mathbb{R}, \mathsf{a} + \mathsf{c} > 0. \tag{1.4}$$

Namely, the countable set $\mathcal{M}^{(L)} = \bigcup_{i,j \geq 0} \mathcal{M}_{i,j}^{(L)}$ becomes a probability space with the discrete probability measure $\mathbb{P}_L$ determined by

$$\mathbb{P}_L(\gamma) \equiv \mathbb{P}_{\mathsf{a},\mathsf{c},\sigma,L}(\gamma) \equiv \mathbb{P}_{\mathsf{a},\mathsf{c},\sigma,L}(\{\gamma\}) = \frac{\alpha_{L,\gamma_0}\beta_{L,\gamma_L}}{\mathfrak{C}_L} w(\gamma), \text{ for all } \gamma \in \mathcal{M}^{(L)}, \tag{1.5}$$

with

$$\mathfrak{C}_L := \sum_{i,j \geq 0} \alpha_{L,i} \mathfrak{W}_{i,j}^{(L)} \beta_{L,j} < \infty.$$

Note that throughout for finite L implicitly we assume L is large enough so that $\rho_{L,0}\rho_{L,1} \in (0,1)$ and hence $\mathbb{P}_L$ is a well-defined probability measure. In our previous work Bryc and Wang (2019a, Theorem 1.1) we proved that if γ is selected uniformly from $\mathcal{M}_{0,0}^{(L)}$, then

$$\frac{1}{\sqrt{2L}} \left\{ U_L(x) - \frac{\lfloor Lx \rfloor}{3}, H_L(x) - \frac{\lfloor Lx \rfloor}{3}, D_L(x) - \frac{\lfloor Lx \rfloor}{3} \right\}_{x \in [0,1]} \xrightarrow{f.d.d.} \left\{ \frac{1}{2\sqrt{3}} B_x^{ex} + \frac{1}{6} B_x, -\frac{1}{3} B_x, \frac{1}{6} B_x - \frac{1}{2\sqrt{3}} B_x^{ex} \right\}_{x \in [0,1]},$$

where $(B_x)_{x \in [0,1]}$ is a Brownian motion, $(B_x^{ex})_{x \in [0,1]}$ is a Brownian excursion, and the processes $(B_x)_{x \in [0,1]}$ and $(B_x^{ex})_{x \in [0,1]}$ are independent. Formally, this model corresponds to the choice of $\rho_{L,0} = 0, \rho_{L,1} = 0, \sigma = 1$.

Now, with more general end-point weights that vary with L, the asymptotics of (1.2) relies on another Markov process instead of the Brownian excursion. Let

$$\mathsf{q}_t(x,y) := \frac{1}{\sqrt{2\pi t}} \left[\exp\left(-\frac{1}{2t}(x-y)^2 \right) - \exp\left(-\frac{1}{2t}(x+y)^2 \right) \right] \mathbf{1}_{x>0,y>0}, \quad t > 0, \tag{1.6}$$

denote the transition kernel of the Brownian motion killed at hitting zero. Consider the Markov process $\left(\widetilde{\eta}^{(\mathsf{a},\mathsf{c})}\right)_{x\in[0,1]}$ with joint probability density function at points $0 = x_0 < x_1 < \cdots < x_d = 1$ given by

$$\widetilde{p}^{(\mathsf{a},\mathsf{c})}_{x_0,\ldots,x_d}(y_0,\ldots,y_d) := \frac{1}{\mathfrak{C}_{\mathsf{a},\mathsf{c}}} e^{-(\mathsf{c}y_0+\mathsf{a}y_d)/\sqrt{2}} \prod_{k-1}^{d} \mathsf{q}_{x_k-x_{k-1}}(y_{k-1},y_k), \quad y_0,\ldots,y_d > 0, \tag{1.7}$$

with the normalizing constant

$$\mathfrak{C}_{\mathsf{a},\mathsf{c}} = \int_{\mathbb{R}^2_+} e^{-(\mathsf{c}x+\mathsf{a}y)/\sqrt{2}} \mathsf{q}_1(x,y)dxdy, \tag{1.8}$$

given by the explicit expression (A.1). Let $\eta^{(\mathsf{a},\mathsf{c})}$ denote the increment process

$$\eta^{(\mathsf{a},\mathsf{c})}_x := \widetilde{\eta}^{(\mathsf{a},\mathsf{c})}_x - \widetilde{\eta}^{(\mathsf{a},\mathsf{c})}_0, x \in [0,1]. \tag{1.9}$$

Recall that for each L fixed we let $(\gamma_0,\ldots,\gamma_L)$ denote a sequence from $\mathcal{M}^{(L)}$ sampled from $\mathbb{P}_L$ given in (1.5), including in particular the left-hand side of (1.11), and the counting processes U_L, H_L, D_L depend on $(\gamma_0,\ldots,\gamma_L)$ as in (1.2). Our main result is the following.

Theorem 1.1. *Assume* $\mathsf{a},\mathsf{c} \in \mathbb{R}, \mathsf{a}+\mathsf{c} > 0$ *and* $\sigma > 0$. *Set*

$$\mathsf{a}' = \frac{2\mathsf{a}}{\sqrt{2+\sigma}}, \quad \mathsf{c}' = \frac{2\mathsf{c}}{\sqrt{2+\sigma}}. \tag{1.10}$$

Then the following convergence holds.

(i) As $L \to \infty$, *we have*

$$\sqrt{\frac{2+\sigma}{2L}} \left(\gamma_{\lfloor Lx \rfloor}\right)_{x\in[0,1]} \xrightarrow{f.d.d.} \left(\widetilde{\eta}^{(\mathsf{a}',\mathsf{c}')}_x\right)_{x\in[0,1]}. \tag{1.11}$$

(ii) As $L \to \infty$, *we have*

$$\begin{aligned}&\frac{1}{\sqrt{2L}} \left\{ U_L(x) - \frac{1}{2+\sigma}\lfloor Lx \rfloor,\ H_L(x) - \frac{\sigma}{2+\sigma}\lfloor Lx \rfloor,\ D_L(x) - \frac{1}{2+\sigma}\lfloor Lx \rfloor \right\}_{x\in[0,1]} \\ &\xrightarrow{f.d.d.} \left\{ \frac{1}{2\sqrt{2+\sigma}}\eta^{(\mathsf{a}',\mathsf{c}')}_x + \frac{\sqrt{\sigma}}{2(2+\sigma)}B_x,\ -\frac{\sqrt{\sigma}}{2+\sigma}B_x,\ \frac{\sqrt{\sigma}}{2(2+\sigma)}B_x - \frac{1}{2\sqrt{2+\sigma}}\eta^{(\mathsf{a}',\mathsf{c}')}_x \right\}_{x\in[0,1]},\end{aligned} \tag{1.12}$$

where $(B_x)_{x\in[0,1]}$ *is a Brownian motion,* $(\eta^{(\mathsf{a}',\mathsf{c}')}_x)_{x\in[0,1]}$ *is given by* (1.9), *and the processes* $(B_x)_{x\in[0,1]}$ *and* $(\eta^{(\mathsf{a}',\mathsf{c}')}_x)_{x\in[0,1]}$ *are independent.*

Remark 1.2. Note that as a corollary of Theorem 1.1(ii), using $U_L(x) - D_L(x) = \gamma_{\lfloor Lx \rfloor} - \gamma_0$, we have

$$\frac{1}{\sqrt{2L}}\{\gamma_{\lfloor Lx \rfloor} - \gamma_0\}_{x\in[0,1]} \xrightarrow{f.d.d.} \frac{1}{\sqrt{2+\sigma}}\eta^{(\mathsf{a}',\mathsf{c}')} \quad \text{as } L \to \infty.$$

This result, in fact, can be obtained directly by a soft argument and in a stronger convergence mode, as shown in the next proposition. We thank an anonymous referee for this observation. It is plausible that convergence in (1.11) and (1.12) can also be strengthened to convergence in $D[0,1]$.

Proposition 1.3. *Under the assumptions of Theorem 1.1, with*

$$\xi_L(x) := \gamma_{\lfloor Lx \rfloor} - \gamma_0, x \in [0,1],$$

we have

$$\left\{\frac{1}{\sqrt{L}}\xi_L(x)\right\}_{x\in[0,1]} \Rightarrow \left\{\sqrt{\frac{2}{2+\sigma}}\eta^{(\mathsf{a}',\mathsf{c}')}_x\right\}_{x\in[0,1]}, \tag{1.13}$$

as $L \to \infty$ *in Skorohod's space of càdlàg functions* $D[0,1]$.

Remark 1.4. One can also work with random Motzkin paths with fixed left-end point zero and geometric weights for the right-end point, and obtain a corresponding joint convergence with a 'randomized' Brownian meander (with joint probability density function proportional to $e^{-\mathsf{a}'y/\sqrt{2}}\prod_{k-1}^d \mathsf{q}_{x_k-x_{k-1}}(y_{k-1},y_k)$ for $\mathsf{a}'>0$) in place of B^{ex} and $\eta^{(\mathsf{a}',\mathsf{c}')}$ above. Both Brownian excursion and randomized Brownian meanders showed up already in Bryc et al. (2023b) in the study of limit fluctuations of height functions for open ASEP, denoted by $\eta^{(\infty,\infty)},\eta^{(\mathsf{a}',\infty)}$ therein. We omit the details for this case.

1.2. *Motivation.* Process $(\eta_x^{(\mathsf{a},\mathsf{c})})_{x\in[0,1]}$ from (1.9) has recently appeared in investigations of non-equilibrium systems in the mathematical physics literature.

First, it was shown in Barraquand and Le Doussal (2022) and Bryc and Kuznetsov (2022) that $\eta^{(\mathsf{a},\mathsf{c})}$ can be obtained as a re-scaling of the processes that appeared in the description of the *stationary measure of open KPZ* (on an interval), recently identified in Corwin and Knizel (2024+), Bryc et al. (2023a) and Barraquand and Le Doussal (2022). Namely, one can represent the stationary measure of the open KPZ equation on an interval $[0,\tau]$ as

$$\left\{\frac{1}{\sqrt{2}}B_x\right\}_{x\in[0,\tau]}+\left\{Y_x^{(\mathsf{a},\mathsf{c})}-Y_0^{(\mathsf{a},\mathsf{c})}\right\}_{x\in[0,\tau]}, \tag{1.14}$$

where B is a Brownian motion, and processes B and Y are independent. As $\tau\to\infty$, we then have

$$\left\{\frac{1}{\sqrt{\tau}}Y_{x\tau}^{(\mathsf{a}/\sqrt{\tau},\mathsf{c}/\sqrt{\tau})}\right\}_{x\in[0,1]}\xrightarrow{f.d.d.}\left\{\frac{1}{\sqrt{2}}\widetilde{\eta}_x^{(\mathsf{a},\mathsf{c})}\right\}_{x\in[0,1]},$$

and hence

$$\left\{\frac{1}{\sqrt{\tau}}\left(Y_{x\tau}^{(\mathsf{a}/\sqrt{\tau},\mathsf{c}/\sqrt{\tau})}-Y_0^{(\mathsf{a}/\sqrt{\tau},\mathsf{c}/\sqrt{\tau})}\right)\right\}_{x\in[0,1]}\xrightarrow{f.d.d.}\left\{\frac{1}{\sqrt{2}}\eta_x^{(\mathsf{a},\mathsf{c})}\right\}_{x\in[0,1]}\quad\text{as }\tau\to\infty. \tag{1.15}$$

(The process denoted by $\widetilde{\eta}$ in Bryc and Kuznetsov (2022, Theorem 2.1) is $\frac{1}{\sqrt{2}}\widetilde{\eta}^{(\mathsf{a},\mathsf{c})}$ here.) The identification of the process $Y^{(\mathsf{a},\mathsf{c})}$ is a recent groundbreaking work. It is a Markov process with transitional law determined by a Doob's h-transform applied to the Yakubovich heat kernel; see Bryc et al. (2023a) for details. The process (1.14) arises in the scaling limit of height function of particle densities of open ASEP with five parameters $\alpha_n,\beta_n,\gamma_n,\delta_n,q_n$ all depending on the size n of the system and appropriately chosen (known as the Liggett's condition). It was conjectured by Barraquand and Le Doussal (2022) that $\eta^{(\mathsf{a},\mathsf{c})}$ appears in the description of the stationary measure of open KPZ fixed point, a space-time Markov process that has not been rigorously defined yet in the literature. Note that the limit theorem (1.15) leading to $\eta^{(\mathsf{a},\mathsf{c})}$ as summarized above can be understood as a double-limit theorem (first the convergence from height function of open ASEP to $\{Y_x^{(\mathsf{a},\mathsf{c})}-Y_0^{(\mathsf{a},\mathsf{c})}\}_{x\in[0,\tau]}$, and then the second convergence (1.15)).

Second, it was later shown by Bryc et al. (2023b) that with parameters $\alpha_n,\beta_n,\gamma_n,\delta_n$ appropriately chosen and $q\in[0,1)$ fixed, the process $(\eta^{(\mathsf{a},\mathsf{c})}+B)/\sqrt{2}$, where B is an independent Brownian motion, arises directly as the scaling limit of height function of particle densities. This convergence, in contrast to the first case, can be understood as a single-limit theorem.

The contribution of this paper is a third limit theorem for the process $\widetilde{\eta}^{(\mathsf{a},\mathsf{c})}$. We show that this process arises as the scaling limit of random Motzkin paths. Our model and analysis is considerably simpler than the open ASEP, and therefore the limit theorem provides a quick access to the process $\widetilde{\eta}^{(\mathsf{a},\mathsf{c})}$. At the same time, we emphasize that we focus on the stationary measure of conjectured open KPZ fixed point, instead of the dynamics of the model (say starting from an arbitrary initial configuration).

The paper is organized as follows. Section 2 provides matrix and Markov representations for a larger class of random Motzkin paths including the one in Theorem 1.1. Section 3 provides the proof of Theorem 1.1. Section 4 provides the proof of Proposition 1.3.

2. Matrix and Markov representations for random Motzkin paths

Our method is based on the fact that explicit integral representations of statistics of interest are available in closed form, and moreover they are convenient for asymptotic analysis. We shall establish these representations for a larger class of random Motzkin paths than those considered in Theorem 1.1 (which corresponds to taking $\boldsymbol{a} = \boldsymbol{c} = (1, 1, \dots)$ and $\boldsymbol{b} = (\sigma, \sigma, \dots)$ below).

Throughout this section, the length of the Motzkin paths L is fixed. We first construct the weights of edges from three sequences

$$\boldsymbol{a} = (a_j)_{j\ge 0}, \quad \boldsymbol{b} = (b_j)_{j\ge 0}, \quad \boldsymbol{c} = (c_j)_{j\ge 1},$$

of real numbers, where we assume that $a_0, a_1, \dots > 0$, $b_0, b_1, \dots \ge 0$, and $c_1, c_2, \dots > 0$. For each path

$$\gamma = (\gamma_0 = i, \gamma_1, \dots, \gamma_{L-1}, \gamma_L = j) \in \mathcal{M}_{i,j}^{(L)},$$

we define its weight

$$w(\gamma) \equiv w_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},L}(\gamma) = \prod_{k=1}^{L} a_{\gamma_{k-1}}^{\varepsilon_k^+} b_{\gamma_{k-1}}^{\varepsilon_k^0} c_{\gamma_{k-1}}^{\varepsilon_k^-}, \quad \gamma \in \mathcal{M}_{i,j}^{(L)}, L \in \mathbb{N}.$$

That is, we take $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as the weights of the up steps, horizontal steps and down steps, and the weight of a step depends also on the altitude of the left-end of an edge. Since $\mathcal{M}_{i,j}^{(L)}$ is a finite set, the normalization constants

$$\mathfrak{W}_{i,j}^{(L)} = \sum_{\gamma \in \mathcal{M}_{i,j}^{(L)}} w(\gamma)$$

are well defined for all $i, j \ge 0$.

In addition to the weights of the edges, we wish to also weight the end-points, i.e. the initial and the final altitudes of a Motzkin path. To this end we choose two additional non-negative sequences $\boldsymbol{\alpha} = (\alpha_i)_{i\ge 0}$ and $\boldsymbol{\beta} = (\beta_i)_{i\ge 0}$ such that

$$\mathfrak{C}_L \equiv \mathfrak{C}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},L} := \sum_{i,j\ge 0} \alpha_i \mathfrak{W}_{i,j}^{(L)} \beta_j < \infty. \tag{2.1}$$

Most of the time, for the sake of simplicity we drop the dependence on the boundary-weight parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and edge-weight parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, but keep the dependence on the length L.

Note that $\mathfrak{W}_{i,j}^{(L)} = 0$ for $|j - i| > L$. So if the sequences $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are bounded, then $\mathfrak{W}_{i,j}^{(L)}$ are also bounded, and (2.1) is finite if $\sum_{n\ge 0} \alpha_n \beta_{n+j} < \infty$ for $-L \le j \le L$. With finite normalizing constant (2.1), the countable set $\mathcal{M}^{(L)} = \bigcup_{i,j\ge 0} \mathcal{M}_{i,j}^{(L)}$ becomes a probability space with the discrete probability measure

$$\mathbb{P}_L(\gamma) \equiv \mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},L}(\gamma) = \frac{\alpha_{\gamma_0}\beta_{\gamma_L}}{\mathfrak{C}_L} w(\gamma), \text{ for all } \gamma \in \mathcal{M}^{(L)}.$$

By a random Motzkin path of length L, we refer to the random element in $\mathcal{M}^{(L)}$ with law $\mathbb{P}_L$.

Such a construction seems to be a folklore. The case $\alpha = (1, 0, 0 \dots), \beta = (1, 0, 0, \dots)$ and $\boldsymbol{a} = \boldsymbol{b} = (1, 1, \dots)$, $\boldsymbol{c} = (1, 1, \dots)$ recovers the uniform choice of Motzkin paths from $\mathcal{M}_{0,0}^{(L)}$ that we considered in Bryc and Wang (2019a, Theorem 1.1). Of our special interest is the example with bounded $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and geometric weights

$$\alpha_n = \rho_0^n, \beta_n = \rho_1^n, n \ge 0, \quad \text{for some} \quad \rho_0, \rho_1 > 0, \rho_0\rho_1 < 1. \tag{2.2}$$

In this case, the normalizing constant (2.1) is finite when the product $\rho_0\rho_1 < 1$.

For non-uniform laws and geometric boundary weights, we mention an example that motivated our framework here.

Example 2.1. Flajolet and Sedgewick (2009, Section V.4) and and Viennot (1985) consider the case of equal weights $\boldsymbol{a} = (1, 1, \dots)$ for the up-steps, with varying weights of horizontal and down steps. The choice

$$\alpha_n = \left(\frac{1-\alpha}{\alpha}\right)^{n+1}, \ \beta_n = \left(\frac{1-\beta}{\beta}\right)^{n+1}, \ \boldsymbol{a} = \boldsymbol{c} = (1, 1, \dots), \ \boldsymbol{b} = (2, 2, \dots),$$

with $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta > 1$ recovers Motzkin paths that appear in the analysis of open TASEP in Derrida et al. (2004, Section 2.2) (after shifting their paths down by one unit).

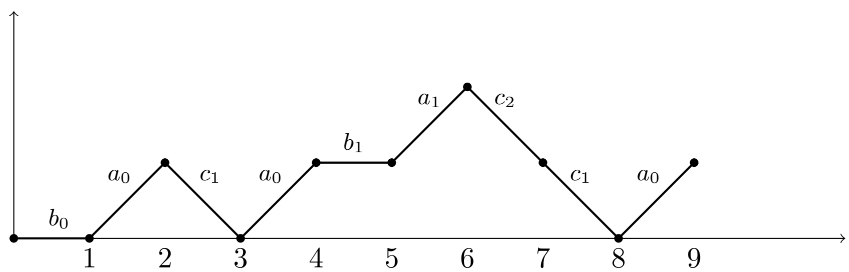

FIGURE 2.1. Motzkin path $\gamma = (0, 0, 1, 0, 1, 1, 2, 1, 0, 1) \in \mathcal{M}^{(9)}$ with weight contributions marked at the edges. The probability of selecting this path from $\mathcal{M}^{(9)}$ is $\mathbb{P}(\gamma) = \frac{\alpha_0\beta_1}{\mathfrak{C}_9} b_0 b_1 a_0^3 a_1 c_1^2 c_2$. The total number of horizontal steps is $H_9(1) = 2$ and the total number of up steps is $U_9(1) = 4$.

Recall that the general framework of the random Motzkin paths depending on the edge-weight parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and boundary-weight parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$. For such a random Motzkin path with length L, we let

$$\mathbb{P}_L \equiv \mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},L}$$

denote its law (a probability measure on $\mathcal{M}^{(L)}$), and $\mathbb{E}_L$ the expectation with respect to $\mathbb{P}_L$.

2.1. *Matrix representation.* We first start with a matrix representation, known as the matrix ansatz in the literature. Introduce

$$\boldsymbol{A} = \begin{bmatrix} 0 & a_0 & 0 & 0 & \dots \\ 0 & 0 & a_1 & 0 & \\ 0 & 0 & 0 & a_2 & \\ \vdots & & & & \ddots \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} b_0 & 0 & 0 & 0 & \dots \\ 0 & b_1 & 0 & 0 & \\ 0 & 0 & b_2 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ c_1 & 0 & 0 & 0 & \\ 0 & c_2 & 0 & 0 & \\ \vdots & & \ddots & & \end{bmatrix}.$$

Furthermore, introduce two vectors

$$\langle W_{\boldsymbol{\alpha}}(z)| = \begin{bmatrix} \alpha_0 & \alpha_1 z & \alpha_2 z^2 & \dots \end{bmatrix}, \quad |V_{\boldsymbol{\beta}}(z)\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 z \\ \beta_2 z^2 \\ \vdots \end{bmatrix},$$

which are viewed as functions of z. Recall the "decomposition" of a Motzkin path defined in (1.1). Throughout, for product of matrices $M_1, \dots, M_L$, we take the convention $\prod_{k=1}^L M_k = M_1 M_2 \cdots M_L$.

Lemma 2.2. *Under assumption* (2.1), *given* $s_k, t_k, u_k > 0$ *and* $z_0, z_1 \in (0,1]$, *we have*

$$\sum_{\gamma\in\mathcal{M}^{(L)}} z_0^{\gamma_0}\prod_{k=1}^L\left(s_k^{\varepsilon_k^+}t_k^{\varepsilon_k^-}u_k^{\varepsilon_k^0}\right)z_1^{\gamma_L}\alpha_{\gamma_0}w(\gamma)\beta_{\gamma_L} = \left\langle W_{\boldsymbol\alpha}(z_0)\middle|\prod_{k=1}^L(s_k\boldsymbol{A}+t_k\boldsymbol{C}+u_k\boldsymbol{B})\middle|V_{\boldsymbol\beta}(z_1)\right\rangle,$$

$$\mathfrak{C}_L = \sum_{\gamma\in\mathcal{M}^{(L)}}\alpha_{\gamma_0}w(\gamma)\beta_{\gamma_L} = \langle W_{\boldsymbol\alpha}(1)|(\boldsymbol{A}+\boldsymbol{C}+\boldsymbol{B})^L|V_{\boldsymbol\beta}(1)\rangle. \tag{2.3}$$

In particular,

$$\mathbb{E}_L\left[z_0^{\gamma_0}\prod_{k=1}^L s_k^{\varepsilon_k^+}t_k^{\varepsilon_k^-}u_k^{\varepsilon_k^0}z_1^{\gamma_L}\right] = \frac{1}{\mathfrak{C}_L}\left\langle W_{\boldsymbol\alpha}(z_0)\middle|\prod_{k=1}^L(s_k\boldsymbol{A}+t_k\boldsymbol{C}+u_k\boldsymbol{B})\middle|V_{\boldsymbol\beta}(z_1)\right\rangle. \tag{2.4}$$

Proof: We first notice that, by definition,

$$\mathbb{E}_L\left[z_0^{\gamma_0}\prod_{k=1}^L s_k^{\varepsilon_k^+}t_k^{\varepsilon_k^-}u_k^{\varepsilon_k^0}z_1^{\gamma_L}\right] = \frac{\sum_{i,j\ge0}\alpha_i z_0^i\beta_j z_1^j\sum_{\gamma\in\mathcal{M}_{i,j}^{(L)}}\prod_{k=1}^L(s_k^{\varepsilon_k^+}t_k^{\varepsilon_k^-}u_k^{\varepsilon_k^0})w(\gamma)}{\sum_{i,j\ge0}\alpha_i\beta_j\sum_{\gamma\in\mathcal{M}_{i,j}^{(L)}}w(\gamma)}, \tag{2.5}$$

and the denominator on the right-hand side is nothing but $\mathfrak{C}_L$ in (2.1). Recall that ε's are functions of γ.

We start by proving the formula for $\mathfrak{C}_L$. First recall the following well-known fact. Consider a finite (say n) state Markov chain. Let $P = (P_{i,j})_{i,j=1,\dots,n}$ be its transitional probability matrix, so that $P_{i,j}$ is the probability of transitioning from state i to j in one step. Let $\vec\pi$, a vertical vector in $\mathbb{R}^n$, represent a marginal law of the Markov chain. Then, $\vec\pi^TP^k$ represents the marginal law of the Markov chain starting from the law represented by $\vec\pi$ in k steps. This representation can be extended to Markov chain with countably infinite states, and also to inhomogeneous ones.

Moreover, this presentation can be further extended to the situation where P is replaced by a weight matrix (each entry is non-negative but the sum of each row is not necessarily one), and also that the sum of entries in $\vec\pi$ is not necessarily one. In this case, $\vec\pi$ and $\vec\pi^TP^k$ are no longer interpreted as probability laws. However, by the proof behind the interpretation of $\vec\pi^TP^k$ above, it is readily checked that $(\vec\pi^TP^k)_i$ is the total weights of all paths (understood in the obvious way) ending at location i in k steps, with unit weight assigned to the location i.

The above discussion provides an interpretation of $\langle W_{\boldsymbol\alpha}(1)|(\boldsymbol{A}+\boldsymbol{C}+\boldsymbol{B})^L|$, with $P=\boldsymbol{A}+\boldsymbol{C}+\boldsymbol{B}$ and $\vec\pi^T = \langle W_{\boldsymbol\alpha}(1)|$, as the total weights of L step paths with initial weight $\vec\pi$ and weight matrix P, and *uniform weight on the end points*. Now, the right-hand side of (2.3), with the extra factor $|V_{\boldsymbol\beta}(1)\rangle$ on the right, can be interpreted similarly, with weights $\boldsymbol\beta$ assigned additionally to the end locations. Therefore (2.3) follows.

For the numerator on the right-hand side of (2.5), notice that one can write

$$\sum_{\gamma\in\mathcal{M}_{i,j}^{(L)}}\prod_{k=1}^L s_k^{\varepsilon_k^+}t_k^{\varepsilon_k^-}u_k^{\varepsilon_k^0}w(\gamma) = \sum_{\gamma\in\mathcal{M}_{i,j}^{(L)}}\widetilde w(\gamma) \quad \text{with} \quad \widetilde w(\gamma) = \prod_{k=1}^L(s_ka_{\gamma_{k-1}})^{\varepsilon_k^+}(u_kb_{\gamma_{k-1}})^{\varepsilon_k^0}(t_kc_{\gamma_{k-1}})^{\varepsilon_k^-}.$$

So, again by the same interpretation before but now for inhomogeneous weight matrices $(s_k\boldsymbol{A}+t_k\boldsymbol{C}+u_k\boldsymbol{B})_{k=1,\dots,L}$, we see

$$\left\langle W_{\boldsymbol\alpha}(z_0)\middle|\prod_{k=1}^L(s_k\boldsymbol{A}+t_k\boldsymbol{C}+u_k\boldsymbol{B})\middle|V_{\boldsymbol\beta}(z_1)\right\rangle = \sum_{i,j\ge0}\alpha_iz_0^i\beta_jz_1^j\sum_{\gamma\in\mathcal{M}_{i,j}^{(L)}}\widetilde w(\gamma).$$

This completes the proof. □

The left-hand side of (2.4) can be related to the joint Laplace transform of finite-dimensional distributions of the random Motzkin paths. However, the matrix representation on the right-hand side is not always convenient for asymptotic analysis.

2.2. *Markov representation.* The next step is to re-express the matrix representation in terms of integrals (expectations) involving certain Markov process. Also in this step, we eliminate one of the 3 variables by the relation $\varepsilon_k^+ + \varepsilon_k^- + \varepsilon_k^0 = 1$. That is, we shall be interested here in (2.4) with $s_k = 1$.

First, for $t > 0$ consider a family of orthogonal polynomials $\{p_n(x;t)\}_{n\geq 0}$ with Jacobi matrix $\boldsymbol{A} + t\boldsymbol{C}$. That is, with

$$\vec{\boldsymbol{p}}(x;t) = \begin{bmatrix} p_0(x;t) \\ p_1(x;t) \\ p_2(x;t) \\ \vdots \end{bmatrix},$$

the orthogonal polynomials are determined by

$$x\vec{\boldsymbol{p}}(x;t) = (\boldsymbol{A} + t\boldsymbol{C})\vec{\boldsymbol{p}}(x;t), t > 0,$$

or equivalently,

$$xp_n(x;t) = a_n p_{n-1}(x;t) + tc_n p_{n+1}(x;t), n \geq 0, \tag{2.6}$$

with $p_0(x;t) = 1, p_{-1}(x;t) = 0$. For each $t > 0$ let ν_t denote the associated orthogonal measure.

Assumption 2.1. Consider $\vec{\boldsymbol{p}}$ and $\{\nu_t\}_{t\geq 0}$ as above for $\boldsymbol{A}$ and $\boldsymbol{C}$ given. We assume that there exists a Markov process $(X_t)_{t>0}$ such that the law of X_t is ν_t and furthermore that for each $n \geq 0$, the stochastic process $\{p_n(X_t;t)\}_{t>0}$ is a martingale polynomial in the sense that

$$\mathbb{E}(p_n(X_t;t)|X_s) = p_n(X_s, s) \text{ for all } 0 \leq s \leq t.$$

Some general conditions in terms of matrices $\boldsymbol{A}$ and $\mathbf{C}$ for the existence of such Markov process could be read out from Bryc et al. (2007); there are also many classical as well as less-classical examples, see e.g. Bryc and Wesołowski (2005, 2010).

It is easy to check that with

$$p_n(x) := p_n(x;1),$$

the solution of the three step recursion (2.6) is

$$p_n(x;t) = t^{n/2} p_n(x/\sqrt{t}). \tag{2.7}$$

so measure ν_t is just a dilation of measure $\nu \equiv \nu_1$, in the sense that $\nu_t(\cdot) = \nu(\sqrt{t}\cdot)$. It is also well known (Askey and Wilson (1985, (1.23))) that that

$$\|p_n(\cdot;t)\|^2_{L^2(\nu_t)} := \int_{\mathbb{R}} p_n^2(x;t)\nu_t(dx) = \prod_{k=1}^n \frac{tc_k}{a_{k-1}} = t^n \|p_n\|^2_{L^2(\nu)}.$$

We next introduce two generating functions

$$\phi_{\boldsymbol{\alpha}}(x,z) = \sum_{n=0}^\infty \alpha_n z^n p_n(x) \quad \text{ and } \quad \psi_{\boldsymbol{\beta}}(x,z) = \sum_{n=0}^\infty \beta_n z^n \frac{p_n(x)}{\|p_n\|^2_{L^2(\nu)}}.$$

We assume that both series converge absolutely at $z = 1$ on the support of probability measure ν, and that the product of sums of the absolute values are integrable with respect to the measure $|x|^L\nu(dx)$ so that Fubini's theorem can be used in the proof below. That is, we need to assume a stronger property that

$$\int_{\mathbb{R}} \sum_{m,n=0}^\infty \alpha_n \frac{\beta_m}{\|p_m\|_2^2} \left|x^L p_n(x)p_m(x)\right| \nu(dx) < \infty. \tag{2.8}$$

In view of (2.7), we have

$$\sum_{n=0}^{\infty} \alpha_n z^n p_n(x;t) = \phi_{\boldsymbol{\alpha}}\left(\frac{x}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right) \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n z^n \frac{p_n(x;t)}{\|p_n(\cdot;t)\|^2_{L^2(\nu_t)}} = \psi_{\boldsymbol{\beta}}\left(\frac{x}{\sqrt{t}}, \frac{z}{\sqrt{t}}\right).$$

Proposition 2.3. *Consider fixed parameters*

$$t_1 \ge t_2 \ge \cdots \ge t_L > 0 \quad \textit{and} \quad |z_0|^2 t_1, |z_1|^2/t_L < 1.$$

We have

$$\sum_{\gamma\in\mathcal{M}^{(L)}} \left[z_0^{\gamma_0} z_1^{\gamma_L} \prod_{k=1}^{L} t_k^{\varepsilon_k^-} u_k^{\varepsilon_k^0}\right] = \mathbb{E}\left[\phi_{\boldsymbol{\alpha}}\left(\frac{X_{t_1}}{\sqrt{t_1}}, z_0\sqrt{t_1}\right) \psi_{\boldsymbol{\beta}}\left(\frac{X_{t_L}}{\sqrt{t_L}}, \frac{z_1}{\sqrt{t_L}}\right) \prod_{k=1}^{L}(\sigma u_k + X_{t_k})\right]. \tag{2.9}$$

In particular,

$$\mathbb{E}_L\left[z_0^{\gamma_0} z_1^{\gamma_L} \prod_{k=1}^{L} t_k^{\varepsilon_k^-} u_k^{\varepsilon_k^0}\right] = \frac{\mathbb{E}\left[\phi_{\boldsymbol{\alpha}}(X_{t_1}/\sqrt{t_1}, z_0\sqrt{t_1})\psi_{\boldsymbol{\beta}}(X_{t_L}/\sqrt{t_L}, z_1/\sqrt{t_L}) \prod_{k=1}^{L}(\sigma u_k + X_{t_k})\right]}{\mathbb{E}\left[\phi_{\boldsymbol{\alpha}}(X_1,1)\psi_{\boldsymbol{\beta}}(X_1,1)(\sigma+X_1)^L\right]}. \tag{2.10}$$

The proof is based on the ideas in Bryc and Wesołowski (2010), but there are also significant differences. In particular we do not rely on q-commutation equations or quadratic harnesses.

Proof: Denote

$$\vec{\boldsymbol{p}}(x;t) = \begin{bmatrix} p_0(x;t) \\ p_1(x;t) \\ p_2(x;t) \\ \vdots \end{bmatrix}.$$

First, notice that by orthogonality, and Fubini's theorem justified by (2.8)

$$|V_{\boldsymbol{\beta}}(z_1)\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 z_1 \\ \beta_2 z_1^2 \\ \vdots \end{bmatrix} = \mathbb{E}_L\left[\sum_{n=0}^{\infty} \beta_n z_1^n \frac{p_n(X_{t_L};t_L)}{\|p_n(\cdot;t)\|^2_{L^2(\nu_t)}} \vec{\boldsymbol{p}}(X_{t_L};t_L)\right] = \mathbb{E}_L\left[\psi_{\boldsymbol{\beta}}\left(\frac{X_{t_L}}{\sqrt{t_L}}, \frac{z_1}{\sqrt{t_L}}\right)\vec{\boldsymbol{p}}(X_{t_L};t_L)\right].$$

Note also that

$$(\boldsymbol{A} + u\boldsymbol{B} + t\boldsymbol{C})\vec{\boldsymbol{p}}(x;t) = (x+\sigma u)\vec{\boldsymbol{p}}(x;t).$$

So

$$\begin{aligned}(\boldsymbol{A} + t_L\boldsymbol{C} + u_L\boldsymbol{B})|V_{\boldsymbol{\beta}}(z_1)\rangle &= \mathbb{E}\left[\psi_{\boldsymbol{\beta}}(X_{t_L}/\sqrt{t_L}, z_1/\sqrt{t_L})(\boldsymbol{A} + t_L\boldsymbol{C} + u_L\boldsymbol{B})\vec{\boldsymbol{p}}(X_{t_L};t_L)\right] \\ &= \mathbb{E}\left[\psi_{\boldsymbol{\beta}}(X_{t_L}/\sqrt{t_L}, z_1/\sqrt{t_L})(X_{t_L}+\sigma u_L)\vec{\boldsymbol{p}}(X_{t_L};t_L)\right] \\ &= \mathbb{E}\left[\psi_{\boldsymbol{\beta}}(X_{t_L}/\sqrt{t_L}, z_1/\sqrt{t_L})(X_{t_L}+\sigma u_L)\vec{\boldsymbol{p}}(X_{t_{L-1}};t_{L-1})\right],\end{aligned}$$

where in the last step we used the fact that $t_{L-1} > t_L$, and that if $\{M_t\}_{t\ge 0}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\ge0}$, then for all Y measurable with respect to $\mathcal{F}_s$, and $0 \le s < t$, $\mathbb{E}(YM_s) = \mathbb{E}(YM_t)$ provided that integrability is guaranteed.

Hence

$$\begin{aligned}&(\boldsymbol{A} + t_{L-1}\boldsymbol{C} + u_{L-1}\boldsymbol{B})(\boldsymbol{A} + t_L\boldsymbol{C} + u_L\boldsymbol{B})|V_{\boldsymbol{\beta}}(z_1)\rangle \\ &\qquad = \mathbb{E}\left[\psi_{\boldsymbol{\beta}}(X_{t_L}/\sqrt{t_L}, z_1/\sqrt{t_L})(X_{t_L}+\sigma u_L)(\boldsymbol{A} + t_{L-1}\boldsymbol{C} + u_{L-1}\boldsymbol{B})\vec{\boldsymbol{p}}(X_{t_{L-1}};t_{L-1})\right] \\ &\qquad = \mathbb{E}\left[\psi_{\boldsymbol{\beta}}(X_{t_L}/\sqrt{t_L}, z_1/\sqrt{t_L})(X_{t_L}+\sigma u_L)(X_{t_{L-1}}+\sigma u_{L-1})\vec{\boldsymbol{p}}(X_{t_{L-1}};t_{L-1})\right] \\ &\qquad = \mathbb{E}\left[\psi_{\boldsymbol{\beta}}(X_{t_L}/\sqrt{t_L}, z_1/\sqrt{t_L})(X_{t_L}+\sigma u_L)(X_{t_{L-1}}+\sigma u_{L-1})\vec{\boldsymbol{p}}(X_{t_{L-2}};t_{L-2})\right],\end{aligned}$$

where in the last step we used the martingale property and $t_{L-2} > t_{L-1}$. Proceeding recurrently, we get

$$\prod_{k-1}^{L}(\boldsymbol{A}+t_k\boldsymbol{C}+u_k\boldsymbol{B})|V_{\boldsymbol{\beta}}(z_1)\rangle = \mathbb{E}\left[\prod_{k=1}^{L}(X_{t_k}+\sigma u_k)\vec{\boldsymbol{p}}(X_{t_1};t_1)\psi_{\boldsymbol{\beta}}(X_{t_L}/\sqrt{t_L}, z_1/\sqrt{t_L})\right].$$

Since

$$\langle W_{\boldsymbol{\alpha}}(z_0)|\vec{\boldsymbol{p}}(X_{\tau_0};\tau_0) = \sum_{n=0}^{\infty}\alpha_n z_0^n(\tau_0)^{n/2}p_n(X_{\tau_0}/\sqrt{\tau_0}) = \phi_{\boldsymbol{\alpha}}(X_{\tau_0}/\sqrt{\tau_0}, z_0\sqrt{\tau_0}),$$

this ends the proof of (2.9). For the denominator in (2.10), it suffices to take $t_1 = \cdots = t_L = 1$ and $z_0 = z_1 = 1$. □

2.3. *Formulae with constant step weights and geometric boundary weights.* We have shown in Proposition 2.3 how to represent the probability generating function in terms of expectations of certain Markov processes. To make use of such a representation, we would like to work with Markov processes with explicit formulae, and also the appropriate choice of boundary weights $\boldsymbol{\alpha}, \boldsymbol{\beta}$ so that the introduced functions $\phi_{\boldsymbol{\alpha}}, \psi_{\boldsymbol{\beta}}$ has simple formulae.

From now on we restrict to constant step weights and geometric boundary weights. For convenience we recall them here:

$$\boldsymbol{a} = (1,1,\dots), \quad \boldsymbol{b} = (\sigma,\sigma,\dots), \quad \boldsymbol{c} = (1,1,\dots), \quad \alpha_n = \rho_0^n, \quad \beta_n = \rho_1^n, \tag{2.11}$$

with $\sigma > 0, \rho_0, \rho_1 > 0, \rho_0\rho_1 < 1$. The corresponding orthogonal polynomials (depending on $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ alone) are determined by

$$xp_n(x) = p_{n+1}(x) + p_{n-1}(x),$$

we are now dealing directly with the Chebyshev polynomials of the second kind. It is well known that the associated measure is the semi-circular law

$$\nu(dx) = \frac{\sqrt{4-x^2}}{2\pi}\mathbf{1}_{\{|x|\le 2\}}dx.$$

It is also well known that $|p_n(x)| \le n+1$ on the support $[-2,2]$ of ν, that $\|p_n\|_2^2 = 1$, and that the generating function is

$$\phi(x,z) := \sum_{n=0}^{\infty} z^n p_n(x) = \frac{1}{1-xz+z^2}, \ |z|<1, x\in[-2,2].$$

(The above formulas follow from Ismail (2009, (4.5.28), (4.5.20), (4.5.23)) by a change of variable x to $x/2$.)

The Markov processes and orthogonal martingale polynomials in Assumption 2.1 in this case have been studied. It is known (Bryc and Wesołowski, 2005, Remark 4.1) that the functions $\{p_n(x;t)\}_{t\ge 0}$ defined by (2.7) are then orthogonal martingale polynomials for a Markov process $(X_t)_{t\ge 0}$ with univariate distributions $P(X_t \in dx) = p_t(x)dx$ given by

$$p_t(x) = \frac{\sqrt{4t-x^2}}{2\pi t}1_{\{|x|\le 2\sqrt{t}\}}, \quad t>0,$$

or $p_t = \nu_t$ with ν_t determined by a dilation of ν, and with transition probabilities $P(X_t \in dy \mid X_s = x) = p_{s,t}(x,y)dy$ for $0 \le s < t$ given by

$$p_{s,t}(x,y) = \frac{1}{2\pi}\frac{(t-s)\sqrt{4t-y^2}}{tx^2+sy^2-(s+t)xy+(t-s)^2} \ \text{ for } |x|\le 2\sqrt{s}, |y|\le 2\sqrt{t}.$$

With geometric weights (2.2), the functions $\phi_{\boldsymbol{\alpha}}, \psi_{\boldsymbol{\beta}}$ now can be expressed as,

$$\phi_{\boldsymbol{\alpha}}(x,z) = \phi(x,z\rho_0) \quad \text{and} \quad \psi_{\boldsymbol{\beta}}(x,z) = \phi(x,z\rho_1),$$

for z such that $|z\rho_0| < 1$ and $|z\rho_1| < 1$, respectively.

Combining the above with Proposition 2.3, we have arrived at the following. Note that $\mathbb{E}_L$, the probability measure on $\mathcal{M}^{(L)}$, depends now on σ, ρ_0, ρ_1. We let $\mathbb{E}$ also denote the expectation for functionals of the associated Markov process $\{X_t\}_{t\geq 0}$.

Proposition 2.4. *Assume (2.11). If $\rho_0\rho_1 < 1$, $t_1 \geq t_2 \geq \cdots \geq t_L > 0$ and z_0, z_1 are close enough to 0 so that*

$$\rho_0|z_0|\sqrt{t_1} < 1 \quad \text{and} \quad \frac{\rho_1|z_1|}{\sqrt{t_L}} < 1, \tag{2.12}$$

then

$$\mathbb{E}_L\left[z_0^{\gamma_0}\prod_{k=1}^L t_k^{\varepsilon_k^-}u_k^{\varepsilon_k^0}z_1^{\gamma_L}\right] = \frac{1}{\mathfrak{C}_L}\mathbb{E}\left[\frac{\prod_{k=1}^L(\sigma u_k + X_{t_k})}{(1-\rho_0 z_0 X_{t_1} + \rho_0^2 z_0^2 t_1)(1-\rho_1 z_1 X_{t_L}/t_L + \rho_1^2 z_1^2/t_L)}\right]. \tag{2.13}$$

Here $\mathfrak{C}_L$ is the normalization constant (2.1).

Remark 2.5. If one is interested only in Theorem 1.1(i), then (2.13) can be simplified as follows. With z_0 and $\tau_1 < \tau_2 < \cdots < \tau_L$ small enough we have

$$\mathbb{E}_L\left[z_0^{\gamma_0}\prod_{k=1}^L \tau_k^{\gamma_k-\gamma_{k-1}}\right] = \frac{1}{\mathfrak{C}_L}\mathbb{E}\left[\frac{\prod_{k=1}^L(\sigma + \tau_k X_{1/\tau_k^2})}{(1-\rho_0 z_0 X_{1/\tau_1^2} + \rho_0^2 z_0^2/\tau_1^2)(1-\rho_1\tau_L^2 X_{1/\tau_L^2} + \rho_1^2\tau_L^2)}\right]. \tag{2.14}$$

To see this, we use (2.13) with $z_0 = z_0$, $z_1 = 1$, we take $t_k = 1/\tau_k^2$ and $u_k = 1/\tau_k$. After multiplying both sides by $\tau_1 \dots \tau_L$, on the left hand side of (2.14) we get $\mathbb{E}_L\left[z_0^{\gamma_0}\prod_{k=1}^L z_k^{1-2\varepsilon_k^- - \varepsilon_k^0}\right]$. To complete the derivation we note that $1 - 2\varepsilon_k^- - \varepsilon_k^0 = (\varepsilon_k^+ + \varepsilon_k^0 + \varepsilon_k^-) - 2\varepsilon_k^- - \varepsilon_k^0 = \varepsilon_k^+ - \varepsilon_k^- = \gamma_k - \gamma_{k-1}$.

The integral formula for the normalization constant $\mathfrak{C}_L$, however, will require additional effort as we want to include the case where ρ_1 can be larger than 1 in our asymptotic analysis. In particular the following representation of $\mathfrak{C}_L$ will be useful.

Proposition 2.6. *Assume $\rho_0 \in (0,1), \rho_0\rho_1 \in (0,1)$. Then,*

$$\mathfrak{C}_L = \int_{\mathbb{R}} \frac{(x+\sigma)^L}{1 - x\rho_0 + \rho_0^2}\mu_{\rho_1}(dx), \tag{2.15}$$

where the probability measure μ_{ρ_1} of a possibly mixed type is given by

$$\mu_\rho(dx) = \frac{1}{2\pi}\frac{\sqrt{4-x^2}}{1-x\rho+\rho^2}\mathbf{1}_{\{|x|<2\}}dx + \left(1 - \frac{1}{\rho^2}\right)_+ \delta_{\rho+\frac{1}{\rho}}(dx). \tag{2.16}$$

(Here $x_+ := \max\{0, x\}$.)

We remark that measure (2.16) is a shifted Marchenko–Pastur law.

Proof: We first note that the result holds if both $\rho_0, \rho_1 < 1$. Indeed, in this case, we can apply (2.13) with $z_0 = z_1 = t_k = u_k = 1$. Then the left hand side of (2.13) is 1, so the right hand side gives the integral formula for $\mathfrak{C}_L$ that we want. We now fix $\rho_0 \in (0,1)$. As a function of ρ_1, this explicit integral formula extends analytically to complex argument, defining a function

$$f(\rho) = \frac{1}{2\pi}\int_{-2}^{2}\frac{(\sigma+x)^L}{(1-x\rho_0+\rho_0^2)(1-x\rho+\rho^2)}\sqrt{4-x^2}dx, \tag{2.17}$$

which is analytic in the complex unit disk $|\rho| < 1$.

Next we note that since the edge-weights are bounded by $\max\{\sigma, 1\}$, the function

$$\mathfrak{C}(\rho) = \sum_{i,j=0}^{\infty}\rho_0^i\rho^j\mathfrak{W}_{i,j}^{(L)}, \tag{2.18}$$

is analytic in the complex disk $|\rho| < 1/\rho_0$ (see (2.1)). Since we deduced from (2.13) that $f(\rho) = \mathfrak{C}(\rho)$ for $\rho \in (0,1)$, expression (2.18) coincides with (2.17) for $|\rho| < 1$ and is its analytic extension to the complex disk $|\rho| < 1/\rho_0$.

Our goal is to extend the integral representation (2.17) to a larger domain by explicit analytic continuation. We first re-write (2.17) as a complex integral. Substituting $x = 2\cos\theta$, and then $z = e^{i\theta}$, in (2.17) we get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-2}^{2} \frac{(\sigma+x)^L}{(1-x\rho_0+\rho_0^2)(1-x\rho+\rho^2)}\sqrt{4-x^2}dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \frac{4\sin^2\theta(\sigma+2\cos\theta)^L}{(1-2\rho_0\cos\theta+\rho_0^2)(1-2\rho\cos\theta+\rho^2)}d\theta \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{4\sin^2\theta(\sigma+2\cos\theta)^L}{(1-2\rho_0\cos\theta+\rho_0^2)(1-2\rho\cos\theta+\rho^2)}d\theta \\
&= \left(-\frac{1}{2}\right)\cdot\frac{1}{2\pi i}\oint_{|z|=1} \frac{(z^2-1)^2\left(\sigma+z+\frac{1}{z}\right)^L}{(1-\rho_0 z)(z-\rho_0)(1-\rho z)(z-\rho)}\frac{dz}{z},
\end{aligned} \tag{2.19}$$

which is valid for all $|\rho| < 1$. Consider now $|\rho| \in (\rho_0, 1)$. In the last line above one can replace the contour $|z| = 1$ by $|z| = r$, and this replacement is valid as long as the circle does not cross any pole of the integrand, that is, for $r \in (1, 1/|\rho|)$. Next, let the contour cross the pole at $1/\rho$ and (because of the additional factor $-1/2$ in front of the integral) add half of the residue at $z = 1/\rho$. We arrive at

$$f_1(\rho) = -\frac{1}{4\pi i}\oint_{|z|=r} \frac{(z^2-1)^2\left(\sigma+z+\frac{1}{z}\right)^L}{(1-\rho_0 z)(z-\rho_0)(1-\rho z)(z-\rho)}\frac{dz}{z} + \frac{1}{2}\frac{(\rho^2-1)\left(\rho+\frac{1}{\rho}+\sigma\right)^L}{\rho(\rho-\rho_0)(1-\rho_0\rho)}, \tag{2.20}$$

which coincides with $f(\rho)$ for $|\rho| \in (\rho_0, 1)$, by deformation of contour as explained above, and, with r fixed, gives the analytic extension of f to all ρ such that $1/r < |\rho| < r$. In particular, $\mathfrak{C}(\rho) = f_1(\rho)$ for $1/r < |\rho| < r$. Note that r can be taken arbitrarily close to $1/\rho_0$.

Next, consider the expression (2.20) for $r \in (1, 1/\rho_0)$ and ρ such that $|\rho| \in (1, r)$, and deform the contour of integration back to $|z| = 1$. This subtracts half of the residue of the integrand at $z = \rho$. Since r can be taken arbitrarily close to $1/\rho_0$, $f_1(\rho)$ is equal to

$$f_2(\rho) = -\frac{1}{4\pi i}\oint_{|z|=1} \frac{(z^2-1)^2\left(\sigma+z+\frac{1}{z}\right)^L}{(1-\rho_0 z)(z-\rho_0)(1-\rho z)(z-\rho)}\frac{dz}{z} + \frac{(\rho^2-1)\left(\rho+\frac{1}{\rho}+\sigma\right)^L}{\rho(\rho-\rho_0)(1-\rho_0\rho)} \tag{2.21}$$

for all ρ such that $|\rho| \in (1, 1/\rho_0)$. In particular, $\mathfrak{C}(\rho) = f_2(\rho)$ for $|\rho| \in (1, 1/\rho_0)$. Returning back to the real arguments, we see that (2.17), (2.19) and (2.21) can be combined together into a single formula which gives

$$\begin{aligned}
\mathfrak{C}(\rho) = \frac{1}{2\pi}\int_{-2}^{2} &\frac{(\sigma+x)^L}{(1-x\rho_0+\rho_0^2)(1-x\rho+\rho^2)}\sqrt{4-x^2}dx \\
&+ \frac{\left(1-\frac{1}{\rho^2}\right)_+\rho\left(\rho+\frac{1}{\rho}+\sigma\right)^L}{(\rho-\rho_0)(1-\rho_0\rho)}, \quad \rho \in (0, 1/\rho_0), \rho \neq 1.
\end{aligned}$$

This formula extends to $\rho = 1$ by continuity, establishing (2.16) for $\rho_1 \leq 1$. To prove (2.16) for $\rho_1 \in (1, 1/\rho_0)$, we note that by an elementary calculation the contribution of the atom of μ_{ρ_1}

matches the additional term arising from the residua in (2.21):

$$\int_{\mathbb{R}} \frac{(x+\sigma)^L}{1-x\rho_0+\rho_0^2}\left(1-\frac{1}{\rho^2}\right)\delta_{\rho+1/\rho}(dx) = \left(1-\frac{1}{\rho^2}\right)\frac{(\sigma+x)^L}{1-x\rho_0+\rho_0^2}\Big|_{x=\rho+1/\rho}$$
$$= \left(1-\frac{1}{\rho^2}\right)\frac{\rho\left(\rho+\frac{1}{\rho}+\sigma\right)^L}{(\rho-\rho_0)(1-\rho\rho_0)}.$$

□

3. Proof of Theorem 1.1

We first note that since $U_L(x)+H_L(x)+D_L(x)=\lfloor Lx\rfloor$, it is enough to prove joint convergence of two of the three processes. We will show that

$$\frac{1}{\sqrt{L}}\left\{\left(\gamma_{\lfloor Lx\rfloor}, H_L(x)-\frac{\sigma}{2+\sigma}\lfloor Lx\rfloor\right)\right\}_{x\in[0,1]} \xrightarrow{f.d.d.} \left\{\left(\frac{\sqrt{2}}{\sqrt{2+\sigma}}\widetilde{\eta}_x^{(\mathsf{a}',\mathsf{c}')}, \frac{\sqrt{2\sigma}}{2+\sigma}B_x\right)\right\}_{x\in[0,1]}, \tag{3.1}$$

where $\widetilde{\eta}^{(\mathsf{a}',\mathsf{c}')}$, B are independent and a',c' are given by (1.10). The above implies the desired joint convergence of $U_L(x), H_L(x), D_L(x)$, as

$$U_L(x)+D_L(x)=\lfloor Lx\rfloor - H_L(x),$$
$$U_L(x)-D_L(x)=\gamma_{\lfloor Lx\rfloor}-\gamma_0.$$

To prove (3.1), we fix $d\in\mathbb{N}$, and $\boldsymbol{x}=(x_1,\dots,x_d)$ with $x_0:=0<x_1<\dots<x_d=1$. Denoting

$$h_k = H_L(x_k)-\frac{\sigma}{2+\sigma}\lfloor Lx_k\rfloor,$$

we introduce the Laplace transform Φ_L by the formula

$$\Phi_L(\boldsymbol{c},\boldsymbol{\theta}) := \mathbb{E}_L\left[\exp\left(-\sum_{k=0}^d c_k\gamma_{\lfloor Lx_k\rfloor}+\sum_{k=1}^d\theta_k h_k\right)\right]. \tag{3.2}$$

In this section, recall that $\mathbb{E}_L$ is the expected value with respect to probability measure $\mathbb{P}_L$ on $\mathcal{M}^{(L)}$ defined by formula (1.5) with parameters (1.4). Since $\boldsymbol{x}$ is fixed throughout this proof, we suppress dependence of Φ_L on $\boldsymbol{x}=(x_0,\dots,x_d)$ in our notation.

Our goal is to compute $\Phi_L(L^{-1/2}\boldsymbol{c}, L^{-1/2}\boldsymbol{\theta})$ and identify the limit. The main step in the proof is to show that the expression for the limiting Laplace transform factors and takes the following form.

Proposition 3.1. *If $\theta_1,\theta_2,\dots,\theta_d\in\mathbb{R}$ and $c_0,c_1,\dots,c_d>0$ are such that*

$$\mathsf{c}+c_0>0 \quad \text{and} \quad c_d+\mathsf{a}>0, \tag{3.3}$$

then

$$\lim_{L\to\infty}\Phi_L\left(\frac{\boldsymbol{c}}{\sqrt{L}},\frac{\boldsymbol{\theta}}{\sqrt{L}}\right)=\Psi(\boldsymbol{c})\cdot\exp\left(\frac{\sigma}{(2+\sigma)^2}\sum_{k=1}^d(x_k-x_{k-1})\widetilde{s}_k^2\right), \tag{3.4}$$

with $\widetilde{s}_k=\sum_{j=k}^d\theta_j$, $k=1,\dots,d$, and

$$\Psi(\boldsymbol{c})=\frac{\sqrt{2+\sigma}}{\sqrt{2}\pi\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}}\int_{\mathbb{R}_+^d} e^{-\frac{1}{2+\sigma}\sum_{k=1}^d(x_k-x_{k-1})u_k} f(u_1)g(u_d)\prod_{k=1}^{d-1}\mathsf{p}_{c_k}(u_k,u_{k+1})\,d\boldsymbol{u}, \tag{3.5}$$

with

$$f(u_1)=\frac{\sqrt{u_1}}{(\mathsf{c}+c_0)^2+u_1},\qquad g(u_d)=\frac{1}{(\mathsf{a}+c_d)^2+u_d},$$

and

$$\mathsf{p}_t(x,y)=\frac{2}{\pi}\cdot\frac{t\sqrt{y}}{t^4+(y-x)^2+2(y+x)t^2}. \tag{3.6}$$

The proof of this proposition is postponed to Section 3.1. The transition probability density (3.6) appeared as the tangent process in Bryc and Wang (2016), and later as the square of the radial part of a 3-dimensional Cauchy process in Kyprianou and O'Connell (2021, Corollary 1).

The second step in the proof of Theorem 1.1 is to use the following re-write of Bryc et al. (2023b, Proposition 4.10) which uses change of variables and self-similarity $\mathsf{p}_{at}(a^2x,a^2y)=\mathsf{p}_t(x,y)/a^2$, $\mathsf{q}_{a^2x}(az,az')=\mathsf{q}_x(z,z')/a$, $a>0$, of kernels (3.6) and (1.6) to insert an auxiliary parameter $\tau>0$ into the formula.

Proposition 3.2. *Let f,g be two measurable functions on $\mathbb{R}_+$. With $\tau,c_1,\cdots,c_{d-1}>0$ and $0=x_0<x_1<\cdots<x_d\le 1$, we have*

$$\int_{\mathbb{R}_+^d} e^{-\tau\sum_{k=1}^d (x_k-x_{k-1})u_k} f(u_1)\left(\prod_{k=1}^{d-1}\mathsf{p}_{c_k}(u_k,u_{k+1})\right)g(u_d)d\boldsymbol{u}$$
$$=\frac{4}{\pi}\int_{\mathbb{R}_+^{d-1}} e^{-\sum\limits_{k=1}^{d-1}c_kz_k}\widehat{f}(z_1)\left(\prod_{k=2}^{d-1}\mathsf{q}_{2\tau(x_k-x_{k-1})}(z_{k-1},z_k)\right)\widehat{g}(z_{d-1})d\boldsymbol{z}, \tag{3.7}$$

where

$$\widehat{f}(z):=\int_{\mathbb{R}_+} f(u^2)\sin(uz)e^{-\tau x_1u^2}du, \tag{3.8}$$

$$\widehat{g}(z):=\int_{\mathbb{R}_+} g(u^2)u\sin(uz)e^{-\tau(x_d-x_{d-1})u^2}du, \tag{3.9}$$

provided that the functions under the multiple integrals in (3.7) are absolutely integrable.

We apply Proposition 3.2 to (3.5), using an auxiliary Markov processes ζ with transition probabilities $\mathbb{P}(\zeta_t\in dy|\zeta_s=x)=\mathsf{p}_{t-s}(x,y)dy$ for $s<t$ with density (3.6).

Proposition 3.3.

$$\frac{\sqrt{2+\sigma}}{\sqrt{2}}\frac{1}{\pi\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}}\int_0^\infty \mathbb{E}\left[\frac{e^{-\frac{1}{2+\sigma}\sum_{k=1}^d(x_k-x_{k-1})\zeta_{s_k}}}{\zeta_{s_1}+(\mathsf{c}+c_0)^2}\middle|\zeta_{s_d}=u\right]\frac{\sqrt{u}\,du}{(\mathsf{a}+c_d)^2+u}$$
$$=\mathbb{E}\left[e^{\frac{1}{2+\sigma}\sum_{k=0}^d c_k\widetilde{\eta}_{x_k}^{(\mathsf{a}',\mathsf{c}')}}\right]. \tag{3.10}$$

Proof: We use Proposition 3.2 with $\tau=1/(2+\sigma)$, $f(u)=\sqrt{u}/((\mathsf{c}+c_0)^2+u)$ and $g(u)=1/((\mathsf{a}+c_d)^2+u)$. Since $\int_0^\infty e^{-sz}\sin(uz)dz=\frac{u}{s^2+u^2}$, we have

$$f(u^2)=\frac{u}{(\mathsf{c}+c_0)^2+u^2}=\int_0^\infty e^{-(\mathsf{c}+c_0)z_0}\sin(z_0u)dz_0,$$
$$ug(u^2)=\frac{u}{(\mathsf{a}+c_d)^2+u^2}=\int_0^\infty e^{-(\mathsf{a}+c_d)z_d}\sin(z_du)dz_d.$$

Formulas (3.8) and (3.9) become

$$\widehat{f}(z_1)=\int_{\mathbb{R}_+} f(u^2)\sin(uz_1)e^{-\tau x_1u^2}du=\int_0^\infty e^{-(\mathsf{c}+c_0)z_0}\int_{\mathbb{R}_+} e^{-\tau x_1u^2}\sin(uz_1)\sin(z_0u)dudz_0$$
$$=\frac{\pi}{2}\int_{\mathbb{R}_+} e^{-(\mathsf{c}+c_0)z_0}\mathsf{q}_{2\tau x_1}(z_0,z_1)dz_0,$$

and

$$\begin{aligned}\widehat{g}(z_{d-1}) &= \int_{\mathbb{R}_+} u g(u^2)\sin(uz_d)e^{-\tau(x_d-x_{d-1})u^2}du\\ &= \int_{\mathbb{R}_+}\left(\int_0^\infty e^{-(\mathsf{a}+c_d)z_d}\sin(z_d u)dz_d\right)\sin(uz_d)e^{-\tau(x_d-x_{d-1})u^2}du\\ &= \int_{\mathbb{R}_+} e^{-\tau(x_d-x_{d-1})u^2}\sin(z_d u)\sin(uz_d)du dz_d\\ &= \frac{\pi}{2}\int_{\mathbb{R}_+} e^{-(\mathsf{a}+c_d)z_d}\mathsf{q}_{2\tau(x_d-x_{d-1})}(z_{d-1},z_d)dz_d.\end{aligned}$$

Thus by (3.7), the left hand side of (3.10) becomes

$$\begin{aligned}&\frac{\sqrt{2+\sigma}}{\sqrt{2}}\frac{1}{\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}}\int_{\mathbb{R}_+^{d+1}} e^{-\sum_{k=0}^d c_k z_k}e^{-\mathsf{c}z_0}\left(\prod_{k=1}^d \mathsf{q}_{2\tau(x_k-x_{k-1})}(z_{k-1},z_k)\right)e^{-\mathsf{a}z_d}d\boldsymbol{z}\\ &\quad= \frac{\sqrt{2+\sigma}}{\sqrt{2}}\frac{1}{\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}}\int_{\mathbb{R}_+^{d+1}} e^{-\sum_{k=0}^d c_k z_k}e^{-\mathsf{c}z_0}\left(\prod_{k=1}^d \mathsf{q}_{2\tau(x_k-x_{k-1})}(z_{k-1},z_k)\right)e^{-\mathsf{a}z_d}d\boldsymbol{z}\\ &\quad= \frac{\sqrt{2+\sigma}}{\sqrt{2}}\frac{1}{\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}}(2\tau)^{-d/2}\int_{\mathbb{R}_+^{d+1}} e^{-\sum_{k=0}^d c_k z_k}e^{-\mathsf{c}z_0}\left(\prod_{k=1}^d \mathsf{q}_{x_k-x_{k-1}}(z_{k-1}/\sqrt{2\tau},z_k/\sqrt{2\tau})\right)e^{-\mathsf{a}z_d}d\boldsymbol{z},\end{aligned}$$

where we used scaling $\mathsf{q}_{2\tau x}(z,z') = \frac{1}{\sqrt{2\tau}}\mathsf{q}_x(z/\sqrt{2\tau},z'/\sqrt{2\tau})$. Substituting $z_k' = z_k/\sqrt{2\tau}$ into the integral and dropping the primes on z_k', we get

$$\frac{1}{\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}}\int_{\mathbb{R}_+^{d+1}} e^{-\sum_{k=0}^d c_k z_k\sqrt{2\tau}}e^{-\mathsf{c}\sqrt{2\tau}z_0}\left(\prod_{k=1}^d \mathsf{q}_{x_k-x_{k-1}}(z_{k-1},z_k)\right)e^{-\mathsf{a}\sqrt{2\tau}z_d}d\boldsymbol{z},$$

which we recognize as the desired right-hand side of (3.10). ☐

Proof of Theorem 1.1: By Proposition 3.1, the limiting Laplace transform factors. Proposition 3.3 identifies the first factor in (3.4) as the Laplace transform of the first component of the process in (3.1). We recognize the second factor in (3.4) as the Laplace transform of the second component of the process in (3.1). To see this, we write it as

$$\mathbb{E}\left[e^{\frac{\sqrt{2\sigma}}{2+\sigma}\sum_{k=1}^d \widetilde{s}_k(B_{x_k}-B_{x_{k-1}})}\right] = \mathbb{E}\left[e^{\frac{\sqrt{2\sigma}}{2+\sigma}\sum_{k=1}^d \theta_k B_{x_k}}\right].$$

This identifies the limit of the Laplace transforms (3.4) as a Laplace transform of a probability measure. To conclude the proof we invoke Bryc and Wang (2019b, Theorem A.1), which asserts that convergence of Laplace transforms on an open set to a Laplace transform of a probability measure implies convergence in distribution. ☐

3.1. *Proof of Proposition 3.1.* By symmetry, we assume $\mathsf{c} > 0$. We start by rewriting the expression (3.2) solely in terms of ε_k^- and ε_k^0. The first step is to write (recall that $h_0 = 0$)

$$\begin{aligned}\Phi_L(\boldsymbol{c},\boldsymbol{\theta}) &= \mathbb{E}_L\left[e^{-\gamma_0\sum_{j=0}^d c_j}\exp\left(-\sum_{k=1}^d(\gamma_{L_k}-\gamma_{L_{k-1}})\sum_{j=k}^d c_j + \sum_{k=1}^d(h_k-h_{k-1})\sum_{j=k}^d\theta_j\right)\right]\\ &= \mathbb{E}_L\left[e^{-s_0\gamma_0}\exp\left(-\sum_{k=1}^d s_k\sum_{j=L_{k-1}+1}^{L_k}(\varepsilon_j^+-\varepsilon_j^-) + \sum_{k=1}^d \widetilde{s}_k\sum_{j=L_{k-1}+1}^{L_k}\left(\varepsilon_j^0 - \frac{\sigma}{2+\sigma}\right)\right)\right],\end{aligned}$$

with

$$s_k = \sum_{j=k}^{d} c_j, k = 0, \dots, d, \quad \widetilde{s}_k = \sum_{j=k}^{d} \theta_j, \quad k = 1, \dots, d.$$

Since $\varepsilon_k^+ - \varepsilon_k^- = 1 - \varepsilon_k^0 - 2\varepsilon_k^-$, we get

$$\Phi_L(\boldsymbol{c}, \boldsymbol{\theta}) = e^{-\sum_{k=1}^d (L_k - L_{k-1})(s_k + \sigma \widetilde{s}_k/(2+\sigma))} \times \mathbb{E}_L \left[e^{-s_0\gamma_0} \exp\left(2\sum_{k=1}^d s_k \sum_{j=L_{k-1}+1}^{L_k} \varepsilon_j^- + \sum_{k=1}^d (\widetilde{s}_k + s_k) \sum_{j=L_{k-1}+1}^{L_k} \varepsilon_j^0 \right) \right].$$

We therefore get

$$\Phi_L\left(\frac{\boldsymbol{c}}{\sqrt{L}}, \frac{\boldsymbol{\theta}}{\sqrt{L}}\right) = \prod_{k=1}^d t_{L,k}^{-(L_k - L_{k-1})/2} \prod_{k=1}^d v_{L,k}^{-(L_k - L_{k-1})\frac{\sigma}{2+\sigma}} \mathbb{E}_L \left[z_{L,0}^{\gamma_0} \left(\prod_{k=1}^d t_{L,k}^{\sum_{j=L_{k-1}+1}^{L_k} \varepsilon_j^-} u_{L,k}^{\sum_{j=L_{k-1}+1}^{L_k} \varepsilon_j^0} \right) \right],$$

with

$$z_{L,0} = e^{-s_0/\sqrt{L}}, \quad t_{L,k} = e^{2s_k/\sqrt{L}}, \quad v_{L,k} = e^{\widetilde{s}_k/\sqrt{L}}, \quad u_{L,k} = \sqrt{t_{L,k}} v_{L,k}.$$

Next, we apply the Markov representation (2.13), but before that we verify that (2.12) holds. We note that $t_{L,1} \geq t_{L,2} \geq \cdots \geq t_{L,d}$ and that our assumptions on the coefficients $c_0, \dots, c_d$ in (3.3) guarantee that

$$\rho_{L,0} |z_{L,0}| \sqrt{t_{L,1}} = \left(1 - \frac{\mathsf{c}}{\sqrt{L}}\right) e^{-c_0/\sqrt{L}} < 1 \quad \text{and} \quad \frac{\rho_{L,1}}{\sqrt{t_{L,d}}} = \left(1 - \frac{\mathsf{a}}{\sqrt{L}}\right) e^{-c_d/\sqrt{L}} < 1$$

for L large enough. We assume implicitly L large enough so the above holds from now on. Thus after some rewriting we have

$$\Phi_L\left(\frac{\boldsymbol{c}}{\sqrt{L}}, \frac{\boldsymbol{\theta}}{\sqrt{L}}\right) = \frac{\prod_{k=1}^d v_{L,k}^{-\sigma(L_k - L_{k-1})/(2+\sigma)}}{\mathfrak{C}_L} \times \mathbb{E}\left[\frac{\prod_{k=1}^d (\sigma v_{L,k} + X_{t_{L,k}}/\sqrt{t_{L,k}})^{L_k - L_{k-1}}}{(1 - \rho_{L,0} z_{L,0} X_{t_{L,1}} + \rho_{L,0}^2 z_{L,0}^2 t_{L,1})(1 - \rho_{L,1} X_{t_{L,d}}/t_{L,d} + \rho_{L,1}^2/t_{L,d})} \right].$$

Lemma 3.4. *The normalizing constant satisfies:*

$$\mathfrak{C}_L \sim (2+\sigma)^L L^{1/2} \cdot \frac{\sqrt{2}}{\sqrt{2+\sigma}} \cdot \mathfrak{C}_{\mathsf{a}',\mathsf{c}'} \text{ as } L \to \infty, \tag{3.11}$$

where $\mathfrak{C}_{\mathsf{a},\mathsf{c}}$ *is given by* (A.1) *and* a', c' *are from* (1.10)*.*

Proof: We use the explicit form of expression (2.15):

$$\begin{aligned}\mathfrak{C}_L &= \frac{1}{2\pi} \int_{-2}^{2} \frac{(\sigma + x)^L \sqrt{4 - x^2}}{(1 - \rho_{L,0} x + \rho_{L,0}^2)(1 - \rho_{L,1} x + \rho_{L,1}^2)} dx + \left(\rho_{L,1} - \frac{1}{\rho_{L,1}}\right)_+ \frac{\left(\rho_{L,1} + \frac{1}{\rho_{L,1}} + \sigma\right)^L}{(\rho_{L,1} - \rho_{L,0})(1 - \rho_{L,0}\rho_{L,1})} \\ &=: I_L + D_L. \end{aligned} \tag{3.12}$$

The dominant term in the integral I_L comes from the integral over $[0, 2]$. This is easy to see as $|x + \sigma|^L \leq (max\{2 - \sigma, \sigma\})^L = o((2 + \sigma)^L)$ for $-2 \leq x \leq 0$ when $\sigma > 0$.

The argument for the asymptotics of the integral over $[0,2]$ relies on the substitution $x = 2-u^2/L$ that appeared in similar context in paper Bryc and Wang (2016) and later in Bryc and Wang (2019b), Bryc et al. (2023b).

$$I_L \sim \frac{1}{2\pi}\int_0^2 \frac{(\sigma+x)^L\sqrt{4-x^2}}{(1-\rho_{L,0}x+\rho_{L,0}^2)(1-\rho_{L,1}x+\rho_{L,1}^2)}dx$$
$$\sim \frac{(2+\sigma)^L}{2\pi}\int_0^{\sqrt{2L}} \frac{\left(1-\frac{u^2}{(2+\sigma)L}\right)^L}{\left((\frac{\mathsf{c}}{\sqrt{L}})^2+(1-\frac{\mathsf{c}}{\sqrt{L}})\frac{u^2}{L}\right)\left((\frac{\mathsf{a}}{\sqrt{L}})^2+(1-\frac{\mathsf{a}}{\sqrt{L}})\frac{u^2}{L}\right)}\sqrt{4-\frac{u^2}{L}}\,\frac{2}{\sqrt{L}}u^2\frac{du}{L}$$
$$\sim \frac{2}{\pi}(2+\sigma)^L L^{1/2}\int_0^{\sqrt{2L}} \frac{\left(1-\frac{u^2}{(2+\sigma)L}\right)^L}{(\mathsf{c}^2+u^2)\,(\mathsf{a}^2+u^2)}u^2du \sim (2+\sigma)^L L^{1/2}\frac{2}{\pi}\int_0^\infty \frac{u^2e^{-\frac{u^2}{2+\sigma}}du}{(\mathsf{c}^2+u^2)\,(\mathsf{a}^2+u^2)}.$$

The integral is an explicit expression (A.2), compare Bryc et al. (2023b, (4.38) and Lemma A.2).

For $\mathsf{a}\neq\mathsf{c}$ we get

$$I_L \sim (2+\sigma)^L\sqrt{L}\cdot\frac{2}{\sqrt{2+\sigma}}\frac{|\mathsf{c}'|H(|\mathsf{c}'|/2)-|\mathsf{a}'|H(|\mathsf{a}'|/2)}{\mathsf{c}'^2-\mathsf{a}'^2} = (2+\sigma)^L\sqrt{L}\cdot\frac{\sqrt{2}}{\sqrt{2+\sigma}}\mathfrak{C}_{|\mathsf{a}'|,|\mathsf{c}'|}. \tag{3.13}$$

This proves (3.11) for $\mathsf{a}\neq\mathsf{c}$. For $\mathsf{a}=\mathsf{c}$ we get

$$I_L \sim (2+\sigma)^L\sqrt{L}\cdot\frac{\sqrt{\pi}\left(\mathsf{a}'^2+2\right)H\left(\frac{\mathsf{a}'}{2}\right)-2\mathsf{a}'}{2\sqrt{\pi}\mathsf{a}'\sqrt{\sigma+2}} = (2+\sigma)^L\sqrt{L}\cdot\frac{\sqrt{2}}{\sqrt{2+\sigma}}\mathfrak{C}_{\mathsf{a}',\mathsf{a}'}.$$

Thus (3.11) holds also for $\mathsf{a}=\mathsf{c}>0$.

When $\mathsf{a}<0,\mathsf{c}>0$ (but $\mathsf{a}+\mathsf{c}>0$) we need to include the contribution of the discrete part. It is easy to see that with $\rho_{L,1}=1-\mathsf{a}/\sqrt{L}>1$ the discrete part in (3.12) is

$$D_L = \frac{\mathsf{a}L\left(2\sqrt{L}-\mathsf{a}\right)\left(-\frac{\mathsf{a}}{\sqrt{L}}+\frac{1}{1-\frac{\mathsf{a}}{\sqrt{L}}}+\sigma+1\right)^L}{(\mathsf{a}-\mathsf{c})\left(\sqrt{L}-\mathsf{a}\right)\left(\sqrt{L}(\mathsf{a}+\mathsf{c})-\mathsf{ac}\right)} \sim \sqrt{L}(2+\sigma)^L\cdot\frac{2\mathsf{a}\left(1+\frac{\mathsf{a}^2}{L(\sigma+2)}\right)^L}{\mathsf{a}^2-\mathsf{c}^2}$$
$$\sim \sqrt{L}(2+\sigma)^L\cdot\frac{2\mathsf{a}e^{\frac{\mathsf{a}^2}{\sigma+2}}}{\mathsf{a}^2-\mathsf{c}^2} = \sqrt{L}(2+\sigma)^L\cdot\frac{\sqrt{2}}{\sqrt{2+\sigma}}\cdot\frac{2\sqrt{2}\mathsf{a}'e^{\frac{\mathsf{a}'^2}{4}}}{\mathsf{a}'^2-\mathsf{c}'^2}.$$

Combining this with (3.13) we see that for $\mathsf{a}<0$, we have

$$\mathfrak{C}_L \sim \sqrt{L}(2+\sigma)^L\cdot\frac{\sqrt{2}}{\sqrt{2+\sigma}}\left(\mathfrak{C}_{-\mathsf{a}',\mathsf{c}'}+\frac{2\sqrt{2}\,\mathsf{a}'e^{\frac{\mathsf{a}'^2}{4}}}{\mathsf{a}'^2-\mathsf{c}'^2}\right).$$

We now use the identity $\operatorname{erfc}(x)+\operatorname{erfc}(-x)=2$ to verify that

$$\mathfrak{C}_{-\mathsf{a}',\mathsf{c}'}+\frac{2\sqrt{2}\,\mathsf{a}'e^{\frac{\mathsf{a}'^2}{2}}}{\mathsf{a}'^2-\mathsf{c}'^2} = \mathfrak{C}_{\mathsf{a}',\mathsf{c}'}.$$

This completes the proof. □

It turns out that it suffices to restrict the expectation to the event $\{X_{t_k}\geq 0\}$, as shown below.

Lemma 3.5. *If $\sigma > 0$ then*

$$\Phi_L\left(\frac{\boldsymbol{c}}{\sqrt{L}}, \frac{\boldsymbol{\theta}}{\sqrt{L}}\right) \sim \frac{\prod_{k=1}^d v_{L,k}^{-\sigma(L_k-L_{k-1})/(2+\sigma)}}{\mathfrak{C}_L} \times \mathbb{E}\left[\frac{\prod_{k=1}^d ((\sigma v_{L,k} + X_{t_{L,k}}/\sqrt{t_{L,k}})^{L_k-L_{k-1}} \mathbf{1}_{\left\{X_{t_{L,k}}\geq 0\right\}})}{(1-\rho_{L,0}z_{L,0}X_{t_1} + \rho_{L,0}^2 z_{L,0}^2 t_{L,1})(1-\rho_{L,1}X_{t_{L,d}}/t_{L,d} + \rho_{L,1}^2/t_{L,d})}\right]. \quad (3.14)$$

Proof: Since $-2\sqrt{t} \leq X_t \leq 2\sqrt{t}$ for $t > 0$, $x_j - x_{j-1} > 0$ and $t_{L,d} \geq 1$, the expected value over the set $X_{t_{L,j}} < 0$ is bounded by a factor

$$\frac{(\max\{\sigma, 2-\sigma\})^{L_j-L_{j-1}}(2+\sigma)^{L-(L_j-L_{j-1})}}{(1-\rho_{L,0}z_{L,0}\sqrt{t_{L,1}})^2(1-\rho_{L,1}/\sqrt{t_{L,d}})^2} \sim C(2+\sigma)^L L^2 \left(\frac{\max\{\sigma, 2-\sigma\}}{2+\sigma}\right)^{L(x_j-x_{j-1})} = o\left((2+\sigma)^L\sqrt{L}\right) \text{ as } L\to\infty$$

By (3.11) this proves (3.14). ∎

To determine the asymptotics of the expectation, introduce

$$U_s := e^{-s}X_{e^{2s}}, s \in \mathbb{R}.$$

This is a stationary $[-2,2]$-valued Markov process with univariate probabilities

$$\mathbb{P}(U_s = dy) = \frac{\sqrt{4-y^2}}{2\pi}\mathbf{1}_{\{|y|\leq 2\}}dy \quad (3.15)$$

and transition probabilities

$$\mathbb{P}(U_s = dy | U_{s'} = y') = \frac{\sqrt{4-y^2}}{2\pi} \frac{e^{2(s-s')}-1}{-2yy'\cosh(s-s') + 2\cosh(2(s-s')) + y^2 + y'^2 - 2}dy, \quad s' < s,\ y, y' \in [-2,2]. \quad (3.16)$$

So we arrive at

$$\Phi_L\left(\frac{\boldsymbol{c}}{\sqrt{L}}, \frac{\boldsymbol{\theta}}{\sqrt{L}}\right) \sim \frac{\prod_{k=1}^d v_{L,k}^{-\sigma(L_k-L_{k-1})/(2+\sigma)}}{\mathfrak{C}_L} \times \mathbb{E}\left[\frac{\prod_{k=1}^d ((\sigma v_{L,k} + U_{s_k/\sqrt{L}})^{L_k-L_{k-1}} \mathbf{1}_{U_{s_k/\sqrt{L}}>0})}{(1-\rho_{L,0}z_{L,0}\sqrt{t_{L,1}}U_{s_1/\sqrt{L}} + \rho_{L,0}^2 z_{L,0}^2 t_{L,1})(1-\rho_{L,1}U_{s_d}/\sqrt{t_{L,d}} + \rho_{L,1}^2/t_{L,d})}\right]. \quad (3.17)$$

Introduce

$$Y_L(s) := L\left(2 - U_{s/\sqrt{L}}\right).$$

This is a well-studied Markov process, and we shall explain it later. Now, we re-write the expectation on the right-hand side of (3.17) as

$$\prod_{k=1}^d (2+\sigma v_{L,k})^{L_k-L_{k-1}} \times \mathbb{E}\left[\frac{\prod_{k=1}^d (1-\frac{Y_L(s_k)}{(2+\sigma v_{L,k})L})^{L_k-L_{k-1}}\mathbf{1}_{\{Y_L(s_k)<2L\}}}{((1-\rho_{L,0}z_{L,0}\sqrt{t_{L,1}})^2 + \rho_{L,0}z_{L,0}Y_L(s_1)/L) \times ((1-\rho_{L,1}/\sqrt{t_{L,d}})^2 + \frac{\rho_{L,1}Y_L(c_d)}{\sqrt{t_{L,d}}L})}\right].$$

Grouping the first product above with the first product on the right-hand side of (3.17), we arrive at

$$\Phi_L\left(\frac{\boldsymbol{c}}{\sqrt{L}},\frac{\boldsymbol{\theta}}{\sqrt{L}}\right)\sim\widetilde{\psi}_L\left(\frac{\boldsymbol{\theta}}{L^{1/2}}\right)\psi_L\left(\frac{\boldsymbol{c}}{L^{1/2}},\frac{\boldsymbol{\theta}}{L^{1/2}}\right), \tag{3.18}$$

where

$$\widetilde{\psi}_L\left(\frac{\boldsymbol{\theta}}{L^{1/2}}\right)=\prod_{k=1}^d\left(\frac{\sigma}{2+\sigma}v_{L,k}^{2/(2+\sigma)}+\frac{2}{2+\sigma}v_{L,k}^{-\sigma/(2+\sigma)}\right)^{L_k-L_{k-1}},$$

$$\psi_L\left(\frac{\boldsymbol{c}}{L^{1/2}},\frac{\boldsymbol{\theta}}{L^{1/2}}\right)=\frac{L^2(2+\sigma)^L}{\mathfrak{C}_L}\mathbb{E}\left[G_L\left(Y_L(s_1),\dots,Y_L(s_d)\right)\right], \tag{3.19}$$

with

$$G_L(y_1,\dots,y_d):=\frac{\prod_{j=1}^d((1-\frac{y_j}{(2+\sigma v_{L,j})L})^{L_j-L_{j-1}}\mathbf{1}_{\{|y_j|\le 2L\}})}{\left(L(1-\rho_{L,0}z_0\sqrt{t_{L,1}})^2+\rho_{L,0}z_{L,0}y_1\right)\left(L(1-\rho_{L,1}/\sqrt{t_{L,d}})^2+\rho_{L,1}y_d/\sqrt{t_{L,d}}\right)}.$$

It is clear that

$$\lim_{L\to\infty}\widetilde{\psi}_L\left(\frac{\boldsymbol{\theta}}{L^{1/2}}\right)=e^{\frac{\sigma}{(2+\sigma)^2}\sum_{k=1}^d(x_k-x_{k-1})\widetilde{s}_k^2}. \tag{3.20}$$

This determines the second factor on the right hand side of (3.4). For (3.19), since $v_{L,k}=1+\widetilde{s}_k/\sqrt{L}+O(L^{-1})$,

$$(L(1-\rho_{L,0}z_{L,0}\sqrt{t_1})^2+\rho_{L,0}z_{L,0}y_1)=\left((\mathsf{c}+c_0)^2+y_1\right)+O(L^{-1/2}),$$
$$L(1-\rho_{L,1}/\sqrt{t_{L,d}})^2+\rho_{L,1}y_d/\sqrt{t_{L,d}}=\left((\mathsf{a}+c_d)^2+y_d\right)+O(L^{-1/2}),$$

we have

$$\lim_{L\to\infty}G_L(y_1,\dots,y_d)=G(y_1,\dots,y_d):=\frac{\exp(-\sum_{j=1}^d\frac{x_j-x_{j-1}}{2+\sigma}y_j)}{\left((\mathsf{c}+c_0)^2+y_1\right)\left((\mathsf{a}+c_d)^2+y_d\right)}.$$

Let also $\pi_L(u)$ denote the marginal density of $Y_L(c_d)$. We have

$$\begin{aligned}\psi_L\left(\frac{\boldsymbol{c}}{L^{1/2}},\frac{\boldsymbol{\theta}}{L^{1/2}}\right)&=\frac{L^2(2+\sigma)^L}{\mathfrak{C}_L}\mathbb{E}\left[G_L\left(Y_L(s_1),\dots,Y_L(s_d)\right)\right]\\&\sim\frac{L^{3/2}\sqrt{2+\sigma}}{\sqrt{2}\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}}\mathbb{E}\left[G_L\left(Y_L(s_1),\dots,Y_L(s_d)\right)\right]\\&=\frac{\sqrt{2+\sigma}}{\sqrt{2}\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}}\int_0^{2L}\mathbb{E}\left[G_L(Y_n(s_1),\dots,Y_n(s_{d-1}),u)|Y_L(c_d)=u\right]\ L^{3/2}\pi_L(u)\,du.\end{aligned}$$

Now we take a closer look at the process $\{Y_L(s)\}_{s>0}$. This is a Markov process with the univariate law that can be computed from (3.15) such that

$$\mathbb{P}(Y_L(s)=dv)=\pi_L(v)dv=\frac{\sqrt{v(4L-v)}}{2\pi L^2}dv,$$

compare Bryc and Wang (2019b, Lemma 4.2), and transition probabilities for $s_k>s_{k+1}$ that can be computed from (3.16). Moreover, it is known (Bryc and Wang, 2016) that as $L\to\infty$,

$$\mathcal{L}\left(\left(Y_L(s)\right)_{s\ge c_d}\,\middle|\,Y_L(c_d)=u\right)\xrightarrow{f.d.d.}\mathcal{L}\left(\left(\zeta_s\right)_{s\ge c_d}\,\middle|\,\zeta_{c_d}=u\right)$$

where we let ζ denote the Markov process with transition probabilities $\mathbb{P}(\zeta_t\in dy|\zeta_s=x)=\mathsf{p}_{t-s}(x,y)dy$ given in (3.6). In the above, $\mathcal{L}(\cdot\mid\cdot)$ is understood as the law induced by the conditional law of the corresponding Markov process starting at fixed time from a fixed point u.

In view of the bound $(1-y/((2+\sigma)L))^{xL} \leq \exp(-xy/(2+\sigma))$ which is valid for $0 \leq y \leq 2L$, we see that

$$G_L(y_1,\dots,y_d) \leq \frac{C}{(\mathsf{c}+c_0)^2(\mathsf{a}+c_d)^2} \exp\left(-\sum_{k=1}^d (x_k - x_{k-1})y_k/(2+\sigma)\right)$$

for some C and large L. (More precisely, there is L_0 and C such that this bound holds for all $L \geq L_0$ and all $0 \leq y_k \leq 2L$, but the bound extends to all $0 \leq y_k < \infty$ as $G_L(\boldsymbol{y}) = 0$ when some $y_k > 2L$.) So either invoking Billingsley (1999, Exercise 6.6) or the dominated convergence theorem, we see that

$$\lim_{L\to\infty} \psi_L\left(\frac{\boldsymbol{c}}{L^{1/2}}, \frac{\boldsymbol{\theta}}{L^{1/2}}\right) = \frac{\sqrt{2+\sigma}}{\sqrt{2}\pi \mathfrak{C}_{\mathsf{a}',\mathsf{c}'}} \int_0^\infty \mathbb{E}\left[G(\zeta_{s_1},\dots,\zeta_{s_d}) | \zeta_{s_d} = u\right] \sqrt{u}\, du = \psi(\boldsymbol{c}).$$

Combined with (3.18) and (3.20), this completes the proof of (3.4).

4. Proof of Proposition 1.3

To avoid cumbersome notation and additional technicalities, we prove Proposition 1.3 for

$$\rho_{L,0} = e^{-\mathsf{c}/\sqrt{L}}, \ \rho_{L,1} = e^{-\mathsf{a}/\sqrt{L}} \tag{4.1}$$

instead of the asymptotically equivalent expression (1.4).

First, it is known, see Barraquand and Le Doussal (2022), that the law $\mathbb{P}_{\eta^{(\mathsf{a},\mathsf{c})}/\sqrt{2}}$ on $C[0,1]$ of process $\eta^{(\mathsf{a},\mathsf{c})}/\sqrt{2}$ is absolutely continuous with respect to the law $\mathbb{P}_B$ of the Brownian motion of variance $1/2$ with the Radon-Nikodym derivative

$$\frac{d\mathbb{P}_{\eta^{(\mathsf{a},\mathsf{c})}/\sqrt{2}}}{d\mathbb{P}_B} = \frac{2}{(\mathsf{a}+\mathsf{c})\mathfrak{C}_{\mathsf{a},\mathsf{c}}} e^{(\mathsf{a}+\mathsf{c})\min_{x\in[0,1]} B_x - \mathsf{a}B_1}, \tag{4.2}$$

where $\mathfrak{C}_{\mathsf{a},\mathsf{c}}$ is the normalizing constant (1.8).

Next, denote by $\boldsymbol{S} = \{S_1, S_2, \dots, S_L\}$ a random walk starting at 0 with i.i.d increments taking values in $\{\pm 1, 0\}$ with probabilities $1/(2+\sigma)$ and $\sigma/(2+\sigma)$ respectively. Introduce its partial-sum process

$$\zeta_L(t) := S_{\lfloor Lt \rfloor}, t \in [0,1]. \tag{4.3}$$

The law of ξ_L is absolutely continuous with respect to the law of ζ_L on $D[0,1]$, denoted by $\mathbb{P}_{\xi_L}, \mathbb{P}_{\zeta_L}$ respectively. To see this, it suffices to compare the laws of the vectors $\boldsymbol{\gamma}_L^\circ = \{\gamma_k - \gamma_0\}_{k=1,\dots,L}$ and $\boldsymbol{S}_L = (S_1,\dots,S_L)$ on $\mathbb{Z}^L$, denoted by $\mathbb{P}_{\boldsymbol{\gamma}_L^\circ}, \mathbb{P}_{\boldsymbol{S}_L}$ respectively. By summing over the values of $\gamma_0 \in \mathbb{Z}_{\geq 0}$ such that $\min_{0\leq k\leq L}\{\gamma_k\} \geq 0$, for $\boldsymbol{s} = \{s_1,\dots,s_L\}$ in the support of $\boldsymbol{S}_L$, the Radon-Nikodym derivative is

$$\frac{d\mathbb{P}_{\boldsymbol{\gamma}_L^\circ}}{d\mathbb{P}_{\boldsymbol{S}_L}}(\boldsymbol{s}) = \frac{1}{\mathsf{C}_L} (\rho_{L,0}\rho_{L,1})^{-\min_{k=0,\dots,L} s_k} \rho_{L,1}^{s_L},$$

where C_L is the normalizing constant and $s_0 = 0$. It then follows that with $\omega = \{\omega_t\}_{t\in[0,1]} \in D[0,1]$ we have

$$\frac{d\mathbb{P}_{\xi_L}}{d\mathbb{P}_{\zeta_L}}(\omega) = \frac{1}{\mathsf{C}_L}(\rho_{L,0}\rho_{L,1})^{-\min_{x\in[0,1]}\omega_x} \rho_{L,1}^{\omega_1} = \frac{1}{\mathsf{C}_L}\mathcal{E}(\omega/\sqrt{L}), \tag{4.4}$$

where we used (4.1) and denoted $\mathcal{E}(\omega) := \exp\left((\mathsf{a}+\mathsf{c})\inf_{x\in[0,1]}\omega_x - \mathsf{a}\omega_1\right)$. Formula (4.4) implies that for any bounded continuous function $\Phi : D[0,1] \to \mathbb{R}$ we have

$$\mathbb{E}\left[\Phi\left(\frac{\xi_L}{\sqrt{L}}\right)\right] = \frac{1}{\mathsf{C}_L}\mathbb{E}\left[\Phi\left(\frac{\zeta_L}{\sqrt{L}}\right)\mathcal{E}\left(\frac{\zeta_L}{\sqrt{L}}\right)\right], \tag{4.5}$$

Since the increments of $\boldsymbol{S}$ have mean zero and variance $2/(2+\sigma)$, by Donsker's theorem

$$\frac{1}{\sqrt{L}}\{\zeta_L(x)\}_{x\in[0,1]} \Rightarrow \frac{2}{\sqrt{2+\sigma}}\{B_x\}_{x\in[0,1]}$$

in $D[0,1]$. Since $\sup_{L=1,2,\dots} \mathbb{E}\left[\mathcal{E}\left(\frac{\zeta_L}{\sqrt{L}}\right)^2\right] < \infty$ and Φ is bounded, the real random variables $\Phi\left(\frac{\zeta_L}{\sqrt{L}}\right)\mathcal{E}\left(\frac{\zeta_L}{\sqrt{L}}\right)$, $L = 1, 2 \dots$ are uniformly integrable. Uniform integrability and weak convergence imply convergence of expectations (Billingsley (1999, Theorem 3.5)), so it follows that

$$\begin{aligned}
\lim_{L\to\infty} \mathbb{E}\left[\Phi\left(\frac{\zeta_L}{\sqrt{L}}\right)\mathcal{E}_L\left(\frac{\zeta_L}{\sqrt{L}}\right)\right] &= \mathbb{E}\left[\Phi\left(\frac{2}{\sqrt{2+\sigma}}B\right)\mathcal{E}\left(\frac{2}{\sqrt{2+\sigma}}B\right)\right] \\
&= \mathbb{E}\left[\Phi\left(\frac{2}{\sqrt{2+\sigma}}B\right)e^{(\mathsf{a}+\mathsf{c})\inf_{t\in[0,1]}\frac{2}{\sqrt{2+\sigma}}B_x - \mathsf{a}\frac{2}{\sqrt{2+\sigma}}B_1}\right] \\
&= \mathbb{E}\left[\Phi\left(\frac{2}{\sqrt{2+\sigma}}B\right)e^{(\mathsf{a}'+\mathsf{c}')\min_{0\le x\le 1} B_x - \mathsf{a}' B_1}\right]. \qquad (4.6)
\end{aligned}$$

In particular, (4.6) with $\Phi \equiv 1$ implies that the normalizing constants converge, $\mathsf{C}_L \to (\mathsf{a}' + \mathsf{c}')\mathfrak{C}_{\mathsf{a}',\mathsf{c}'}/2$. Dividing (4.6) by these normalizing constantsand using formulas (4.5) and (4.2) we get

$$\lim_{L\to\infty} \mathbb{E}\left[\Phi\left(\frac{\xi_L}{\sqrt{L}}\right)\right] = \mathbb{E}\left[\Phi\left(\sqrt{\frac{2}{2+\sigma}}\eta^{(\mathsf{a}',\mathsf{c}')}\right)\right],$$

for all continuous and bounded functions Φ from $D[0,1]$ to $\mathbb{R}$. This completes the proof of (1.13) under assumption (4.1). We omit the proof of (1.13) under assumption (1.4), as it requires cumbersome notation and additional steps.

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Appendix A. Auxiliary formulas

The following integral is known, for a derivation, see for example Bryc et al. (2023b, Lemma A.2).

Lemma A.1. *The normalizing constant*

$$\mathfrak{C}_{\mathsf{a},\mathsf{c}} = \int_{\mathbb{R}_+^2} e^{-(\mathsf{c}x+\mathsf{a}y)/\sqrt{2}}\mathsf{q}_1(x,y)\,dx\,dy$$

is given by the expression

$$\mathfrak{C}_{\mathsf{a},\mathsf{c}} = \begin{cases} \sqrt{2}\cdot\dfrac{\mathsf{a}H(\mathsf{a}/2) - \mathsf{c}H(\mathsf{c}/2)}{\mathsf{a}^2-\mathsf{c}^2}, & \text{if } \mathsf{a}\neq\mathsf{c},\ \mathsf{a}+\mathsf{c}>0, \\[2ex] \dfrac{2+\mathsf{a}^2}{2\sqrt{2}\mathsf{a}}\cdot H(\mathsf{a}/2) - \dfrac{1}{\sqrt{2\pi}}, & \text{if } \mathsf{a}=\mathsf{c}>0, \end{cases} \qquad (A.1)$$

where for $x \in \mathbb{R}$,

$$H(x) = e^{x^2}\operatorname{erfc}(x) \quad \text{with} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt.$$

The following is a minor re-write of known integrals, see Bryc and Kuznetsov (2022, Lemma 4.5) or Bryc et al. (2023b, formula after (4.38)).

Lemma A.2. *For $\mathsf{a}+\mathsf{c}>0$ and $\tau>0$, we have*

$$\frac{1}{2\pi}\int_0^\infty e^{-\tau v^2/2}\frac{4v^2}{(\mathsf{a}^2+v^2)(\mathsf{c}^2+v^2)}\,dv = \sqrt{\tau}\,\mathfrak{C}_{|\mathsf{a}|\sqrt{2\tau},|\mathsf{c}|\sqrt{2\tau}}. \qquad (A.2)$$

References

Askey, R. and Wilson, J. Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Mem. Amer. Math. Soc.*, **54** (319), iv+55 (1985). MR783216.

Barraquand, G. and Le Doussal, P. Steady state of the KPZ equation on an interval and Liouville quantum mechanics. *EPL*, **137** (6), 61003 (2022). DOI: 10.1209/0295-5075/ac25a9 ArXiv preprint with Supplementary material: arXiv: 2105.15178.

Billingsley, P. *Convergence of probability measures.* Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition (1999). ISBN 0-471-19745-9. MR1700749.

Bryc, W. and Kuznetsov, A. Markov limits of steady states of the KPZ equation on an interval. *ALEA Lat. Am. J. Probab. Math. Stat.*, **19** (2), 1329–1351 (2022). MR4512149.

Bryc, W., Kuznetsov, A., Wang, Y., and Wesołowski, J. Markov processes related to the stationary measure for the open KPZ equation. *Probab. Theory Related Fields*, **185** (1-2), 353–389 (2023a). MR4528972.

Bryc, W., Matysiak, W., and Wesołowski, J. Quadratic harnesses, q-commutations, and orthogonal martingale polynomials. *Trans. Amer. Math. Soc.*, **359** (11), 5449–5483 (2007). MR2327037.

Bryc, W. and Wang, Y. The local structure of q-Gaussian processes. *Probab. Math. Statist.*, **36** (2), 335–352 (2016). MR3593028.

Bryc, W. and Wang, Y. Fluctuations of random Motzkin paths. *Adv. in Appl. Math.*, **106**, 96–116 (2019a). MR3915365.

Bryc, W. and Wang, Y. Limit fluctuations for density of asymmetric simple exclusion processes with open boundaries. *Ann. Inst. Henri Poincaré Probab. Stat.*, **55** (4), 2169–2194 (2019b). MR4029151.

Bryc, W., Wang, Y., and Wesołowski, J. From the asymmetric simple exclusion processes to the stationary measures of the KPZ fixed point on an interval. *Ann. Inst. Henri Poincaré Probab. Stat.*, **59** (4), 2257–2284 (2023b). MR4663522.

Bryc, W. and Wesołowski, J. Conditional moments of q-Meixner processes. *Probab. Theory Related Fields*, **131** (3), 415–441 (2005). MR2123251.

Bryc, W. and Wesołowski, J. Askey-Wilson polynomials, quadratic harnesses and martingales. *Ann. Probab.*, **38** (3), 1221–1262 (2010). MR2674998.

Corwin, I. and Knizel, A. Stationary measure for the open KPZ equation (2024+). To appear in *Comm. Pure Appl. Math.*. DOI: 10.1002/cpa.22174.

Derrida, B., Enaud, C., and Lebowitz, J. L. The asymmetric exclusion process and Brownian excursions. *J. Statist. Phys.*, **115** (1-2), 365–382 (2004). MR2070099.

Flajolet, P. and Sedgewick, R. *Analytic combinatorics.* Cambridge University Press, Cambridge (2009). ISBN 978-0-521-89806-5. MR2483235.

Ismail, M. E. H. *Classical and quantum orthogonal polynomials in one variable*, volume 98 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge (2009). ISBN 978-0-521-14347-9. MR2542683.

Kyprianou, A. E. and O'Connell, N. The Doob-McKean identity for stable Lévy processes. In *A lifetime of excursions through random walks and Lévy processes—a volume in honour of Ron Doney's 80th birthday*, volume 78 of *Progr. Probab.*, pp. 269–282. Birkhäuser/Springer, Cham (2021). MR4425796.

Viennot, G. A combinatorial theory for general orthogonal polynomials with extensions and applications. In *Orthogonal polynomials and applications (Bar-le-Duc, 1984)*, volume 1171 of *Lecture Notes in Math.*, pp. 139–157. Springer, Berlin (1985). MR838979.