ALEA, Lat. Am. J. Probab. Math. Stat. 21, 179–214 (2024)
DOI: 10.30757/ALEA.v21-08

Optimal Rate of Convergence for Vector-valued Wiener-Itô Integral

Huiping Chen

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China
E-mail address: chenhp@pku.edu.cn

Abstract. We investigate the optimal rate of convergence in the multidimensional normal approximation of vector-valued Wiener-Itô integrals whose components all belong to a fixed Wiener chaos with coinciding orders. By combining Malliavin calculus, Stein's method for normal approximation and the method of cumulants, we obtain the optimal rate of convergence with respect to a suitable smooth distance. As applications, we derive the optimal rates of convergence for complex Wiener-Itô integrals, vector-valued Wiener-Itô integrals with kernels of step functions and vector-valued Toeplitz quadratic functionals.

1. Introduction

On a complete probability space $(\Omega, \mathcal{F}, P)$, let $X = \{X(h) : h \in \mathfrak{H}\}$ be an isonormal Gaussian process over a real separable Hilbert space $\mathfrak{H}$. That is, X is a Gaussian family of centered random variables such that $\mathrm{E}[X(h)X(g)] = \langle h, g\rangle_{\mathfrak{H}}$ for any $h, g \in \mathfrak{H}$, where $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$ denotes the inner product in $\mathfrak{H}$. Let $\{G_n\}_{n\geq 1}$ be a sequence of random variables living in a fixed Wiener chaos of X with unit variance. The research associated with the normal approximation of $\{G_n\}_{n\geq 1}$ has been concerned in recent years. The seminal article Nualart and Peccati (2005) first proved the Fourth Moment Theorem, which shows that $\{G_n\}_{n\geq 1}$ converges to a standard normal variable $N \sim \mathcal{N}(0,1)$ if and only if $\mathrm{E}\left[G_n^4\right] \to 3$ as $n \to \infty$. Shortly afterwards, Peccati and Tudor (2005) provided a multidimensional version of this characterization. By using techniques of Malliavin calculus, Nualart and Ortiz-Latorre (2008) proposed a new proof of the Fourth Moment Theorem. Further, Nourdin and Peccati (2009b) combined Malliavin calculus with Stein's method to derive quantitative and explicit upper bounds in the normal approximation of $\{G_n\}_{n\geq 1}$. This approach of combining Malliavin calculus and Stein's method has also been developed to obtain explicit upper bounds in the multidimensional normal approximation of functionals of Gaussian fields, see Noreddine and Nourdin (2011); Nourdin and Peccati (2010b); Nourdin et al. (2010b) for instance. Fix an integer

Received by the editors May 3rd, 2023; accepted January 3rd, 2024.

2010 *Mathematics Subject Classification.* 60F05, 60G15, 60H05, 60H07.

Key words and phrases. Optimal rate of convergence, vector-valued Wiener-Itô integral, Malliavin calculus, Stein's method, method of cumulants.

This work is supported by NSFC (No. 11731009, No. 12231002) and Center for Statistical Science, PKU.

$d \geq 2$. Let $\{F_n = (F_{n,1}, \ldots, F_{n,d})\}_{n \geq 1}$ be a sequence of d-dimensional random vectors, with components living in some Wiener chaos of X. Suppose that the covariance matrix of F_n is C. In this paper, we focus on the optimal rate of convergence with respect to a suitable distance under the assumption that F_n converges in distribution to a d-dimensional normal vector $Z \sim \mathcal{N}_d(0, C)$. We say that a positive sequence $\{\varphi(n)\}_{n\geq 1}$ converging to zero provides an optimal rate of convergence with respect to some distance $d(\cdot, \cdot)$, if $d(F_n, N) \asymp \varphi(n)$. Here, for two numerical sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, we write $a_n \asymp b_n$ if there exist two constants $0 < c_1 < c_2 < \infty$ not depending on n such that $c_1 b_n \leq a_n \leq c_2 b_n$ for n sufficiently large. Throughout the paper, we denote by c_1 and c_2 two finite positive constants that do not depend on n and can vary from line to line. We denote by $C^k(\mathbb{R}^d)$ the space of k-times continuously differentiable real-valued functions on $\mathbb{R}^d$ for $k \geq 1$.

Fix an integer $q \geq 2$. Let $\mathfrak{H}^{\odot q}$ denote the q-th symmetric tensor product of $\mathfrak{H}$, and let $I_q(f)$ denote the real q-th Wiener-Itô integral of $f \in \mathfrak{H}^{\odot q}$ with respect to X (see Section 2.1 for the definition). For a sequence of random variables $\{G_n = I_q(g_n)\}_{n\geq 1}$ with unit variance, where all $g_n \in \mathfrak{H}^{\odot q}$, we assume that G_n converges in distribution to a standard normal variable $N \sim \mathcal{N}(0, 1)$. There are complete characterisations of the optimal rate of convergence for G_n with respect to some suitable distance.

Nourdin and Peccati (2009a) demonstrated how to determine optimal Berry-Esseen bound in the normal approximation of functionals of X and further refined the main results they proven in Nourdin and Peccati (2009b). Specifically, they assumed additionally that the two-dimensional random vector

$$\left(G_n, \frac{1 - q^{-1} \|DG_n\|_{\mathfrak{H}}^2}{\chi(n)} \right) \xrightarrow{d} (N_1, N_2), \quad n \to \infty, \tag{1.1}$$

where $\chi(n) := \sqrt{\mathrm{Var}(q^{-1} \|DG_n\|_{\mathfrak{H}}^2)}$, DG_n is the Malliavin derivative of G_n (see (2.7) for the definition) and is a $\mathfrak{H}$-valued random element, the notation $\xrightarrow{d}$ represents convergence in distribution, and (N_1, N_2) is a centered two-dimensional Gaussian vector satisfying $\mathrm{E}\left(N_1^2\right) = \mathrm{E}\left(N_2^2\right) = 1$ and $\mathrm{E}\left(N_1 N_2\right) = \rho$. If $\rho \neq 0$, then for every $z \in \mathbb{R}$,

$$\frac{P(G_n \leq z) - P(N \leq z)}{\chi(n)}$$

converges to a nonzero limit. Therefore, by Nourdin and Peccati (2009a, Theorem 3.1, Proposition 3.3) and the fact that $\chi(n) \asymp \sqrt{\mathrm{E}\left[G_n^4\right] - 3}$ (see Nourdin and Peccati (2012, Lemma 5.2.4)),

$$d_{\mathrm{Kol}}\left(G_n, N\right) \asymp \chi(n) \asymp \left|\mathrm{E}\left[G_n^3\right]\right| \asymp \sqrt{\mathrm{E}\left[G_n^4\right] - 3}, \tag{1.2}$$

where $d_{\mathrm{Kol}}\left(G_n, N\right)$ is the Kolmogorov distance defined as

$$d_{\mathrm{Kol}}\left(G_n, N\right) = \sup_{z \in \mathbb{R}} \left|P(G_n \leq z) - P(N \leq z)\right|.$$

Nourdin and Peccati (2009a, Proposition 3.6) proposed that if q is even, sufficient conditions for (1.1) are as $n \to \infty$,

$$\sum_{r=1}^{q-1} \sum_{l=1}^{2(q-r)-1} \chi(n)^{-2} \left\| \left(g_n \tilde{\otimes}_r g_n\right) \otimes_l \left(g_n \tilde{\otimes}_r g_n\right) \right\|_{\mathfrak{H}^{\otimes 2(2(q-r)-l)}} \to 0, \tag{1.3}$$

and

$$-q q!(q/2 - 1)! \binom{q-1}{q/2 - 1}^2 \chi(n)^{-1} \left\langle g_n, g_n \tilde{\otimes}_{q/2} g_n \right\rangle_{\mathfrak{H}^{\otimes q}} \to \rho. \tag{1.4}$$

Here, given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$ with $p, q \geq 1$, for $r = 0, \ldots, p \wedge q$, where $p \wedge q$ denotes the minimum of p and q, $f \otimes_r g$ is the r-th contraction of f and g (see (2.2) for the definition), and $f \tilde{\otimes}_r g$ denotes its symmetrization. In this case, if $\rho \neq 0$, then (1.2) is valid, that is, $\chi(n)$ is the optimal rate of convergence for G_n with respect to $d_{\mathrm{Kol}}(\cdot, \cdot)$. However, when q is even and $\rho = 0$, or

q is odd and (1.3) is satisfied (which imply (1.1) with $\rho = 0$), the optimal rate of convergence with respect to $d_{\mathrm{Kol}}(\cdot,\cdot)$ is unknown.

In Biermé et al. (2012), Biermé, Bonami, Nourdin and Peccati gave a complete solution to the optimal rate of convergence with respect to a suitable smooth distance. They proved that

$$d(G_n, N) \asymp \widetilde{M}(G_n) := \max\left(\left|\mathrm{E}\left[G_n^3\right]\right|, \mathrm{E}\left[G_n^4\right] - 3\right), \tag{1.5}$$

where $d(G_n, N) = \sup|\mathrm{E}[h(G_n)] - \mathrm{E}[h(N)]|$, and h runs over the class of all real-valued functions $h \in C^2(\mathbb{R})$ with the second derivative bounded by one. Note that, it is shown in Nourdin and Peccati (2010b, Proposition 3.1) that

$$\left|\mathrm{E}\left[G_n^3\right]\right| \le c\sqrt{\mathrm{E}\left[G_n^4\right] - 3},$$

where c is constant only depending on q. Furthermore, by using Lusin's Theorem, Nourdin and Peccati (2015) dealt with irregular test functions and obtained that $\widetilde{M}(G_n)$ also provides an optimal rate of convergence in the total variation distance $d_{\mathrm{TV}}(\cdot,\cdot)$, a non-smooth distance. That is,

$$d_{\mathrm{TV}}(G_n, N) := \sup_{A \in \mathcal{B}(\mathbb{R})} |P(G_n \in A) - P(N \in A)| \asymp \widetilde{M}(G_n).$$

As far as we know, there are few references studying the optimal rate of convergence for a sequence of random vectors whose components are functionals of some isonormal Gaussian process. Campese (2013) extended the results of Nourdin and Peccati (2009a) to the multidimensional case and developed techniques for determining the exact asymptotic speed of convergence for multidimensional smooth functionals of some isonormal Gaussian process. Let $\{F_n = (F_{n,1}, \ldots, F_{n,d})\}_{n\ge1}$ be a sequence of d-dimensional random vectors, where $F_{n,i} = I_{q_i}(f_{n,i})$, $q_i \ge 2$, and $f_{n,i} \in \mathfrak{H}^{\odot q_i}$ for $1 \le i \le d$. Suppose that the covariance matrix of F_n is $C = (C_{ij})_{1\le i,j\le d}$ and F_n converges in distribution to a d-dimensional normal vector $Z \sim \mathcal{N}_d(0, C)$. It is shown in Nourdin et al. (2010b) that

$$\Delta(F_n) := \sqrt{\sum_{i,j=1}^{d} \mathrm{Var}\left(q_j^{-1}\langle DF_{n,i}, DF_{n,j}\rangle_{\mathfrak{H}}\right)} \tag{1.6}$$

provides an upper bound in the multidimensional normal approximation of $\{F_n\}_{n\ge1}$ with respect to the Wasserstein distance. In Campese (2013), Campese aimed to establish conditions on $\{F_n\}_{n\ge1}$ such that $\Delta(F_n)$ provides an optimal rate of convergence for F_n with respect to some suitable distance. Let $g \in C^3(\mathbb{R}^d)$ with bounded derivatives up to order three. By Stein's method as introduced in Section 2.4 and Lemma 2.1,

$$\begin{aligned} \mathrm{E}[g(F_n)] - \mathrm{E}[g(Z)] &= \mathrm{E}\left[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d} - \langle C, \mathrm{Hess}\, U_{g,C}(F_n)\rangle_{\mathrm{HS}}\right] \\ &= \sum_{i,j=1}^{d} \mathrm{E}\left[\partial_{ij}^2 U_{g,C}(F_n)\left(q_j^{-1}\langle DF_{n,i}, DF_{n,j}\rangle_{\mathfrak{H}} - C_{ij}\right)\right], \end{aligned} \tag{1.7}$$

where $U_{g,C}$ defined as (2.17) satisfies the multidimensional Stein's equation (2.16), $\mathrm{Hess}\, U_{g,C}$ is a $d\times d$ matrix with entries given by $(\mathrm{Hess}\, U_{g,C})_{ij} = \partial_{ij}^2 U_{g,C}$, $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ denotes the Euclidean inner product and $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ denotes the Hilbert-Schmidt inner product (see Section 2.4 for specific definitions). If the sequence

$$\left\{\frac{\sum_{i,j=1}^{d} \mathrm{E}\left[\partial_{ij}^2 U_{g,C}(F_n)\left(q_j^{-1}\langle DF_{n,i}, DF_{n,j}\rangle_{\mathfrak{H}} - C_{ij}\right)\right]}{\Delta(F_n)}\right\}_{n\ge1} \tag{1.8}$$

which involves (1.6) and the right-hand side of (1.7), converges to a nonzero limit, then $\Delta(F_n)$ exactly characterizes the rate of convergence of $\mathrm{E}[g(F_n)] - \mathrm{E}[g(Z)]$. To prove the convergence of

(1.8), Campese assumed that, for $1 \le i, j \le d$ such that

$$\Delta_{ij}(F_n) := \sqrt{\operatorname{Var}\left(q_j^{-1}\left\langle DF_{n,i}, DF_{n,j}\right\rangle_{\mathfrak{H}}\right)} \asymp \Delta(F_n) \tag{1.9}$$

holds, the $(d+1)$-dimensional random vector sequence

$$\left\{\left(F_n, \frac{q_j^{-1}\left\langle DF_{n,i}, DF_{n,j}\right\rangle_{\mathfrak{H}} - C_{ij}}{\Delta_{ij}(F_n)}\right)\right\}_{n\ge 1} \tag{1.10}$$

converges in distribution to a centered $(d+1)$-dimensional Gaussian vector $\left(Z, \tilde{Z}_{ij}\right)$. Under such assumptions, Campese (2013, Theorem 3.4, Corollary 3.6) showed that the lim inf and lim sup of the sequence

$$\left\{\frac{\mathrm{E}\left[g\left(F_n\right)\right] - \mathrm{E}[g(Z)]}{\Delta(F_n)}\right\}_{n\ge 1}$$

coincide with those of

$$\left\{\frac{1}{3}\sum_{i,j,k=1}^{d} \frac{\Delta_{ij}(F_n)}{\Delta(F_n)} \rho_{ijk} \mathrm{E}\left[\partial_{ijk}^3 g(Z)\right]\right\}_{n\ge 1}, \tag{1.11}$$

where the constants ρ_{ijk} are defined by $\rho_{ijk} = \mathrm{E}\left[\tilde{Z}_{ij} Z_k\right]$ for $1 \le i, j, k \le d$ such that (1.9) is true, and $\rho_{ijk} = 0$ otherwise. If the lim inf and lim sup of (1.11) are not equal to zero and finite, then $\Delta(F_n)$ provides an optimal rate of convergence for F_n with respect to the distance defined as

$$d(F_n, Z) = \sup\left\{\left|\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right]\right|\right\},$$

where g runs over the class of all functions $g \in C^3(\mathbb{R}^d)$ with bounded derivatives up to order three. Sufficient conditions analogously to (1.3) and (1.4), for the convergence in distribution of the random sequence (1.10) to a centered Gaussian vector, are established in Campese (2013, Proposition 4.2).

The techniques developed in Campese (2013) are extensive and heuristic. In his framework, smooth functionals of a Gaussian process, whose components do not necessarily belong to Wiener chaoses, are considered, and covariances of these functionals are allowed to fluctuate. However, due to the assumption that the random sequence (1.10) converges in distribution, it seems that Campese's findings do not provide a complete characterization of the optimal rate of convergence for F_n. Specially, Campese's techniques will fail when the limit of (1.11) is equal to zero, see Campese (2013, Section 5.1) or Section 4.2 in this paper for example. Note that, in this example, all components of F_n belong to the second Wiener chaos of some isonormal Gaussian process. In addition, Campese (2013, Section 5.4) remarked that for a non-trivial application of the results he obtained in the Breuer-Major central limit theorem, at least one of the integers q_i, the order of $F_{n,i}$, should be even.

In this paper, we consider a sequence of d-dimensional random vectors $\{F_n = (F_{n,1}, \dots, F_{n,d})\}_{n\ge 1}$ whose components all belong to the q-th Wiener chaos, where $q \ge 2$. We still assume that the covariance matrix of F_n is C and F_n converges in distribution to a d-dimensional normal vector $Z \sim \mathcal{N}_d(0, C)$. Without any additional assumptions, we exhaustively investigate the optimal rate of convergence with respect to the smooth distance $\rho_4(\cdot,\cdot)$ defined as

$$\rho_4(F, G) = \sup\left\{\left|\mathrm{E}\left[g(F)\right] - \mathrm{E}\left[g(G)\right]\right|\right\},$$

where g ranges over the class of all functions $g \in C^4(\mathbb{R}^d)$ such that $M_j(g) \le 1$ for all $0 \le j \le 4$ (that is, g and all its derivatives up to order four are bounded by one, see (2.14) for the precise definition of $M_j(g)$), and F, G are two d-dimensional random vectors. Moreover, if C is positive definite, we improve the distance to

$$\rho_3(F, G) = \sup\left\{\left|\mathrm{E}\left[g(F)\right] - \mathrm{E}\left[g(G)\right]\right|\right\},$$

where g runs over the class of all functions $g \in C^3(\mathbb{R}^d)$ such that $M_j(g) \leq 1$ for all $0 \leq j \leq 3$. Specifically, in Theorem 3.1, we obtain that

$$\rho_4(F_n, Z) \asymp M(F_n) := \max\left\{\sum_{|m|=3} |\kappa_m(F_n)|, \sum_{i=1}^d \kappa_{4e_i}(F_n)\right\},$$

and furthermore, if C is positive definite,

$$\rho_3(F_n, Z) \asymp M(F_n).$$

Here, for a multi-index $m \in \mathbb{N}_0^d = (\mathbb{N} \cup \{0\})^d$, $\kappa_m(F_n)$ is the cumulant of order m of F_n (see Definition 2.2). For example, $\kappa_{4e_i}(F_n) = \mathrm{E}[F_{n,i}^4] - 3(\mathrm{E}[F_{n,i}^2])^2$, where for $1 \leq i \leq d$, $e_i = (e_{i1}, \ldots, e_{id}) \in \mathbb{N}_0^d$ is defined by $e_{ij} = \delta_{ij}$ for $1 \leq j \leq d$, and δ_{ij} is the Kronecker symbol defined by $\delta_{ij} = 1$ if $j = i$ and $\delta_{ij} = 0$ otherwise. In other words, the concise expression $M(F_n)$ is the optimal rate of convergence with respect to the smooth distance $\rho_4(\cdot, \cdot)$, and $\rho_3(\cdot, \cdot)$ if C is positive definite. It can be shown that, there exists a constant c depending only on q and C such that

$$\sum_{|m|=3} |\kappa_m(F_n)| \leq c\sqrt{\sum_{i=1}^d \kappa_{4e_i}(F_n)},$$

by combining interpolation techniques (see Nourdin and Peccati (2010b, Theorem 4.2) or Nourdin et al. (2010a, Theorem 7.2)) and Nourdin and Peccati (2012, Equation (6.2.6)). This is an extension of Nourdin and Peccati (2010b, Proposition 3.1) to the multidimensional case. Note that $M(F_n)$ can be determined by either one of the two quantities $\sum_{|m|=3} |\kappa_m(F_n)|$ and $\sum_{i=1}^d \kappa_{4e_i}(F_n)$ (see Section 4 for examples of both cases).

Although focusing on the case where the components of F_n have coinciding orders, we improve upon the work in Campese (2013) in two main aspects. First, the optimal rate of convergence we obtained is more explicit and concise. Second, our conclusions can be applied to determine the optimal rate of convergence when Campese's techniques can not work, as discussed in Section 4.2.

There are two main differences in strategies between Campese (2013) and this paper. The first difference is that we expand $\mathrm{E}[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d}]$ appeared in (1.7) to a higher order by using Proposition 3.7 to obtain a more refined bound. In (1.7), Campese expanded $\mathrm{E}[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d}]$ as

$$\begin{aligned}\mathrm{E}[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d}] &= \sum_{i,j=1}^d \mathrm{E}\left[\partial_{ij}^2 U_{g,C}(F_n) q_j^{-1} \langle DF_{n,i}, DF_{n,j}\rangle_{\mathfrak{H}}\right] \\ &= \sum_{m=e_i+e_j, 1\leq i,j\leq d} \mathrm{E}\left[\Gamma_{e_i,e_j}(F_n)\partial^m U_{g,C}(F_n)\right],\end{aligned}$$

which means that Campese took $M = 2$ in Proposition 3.7. Please refer to Definition (2.3) for the definition of Γ-random variable. While we take $M = 4$ and expand $\mathrm{E}[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d}]$ as

$$\begin{aligned}\mathrm{E}[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d}] = &\sum_{i,j=1}^d C_{ij}\mathrm{E}\left[\partial_{ij}^2 U_{g,C}(F_n)\right] + \frac{1}{2}\sum_{\substack{m=e_i+e_j+e_k,\\ 1\leq i,j,k\leq d}} \kappa_m(F_n)\mathrm{E}\left[\partial^m U_{g,C}(F_n)\right] \\ &+ \sum_{\substack{m=e_i+e_j+e_k+e_l,\\ 1\leq i,j,k,l\leq d}} \mathrm{E}\left[\Gamma_{e_i,e_j,e_k,e_l}(F_n)\partial^m U_{g,C}(F_n)\right], \qquad (1.12)\end{aligned}$$

to achieve the optimal convergence of rate. Second, besides Malliavin calculus and Stein's method for normal approximation, which are the main techniques used in Campese (2013), we also make full use of the method of cumulants to estimate the reminder terms of $\mathrm{E}[g(F_n)] - \mathrm{E}[g(Z)]$, namely the second and third terms on the right-hand side of (1.12). More precisely, on the one hand, combining the higher-order expansion of $\mathrm{E}[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d}]$ and technical estimates of Γ-random variables (see Proposition 3.9), we prove the upper bound. Namely, there exists a constant $0 < c_2 < \infty$ such that for n large enough,

$$\rho_k\left(F_n, Z\right) \leq c_2 \max\left\{\sum_{|m|=3}\left|\kappa_m(F_n)\right|, \sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right\},$$

for $k = 4$, and $k = 3$ if C is positive definite. On the other hand, we delicately set up several specific test functions (see Lemma 3.11) to obtain the lower bound. That is, there exists a constant $0 < c_1 < \infty$ such that for n large enough, for $k = 3, 4$,

$$\rho_k\left(F_n, Z\right) \geq c_1 \max\left\{\sum_{|m|=3}\left|\kappa_m(F_n)\right|, \sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right\}.$$

Note that under the assumption that all components of F_n belong to the same fixed Wiener chaos, the optimal rate of convergence we obtained is comparatively concise. To some extent, this result is consistent with and improves Noreddine and Nourdin (2011, Theorem 1.5), in which it is proved that

$$\sup\left\{\left|\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right]\right|\right\} \leq c_1 \sum_{i=1}^{d}\sqrt{\kappa_{4e_i}(F_n)},$$

where g runs over the class of all functions $g \in C^2(\mathbb{R}^d)$ with the second derivatives bounded by one.

We assume coinciding orders in this paper because the critical estimates of cumulants and related Γ-random variables (see Proposition 3.9) that we use in the proof of Theorem 3.1 may not be valid when the orders are different. Specifically, the definition of Γ-random variable (see Definition 2.3) implies that the essence of Proposition 3.9 is to estimate the variance of the random variable $\langle DF, DG\rangle_{\mathfrak{H}}$, where $F = I_p(f)$ and $G = I_q(g)$ with $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, $p, q \geq 1$. By Nourdin and Peccati (2012, Lemma 6.2.1), we have that if $p = q$, then

$$\operatorname{Var}\left(\langle DF, DG\rangle_{\mathfrak{H}}\right) \leq \frac{p^4}{2}\sum_{r=1}^{p-1}(r-1)!^2\binom{p-1}{r-1}^4(2p-2r)!\left(\|f\otimes_{p-r}f\|^2_{\mathfrak{H}^{\otimes 2r}} + \|g\otimes_{p-r}g\|^2_{\mathfrak{H}^{\otimes 2r}}\right),$$

and if $p < q$, then

$$\begin{aligned}
&\operatorname{Var}\left(\langle DF, DG\rangle_{\mathfrak{H}}\right)\\
&\leq q^2 p!^2\binom{q-1}{p-1}^2(q-p)!\|f\|^2_{\mathfrak{H}^{\otimes p}}\|g\otimes_{q-p}g\|_{\mathfrak{H}^{\otimes 2p}}\\
&\quad + \frac{p^2q^2}{2}\sum_{r=1}^{p-1}(r-1)!^2\binom{p-1}{r-1}^2\binom{q-1}{r-1}^2(p+q-2r)!\left(\|f\otimes_{p-r}f\|^2_{\mathfrak{H}^{\otimes 2r}} + \|g\otimes_{q-r}g\|^2_{\mathfrak{H}^{\otimes 2r}}\right).
\end{aligned} \tag{1.13}$$

It is obvious that when $p = q$, we have that

$$\operatorname{Var}\left(\langle DF, DG\rangle_{\mathfrak{H}}\right) \leq c(p)\left(\kappa_4(F) + \kappa_4(G)\right), \tag{1.14}$$

where $c(p)$ is a constant depending on p, $\kappa_4(F) := \mathrm{E}\left[F^4\right] - 3\left(\mathrm{E}\left[F^2\right]\right)^2$ and the definition of $\kappa_4(G)$ is the same as that of $\kappa_4(F)$. However, when $p < q$, one can only obtain that

$$\operatorname{Var}\left(\langle DF, DG\rangle_{\mathfrak{H}}\right) \leq c\left(p, \mathrm{E}\left[F^2\right]\right)\left(\kappa_4(F) + \kappa_4(G) + \sqrt{\kappa_4(G)}\right), \tag{1.15}$$

where $c\left(p, \mathrm{E}\left[F^2\right]\right)$ is a constant that depends on p and $\mathrm{E}\left[F^2\right]$, and $\sqrt{\kappa_4(G)}$ is from the first term on the right-hand side of (1.13). A comparison of the estimates (1.14) and (1.15) shows that the estimate obtained when $p = q$ is better than when $p < q$. As a result, the techniques we used are insufficient to solve the case of different orders. This improving topic will be investigated in other works.

Compared to one-dimensional case, the test functions in the definition of the distance between two random vectors in multidimensional case need to be smoother. There are two reasons for this. First, second-order differential operators are involved in the multidimensional Stein's equation (see (2.16)), unlike in one-dimensional case (see Nourdin and Peccati (2012, Definition 3.2.1)). Second, given a test function g, the solution of the one-dimensional Stein's equation associated with g has better regularity than the solution of the multidimensional Stein's equation, see Biermé et al. (2012, Proposition 3.2) and Chen et al. (2011, Lemma 2.6) respectively. In Biermé et al. (2012), Biermé, Bonami, Nourdin and Peccati applied the chain rule (2.8) three times for the solution of the one-dimensional Stein's equation. Combining the fact that the solution of Stein's equation is smoother than the test function (see Biermé et al. (2012, Proposition 3.2)), the authors in Biermé et al. (2012) required the test functions to be smooth up to order two and then derived the optimal rate of convergence in the one-dimensional case. After that, Nourdin and Peccati (2015) dealt with irregular test functions by using Lusin's Theorem and obtained the optimal rate of convergence in the total variation distance, a non-smooth distance. In the multidimensional case, less smoothness of test functions means less use of the chain rule (2.8) and then less expansion for the quantity $\mathrm{E}[\langle F, \nabla f(F)\rangle_{\mathbb{R}^d}]$ in Proposition 3.7. This usually leads to a suboptimal rate of convergence, where only an upper bound can be obtained. In this paper, we define the distance of two random vectors as ρ_4 to achieve the optimal rate of convergence for a general non-negative definite covariance matrix C. When C is positive definite, we combine the smoothing argument and the regularity of the solution of the multidimensional Stein's equation (see Lemma 2.5) to improve the distance to ρ_3. It is reasonable to require more smoothness of the test functions when C is singular. See Chatterjee and Meckes (2008); Dung (2019); Fang and Koike (2022); Meckes (2009); Nourdin et al. (2010b); Peccati and Zheng (2010); Reinert and Röllin (2009) for more discussions on Stein's method and smoothness requirements for test functions in multidimensional normal approximation.

As an application, we first consider a sequence of complex Wiener-Itô integrals $\{F_n\}_{n\geq 1}$ in Section 4.1. Assume that F_n converges in distribution to a complex normal variable Z with the same covariance matrix as F_n. Combining Theorem 3.1 and the fact that the real and imaginary parts of a complex Wiener-Itô integral can be expressed as a real Wiener-Itô integral respectively (see Chen and Liu (2017, Theorem 3.3)), we yield Theorem 4.1, which states that for $k=4$, and $k=3$ if the covariance matrix of F_n is positive definite,

$$\rho_k\left(F_n, Z\right) \asymp \max\left\{\left|\mathrm{E}\left[F_n^3\right]\right|, \left|\mathrm{E}\left[F_n^2\overline{F_n}\right]\right|, \mathrm{E}\left[|F_n|^4\right] - 2\left(\mathrm{E}\left[|F_n|^2\right]\right)^2 - \left|\mathrm{E}\left[F_n^2\right]\right|^2\right\}.$$

As an example, we get the optimal rate of convergence for a statistic associated with the least squares estimator of the drift coefficient for the complex-valued Ornstein-Uhlenbeck process. In Section 4.2, we consider the counterexample provided in Campese (2013, Section 5.1) and apply our conclusion to derive the optimal rate of convergence for a sequence of vector-valued Wiener-Itô integrals with kernels of step functions. In Section 4.3, by combining our techniques and some results from the literature such as Campese (2013); Ginovian (1994); Ginovyan and Sahakyan (2007), we get the optimal rate of convergence in the multidimensional normal approximation of vector-valued Toeplitz quadratic functionals.

The paper is organized as follows. Section 2 introduces some elements of the isonormal Gaussian process, Malliavin calculus, the method of cumulants and multidimensional Stein's method for normal approximation. In Section 3, we obtain the optimal rate of convergence for a sequence of vector-valued Wiener-Itô integrals with respect to smooth distance $\rho_k(\cdot,\cdot)$, where $k=3$ or 4. In Section 4, we apply the main results we proved in Section 3 to derive the optimal rates of convergence for a sequence of complex Wiener-Itô integrals, vector-valued Wiener-Itô integrals with kernels of step functions and vector-valued Toeplitz quadratic functionals.

2. Preliminaries

In this section, we briefly introduce some basic theories of the isonormal Gaussian process, Malliavin calculus, cumulants and multidimensional Stein's method. See Chen and Liu (2019); Itô (1952); Nourdin and Peccati (2012); Nualart (2006); Nualart and Nualart (2018) for more details.

2.1. *Isonormal Gaussian process.* Suppose that $\mathfrak{H}$ is a real separable Hilbert space with the inner product denoted by $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$. Let $\|h\|_{\mathfrak{H}}$ denote the norm of $h\in\mathfrak{H}$. Consider a real isonormal Gaussian process $X=\{X(h):h\in\mathfrak{H}\}$ defined on a complete probability space $(\Omega,\mathcal{F},P)$. That is, X is a Gaussian family of centered random variables such that $\mathrm{E}\left[X(h)X(g)\right]=\langle h,g\rangle_{\mathfrak{H}}$ for any $h,g\in\mathfrak{H}$.

For $q\geq 0$, the q-th Wiener-Itô chaos $\mathcal{H}_q(X)$ of X is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)):h\in\mathfrak{H},\|h\|_{\mathfrak{H}}=1\}$, where $H_q(x)$ is the Hermite polynomial of degree q defined by the equality

$$\exp\left\{tx-\frac{1}{2}t^2\right\}=\sum_{q=0}^{\infty}\frac{t^q}{q!}H_q(x),\ t\in\mathbb{R}.$$

Let $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$ denote the q-th tensor product and the q-th symmetric tensor product of $\mathfrak{H}$, respectively. For any $q\geq 1$, the mapping $I_q\left(h^{\otimes q}\right)=H_q(X(h))$ for $\|h\|_{\mathfrak{H}}=1$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$, equipped with the norm $\sqrt{q!}\|\cdot\|_{\mathfrak{H}^{\otimes q}}$, and the q-th Wiener-Itô chaos $\mathcal{H}_q(X)$. For $q=0$, we write $I_0(c)=c$ for $c\in\mathbb{R}$. For any $f\in\mathfrak{H}^{\odot q}$, the random variable $I_q(f)$ is called the real q-th Wiener-Itô integral of f with respect to X. Wiener chaoses of different orders are orthogonal, that is, for $f\in\mathfrak{H}^{\odot p}$ and $g\in\mathfrak{H}^{\odot q}$, where $p,q\geq 1$,

$$\mathrm{E}\left[I_p(f)I_q(g)\right]=\begin{cases}p!\left\langle f,g\right\rangle_{\mathfrak{H}^{\otimes p}}, & p=q,\\ 0, & p\neq q.\end{cases}\tag{2.1}$$

The Wiener-Itô chaos decomposition of $L^2(\Omega,\sigma(X),P)$ implies that $L^2(\Omega,\sigma(X),P)$ can be decomposed into an infinite orthogonal sum of the spaces $\mathcal{H}_q(X)$. That is, any random variable $F\in L^2(\Omega,\sigma(X),P)$ admits a unique expansion of the form

$$F=\sum_{q=0}^{\infty}I_q\left(f_q\right),$$

where $f_0=\mathrm{E}[F]$, and $f_q\in\mathfrak{H}^{\odot q}$ for $q\geq 1$ are uniquely determined by F.

Let $\{\eta_k\}_{k\geq 1}$ be a complete orthonormal system in $\mathfrak{H}$. We define $a\wedge b$ as the minimum of $a,b\in\mathbb{R}$. Given $f\in\mathfrak{H}^{\odot p}$, $g\in\mathfrak{H}^{\odot q}$, for $r=0,\dots,p\wedge q$, the r-th contraction of f and g is an element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined as

$$f\otimes_r g=\sum_{i_1,\dots,i_r=1}^{\infty}\left\langle f,\eta_{i_1}\otimes\cdots\otimes\eta_{i_r}\right\rangle_{\mathfrak{H}^{\otimes r}}\otimes\left\langle g,\eta_{i_1}\otimes\cdots\otimes\eta_{i_r}\right\rangle_{\mathfrak{H}^{\otimes r}}.\tag{2.2}$$

Notice that $f\otimes_r g$ is not necessarily symmetric, and we denote by $f\tilde{\otimes}_r g$ its symmetrization. The product formula for real multiple Wiener-Itô integral, as shown in Nourdin and Peccati (2012, Proposition 2.7.10), states that for $f\in\mathfrak{H}^{\odot p}$ and $g\in\mathfrak{H}^{\odot q}$ with $p,q\geq 0$,

$$I_p(f)I_q(g)=\sum_{r=0}^{p\wedge q}r!\binom{p}{r}\binom{q}{r}I_{p+q-2r}(f\tilde{\otimes}_r g).\tag{2.3}$$

Next, we introduce the complex isonormal Gaussian process. We complexify $\mathfrak{H}$ and $L^2(\Omega)$ in the usual way, denoted as $\mathfrak{H}_{\mathbb{C}}$ and $L^2_{\mathbb{C}}(\Omega)$, respectively. Suppose $\mathfrak{h}=f+\mathrm{i}g\in\mathfrak{H}_{\mathbb{C}}$ with $f,g\in\mathfrak{H}$, we define

$$X_{\mathbb{C}}(\mathfrak{h}):=X(f)+\mathrm{i}X(g),$$

which satisfies $\mathrm{E}\left[X_{\mathbb{C}}(\mathfrak{h})\overline{X_{\mathbb{C}}(\mathfrak{h}')}\right] = \langle \mathfrak{h}, \mathfrak{h}'\rangle_{\mathfrak{H}_{\mathbb{C}}}$ with $\mathfrak{h}' \in \mathfrak{H}_{\mathbb{C}}$. Here, for $\mathfrak{h} = f + \mathrm{i}g, \mathfrak{h}' = f' + \mathrm{i}g' \in \mathfrak{H}_{\mathbb{C}}$, where $f, g, f', g' \in \mathfrak{H}$,

$$\langle \mathfrak{h}, \mathfrak{h}'\rangle_{\mathfrak{H}_{\mathbb{C}}} = \langle f + \mathrm{i}g, f' + \mathrm{i}g'\rangle_{\mathfrak{H}_{\mathbb{C}}} = \left(\langle f, f'\rangle_{\mathfrak{H}} + \langle g, g'\rangle_{\mathfrak{H}}\right) + \mathrm{i}\left(-\langle f, g'\rangle_{\mathfrak{H}} + \langle g, f'\rangle_{\mathfrak{H}}\right).$$

Let $Y = \{Y(h) : h \in \mathfrak{H}\}$ be an independent copy of the isonormal Gaussian process X over $\mathfrak{H}$. We define $Y_{\mathbb{C}}(\mathfrak{h})$ for $\mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}$ in the same way as $X_{\mathbb{C}}(\mathfrak{h})$. Now, we define the complex isonormal Gaussian process $Z = \{Z(\mathfrak{h}) : \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}\}$ over $\mathfrak{H}_{\mathbb{C}}$ as

$$Z(\mathfrak{h}) := \frac{X_{\mathbb{C}}(\mathfrak{h}) + \mathrm{i}Y_{\mathbb{C}}(\mathfrak{h})}{\sqrt{2}},\ \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}.$$

Note that Z is a centered complex Gaussian family and satisfies

$$\mathrm{E}[Z(\mathfrak{h})^2] = 0, \quad \mathrm{E}[Z(\mathfrak{h})\overline{Z(\mathfrak{h}')}] = \langle \mathfrak{h}, \mathfrak{h}'\rangle_{\mathfrak{H}_{\mathbb{C}}},\ \forall \mathfrak{h}, \mathfrak{h}' \in \mathfrak{H}_{\mathbb{C}}.$$

For each $p, q \geq 0$, let $J_{p,q}(z)$ be the complex Hermite polynomial, or Hermite-Laguerre-Itô polynomial, given by

$$\exp\left\{\lambda\overline{z} + \overline{\lambda}z - 2|\lambda|^2\right\} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\overline{\lambda}^p\lambda^q}{p!q!}J_{p,q}(z),\ \lambda \in \mathbb{C}.$$

For example, $J_{p,0}(z) = z^p$ for $p \geq 0$, $J_{0,q}(z) = \overline{z}^q$ for $q \geq 0$, $J_{1,1}(z) = |z|^2 - 2$, $J_{1,2}(z) = \overline{z}\left(|z|^2 - 4\right)$, and $J_{2,2}(z) = |z|^4 - 8|z|^2 + 8$. Let $\mathcal{H}_{p,q}(Z)$ be the closed linear subspace of $L^2_{\mathbb{C}}(\Omega)$ generated by the random variables

$$\left\{J_{p,q}(Z(\mathfrak{h})) : \mathfrak{h} \in \mathfrak{H}_{\mathbb{C}}, \|\mathfrak{h}\|_{\mathfrak{H}_{\mathbb{C}}} = \sqrt{2}\right\}.$$

The space $\mathcal{H}_{p,q}(Z)$ is called the (p,q)-th Wiener-Itô chaos of Z.

Take a complete orthonormal system $\{\xi_k\}_{k\geq 1}$ in $\mathfrak{H}_{\mathbb{C}}$. Let Λ be the set of all sequences $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$ of non-negative integers with only finitely many nonzero components and set $|\mathbf{a}| := \sum_{k=1}^{\infty} a_k$. Let $\mathrm{symm}\,(h)$ denote the symmetrization of $h \in \mathfrak{H}_{\mathbb{C}}^{\otimes p}$ for some $p \geq 1$. For two sequences $\mathbf{p} = \{p_k\}_{k=1}^{\infty}, \mathbf{q} = \{q_k\}_{k=1}^{\infty} \in \Lambda$ satisfying $|\mathbf{p}| = p$ and $|\mathbf{q}| = q$, the linear mapping

$$\mathcal{I}_{p,q}\left(\mathrm{symm}\left(\otimes_{\mathrm{k}=1}^{\infty}\xi_{\mathrm{k}}^{\otimes \mathrm{p_k}}\right) \otimes \mathrm{symm}\left(\otimes_{\mathrm{k}=1}^{\infty}\overline{\xi_{\mathrm{k}}}^{\otimes \mathrm{q_k}}\right)\right) := \prod_{k=1}^{\infty}\frac{1}{\sqrt{2^{p_k+q_k}}}J_{p_k,q_k}\left(\sqrt{2}Z\left(\xi_k\right)\right), \tag{2.4}$$

provides an isometry from the tensor product $\mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q}$, equipped with the norm $\sqrt{p!q!}\|\cdot\|_{\mathfrak{H}_{\mathbb{C}}^{\otimes(p+q)}}$, onto the (p,q)-th Wiener-Itô chaos $\mathcal{H}_{p,q}(Z)$. Itô proved (2.4) in Itô (1952, Theorem 13.2). For any $f \in \mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q}$, $\mathcal{I}_{p,q}(f)$ is called the complex (p,q)-th Wiener-Itô integral of f with respect to Z. Itô (1952, Theorem 7) showed that complex Wiener-Itô integrals satisfy the isometry property. That is, for $f \in \mathfrak{H}^{\odot a} \otimes \mathfrak{H}^{\odot b}$ and $g \in \mathfrak{H}^{\odot c} \otimes \mathfrak{H}^{\odot d}$,

$$\mathrm{E}\left[\mathcal{I}_{a,b}(f)\overline{\mathcal{I}_{c,d}(g)}\right] = \begin{cases} a!b!\,\langle f, g\rangle_{\mathfrak{H}_{\mathbb{C}}^{\otimes(a+b)}}, & a = c, b = d, \\ 0, & otherwise. \end{cases} \tag{2.5}$$

The complex Wiener-Itô chaos decomposition of $L^2_{\mathbb{C}}(\Omega, \sigma(Z), P)$ implies that $L^2_{\mathbb{C}}(\Omega, \sigma(Z), P)$ can be decomposed into an infinite orthogonal sum of the spaces $\mathcal{H}_{p,q}(Z)$. That is, any random variable $F \in L^2_{\mathbb{C}}(\Omega, \sigma(Z), P)$ admits a unique expansion of the form

$$F = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\mathcal{I}_{p,q}\left(f_{p,q}\right),$$

where $f_{0,0} = \mathrm{E}[F]$, and $f_{p,q} \in \mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q}$ for $p + q \geq 1$ are uniquely determined by F.

Given $f \in \mathfrak{H}_{\mathbb{C}}^{\odot a} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot b}$ and $g \in \mathfrak{H}_{\mathbb{C}}^{\odot c} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot d}$, for $i = 0, \dots, a \wedge d$, $j = 0, \dots, b \wedge c$, the (i,j)-th contraction of f and g is an element of $\mathfrak{H}_{\mathbb{C}}^{\otimes(a+c-i-j)} \otimes \mathfrak{H}_{\mathbb{C}}^{\otimes(b+d-i-j)}$ defined as

$$f \otimes_{i,j} g = \sum_{l_1,\dots,l_{i+j}=1}^{\infty} \left\langle f, \xi_{l_1} \otimes \cdots \otimes \xi_{l_i} \otimes \overline{\xi_{l_{i+1}}} \otimes \cdots \otimes \overline{\xi_{l_{i+j}}} \right\rangle \otimes \left\langle g, \xi_{l_{i+1}} \otimes \cdots \otimes \xi_{l_{i+j}} \otimes \overline{\xi_{l_1}} \otimes \cdots \otimes \overline{\xi_{l_i}} \right\rangle,$$

and by convention, $f \otimes_{0,0} g = f \otimes g$ denotes the tensor product of f and g. The product formula for complex Wiener-Itô integral, as presented in Chen (2017, Theorem 2.1) and Hoshino et al. (2017, Theorem A.1), states that for $f \in \mathfrak{H}^{\odot a} \otimes \mathfrak{H}^{\odot b}$ and $g \in \mathfrak{H}^{\odot c} \otimes \mathfrak{H}^{\odot d}$, where $a, b, c, d \geq 0$,

$$\mathcal{I}_{a,b}(f)\mathcal{I}_{c,d}(g) = \sum_{i=0}^{a\wedge d} \sum_{j=0}^{b\wedge c} \binom{a}{i}\binom{d}{i}\binom{b}{j}\binom{c}{j} i! j! \mathcal{I}_{a+c-i-j,b+d-i-j}\left(f \otimes_{i,j} g\right). \tag{2.6}$$

2.2. *Malliavin calculus.* Let $\mathcal{S}$ denote the class of smooth random variables given by

$$F = f\left(X(h_1), \dots, X(h_d)\right),$$

where $h_1, \dots, h_d \in \mathfrak{H}$, $d \geq 1$ and $f \in C_p^{\infty}(\mathbb{R}^d)$. Here, $C_p^{\infty}(\mathbb{R}^d)$ is the set of all infinitely differentiable real-valued functions such that all its partial derivatives have polynomial growth. Given $F \in \mathcal{S}$, the Malliavin derivative DF is a $\mathfrak{H}$-valued random element defined as

$$DF = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\left(X\left(h_1\right), \dots, X\left(h_d\right)\right) h_i.$$

The derivative operator D is a closable and unbounded operator from $L^p(\Omega)$ to $L^p(\Omega; \mathfrak{H})$ for any $p \geq 1$. By iteration, for $k \geq 2$, one can define the k-th derivative $D^k F \in L^p(\Omega; \mathfrak{H}^{\otimes k})$. For any $p \geq 1$ and $k \geq 0$, we denote $\mathbb{D}^{k,p}$ as the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$ given by

$$\|F\|_{k,p}^p = \sum_{i=0}^{k} \mathrm{E}\left(\left\|D^i F\right\|_{\mathfrak{H}^{\otimes i}}^p\right).$$

For any $p \geq 1$ and $k \geq 0$, we set $\mathbb{D}^{\infty,p} = \bigcap_{k\geq 0} \mathbb{D}^{k,p}$, $\mathbb{D}^{k,\infty} = \bigcap_{p \geq 1} \mathbb{D}^{k,p}$ and $\mathbb{D}^{\infty} = \bigcap_{k\geq 0} \mathbb{D}^{k,\infty}$. It is known that for $f \in \mathfrak{H}^{\odot p}$, $I_p(f) \in \mathbb{D}^{\infty}$ and for any $k \geq 0$,

$$D^k I_p(f) = \begin{cases} \frac{p!}{(p-k)!} I_{p-k}(f), & k \leq p, \\ 0, & k > p. \end{cases} \tag{2.7}$$

The derivative operator D satisfies the chain rule. Specifically, if $\varphi : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and $F = (F_1, \dots, F_d)$ is a vector with components belonging to $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^{d} \frac{\partial \varphi}{\partial x_i}(F) DF_i. \tag{2.8}$$

The chain rule still holds if $F_i \in \mathbb{D}^{\infty}$ for $1 \leq i \leq d$ and φ has continuous partial derivatives with at most polynomial growth.

We denote by δ the divergence operator, defined as the adjoint operator of D, which is an unbounded operator from a domain in $L^2(\Omega; \mathfrak{H})$ to $L^2(\Omega)$. A random element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of δ, denoted $\mathrm{Dom}\delta$, if and only if it verifies

$$\left|\mathrm{E}\left[\langle DF, u \rangle_{\mathfrak{H}}\right]\right| \leq c_u \sqrt{\mathrm{E}\left[F^2\right]},$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. In particular, if $u \in \mathrm{Dom}\delta$, then $\delta(u)$ is characterized by the following duality relationship

$$\mathrm{E}(F\delta(u)) = \mathrm{E}\left(\langle DF, u \rangle_{\mathfrak{H}}\right), \tag{2.9}$$

for any $F \in \mathbb{D}^{1,2}$.

The operator L defined as $L = -\sum_{p=0}^{\infty} pJ_p$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $T_t = \sum_{p=0}^{\infty} e^{-pt} J_p$, where J_p denotes the orthogonal projection on the p-th Wiener chaos. Its domain in $L^2(\Omega)$ is

$$\mathrm{Dom}L = \left\{ F \in L^2(\Omega) : \sum_{p=1}^{\infty} p^2 \left\| J_p F \right\|_2^2 < \infty \right\}.$$

By Nualart (2006, Proposition 1.4.3), we know that the operators D, δ and L satisfy the following relation. For $F \in L^2(\Omega)$, $F \in \mathrm{Dom}L$ if and only if $F \in \mathrm{Dom}(\delta D)$, and in this case

$$\delta DF = -LF. \tag{2.10}$$

For any $F \in L^2(\Omega)$, we also define $L^{-1}F = -\sum_{p=1}^{\infty} \frac{1}{p} J_p(F)$. The operator L^{-1} is called the pseudo-inverse of L. For any $F \in L^2(\Omega)$, we have that $L^{-1}F \in \mathrm{Dom}L$, and

$$LL^{-1}F = L^{-1}LF = F - \mathrm{E}[F]. \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we can get the following useful lemma.

Lemma 2.1 (Nourdin and Peccati (2010a) Lemma 3.1)**.** *Suppose that $F \in \mathbb{D}^{1,2}$ and $G \in L^2(\Omega)$. Then, $L^{-1}G \in \mathbb{D}^{2,2}$ and*

$$\mathrm{E}[FG] = \mathrm{E}[F]\mathrm{E}[G] + \mathrm{E}\left[\left\langle DF, -DL^{-1}G \right\rangle_{\mathfrak{H}} \right].$$

2.3. *Cumulants.* First, we recall some standard multi-index notations. A multi-index is defined as a d-dimensional vector $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d = (\mathbb{N} \cup \{0\})^d$. For ease of notations, we write $|m| = \sum_{i=1}^{d} m_i$, $\partial_i = \frac{\partial}{\partial x_i}$, $\partial^m = \partial_1^{m_1} \cdots \partial_d^{m_d}$, $x^m = \prod_{i=1}^{d} x_i^{m_i}$ and $|x|^m = \prod_{i=1}^{d} |x_i|^{m_i}$, where $x \in \mathbb{R}^d$. By convention, we have $0^0 = 1$. For any $i = 1, \ldots, d$, we denote by $e_i = (e_{i1}, \ldots, e_{id}) \in \mathbb{N}_0^d$ the multi-index defined by $e_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. We can write every multi-index m as a sum of $|m|$ multi-indices $l_1, \ldots, l_{|m|} \in \{e_1, \ldots, e_d\}$, and this sum is unique up to the order of the summands. For instance, the elementary decomposition for the multi-index $(1, 2, 0)$ is $\{(1, 0, 0), (0, 1, 0), (0, 1, 0)\}$. We denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ the Euclidean inner product.

Definition 2.2. Let $F = (F_1, \ldots, F_d)$ be a d-dimensional random vector satisfying $\mathrm{E}|F|^m < \infty$ for some $m \in \mathbb{N}_0^d \backslash \{0\}$. The characteristic function of F is denoted by $\phi_F(t) = \mathrm{E}\left[\mathrm{e}^{\mathrm{i}\langle t, F \rangle_{\mathbb{R}^d}} \right]$ for $t \in \mathbb{R}^d$. Then the cumulant of order m of F is defined as

$$\kappa_m(F) = (-\mathrm{i})^{|m|} \partial^m \log \phi_F(t) \Big|_{t=0}.$$

For example, if $F_i, F_j \in L^2(\Omega)$, then $\kappa_{e_i}(F) = \mathrm{E}[F_i]$ and $\kappa_{e_i+e_j}(F) = \mathrm{E}[F_i F_j] - \mathrm{E}[F_i]\,\mathrm{E}[F_j]$.

Definition 2.3. Let $F = (F_1, \ldots, F_d)$ be a d-dimensional random vector with $F_i \in \mathbb{D}^{1,2}$ for $1 \leq i \leq d$. Suppose that $l_1, l_2, \ldots$ is a sequence taking values in $\{e_1, \ldots, e_d\}$. Set $\Gamma_{l_1}(F) := F^{l_1} = F_k$, if $l_1 = e_k$ for some $1 \leq k \leq d$. If the random variable $\Gamma_{l_1,\ldots,l_k}(F)$ is a well-defined element of $L^2(\Omega)$ for some $k \geq 1$, we set

$$\Gamma_{l_1,\ldots,l_{k+1}}(F) = \left\langle DF^{l_{k+1}}, -DL^{-1}\Gamma_{l_1,\ldots,l_k}(F) \right\rangle_{\mathfrak{H}}.$$

As shown in Noreddine and Nourdin (2011, Lemma 4.3), if $F_i \in \mathbb{D}^\infty$ for $1 \leq i \leq d$, then for any $k \geq 1$, the random variable $\Gamma_{l_1,\ldots,l_k}(F) \in \mathbb{D}^\infty$. The following theorem tells us the relation between the cumulant $\kappa_m(F)$ and the random variable $\Gamma_{l_1,\ldots,l_{|m|}}(F)$ with $m = l_1 + \ldots + l_{|m|}$.

Theorem 2.4 (Noreddine and Nourdin (2011) Theorem 4.4)**.** *Let $m \in \mathbb{N}_0^d \backslash \{0\}$. Write $m = l_1 + \cdots + l_{|m|}$, where $l_i \in \{e_1, \ldots, e_d\}$ for $1 \le i \le d$. Suppose that the random vector $F = (F_1, \ldots, F_d)$ is such that $F_i \in \mathbb{D}^{|m|, 2^{|m|}}$ for all $1 \le i \le d$. Then, we have*

$$\kappa_m(F) = (|m|-1)!\mathrm{E}\left[\Gamma_{l_1,\ldots,l_{|m|}}(F)\right].$$

If the components of random vector $F = (F_1, \ldots, F_d)$ are all multiple Wiener-Itô integrals, namely, $F = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d))$, where each $f_i \in \mathfrak{H}^{\odot q_i}$, the cumulant $\kappa_m(F)$ and the random variable $\Gamma_{l_1,\ldots,l_{|m|}}(F)$ with $m = l_1 + \cdots + l_{|m|}$ and $l_i \in \{e_1, \ldots, e_d\}$ for each i can be expressed more clearly, see Noreddine and Nourdin (2011, Theorem 4.6, Equation (4.29)). Specifically, set $\lambda_k = j$ when $l_k = e_j$. For simplicity, we drop the brackets and write $f_{\lambda_1} \tilde{\otimes}_{r_2} \cdots \tilde{\otimes}_{r_{|m|-1}} f_{\lambda_{|m|-1}}$ to implicitly assume that this quantity is defined iteratively from left to right. For instance, $f \tilde{\otimes}_\alpha g \tilde{\otimes}_\beta h$ actually means $\left(f \tilde{\otimes}_\alpha g\right) \tilde{\otimes}_\beta h$. Then

$$\begin{aligned}
&\Gamma_{l_1,\ldots,l_{|m|}}(F) \\
&= \sum_{r_2=1}^{q_{\lambda_1} \wedge q_{\lambda_2}} \cdots \sum_{r_{|m|}=1}^{[q_{\lambda_1}+\cdots+q_{\lambda_{|m|-1}}-2r_2-\cdots-2r_{|m|-1}] \wedge q_{\lambda_{|m|}}} c_{q,l}\left(r_2, \ldots, r_{|m|}\right) \mathbf{1}_{\left\{r_2 < \frac{q_{\lambda_1}+q_{\lambda_2}}{2}\right\}} \times \cdots \\
&\quad \times \mathbf{1}_{\left\{r_2+\cdots+r_{|m|-1} < \frac{q_{\lambda_1}+\cdots+q_{\lambda_{|m|-1}}}{2}\right\}} I_{q_{\lambda_1}+\cdots+q_{\lambda_{|m|}}-2r_2-\cdots-2r_{|m|}}\left(f_{\lambda_1} \tilde{\otimes}_{r_2} f_{\lambda_2} \ldots \tilde{\otimes}_{r_{|m|}} f_{\lambda_{|m|}}\right).
\end{aligned} \tag{2.12}$$

And

$$\kappa_m(F) = q_{\lambda_{|m|}}!(|m|-1)! \sum c_{q,l}\left(r_2, \ldots, r_{|m|-1}\right) \left\langle f_{\lambda_1} \tilde{\otimes}_{r_2} f_{\lambda_2} \cdots \tilde{\otimes}_{r_{|m|-1}} f_{\lambda_{|m|-1}}, f_{\lambda_{|m|}} \right\rangle_{\mathfrak{H}^{\otimes q_{\lambda_{|m|}}}},$$

where the sum $\sum$ runs over all collections of integers $r_2, \ldots, r_{|m|-1}$ such that

(1) $1 \le r_i \le q_{\lambda_i}$ for all $i = 2, \ldots, |m|-1$;
(2) $r_2 + \cdots + r_{|m|-1} = \frac{q_{\lambda_1}+\cdots+q_{\lambda_{|m|-1}}-q_{\lambda_{|m|}}}{2}$;
(3) $r_2 < \frac{q_{\lambda_1}+q_{\lambda_2}}{2}, \ldots, r_2 + \cdots + r_{|m|-2} < \frac{q_{\lambda_1}+\cdots+q_{\lambda_{|m|-2}}}{2}$;
(4) $r_3 \le q_{\lambda_1} + q_{\lambda_2} - 2r_2, \ldots, r_{|m|-1} \le q_{\lambda_1} + \cdots + q_{\lambda_{|m|-2}} - 2r_2 - \cdots - 2r_{|m|-2}$.

Here, the combinatorial constants $c_{q,l}(r_2, \ldots, r_s)$ are recursively defined by the relations

$$c_{q,l}(r_2) = q_{\lambda_2}(r_2-1)!\binom{q_{\lambda_1}-1}{r_2-1}\binom{q_{\lambda_2}-1}{r_2-1},$$

and, for $s \ge 3$,

$$\begin{aligned}
c_{q,l}(r_2, \ldots, r_s) = q_{\lambda_s}(r_s-1)!&\binom{q_{\lambda_1}+\cdots+q_{\lambda_{s-1}}-2r_2-\cdots-2r_{s-1}-1}{r_s-1} \\
&\binom{q_{\lambda_s}-1}{r_s-1} c_{q,l}(r_2, \ldots, r_{s-1}).
\end{aligned}$$

In particular, if $q_1 = \cdots = q_d = 2$, then the only possible integers $r_2, \ldots, r_{|m|-1}$ satisfying (1)-(4) are $r_2 = \cdots = r_{|m|-1} = 1$. Calculating directly, one can get that $c_{q,l}(r_2, \ldots, r_s) = 2^{s-1}$ for $s \ge 2$. Therefore, for any $f_1, \ldots, f_d \in \mathfrak{H}^{\odot 2}$ and any $m \in \mathbb{N}_0^d \backslash \{0\}$ with $|m| \ge 3$, we have

$$\kappa_m(I_2(f_1), \ldots, I_2(f_d)) = 2^{|m|-1}(|m|-1)! \left\langle f_{\lambda_1} \tilde{\otimes}_1 \cdots \tilde{\otimes}_1 f_{\lambda_{|m|-1}}, f_{\lambda_{|m|}} \right\rangle_{\mathfrak{H}^{\otimes 2}}. \tag{2.13}$$

2.4. *Multidimensional Stein's method for normal approximations.* We first introduce some notations. We denote by $\mathcal{M}_d(\mathbb{R})$ the collection of all real $d \times d$ matrices. The Hilbert-Schmidt inner product and the Hilbert-Schmidt norm on $\mathcal{M}_d(\mathbb{R})$, denoted respectively by $\langle \cdot, \cdot \rangle_{\mathrm{HS}}$ and $\| \cdot \|_{\mathrm{HS}}$, are defined as

$$\langle A, B \rangle_{\mathrm{HS}} = \mathrm{Tr}\left(AB^T\right), \quad \|A\|_{\mathrm{HS}} = \sqrt{\langle A, A \rangle_{\mathrm{HS}}}, \quad A, B \in \mathcal{M}_d(\mathbb{R}),$$

where $\mathrm{Tr}(\cdot)$ and $(\cdot)^T$ denote the usual trace and transposition operators, respectively. We denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ the Euclidean inner product and $\|\cdot\|$ the Euclidean norm. The k-th derivative $f^{(k)}(x)$ of a function $f \in C^k\left(\mathbb{R}^d\right)$ is a k-linear form on $\mathbb{R}^d$, given by

$$\left\langle f^{(k)}(x), (v_1, \dots, v_k) \right\rangle = \sum_{i_1, \dots, i_k = 1}^{d} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(x) \, (v_1)_{i_1} \cdots (v_k)_{i_k} \, ,$$

where $(v_i)_j$ denotes the j-th component of the vector v_i. Define the operator norm of $f^{(k)}(x)$ as

$$\|f^{(k)}(x)\|_{op} = \sup\left\{ \left| \left\langle f^{(k)}(x), (v_1, \dots, v_k) \right\rangle \right| : \|v_1\| = \cdots = \|v_k\| = 1 \right\}.$$

For $f \in C^k(\mathbb{R}^d)$, $k \geq 1$, let

$$M_k(f) := \sup_{x \in \mathbb{R}^d} \left\| f^{(k)}(x) \right\|_{op}, \tag{2.14}$$

and

$$\left\| f^{(k)} \right\|_{\infty} := \max_{1 \leq i_1, \dots, i_k \leq d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} f(x) \right|.$$

We write $M_0(f) = \|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$. Note that

$$\left\| f^{(k)} \right\|_{\infty} \leq M_k(f) \leq d^{k/2} \left\| f^{(k)} \right\|_{\infty}, \tag{2.15}$$

and

$$M_k(f) = \sup_{x \neq y} \frac{\left\| f^{(k-1)}(x) - f^{(k-1)}(y) \right\|_{op}}{\|x - y\|},$$

that is, $M_k(f)$ is the Lipschitz constant of the $(k-1)$-th derivative of f.

Let $C = (C_{ij})_{1 \leq i,j \leq d} \in \mathcal{M}_d(\mathbb{R})$ be a non-negative definite and symmetric matrix. We denote by $\mathcal{N}_d(0, C)$ the law of a d-dimensional Gaussian vector with zero mean and covariance matrix C. Multidimensional Stein's lemma (see Nourdin and Peccati (2012, Lemma 4.1.3)) shows that a random vector $N = (N_1, \dots, N_d) \sim \mathcal{N}_d(0, C)$ if and only if

$$\mathrm{E}\left[\langle N, \nabla f(N) \rangle_{\mathbb{R}^d}\right] = \mathrm{E}\left[\langle C, \mathrm{Hess}\, f(N) \rangle_{\mathrm{HS}}\right],$$

for every $f \in C^2(\mathbb{R}^d)$ having bounded first and second derivatives. Here $\mathrm{Hess}\, f$ denotes the Hessian of f, a $d \times d$ matrix with entries given by $(\mathrm{Hess}\, f)_{ij} = \partial^2_{ij} f$.

Suppose F is a d-dimensional random vector such that the expectation

$$\mathrm{E}\left[\langle F, \nabla f(F) \rangle_{\mathbb{R}^d} - \langle C, \mathrm{Hess}\, f(F) \rangle_{\mathrm{HS}}\right],$$

is close to zero for a large class of smooth functions f. In view of Stein's Lemma, it is possible to conclude that the law of F is close to the law of N. To provide a quantitative version of Stein's lemma, we introduce the definition of Stein's equation. Suppose the random vector $Z \sim \mathcal{N}_d(0, C)$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be such that $\mathrm{E}|g(Z)| < \infty$. The Stein's equation associated with g and Z is the partial differential equation

$$\langle x, \nabla f(x) \rangle_{\mathbb{R}^d} - \langle C, \mathrm{Hess}\, f(x) \rangle_{\mathrm{HS}} = g(x) - \mathrm{E}[g(Z)]. \tag{2.16}$$

A solution to Equation (2.16) is a function $f \in C^2(\mathbb{R}^d)$ satisfying (2.16) for every $x \in \mathbb{R}^d$.

Given a Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$ with at most polynomial growth, we define $U_{g,C} : \mathbb{R}^d \to \mathbb{R}$ by

$$U_{g,C}(x) = \int_0^1 \frac{1}{2t} \left(\mathrm{E}\left[g\left(\sqrt{t}x + \sqrt{1-t}N \right) \right] - \mathrm{E}[g(N)] \right) \mathrm{d}t, \tag{2.17}$$

where $N \sim \mathcal{N}_d(0, C)$ is independent of Z. As shown in Campese (2013, Lemma 2.4), $U_{g,C}$ defined as (2.17) satisfies the multidimensional Stein's equation (2.16). Moreover, if $g \in C^k(\mathbb{R}^d)$ with bounded derivatives up to order k, the same is true for $U_{g,C}$. In this case, for any $m \in \mathbb{N}_0^d$ with $|m| \le k$, the derivative is given by

$$\partial^m U_{g,C}(x) = \int_0^1 \frac{1}{2} t^{\frac{|m|}{2}-1} \mathrm{E}\left[\partial^m g\left(\sqrt{t}x + \sqrt{1-t}N \right) \right] \mathrm{d}t, \tag{2.18}$$

and it holds that

$$\mathrm{E}\left[\partial^m U_{g,C}(Z) \right] = \frac{1}{|m|} \mathrm{E}\left[\partial^m g(Z) \right]. \tag{2.19}$$

The following lemma shows some estimates of the derivatives of $U_{g,C}$.

Lemma 2.5. *For $g \in C^4\left(\mathbb{R}^d\right)$ given, $U_{g,C}$ defined as (2.17) satisfies that for all $1 \le k \le 4$,*

$$M_k\left(U_{g,C}\right) \le \frac{1}{k} M_k(g). \tag{2.20}$$

In addition, if C is positive definite, then

$$M_k\left(U_{g,C}\right) \le c_k d \left\| C^{-1/2} \right\|_{op} M_{k-1}(g), \quad 1 \le k \le 4, \tag{2.21}$$

where $c_1 = \sqrt{\pi/2}$, $c_2 = \sqrt{2/\pi}$, $c_3 = \sqrt{2\pi}/4$ and $c_4 = (2\sqrt{2})/(3\sqrt{\pi})$.

Proof: We prove (2.21) for $k = 4$. See Meckes (2009, Lemma 2) for the proofs of (2.20) and (2.21) for $1 \le k \le 3$. By the formula (2.17) for $U_{g,C}$ and formula of integration by parts,

$$\begin{aligned} \frac{\partial^3 U_{g,C}}{\partial x_i \partial x_j \partial x_k}(x) &= \int_0^1 \frac{\sqrt{t}}{2} \mathrm{E}\left[\frac{\partial^3 g}{\partial x_i \partial x_j \partial x_k} \left(\sqrt{t}x + \sqrt{1-t}N \right) \right] \mathrm{d}t \\ &= \int_0^1 \frac{\sqrt{t}}{2\sqrt{1-t}} \mathrm{E}\left[\left[C^{-1}N \right]_i \frac{\partial^2 g}{\partial x_j \partial x_k} \left(\sqrt{t}x + \sqrt{1-t}N \right) \right] \mathrm{d}t. \end{aligned}$$

Then

$$\begin{aligned} M_4(U_{g,C}) &= \sup_{x \ne y} \frac{1}{\|x-y\|} \left\| U_{g,C}^{(3)}(x) - U_{g,C}^{(3)}(y) \right\|_{op} \\ &= \sup_{x \ne y} \sup_{\|u\|=\|v\|=\|w\|=1} \frac{1}{\|x-y\|} \left| \sum_{i,j,k=1}^d \left(\frac{\partial^3 U_{g,C}}{\partial x_i \partial x_j \partial x_k}(x) - \frac{\partial^3 U_{g,C}}{\partial x_i \partial x_j \partial x_k}(y) \right) u_i v_j w_k \right| \\ &\le M_3(g) \int_0^1 \frac{t}{2\sqrt{1-t}} \mathrm{d}t \mathrm{E}\left[\sum_{i=1}^d \left| \left(C^{-1}N \right)_i u_i \right| \right]. \end{aligned}$$

By Cauchy-Schwarz inequality, and the fact that $\int_0^1 \frac{t}{2\sqrt{1-t}} \mathrm{d}t = 2/3$ and $C^{-1/2}N \sim \mathcal{N}_d(0, \mathrm{I}_d)$, where I_d is the $d \times d$ identity matrix, we have

$$\begin{aligned} \mathrm{E}\left[\sum_{i=1}^d \left| \left(C^{-1}N \right)_i u_i \right| \right] &\le \mathrm{E}\left[\sqrt{ \sum_{i=1}^d \left| (C^{-1}N)_i \right|^2 } \right] = \mathrm{E}\left[\left\| C^{-1}N \right\| \right] \\ &\le \left\| C^{-1/2} \right\|_{op} \mathrm{E}\left[\left\| C^{-1/2}N \right\| \right] \le d\sqrt{\frac{2}{\pi}} \left\| C^{-1/2} \right\|_{op}. \end{aligned}$$

Then we finish the proof. □

3. Optimal rate of convergence for vector-valued Wiener-Itô integral

Let $\{F_n = (F_{n,1}, \ldots, F_{n,d})\}_{n\geq 1}$ be a sequence of random vectors whose components all belong to the q-th Wiener chaos, where $q \geq 2$. Suppose that F_n converges in distribution to a d-dimensional normal vector Z. Define

$$M(F_n) = \max\left\{\sum_{|m|=3} |\kappa_m(F_n)|\,, \sum_{i=1}^{d} \kappa_{4e_i}(F_n)\right\}. \tag{3.1}$$

Note that $M(F_n) \geq \sum_{i=1}^{d} \kappa_{4e_i}(F_n) > 0$ (see Nourdin and Peccati (2012, Lemma 5.2.4)) and $M(F_n) \to M(Z) = 0$ under the assumption that F_n converges in distribution to Z.

For $k = 3, 4$, we define two distances between the distributions of two d-dimensional random vectors as

$$\rho_k(F, G) = \sup\left\{\left|\mathrm{E}\left[g(F)\right] - \mathrm{E}\left[g(G)\right]\right|\right\}, \tag{3.2}$$

where g runs over the class of all functions $g \in C^k(\mathbb{R}^d)$ such that $M_j(g) \leq 1$ for all $0 \leq j \leq k$. Note that it is sufficient to assume that $M_j(g)$, $0 \leq j \leq k$, are bounded. Here, we require $M_j(g) \leq 1$ for all $0 \leq j \leq k$ for the convenience of calculation.

Theorem 3.1. *Fix $q \geq 2$. Let $\{F_n = (F_{n,1}, \ldots, F_{n,d})\}_{n\geq 1}$ be a sequence of random vectors whose components live in the q-th Wiener chaos. Suppose that the covariance matrix of F_n is C and F_n converges in distribution to $Z \sim \mathcal{N}_d(0, C)$. Then there exist two finite constants $0 < c_1 < c_2$ not depending on n such that for n large enough,*

$$c_1 M\left(F_n\right) \leq \rho_4\left(F_n, Z\right) \leq c_2 M\left(F_n\right).$$

Moreover, if C is positive definite, then

$$c_1 M\left(F_n\right) \leq \rho_3\left(F_n, Z\right) \leq c_2 M\left(F_n\right).$$

Remark 3.2. From the proof of Theorem 3.1, one can see that the upper bound, namely $\rho_k\left(F_n, Z\right) \leq c_2 M\left(F_n\right)$, $k = 3, 4$, still holds without the assumption that F_n converges in distribution to $Z \sim \mathcal{N}_d(0, C)$.

Remark 3.3. There are two reasons why we consider $M(F_n)$ as the optimal rate of convergence and require the smoothness of the test function g in (3.2) to be of order four if C is not positive definite. First, combining Proposition 3.7 below and Stein's method, the test function g should be at least continuously differentiable up to order three. However, if we take $M = 3$ in Proposition 3.7, the remainder term

$$\sum_{m=e_i+e_j+e_k, 1\leq i,j,k\leq d} \mathrm{E}\left[\Gamma_{e_i,e_j,e_k}(F)\partial^m f(F)\right]$$

is bounded by $\max\left\{\sum_{|m|=3} |\kappa_m(F_n)|\,, \sum_{i=1}^{d} \kappa_{4e_i}(F_n)^{\frac{3}{4}}\right\}$ according to Equation (3.4). The convergence rate of this bound is slower than $M(F_n)$, the upper bound we get in Theorem 3.1 by taking $M = 4$ in Proposition 3.7. Second, if $M \geq 5$, the reminder term

$$\sum_{s=3}^{M-1} \sum_{\substack{m=e_{j_1}+\cdots+e_{j_s},\\ 1\leq j_k\leq d, 1\leq k\leq s}} \frac{\kappa_m(F)}{(s-1)!}\mathrm{E}\left[\partial^m f(F)\right] + \sum_{\substack{m=e_{j_1}+\cdots+e_{j_M},\\ 1\leq j_k\leq d, 1\leq k\leq M}} \mathrm{E}\left[\Gamma_{e_{j_1},\ldots,e_{j_M}}(F)\partial^m f(F)\right]$$

is still bounded by $M(F_n)$. For example, taking $M = 5$, the reminder term is bounded by

$$\max\left\{\sum_{|m|=3} |\kappa_m(F_n)|\,, \sum_{i=1}^{d} \kappa_{4e_i}(F_n), \sum_{i=1}^{d} \kappa_{4e_i}(F_n)^{\frac{5}{4}}\right\} = M(F_n)$$

according to Proposition 3.9. The above two points are the reasons why we define the optimal rate of convergence as (3.1) and the distance as (3.2).

Remark 3.4. Compared to the optimal rate of convergence contributed by each component $F_{n,i}$ of F_n, namely

$$\sum_{i=1}^{d} \widetilde{M}(F_{n,i}) = \max\left\{\sum_{i=1}^{d} \left|\mathrm{E}\left[F_{n,i}^3\right]\right|, \sum_{i=1}^{d} \kappa_{4e_i}(F_n)\right\},$$

where $\widetilde{M}(\cdot)$ is defined as (1.5), one can see that the mixed moments of order three $|\mathrm{E}\left[F_{n,i}F_{n,j}F_{n,k}\right]|$, where $1 \leq i,j,k \leq d$ such that they are not all equal, also contribute to $M(F_n)$, the optimal rate of convergence of F_n. This is one of the manifestations of complexity in the multivariate setting compared to the univariate setting.

Remark 3.5. To achieve the optimal convergence of rate, we set $M = 4$ in Proposition 3.7 and apply the chain rule (2.8) three times for $\partial_i f$ with $1 \leq i \leq d$. Consequently, the test function g in (3.2) needs to be smooth up to order four. This distance ρ_4 is also used in Krokowski et al. (2016); Krokowski and Thäle (2017) to study multivariate limit theorems for functionals of Rademacher sequences. Moreover, if the covariance matrix C is positive definite, then the required smoothness of the test function can be reduced to three. It is reasonable to require test functions to be smoother when C is singular.

In Theorem 3.1, we consider the sequence of vector-valued Wiener-Itô integrals $\{F_n\}_{n\geq 1}$ with a deterministic covariance matrix C. The conclusion can be extended to the case where the covariance matrix of F_n, denoted by C_n, converges to C in the sense that $\|C_n - C\|_{\mathrm{HS}} \to 0$ as $n \to \infty$. We introduce the definition of being asymptotically close to normal. We say that $\{F_n\}_{n\geq 1}$ is asymptotically close to normal if $\rho_4(F_n, Z_n) \to 0$, where Z_n is a d-dimensional Gaussian vector with the covariance matrix C_n. This definition was introduced in Campese (2013, Definition 2.3) with respect to the Prokhorov distance β, which is equivalent to convergence in distribution in the sense that $\beta(F_n, Z) \to 0 \Leftrightarrow F_n \xrightarrow{d} Z$ as $n \to \infty$. Here, we adopt the distance $\rho_4(\cdot,\cdot)$ (see Definition 3.2), which is also equivalent to convergence in distribution, meaning that $\rho_4(F_n, Z) \to 0 \Leftrightarrow F_n \xrightarrow{d} Z$ as $n \to \infty$. Note that if the test function g in the definition of the distance $\rho_k(\cdot,\cdot)$, $k = 3,4$, is not necessarily bounded, then the topology induced by $\rho_k(\cdot,\cdot)$ is stronger than that of the convergence in distribution. Using a similar argument as in the proof of Theorem 3.1, we can obtain the following proposition.

Proposition 3.6. *Fix $q \geq 2$. Let $\{F_n = (F_{n,1},\ldots,F_{n,d})\}_{n\geq 1}$ be a sequence of random vectors whose components live in the q-th Wiener chaos. Suppose that $\|C_n - C\|_{\mathrm{HS}} \to 0$, as $n \to \infty$.*

(1) *If C is invertible, we set $F_n^{'} = C^{\frac{1}{2}} C_n^{-\frac{1}{2}} F_n$ and assume that $F_n^{'}$ converges in distribution to $Z \sim \mathcal{N}_d(0, C)$. Then for n large enough,*
$$\rho_3(F_n^{'}, Z) \asymp M(F_n^{'}).$$
(2) *If C is not invertible, suppose that $\{F_n\}_{n\geq 1}$ is asymptotically close to normal. That is, $\rho_4(F_n, Z_n) \to 0$, where Z_n is a d-dimensional Gaussian vector with covariance matrix C_n. Then for n large enough,*
$$\rho_4(F_n, Z_n) \asymp M(F_n).$$

To prove Theorem 3.1, we need several results as follows. In order to analyze $\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right]$, which is equal to

$$\mathrm{E}\left[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d} - \langle C, \mathrm{Hess}\, U_{g,C}(F_n)\rangle_{\mathrm{HS}}\right],$$

by Stein's method as introduced in Section 2.4, we first establish Proposition 3.7 to expand

$$\mathrm{E}[\langle F, \nabla f(F)\rangle_{\mathbb{R}^d}],$$

for a general function f satisfying certain smoothness requirements. This expansion is expressed as a sum associated with cumulants and related Γ-random variables. Proposition 3.7 is proven by utilizing the formula of integration by parts (see Lemma 2.1), the chain rule (2.8) and the relation between cumulant and related Γ-random variable (see Theorem 2.4).

After expanding $\mathrm{E}[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d}]$ as (1.12) by using Proposition 3.7, we derive technical estimates of Γ-random variables in Proposition 3.9 to bound the remainder term, namely the second and third terms on the right-hand side of (1.12). Proposition 3.7 is proven by using the definition of Γ-random variable (Definition 2.3) and generalized Cauchy-Schwarz inequality (Biermé et al. (2012, Lemma 4.1)).

Combining Proposition 3.7 with Proposition 3.9, we obtain the upper bound in Theorem 3.1. To establish the lower bound, we delicately set up several specific test functions in Lemma 3.11.

Proposition 3.7. *Let $F = (F_1, \dots, F_d)$ with $F_i \in \mathbb{D}^\infty$, $1 \le i \le d$. Then, for every $M \ge 2$ and every function $f \in C^M(\mathbb{R}^d)$ with derivatives having at most polynomial growth, we have*

$$\begin{aligned}\mathrm{E}[\langle F, \nabla f(F)\rangle_{\mathbb{R}^d}] = &\sum_{s=1}^{M-1} \sum_{\substack{m=e_{j_1}+\cdots+e_{j_s},\\ 1\le j_k\le d, 1\le k\le s}} \frac{\kappa_m(F)}{(s-1)!} \mathrm{E}\left[\partial^m f(F)\right] \\ &+ \sum_{\substack{m=e_{j_1}+\cdots+e_{j_M},\\ 1\le j_k\le d, 1\le k\le M}} \mathrm{E}\left[\Gamma_{e_{j_1},\dots,e_{j_M}}(F)\partial^m f(F)\right].\end{aligned} \tag{3.3}$$

Remark 3.8. Proposition 3.7 can be seen as an extension of Biermé et al. (2012, Proposition 3.11) to the multidimensional case. For $d \ge 2$, Equation (3.3) is new as far as we know.

Proof: Using the formula of integration by parts (see Lemma 2.1), the chain rule (2.8) and the relation between cumulant and related Γ-random variable (see Theorem 2.4) repeatedly, we obtain

$$\begin{aligned}&\mathrm{E}\left[F_{j_1}\partial_{j_1} f(F)\right] = \mathrm{E}\left[F_{j_1}\right]\mathrm{E}\left[\partial_{j_1} f(F)\right] + \mathrm{E}\left[\left\langle D\partial_{j_1} f(F), -DL^{-1}F_{j_1}\right\rangle_{\mathfrak{H}}\right]\\
&= \kappa_{e_{j_1}}(F)\mathrm{E}\left[\partial_{j_1} f(F)\right] + \sum_{j_2=1}^d \mathrm{E}\left[\partial^2_{j_1,j_2} f(F)\left\langle DF_{j_2}, -DL^{-1}F_{j_1}\right\rangle_{\mathfrak{H}}\right]\\
&= \kappa_{e_{j_1}}(F)\mathrm{E}\left[\partial_{j_1} f(F)\right] + \sum_{j_2=1}^d \mathrm{E}\left[\partial^2_{j_1,j_2} f(F)\right]\mathrm{E}\left[\Gamma_{e_{j_1},e_{j_2}}(F)\right] + \sum_{j_2,j_3=1}^d \mathrm{E}\left[\partial^3_{j_1,j_2,j_3} f(F)\Gamma_{e_{j_1},e_{j_2},e_{j_3}}(F)\right]\\
&= \cdots = \kappa_{e_{j_1}}(F)\mathrm{E}\left[\partial_{j_1} f(F)\right] + \sum_{j_2=1}^d \kappa_{e_{j_1}+e_{j_2}}(F)\mathrm{E}\left[\partial^2_{j_1,j_2} f(F)\right] + \cdots\\
&\quad + \sum_{j_2,\dots,j_{M-1}=1}^d \frac{\kappa_{e_{j_1}+\cdots+e_{j_{M-1}}}(F)}{(M-2)!}\mathrm{E}\left[\partial^{M-1}_{j_1,\dots,j_{M-1}} f(F)\right] + \sum_{j_2,\dots,j_M=1}^d \mathrm{E}\left[\partial^M_{j_1,\dots,j_M} f(F)\Gamma_{e_{j_1},\dots,e_{j_M}}(F)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{E}[\langle F, \nabla f(F)\rangle_{\mathbb{R}^d}] = \sum_{j_1=1}^d \mathrm{E}\left[F_{j_1}\partial_{j_1} f(F)\right]\\
&= \sum_{s=1}^{M-1} \sum_{\substack{m=e_{j_1}+\cdots+e_{j_s},\\ 1\le j_k\le d, 1\le k\le s}} \frac{\kappa_m(F)}{(s-1)!}\mathrm{E}\left[\partial^m f(F)\right] + \sum_{\substack{m=e_{j_1}+\cdots+e_{j_M},\\ 1\le j_k\le d, 1\le k\le M}} \mathrm{E}\left[\Gamma_{e_{j_1},\dots,e_{j_M}}(F)\partial^m f(F)\right].\end{aligned}$$

□

Proposition 3.9. *For each integer $q \geq 2$, there exist positive constants $c_1(q), c_2(q), c_3(q)$ only depending on q such that, for all $F = (I_q(f_1), \ldots, I_q(f_d))$ with $f_i \in \mathfrak{H}^{\odot q}$ and $1 \leq i \leq d$, we have*

$$\mathrm{E}\left[\left|\Gamma_{e_i,e_j,e_k}(F) - \frac{1}{2}\kappa_{e_i+e_j+e_k}(F)\right|\right] \leq c_1(q) \max_{1\leq i\leq d}\left\{\kappa_{4e_i}(F)^{\frac{3}{4}}\right\}, \tag{3.4}$$

$$\mathrm{E}\left[\left|\Gamma_{e_i,e_j,e_k,e_l}(F)\right|\right] \leq c_2(q) \max_{1\leq i\leq d}\left\{\kappa_{4e_i}(F)\right\}, \tag{3.5}$$

$$\mathrm{E}\left[\left|\Gamma_{e_i,e_j,e_k,e_l,e_s}(F)\right|\right] \leq c_3(q) \max_{1\leq i\leq d}\left\{\kappa_{4e_i}(F)^{\frac{5}{4}}\right\}, \tag{3.6}$$

for any $1 \leq i, j, k, l, s \leq d$.

Remark 3.10. See Biermé et al. (2012, Proposition 4.3) for the estimates of cumulants and related Γ-random variables for $d = 1$.

Proof: By making suitable modifications to the proof of Biermé et al. (2012, Proposition 4.3), we can get the conclusion. We show (3.4) as an illustrative example here and prove (3.5) and (3.6) in Appendix. According to Equation (2.12), we have

$$\Gamma_{e_i,e_j,e_k}(F) = \sum_{r_2=1}^{q-1}\sum_{r_3=1}^{(2q-2r_2)\wedge q} c_{q,l}\left(r_2, r_3\right) I_{3q-2r_2-2r_3}\left(\left(f_i \tilde{\otimes}_{r_2} f_j\right)\tilde{\otimes}_{r_3} f_k\right), \tag{3.7}$$

where $c_{q,l}\left(r_2, r_3\right)$, defined as in (2.3), is a constant depending only on q, r_2 and r_3. Utilizing Theorem 2.4, we obtain

$$\mathrm{E}\left[\Gamma_{e_i,e_j,e_k}(F)\right] = \frac{1}{2}\kappa_{e_i+e_j+e_k}(F).$$

Consequently, the random variable $\Gamma_{e_i,e_j,e_k}(F) - \frac{1}{2}\kappa_{e_i+e_j+e_k}(F)$ is derived by restricting the sum in (3.7) to terms satisfying $2r_2 + 2r_3 < 3q$. Combining the fact that there exists a constant $c(q)$ depending only on q such that

$$\max_{1\leq r\leq q-1}\|f_i \otimes_r f_i\|^2_{\mathfrak{H}^{\otimes(2q-2r)}} \leq c(q)\kappa_{4e_i}(F), \tag{3.8}$$

which is derived from Nourdin and Peccati (2012, Equation (5.2.6)), it suffices to demonstrate that for r_2 and r_3 satisfying $1 \leq r_2 \leq q-1$, $1 \leq r_3 \leq (2q-2r_2) \wedge q$ and $2r_2 + 2r_3 < 3q$,

$$\left\|\left(f_i \tilde{\otimes}_{r_2} f_j\right)\tilde{\otimes}_{r_3} f_k\right\|_{\mathfrak{H}^{\otimes(3q-2r_2-2r_3)}} \leq \max_{1\leq i\leq d}\max_{1\leq r\leq q-1}\|f_i \otimes_r f_i\|^{\frac{3}{2}}_{\mathfrak{H}^{\otimes(2q-2r)}}. \tag{3.9}$$

Without loss of generality, in the proof, we assume that $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space, and μ is a σ-finite and non-atomic measure. For a vector $\mathbf{z} = (z_1, \ldots, z_d)$ and a permutation σ of $\{1, \ldots, d\}$, we write $\sigma(\mathbf{z}) = (z_{\sigma(1)}, \ldots, z_{\sigma(d)})$. For vectors $\mathbf{z} = (z_1, \ldots, z_j)$ and $\mathbf{y} = (y_1, \ldots, y_k)$, we set $\mathbf{z} \cup \mathbf{y}$ to be the vector of dimension $j + k$ obtained by juxtaposing $\mathbf{z}$ and $\mathbf{y}$, that is, $\mathbf{z} \cup \mathbf{y} = (z_1, \ldots, z_j, y_1, \ldots, y_k)$. We identify vectors of dimension zero with the empty set. That is, if $\mathbf{z}$ has dimension zero, then integration with respect to $\mathbf{z}$ is removed by convention.

First, we assume that $r_3 < q$, then both $q - r_2$ and $q - r_3$ belong to $\{1, \ldots, q-1\}$. By Cauchy-Schwarz inequality (or see Biermé et al. (2012, Equation (4.3), Equation (4.4))), we get that

$$\begin{aligned}
&\left\|\left(f_i \tilde{\otimes}_{r_2} f_j\right)\tilde{\otimes}_{r_3} f_k\right\|_{\mathfrak{H}^{\otimes(3q-2r_2-2r_3)}} \\
\leq& \left\|\left(f_i \tilde{\otimes}_{r_2} f_j\right)\otimes_{r_3} f_k\right\|_{\mathfrak{H}^{\otimes(3q-2r_2-2r_3)}} \\
\leq& \left\|f_i \tilde{\otimes}_{r_2} f_j\right\|_{\mathfrak{H}^{\otimes(2q-2r_2)}}\sqrt{\|f_k \otimes_{q-r_3} f_k\|_{\mathfrak{H}^{\otimes 2r_3}}} \\
\leq& \sqrt{\|f_i \otimes_{q-r_2} f_i\|_{\mathfrak{H}^{\otimes 2r_2}}}\sqrt{\|f_j \otimes_{q-r_2} f_j\|_{\mathfrak{H}^{\otimes 2r_2}}}\sqrt{\|f_k \otimes_{q-r_3} f_k\|_{\mathfrak{H}^{\otimes 2r_3}}} \\
\leq& \max_{1\leq i\leq d}\max_{1\leq r\leq q-1}\|f_i \otimes_r f_i\|^{\frac{3}{2}}_{\mathfrak{H}^{\otimes(2q-2r)}}.
\end{aligned}$$

Now we consider the case when $r_3 = q$ and $1 \le r_2 < \frac{q}{2}$. In this case,

$$\left(f_i \tilde{\otimes}_{r_2} f_j\right) \tilde{\otimes}_{r_3} f_k = \left\langle f_i \tilde{\otimes}_{r_2} f_j, f_k \right\rangle_{\mathfrak{H}^{\otimes q}}$$

defines a function of $q - 2r_2$ variables. Considering the symmetry of f_i for $1 \le i \le d$ and the symmetrization of contractions, such a function can be expressed as a finite linear combination of functions of the form

$$F(\mathbf{t}) = \int f_i(\mathbf{t}_1, \mathbf{x}_1, \mathbf{w}) f_j(\mathbf{t}_2, \mathbf{x}_2, \mathbf{w}) f_k(\mathbf{x}_1, \mathbf{x}_2) \, \mathrm{d}\mu^{q+r_2}(\mathbf{w}, \mathbf{x}_1, \mathbf{x}_2),$$

where $\mathbf{w}$ has length r_2, $\mathbf{t}_1 \cup \mathbf{t}_2 = \sigma(\mathbf{t})$ for some permutation σ of $\{1, \ldots, q - 2r_2\}$ and $\mathbf{t} = (t_1, \ldots, t_{q-2r_2})$. Without loss of generality, we assume that $\mathbf{t}_1$ has positive length (recall that $1 \le r_2 < q/2$ and thus $q - 2r_2 > 0$). We denote by s_j the length of the vector $\mathbf{x}_j$. Then by $r_2 \ge 1$ and $s_1 + s_2 = q$, we have $1 \le s_1 < q - r_2$ and $r_2 < s_2 \le q - 1$. Exchanging the order of integrations, we get

$$F(\mathbf{t}) = \int f_i(\mathbf{t}_1, \mathbf{x}_1, \mathbf{w}) \left(f_j \otimes_{s_2} f_k\right)(\mathbf{t}_2, \mathbf{x}_1, \mathbf{w}) \, \mathrm{d}\mu^{r_2+s_1}(\mathbf{w}, \mathbf{x}_1).$$

Then

$$\begin{aligned}
\|F\|^2_{\mathfrak{H}^{\otimes(q-2r_2)}} &= \int f_i(\mathbf{t}_1, \mathbf{x}_1, \mathbf{w}) \left(f_j \otimes_{s_2} f_k\right)(\mathbf{t}_2, \mathbf{x}_1, \mathbf{w}) \\
&\qquad f_i(\mathbf{t}_1, \tilde{\mathbf{x}}_1, \tilde{\mathbf{w}}) \left(f_j \otimes_{s_2} f_k\right)(\mathbf{t}_2, \tilde{\mathbf{x}}_1, \tilde{\mathbf{w}}) \, \mathrm{d}\mu^{q+2s_1}(\mathbf{w}, \mathbf{x}_1, \tilde{\mathbf{w}}, \tilde{\mathbf{x}_1}, \mathbf{t}_1, \mathbf{t}_2) \\
&= \int f_i \otimes_{q-r_2-s_1} f_i(\mathbf{x}_1, \mathbf{w}, \tilde{\mathbf{x}}_1, \tilde{\mathbf{w}}) \left(f_j \otimes_{s_2} f_k\right)(\mathbf{t}_2, \mathbf{x}_1, \mathbf{w}) \\
&\qquad \left(f_j \otimes_{s_2} f_k\right)(\mathbf{t}_2, \tilde{\mathbf{x}}_1, \tilde{\mathbf{w}}) \, \mathrm{d}\mu^{3s_1+r_2}(\mathbf{w}, \mathbf{x}_1, \tilde{\mathbf{w}}, \tilde{\mathbf{x}_1}, \mathbf{t}_2).
\end{aligned}$$

Applying generalized Cauchy-Schwarz inequality (Biermé et al. (2012, Lemma 4.1)), we obtain

$$\|F\|^2_{\mathfrak{H}^{\otimes(q-2r_2)}} \le \|f_i \otimes_{q-r_2-s_1} f_i\|_{\mathfrak{H}^{\otimes 2(r_2+s_1)}} \|f_j \otimes_{s_2} f_k\|^2_{\mathfrak{H}^{\otimes 2(q-s_2)}}.$$

Using Cauchy-Schwarz inequality (or see Biermé et al. (2012, Equation (4.4))) and the fact that $0 < q - r_2 - s_1 \le q - 1$, $1 \le r_2 < s_2 \le q - 1$, we deduce

$$\begin{aligned}
\|F\|_{\mathfrak{H}^{\otimes(q-2r_2)}} &\le \max_{1 \le r \le q-1} \sqrt{\|f_i \otimes_r f_i\|_{\mathfrak{H}^{\otimes(2q-2r)}}} \sqrt{\|f_j \otimes_r f_j\|_{\mathfrak{H}^{\otimes(2q-2r)}}} \sqrt{\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}}} \\
&\le \max_{1 \le i \le d} \max_{1 \le r \le q-1} \|f_i \otimes_r f_i\|^{\frac{3}{2}}_{\mathfrak{H}^{\otimes(2q-2r)}}.
\end{aligned}$$

This completes the proof of (3.4). □

Inspired by Biermé et al. (2012); Nourdin and Peccati (2015), we construct several specific test functions for use in proving the lower bound in Theorem 3.1. Define

$$g_t(x) = a \exp\left\{\frac{1}{2} t^T C t\right\} \sin\left(\langle t, x\rangle_{\mathbb{R}^d}\right), \quad h_t(x) = a \exp\left\{\frac{1}{2} t^T C t\right\} \cos\left(\langle t, x\rangle_{\mathbb{R}^d}\right),$$

where a is defined as

$$a = \exp\left\{-\frac{1}{2} \max_{t \in \{-1,0,1\}^d} t^T C t\right\},$$

to ensure that g_t and h_t are bounded by one.

Lemma 3.11. *Fix $1 \le i, j, k \le d$ satisfying $i \ne j, k$ and $j \ne k$, define $h_i(x), g_i(x), g_{ij}(x), g_{ijk}(x) : \mathbb{R}^d \to \mathbb{R}$ as*

$$\begin{aligned}
h_i(x) &= h_{e_i}(x) = a e^{\frac{1}{2} C_{ii}} \cos x_i, \\
g_i(x) &= g_{e_i}(x) = a e^{\frac{1}{2} C_{ii}} \sin x_i,
\end{aligned}$$

$$g_{ij}(x) = \frac{1}{4}\left(g_{e_i-e_j}(x) - g_{e_i+e_j}(x) + 2g_{e_j}(x)\right),$$
$$g_{ijk}(x) = \frac{1}{12}\left(g_{e_i+e_j-e_k}(x) - g_{e_i+e_j+e_k}(x) - 4g_{ik}(x) - 4g_{jk}(x) + 2g_{e_k}(x)\right).$$

Then $h_i(x), g_i(x), g_{ij}(x), g_{ijk}(x)$ are bounded by one and infinitely differentiable with all derivatives bounded by one under the norm $\|\cdot\|_\infty$, and satisfy

$$\mathrm{E}\left[\partial^m U_{h_i,C}(Z)\right] = \begin{cases} \frac{a}{|m|}(-1)^{\frac{|m|}{2}}, & m = |m|e_i, |m| = 0,2,4,\dots, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.10}$$

$$\mathrm{E}\left[\partial^m U_{g_i,C}(Z)\right] = \begin{cases} \frac{a}{|m|}(-1)^{\frac{|m|-1}{2}}, & m = |m|e_i, |m| = 1,3,5,\dots, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.11}$$

$$\mathrm{E}\left[\partial^m U_{g_{ij},C}(Z)\right] = \begin{cases} \frac{a}{2|m|}(-1)^{\frac{|m|+1}{2}}, & m = m_ie_i + m_je_j, m_i > 0, m_j \textit{ and } |m| \textit{ are odd}, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.12}$$

and

$$\mathrm{E}\left[\partial^m U_{g_{ijk},C}(Z)\right] = \begin{cases} \frac{a}{6|m|}(-1)^{\frac{|m|+1}{2}}, & m = m_ie_i + m_je_j + m_ke_k, m_i, m_j > 0, m_k \textit{ and } |m| \textit{ are odd}, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.13}$$

where $Z \sim \mathcal{N}_d(0, C)$ and $U_{g,C}(x)$ is defined as (2.17) for a general smooth function g.

Proof: Firstly, it is obvious that $h_i(x), g_i(x), g_{ij}(x), g_{ijk}(x)$ are bounded by one and infinitely continuously differentiable with all derivatives bounded by one under the norm $\|\cdot\|_\infty$.

For $Z = (Z_1, \dots, Z_d) \sim \mathcal{N}_d(0, C)$ and for any $t \in \mathbb{R}^d$, we have

$$e^{-\frac{1}{2}t^TCt} = \mathrm{E}\left[e^{\mathrm{i}\langle t,Z\rangle_{\mathbb{R}^d}}\right] = \mathrm{E}\left[\cos\left(\langle t,Z\rangle_{\mathbb{R}^d}\right)\right] + \mathrm{i}\mathrm{E}\left[\sin\left(\langle t,Z\rangle_{\mathbb{R}^d}\right)\right],$$

which implies, for any $t \in \mathbb{R}^d$,

$$\mathrm{E}\left[\sin\left(\langle t,Z\rangle_{\mathbb{R}^d}\right)\right] = 0, \quad \mathrm{E}\left[\cos\left(\langle t,Z\rangle_{\mathbb{R}^d}\right)\right] = e^{-\frac{1}{2}t^TCt}.$$

Fix $1 \le i \le d$, let $h_i(x) : \mathbb{R}^d \to \mathbb{R}, h_i(x) = h_{e_i}(x) = ae^{\frac{1}{2}C_{ii}}\cos x_i$,

$$\partial^m h_i(x) = \begin{cases} ae^{\frac{1}{2}C_{ii}}(-1)^{\frac{|m|+1}{2}}\sin x_i, & m = |m|e_i, |m| = 1,3,5,\dots, \\ ae^{\frac{1}{2}C_{ii}}(-1)^{\frac{|m|}{2}}\cos x_i, & m = |m|e_i, |m| = 0,2,4,\dots, \\ 0, & \textit{otherwise}. \end{cases}$$

Then, by (2.19),

$$\mathrm{E}\left[\partial^m U_{h_i,C}(Z)\right] = \frac{1}{|m|}\mathrm{E}\left[\partial^m h_i(Z)\right] = \begin{cases} \frac{a}{|m|}(-1)^{\frac{|m|}{2}}, & m = |m|e_i, |m| = 0,2,4,\dots, \\ 0, & \textit{otherwise}. \end{cases}$$

By a similar argument, we get (3.11) and for fixed $1 \le i, j \le d$ satisfying $i \ne j$,

$$\mathrm{E}\left[\partial^m U_{g_{e_i-e_j},C}(Z)\right] = \frac{1}{|m|}\mathrm{E}\left[\partial^m g_{e_i-e_j}(Z)\right] = \begin{cases} \frac{a}{|m|}(-1)^{\frac{|m|-1}{2}+m_j}, & m = m_ie_i + m_je_j, |m| \text{ is odd}, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\mathrm{E}\left[\partial^m U_{g_{e_i+e_j},C}(Z)\right] = \frac{1}{|m|}\mathrm{E}\left[\partial^m g_{e_i+e_j}(Z)\right] = \begin{cases} \frac{a}{|m|}(-1)^{\frac{|m|-1}{2}}, & m = m_ie_i + m_je_j, |m| \text{ is odd}, \\ 0, & \textit{otherwise}. \end{cases}$$

Then for $g_{ij}(x) = \frac{1}{4}\left(g_{e_i-e_j}(x) - g_{e_i+e_j}(x) + 2g_{e_j}(x)\right)$, by (2.18),

$$\mathrm{E}\left[\partial^m U_{g_{ij},C}(Z)\right] = \frac{1}{4}\left(\mathrm{E}\left[\partial^m U_{g_{e_i-e_j},C}(Z)\right] - \mathrm{E}\left[\partial^m U_{g_{e_i+e_j},C}(Z)\right] + 2\mathrm{E}\left[\partial^m U_{g_j,C}(Z)\right]\right)$$

$$= \begin{cases} \frac{a}{2|m|}(-1)^{\frac{|m|+1}{2}}, & m = m_i e_i + m_j e_j, m_i > 0, m_j \text{ is odd}, |m| \text{ is odd}, \\ 0, & otherwise. \end{cases}$$

Similarly, we can obtain (3.13). □

We now turn to the proof of Theorem 3.1.

Proof: Upper bound. Taking $g \in C^M(\mathbb{R}^d)$ such that $M_j(g) \leq 1$ for all $0 \leq j \leq M$, by Stein's equation (2.16) and Proposition 3.7,

$$\begin{aligned} &\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right] = \mathrm{E}\left[\langle F_n, \nabla U_{g,C}(F_n)\rangle_{\mathbb{R}^d}\right] - \mathrm{E}\left[\langle C, \operatorname{Hess} U_{g,C}(F_n)\rangle_{\mathrm{HS}}\right] \\ &= \sum_{s=3}^{M-1} \sum_{\substack{m=e_{j_1}+\cdots+e_{j_s}, \\ 1\leq j_k\leq d, 1\leq k\leq s}} \frac{\kappa_m(F_n)}{(s-1)!} \mathrm{E}\left[\partial^m U_{g,C}(F_n)\right] + \sum_{\substack{m=e_{j_1}+\cdots+e_{j_M}, \\ 1\leq j_k\leq d, 1\leq k\leq M}} \mathrm{E}\left[\Gamma_{e_{j_1},\ldots,e_{j_M}}(F_n)\partial^m U_{g,C}(F_n)\right]. \end{aligned} \tag{3.14}$$

Take $M = 4$,

$$\begin{aligned} &\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right] \\ &= \frac{1}{2} \sum_{\substack{m=e_i+e_j+e_k, \\ 1\leq i,j,k\leq d}} \kappa_m(F_n) \mathrm{E}\left[\partial^m U_{g,C}(F_n)\right] + \sum_{\substack{m=e_i+e_j+e_k+e_l, \\ 1\leq i,j,k,l\leq d}} \mathrm{E}\left[\Gamma_{e_i,e_j,e_k,e_l}(F_n)\partial^m U_{g,C}(F_n)\right]. \end{aligned} \tag{3.15}$$

Combining (2.15), (2.20) and Proposition 3.9,

$$\begin{aligned} &\left|\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right]\right| \\ &\leq \frac{1}{2} \left\|U^{(3)}_{g,C}\right\|_\infty \sum_{m=e_i+e_j+e_k} |\kappa_m(F_n)| + \left\|U^{(4)}_{g,C}\right\|_\infty \sum_{m=e_i+e_j+e_k+e_l} \mathrm{E}\left[\left|\Gamma_{e_i,e_j,e_k,e_l}(F_n)\right|\right] \\ &\leq \frac{d^3}{6} \sum_{|m|=3} |\kappa_m(F_n)| + \frac{d^4}{4} c_2(q) \sum_{i=1}^d \kappa_{4e_i}(F_n) \\ &\leq \max\left\{\frac{d^3}{3}, \frac{d^4 c_2(q)}{2}\right\} \max\left\{\sum_{|m|=3} |\kappa_m(F_n)|, \sum_{i=1}^d \kappa_{4e_i}(F_n)\right\}. \end{aligned}$$

That is,

$$\rho_4(F_n, Z) \leq \max\left\{\frac{d^3}{3}, \frac{d^4 c_2(q)}{2}\right\} M(F_n).$$

Moreover, if C is positive definite, then by (2.21) and Proposition 3.9, for $g \in C^4(\mathbb{R}^d)$,

$$\begin{aligned} &\left|\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right]\right| \\ &\leq \frac{1}{6} M_3(g) \sum_{m=e_i+e_j+e_k} |\kappa_m(F_n)| + \frac{2d}{3}\sqrt{\frac{2}{\pi}} \left\|C^{-1/2}\right\|_{op} M_3(g) \sum_{m=e_i+e_j+e_k+e_l} \mathrm{E}\left[\left|\Gamma_{e_i,e_j,e_k,e_l}(F_n)\right|\right] \\ &\leq \max\left\{\frac{d^3}{3}, \frac{4d^4}{3}\sqrt{\frac{2}{\pi}} \left\|C^{-1/2}\right\|_{op}\right\} M_3(g) \max\left\{\sum_{|m|=3} |\kappa_m(F_n)|, \sum_{i=1}^d \kappa_{4e_i}(F_n)\right\}. \end{aligned}$$

Then for $g \in C^3(\mathbb{R}^d)$ satisfying $M_j(g) \leq 1$ for all $0 \leq j \leq 3$, let $g_\epsilon(x) = \mathrm{E}\left[g(x + \sqrt{\epsilon}Y)\right]$, where $Y \sim \mathcal{N}_d(0, \mathrm{I}_d)$. Then,

(1) for each $\epsilon > 0$, $M_j(g_\epsilon)$ for all $0 \leq j \leq 4$ are bounded. Specifically, $M_j(g_\epsilon) \leq M_j(g) \leq 1$ for $0 \leq j \leq 3$. And by using a similar argument in the proof of Lemma 2.5, $M_4(g_\epsilon) \leq \frac{d}{\sqrt{\epsilon}}\sqrt{\frac{2}{\pi}} M_3(g)$,

(2) as $\epsilon \to 0$, $\|g_\epsilon - g\|_\infty \to 0$.

This similar smoothing argument can be found in the proof of Nourdin et al. (2010b, Theorem 3.5). Thus we have

$$\begin{aligned}
&|\mathrm{E}\left[g_\epsilon(F_n)\right] - \mathrm{E}\left[g_\epsilon(Z)\right]| \\
&\le \max\left\{\frac{d^3}{3}, \frac{4d^4}{3}\sqrt{\frac{2}{\pi}}\left\|C^{-1/2}\right\|_{op}\right\} M_3(g_\epsilon) \max\left\{\sum_{|m|=3}|\kappa_m(F_n)|, \sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right\} \\
&\le \max\left\{\frac{d^3}{3}, \frac{4d^4}{3}\sqrt{\frac{2}{\pi}}\left\|C^{-1/2}\right\|_{op}\right\} \max\left\{\sum_{|m|=3}|\kappa_m(F_n)|, \sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right\}.
\end{aligned}$$

By the dominated convergence theorem, let $\epsilon \to 0$, we get that

$$|\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right]| \le \max\left\{\frac{d^3}{3}, \frac{4d^4}{3}\sqrt{\frac{2}{\pi}}\left\|C^{-1/2}\right\|_{op}\right\} \max\left\{\sum_{|m|=3}|\kappa_m(F_n)|, \sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right\}.$$

That is,

$$\rho_3(F_n, Z) \le \max\left\{\frac{d^3}{3}, \frac{4d^4}{3}\sqrt{\frac{2}{\pi}}\left\|C^{-1/2}\right\|_{op}\right\} \max\left\{\sum_{|m|=3}|\kappa_m(F_n)|, \sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right\}.$$

Lower bound. Take $M = 5$ in (3.14), we have

$$\begin{aligned}
\mathrm{E}\left[g(F_n)\right] - \mathrm{E}\left[g(Z)\right] &= \frac{1}{2}\sum_{\substack{m=e_i+e_j+e_k,\\ 1\le i,j,k\le d}} \kappa_m(F_n)\mathrm{E}\left[\partial^m U_{g,C}(F_n)\right] \\
&+ \frac{1}{6}\sum_{\substack{m=e_i+e_j+e_k+e_l,\\ 1\le i,j,k,l\le d}} \kappa_m(F_n)\mathrm{E}\left[\partial^m U_{g,C}(F_n)\right] \sum_{\substack{m=e_{j_1}+\cdots+e_{j_5},\\ 1\le j_k\le d, 1\le k\le 5}} \mathrm{E}\left[\Gamma_{e_{j_1},\ldots,e_{j_5}}(F_n)\partial^m U_{g,C}(F_n)\right].
\end{aligned}$$

Replace the test function g with h_i, then by Proposition 3.9 and Lemma 3.11, we get that

$$\begin{aligned}
&\left|\mathrm{E}\left[h_i(F_n)\right] - \mathrm{E}\left[h_i(Z)\right] - \frac{a}{24}\kappa_{4e_i}(F_n)\right| \\
=&\left|\mathrm{E}\left[h_i(F_n)\right] - \mathrm{E}\left[h_i(Z)\right] - \sum_{m=e_i+e_j+e_k+e_l}\frac{\kappa_m(F_n)}{6}\mathrm{E}\left[\partial^m U_{h_i,C}(Z)\right]\right| \\
=&\left|\sum_{\substack{m=e_i+e_j+e_k,\\ 1\le i,j,k\le d}}\frac{\kappa_m(F_n)}{2}\mathrm{E}\left[\partial^m U_{h_i,C}(F_n)\right] + \sum_{\substack{m=e_i+e_j+e_k+e_l,\\ 1\le i,j,k,l\le d}}\frac{\kappa_m(F_n)}{6}\left(\mathrm{E}\left[\partial^m U_{h_i,C}(F_n)\right] - \mathrm{E}\left[\partial^m U_{h_i,C}(Z)\right]\right)\right. \\
&\left.+\sum_{\substack{m=e_{j_1}+\cdots+e_{j_5},\\ 1\le j_k\le d, 1\le k\le 5}}\mathrm{E}\left[\Gamma_{e_{j_1},\ldots,e_{j_5}}(F_n)\partial^m U_{h_i,C}(F_n)\right]\right| \\
\le&\frac{1}{2}\sum_{m=e_i+e_j+e_k}|\kappa_m(F_n)|\left|\mathrm{E}\left[\partial^m U_{h_i,C}(F_n)\right]\right| + + \sum_{\substack{m=e_{j_1}+\cdots+e_{j_5},\\ 1\le j_k\le d, 1\le k\le 5}}\left\|\partial^m U_{h_i,C}\right\|_\infty \mathrm{E}\left[\left|\Gamma_{e_{j_1},\ldots,e_{j_5}}(F_n)\right|\right] \\
&+\frac{1}{6}\sum_{m=e_i+e_j+e_k+e_l}|\kappa_m(F_n)|\left|\mathrm{E}\left[\partial^m U_{h_i,C}(F_n)\right] - \mathrm{E}\left[\partial^m U_{h_i,C}(Z)\right]\right|
\end{aligned}$$

$$\leq \max\left\{\sum_{|m|=3}|\kappa_m(F_n)|, \sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right\}\left[\frac{1}{2}\sum_{m=e_i+e_j+e_k}|\mathrm{E}\left[\partial^m U_{h_i,C}(F_n)\right]|\right.$$
$$\left.+c_2(q)\sum_{m=e_i+e_j+e_k+e_l}|\mathrm{E}\left[\partial^m U_{h_i,C}(F_n)\right]-\mathrm{E}\left[\partial^m U_{h_i,C}(Z)\right]| + \frac{c_3(q)d^5}{5}\left(\sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right)^{\frac{1}{4}}\right].$$

As $n\to\infty$, we have $\mathrm{E}\left[\partial^m U_{h_i,C}(F_n)\right]\to\mathrm{E}\left[\partial^m U_{h_i,C}(Z)\right]=0$ for $|m|=3$, $\mathrm{E}\left[\partial^m U_{h_i,C}(F_n)\right]-\mathrm{E}\left[\partial^m U_{h_i,C}(Z)\right]\to 0$, and $\sum_{i=1}^{d}\kappa_{4e_i}(F_n)\to 0$. Therefore, set

$$\bar{d}=2d+d(d-1)+\frac{d(d-1)(d-2)}{6},\quad c_1=\frac{a}{36\left(\bar{d}+1\right)},$$

we have that for n large enough,

$$\left|\mathrm{E}\left[h_i(F_n)\right]-\mathrm{E}\left[h_i(Z)\right]-\frac{a}{24}\kappa_{4e_i}(F_n)\right|\leq\frac{c_1}{\bar{d}}M(F_n),$$

which implies that

$$|\mathrm{E}\left[h_i(F_n)\right]-\mathrm{E}\left[h_i(Z)\right]|\geq\frac{a}{24}\kappa_{4e_i}(F_n)-\frac{c_1}{\bar{d}}M(F_n),\quad 1\leq i\leq d.$$

Similarly, for $1\leq i,j,k\leq d$,

$$\begin{aligned}
|\mathrm{E}\left[g_i(F_n)\right]-\mathrm{E}\left[g_i(Z)\right]|&\geq\frac{a}{6}\left|\kappa_{3e_i}(F_n)\right|-\frac{c_1}{\bar{d}}M(F_n),\\
|\mathrm{E}\left[g_{ij}(F_n)\right]-\mathrm{E}\left[g_{ij}(Z)\right]|&\geq\frac{a}{12}\left|\kappa_{2e_i+e_j}(F_n)\right|-\frac{c_1}{\bar{d}}M(F_n),\ i\neq j,\\
|\mathrm{E}\left[g_{ijk}(F_n)\right]-\mathrm{E}\left[g_{ijk}(Z)\right]|&\geq\frac{a}{36}\left|\kappa_{e_i+e_j+e_k}(F_n)\right|-\frac{c_1}{\bar{d}}M(F_n),\ i\neq j,k \text{ and } j\neq k.
\end{aligned}$$

Then for $k=3,4$,

$$\begin{aligned}
\bar{d}\rho_k(F_n,Z)&\geq\sum_{i=1}^{d}|\mathrm{E}\left[h_i(F_n)\right]-\mathrm{E}\left[h_i(Z)\right]|+\sum_{i=1}^{d}|\mathrm{E}\left[g_i(F_n)\right]-\mathrm{E}\left[g_i(Z)\right]|\\
&\quad+\sum_{i=1}^{d}\sum_{j\neq i}|\mathrm{E}\left[g_{ij}(F_n)\right]-\mathrm{E}\left[g_{ij}(Z)\right]|+\sum_{i=1}^{d}\sum_{j=i+1}^{d}\sum_{k=j+1}^{d}|\mathrm{E}\left[g_{ijk}(F_n)\right]-\mathrm{E}\left[g_{ijk}(Z)\right]|\\
&\geq\left(\bar{d}+1\right)c_1\left(\sum_{|m|=3}|\kappa_m(F_n)|+\sum_{i=1}^{d}\kappa_{4e_i}(F_n)\right)-c_1M(F_n)\\
&\geq\bar{d}c_1M(F_n).
\end{aligned}$$

That is, for $k=3,4$,

$$\rho_k(F_n,Z)\geq c_1M(F_n).$$

□

4. Applications

4.1. *Application for complex Wiener-Itô integral.* We identify the distribution of a complex random variable $F=F_1+\mathrm{i}F_2$ as the distribution of a two-dimensional random vector (F_1,F_2). Then the distance between the distributions of two complex random variables $F=F_1+\mathrm{i}F_2$ and $G=G_1+\mathrm{i}G_2$ is actually the distance between the distributions of two two-dimensional random vectors (F_1,F_2) and (G_1,G_2). Namely, we take $d=2$ in (3.2), and for $k=3,4$, we define

$$\rho_k(F,G)=\sup\left\{|\mathrm{E}\left[g(F_1,F_2)\right]-\mathrm{E}\left[g(G_1,G_2)\right]|\right\},$$

where g runs over the class of all functions belonging to $C^k(\mathbb{R}^2)$ such that $M_j(g) \le 1$ for all $0 \le j \le k$. Define the covariance matrix of the complex random variable $F = F_1 + \mathrm{i}F_2$ as the covariance matrix of the two-dimensional random vector (F_1, F_2). We write AF to denote $A\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ for any 2×2 matrix A. For a sequence of complex random variables $\{F_n = F_{n,1} + \mathrm{i}F_{n,2}\}_{n\ge1}$, let

$$M^{'}(F_n) = \max\left\{\left|\mathrm{E}\left[F_n^3\right]\right|, \left|\mathrm{E}\left[F_n^2\overline{F_n}\right]\right|, \mathrm{E}\left[|F_n|^4\right] - 2\left(\mathrm{E}\left[|F_n|^2\right]\right)^2 - \left|\mathrm{E}\left[F_n^2\right]\right|^2\right\}.$$

Theorem 4.1. *Consider a sequence of complex Wiener-Itô integrals $\{F_n = \mathcal{I}_{p,q}(f_n)\}_{n\ge1}$, where $f_n \in \mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q}$ and $p + q \ge 2$. Suppose that F_n converges in distribution to a complex normal variable Z with the same covariance matrix as F_n. Then there exist two finite constants $0 < c_1 < c_2$ not depending on n such that for n large enough,*

$$c_1 M^{'}(F_n) \le \rho_4(F_n, Z) \le c_2 M^{'}(F_n).$$

Moreover, if the covariance matrix of F_n is positive definite, then

$$c_1 M^{'}(F_n) \le \rho_3(F_n, Z) \le c_2 M^{'}(F_n).$$

Proof: Assume $F_n = F_{n,1} + \mathrm{i}F_{n,2}$. According to Chen and Liu (2017, Theorem 3.3), $\{(F_{n,1}, F_{n,2})\}_{n\ge1}$ is actually a sequence of two-dimensional random vectors whose components live in the $(p + q)$-th Wiener chaos of the real isonormal Gaussian process over $\mathfrak{H} \oplus \mathfrak{H}$. Combining Theorem 3.1 and the fact that

$$M((F_{n,1}, F_{n,2})) \asymp M^{'}(F_n),$$

which is derived from the following Lemma 4.3 and Lemma 4.4, we obtain the conclusion. □

Using the similar argument as in the proof of Theorem 4.1, we can extend Theorem 4.1 to the case where the covariance matrix of F_n, denoted by C_n, converges to C in the sense of $\|C_n - C\|_{\mathrm{HS}} \to 0$ as $n \to \infty$.

Proposition 4.2. *Let $\{F_n = \mathcal{I}_{p,q}(f_n)\}_{n\ge1}$ be a sequence of complex Wiener-Itô integrals, where $f_n \in \mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q}$ and $p + q \ge 2$. Suppose that $\|C_n - C\|_{\mathrm{HS}} \to 0$, as $n \to \infty$.*

(1) *If C is invertible, we set $F_n^{'} = C^{\frac{1}{2}} C_n^{-\frac{1}{2}} F_n$ and assume that $F_n^{'}$ converges in distribution to a complex normal variable Z with covariance matrix C. Then for n large enough,*

$$\rho_3(F_n^{'}, Z) \asymp M^{'}(F_n^{'}).$$

(2) *If C is not invertible, suppose that $\{F_n\}_{n\ge1}$ is asymptotically close to normal. That is, $\rho_4(F_n, Z_n) \to 0$, where Z_n is a complex normal variable with covariance matrix C_n. Then for n large enough,*

$$\rho_4(F_n, Z_n) \asymp M^{'}(F_n).$$

In the following Lemma 4.3 and Lemma 4.4, we prove that

$$M((F_{n,1}, F_{n,2})) \asymp M^{'}(F_n)$$

for $F_n = \mathcal{I}_{p,q}(f_n) = F_{n,1} + \mathrm{i}F_{n,2}$ with $f_n \in \mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q}$ and $p + q \ge 2$.

Lemma 4.3. *For a complex Wiener-Itô integral $F = \mathcal{I}_{p,q}(f) = F_1 + \mathrm{i}F_2$ with $f \in \mathfrak{H}_{\mathbb{C}}^{\odot p} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot q}$ and $p + q \ge 2$, denote $\tilde{F}$ the two-dimensional random vector (F_1, F_2). Then*

$$\sum_{i=1}^{2} \kappa_{4e_i}(\tilde{F}) \le \mathrm{E}\left[|F|^4\right] - 2\left(\mathrm{E}\left[|F|^2\right]\right)^2 - \left|\mathrm{E}\left[F^2\right]\right|^2 \le c\sum_{i=1}^{2} \kappa_{4e_i}(\tilde{F}), \tag{4.1}$$

where $\sum_{i=1}^{2} \kappa_{4e_i}(\tilde{F}) = \sum_{i=1}^{2} \mathrm{E}\left[F_i^4\right] - 3\left(\mathrm{E}\left[F_i^2\right]\right)^2$ and $c > 1$ is a constant depending only on $p + q$.

Proof: Calculating directly, we get that

$$\mathrm{E}\left[|F|^4\right]-2\left(\mathrm{E}\left[|F|^2\right]\right)^2-\left|\mathrm{E}\left[F^2\right]\right|^2=\sum_{i=1}^{2}\kappa_{4e_i}(\tilde{F})+2\left(\mathrm{E}\left[F_1^2F_2^2\right]-\mathrm{E}\left[F_1^2\right]\mathrm{E}\left[F_2^2\right]-2\left(\mathrm{E}\left[F_1F_2\right]\right)^2\right).$$

To prove (4.1), it suffices to show that

$$0\le 2\left(\mathrm{E}\left[F_1^2F_2^2\right]-\mathrm{E}\left[F_1^2\right]\mathrm{E}\left[F_2^2\right]-2\left(\mathrm{E}\left[F_1F_2\right]\right)^2\right)\le \tilde{c}\sum_{i=1}^{2}\kappa_{4e_i}(\tilde{F}), \tag{4.2}$$

where $\tilde{c}$ is a positive constant depending only on $p+q$. The first inequality in (4.2) can be obtained by Chen and Liu (2017, Theorem 3.3, Lemma 4.8). On the other hand, by Theorem 2.4 and Equation (3.5),

$$\begin{aligned}\mathrm{E}\left[F_1^2F_2^2\right]-\mathrm{E}\left[F_1^2\right]\mathrm{E}\left[F_2^2\right]-2\left(\mathrm{E}\left[F_1F_2\right]\right)^2&=\kappa_{2e_1+2e_2}(\tilde{F})=6\mathrm{E}\left[\Gamma_{e_1,e_1,e_2,e_2}(\tilde{F})\right]\\&\le 6\mathrm{E}\left[\left|\Gamma_{e_1,e_1,e_2,e_2}(\tilde{F})\right|\right]\le 6c_2(p+q)\sum_{i=1}^{2}\kappa_{4e_i}(\tilde{F}).\end{aligned}$$

Then we complete the proof. □

Lemma 4.4. *For a complex random variable $F=F_1+\mathrm{i}F_2$, it holds that*

$$\begin{aligned}\frac{1}{4}\left(\left|\mathrm{E}\left[F^3\right]\right|+\left|\mathrm{E}\left[F^2\overline{F}\right]\right|\right)&\le\left|\mathrm{E}\left[F_1^3\right]\right|+\left|\mathrm{E}\left[F_2^3\right]\right|+\left|\mathrm{E}\left[F_1^2F_2\right]\right|+\left|\mathrm{E}\left[F_1F_2^2\right]\right|\\&\le\sqrt{2}\left(\left|\mathrm{E}\left[F^3\right]\right|+\left|\mathrm{E}\left[F^2\overline{F}\right]\right|\right).\end{aligned}$$

Proof: Calculating directly, we have that

$$\mathrm{E}\left[F^3\right]=\tau+\mathrm{i}\nu,\quad \mathrm{E}\left[F^2\overline{F}\right]=\tilde{\tau}+\mathrm{i}\tilde{\nu},$$

where

$$\begin{aligned}\tau&=\mathrm{E}\left[F_1^3\right]-3\mathrm{E}\left[F_1F_2^2\right],\quad \nu=-\mathrm{E}\left[F_2^3\right]+3\mathrm{E}\left[F_1^2F_2\right],\\ \tilde{\tau}&=\mathrm{E}\left[F_1^3\right]+\mathrm{E}\left[F_1F_2^2\right],\quad \tilde{\nu}=\mathrm{E}\left[F_2^3\right]+\mathrm{E}\left[F_1^2F_2\right].\end{aligned}$$

Then, by the triangle inequality $|x\pm y|\le|x|+|y|$ for $x,y\in\mathbb{R}$,

$$\begin{aligned}\left|\mathrm{E}\left[F^3\right]\right|+\left|\mathrm{E}\left[F^2\overline{F}\right]\right|&\le|\tau|+|\nu|+|\tilde{\tau}|+|\tilde{\nu}|\\&\le 4\left(\left|\mathrm{E}\left[F_1^3\right]\right|+\left|\mathrm{E}\left[F_2^3\right]\right|+\left|\mathrm{E}\left[F_1^2F_2\right]\right|+\left|\mathrm{E}\left[F_1F_2^2\right]\right|\right).\end{aligned}$$

On the other hand, note that

$$\begin{aligned}\left|\mathrm{E}\left[F_1^3\right]\right|&=\frac{1}{4}\left|\tau+3\tilde{\tau}\right|,\quad \left|\mathrm{E}\left[F_2^3\right]\right|=\frac{1}{4}\left|\nu-3\tilde{\nu}\right|,\\ \left|\mathrm{E}\left[F_1^2F_2\right]\right|&=\frac{1}{4}\left|\nu+\tilde{\nu}\right|,\quad \left|\mathrm{E}\left[F_1F_2^2\right]\right|=\frac{1}{4}\left|\tau-\tilde{\tau}\right|.\end{aligned}$$

Then

$$\begin{aligned}\left|\mathrm{E}\left[F_1^3\right]\right|+\left|\mathrm{E}\left[F_2^3\right]\right|+\left|\mathrm{E}\left[F_1^2F_2\right]\right|+\left|\mathrm{E}\left[F_1F_2^2\right]\right|&\le\frac{1}{2}\left(|\tau|+|\nu|\right)+\left(|\tilde{\tau}|+|\tilde{\nu}|\right)\\&\le\sqrt{2}\left(\left|\mathrm{E}\left[F^3\right]\right|+\left|\mathrm{E}\left[F^2\overline{F}\right]\right|\right),\end{aligned}$$

where the first inequality is from the triangle inequality $|x\pm y|\le|x|+|y|$ for $x,y\in\mathbb{R}$, and the second inequality is by the fact that $|x|+|y|\le\sqrt{2}\sqrt{x^2+y^2}=\sqrt{2}|z|$ for a complex number $z=x+\mathrm{i}y$. □

As an example, we consider a complex-valued Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$\mathrm{d}Z_t = -\gamma Z_t \mathrm{d}t + \mathrm{d}\zeta_t, \quad t \geq 0, \tag{4.3}$$

where $Z_0 = 0$, $\gamma \in \mathbb{C}$ is unknown, and ζ_t is a complex Brownian motion. That is $\zeta_t = \frac{1}{\sqrt{2}}\left(B_t^1 + \mathrm{i}B_t^2\right)$, where $(B_t^1, B_t^2)_{t\geq 0}$ is a two-dimensional standard Brownian motion. Suppose that only one trajectory $(Z_t)_{0\leq t\leq T}$ for $T > 0$ can be observed. Motivated by the work of Hu and Nualart (2010), Chen, Hu and Wang in Chen et al. (2017) considered a least squares estimator of γ defined as follows by minimizing $\int_0^T \left|\dot{Z}_t + \gamma Z_t\right|^2 \mathrm{d}t$,

$$\hat{\gamma}_T = -\frac{\int_0^T \overline{Z_t}\mathrm{d}Z_t}{\int_0^T |Z_t|^2 \mathrm{d}t} = \gamma - \frac{\int_0^T \overline{Z_t}\mathrm{d}\zeta_t}{\int_0^T |Z_t|^2 \mathrm{d}t}.$$

They proved that $\sqrt{T}\left(\hat{\gamma}_T - \gamma\right)$ is asymptotically normal. Namely, as $T \to \infty$,

$$\sqrt{T}\left[\hat{\gamma}_T - \gamma\right] = -\frac{\frac{1}{\sqrt{T}}\int_0^T \overline{Z_t}\mathrm{d}\zeta_t}{\frac{1}{T}\int_0^T |Z_t|^2 \mathrm{d}t} \xrightarrow{d} \mathcal{N}_2\left(0, \lambda \mathrm{Id}_2\right),$$

where $\lambda > 0$ is the real part of γ and Id_2 denotes the 2×2 identity matrix. They showed that the denominator satisfies

$$\frac{1}{T}\int_0^T |Z_t|^2 \mathrm{d}t \xrightarrow{a.s.} \frac{1}{2\lambda},$$

where the notation $\xrightarrow{a.s.}$ denotes convergence almost surely, and for the numerator $F_T := \frac{1}{\sqrt{T}}\int_0^T \overline{Z_t}\mathrm{d}\zeta_t$,

$$\left(F_{T,1}, F_{T,2}\right) \xrightarrow{d} \mathcal{N}_2\left(0, \frac{1}{4\lambda}\mathrm{Id}_2\right),$$

where $F_{T,1}$ and $F_{T,1}$ are the real and imaginary parts of F_T respectively. Then the asymptotic normality of the estimator $\hat{\gamma}_T$ is obtained. One should note that, in Chen et al. (2017), the noise considered by Chen, Hu and Wang is a complex fractional Brownian motion with a Hurst parameter belonging to $[1/2, 3/4)$. This case involves more complicated calculations and more precise estimations. Here, to demonstrate the availability of our techniques, we focus on the case in which the noise is a complex standard Brownian motion.

Next we will derive that $T^{-1/2}$ is the optimal rate of convergence for the numerator F_T. We have no idea how to handle the optimal rate of convergence for the statistic $\sqrt{T}\left[\hat{\gamma}_T - \gamma\right]$, although we conjecture that it is still $T^{-1/2}$. Note that Kim and Park (2017a,b) obtained that $T^{-1/2}$ is the optimal Berry-Esseen bound for normal approximation of the least squares estimator of the drift coefficient of the real-valued one-dimensional Ornstein-Uhlenbeck process driven by a standard Brownian motion. As they stated in Kim and Park (2017b), in many situations encountered in statistics, one needs to consider the rate of convergence for the sequence $\{F_n/G_n\}_{n\geq 1}$ with $G_n > 0$ almost surely (such as $\sqrt{T}\left[\hat{\gamma}_T - \gamma\right]$). Therefore, we shall deal with the optimal rate of convergence for the statistic $\sqrt{T}\left[\hat{\gamma}_T - \gamma\right]$ in a separate project.

Define the Hilbert space $\mathfrak{H} = L^2\left([0, +\infty)\right)$ with the inner product $\langle f, g\rangle_{\mathfrak{H}} = \int_0^\infty f(t)g(t)\mathrm{d}t$. We complexify $\mathfrak{H}$ in the usual way and denote it by $\mathfrak{H}_{\mathbb{C}}$. For any $f, g \in \mathfrak{H}_{\mathbb{C}}$, $\langle f, g\rangle_{\mathfrak{H}_{\mathbb{C}}} = \int_0^\infty f(t)\overline{g(t)}\mathrm{d}t$. Given $f \in \mathfrak{H}_{\mathbb{C}}^{\odot a} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot b}$, $g \in \mathfrak{H}_{\mathbb{C}}^{\odot c} \otimes \mathfrak{H}_{\mathbb{C}}^{\odot d}$, for $i = 0, \ldots, a\wedge d$, $j = 0, \ldots, b\wedge c$, the (i,j)-th contraction of f and g is an element of $\mathfrak{H}_{\mathbb{C}}^{\otimes(a+c-i-j)} \otimes \mathfrak{H}_{\mathbb{C}}^{\otimes(b+d-i-j)}$ defined as

$$\begin{aligned}
& f \otimes_{i,j} g\left(t_1, \ldots, t_{a+c-i-j}; s_1, \ldots, s_{b+d-i-j}\right) \\
= & \int_{[0,+\infty)^{2l}} f\left(t_1, \ldots, t_{a-i}, u_1, \ldots, u_i; s_1 \ldots, s_{b-j}, v_1, \ldots, v_j\right)
\end{aligned}$$

$$g\left(t_{a-i+1},\ldots,t_{p-l},v_1,\ldots,v_j;s_{b-j+1},\ldots,s_{q-l},u_1,\ldots,u_i\right)\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v},$$

where $l=i+j$, $\mathbf{u}=(u_1,\ldots,u_i)$ and $\mathbf{v}=(v_1,\ldots,v_j)$.

According to (4.3), let

$$\psi_T(t;s)=\frac{1}{\sqrt{T}}e^{-\overline{\gamma}(t-s)}\mathbf{1}_{\{0\le s\le t\le T\}},\quad h_T(t;s)=\overline{\psi_T(s;t)}=\frac{1}{\sqrt{T}}e^{-\gamma(s-t)}\mathbf{1}_{\{0\le t\le s\le T\}},$$

where $\mathbf{1}_E$ is the indicator function of a set E, we know that

$$\begin{aligned}F_T&=\frac{1}{\sqrt{T}}\int_0^T\overline{Z_t}\mathrm{d}\zeta_t=\frac{1}{\sqrt{T}}\int_0^T\int_0^T e^{-\overline{\gamma}(t-s)}\mathbf{1}_{\{0\le s\le t\le T\}}\mathrm{d}\zeta_t\mathrm{d}\overline{\zeta_s}=\mathcal{I}_{1,1}(\psi_T(t;s)),\\ \overline{F_T}&=\mathcal{I}_{1,1}(h_T(t;s)).\end{aligned}\tag{4.4}$$

By the isometry property of complex Wiener-Itô integral (2.5), we obtain that

$$\mathrm{E}\left[F_T^2\right]=\langle\psi_T,h_T\rangle_{\mathfrak{H}_{\mathbb{C}}^{\otimes 2}}=\int_0^\infty\int_0^\infty\psi_T\left(t;s\right)\overline{h_T\left(t;s\right)}\mathrm{d}t\mathrm{d}s=0,$$

and as $T\to\infty$,

$$\mathrm{E}\left[|F_T|^2\right]=\langle\psi_T,\psi_T\rangle_{\mathfrak{H}_{\mathbb{C}}^{\otimes 2}}=\frac{1}{T}\int_0^T\int_0^t e^{-2\lambda(t-s)}\mathrm{d}s\mathrm{d}t=\frac{1}{2\lambda}+\frac{1}{4\lambda^2T}e^{-2\lambda T}-\frac{1}{4\lambda^2T}\to\frac{1}{2\lambda}.$$

Since $\lim\limits_{T\to\infty}\left(1+\frac{1}{2\lambda T}e^{-2\lambda T}-\frac{1}{2\lambda T}\right)=1$, for sufficiently large T, $1+\frac{1}{2\lambda T}e^{-2\lambda T}-\frac{1}{2\lambda T}>0$. Consider

$$F_T^{'}=\left(1+\frac{1}{2\lambda T}e^{-2\lambda T}-\frac{1}{2\lambda T}\right)^{-\frac{1}{2}}F_T.$$

Then the covariance matrix of $F_T^{'}$ is equal to $\frac{1}{4\lambda}\mathrm{Id}_2$. Now we consider the optimal rate of convergence of $F_T^{'}$ to a complex normal variable Z with the covariance matrix $\frac{1}{4\lambda}\mathrm{Id}_2$ under the distance $\rho_3(F_T^{'},Z)$ as $T\to\infty$.

Theorem 4.5. *$F_T^{'}$ converges in distribution to a complex normal variable Z with the covariance matrix $\frac{1}{4\lambda}\mathrm{Id}_2$, and there exist two finite constants $0<c_1<c_2$ not depending on T such that for T large enough,*

$$c_1\frac{1}{\sqrt{T}}\le\rho_3\left(F_T^{'},Z\right)\le c_2\frac{1}{\sqrt{T}}.$$

Proof: By Theorem 4.1, it suffices to show that

$$M^{'}\left(\left(1+\frac{1}{2\lambda T}e^{-2\lambda T}-\frac{1}{2\lambda T}\right)^{-\frac{1}{2}}F_T\right)\asymp\frac{1}{\sqrt{T}}.$$

Equivalently, we need to prove that

$$M^{'}(F_T)=\max\left\{\left|\mathrm{E}\left[F_T^3\right]\right|,\left|\mathrm{E}\left[F_T^2\overline{F_T}\right]\right|,\mathrm{E}\left[|F_T|^4\right]-2\left(\mathrm{E}\left[|F_T|^2\right]\right)^2-\left|\mathrm{E}\left[F_T^2\right]\right|^2\right\}\asymp\frac{1}{\sqrt{T}}.\tag{4.5}$$

Combining the following Lemma 4.6 and Lemma 4.7, we get (4.5). Then the proof is finished. □

Lemma 4.6. *F_T is defined as (4.4), then*

$$\left|\mathrm{E}\left[F_T^3\right]\right|=0,\quad\left|\mathrm{E}\left[F_T^2\overline{F_T}\right]\right|\asymp\frac{1}{\sqrt{T}}.$$

Proof: According to the product formula of complex Wiener-Itô integral (2.6), we obtain that

$$F_T^3=\sum_{i,j=0}^{1}\sum_{m,n=0}^{1\wedge(2-i-j)}\binom{2-i-j}{m}\binom{2-i-j}{n}\mathcal{I}_{3-i-j-m-n,3-i-j-m-n}\left(\left(\psi_T\otimes_{i,j}\psi_T\right)\otimes_{m,n}\psi_T\right),$$

$$F_T^2\overline{F_T}=\sum_{i,j=0}^{1}\sum_{m,n=0}^{1\wedge(2-i-j)}\binom{2-i-j}{m}\binom{2-i-j}{n}\mathcal{I}_{3-i-j-m-n,3-i-j-m-n}\left(\left(\psi_T\otimes_{i,j}\psi_T\right)\otimes_{m,n}h_T\right).$$

Taking the expectation, we have that

$$\begin{aligned}\mathrm{E}\left[F_T^3\right]&=\left(\psi_T\otimes_{1,0}\psi_T\right)\otimes_{1,1}\psi_T+\left(\psi_T\otimes_{0,1}\psi_T\right)\otimes_{1,1}\psi_T=2\left(\psi_T\otimes_{1,0}\psi_T\right)\otimes_{1,1}\psi_T\\&=2\int_0^\infty\int_0^\infty\frac{1}{T^{3/2}}\mathbf{1}_{\{0\le s\le t\le T\}}\left(t-s\right)e^{-\overline{\gamma}(t-s)}e^{-\overline{\gamma}(s-t)}\mathbf{1}_{\{0\le t\le s\le T\}}\mathrm{d}s\mathrm{d}t=0,\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}\left[F_T^2\overline{F_T}\right]&=\left(\psi_T\otimes_{1,0}\psi_T\right)\otimes_{1,1}h_T+\left(\psi_T\otimes_{0,1}\psi_T\right)\otimes_{1,1}h_T=2\left(\psi_T\otimes_{1,0}\psi_T\right)\otimes_{1,1}h_T\\&=2\int_0^\infty\int_0^\infty\frac{1}{T^{3/2}}\mathbf{1}_{\{0\le s\le t\le T\}}\left(t-s\right)e^{-\overline{\gamma}(t-s)}e^{-\gamma(t-s)}\mathbf{1}_{\{0\le s\le t\le T\}}\mathrm{d}s\mathrm{d}t\\&=\frac{1}{2\lambda^2\sqrt{T}}e^{-2\lambda T}\left(1+\frac{1}{\lambda T}\right)-\frac{1}{2\lambda^3T^{3/2}}+\frac{1}{4\lambda^2\sqrt{T}}\asymp\frac{1}{\sqrt{T}}.\end{aligned}$$

Then we get the conclusion. □

Lemma 4.7. *F_T is defined as (4.4), then*

$$\mathrm{E}\left[|F_T|^4\right]-2\left(\mathrm{E}\left[|F_T|^2\right]\right)^2-\left|\mathrm{E}\left[F_T^2\right]\right|^2\asymp\frac{1}{T}.$$

Proof: By Chen et al. (2017, Lemma 2.3), we have that

$$\begin{aligned}&\mathrm{E}\left[|F_T|^4\right]-2\left(\mathrm{E}\left[|F_T|^2\right]\right)^2-\left|\mathrm{E}\left[F_T^2\right]\right|^2\\&=\|\psi_T\otimes_{0,1}\psi_T\|^2_{\mathfrak{H}_{\mathbb{C}}^{\otimes 2}}+\|\psi_T\otimes_{1,0}\psi_T\|^2_{\mathfrak{H}_{\mathbb{C}}^{\otimes 2}}+\|\psi_T\otimes_{0,1}h_T+\psi_T\otimes_{1,0}h_T\|^2_{\mathfrak{H}_{\mathbb{C}}^{\otimes 2}}.\end{aligned}$$

Calculating directly, we get that

$$\psi_T\otimes_{0,1}\psi_T(t;s)=\psi_T\otimes_{1,0}\psi_T(t;s)=\int_0^\infty\psi_T(t;u)\psi_T(u;s)\mathrm{d}u=\frac{1}{T}\mathbf{1}_{\{0\le s\le t\le T\}}\left(t-s\right)e^{-\overline{\gamma}(t-s)}.$$

Then

$$\begin{aligned}\|\psi_T\otimes_{1,0}\psi_T\|^2_{\mathfrak{H}_{\mathbb{C}}^{\otimes 2}}&=\|\psi_T\otimes_{0,1}\psi_T\|^2_{\mathfrak{H}_{\mathbb{C}}^{\otimes 2}}=\frac{1}{T^2}\int_0^\infty\int_0^\infty\mathbf{1}_{\{0\le s\le t\le T\}}\left(t-s\right)^2e^{-\gamma(t-s)}e^{-\overline{\gamma}(t-s)}\mathrm{d}t\mathrm{d}s\\&=\frac{1}{2\lambda^2}e^{-2\lambda T}\left(\frac{1}{2}+\frac{1}{\lambda T}+\frac{3}{4\lambda^2T^2}\right)-\frac{3}{8\lambda^4T^2}+\frac{1}{4\lambda^3T}\asymp\frac{1}{T}.\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}\psi_T\otimes_{0,1}h_T\left(t;s\right)&=\frac{1}{2\lambda T}\mathbf{1}_{\{0\le s,t\le T\}}e^{-\overline{\gamma}t-\gamma s}\left(e^{2\lambda(t\wedge s)}-1\right),\\\psi_T\otimes_{1,0}h_T\left(t;s\right)&=\frac{1}{2\lambda T}\mathbf{1}_{\{0\le s,t\le T\}}e^{\overline{\gamma}s+\gamma t}\left(e^{-2\lambda(t\vee s)}-e^{-2\lambda T}\right),\end{aligned}$$

where $a\vee b$ denotes the maximum of $a,b\in\mathbb{R}$, and

$$\begin{aligned}&\|\psi_T\otimes_{0,1}h_T+\psi_T\otimes_{1,0}h_T\|^2_{\mathfrak{H}_{\mathbb{C}}^{\otimes 2}}\\&=\frac{1}{4\lambda^2}e^{-2\lambda T}\left(2+\frac{8}{\lambda T}+\frac{5}{\lambda^2T^2}+\frac{1}{2\lambda^2T^2}e^{-2\lambda T}\right)-\frac{11}{8\lambda^4T^2}+\frac{1}{\lambda^3T}\asymp\frac{1}{T}.\end{aligned}$$

Then the proof is finished. □

4.2. *Application for Wiener-Itô integrals with kernels of step functions.* In Campese (2013, Section 5.1), Campese proposed a counterexample to explain that his techniques sometimes are not applicable. In this section, for this example, we apply our conclusions to get the optimal rate of convergence with respect to the distance $\rho_3(\cdot,\cdot)$. Specifically, let $\mathfrak{H} = L^2([0,1),\mu)$, where μ is the Lebesgue measure on $[0,1)$, and partition $[0,1)$ into N equidistant intervals $\alpha_1, \alpha_2, \ldots, \alpha_N$, where $\alpha_k = \left[\frac{k-1}{N}, \frac{k}{N}\right)$ for $k = 1, \ldots, N$. Define $f \in \mathfrak{H}^{\odot 2}$ as

$$f(x,y) = \sum_{i,j=1}^{N} a_{ij} \mathbf{1}_{\alpha_i}(x) \mathbf{1}_{\alpha_j}(y), \tag{4.6}$$

where $a_{ij} \in \mathbb{R}$, $a_{ij} = a_{ji}$ for $1 \le i,j \le d$. It is obvious that f is uniquely determined by the symmetric matrix $A = (a_{ij})_{1\le i,j\le N}$. If g is another kernel of the type (4.6), given by a matrix $B = (b_{ij})_{1\le i,j\le N}$, we have

$$\begin{aligned}
(f \otimes_1 g)(x,y) &= \int_0^1 \left(\sum_{i,j=1}^{N} a_{ij} \mathbf{1}_{\alpha_i}(x) \mathbf{1}_{\alpha_j}(t) \right) \left(\sum_{k,l=1}^{N} b_{kl} \mathbf{1}_{\alpha_k}(y) \mathbf{1}_{\alpha_l}(t) \right) \mathrm{d}\mu(t) \\
&= \sum_{i,j,k=1}^{N} a_{ij} b_{kj} \mu(\alpha_j) \mathbf{1}_{\alpha_i}(x) \mathbf{1}_{\alpha_k}(y) = \frac{1}{N} \sum_{i,j,k=1}^{N} a_{ij} b_{jk} \mathbf{1}_{\alpha_i}(x) \mathbf{1}_{\alpha_k}(y),
\end{aligned}$$

and

$$\left(f \tilde{\otimes}_1 g\right)(x,y) = \frac{1}{2N} \sum_{i,j,k=1}^{N} (a_{ij} b_{jk} + a_{kj} b_{ji}) \mathbf{1}_{\alpha_i}(x) \mathbf{1}_{\alpha_k}(y).$$

Therefore, $f \otimes_1 g$ can be identified with the matrix $C = \frac{1}{N} AB$ and $f \tilde{\otimes}_1 g$ with $\frac{1}{2}\left(C + C^T\right)$. Similarly, one can show that

$$\langle f, g \rangle_{\mathfrak{H}^{\otimes 2}} = \frac{1}{N^2} \langle A, B \rangle_{\mathrm{HS}} = \frac{1}{N^2} \mathrm{tr}\left(AB^T\right). \tag{4.7}$$

For simplicity, we fix $d = 2$. We define two-dimensional random vectors $F_n = (I_2(f_{n,1}), I_2(f_{n,2}))$ for $n \ge 1$, where the kernels $f_{n,1}$ and $f_{n,2}$ are given by $(3n) \times (3n)$ matrices

$$A_{n,1} = \sqrt{n} \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_n & \tilde{\mathbf{1}}_n \\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n \\ \tilde{\mathbf{1}}_n & \mathbf{0}_n & \mathbf{0}_n \end{pmatrix} \text{ and } A_{n,2} = \sqrt{n} \begin{pmatrix} \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & \tilde{\mathbf{1}}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n \end{pmatrix},$$

respectively. Here, we denote by $\mathbf{0}_n$ the $n \times n$ matrix with all entries equal to zero, and $\tilde{\mathbf{1}}_n$ the $n \times n$ matrix with entries on the anti-diagonal equal to one and other entries equal to zero.

According to (2.13) and (4.7), for $1 \le i,j,k \le 2$, we have that

$$\begin{aligned}
\kappa_{e_i+e_j}(F_n) &= 2 \langle f_{n,i}, f_{n,j} \rangle_{\mathfrak{H}^{\otimes 2}} = \frac{2}{9n^2} \mathrm{Tr}\left(A_{n,i} A_{n,j}\right) = \begin{cases} \frac{4}{9}, & i = j = 1, \\ \frac{2}{9}, & i = j = 2, \\ 0, & i \ne j, \end{cases} \\
\kappa_{e_i+e_j+e_k}(F_n) &= 2^2 \cdot 2! \left\langle f_{n,i} \tilde{\otimes}_1 f_{n,j}, f_{n,k} \right\rangle_{\mathfrak{H}^{\otimes 2}} = \frac{8}{9n^2} \mathrm{Tr}\left(\frac{1}{6n} \left(A_{n,i} A_{n,j} + A_{n,j} A_{n,i}\right) A_{n,k} \right) \\
&= \frac{8}{27n^3} \mathrm{Tr}\left(A_{n,i} A_{n,j} A_{n,k}\right) = \begin{cases} \frac{8}{27n^{3/2}}, & n \text{ is odd and } i = j = k = 2, \\ 0, & \textit{otherwise}. \end{cases}
\end{aligned} \tag{4.8}$$

By a similar argument, we know that

$$\kappa_{4e_i}(F_n) = \frac{2^3 \cdot 3!}{(3n)^4} \mathrm{Tr}\left(A_{n,i}^4\right) = \begin{cases} \frac{32}{27n}, & i = 1, \\ \frac{16}{27n}, & i = 2. \end{cases}$$

Therefore, F_n converges in distribution to a two-dimensional normal vector $Z \sim \mathcal{N}_2\left(0, \frac{2}{9}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right)$ as $n \to \infty$ by the multidimensional Fourth Moment Theorem (see Peccati and Tudor (2005, Theorem 1)), and

$$M(F_n) = \max\left\{ \sum_{|m|=3} |\kappa_m(F_n)|, \sum_{i=1}^{d} \kappa_{4e_i}(F_n) \right\} \asymp \frac{1}{n}.$$

Then we obtain the following theorem.

Theorem 4.8. *For $n \geq 1$, define F_n as above. Then F_n converges in distribution to $Z \sim \mathcal{N}_2\left(0, \frac{2}{9}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right)$ as $n \to \infty$, and there exist two finite constants $0 < c_1 < c_2$ not depending on n such that for n large enough,*

$$c_1 \frac{1}{n} \leq \rho_3(F_n, Z) < c_2 \frac{1}{n}.$$

Remark 4.9. We now point out that Campese's techniques in Campese (2013) fail to provide the optimal rate of convergence since the limit of (1.11) is equal to zero in the above example. By using the orthogonality property (2.1) and the product formula (2.3) of multiple Wiener-Itô integrals, we have that

$$\begin{aligned}
\Delta_{ij}(F_n) &= \left(\mathrm{Var}\left(\frac{1}{2}\left\langle DI_2(f_{n,i}), DI_2(f_{n,j})\right\rangle_{\mathfrak{H}}\right)\right)^{\frac{1}{2}} = 2\sqrt{2}\left\| f_{n,i}\tilde{\otimes}_1 f_{n,j}\right\|_{\mathfrak{H}^{\otimes 2}} \\
&= \frac{2\sqrt{2}}{3n}\left[\mathrm{Tr}\left(\left(\frac{1}{6n}\left(A_{n,i}A_{n,j} + A_{n,j}A_{n,i}\right)\right)^2\right)\right]^{\frac{1}{2}} = \begin{cases} \frac{4}{9\sqrt{n}}, & i = j = 1, \\ \frac{2\sqrt{2}}{9\sqrt{n}}, & i = j = 2, \\ 0, & i \neq j. \end{cases}
\end{aligned}$$

This implies that only when $i = j = 1$ or $i = j = 2$, (1.9) is valid. Thus $\rho_{ijk} = 0$ for $1 \leq i, j, k \leq 2$ and $i \neq j$ in (1.11). For $1 \leq i, k \leq 2$, by the proof of Campese (2013, Theorem 3.7) and (4.8),

$$\rho_{iik} = \frac{1}{2}\lim_{n\to\infty}\frac{\kappa_{2e_i+e_k}(F_n)}{\Delta_{ij}(F_n)} = 0.$$

Therefore, the limit of (1.11) is equal to zero due to $\rho_{ijk} = 0$ for all $1 \leq i, j, k \leq 2$ and Campese's techniques in Campese (2013) fail to provide the optimal rate of convergence.

4.3. *Application for vector-valued Toeplitz quadratic functional.* Let $X = (X_t)_{t\in\mathbb{R}}$ be a centered real-valued stationary Gaussian process with a covariance function $r(t): \mathbb{R} \to \mathbb{R}$ and an integrable and even spectral density $f(\lambda): \mathbb{R} \to \mathbb{R}$. This is, for every $u, t \in \mathbb{R}$, one has

$$\mathrm{E}\left(X_u X_{u+t}\right) := r(t) = \hat{f}(t) := \int_{-\infty}^{+\infty} e^{\mathrm{i}\lambda t} f(\lambda)\mathrm{d}\lambda.$$

We consider the normalized random variable

$$\tilde{Q}_{g,T} = \frac{Q_{g,T} - \mathrm{E}\left(Q_{g,T}\right)}{\sqrt{T}},$$

where $Q_{g,T}$ is the Toeplitz quadratic functional of the process X associated with some integrable even function g and $T > 0$, defined as

$$Q_{g,T} = \int_0^T \int_0^T \hat{g}(t-s)X(t)X(s)\mathrm{d}t\mathrm{d}s.$$

Given $T>0$ and $\psi \in L^1(\mathbb{R})$, we denote by $B_T(\psi)$ the truncated Toeplitz operator associated with ψ and T, defined on $L^2(\mathbb{R})$ as

$$B_T(\psi)(u)(t)=\int_0^T u(x)\hat{\psi}(t-x)\mathrm{d}x, \quad t\in\mathbb{R}.$$

Given $\psi,\gamma \in L^1(\mathbb{R})$, let $B_T(\psi)B_T(\gamma)$ be the product of the two operators $B_T(\psi)$ and $B_T(\gamma)$.

We refer readers to Avram (1988); Fox and Taqqu (1987); Ginovian (1994); Ginovyan and Sahakyan (2005, 2007); Giraitis and Surgailis (1990); Grenander and Szegö (1958) for central limit theorems regarding Toeplitz quadratic functionals of discrete-time and continuous-time stationary Gaussian processes. Choosing even functions $g_1,\ldots,g_d \in L^1(\mathbb{R})$, we consider the random vector $G_T=(G_{T,1},\ldots,G_{T,d})$ defined by setting $G_{T,i}=\tilde{Q}_{g_i,T}$ for $1\le i\le d$ and $T>0$.

Theorem 4.10 (Campese (2013) Theorem 5.3)**.** *Let $m\in\mathbb{N}_0^d$ be a multi-index with $|m|\ge 2$ and elementary decomposition $\{l_1,\ldots,l_{|m|}\}$. For $1\le i\le |m|$, let $g_{l_i}=g_j$ if $l_i=e_j$ for some $1\le j\le d$. Then the following is true.*

(1) *The cumulant $\kappa_m(G_T)$ is given by*

$$\kappa_m(G_T)=T^{-|m|/2}2^{|m|-1}(|m|-1)!\mathrm{Tr}\left[B_T(f)^{|m|}\prod_{i=1}^{|m|}B_T(g_{l_i})\right].$$

(2) *If $f\in L^1(\mathbb{R})\cap L^{q_0}(\mathbb{R})$ and $g_i\in L^1(\mathbb{R})\cap L^{q_i}(\mathbb{R})$ such that $1/q_0+1/q_i\le 1/|m|$ for $1\le i\le d$, then*

$$\lim_{T\to\infty}T^{|m|/2-1}\kappa_m(G_T)=2^{|m|-1}(|m|-1)!(2\pi)^{2|m|-1}\int_{-\infty}^{\infty}f^{|m|}(x)\prod_{i=1}^{|m|}g_{l_i}(x)\mathrm{d}x.$$

(3) *If $f\in L^1(\mathbb{R})\cap L^{q_0}(\mathbb{R})$ and $g_i\in L^1(\mathbb{R})\cap L^{q_i}(\mathbb{R})$ such that $1/q_0+1/q_i\le 1/2$ for $1\le i\le d$, then*

$$G_T\xrightarrow{d}Z\sim\mathcal{N}_d(0,C), \quad T\to\infty,$$

where the covariance matrix $C=(C_{ij})_{1\le i,j\le d}$ is given by

$$C_{ij}=16\pi^3\int_0^{\infty}f^2(x)g_i(x)g_j(x)\mathrm{d}x.$$

Suppose that C is invertible. We denote by C_T the covariance matrix of G_T. Then for T large enough, C_T is invertible. We now consider the random vector $G_T^{'}=C^{\frac12}C_T^{-\frac12}G_T$. Note that each component $G_{T,i}$ of G_T can be represented as a double Wiener-Itô integral with respect to X. Combining Theorem 3.1 and Theorem 4.10, we obtain the optimal rate of convergence of $G_T^{'}$ to the normal vector $Z\sim\mathcal{N}_d(0,C)$ under the distance $\rho_3(G_T^{'},Z)$ as $T\to\infty$. We point out that the optimal rate of convergence given in Theorem 4.11 is more explicit compared to Campese (2013, Proposition 5.3).

Theorem 4.11. *If $f\in L^1(\mathbb{R})\cap L^{q_0}(\mathbb{R})$ and $g_i\in L^1(\mathbb{R})\cap L^{q_i}(\mathbb{R})$ such that $1/q_0+1/q_i\le 1/4$ for $1\le i\le d$, then $G_T^{'}\xrightarrow{d}Z\sim\mathcal{N}_d(0,C)$ as $T\to\infty$. Moreover,*

(1) *If $\int_{-\infty}^{\infty}f^3(x)\prod_{i=1}^3 g_{l_i}(x)\mathrm{d}x\ne 0$ for some multi-index m with $|m|=3$ and elementary decomposition $\{l_1,l_2,l_3\}$, then there exist two finite constants $0<c_1<c_2$ not depending on T such that for T large enough,*

$$c_1\frac{1}{\sqrt{T}}\le\rho_3(G_T^{'},Z)\le c_2\frac{1}{\sqrt{T}}.$$

(2) *If* $\int_{-\infty}^{\infty} f^4(x) g_i^4(x)\mathrm{d}x \neq 0$ *for some* $1 \le i \le d$, *and* $\lim_{T\to\infty} \frac{1}{\sqrt{T}} \mathrm{Tr}\left[B_T(f)^{|m|} \prod_{i=1}^{|m|} B_T(g_{l_i})\right] < \infty$ *for any multi-index* m *with* $|m| = 3$ *and elementary decomposition* $\{l_1, l_2, l_3\}$, *then there exist two finite constants* $0 < c_1 < c_2$ *not depending on* T *such that for* T *large enough,*

$$c_1 \frac{1}{T} \le \rho_3(G_T', Z) \le c_2 \frac{1}{T}.$$

Acknowledgement

The author expresses gratitude to an anonymous referee for providing insightful and constructive suggestions, and to Prof. Yong Chen and Prof. Yong Liu for their valuable comments and discussions.

Appendix

Proof of (3.5)*:* According to Equation (2.12),

$$\Gamma_{e_i,e_j,e_k,e_l}(F) = \sum_{r_2=1}^{q-1} \sum_{r_3=1}^{(2q-2r_2)\wedge q} \sum_{r_4=1}^{(3q-2r_2-2r_3)\wedge q} c_{q,l}(r_2,r_3,r_4) \mathbf{1}_{\left\{r_2+r_3<\frac{3q}{2}\right\}} I_{4q-2(r_2+r_3+r_4)}\left(f_i \tilde{\otimes}_{r_2} f_j \tilde{\otimes}_{r_3} f_k \tilde{\otimes}_{r_4} f_l\right). \tag{4.9}$$

To prove (3.5), by (3.8), it suffices to demonstrate that for any choice of (r_2, r_3, r_4) in the sum (4.9), the inequality

$$\left\| \left(\left(f_i \tilde{\otimes}_{r_2} f_j \right) \tilde{\otimes}_{r_3} f_k \right) \tilde{\otimes}_{r_4} f_l \right\|_{\mathfrak{H}^{\otimes(4q-2r_2-2r_3-2r_4)}} \le \max_{1\le i\le d} \max_{1\le r\le q-1} \|f_i \otimes_r f_i\|^2_{\mathfrak{H}^{\otimes(2q-2r)}}, \tag{4.10}$$

holds. Note that $f_i \tilde{\otimes}_{r_2} f_j \tilde{\otimes}_{r_3} f_k$ has already been discussed when proving (3.4), due to the assumption that $r_2 + r_3 < \frac{3q}{2}$. Using the previous estimate and Cauchy-Schwarz inequality (or see Biermé et al. (2012, Equation (4.3), Equation (4.4))), we conclude directly for $r_4 < q$. It remains to consider the case when $r_4 = q$.

As before, taking into account the symmetry of f_i for $1 \le i \le d$ and the symmetrization of contractions, it is sufficient to consider the function of $2(q - r_2 - r_3)$ variables of the type

$$F(\mathbf{t}) = \int f_i(\mathbf{t}_1, \mathbf{x}_1, \mathbf{a}_1, \mathbf{w}) f_j(\mathbf{t}_2, \mathbf{x}_2, \mathbf{a}_2, \mathbf{w}) f_k(\mathbf{t}_3, \mathbf{x}_3, \mathbf{a}_1, \mathbf{a}_2) f_l(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \mathrm{d}\mu^{q+r_2+r_3}(\mathbf{w}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3),$$

where $\mathbf{w}$ has length r_2, $\mathbf{a}_1 \cup \mathbf{a}_2$ has length r_3, $\mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \sigma(\mathbf{t})$ for some permutation σ of $\{1, \ldots, 2(q - r_2 - r_3)\}$ and $\mathbf{t} = \left(t_1, \ldots, t_{2(q-r_2-r_3)}\right)$. Now we consider two cases.

(1) The length of $\mathbf{x}_3$, denoted by s_3, is not zero. Then

$$\|F\|^2_{\mathfrak{H}^{\otimes 2(q-r_2-r_3)}} = \int (f_i \otimes_{r_2} f_j)(\mathbf{t}_1, \mathbf{x}_1, \mathbf{a}_1, \mathbf{t}_2, \mathbf{x}_2, \mathbf{a}_2)(f_i \otimes_{r_2} f_j)(\mathbf{t}_1, \tilde{\mathbf{x}}_1, \tilde{\mathbf{a}}_1, \mathbf{t}_2, \tilde{\mathbf{x}}_2, \tilde{\mathbf{a}}_2) (f_k \otimes_{s_3} f_l)(\mathbf{t}_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{x}_1, \mathbf{x}_2)(f_k \otimes_{s_3} f_l)(\mathbf{t}_3, \tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) \mathrm{d}\mu^{4q-2r_2-2s_3}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{x}_1, \mathbf{x}_2, \tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3).$$

By generalized Cauchy-Schwarz inequality (Biermé et al. (2012, Lemma 4.1)), we get

$$\|F\|^2_{\mathfrak{H}^{\otimes 2(q-r_2-r_3)}} \le \|f_i \otimes_{r_2} f_j\|^2_{\mathfrak{H}^{\otimes 2(q-r_2)}} \|f_k \otimes_{s_3} f_l\|^2_{\mathfrak{H}^{\otimes 2(q-s_3)}}.$$

By the fact that $1 \le r_2 \le q-1$, $1 \le s_3 \le q - r_3 \le q-1$ and Cauchy-Schwarz inequality,

$$\|F\|_{\mathfrak{H}^{\otimes 2(q-r_2-r_3)}} \le \max_{1\le i\le d} \max_{1\le r\le q-1} \|f_i \otimes_r f_i\|^2_{\mathfrak{H}^{\otimes(2q-2r)}}.$$

(2) The length of $\mathbf{x}_3$ is zero. Then both $\mathbf{x}_1$ and $\mathbf{x}_2$ are not empty since $r_2 \geq 1$. We denote by s_2 the length of $\mathbf{x}_2$. Note that either $\mathbf{a}_1$ or $\mathbf{a}_2$ is not empty since $r_3 \geq 1$. Without loss of generality, we assume that the length of $\mathbf{a}_1$, denoted by τ_1, is not zero. Then

$$\|F\|^2_{\mathfrak{H}^{\otimes 2(q-r_2-r_3)}} = \int (f_i \otimes_{\tau_1} f_k)(\mathbf{t}_1, \mathbf{x}_1, \mathbf{w}, \mathbf{t}_3, \mathbf{x}_3, \mathbf{a}_2)(f_i \otimes_{\tau_1} f_k)(\mathbf{t}_1, \tilde{\mathbf{x}}_1, \tilde{\mathbf{w}}, \mathbf{t}_3, \tilde{\mathbf{x}}_3, \tilde{\mathbf{a}}_2)$$
$$(f_j \otimes_{s_2} f_l)(\mathbf{t}_2, \mathbf{a}_2, \mathbf{w}, \mathbf{x}_1, \mathbf{x}_3)(f_j \otimes_{s_2} f_l)(\mathbf{t}_2, \tilde{\mathbf{a}}_2, \tilde{\mathbf{w}}, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_3)$$
$$\mathrm{d}\mu^{4q-2\tau_1-2s_2}(\mathbf{w}, \tilde{\mathbf{w}}, \mathbf{a}_2, \mathbf{x}_1, \mathbf{x}_3, \tilde{\mathbf{a}}_2, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_3, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3),$$

By generalized Cauchy-Schwarz inequality (Bierme et al. (2012, Lemma 4.1)), we get that

$$\|F\|^2_{\mathfrak{H}^{\otimes 2(q-r_2-r_3)}} \leq \|f_i \otimes_{\tau_1} f_k\|^2_{\mathfrak{H}^{\otimes 2(q-r_2)}} \|f_j \otimes_{s_2} f_l\|^2_{\mathfrak{H}^{\otimes 2(q-s_3)}}.$$

By the fact that $1 \leq \tau_1 \leq q - r_2 \leq q-1$, $1 \leq s_2 \leq q - r_2 \leq q-1$ and Cauchy-Schwarz inequality,

$$\|F\|_{\mathfrak{H}^{\otimes 2(q-r_2-r_3)}} \leq \max_{1\leq i\leq d} \max_{1\leq r\leq q-1} \|f_i \otimes_r f_i\|^2_{\mathfrak{H}^{\otimes (2q-2r)}}.$$

Then we finish the proof of (3.5). □

Proof of (3.6)*:* According to Equation (2.12),

$$\Gamma_{e_i,e_j,e_k,e_l,e_s}(F) = \sum_{r_2=1}^{q-1} \sum_{r_3=1}^{(2q-2r_2)\wedge q} \sum_{r_4=1}^{(3q-2r_2-2r_3)\wedge q} \sum_{r_5=1}^{(4q-2r_2-2r_3-2r_4)\wedge q} c_{q,l}(r_2, r_3, r_4, r_5)$$
$$\mathbf{1}_{\left\{r_2+r_3<\frac{3q}{2}\right\}} \mathbf{1}_{\{r_2+r_3+r_4<2q\}} I_{5q-2(r_2+r_3+r_4+r_5)}\left(f_i \tilde{\otimes}_{r_2} f_j \tilde{\otimes}_{r_3} f_k \tilde{\otimes}_{r_4} f_l \tilde{\otimes}_{r_5} f_s\right). \quad (4.11)$$

To prove (3.6), by (3.8), it suffices to demonstrate that for any choice of (r_2, r_3, r_4, r_5) in the sum (4.11), the inequality

$$\left\|\left(\left(\left(f_i \tilde{\otimes}_{r_2} f_j\right) \tilde{\otimes}_{r_3} f_k\right) \tilde{\otimes}_{r_4} f_l\right) \tilde{\otimes}_{r_5} f_s\right\|_{\mathfrak{H}^{\otimes(5q-2(r_2+r_3+r_4+r_5))}} \leq \max_{1\leq i\leq d} \max_{1\leq r\leq q-1} \|f_i \otimes_r f_i\|^{\frac{5}{2}}_{\mathfrak{H}^{\otimes(2q-2r)}},$$

holds. Note that $\left(\left(f_i \tilde{\otimes}_{r_2} f_j\right) \tilde{\otimes}_{r_3} f_k\right) \tilde{\otimes}_{r_4} f_l$ has already been considered when proving (3.5). Using the previous estimate and Cauchy-Schwarz inequality (or see Bierme et al. (2012, Equation (4.3), Equation (4.4))), we conclude directly for $r_5 < q$. It remains to consider the case when $r_5 = q$.

As before, taking into account the symmetry of f_i for $1 \leq i \leq d$ and the symmetrization of contractions, it suffices to consider the function of $3q - 2(r_2 + r_3 + r_4)$ variables of the type

$$F(\mathbf{t}) = \int f_i(\mathbf{t}_1, \mathbf{x}_1, \mathbf{b}_1, \mathbf{a}_1, \mathbf{w}) f_j(\mathbf{t}_2, \mathbf{x}_2, \mathbf{b}_2, \mathbf{a}_2, \mathbf{w}) f_k(\mathbf{t}_3, \mathbf{x}_3, \mathbf{b}_3, \mathbf{a}_1, \mathbf{a}_2)$$
$$f_l(\mathbf{t}_4, \mathbf{x}_4, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) f_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\, \mathrm{d}\mu^{q+r_2+r_3+r_4}(\mathbf{w}, \mathbf{a}, \mathbf{b}, \mathbf{x}),$$

where $\mathbf{w}$ has length r_2, $\mathbf{a} = \mathbf{a}_1 \cup \mathbf{a}_2$ has length r_3, $\mathbf{b} = \mathbf{b}_1 \cup \mathbf{b}_2 \cup \mathbf{b}_3$ has length r_4, $\mathbf{x} = \mathbf{x}_1 \cup \mathbf{x}_2 \cup \mathbf{x}_3 \cup \mathbf{x}_4$, $\mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 \cup \mathbf{t}_4 = \sigma(\mathbf{t})$ for some permutation σ of $\{1, \ldots, 3q - 2(r_2 + r_3 + r_4)\}$ and $\mathbf{t} = \left(t_1, \ldots, t_{3q-2(r_2+r_3+r_4)}\right)$. Following the proof of Bierme et al. (2012, Equation (4.8)), we consider the following five cases.

(1) The length of $\mathbf{x}_4$, denoted by s_4, is not zero. Then

$$\|F\|^2_{\mathfrak{H}^{\otimes(3q-2(r_2+r_3+r_4))}} = \int (f_i \otimes_{r_2} f_j) \otimes_{r_3} f_k(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{b})\, f_l \otimes_{s_4} f_s(\mathbf{t}_4, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{b})$$
$$(f_i \otimes_{r_2} f_j) \otimes_{r_3} f_k\left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3, \tilde{\mathbf{b}}\right) f_l \otimes_{s_4} f_s\left(\mathbf{t}_4, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3, \tilde{\mathbf{b}}\right)$$
$$\mathrm{d}\mu^{5q-2(r_2+r_3+s_4)}\left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3, \mathbf{b}, \tilde{\mathbf{b}}\right).$$

(2) The length of $\mathbf{x}_4$ is zero but the length of $\mathbf{t}_4$ is not zero. Then

$$\|F\|^2_{\mathfrak{H}^{\otimes(3q-2(r_2+r_3+r_4))}} = \int ((f_i \otimes_{r_2} f_j) \otimes_{r_3} f_k) \otimes_q f_s (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{b}) f_l \otimes_{q-r_4} f_l \left(\mathbf{b}, \tilde{\mathbf{b}}\right)$$
$$((f_i \otimes_{r_2} f_j) \otimes_{r_3} f_k) \otimes_q f_s \left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \tilde{\mathbf{b}}\right) \mathrm{d}\mu^{2q-2(r_2+r_3)+r_4} \left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{b}, \tilde{\mathbf{b}}\right).$$

(3) The lengths of $\mathbf{x}_4$ and $\mathbf{t}_4$ are zero, but the length of $\mathbf{x}_3$, denoted by s_3, is not zero. In this case, $\mathbf{b}_1 \cup \mathbf{b}_2$ is not empty and we denote by k_{12} the length of $\mathbf{b}_1 \cup \mathbf{b}_2$. Then

$$\|F\|^2_{\mathfrak{H}^{\otimes(3q-2(r_2+r_3+r_4))}} = \int (f_i \otimes_{r_2} f_j) \otimes_{k_{12}} f_l (\mathbf{t}_1, \mathbf{t}_2, \mathbf{x}_1, \mathbf{x}_2, \mathbf{a}, \mathbf{b}_3) f_k \otimes_{s_3} f_s (\mathbf{t}_3, \mathbf{x}_1, \mathbf{x}_2, \mathbf{a}, \mathbf{b}_3)$$
$$(f_i \otimes_{r_2} f_j) \otimes_{k_{12}} f_l \left(\mathbf{t}_1, \mathbf{t}_2, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{a}}, \tilde{\mathbf{b}}_3\right) f_k \otimes_{s_3} f_s \left(\mathbf{t}_3, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{a}}, \tilde{\mathbf{b}}_3\right)$$
$$\mathrm{d}\mu^{5q-2(r_2+k_{12}+s_3)} \left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{x}_1, \mathbf{x}_2, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \mathbf{a}, \tilde{\mathbf{a}}, \mathbf{b}_3, \tilde{\mathbf{b}}_3\right).$$

(4) The lengths of $\mathbf{x}_3$, $\mathbf{x}_4$ and $\mathbf{t}_4$ are zero, but the length of $\mathbf{t}_3$, denoted by ω_3, is not zero. Note that $\mathbf{b}_1 \cup \mathbf{b}_2$ is not empty and we still denote by k_{12} the length of $\mathbf{b}_1 \cup \mathbf{b}_2$. Then

$$\|F\|^2_{\mathfrak{H}^{\otimes(3q-2(r_2+r_3+r_4))}} = \int ((f_i \otimes_{r_2} f_j) \otimes_{k_{12}} f_l) \otimes_q f_s (\mathbf{t}_1, \mathbf{t}_2, \mathbf{a}, \mathbf{b}_3)$$
$$((f_i \otimes_{r_2} f_j) \otimes_{k_{12}} f_l) \otimes_q f_s \left(\mathbf{t}_1, \mathbf{t}_2, \tilde{\mathbf{a}}, \tilde{\mathbf{b}}_3\right) f_k \otimes_{\omega_3} f_k \left(\mathbf{b}_3, \tilde{\mathbf{b}}_3, \mathbf{a}, \tilde{\mathbf{a}}\right)$$
$$\mathrm{d}\mu^{3q-2(r_2+k_{12})-\omega_3} \left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{a}, \tilde{\mathbf{a}}, \mathbf{b}_3, \tilde{\mathbf{b}}_3\right).$$

(5) The lengths of $\mathbf{x}_3$, $\mathbf{x}_4$, $\mathbf{t}_3$ and $\mathbf{t}_4$ are zero. In this case, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{b}_2 \cup \mathbf{b}_3$ are not empty. We denote by s_1 the length of $\mathbf{x}_1$ and by k_{23} the length of $\mathbf{b}_2 \cup \mathbf{b}_3$. Without loss of generality, we assume that the length of $\mathbf{a}_2$, denoted by τ_2, is not zero. Then

$$\|F\|^2_{\mathfrak{H}^{\otimes(3q-2(r_2+r_3+r_4))}} = \int (f_j \otimes_{\tau_2} f_k) \otimes_{k_{23}} f_l (\mathbf{t}_2, \mathbf{x}_2, \mathbf{a}_1, \mathbf{b}_1, \mathbf{w}) f_i \otimes_{s_1} f_s (\mathbf{t}_1, \mathbf{x}_2, \mathbf{a}_1, \mathbf{b}_1, \mathbf{w})$$
$$(f_j \otimes_{\tau_2} f_k) \otimes_{k_{23}} f_l \left(\mathbf{t}_2, \tilde{\mathbf{x}}_2, \tilde{\mathbf{a}}_1, \tilde{\mathbf{b}}_1, \tilde{\mathbf{w}}\right) f_i \otimes_{s_1} f_s \left(\mathbf{t}_1, \tilde{\mathbf{x}}_2, \tilde{\mathbf{a}}_1, \tilde{\mathbf{b}}_1, \tilde{\mathbf{w}}\right)$$
$$\mathrm{d}\mu^{5q-2(\tau_2+k_{23}+s_1)} \left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{x}_2, \tilde{\mathbf{x}}_2, \mathbf{a}_1, \tilde{\mathbf{a}}_1, \mathbf{b}_1, \tilde{\mathbf{b}}_1, \mathbf{w}, \tilde{\mathbf{w}}\right).$$

By using generalized Cauchy-Schwarz inequality (Biermé et al. (2012, Lemma 4.1)), and the estimate (3.9) in cases (1), (3) and (5) or the estimate (4.10) in cases (2) and (4), we can get

$$\|F\|_{\mathfrak{H}^{\otimes(3q-2(r_2+r_3+r_4))}} \leq \max_{1\leq i\leq d} \max_{1\leq r\leq q-1} \|f_i \otimes_r f_i\|^{\frac{5}{2}}_{\mathfrak{H}^{\otimes(2q-2r)}}.$$

Then the proof of (3.6) is finished. □

References

Avram, F. On bilinear forms in Gaussian random variables and Toeplitz matrices. *Probab. Theory Related Fields*, **79** (1), 37–45 (1988). MR952991.

Biermé, H., Bonami, A., Nourdin, I., and Peccati, G. Optimal Berry-Esseen rates on the Wiener space: the barrier of third and fourth cumulants. *ALEA Lat. Am. J. Probab. Math. Stat.*, **9** (2), 473–500 (2012). MR3069374.

Campese, S. Optimal convergence rates and one-term edge-worth expansions for multidimensional functionals of Gaussian fields. *ALEA Lat. Am. J. Probab. Math. Stat.*, **10** (2), 881–919 (2013). MR3149458.

Chatterjee, S. and Meckes, E. Multivariate normal approximation using exchangeable pairs. *ALEA Lat. Am. J. Probab. Math. Stat.*, **4**, 257–283 (2008). MR2453473.

Chen, L. H. Y., Goldstein, L., and Shao, Q.-M. *Normal approximation by Stein's method*. Probability and its Applications (New York). Springer, Heidelberg (2011). ISBN 978-3-642-15006-7. MR2732624.

Chen, Y. Product formula and independence for complex multiple Wiener-Itô integrals. *Adv. Math. (China)*, **46** (6), 819–827 (2017). MR3778509.

Chen, Y., Hu, Y., and Wang, Z. Parameter estimation of complex fractional Ornstein-Uhlenbeck processes with fractional noise. *ALEA Lat. Am. J. Probab. Math. Stat.*, **14** (1), 613–629 (2017). MR3680192.

Chen, Y. and Liu, Y. On the fourth moment theorem for complex multiple Wiener-Itô integrals. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **20** (1), 1750005, 24 (2017). MR3623878.

Chen, Y. and Liu, Y. Complex Wiener-Itô chaos decomposition revisited. *Acta Math. Sci. Ser. B (Engl. Ed.)*, **39** (3), 797–818 (2019). MR4066506.

Dung, N. T. Explicit rates of convergence in the multivariate CLT for nonlinear statistics. *Acta Math. Hungar.*, **158** (1), 173–201 (2019). MR3950207.

Fang, X. and Koike, Y. New error bounds in multivariate normal approximations via exchangeable pairs with applications to Wishart matrices and fourth moment theorems. *Ann. Appl. Probab.*, **32** (1), 602–631 (2022). MR4386537.

Fox, R. and Taqqu, M. S. Central limit theorems for quadratic forms in random variables having long-range dependence. *Probab. Theory Related Fields*, **74** (2), 213–240 (1987). MR871252.

Ginovian, M. S. On Toeplitz type quadratic functionals of stationary Gaussian processes. *Probab. Theory Related Fields*, **100** (3), 395–406 (1994). MR1305588.

Ginovyan, M. S. and Sahakyan, A. A. On the central limit theorem for Toeplitz quadratic forms of stationary sequences. *Theory Probab. Appl.*, **49** (4), 612–628 (2005). MR2142560.

Ginovyan, M. S. and Sahakyan, A. A. Limit theorems for Toeplitz quadratic functionals of continuous-time stationary processes. *Probab. Theory Related Fields*, **138** (3-4), 551–579 (2007). MR2299719.

Giraitis, L. and Surgailis, D. A central limit theorem for quadratic forms in strongly dependent linear variables and its application to asymptotical normality of Whittle's estimate. *Probab. Theory Related Fields*, **86** (1), 87–104 (1990). MR1061950.

Grenander, U. and Szegö, G. *Toeplitz forms and their applications*. California Monographs in Mathematical Sciences. University of California Press, Berkeley-Los Angeles (1958). MR94840.

Hoshino, M., Inahama, Y., and Naganuma, N. Stochastic complex Ginzburg-Landau equation with space-time white noise. *Electron. J. Probab.*, **22**, Paper No. 104, 68 (2017). MR3742401.

Hu, Y. and Nualart, D. Parameter estimation for fractional Ornstein-Uhlenbeck processes. *Statist. Probab. Lett.*, **80** (11-12), 1030–1038 (2010). MR2638974.

Itô, K. Complex multiple Wiener integral. *Jpn. J. Math.*, **22**, 63–86 (1953) (1952). MR63609.

Kim, Y. T. and Park, H. S. Optimal Berry-Esseen bound for an estimator of parameter in the Ornstein-Uhlenbeck process. *J. Korean Statist. Soc.*, **46** (3), 413–425 (2017a). MR3685579.

Kim, Y. T. and Park, H. S. Optimal Berry-Esseen bound for statistical estimations and its application to SPDE. *J. Multivariate Anal.*, **155**, 284–304 (2017b). MR3607896.

Krokowski, K., Reichenbachs, A., and Thäle, C. Berry-Esseen bounds and multivariate limit theorems for functionals of Rademacher sequences. *Ann. Inst. Henri Poincaré Probab. Stat.*, **52** (2), 763–803 (2016). MR3498009.

Krokowski, K. and Thäle, C. Multivariate central limit theorems for Rademacher functionals with applications. *Electron. J. Probab.*, **22**, Paper No. 87, 30 (2017). MR3718714.

Meckes, E. On Stein's method for multivariate normal approximation. In *High dimensional probability V: the Luminy volume*, volume 5 of *Inst. Math. Stat. (IMS) Collect.*, pp. 153–178. Inst. Math. Statist., Beachwood, OH (2009). MR2797946.

Noreddine, S. and Nourdin, I. On the Gaussian approximation of vector-valued multiple integrals. *J. Multivariate Anal.*, **102** (6), 1008–1017 (2011). MR2793872.

Nourdin, I. and Peccati, G. Stein's method and exact Berry-Esseen asymptotics for functionals of Gaussian fields. *Ann. Probab.*, **37** (6), 2231–2261 (2009a). MR2573557.

Nourdin, I. and Peccati, G. Stein's method on Wiener chaos. *Probab. Theory Related Fields*, **145** (1-2), 75–118 (2009b). MR2520122.

Nourdin, I. and Peccati, G. Cumulants on the Wiener space. *J. Funct. Anal.*, **258** (11), 3775–3791 (2010a). MR2606872.

Nourdin, I. and Peccati, G. Stein's method meets Malliavin calculus: a short survey with new estimates. In *Recent development in stochastic dynamics and stochastic analysis*, volume 8 of *Interdiscip. Math. Sci.*, pp. 207–236. World Sci. Publ., Hackensack, NJ (2010b). MR2807823.

Nourdin, I. and Peccati, G. *Normal approximations with Malliavin calculus. From Stein's method to universality*, volume 192 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge (2012). ISBN 978-1-107-01777-1. MR2962301.

Nourdin, I. and Peccati, G. The optimal fourth moment theorem. *Proc. Amer. Math. Soc.*, **143** (7), 3123–3133 (2015). MR3336636.

Nourdin, I., Peccati, G., and Reinert, G. Invariance principles for homogeneous sums: universality of Gaussian Wiener chaos. *Ann. Probab.*, **38** (5), 1947–1985 (2010a). MR2722791.

Nourdin, I., Peccati, G., and Réveillac, A. Multivariate normal approximation using Stein's method and Malliavin calculus. *Ann. Inst. Henri Poincaré Probab. Stat.*, **46** (1), 45–58 (2010b). MR2641769.

Nualart, D. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition (2006). ISBN 978-3-540-28328-7; 3-540-28328-5. MR2200233.

Nualart, D. and Nualart, E. *Introduction to Malliavin calculus*, volume 9 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge (2018). ISBN 978-1-107-61198-6; 978-1-107-03912-4. MR3838464.

Nualart, D. and Ortiz-Latorre, S. Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Process. Appl.*, **118** (4), 614–628 (2008). MR2394845.

Nualart, D. and Peccati, G. Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.*, **33** (1), 177–193 (2005). MR2118863.

Peccati, G. and Tudor, C. A. Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII*, volume 1857 of *Lecture Notes in Math.*, pp. 247–262. Springer, Berlin (2005). MR2126978.

Peccati, G. and Zheng, C. Multi-dimensional Gaussian fluctuations on the Poisson space. *Electron. J. Probab.*, **15**, no. 48, 1487–1527 (2010). MR2727319.

Reinert, G. and Röllin, A. Multivariate normal approximation with Stein's method of exchangeable pairs under a general linearity condition. *Ann. Probab.*, **37** (6), 2150–2173 (2009). MR2573554.