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Fluctuation bounds for first-passage percolation on the square, tube, and torus

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Abstract. In first-passage percolation, one assigns i.i.d. nonnegative weights (t_e) to the edges of $\mathbb{Z}^d$ and studies the induced distance (passage time) $T(x,y)$ between vertices x and y. It is known that for $d=2$, the fluctuations of $T(x,y)$ are at least order $\sqrt{\log|x-y|}$ under mild assumptions on t_e. We study the question of fluctuation lower bounds for T_n, the minimal passage time between two opposite sides of an n by n square. The main result is that, under a curvature assumption, this quantity has fluctuations at least of order $n^{1/8-\epsilon}$ for any $\epsilon>0$ when the t_e are exponentially distributed. As previous arguments to bound the fluctuations of $T(x,y)$ only give a constant lower bound for those of T_n (even assuming curvature), a different argument, representing T_n as a minimum of cylinder passage times, and deriving more detailed information about the distribution of cylinder times using the Markov property, is developed. As a corollary, we obtain the first polynomial lower bounds on higher central moments of the discrete torus passage time, under the same curvature assumption. A major tool in the proof is a new bound on the fluctuations of the minimum of independent cylinder passage times. This result is proved without the curvature assumption.

1. Introduction

In this paper, we study first-passage percolation (FPP) on the discrete square, tube, and torus. The square is defined as $B(n)=\mathbb{Z}^2\cap[0,n]^2$ with edge set $E(n)=\{\{x,y\}:x,y\in B(n),|x-y|=1\}$. Let (t_e) be a family of i.i.d. exponential random variables with mean 1, indexed by all nearest neighbor edges of $\mathbb{Z}^2$. For $x,y\in B(n)$, we set

$$T_n^{\mathrm{sq}}(x,y)=\inf_{\Gamma:x\to y}T(\Gamma). \tag{1.1}$$

Here, the minimum is taken over all vertex self-avoiding paths starting at x, ending at y, and taking edges in $E(n)$, and $T(\Gamma)=\sum_{e\in\Gamma}t_e$. Then we define the passage time between the left and right

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sides of the square as

$$T_n^{\mathrm{sq}} = \inf_{\substack{x \in B(n)\,:\, x\cdot \mathbf{e_1}=0 \\ y \in B(n)\,:\, y\cdot \mathbf{e_1}=n}} T_n^{\mathrm{sq}}(x,y), \tag{1.2}$$

where $\mathbf{e_i}$ stands for the i-th coordinate vector.

To define the tube and torus passage times, we represent them on $\mathbb{Z}^2$ using periodic edge-weights. Let $E(n)^o$ be the set of edges $\{x,y\} \in E(n)$ such that at least one of x or y is in $B(n-1)$. Let $(t_e^{(n)})_{e\in E(n)^o}$ be a family of i.i.d. exponential random variables with mean 1 assigned to the edges of $E(n)^o$ and extend the definition to all nearest neighbor edges of $\mathbb{Z}^2$ by defining $t_{e+nz}^{(n)} = t_e^{(n)}$ for $z \in \mathbb{Z}^2$. Observe that the distribution of $(t_e^{(n)})$ is invariant under integer translations of $\mathbb{Z}^2$. The tube will use the vertical strip $S(n) = \{x \in \mathbb{Z}^2 : 0 \le x \cdot \mathbf{e}_1 \le n\}$, and we accordingly define the passage time as

$$T_n^{\mathrm{tube}} = \inf_{\substack{x: x\cdot \mathbf{e}_1=0 \\ y: y\cdot \mathbf{e}_1=n}} T_n^{\mathrm{tube}}(x,y). \tag{1.3}$$

Here, $T_n^{\mathrm{tube}}(x,y)$ is defined for $x,y \in S(n)$ as

$$T_n^{\mathrm{tube}}(x,y) = \inf_{\Gamma: x\to y} T^{(n)}(\Gamma),$$

with the infimum over all vertex self-avoiding paths connecting x and y and using vertices in $S(n)$. The term $T^{(n)}(\Gamma)$ is the passage time of Γ computed with the weights $(t_e^{(n)})$: $T^{(n)}(\Gamma) = \sum_{e\in\Gamma} t_e^{(n)}$. Last, the torus passage time is defined as

$$T_n^{\mathrm{tor}} = \inf_{x: x\cdot \mathbf{e}_1=0} T^{(n)}(x, x+n\mathbf{e}_1), \tag{1.4}$$

where $T^{(n)}(x,y)$ for $x,y \in \mathbb{Z}^2$ is the minimum of $T^{(n)}(\Gamma)$ over all paths connecting x and y (not necessarily staying in $S(n)$). As defined, the tube passage time is the same as the first passage time between the left and right sides of $B(n)$ (using weights $(t_e^{(n)})$) after we have identified the top and bottom sides, turning $B(n)$ into a tube. The torus passage time is the same as the first passage time among all paths that wind once in the $\mathbf{e}_1$ direction around a torus obtained from identifying the left and right sides of $B(n)$, as well as the top and bottom sides. See Figure 1.1.

Our main goal is to find lower bounds for the *fluctuations* of these variables. We use the following definition of fluctuations, similar to that taken in Damron et al. (2020).

Definition 1.1. Let $\{X_n\}_{n=1}^\infty$ be a sequence of real-valued random variables. $\{X_n\}_{n=1}^\infty$ is said to have fluctuations of at least order $f(n)$ if there exist reals a_n, b_n and a number $c > 0$ such that for all large n, $b_n - a_n \ge cf(n)$,

$$\mathbf{P}(X_n \le a_n) > c, \text{ and } \mathbf{P}(X_n \ge b_n) > c.$$

Observe that if (X_n) has fluctuations of at least order $f(n)$, then $\liminf_{n\to\infty} \frac{\mathrm{Var}(X_n)}{f(n)^2} > 0$. However, the converse may fail. For example, the sequence defined by

$$X_n = \begin{cases} 0 & \text{with probability } 1 - \frac{1}{n} \\ n & \text{with probability } \frac{1}{n} \end{cases}$$

has diverging variance, but its fluctuations are not at least order constant.

A logarithmic lower bound for the variance of the point-to-point minimal passage time $T(x,y)$ in the standard model (FPP on the infinite discrete lattice $\mathbb{Z}^2$) is well-known for a large class of distributions of edge weights; see Auffinger et al. (2017, Sec. 3.3) and Bates and Chatterjee (2020); Damron et al. (2020); Pemantle and Peres (1994) for fluctuation bounds. A lower bound

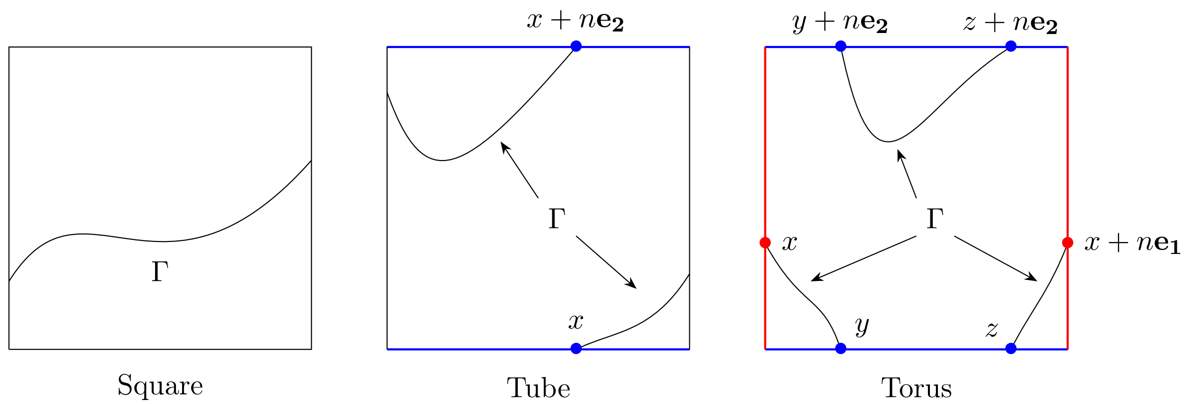

FIGURE 1.1. Left: the square $B(n)$ with an optimal path for T_n^{sq} connecting the left and right sides. Middle: a tube constructed from the square by identifying the top and bottom sides, in blue. An optimal path is shown connecting the left and right sides, traveling through the top boundary and coming out the bottom. Right: a torus constructed from the square by identifying the top and bottom sides (in blue), as well as the left and right sides (in red). The optimal path connects the left and right sides but its initial and final points, x and $x + n\mathbf{e}_1$ are identified, so the path forms a circuit that winds once around the torus in the $\mathbf{e}_1$-direction.

of polynomial order can even be shown under the curvature assumption we make below. Both of these arguments use an inequality of the form

$$\mathrm{Var}\ T(x,y) \geq c \sum_e \mathbf{P}(e \text{ is in an optimal path from } x \text{ to } y)^2,$$

and this sum has at least logarithmic growth in $|x-y|$ (polynomial growth under the curvature assumption). Unfortunately, when one uses a similar bound for T_n, T_n^{tube}, or T_n^{tor}, one obtains a sum of constant order. For example, the probability in the sum corresponding to T_n^{tor} has order n^{-1} by translation invariance. The main difference is that $T(x,y)$ is a passage time between fixed points, whereas the others are passage times between large sets of vertices. There are currently no general methods to analyze the variance which do not rely on bounding terms of this form, and therefore there are currently no nonconstant lower bounds for the variance of these three variables, even assuming curvature. For this reason, we will use a different method, based on finer information about the distribution of cylinder passage times that comes from the memoryless property of exponential weights.

We first define the limiting shape to state the curvature assumption. It uses the point-to-point passage time $T(x,y)$ mentioned in the previous paragraph, so we begin with a rigorous definition of $T(x,y)$. Similar to (1.1), we define the passage time between $x, y \in \mathbb{Z}^2$ using all nearest neighbor edges. It is

$$T(x,y) = \min_{\Gamma : x \to y} T(\Gamma), \tag{1.5}$$

where the minimum is over all self-avoiding paths Γ starting from vertex x and ending at y, and $T(\Gamma) = \sum_{e \in \Gamma} t_e$. (We recall that (t_e) is the i.i.d. family indexed by all nearest neighbor edges of $\mathbb{Z}^2$.) By the subadditive ergodic theorem (see Auffinger et al. (2017, Theorem 2.1)), it can be shown that there exists a norm $g : \mathbf{R}^2 \to \mathbf{R}_+$ such that a.s.,

$$g(x) = \lim_{n\to\infty} \frac{T(0,nx)}{n} \text{ for all } x \in \mathbb{Z}^2. \tag{1.6}$$

Next, we define

$$B(t) = \{x \in \mathbb{Z}^2 : T(0,x) \le t\}$$

and set $\widetilde{B}(t)$ equal to the sum set $B(t) + [0,1)^2$. Considering the passage time in all directions simultaneously, the shape theorem Cox and Durrett (1981) states that there exists a deterministic, convex, compact set $\mathcal{B} \subset \mathbf{R}^2$ with nonempty interior and the symmetries of $\mathbb{Z}^2$ that fix the origin such that for all $\epsilon > 0$,

$$\mathbf{P}((1-\epsilon)\mathcal{B} \subset \widetilde{B}(t)/t \subset (1+\epsilon)\mathcal{B} \text{ for all large } t) = 1. \tag{1.7}$$

We can express the limit shape as

$$\mathcal{B} = \left\{x \in \mathbf{R}^2 : g(x) \le 1\right\}.$$

Next, we state our curvature assumption, which is about the right extreme of the limit shape. It has not been verified for any edge-weight distribution, but it is strongly believed to hold; see Auffinger et al. (2017, Sec. 2.8).

Assumption 1.1. (Curvature assumption in the direction $\mathbf{e_1}$) There are constants $\epsilon_0, c_0 > 0$ such that for all $\beta \in (-\epsilon_0, \epsilon_0)$,

$$g(\mathbf{e_1} + \beta\mathbf{e}_2) - g(\mathbf{e}_1) \ge c_0\beta^2. \tag{1.8}$$

Now we are ready to state our result.

Theorem 1.2. *Let T_n^{sq} and T_n^{tube} be passage times defined by* (1.2) *and* (1.3)*, and let $\epsilon > 0$. Under Assumption 1.1, T_n^{sq} and T_n^{tube} both have fluctuations of at least order $n^{1/8-\epsilon}$.*

Remark 1.3. If we assume instead that (1.8) holds with the exponent 2 replaced by $\kappa \ge 1$, then the estimate of Theorem 1.2 changes from $n^{1/8-\epsilon}$ to $n^{1/(4\kappa)-\epsilon}$.

Remark 1.4. A result analogous to Theorem 1.2 holds for squares that are oriented in a direction $\mathbf{u}$ with $|\mathbf{u}| = 1$. In this case, we must make a curvature assumption like (1.8) in direction $\mathbf{u}$ instead of in direction $\mathbf{e}_1$.

The strategy for the proof of Theorem 1.2 is to split the tube (or square) into non-overlapping cylinders of length n and height n^α, where $\alpha > 3/4$. We first show that, because of Assumption 1.1, any optimal path must with high probability be contained in one of these cylinders (or a shifted version of these cylinders). This reduces the problem to finding the order of fluctuations of $n^{1-\alpha}$ many independent cylinder passage times. Cylinder times have been studied in Ahlberg (2015); Chatterjee and Dey (2013); Damron et al. (2020), but those works only provide lower bounds for the fluctuations of order $n^{(1-\alpha)/2}$ for a single time. To extend this to a minimum of many cylinder times, we need much more precise information about their distributions, in particular estimates for their $1/n^{1-\alpha}$ quantile. This is the main contribution of the present article. Using the memoryless property of exponentials, we represent the cylinder process as a Richardson-type growth model and prove that, conditional on geometric information of the growth, the times satisfy an entropic central limit theorem with bounds on the rate of convergence. Consequently, we can (conditionally) couple the cylinder times to independent normal variables. In the appendix, we derive a result that bounds the fluctuations of independent normal variables from below by the fluctuations of i.i.d. normal variables. From this bound, we conclude that the fluctuations of the minimum of cylinder times are at least order $n^{(1-\alpha)/2}/\sqrt{\log n}$. See Section 1.1 for an outline of the argument.

The torus passage time has particular difficulties which do not allow an easy comparison with the passage times for the square and the tube. But we can show a lower bound for the higher moments of the torus passage time as a corollary to our main theorem. It is an open problem Auffinger et al. (2017, Question 16) to show a diverging lower bound on the variance of T_n^{tor}, even under a curvature assumption. The corollary below shows that for any real $k > 12$, the k-th central moment diverges.

Corollary 1.5. *Let $\epsilon > 0$ and T_n^{tor} be the torus passage time as defined in (1.4). Under Assumption 1.1, there exists $c > 0$ such that for all n and k,*

$$\mathbf{E}\left|T_n^{tor} - \mathbf{E}T_n^{tor}\right|^k \geq cn^{\left(\frac{1}{8}-\epsilon\right)k-\frac{3}{2}}.$$

Last, following a suggestion from a reviewer of an earlier draft of this paper, we highlight one of our central tools, the fluctuation bound for the minimum of independent cylinder passage times. Its proof requires t_e to be exponentially distributed, but does not require the curvature assumption, Assumption 1.1. For $n \geq 1$, pick integers $K_n^{(1)}, \ldots, K_n^{(r_n)} \in [1, n]$ such that $\sum_{j=1}^{r_n} K_n^{(j)} = n$. Define cylinders as $\mathsf{C}^{(1)} = [0, n] \times [0, K_n^{(1)} - 1]$ and for $j = 2, \ldots, r_n$,

$$\mathsf{C}^{(j)} = [0, n] \times [K_n^{(1)} + \cdots + K_n^{(j-1)}, K_n^{(1)} + \cdots + K_n^{(j)} - 1].$$

For each j, we set $\mathcal{T}^{(j)}$ to be the corresponding cylinder passage time. It is the minimal passage time of any path in $\mathsf{C}^{(j)}$ connecting the left and right sides. Because the cylinders are disjoint, the $\mathcal{T}^{(j)}$ are independent. Last, put

$$\mathcal{T}_n = \mathcal{T}_{n,(K_n^{(j)})} = \min\{\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(r_n)}\}.$$

Proposition 1.6. *Suppose that $n^{-1/2} \sum_{j=1}^{r_n} (K_n^{(j)})^{-1/2} \to 0$. Then*

$$(\mathcal{T}_n) \text{ has fluctuations of at least order } \min_{j=1,\ldots,r_n} \sqrt{\frac{n}{K_n^{(j)}(1 + \log r_n)}}.$$

This proposition will be restated and proved as Proposition 3.6.

1.1. *Outline of the paper.* In the next section, we study optimal paths between points in our periodic environment $(t_e^{(n)})$. After showing that they exist in Lemma 2.1, we prove in Proposition 2.3 that optimal paths for T_n^{tube} are contained in horizontal cylinders of height n^α, for any $\alpha > 3/4$, with high probability. Although such a statement is standard in the planar model with i.i.d. weights (under Assumption 1.1), it will take work to establish it in the periodic environment. In Section 3, we study the fluctuations of the passage time across cylinders. First, because of the Markov property, we can represent this time using a Richardson-type growth model, and in Section 3.1, we estimate various quantities (size of the boundary of the growth, and number of steps to reach the opposite side) associated to it. We use these bounds in Section 3.2 to prove that the passage time across a cylinder satisfies a conditional (entropic) CLT, with a total variation bound coming from the estimates from Section 3.1. Using this, in Section 3.3, we prove our main fluctuation result for the minimum of independent cylinder times, Proposition 3.6. This fluctuation result is the main tool in the proofs of Theorem 1.2 and Corollary 1.5, in Section 4. Finally, the appendix serves to relate the fluctuations of the minimum of i.i.d. normal random variables to the fluctuations of the minimum of independent normal random variables with different means and variances. This result is in Theorem A.1 and is an important ingredient in the proof of Proposition 3.6 back in Section 3.3.

2. Coupling and confinement of geodesics

In this section, we focus on the discrete tube and show that the geodesics are with high probability contained in cylinders with height of order n^α for $\alpha > 3/4$. As mentioned, this statement is well-known on $\mathbb{Z}^2$ under the curvature assumption; see Newman and Piza (1995, Theorem 6). To show it for our periodic environment, we will need to couple the periodic model with the full-plane model and use concentration estimates.

It will be useful to observe that although T_n^{tube} is naturally defined using only paths that remain in the strip $S(n)$ by using $T_n^{\text{tube}}(x,y)$, this restriction is not necessary. That is,

$$T_n^{\text{tube}} = \min_{\substack{x\,:\,x\cdot\mathbf{e_1}=0 \\ y\,:\,y\cdot\mathbf{e_1}=n}} T^{(n)}(x,y). \tag{2.1}$$

The inequality $\geq$ holds trivially. For the other inequality, any path Γ between some x and y as above contains a segment Γ' which uses only vertices in $S(n)$. Indeed, we can simply follow Γ until it touches the set $\{z : z\cdot\mathbf{e}_1 = n\}$ first at some point y_0, set this to be the final point of Γ', and let the initial point of Γ' be the last intersection of Γ with $\{z : z\cdot\mathbf{e}_1 = 0\}$ before it touches y_0. Then $T^{(n)}(\Gamma) \geq T^{(n)}(\Gamma') \geq T_n^{\text{tube}}$. Taking minimum over Γ gives the inequality $\leq$ in (2.1).

To state geodesic concentration on the tube, we first need to show that geodesics exist. For $x,y\in\mathbb{Z}^2$, we say a path Γ from x to y is a geodesic for $T^{(n)}(x,y)$ if $T^{(n)}(x,y) = T^{(n)}(\Gamma)$. Similarly, a path Γ from $\{x : x\cdot\mathbf{e}_1 = 0\}$ to $\{x : x\cdot\mathbf{e}_1 = n\}$ is a geodesic for T_n^{tube} if $T^{(n)}(\Gamma) = T_n^{\text{tube}}$. In general, geodesics need not be unique. For instance, with positive probability, there are two geodesics for $T^{(n)}(0, n(\mathbf{e}_1+\mathbf{e}_2))$: one following the $\mathbf{e}_1$-axis from 0 to $n\mathbf{e}_1$ and the proceeding vertically to $n(\mathbf{e}_1+\mathbf{e}_2)$, and one following the $\mathbf{e}_2$-axis from 0 to $n\mathbf{e}_2$ and then proceeding horizontally to $n(\mathbf{e}_1+\mathbf{e}_2)$.

Lemma 2.1. *For any $x,y\in\mathbb{Z}^2$,*

$$\mathbf{P}(\textit{there is a geodesic for } T^{(n)}(x,y) \textit{ from } x \textit{ to } y) = 1.$$

Also

$$\mathbf{P}(\textit{there is a geodesic for } T_n^{\text{tube}}) = 1.$$

Proof: The argument is similar to that for Auffinger et al. (2017, Proposition 4.4). Let

$$\rho^{(n)} = \inf\{T^{(n)}(\gamma) : \gamma \text{ is an infinite edge self-avoiding path from } 0\}.$$

By definition of $(t_e^{(n)})$, we have $\inf_e t_e^{(n)} > 0$ a.s., and so for any infinite self-avoiding γ from 0, we have $T^{(n)}(\gamma) \geq (\inf_e t_e^{(n)})\#\gamma = \infty$. This means that $\rho^{(n)} = \infty$ a.s. The argument in Auffinger et al. (2017, Proposition 4.4) shows that for any outcome such that $\rho^{(n)} = \infty$, there is a geodesic for $T^{(n)}(x,y)$ for all $x,y\in\mathbb{Z}^2$. (The proof is deterministic, so it applies to $T^{(n)}$.) For this reason, we treat only the second statement in more detail.

By periodicity, we may restrict x in the definition of T_n^{tube} to be in $B(n)$. Let σ be the path that starts at the origin and moves n steps along the positive $\mathbf{e}_1$-axis until it ends at $n\mathbf{e}_1$. Fix any outcome for which $\rho^{(n)} = \infty$. Using the fact that $\rho^{(n)} = \lim_{K\to\infty} T^{(n)}(0, \partial[-K,K]^2)$ (from Auffinger et al. (2017, Lemma 4.3)), we can choose $K > n$ such that any path π from $B(n)$ to a vertex in $([-K,K]^2)^c$ satisfies $T^{(n)}(\pi) > T^{(n)}(\sigma)$. Therefore any path that starts in $B(n)$ and leaves $[-K,K]^2$ cannot be a geodesic for T_n^{tube}. This means that the minimum in the definition of T_n^{tube} (when we restrict x to be in $B(n)$) is over a finite set, and there is a minimizer. □

Remark 2.2. A reader of this paper made the following observation: Lemma 2.1 is a deterministic fact that applies whenever the realized edge-weights have only finitely many distinct values. Consider the following reasoning: Let $T(x,y;k)$ be the infimum over paths between x and y that contain at most k positive-weight edges. By assumption, the set $S = \{t_e : t_e > 0\}$ is a finite set. Hence $\inf S$ is positive, and so there must exist some (random) k such that $T(x,y;k) = T(x,y)$. Because $\#S < \infty$, there are only finitely many possible values for $T(\gamma)$ if γ contains at most k positive-weight edges. So this collection of values admits a minimum, which is necessarily $T(x,y)$, meaning the γ achieving said minimum must be a geodesic.

Next is the main result of the section, that for $\alpha > 3/4$, geodesics for T_n^{tube} are with high probability contained in horizontal strips of height n^α.

Proposition 2.3. *Let $\alpha > 3/4$. Under Assumption 1.1, there exists $b > 0$ such that, for all large n, with probability at least $1 - e^{-n^b}$, the following holds. For any geodesic Γ_n for T_n^{tube}, all vertices on Γ_n are contained in the strip $x_0 + (\mathbb{Z} \times [-n^\alpha, n^\alpha])$, where x_0 is the initial point of Γ_n.*

The proof will require the following concentration inequality for the passage time associated with periodic edge weights. Recall that g was defined in (1.6) using the passage time $T(x, y)$ over i.i.d. exponential weights.

Lemma 2.4. *Let $\epsilon > 0$. There exist $c_1, b > 0$ such that for all large n,*

$$\mathbf{P}\left(\text{for all } x, y \in \mathbb{Z}^2 \text{ with } \|x - y\|_\infty \le c_1 n, |T^{(n)}(x, y) - g(x - y)| \le n^{\frac{1}{2}+\epsilon}\right) \ge 1 - e^{-n^b}.$$

Proof: The proof will use similar concentration inequalities for the full-plane model. We may take $\epsilon \in (0, 1]$. By periodicity of $(t_e^{(n)})$, it will suffice to show that for large n,

$$\mathbf{P}(\text{for all } x, y \in \mathbb{Z}^2 \text{ with } x \in B(n) \text{ and } \|x - y\|_\infty \le c_1 n, |T^{(n)}(x, y) - g(x - y)| \le n^{\frac{1}{2}+\epsilon}) \ge 1 - e^{-n^b}. \tag{2.2}$$

Furthermore, putting $x_0 = \left(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor\right)$, it will even suffice to show that

$$\mathbf{P}(\text{for all } y \in \mathbb{Z}^2 \text{ with } \|x_0 - y\|_\infty \le c_1 n, |T^{(n)}(x_0, y) - g(x_0 - y)| \le n^{\frac{1}{2}+\epsilon}) \ge 1 - e^{n^b}. \tag{2.3}$$

This claim follows from translation invariance of the weights and a union bound. Specifically, assuming (2.3) holds, the left side of (2.2) is at least $1 - n^2 e^{-n^b}$, and this implies (2.2) if we replace b with $b/2$.

First we observe that (2.3) holds if we replace $T^{(n)}(x_0, y)$ by $T(x_0, y)$; that is,

$$\mathbf{P}(\text{for all } y \in \mathbb{Z}^2 \text{ with } \|x_0 - y\|_\infty \le c_1 n, |T(x_0, y) - g(x_0 - y)| \le n^{\frac{1}{2}+\epsilon}) \ge 1 - e^{n^b}. \tag{2.4}$$

Inequality (2.4) is standard and follows from two facts. First, Alexander has shown (Alexander, 1997, Theorem 3.2) that for some $C_1 > 0$, we have $\mathbf{E}T(x, y) \le g(x-y) + C_1\sqrt{|x - y|}\log(1+|x-y|)$ for all x, y. Second, an inequality of Talagrand (1995, Proposition 8.3) states that for some $C_2, c_2 > 0$, we have

$$\mathbf{P}(|T(x, y) - \mathbf{E}T(x, y)| \ge u) \le C_2 \exp\left(-c_2 \min\left\{\frac{u^2}{|x - y|}, u\right\}\right) \tag{2.5}$$

for all x, y and all $u \ge 0$. If $\|x_0 - y\|_\infty \le c_1 n$, then $|x_0 - y| \le \sqrt{2} c_1 n$ and so by Alexander's bound,

$$\begin{aligned}
&\mathbf{P}\left(|T(x_0, y) - g(x_0 - y)| \ge n^{\frac{1}{2}+\epsilon}\right) \\
&\le \mathbf{P}\left(|T(x_0, y) - \mathbf{E}T(x_0, y)| \ge n^{\frac{1}{2}+\epsilon} - C_1\sqrt{|x_0 - y|}\log(1 + |x_0 - y|)\right) \\
&\le \mathbf{P}\left(|T(x_0, y) - \mathbf{E}T(x_0, y)| \ge n^{\frac{1}{2}+\frac{\epsilon}{2}}\right),
\end{aligned}$$

so long as n is large. Applying (2.5), this is bounded above for some $c_3 > 0$ by

$$C_2 \exp\left(-c_2 \min\left\{\frac{n^{1+\epsilon}}{|x_0 - y|}, n^{\frac{1}{2}+\frac{\epsilon}{2}}\right\}\right) \le C_2 \exp\left(-c_3 n^\epsilon\right).$$

Therefore by a union bound, the left side of (2.4) is bounded below by $1 - C_2(2c_1 n + 1)^2 e^{-c_3 n^\epsilon}$. This gives (2.4) with $b = \epsilon/2$.

To show (2.3), we need to compare T and $T^{(n)}$. We will assume that the weights (t_e) and $(t_e^{(n)})$ which are used in the definitions of T and $T^{(n)}$ are coupled so that $t_e = t_e^{(n)}$ for all $e \in E(n)^o$, and will prove that for large n,

$$\mathbf{P}(\text{for all } y \in \mathbb{Z}^2 \text{ with } \|x_0 - y\|_\infty \le c_1 n,\ T^{(n)}(x_0, y) = T(x_0, y)) \ge 1 - e^{-n^b}. \tag{2.6}$$

This, along with (2.4), will imply (2.2). To do this, let $c_1 < 1/2$ and consider an outcome in the complement: one for which there is a $y \in \mathbb{Z}^2$ with $\|x_0 - y\|_\infty \le c_1 n$ and such that $T^{(n)}(x_0, y) \neq T(x_0, y)$. We claim that

$$\text{there exists } z \in \mathbb{Z}^2 \text{ with } \|x_0 - z\|_\infty = \lfloor n/2 \rfloor \text{ such that } T(x_0, z) \le T(x_0, y). \tag{2.7}$$

To argue this, choose geodesics γ and $\gamma^{(n)}$ for $T(x_0, y)$ and $T^{(n)}(x_0, y)$ respectively. If both paths use only edges in $E(n)^o$, then their passage times using T or $T^{(n)}$ are the same, and so $T^{(n)}(x_0, y) = T(x_0, y)$, a contradiction. So at least one must use an edge outside $E(n)^o$, and therefore must contain a vertex z with $\|x_0 - z\|_\infty = \lfloor n/2 \rfloor$. First suppose that γ contains such a z and let γ_z be the segment of γ from x_0 to z. Then $T(x_0, z) \le T(\gamma_z) \le T(\gamma) = T(x_0, y)$, so (2.7) holds. The other possibility is that $\gamma^{(n)}$ contains such a z but γ does not. In this case, γ must use only edges in $E(n)^o$, as must $\gamma_z^{(n)}$, the segment of $\gamma^{(n)}$ from x_0 to z, so long as we choose z as the first such z we find as we proceed along $\gamma^{(n)}$ from x_0 to y. Then

$$T(x_0, z) \le T(\gamma_z^{(n)}) = T^{(n)}(\gamma_z^{(n)}) \le T^{(n)}(x_0, y) \le T^{(n)}(\gamma) = T(\gamma) = T(x_0, y).$$

This shows (2.7).

Returning to (2.6), statement (2.7) along with a union bound gives

$$\begin{aligned} &\mathbf{P}(\text{there exists } y \in \mathbb{Z}^2 \text{ with } \|x_0 - y\|_\infty \le c_1 n,\ T^{(n)}(x_0, y) \neq T(x_0, y)) \\ &\le \sum_{\substack{y : \|x_0 - y\|_\infty \le c_1 n \\ z : \|x_0 - z\|_\infty = \lfloor \frac{n}{2} \rfloor}} \mathbf{P}(T(x_0, z) \le T(x_0, y)). \end{aligned} \tag{2.8}$$

If $|T(x_0, y) - g(x_0 - y)|$ and $|T(x_0, z) - g(x_0 - z)|$ were both at most $n^{3/4}$, then we would have

$$T(x_0, z) - T(x_0, y) \ge g(x_0 - z) - g(x_0 - y) - 2n^{\frac{3}{4}},$$

which is positive for large n since $\|x_0 - z\|_\infty = \lfloor n/2 \rfloor$ and $\|x_0 - y\|_\infty \le c_1 n$ (assuming we take c_1 fixed but small). Therefore

$$\mathbf{P}(T(x_0, z) \le T(x_0, y)) \le \mathbf{P}\left(|T(x_0, y) - g(x_0 - y)| \ge n^{\frac{3}{4}}\right) + \mathbf{P}\left(|T(x_0, z) - g(x_0 - z)| \ge n^{\frac{3}{4}}\right).$$

Applying both Alexander's bound and (2.5), we obtain for some $c_4 > 0$

$$\mathbf{P}\left(|T(x_0, y) - g(x_0 - y)| \ge n^{\frac{3}{4}}\right) \le C_2 \exp\left(-c_2 \min\left\{\frac{n^{\frac{3}{2}}}{|x_0 - y|}, n^{\frac{3}{4}}\right\}\right) \le C_2 e^{-c_4 n^{\frac{1}{2}}},$$

with the same bound for $\mathbf{P}(|T(x_0, z) - g(x_0 - z)| \ge n^{3/4})$. Putting this back in (2.8), we find that the left side of (2.6) is bounded below by

$$1 - 2C_2 \sum_{\substack{y : \|y - x_0\|_\infty \le c_1 n \\ z : \|x_0 - z\|_\infty = \lfloor \frac{n}{2} \rfloor}} \exp\left(-c_4 n^{\frac{1}{2}}\right) \ge 1 - C_3 n^3 \exp\left(-c_4 n^{\frac{1}{2}}\right).$$

This is bounded below by the right side of (2.6), if we choose $b < 1/2$. This shows (2.6) and, along with (2.4), completes the proof. □

Proof of Proposition 2.3: The proof will analyze the local behavior of distance-minimizing paths, where there is no difference between the periodic and full-plane weights. Let $\alpha > 3/4$; we may additionally assume that $\alpha < 1$. Choose

$$\epsilon \in \left(0, 2\alpha - \frac{3}{2}\right) \tag{2.9}$$

and fix an outcome in the event in the probability in Lemma 2.4 corresponding to this ϵ. Let Γ_n be a geodesic for T_n^{tube}. We will show that all vertices in Γ_n are contained in the strip $x_0 +$

$(\mathbb{Z} \times [-C_4 n^\alpha, C_4 n^\alpha])$ for $C_4 = 1 + 4/c_1$. (Here, c_1 is from Lemma 2.4.) Because of the bound for the probability in Lemma 2.4, this will prove Proposition 2.3, after slightly increasing α.

To begin, we define a sequence of points (x_i) on Γ_n inductively as follows. Let x_0 be the initial point of Γ_n. For $i \geq 0$, define the rectangle

$$\mathsf{C}_i = x_i + ([-\lfloor n^\alpha \rfloor, \lfloor c_1 n \rfloor] \times [-\lfloor n^\alpha \rfloor, \lfloor n^\alpha \rfloor]),$$

where c_1 is from Lemma 2.4, its boundary

$$\mathsf{B}_i = \{x \in \mathsf{C}_i : \exists y \in \mathbb{Z}^2 \setminus \mathsf{C}_i \text{ with } |x - y| \leq 1\},$$

and its right boundary

$$\mathsf{R}_i = \{x \in \mathsf{B}_i : (x - x_i) \cdot \mathbf{e}_1 = \lfloor c_1 n \rfloor\}.$$

For $i \geq 1$, let x_i be the first vertex of Γ_n after x_{i-1} that lies in B_{i-1}, if one exists. If it does not exist, we set x_i equal to the terminal point of Γ_n. Let N be the first i such that x_i equals this terminal point. We claim that

$$\text{for all } i \in \{1, \ldots, N-1\},\ x_i \in \mathsf{R}_{i-1}. \tag{2.10}$$

To prove (2.10), we use the linear functional $h : \mathbf{R}^2 \to \mathbf{R}$ defined by $h(z) = (z \cdot \mathbf{e}_1) g(\mathbf{e}_1)$. Observe that if we define $\bar{z} = z - 2(z \cdot \mathbf{e}_2)\mathbf{e}_2$, then by symmetry, $g(z) \geq (1/2) g(z + \bar{z}) = (z \cdot \mathbf{e}_1) g(\mathbf{e}_1) = h(z)$. For certain z, though, we have a stronger bound. There exists $c_5 > 0$ such that for large n,

$$g(z) - h(z) \geq c_5 n^{2\alpha - 1} \text{ if } z \in \mathsf{B}_0 \setminus \mathsf{R}_0. \tag{2.11}$$

We first prove this inequality, and then return to the proof of (2.10). Let $z \in \mathsf{B}_0 \setminus \mathsf{R}_0$. Notice that if $z \cdot \mathbf{e}_1 \leq 0$, then $h(z) \leq 0$, so for some $c_6 > 0$, if n is large,

$$g(z) - h(z) \geq g(z) \geq c_6 \|z\|_\infty \geq \frac{c_6}{4} n^\alpha. \tag{2.12}$$

If $z \cdot \mathbf{e}_1 > 0$, then we must have $|z \cdot \mathbf{e}_2| \geq n^\alpha/2$ for large n, and so $|(z \cdot \mathbf{e}_2)/(z \cdot \mathbf{e}_1)| \geq n^{\alpha-1}/(2c_1)$. Write

$$\begin{aligned} g(z) - h(z) &= g((z \cdot \mathbf{e}_1)\mathbf{e}_1 + (z \cdot \mathbf{e}_2)\mathbf{e}_2) - g((z \cdot \mathbf{e}_1)\mathbf{e}_1) \\ &= (z \cdot \mathbf{e}_1) \left(g\left(\mathbf{e}_1 + \frac{z \cdot \mathbf{e}_2}{z \cdot \mathbf{e}_1} \mathbf{e}_2 \right) - g(\mathbf{e}_1) \right). \end{aligned} \tag{2.13}$$

If $|(z \cdot \mathbf{e}_2)/(z \cdot \mathbf{e}_1)| < \epsilon_0$, then we can use Assumption 1.1 for the lower bound

$$g(z) - h(z) \geq c_0 (z \cdot \mathbf{e}_1) \left| \frac{z \cdot \mathbf{e}_2}{z \cdot \mathbf{e}_1} \right|^2 \geq c_0 \frac{n^{\alpha-1}}{2c_1} |z \cdot \mathbf{e}_2| \geq \frac{c_0}{4c_1} n^{2\alpha-1}. \tag{2.14}$$

If $|z \cdot \mathbf{e}_2|/|z \cdot \mathbf{e}_1| \geq \epsilon_0$, then we use a modified curvature inequality: for β such that $|\beta| \geq \epsilon_0$, we have

$$g(\mathbf{e}_1 + \beta \mathbf{e}_2) - g(\mathbf{e}_1) \geq c_0 \epsilon_0 |\beta|. \tag{2.15}$$

To see why this holds, assume by symmetry that $\beta > 0$ and set $\beta' = \epsilon_0$. By convexity of g, $g(\mathbf{e}_1 + \beta' \mathbf{e}_2) \leq (\beta'/\beta) g(\mathbf{e}_1 + \beta \mathbf{e}_2) + (1 - \beta'/\beta) g(\mathbf{e}_1)$, and this gives $g(\mathbf{e}_1 + \beta \mathbf{e}_2) - g(\mathbf{e}_1) \geq (\beta/\beta')(g(\mathbf{e}_1 + \beta' \mathbf{e}_2) - g(\mathbf{e}_1))$. By Assumption 1.1, this implies

$$g(\mathbf{e}_1 + \beta \mathbf{e}_2) - g(\mathbf{e}_1) \geq c_0 \frac{\beta}{\beta'} (\beta')^2 = c_0 \epsilon_0 \beta,$$

which is (2.15). Now, in the case that $|z \cdot \mathbf{e}_2|/|z \cdot \mathbf{e}_1| \geq \epsilon_0$, we apply (2.15) in (2.13) to obtain for large n

$$g(z) - h(z) \geq c_0 \epsilon_0 |z \cdot \mathbf{e}_2| \geq \frac{c_0 \epsilon_0}{2} n^\alpha. \tag{2.16}$$

Combining the three cases (2.12), (2.14), and (2.16), and observing that $\alpha > 2\alpha - 1$, we conclude (2.11).

Having established (2.11), we can return to showing (2.10). Let $I_R = \{i = 1, \ldots, N-1 : x_i \in \mathsf{R}_{i-1}\}$ and $J_R = \{1, \ldots, N-1\} \setminus I_R$; we want to show that $\#J_R = 0$. Write

$$\begin{aligned} T_n^{\text{tube}} - g(n\mathbf{e}_1) &= \sum_{i=1}^{N} (T^{(n)}(x_{i-1}, x_i) - h(x_{i-1} - x_i)) \\ &= \sum_{i=1}^{N} (T^{(n)}(x_{i-1}, x_i) - g(x_{i-1} - x_i)) + \sum_{i=1}^{N} (g(x_{i-1} - x_i) - h(x_{i-1} - x_i)). \end{aligned}$$

Because our outcome is in the event described in Lemma 2.4 and $\|x_i - x_{i-1}\|_\infty \le c_1 n$, we have $|T^{(n)}(x_{i-1}, x_i) - g(x_{i-1} - x_i)| \le n^{1/2+\epsilon}$ for all i. For $i \in J_R$, we apply (2.11), and for $i \notin J_R$, we use $g \ge h$. All together, we obtain

$$T_n^{\text{tube}} - g(n\mathbf{e}_1) \ge -n^{\frac{1}{2}+\epsilon} N + c_5 n^{2\alpha-1} \# J_R.$$

To bound T_n^{tube} from above, we construct a path by starting at 0, moving to $\lfloor c_1 n \rfloor \mathbf{e}_1$ using a geodesic for $T^{(n)}(0, \lfloor c_1 n \rfloor)$, then moving to $2\lfloor c_1 n \rfloor$ using a geodesic for $T^{(n)}(\lfloor c_1 n \rfloor, 2\lfloor c_1 n \rfloor)$, and so on, until we reach the largest multiple of $\lfloor c_1 n \rfloor$ that is at most n. After this, we move to $n\mathbf{e}_1$. Using the fact that g is additive along the $\mathbf{e}_1$-axis, we may apply the condition in Lemma 2.4 at each step to obtain $T_n^{\text{tube}} - g(n\mathbf{e}_1) \le (1/c_1 + 1) n^{1/2+\epsilon}$. Combining with the above produces

$$\begin{aligned} \left(\frac{1}{c_1} + 1\right) n^{\frac{1}{2}+\epsilon} &\ge -n^{\frac{1}{2}+\epsilon} N + c_5 n^{2\alpha-1} \# J_R \\ &= -n^{\frac{1}{2}+\epsilon} (\# I_R + 1) + \left(c_5 n^{2\alpha-1} - n^{\frac{1}{2}+\epsilon}\right) \# J_R. \end{aligned}$$

This implies

$$\# I_R + \frac{1}{c_1} + 2 \ge \left(c_5 n^{2\alpha - \frac{3}{2} - \epsilon} - 1\right) \# J_R. \tag{2.17}$$

To relate $\#I_R$ and $\#J_R$ in a different way, we look at the progression of each segment of Γ_n in the $\mathbf{e_1}$ direction. For each $i \in J_R \cup \{N\}$, we have $(x_i - x_{i-1}) \cdot \mathbf{e}_1 \ge -n^\alpha$ and for each $i \in I_R$, we have $(x_i - x_{i-1}) \cdot \mathbf{e}_1 = \lfloor c_1 n \rfloor$. Therefore

$$n = \sum_{i=1}^{N} (x_i - x_{i-1}) \cdot \mathbf{e_1} \ge \lfloor c_1 n \rfloor \# I_R - n^\alpha (\# J_R + 1). \tag{2.18}$$

For large n, this gives $1 + n^{\alpha-1}(\#J_R + 1) \ge (c_1/2)\#I_R$. Combining this with (2.17), we find

$$\frac{2}{c_1}\left(1 + n^{\alpha-1}(\#J_R + 1)\right) + \frac{1}{c_1} + 2 \ge \left(c_5 n^{2\alpha - \frac{3}{2} - \epsilon} - 1\right) \# J_R.$$

Recall that $\alpha < 1$ but, because of (2.9), we have $2\alpha - 3/2 - \epsilon > 0$. This inequality therefore cannot hold for large n unless $\#J_R = 0$. This proves (2.10).

Now that we have shown (2.10), we can quickly complete the proof of Proposition 2.3. Because $\#J_R = 0$, (2.18) gives $n \ge \lfloor c_1 n \rfloor \# I_R - n^\alpha$, with $\#I_R = N - 1$, and so for large n, we have $n \ge (c_1/2)n(N-1) - n$, or $N \le 1 + 4/c_1$. But then

$$\Gamma_n \subset \cup_{i=0}^{N-1} \mathsf{C}_i \subset x_0 + (\mathbb{Z} \times [-N n^\alpha, N n^\alpha]),$$

and this shows Proposition 2.3. □

3. Asymptotics for cylinder times

Because of the fact, from Proposition 2.3, that geodesics for T_n^{tube} are contained in cylinders, we are led to study passage times across cylinders. Our analysis will crucially rely on the Markov property that comes from our exponential weights. So we begin this section with a description of an alternate representation of the model.

Let $n, K \geq 1$ be integers and consider the cylinder $\mathsf{C}_{n,K} = [0,n] \times [0,K]$ with $K \leq n-1$. For vertices $x, y \in \mathsf{C}_{n,K}$, we define $T_{n,K}(x,y)$ as $\inf_{\Gamma: x \to y} T(\Gamma)$ where the infimum is over all paths with vertices in $\mathsf{C}_{n,K}$ from x to y. (Here we can use the weights (t_e) or $(t_e^{(n)})$ since they have the same distribution in $\mathsf{C}_{n,K}$, but for definiteness we now use (t_e).) Then for $y \in \mathsf{C}_{n,K}$, we put

$$T_{n,K}(y) = \inf_{x \in \mathsf{C}_{n,K} : x \cdot \mathbf{e}_1 = 0} T_{n,K}(x,y)$$

and

$$T_{n,K} = \inf_{y \in \mathsf{C}_{n,K} : y \cdot \mathbf{e}_1 = n} T_{n,K}(y). \tag{3.1}$$

The main observation behind the alternate representation is that, by the memoryless property of the exponential distribution, the sets $\{y \in \mathsf{C}_{n,K} : T_{n,K}(y) \leq t\}$ evolve as a Markov process as t grows. Following the setup of Pemantle and Peres (1994, p. 2), we may build this process in two steps. First we grow a sequence of subgraphs of the cylinder as follows. Let $\mathcal{C}_0 = \{x \in \mathsf{C}_{n,K} : x \cdot \mathbf{e}_1 = 0\}$ with set of boundary edges $\mathcal{B}_0 = \{\{x,y\} : x \in \mathcal{C}_0, y \in \mathsf{C}_{n,K} \setminus \mathcal{C}_0, |x-y| = 1\}$. For $i \geq 0$, choose an edge e_{i+1} uniformly from $\mathcal{B}_i$ and, writing $e_{i+1} = \{x_{i+1}, y_{i+1}\}$, where $x_{i+1} \in \mathcal{C}_i$, set $\mathcal{C}_{i+1} = \mathcal{C}_i \cup \{y_{i+1}\}$ and $\mathcal{B}_{i+1} = \{\{x,y\} : x \in \mathcal{C}_{i+1}, y \in \mathsf{C}_{n,K} \setminus \mathcal{C}_{i+1}, |x-y| = 1\}$. These sequences are defined until the value of $i = n(K+1)$ at which $\mathcal{C}_{n(K+1)} = \mathsf{C}_{n,K} \cap \mathbb{Z}^2$. In words, this is a random growth algorithm (a Richardson-type model Auffinger et al. (2017, Ch. 6)) in which we begin with a seed on the entire left side of the cylinder. At each timestep, the growth absorbs a uniformly chosen edge from its boundary in the cylinder. At some point we have touched the right side: define

$$\mathcal{N} = \min\{i : y_i \cdot \mathbf{e}_1 = n\}.$$

In the second stage of the process, we fix an outcome as above with sets $\mathcal{C}_i, \mathcal{B}_i$, and vertices x_i, y_i. Set $b_i = \#\mathcal{B}_{i-1}$ and let $X_1, X_2, \ldots, X_{\mathcal{N}}$ be independent exponential random variables such that $\mathbf{E}X_i = b_i^{-1}$. Then

$$T_{n,K} \text{ has the same distribution as } X_1 + \cdots + X_{\mathcal{N}}. \tag{3.2}$$

As in Pemantle and Peres (1994), "This is an immediate consequence of the lack of memory of the exponential distribution and of the fact that the minimum of n independent exponentials of mean 1 is an exponential of mean $1/n$." Another fact that follows directly from the representation is that

$$\mathcal{N} \text{ has the same distribution as } \#\{y \in \mathsf{C}_{n,K} : 0 < T_{n,K}(y) \leq T_{n,K}\}. \tag{3.3}$$

In the following two subsections, we first prove bounds on the reals b_i, and then use them, along with an entropic central limit theorem, to bound the rate of convergence of $X_1 + \cdots + X_{\mathcal{N}}$ (given the sequence (b_i) and $\mathcal{N}$) to a standard normal distribution. This will allow us in the third subsection to estimate the fluctuations of the minimum of i.i.d. copies of $T_{n,K}$.

3.1. *Boundary of the growth.* To estimate $\mathcal{N}$ and the b_i's, we will use Kesten's lemma, which can be found in Auffinger et al. (2017, Lemma 4.5).

Lemma 3.1. *There exist $c_7, a > 0$ such that for all $k \geq 1$,*

$$\mathbf{P}(\exists \text{ vertex self-avoiding } \gamma \text{ from } 0 \text{ with } \#\gamma \geq k \text{ but } T(\gamma) \leq ak) \leq e^{-c_7 k}.$$

First we give estimates for $\mathcal{N}$. The upper bound $\mathcal{N} \leq n(K+1)$ is immediate. For a lower bound, we have the following.

Lemma 3.2. *There exists $c_8 > 0$ such that for all large n and all $K \in [1, n-1]$,*

$$\mathbf{P}\left(\mathcal{N} \geq \frac{a}{2} nK\right) \geq 1 - e^{-c_8 n}.$$

Proof: By (3.3), it suffices to show that for large n,

$$\mathbf{P}\left(\#\{y \in \mathsf{C}_{n,K} : 0 < T_{n,K}(y) \leq T_{n,K}\} < \frac{a}{2} nK\right) \leq e^{-c_8 n}. \tag{3.4}$$

If $T_{n,K} \leq an$, then there exists a vertex self-avoiding γ starting from the left side of the cylinder with $\#\gamma \geq n$ but $T(\gamma) \leq an$. By Lemma 3.1 and a union bound,

$$\mathbf{P}(T_{n,K} \leq an) \leq (K+1)e^{-c_7 n}.$$

Therefore

$$\begin{aligned}&\mathbf{P}\left(\#\{y \in \mathsf{C}_{n,K} : 0 < T_{n,K}(y) \leq T_{n,K}\} < \frac{a}{2} nK\right) \\ &\leq (K+1)e^{-c_7 n} + \mathbf{P}\left(\#\{y \in \mathsf{C}_{n,K} : 0 < T_{n,K}(y) \leq an\} < \frac{a}{2} nK\right).\end{aligned} \tag{3.5}$$

We will prove that with high probability, the set of y in (3.5) contains $([1, (a/2)n] \times [0, K]) \cap \mathbb{Z}^2$. Suppose that y is in this latter set, and construct a deterministic path γ_y by starting at y and proceeding to $y - \mathbf{e}_1$, then to $y - 2\mathbf{e}_1$, and so on, in a straight line until we reach $(y \cdot \mathbf{e}_2)\mathbf{e}_2$. Then $\#\gamma_y$, the number of edges in γ_y, satisfies $\#\gamma_y \leq (a/2)n$ and we have $T_{n,K}(y) \leq T(\gamma_y)$, so for $\eta > 0$, we can use Markov's inequality to obtain

$$\begin{aligned}\mathbf{P}(T_{n,K}(y) \geq an) \leq \mathbf{P}\left(e^{\eta T(\gamma_y)} \geq e^{\eta an}\right) &\leq e^{-\eta an}\left(\mathbf{E}e^{\eta t_e}\right)^{\#\gamma_y} \\ &\leq \left(e^{-\eta}\left(\mathbf{E}e^{\eta t_e}\right)^{\frac{1}{2}}\right)^{an} \\ &= \left(\frac{e^{-2\eta}}{1-\eta}\right)^{\frac{a}{2}n}.\end{aligned} \tag{3.6}$$

We fix η to be small so that $b' := e^{-2\eta}/(1-\eta) < 1$ and conclude that by a union bound and the fact that $K \leq n$,

$$\mathbf{P}\left(T_{n,K}(y) \geq an \text{ for some } y \in \left[1, \frac{a}{2}n\right] \times [0, K]\right) \leq (K+1)\left(\frac{a}{2}n\right)(b')^{\frac{a}{2}n} \leq e^{-c_9 n}$$

for some $c_9 > 0$, so long as n is large. The event in the probability above is implied by the event in the probability in (3.5), so we obtain

$$\mathbf{P}\left(\#\{y \in \mathsf{C}_{n,K} : 0 < T_{n,K}(y) \leq T_{n,K}\} < \frac{a}{2} nK\right) \leq (K+1)e^{-c_7 n} + e^{-c_9 n} \leq e^{-c_{10} n}$$

for some $c_{10} > 0$ if n is large. This shows (3.4). □

Next we estimate the boundary sizes b_i.

Lemma 3.3. *For all $i = 1, \ldots, \mathcal{N}$, we have $b_i \geq K+1$. Furthermore, there exists $c_{11} > 0$ such that for all large n and all $K \in [1, n-1]$,*

$$\mathbf{P}\left(\#\left\{i = 1, \ldots, \mathcal{N} : b_i \geq \frac{4}{a}K\right\} \geq \left(1 - \frac{a}{2}\right)\mathcal{N}\right) \leq e^{-c_{11} n}.$$

Proof: For a fixed i and any $m = 0, \ldots, K$, choose $u_m \in \mathcal{C}_{i-1}$ to have $u_m \cdot \mathbf{e}_2 = m$ but with $u_m \cdot \mathbf{e}_1$ maximal. Then $\{u_m, u_m + \mathbf{e}_1\}$ is an edge of $\mathcal{B}_{i-1}$, so $b_i \geq K+1$.

For the other bound, write A for the event in the probability in the statement. We split A according to the passage time T_n^{tube}. Let Ξ be the set of pairs $((d_i), M)$, where $d_i \in \mathbf{N}$ for $i \geq 1$ and $M \in \mathbf{N}$, such that all of the following hold:

- $M \leq n(K+1)$,

- $d_i \geq K+1$ for all i, and
- $\#\{i=1,\ldots,M : d_i \geq 4K/a\} \geq (1-a/2)\,M$.

Then

$$\begin{aligned}\mathbf{P}(A) &\leq \mathbf{P}(A, X_1+\cdots+X_{\mathcal{N}} \geq an) + \mathbf{P}(X_1+\cdots+X_{\mathcal{N}} \leq an)\\ &\leq \sum_{((d_i),M)\in\Xi} \mathbf{P}(X_1+\cdots+X_{\mathcal{N}} \geq an, (b_i)=(d_i), \mathcal{N}=M) + \mathbf{P}(T_{n,K} \leq an).\end{aligned}$$

As in the proof of Lemma 3.2, we have $\mathbf{P}(T_{n,K} \leq an) \leq e^{-c_7 n}$ for large n. Therefore we obtain

$$\begin{aligned}\mathbf{P}(A) &\leq e^{-c_7 n}\\ &\quad + \sum_{((d_i),M)\in\Xi} \mathbf{P}(X_1+\cdots+X_{\mathcal{N}} \geq an \mid (b_i)=(d_i), \mathcal{N}=M)\mathbf{P}((b_i)=(d_i), \mathcal{N}=M). \qquad (3.7)\end{aligned}$$

Conditional on $(b_i)=(d_i)$ and $\mathcal{N}=M$, the variables X_i are independent exponentials with parameters d_i. So, for (Y_i) that are i.i.d. exponential with mean 1, we have

$$\mathbf{P}(X_1+\cdots+X_{\mathcal{N}} \geq an \mid (b_i)=(d_i), \mathcal{N}=M) = \mathbf{P}\left(\frac{Y_1}{d_1}+\cdots+\frac{Y_M}{d_M} \geq an\right).$$

We now split the sum depending on whether indices are in the set $S=\{i=1,\ldots,M : d_i \geq (4/a)K\}$. We obtain

$$\begin{aligned}&\mathbf{P}(X_1+\cdots+X_{\mathcal{N}} \geq an \mid (b_i)=(d_i), \mathcal{N}=M)\\ &= \mathbf{P}\left(\sum_{i\in S}\frac{Y_i}{d_i}+\sum_{i\notin S}\frac{Y_i}{d_i} \geq an\right)\\ &\leq \mathbf{P}\left(\frac{a}{4K}\sum_{i\in S} Y_i \geq \frac{a}{3}n\right) + \mathbf{P}\left(\frac{1}{K+1}\sum_{i\notin S} Y_i \geq \frac{2a}{3}n\right)\\ &\leq \mathbf{P}\left(\sum_{i=1}^{n(K+1)} Y_i \geq \frac{4}{3}nK\right) + \mathbf{P}\left(\sum_{i=1}^{\lceil n(K+1)\frac{a}{2}\rceil} Y_i \geq \frac{2}{3}an(K+1)\right). \qquad (3.8)\end{aligned}$$

By standard large deviation estimates for i.i.d. exponentials, there exists $c_{12}>0$ such that for all $k \geq 1$, we have $\mathbf{P}(Y_1+\cdots+Y_k \geq 7k/6) \leq e^{-c_{12}k}$. So for large n, we can bound (3.8) by

$$e^{-c_{12}n(K+1)} + e^{-c_{12}\lceil n(K+1)\frac{a}{2}\rceil} \leq e^{-c_{13}n},$$

for some $c_{13}>0$. Plug this result back into (3.7) to obtain

$$\mathbf{P}(A) \leq e^{-c_7 n} + \sum_{((d_i),M)\in\Xi} e^{-c_{13}n}\mathbf{P}((b_i)=(d_i), \mathcal{N}=M) \leq e^{-c_7 n}+e^{-c_{13}n}.$$

For large n, this is bounded by $e^{-c_{11}n}$, if $c_{11} < \min\{c_7, c_{13}\}$. This completes the proof. □

3.2. *Conditional CLT.* In this section, we prove a central limit theorem for $T_{n,K}$ conditional on the sequence (b_i) and the number $\mathcal{N}$. The main tool is a theorem from Artstein et al. (2004) which bounds the total variation distance between linear combinations of independent variables and a standard normal variable. To state it, we give some terminology. Recall that an exponential variable X with parameter 1 satisfies a Poincaré inequality with constant 1/4. Namely, for any smooth $f : \mathbf{R} \to \mathbf{R}$,

$$\mathrm{Var}(f(X)) \leq 4\,\mathbf{E}(f'(X)^2).$$

Recall that the entropy of a random variable X with density function f is defined as

$$\mathrm{Ent}(X) := -\int_{\mathbf{R}} f(x)\log f(x)\ \mathrm{d}x,$$

and that

$$\mathrm{Ent}(\mathrm{Exp}(\lambda)) = 1-\log\lambda \quad \text{and} \quad \mathrm{Ent}(Z) = \frac{\log 2\pi + 1}{2}, \tag{3.9}$$

where Z is the standard normal random variable. Last, the total variation distance between two probability measures μ and ν is

$$\mathrm{d_{TV}}(\mu,\nu) = \sup_B |\mu(B)-\nu(B)|,$$

where the supremum is over all Borel sets $B\subset\mathbf{R}$. The total variation distance between two random variables is defined as the total variation distance between their distributions. With these notations, we have the following estimate.

Lemma 3.4. *Let $\{W_i\}_{i=1}^N$ be independent copies of a random variable W which satisfies a Poincaré inequality with some constant $c>0$. Let $\{a_i\}_{i=1}^N$ be such that $\sum_{i=1}^N a_i^2 = 1$ and let $S_N = \sum_{i=1}^N a_i W_i$. Letting Z be a standard normal random variable, we have*

$$\mathrm{d_{TV}}(S_N - \mathbf{E}(S_N), Z)^2 \le 2\frac{\sum_{i=1}^N a_i^4}{\frac{c}{2} + \left(1-\frac{c}{2}\right)\sum_{i=1}^N a_i^4}(\mathrm{Ent}(Z) - \mathrm{Ent}(W)).$$

Proof: This is Artstein et al. (2004, Theorem 1) combined with the inequality $\mathrm{d_{TV}}(S_N - \mathbf{E}S_N, Z)^2 \le 2[\mathrm{Ent}(S_N-\mathbf{E}S_N) - \mathrm{Ent}(Z)]$ in Artstein et al. (2004, Eq. (1)). □

We will apply the lemma to the variables $X_1,\dots,X_{\mathcal{N}}$, but to make them independent, we must condition on (b_i) and $(\mathcal{N})$. For this purpose, we define the admissible set Υ of pairs $((d_i),M)$, where $d_i\in\mathbf{N}$ for $i\ge 1$ and $M\in\mathbf{N}$, by the following conditions:

(1) $anK/2 \le M \le n(K+1)$,
(2) $d_i \ge K+1$ for all i, and
(3) $\#\{i=1,\dots,M : d_i \le 4K/a\} \ge aM/2$.

Summarizing the previous section, if we combine Lemmas 3.2 and 3.3, we find that for all large n and all $K\in[1,n-1]$,

$$\mathbf{P}(((b_i),\mathcal{N})\in\Upsilon) \ge 1-e^{-c_{14}n} \tag{3.10}$$

for $c_{14} = (1/2)\min\{c_8, c_{11}\}$. Next we define the conditional distribution

$$\mu_{n,K}^{(d_i),M}(B) = \mathbf{P}\left(\frac{\sum_{i=1}^{\mathcal{N}}(X_i - b_i^{-1})}{\sqrt{\sum_{i=1}^{\mathcal{N}} b_i^{-2}}} \in B \;\middle|\; (b_i) = (d_i), \mathcal{N} = M\right) \text{ for Borel } B\subset\mathbf{R}.$$

Of course, given $(b_i) = (d_i)$ and $\mathcal{N} = M$, the X_i's are just independent exponentials with mean $b_i^{-1} = d_i^{-1}$.

Proposition 3.5. *For $((d_i),M)\in\Upsilon$, we have*

$$\mathrm{d_{TV}}(\mu_{n,K}^{(d_i),M}, \mu_G)^2 \le (\log 2\pi - 1)\frac{2^{15}}{a^8 n(K+1)},$$

where μ_G is the standard Gaussian distribution. Consequently there exists a probability measure $\mathbf{Q}$ and random variables U,Z on some space such that under $\mathbf{Q}$, U has distribution $\mu_{n,K}^{(d_i),M}$, Z is a standard Gaussian, and

$$\mathbf{Q}(U\ne Z) \le \sqrt{(\log 2\pi - 1)\frac{2^{15}}{a^8 n(K+1)}}.$$

Proof: The second statement is standard and follows from the coupling representation of total variation distance. To show the first statement, we apply Lemma 3.4. Given $(b_i) = (d_i)$ and $\mathcal{N} = M$,

$$\frac{\sum_{i=1}^{\mathcal{N}}(X_i - b_i^{-1})}{\sqrt{\sum_{i=1}^{\mathcal{N}} b_i^{-2}}} = \frac{\sum_{i=1}^{M} \frac{Y_i}{d_i}}{\sqrt{\sum_{i=1}^{M} d_i^{-2}}} - \mathbf{E}\frac{\sum_{i=1}^{M} \frac{Y_i}{d_i}}{\sqrt{\sum_{i=1}^{M} d_i^{-2}}} \text{ in distribution,}$$

where the Y_i are i.i.d. exponential random variables with mean 1. So we set $W_i = Y_i$ and $a_i = d_i^{-1}/\sqrt{\sum_{i=1}^{M} d_i^{-2}}$ in the lemma. By items (1) and (2) in the definition of Υ,

$$\sum_{i=1}^{M} d_i^{-4} \leq \frac{1}{(K+1)^4} \cdot n(K+1) = n(K+1)^{-3}.$$

By items (1) and (3),

$$\left(\sum_{i=1}^{M} d_i^{-2}\right)^2 \geq \left(\frac{a^2}{16K^2} \cdot \frac{aM}{2}\right)^2 \geq \left(\frac{a^4 n}{64K}\right)^2. \tag{3.11}$$

Combining these produces

$$\sum_{i=1}^{M} a_i^4 = \frac{\sum_{i=1}^{M} d_i^{-4}}{\left(\sum_{i=1}^{M} d_i^{-2}\right)^2} \leq \frac{n(K+1)^{-3}}{\left(\frac{a^4 n}{64K}\right)^2} \leq \frac{64^2}{a^8 n(K+1)}.$$

Using Poincaré constant 1/4 in Lemma 3.4, we obtain the following bound for the right side in the lemma:

$$\begin{aligned} 2\frac{\sum_{i=1}^{M} a_i^4}{\frac{c}{2} + \left(1 - \frac{c}{2}\right)\sum_{i=1}^{M} a_i^4}(\mathrm{Ent}(Z) - \mathrm{Ent}(W)) &\leq \frac{\log 2\pi - 1}{2} \cdot \frac{4}{c}\sum_{i=1}^{M} a_i^4 \\ &\leq (\log 2\pi - 1)\frac{2^{15}}{a^8 n(K+1)}. \end{aligned}$$

This completes the proof. □

3.3. *Fluctuation bounds for independent cylinder times.* Here we use the results of the last two subsections to prove a fluctuation lower bound for the minimum of passage times across disjoint cylinders. For $n \geq 1$, pick integers $K_n^{(1)}, \ldots, K_n^{(r_n)} \in [1, n]$ such that $\sum_{j=1}^{r_n} K_n^{(j)} = n$. Define cylinders as $\mathsf{C}^{(1)} = [0, n] \times [0, K_n^{(1)} - 1]$ and for $j = 2, \ldots, r_n$,

$$\mathsf{C}^{(j)} = [0, n] \times [K_n^{(1)} + \cdots + K_n^{(j-1)}, K_n^{(1)} + \cdots + K_n^{(j)} - 1].$$

For each j, we set $\mathcal{T}^{(j)}$ to be the corresponding cylinder passage time. It is the minimal passage time of any path in $\mathsf{C}^{(j)}$ connecting the left and right sides, defined analogously to (3.1). Because the cylinders are disjoint, the $\mathcal{T}^{(j)}$ are independent. Last, put

$$\mathcal{T}_n = \mathcal{T}_{n,(K_n^{(j)})} = \min\{\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(r_n)}\}.$$

Proposition 3.6. *Suppose that $n^{-1/2}\sum_{j=1}^{r_n}(K_n^{(j)})^{-1/2} \to 0$. Then*

$$(\mathcal{T}_n) \textit{ has fluctuations of at least order } \min_{j=1,\ldots,r_n} \sqrt{\frac{n}{K_n^{(j)}(1 + \log r_n)}}.$$

The form of the lower bound can be understood as follows. The j-th cylinder passage time, $\mathcal{T}^{(j)}$, has been shown in Damron et al. (2020) (see also Bates and Chatterjee (2020) for arguments that imply this fact) to have fluctuations at least of order $\sqrt{n/K_n^{(j)}}$. Pretending for the moment that this time is Gaussian distributed, then $\mathcal{T}_n$ would be the minimum of r_n many Gaussian random variables, the j-th of which has variance of order $n/K_n^{(j)}$. In the appendix, we will show that the fluctuations of the minimum of r_n many Gaussian random variables is at least order $(\min_{j=1,\ldots,r_n}\sigma_j)/\sqrt{1+\log r_n}$, where σ_j^2 is the variance of the j-th variable. (The logarithmic factor appears also in the well-known value of the fluctuations of the minimum of i.i.d. Gaussian random variables.) In total, our fluctuation lower bound becomes $(\min_{j=1,\ldots,r_n}\sqrt{n/(K_n^{(j)}(1+\log r_n)})$. Of course, the times $\mathcal{T}^{(j)}$ are not Gaussian distributed, so we need to apply the coupling results we have developed over the previous sections.

Proof: We represent the $\mathcal{T}^{(j)}$ as in (3.2), obtaining boundary sequences $(b_i^{(j)})$ and reals $(\mathcal{N}^{(j)})$ such that the pairs $((b_i^{(j)}),\mathcal{N}^{(j)})$ are independent as j varies. We also find variables $X_i^{(j)}$ for $j=1,\ldots,r_n$ and $i=1,\ldots,\mathcal{N}^{(j)}$ such that conditional on the pairs, the $X_i^{(j)}$ are independent exponentials with means $\mathbf{E}X_i^{(j)}=1/b_i^{(j)}$. Last,

$$\sum_{i=1}^{\mathcal{N}^{(j)}} X_i^{(j)} = \mathcal{T}^{(j)} \text{ in distribution for all } j=1,\ldots,r_n,$$

and so

$$\min_{j=1,\ldots,r_n}\sum_{i=1}^{\mathcal{N}^{(j)}} X_i^{(j)} = \mathcal{T}_n \text{ in distribution.}$$

We define corresponding admissible pairs $\Upsilon^{(j)}$: they are those $((d_i^{(j)}),M^{(j)})$ such that

(1) $anK_n^{(j)}/2 \le M^{(j)} \le n(K_n^{(j)}+1)$,
(2) $d_i^{(j)} \ge K_n^{(j)}+1$ for all i, and
(3) $\#\{i=1,\ldots,M^{(j)} : d_i^{(j)} \le 4K_n^{(j)}/a\} \ge aM^{(j)}/2$.

Then by (3.10), for all large n, and all choices of the $K_n^{(j)}$ as above,

$$\mathbf{P}((b_i^{(j)}),\mathcal{N}^{(j)}) \in \Upsilon^{(j)} \text{ for all } j=1,\ldots,r_n) \ge 1-r_ne^{-c_{14}n}. \tag{3.12}$$

With these definitions, we compute for any Borel set $B\subset\mathbf{R}$

$$\begin{aligned}&\mathbf{P}(\mathcal{T}_n\in B)\\ &\ge \sum_{\Upsilon^{(1)}\times\cdots\times\Upsilon^{(r_n)}}\left[\mathbf{P}\left(\min_{j=1,\ldots,r_n}\sum_{i=1}^{\mathcal{N}^{(j)}}X_i^{(j)}\in B \,\middle|\, ((b_i^{(j)}),\mathcal{N}^{(j)})=((d_i^{(j)}),M^{(j)}) \text{ for all } j\right)\right. \\ &\left.\times\,\mathbf{P}(((b_i^{(j)}),\mathcal{N}^{(j)})=((d_i^{(j)}),M^{(j)}) \text{ for all } j)\right].\end{aligned} \tag{3.13}$$

Write $\mu^{(j)}=\sum_{i=1}^{\mathcal{N}^{(j)}}(b_i^{(j)})^{-1}$ and $\sigma^{(j)}=\sqrt{\sum_{i=1}^{\mathcal{N}^{(j)}}(b_i^{(j)})^{-2}}$, and then set

$$\mathcal{L}^{(j)}=\frac{\sum_{i=1}^{\mathcal{N}^{(j)}}X_i^{(j)}-\mu^{(j)}}{\sigma^{(j)}}.$$

We observe that under the conditional distribution appearing in (3.13), the vector $\left(\mathcal{L}^{(j)}\right)_{j=1}^{r_n}$ has product distribution $\prod_{i=1}^{r_n}\mu_{n,K_n^{(j)}}^{(d_i^{(j)}),M^{(j)}}$. By combining Proposition 3.5 with the elementary fact that

$d_{TV}(\mu_1 \times \cdots \times \mu_m, \nu_1 \times \cdots \times \nu_m) \leq \sum_{i=1}^m d_{TV}(\mu_i, \nu_i)$ for probability measures μ_i, ν_i, we can find a probability measure $\mathbf{Q} = \mathbf{Q}\left(((d_i^{(j)}), M^{(j)})\right)$ and random variables $U_1, \ldots, U_{r_n}, Z_1, \ldots, Z_{r_n}$ defined on some space such that under $\mathbf{Q}$,

- U_j has distribution $\mu_{n,K_n^{(j)}}^{(d_i^{(j)}),M^{(j)}}$,
- Z_j is a standard Gaussian, and
- the pairs $(U_1, Z_1), \ldots, (U_{r_n}, Z_{r_n})$ are independent.

Furthermore, we have the estimate

$$\mathbf{Q}(U_j \neq Z_j \text{ for some } j = 1, \ldots, r_n) \leq \sum_{j=1}^{r_n} \sqrt{(\log 2\pi - 1)\frac{2^{15}}{a^8 n(K_n^{(j)} + 1)}}.$$

By the above remarks and this inequality, we can represent the probability in (3.13) as

$$\begin{aligned}
&\mathbf{P}\left(\min_{j=1,\ldots,r_n} \sum_{i=1}^{\mathcal{N}^{(j)}} X_i^{(j)} \in B \;\middle|\; ((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)}) \text{ for all } j\right) \\
&= \mathbf{Q}\left(\min_{j=1,\ldots,r_n} \left(\mu^{(j)} + \sigma^{(j)} U_j\right) \in B\right) \\
&\geq \mathbf{Q}\left(\min_{j=1,\ldots,r_n} \left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \in B\right) - \mathbf{Q}(U_j \neq Z_j \text{ for some } j) \\
&\geq \mathbf{Q}\left(\min_{j=1,\ldots,r_n} \left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \in B\right) - \frac{C_5}{\sqrt{n}} \sum_{j=1}^{r_n} \frac{1}{\sqrt{K_n^{(j)}}},
\end{aligned}$$

where $C_5 > 0$ is a constant. We plug this back into (3.13) to find

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbf{P}(\mathcal{T}_n \in B) \\
&\geq \liminf_{n\to\infty} \sum_{\Upsilon^{(1)}\times\cdots\times\Upsilon^{(r_n)}} \mathbf{Q}\left(\min_{j=1,\ldots,r_n} \left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \in B\right) \mathbf{P}(((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)}) \forall j) \\
&- \liminf_{n\to\infty} \frac{C_5}{\sqrt{n}} \sum_{j=1}^{r_n} \frac{1}{\sqrt{K_n^{(j)}}} \sum_{\Upsilon^{(1)}\times\cdots\times\Upsilon^{(r_n)}} \mathbf{P}(((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)}) \ \forall j).
\end{aligned}$$

By our assumption in the statement of the proposition, the second term is zero, so we get

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbf{P}(\mathcal{T}_n \in B) \\
&\geq \liminf_{n\to\infty} \sum_{\Upsilon^{(1)}\times\cdots\times\Upsilon^{(r_n)}} \mathbf{Q}\left(\min_{j=1,\ldots,r_n} \left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \in B\right) \mathbf{P}(((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)}) \forall j)
\end{aligned} \tag{3.14}$$

By Theorem A.1 of the appendix, if we write $\sigma = \sigma\left(((d_i^{(j)}), M^{(j)})_{j=1}^{r_n}\right) = \min_{j=1,\ldots,r_n} \sigma^{(j)}$, then we can find reals $a_n = a_n\left(((d_i^{(j)}), M^{(j)})_{j=1}^{r_n}\right)$ and $b_n = b_n\left(((d_i^{(j)}), M^{(j)})_{j=1}^{r_n}\right)$ and a universal constant $c_{15} > 0$ such that

$$\begin{aligned}
&\mathbf{Q}\left(\min_{j=1,\ldots,r_n} \left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \leq a_n\right) \geq c_{15}, \\
&\mathbf{Q}\left(\min_{j=1,\ldots,r_n} \left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \geq b_n\right) \geq c_{15},
\end{aligned} \tag{3.15}$$

and

$$b_n - a_n = \frac{\sigma}{\sqrt{1+\log r_n}}. \tag{3.16}$$

These a_n and b_n can be chosen as measurable functions of $((d_i^{(j)}), M^{(j)})$ by using quantiles of $\min_{j=1,\dots,r_n} \sum_{i=1}^{\mathcal{N}^{(j)}} X_i^{(j)}$ given that $((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)})$ for all j. We would like to set $B = (-\infty, a_n]$ and then $B = [b_n, \infty)$ in (3.14), but since these reals depend on the pairs $((d_i^{(j)}), M^{(j)})$, we must replace them with reals that do not depend on the pairs.

To do this, we recall (3.12), which implies that if n is large enough, then

$$\sum_{\Upsilon^{(1)}\times\cdots\times\Upsilon^{(r_n)}} \mathbf{P}\left(((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)}) \forall j\right) = \mathbf{P}\left(((b_i^{(j)}), \mathcal{N}^{(j)}) \in \Upsilon^{(j)}\ \forall j\right) \geq \frac{1}{2}.$$

The quantities a_n and b_n are random variables when considered as functions of $((b_i^{(j)}), \mathcal{N}^{(j)})_{j=1}^{r_n}$. Because $K(x) = \mathbf{P}\left(a_n \leq x, ((b_i^{(j)}), \mathcal{N}^{(j)}) \in \Upsilon^{(j)}\ \forall j\right)$ is a nondecreasing, right continuous function of x with $\lim_{x\to-\infty} K(x) = 0$ and $\lim_{x\to\infty} K(x) \geq 1/2$, we can find a deterministic number m_n^a such that both of the following hold:

$$\mathbf{P}\left(a_n \leq m_n^a, ((b_i^{(j)}), \mathcal{N}^{(j)}) \in \Upsilon^{(j)}\ \forall j\right) \geq \frac{1}{4}$$
$$\mathbf{P}\left(a_n \geq m_n^a, ((b_i^{(j)}), \mathcal{N}^{(j)}) \in \Upsilon^{(j)}\ \forall j\right) \geq \frac{1}{4}.$$

Observe that if we define

$$\widetilde{\sigma} = \min\{\sigma : ((d_i^{(j)}), M^{(j)}) \in \Upsilon^{(j)} \text{ for all } j\},$$

which, by (3.11), satisfies

$$\widetilde{\sigma} \geq \frac{a^2}{8} \min_{j=1,\dots,r_n} \sqrt{\frac{n}{K_n^{(j)}}}, \tag{3.17}$$

and

$$m_n^b = m_n^a + \frac{\widetilde{\sigma}}{\sqrt{1+\log r_n}}, \tag{3.18}$$

then by (3.16),

$$\mathbf{P}(b_n \geq m_n^b, ((b_i^{(j)}), \mathcal{N}^{(j)}) \in \Upsilon^{(j)}\ \forall j) \geq \mathbf{P}(a_n \geq m_n^a, ((b_i^{(j)}), \mathcal{N}^{(j)}) \in \Upsilon^{(j)}\ \forall j) \geq \frac{1}{4}. \tag{3.19}$$

First we set $B = (-\infty, m_n^a]$ in (3.14) and apply (3.15) to obtain

$$\begin{aligned}
&\sum_{\Upsilon^{(1)}\times\cdots\times\Upsilon^{(r_n)}} \mathbf{Q}\left(\min_{j=1,\dots,r_n}\left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \leq m_n^a\right) \mathbf{P}(((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)}) \forall j) \\
&\geq \sum_{\{a_n \leq m_n^a\}} \mathbf{Q}\left(\min_{j=1,\dots,r_n}\left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \leq a_n\right) \mathbf{P}(((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)}) \forall j) \\
&\geq c_{15} \mathbf{P}(a_n \leq m_n^a, ((b_i^{(j)}), \mathcal{N}^{(j)}) \in \Upsilon^{(j)}\ \forall j) \\
&\geq \frac{c_{15}}{4}.
\end{aligned} \tag{3.20}$$

A similar argument using (3.19) shows that

$$\sum_{\Upsilon^{(1)}\times\cdots\times\Upsilon^{(r_n)}} \mathbf{Q}\left(\min_{j=1,\dots,r_n}\left(\mu^{(j)} + \sigma^{(j)} Z_j\right) \geq m_n^b\right) \mathbf{P}(((b_i^{(j)}), \mathcal{N}^{(j)}) = ((d_i^{(j)}), M^{(j)}) \forall j) \geq \frac{c_{15}}{4}. \tag{3.21}$$

Finally, we put (3.20) into (3.14) to find

$$\liminf_{n\to\infty} \mathbf{P}(\mathcal{T}_n \le m_n^a) \ge \frac{c_{15}}{4}$$

and similarly using (3.21) gives

$$\liminf_{n\to\infty} \mathbf{P}(\mathcal{T}_n \ge m_n^b) \ge \frac{c_{15}}{4}.$$

Because of (3.17) and (3.18), this completes the proof of Proposition 3.6. □

4. Proofs of main results

4.1. *Proof of Theorem 1.2.* We will first bound fluctuations of T_n^{tube} by comparing it to a minimum of cylinder times. Define integers $K_n^{(1)}, \ldots, K_n^{(r_n)} \in [1, n]$ such that $\sum_{j=1}^{r_n} K_n^{(j)} = n$, along with cylinders $\mathsf{C}^{(j)}$, passage times $\mathcal{T}^{(j)}$, and minimum $\mathcal{T}_n$, all as in Sec. 3.3. Although in that section we used the i.i.d. weights (t_e), here we define them using the periodic weights $(t_e^{(n)})$. This does not change the distribution of $\mathcal{T}_n$ because the cylinders are contained in $[0, n] \times [0, n-1]$. For α_1, α_2 such that $3/4 < \alpha_1 < \alpha_2 < 1$, we assume that

$$K_n^{(j)} \in [n^{\alpha_2}, 2n^{\alpha_2}] \text{ for all } j, \text{ so } r_n \le n^{1-\alpha_2}. \tag{4.1}$$

In addition, we shift the cylinders up by $\lfloor n^{\alpha_2}/2 \rfloor$, setting

$$(K_n^{(j)})' = K_n^{(j)} + \left\lfloor \frac{n^{\alpha_2}}{2} \right\rfloor \text{ for } j = 1, \ldots, r_n,$$

with corresponding cylinders $(\mathsf{C}^{(j)})' = \mathsf{C}^{(j)} + \lfloor n^{\alpha_2}/2 \rfloor \mathbf{e}_2$, passage times $(\mathcal{T}^{(j)})'$, and minimum $\mathcal{T}_n'$. Again we use the periodic weights $(t_e^{(n)})$ instead of the i.i.d. weights (t_e). Observe that $\mathcal{T}_n$ and $\mathcal{T}_n'$ have the same distribution, but they are not independent. Last, define $\mathsf{T}_n = \min\{\mathcal{T}_n, \mathcal{T}_n'\}$, and let A_n be the event described in Proposition 2.3:

$$\mathsf{A}_n = \left\{ \text{for any geodesic } \Gamma_n \text{ for } T_n^{\text{tube}},\ \Gamma_n \subset x_0 + (\mathbb{Z} \times [-n^{\alpha_1}, n^{\alpha_1}]) \right\}, \tag{4.2}$$

where x_0 is the initial point of Γ_n. That proposition gives

$$\mathbf{P}(\mathsf{A}_n) \ge 1 - e^{-n^b} \text{ for large } n. \tag{4.3}$$

Any optimal path for T_n connects the sets $\{x : x \cdot \mathbf{e}_1 = 0\}$ and $\{x : x \cdot \mathbf{e}_1 = n\}$, so by (2.1), we have $T_n^{\text{tube}} \le \mathsf{T}_n$. We claim that if n is large, then in fact

$$\text{on } \mathsf{A}_n, \text{ we have } T_n^{\text{tube}} = \mathsf{T}_n. \tag{4.4}$$

To see why this holds, consider an outcome in A_n with n large and let Γ_n be a geodesic for T_n^{tube}. By periodicity of the weights, we may select Γ_n so that its initial point $x_0 = (0, m_0)$ satisfies $m_0 \in \left[\frac{n^{\alpha_2}}{10}, n + \frac{n^{\alpha_2}}{10}\right]$. If the set $[0, n] \times [m_0 - n^{\alpha_1}, m_0 + n^{\alpha_1}]$ is not contained in any of the cylinders $\mathsf{C}^{(j)}$, then either $m_0 \ge n$ or the interval $[m_0 - n^{\alpha_1}, m_0 + n^{\alpha_1}]$ must contain a number of the form $K_n^{(1)} + \cdots + K_n^{(j_0)}$. In the first case, $[0, n] \times [m_0 - n^{\alpha_1}, m_0 + n^{\alpha_1}]$ is contained in $(\mathsf{C}^{(r_n)})'$, and in the second, it is contained in $(\mathsf{C}^{(j_0)})'$. In any case, since A_n occurs, Γ_n must be contained in one of the cylinders $\mathsf{C}^{(j)}$ or $(\mathsf{C}^{(j)})'$, so it is an admissible path for the definition of the corresponding cylinder passage time $\mathcal{T}^{(j)}$ or $(\mathcal{T}^{(j)})'$. Therefore $T_n^{\text{tube}} = T^{(n)}(\Gamma_n) \ge \mathsf{T}_n$.

We estimate from (4.1)

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{r_n} \frac{1}{\sqrt{K_n^{(j)}}} \le n^{\frac{1-3\alpha_2}{2}} \to 0$$

since $\alpha_2 > 1/3$, and

$$\min_{j=1,\dots,r_n} \sqrt{\frac{n}{K_n^{(j)}(1+\log r_n)}} \geq \sqrt{\frac{n^{1-\alpha_2}}{2(1+\log n^{1-\alpha_2})}}.$$

Proposition 3.6 implies the existence of a_n, b_n and a constant $c_{16} > 0$ such that for large n,

$$b_n - a_n \geq c_{16}\sqrt{\frac{n^{1-\alpha_2}}{\log n}} \tag{4.5}$$

and

$$\mathbf{P}(\mathcal{T}_n \geq b_n) \geq c_{16} \text{ and } \mathbf{P}(\mathcal{T}_n \leq a_n) \geq c_{16}, \tag{4.6}$$

with the same statements holding for $\mathcal{T}_n'$. Both $\mathcal{T}_n$ and $\mathcal{T}_n'$ are decreasing functions of the i.i.d. weights $(t_e^{(n)})_{e\in E(n)^o}$, so by Harris's inequality,

$$\mathbf{P}(\mathsf{T}_n \geq b_n) \geq \mathbf{P}(\mathcal{T}_n \geq b_n)^2 \geq c_{16}^2.$$

We also have $\mathbf{P}(\mathsf{T}_n \leq a_n) \geq \mathbf{P}(\mathcal{T}_n \leq a_n) \geq c_{16}$. Last, we use (4.3) and (4.4) to get for large n

$$\mathbf{P}(T_n^{\text{tube}} \geq b_n) \geq \mathbf{P}(\mathsf{T}_n \geq b_n) - e^{-n^b} \geq \frac{c_{16}^2}{2}$$

and similarly $\mathbf{P}(T_n^{\text{tube}} \leq a_n) \geq c_{16}/2$. Therefore (T_n^{tube}) has fluctuations at least of order $\sqrt{n^{1-\alpha_2}/\log n}$. Since α_2 is an arbitrary number bigger than $3/4$, this completes the proof of the first statement of Theorem 1.2.

We move to the other half of Theorem 1.2. To bound fluctuations for T_n^{sq}, we compare this quantity to T_n^{tube}. Because of the result we just proved, if $\epsilon > 0$, we can choose α_2 above (4.1) with $(1-\alpha_2)/2 > 1/8 - \epsilon$ such that the reals a_n, b_n defined in (4.5) satisfy $b_n - a_n \geq n^{1/8-\epsilon}$ for all large n and such that

$$\mathbf{P}(T_n^{\text{tube}} \leq a_n) \geq c_{17} \text{ and } \mathbf{P}(T_n^{\text{tube}} \geq b_n) \geq c_{17} \tag{4.7}$$

for some $c_{17} > 0$ (and similarly for $\mathcal{T}_n$). We first show that

$$\liminf_{n\to\infty} \mathbf{P}(T_n^{\text{sq}} \leq a_n) > 0. \tag{4.8}$$

We will assume that the weights (t_e) and $(t_e^{(n)})$ used in the definitions of T_n^{sq} and T_n^{tube} are coupled so that $t_e = t_e^{(n)}$ for all $e \in E(n)^o$. First, taking the definition of A_n from (4.2), and using (4.7), we obtain

$$\liminf_{n\to\infty} \mathbf{P}(T_n^{\text{tube}} \leq a_n,\ \mathsf{A}_n) > 0. \tag{4.9}$$

Let E_n be the event that some geodesic for T_n^{tube} has initial point in the interval $\{0\} \times [\lfloor n/10\rfloor, \lfloor n/10\rfloor + \lfloor 8n/10\rfloor]$, and let E_n' be the event that some geodesic for T_n^{tube} has initial point in the interval $\{0\} \times [\lfloor n/2\rfloor, \lfloor n/2\rfloor + \lfloor 8n/10\rfloor]$. Then by vertical translation invariance,

$$\begin{aligned}\mathbf{P}(T_n^{\text{tube}} \leq a_n,\ \mathsf{A}_n) &\leq \mathbf{P}(T_n^{\text{tube}} \leq a_n,\ \mathsf{A}_n, \mathsf{E}_n) + \mathbf{P}(T_n^{\text{tube}} \leq a_n,\ \mathsf{A}_n, \mathsf{E}_n') \\ &= 2\mathbf{P}(T_n^{\text{tube}} \leq a_n,\ \mathsf{A}_n, \mathsf{E}_n),\end{aligned}$$

so by (4.9),

$$\liminf_{n\to\infty} \mathbf{P}(T_n^{\text{tube}} \leq a_n,\ \mathsf{A}_n, \mathsf{E}_n) > 0.$$

However, if n is large and $\mathsf{A}_n \cap \mathsf{E}_n$ occurs, then there is a geodesic Γ_n for T_n^{tube} that uses only edges in $E(n)^o$, so it is an admissible path for the definition of T_n^{sq}. Therefore $T_n^{\text{tube}} = T^{(n)}(\Gamma_n) = T(\Gamma_n) \geq T_n^{\text{sq}}$, and so

$$\liminf_{n\to\infty} \mathbf{P}(T_n^{\text{sq}} \leq a_n) \geq \liminf_{n\to\infty} \mathbf{P}(T_n^{\text{tube}} \leq a_n,\ \mathsf{A}_n, \mathsf{E}_n) > 0.$$

To complete the proof, we must show

$$\liminf_{n\to\infty} \mathbf{P}(T_n^{\mathrm{sq}} \ge b_n) > 0. \tag{4.10}$$

To do this, we use three related passage times, all defined with the weights (t_e). Let $T_n(1)$ be the minimum of $T(\Gamma)$ over all Γ with vertices in $B(n)$, connecting $\{0\} \times [0,n]$ to $\{n\} \times [0,n]$, but not using edges with both endpoints in $[0,n] \times \{n\}$ (that is, using only edges in $E(n)^o$). Let $T_n(2)$ be the minimum of $T(\Gamma)$ over all Γ with vertices in $B(n)$, connecting the same two sets, but now not using edges with both endpoints in $[0,n] \times \{0\}$. Last, let $T_n(3)$ be the minimum of $T(\Gamma)$ over all Γ with vertices in $B(n)$, connecting $[0,n] \times \{0\}$ to $[0,n] \times \{n\}$, but not using edges with both endpoints in $\{n\} \times [0,n]$. All the $T_n(i)$ are decreasing functions of the weights (t_e) and are identically distributed. Furthermore, using our coupling of (t_e) and $(t_e^{(n)})$, we have $T_n(1) \ge T_n^{\mathrm{tube}}$. So by Harris's inequality and (4.7),

$$\liminf_{n\to\infty} \mathbf{P}\left(\min_{i=1,2,3} T_n(i) \ge b_n\right) \ge \liminf_{n\to\infty} \mathbf{P}(T_n(1) \ge b_n)^3 \ge \liminf_{n\to\infty} \mathbf{P}(T_n^{\mathrm{tube}} \ge b_n)^3 > 0. \tag{4.11}$$

However,

$$\min_{i=1,2,3} T_n(i) \le T_n^{\mathrm{sq}} \tag{4.12}$$

because of the following argument. Let σ_n be a geodesic for T_n^{sq} (one exists because there are finitely many self-avoiding paths in $B(n)$). If σ_n does not touch $[0,n] \times \{n\}$, then it is an admissible path for $T_n(1)$, so $T_n^{\mathrm{sq}} \ge T_n(1)$. If σ_n does not touch $[0,n] \times \{0\}$, then similarly $T_n^{\mathrm{sq}} \ge T_n(2)$. Last, if σ_n touches both of these sets, let σ_n' be a segment of σ_n that connects them. Because σ_n is a geodesic for T_n^{sq} it a.s. cannot use an edge with both endpoints in $\{n\} \times [0,n]$. Therefore σ_n' cannot either, and so it is an admissible path for $T_n(3)$, implying that $T_n^{\mathrm{sq}} = T(\sigma_n) \ge T(\sigma_n') \ge T_n(3)$. In any of these three cases, (4.12) holds.

Finally, due to (4.12), we have $\mathbf{P}(T_n^{\mathrm{sq}} \ge b_n) \ge \mathbf{P}(\min_{i=1,2,3} T_n(i) \ge b_n)$. This along with (4.11) implies (4.10) and completes the proof of Theorem 1.2.

4.2. *Proof of Corollary 1.5.* The proof of Corollary 1.5 will use the objects defined in the last proof. Specifically, we take α_1, α_2 as above (4.1), the cylinder time $\mathcal{T}_n$, and reals a_n, b_n defined in (4.5). For these reals, we have shown above that we have, in addition to (4.6), similar inequalities for T_n^{tube} and T_n^{sq}. We will also use the event A_n defined in (4.2).

We first prove that for some $c_{18} > 0$, we have for large n

$$\mathbf{P}(T_n^{\mathrm{tube}} \ge b_n') \ge c_{18}, \tag{4.13}$$

where

$$b_n' = \begin{cases} 2b_{\frac{n}{2}} & \text{if } n \text{ is even} \\ b_{\frac{n-1}{2}} + b_{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Because $T_n^{\mathrm{tor}} \ge T_n^{\mathrm{tube}}$, this will also establish $\mathbf{P}(T_n^{\mathrm{tor}} \ge b_n') \ge c_{18}$.

To prove (4.13), first suppose that n is even and consider the three squares

$$\mathsf{S}_i = \left[0, \frac{n}{2}\right]^2 + (i-1)\left\lfloor \frac{n}{3} \right\rfloor \mathbf{e}_2 \text{ for } i = 1,2,3$$

with corresponding square passage times $T_n^{\mathrm{sq}}(i)$ (all using the weights $(t_e^{(n)})$). We also define the shifted squares

$$\bar{\mathsf{S}}_i = \left(\mathsf{S}_i + \frac{n}{2}\mathbf{e}_1\right) + i\left\lfloor \frac{n}{3} \right\rfloor \mathbf{e}_2 \text{ for } i = 1,2,3$$

along with their corresponding square passage times $\bar{T}_n^{\mathrm{sq}}(i)$ (again using $(t_e^{(n)})$). From the version of (4.6) for T_n^{sq} and Harris's inequality, we have

$$\mathbf{P}\left(\min_{i=1,2,3} \min\left\{T_n^{\mathrm{sq}}(i), \bar{T}_n^{\mathrm{sq}}(i)\right\} \geq b_{\frac{n}{2}}\right) \geq c_{16}^6.$$

As a consequence of (4.3), $\mathbf{P}(\mathsf{A}_n) \to 1$, and so

$$\liminf_{n\to\infty} \mathbf{P}\left(\mathsf{A}_n, \min_{i=1,2,3}\left(T_n^{\mathrm{sq}}(i) + \bar{T}_n^{\mathrm{sq}}(i)\right) \geq b_n'\right) > 0. \tag{4.14}$$

However, on the event in this probability, we have $T_n^{\mathrm{tube}} \geq b_n'$. Indeed, choose a geodesic Γ_n for T_n^{tube}. We may assume by periodicity that the initial point x_0 of Γ_n is contained in the interval $\{0\} \times [n/12, n + n/12]$. Then if n is large, there exists some i such that

$$x_0 + ([0,n] \times [-n^{\alpha_1}, n^{\alpha_1}]) \subset \left(\mathsf{S}_i \cup \bar{\mathsf{S}}_i\right).$$

This means that Γ_n contains an initial segment that is an admissible path for $T_n^{\mathrm{sq}}(i)$ and a final (disjoint) segment that is an admissible path for $\bar{T}_n^{\mathrm{sq}}(i)$, so $T_n^{\mathrm{tube}} = T^{(n)}(\Gamma_n) \geq T_n^{\mathrm{sq}}(i) + \bar{T}_n^{\mathrm{sq}}(i) \geq 2b_{n/2} = b_n'$. Because of (4.14), we then conclude that (4.13) holds if n is even. In the case that n is odd, the squares are instead defined as

$$\mathsf{S}_i = \left[0, \frac{n-1}{2}\right]^2 + (i-1)\left\lfloor\frac{n}{3}\right\rfloor \mathbf{e}_2 \text{ for } i = 1,2,3$$

and

$$\bar{\mathsf{S}}_i = \left[\frac{n-1}{2}, n\right] \times \left[0, \frac{n+1}{2}\right] + (i-1)\left\lfloor\frac{n}{3}\right\rfloor \mathbf{e}_2 \text{ for } i = 1,2,3,$$

but the rest of the proof is the same.

The second half of the proof serves to show that if $\beta \in (0,1)$, there is $c_{19} > 0$ such that for large n,

$$\mathbf{P}(T_n^{\mathrm{tor}} \leq a_n') \geq \frac{c_{19}}{n^{1+\alpha_2-2\beta}}, \tag{4.15}$$

where

$$a_n' = \begin{cases} 2a_{\frac{n}{2}} + 8n^\beta & \text{if } n \text{ is even} \\ a_{\frac{n-1}{2}} + a_{\frac{n+1}{2}} + 8n^\beta & \text{if } n \text{ if odd.} \end{cases}$$

We do this by approximately concatenating two low-weight paths in cylinders of height n^{α_1}, one in the left half of the torus and one in the right. Again we will assume that n is even; a similar argument works if n is odd.

First, in addition to the variable $\mathcal{T}_{n/2}$, we define $\bar{\mathcal{T}}_{n/2}$ as the corresponding minimum over the left-right times associated with shifted cylinders $\mathsf{C}^{(j)} + (n/2)\mathbf{e}_1$ and the same integer cylinder-heights $K_{n/2}^{(1)}, \ldots, K_{n/2}^{(r_{n/2})}$ that satisfy equation (4.1) with n replaced by $n/2$. In these definitions, we use the periodic weights $(t_e^{(n)})$. By independence and (4.6), we have

$$\mathbf{P}\left(\max\left\{\mathcal{T}_{\frac{n}{2}}, \bar{\mathcal{T}}_{\frac{n}{2}}\right\} \leq a_{\frac{n}{2}}\right) \geq c_{16}^2.$$

Let G_n be the event that there exists a path γ with initial point x_0 in $\{0\} \times [0,n]$, all of whose edges intersect $(0, n/2) \times [0,n]$, with final point in $\{n/2\} \times [0,n]$, such that $T^{(n)}(\gamma) \leq a_{n/2}$ and $\gamma \subset x_0 + (\mathbb{Z} \times [-2(n/2)^{\alpha_2}, 2(n/2)^{\alpha_2}])$. Similarly, define $\bar{G}_n$ as the event that there is a path $\bar{\gamma}$ with initial point y_0 in $\{n/2\} \times [0,n]$, all of whose edges intersect $(n/2, n) \times [0,n]$, with final point in $\{n\} \times [0,n]$, such that $T^{(n)}(\bar{\gamma}) \leq a_{n/2}$ and $\bar{\gamma} \subset y_0 + (\mathbb{Z} \times [-2(n/2)^{\alpha_2}, 2(n/2)^{\alpha_2}])$. By (4.1), any optimal path in the definition of $\mathcal{T}_{n/2}$ (or $\bar{\mathcal{T}}_{n/2}$) satisfies the properties in the definition of G_n (or $\bar{G}_n$), so for large n

$$\mathbf{P}(G_n \cap \bar{G}_n) \geq c_{16}^2. \tag{4.16}$$

On the event $G_n \cap \bar{G}_n$, we will build an admissible path for T_n^{tor} consisting of γ, $\bar{\gamma}$, and vertical line segments. To do this effectively, we need the endpoints of γ and $\bar{\gamma}$ to be close to each other. For this purpose, we must introduce a few definitions. First, let $\beta \in (0,1)$ and define a family of intervals

$$I_1^1 = \{0\} \times [0, n^\beta], I_2^1 = \{0\} \times [n^\beta, 2n^\beta], \ldots, I_p^1 = \{0\} \times \left[(\lfloor n^{1-\beta} \rfloor - 1)n^\beta, \lfloor n^{1-\beta} \rfloor n^\beta\right],$$

and $I_{p+1}^1 = \{0\} \times [\lfloor n^{1-\beta} \rfloor n^\beta, n]$. We need shifted intervals as well, so we set

$$I_i^2 = I_i^1 + \frac{n}{2}\mathbf{e}_1 \text{ and } I_i^3 = I_i^1 + n\mathbf{e}_1.$$

Also define i_1 as the minimal value of i for which there is a path γ satisfying the conditions of the definition of G_n such that its initial point x_0 is in I_i^1. Set i_2 to be the minimal value of i for which there is a path γ satisfying the conditions of the definition of G_n such that its initial point x_0 is in $I_{i_1}^1$ but its final point is in I_i^2. Next, define i_3 as the minimal value of i for which there is a path $\bar{\gamma}$ satisfying the conditions in the definition of $\bar{G}_n$ such that its initial point y_0 is in I_i^2. Finally, set i_4 to be the minimal value of i for which there is a path $\bar{\gamma}$ satisfying the conditions of the definition of $\bar{G}_n$ such that its initial point y_0 is in $I_{i_3}^2$ and its final point is in I_i^3. If G_n occurs, then i_1 and i_2 are defined, and if $\bar{G}_n$ occurs, then i_3 and i_4 are defined. Whenever any i_j is not defined, we set it to be $+\infty$.

From (4.16), we have

$$\mathbf{P}(i_j < \infty \text{ for } j = 1, \ldots, 4) \geq c_{16}^2. \tag{4.17}$$

Furthermore, by independence,

$$\begin{aligned}
&\mathbf{P}(i_1 = i_4 < \infty, i_2 = i_3 < \infty) \\
&= \sum_{i=1}^{p+1} \left[\sum_{j=1}^{p+1} \mathbf{P}(i_2 = i_3 = j \mid i_1 = i_4 = i) \right] \mathbf{P}(i_1 = i_4 = i) \\
&= \sum_{i=1}^{p+1} \left[\sum_{j=1}^{p+1} \mathbf{P}(i_2 = j \mid i_1 = i)\mathbf{P}(i_3 = j \mid i_4 = i) \right] \mathbf{P}(i_1 = i)\mathbf{P}(i_4 = i).
\end{aligned} \tag{4.18}$$

Observe that if $i_1 = i$, then γ begins at x_0 in the interval I_i^1 but must end at an interval in $x_0 + (\mathbb{Z} \times [-2(n/2)^{\alpha_2}, 2(n/2)^{\alpha_2}])$. Therefore i_2 can take only take values in a set $\mathfrak{S}_i$ which has cardinality at most $8n^{\alpha_2 - \beta}$ if n is large. Using Jensen's inequality and symmetry, the inner sum is

$$\begin{aligned}
\sum_{j \in \mathfrak{S}_i} \mathbf{P}(i_2 = j \mid i_1 = i)\mathbf{P}(i_3 = j \mid i_4 = i) &= \#\mathfrak{S}_i \cdot \frac{1}{\#\mathfrak{S}_i} \sum_{j \in \mathfrak{S}_i} \mathbf{P}(i_2 = j \mid i_1 = i)^2 \\
&\geq \frac{1}{\#\mathfrak{S}_i} \left(\sum_{j \in \mathfrak{S}_i} \mathbf{P}(i_2 = j \mid i_1 = i) \right)^2 \\
&\geq \left(8n^{\alpha_2 - \beta} \right)^{-1}.
\end{aligned}$$

By a similar argument, and now using (4.17), we obtain

$$\sum_{i=1}^{p+1} \mathbf{P}(i_1 = i)\mathbf{P}(i_4 = i) \geq \frac{1}{p+1}\mathbf{P}(i_1 < \infty)^2 \geq \frac{c_{16}^4}{p+1}.$$

Putting these back in (4.18) and using $p + 1 \leq 2n^{1-\beta}$ gives

$$\mathbf{P}(i_1 = i_4 < \infty, i_2 = i_3 < \infty) \geq \frac{c_{16}^4}{16n^{1+\alpha_2-2\beta}}. \tag{4.19}$$

On the event in the probability of (4.19), the endpoints of γ and $\bar\gamma$ are within n^β of each other; now we will connect them with vertical segments. So let H_n be the event that any vertex self-avoiding path π with at most $2n^\beta$ many edges and which is contained in $\cup_{i=1}^3 (\{in/2\} \times [0,n])$ satisfies $T^{(n)}(\pi) \le 4n^\beta$. By the argument of (3.6), any such π satisfies

$$\mathbf{P}(T^{(n)}(\pi) \ge 4n^\beta) \le e^{-4\eta n^\beta} \left(\mathbf{E}e^{\eta t_e}\right)^{\#\pi} \le e^{-c_{20}n^\beta}$$

for some $c_{20} > 0$, and so a union bound produces for large n

$$\mathbf{P}(H_n) \ge 1 - 3(n+1)\cdot 2n^\beta \cdot e^{-c_{20}n^\beta} \ge 1 - e^{-\frac{c_{20}}{2}n^\beta}.$$

So, if n is large, we can combine this with (4.19) for

$$\mathbf{P}(H_n, i_1 = i_4 < \infty, i_2 = i_3 < \infty) \ge \frac{c_{16}^4}{32n^{1+\alpha_2-2\beta}}. \tag{4.20}$$

Now consider an outcome for which H_n occurs, and also $i_1 = i_4 < \infty$, and $i_2 = i_3 < \infty$. Then we may produce a path Γ by starting with γ, following the line $\{n/2\} \times [0,n]$ from the final point of γ to the initial point y_0 of $\bar\gamma$, traversing $\bar\gamma$, and then following the line $\{n\} \times [0,n]$ from the final point of $\bar\gamma$ to $x_0 + n\mathbf{e}_1$, where x_0 is the initial point of γ. In this way we produce an admissible path for T_n^{tor}, and $T^{(n)}(\pi) \le 2(a_{n/2} + 4n^\beta)$. Together with (4.20), this implies (4.15).

Having shown both inequalities (4.13) and (4.15), we complete the proof of Corollary 1.5. For concreteness, we again assume that n is even. For our $\alpha_2 > 3/4$, let $\delta > 0$ and set $\beta = 1/2 - \alpha_2/2 - \delta$. Then by (4.5), for large n,

$$b'_n - a'_n = 2\left(b_{\frac{n}{2}} - a_{\frac{n}{2}}\right) - 8n^\beta \ge 2c_{16}\sqrt{\frac{\left(\frac{n}{2}\right)^{1-\alpha_2}}{\log\frac{n}{2}}} - 8n^{\frac{1-\alpha_2}{2}-\delta} \ge c_{16}\sqrt{\frac{n^{1-\alpha_2}}{\log n}}.$$

Furthermore, by (4.13) and (4.15), for large n,

$$\mathbf{P}(T_n^{\text{tor}} \le a'_n) \ge \frac{c_{19}}{n^{2(\alpha_2+\delta)}}$$

and

$$\mathbf{P}(T_n^{\text{tor}} \ge b'_n) \ge \frac{c_{19}}{n^{2(\alpha_2+\delta)}}.$$

For any random variable Y with finite mean satisfying $\mathbf{P}(Y \le a) \ge c$ and $\mathbf{P}(Y \ge b) \ge c$ for reals $a \le b$, one has $\mathbf{E}|Y - \mathbf{E}Y|^k \ge ((b-a)/2)^k c$. Applying this to T_n^{tor} gives

$$\mathbf{E}|T_n^{\text{tor}} - \mathbf{E}T_n^{\text{tor}}|^k \ge \left(\frac{c_{16}}{2}\sqrt{\frac{n^{1-\alpha_2}}{\log n}}\right)^k \cdot \frac{c_{19}}{n^{2(\alpha_2+\delta)}}.$$

Last, taking $\alpha_2 = 3/4 + \delta$ gives for large n

$$\mathbf{E}|T_n^{\text{tor}} - \mathbf{E}T_n^{\text{tor}}|^k \ge \left(\frac{c_{16}}{2}\cdot\frac{n^{\frac{1}{8}-\frac{\delta}{2}}}{\sqrt{\log n}}\right)^k \cdot \frac{c_{19}}{n^{\frac{3}{2}+4\delta}} \ge c_{19}n^{\left(\frac{1}{8}-\delta\right)k - \frac{3}{2} - 4\delta}.$$

This implies Corollary 1.5 and completes the proof.

Appendix A. Gaussian fluctuation lemmas

This section provides the fluctuation result for normal random variables used to justify (3.15) and (3.16) in the proof of Proposition 3.6. It relates the fluctuations of the minimum of independent normal variables with different means and variances to the fluctuations of the minimum of i.i.d. normal variables. For notational convenience, the result is stated for the maximum, though by symmetry it holds for the minimum.

Theorem A.1. *There exists a universal constant $c_{21} > 0$ such that the following holds. Let $Z_1, \ldots, Z_n$ be independent normal variables with variances $\sigma_1^2, \ldots, \sigma_n^2$ and set $\sigma = \min_{i=1,\ldots,n} \sigma_i$. There exist reals a_n, b_n with*

$$b_n - a_n = \frac{\sigma}{2\sqrt{1+\log n}} \tag{A.1}$$

such that

$$\mathbf{P}\left(\max_{i=1,\ldots,n} Z_i \le a_n\right) \ge c_{21} \text{ and } \mathbf{P}\left(\max_{i=1,\ldots,n} Z_i \ge b_n\right) \ge c_{21}.$$

By Durrett (2019, Ex. 3.2.3), the right side of (A.1) is the order of the fluctuations of the maximum of i.i.d. normal random variables with variance σ^2. Before the proof of Theorem A.1, we state three lemmas. The first establishes a certain logconcavity property of the normal distribution.

Lemma A.2. *Let Φ be the distribution function of the standard normal distribution and f be its density function. Then f/Φ is logconcave.*

Proof: We must show that $(\log(f/\Phi))'' \le 0$, but since $(\log f)'' \equiv -1$, this reduces to $(\log \Phi(x))'' \ge -1$, or

$$f'(x)\Phi(x) - f^2(x) + \Phi^2(x) \ge 0. \tag{A.2}$$

So let $F = f'\Phi - f^2 + \Phi^2$. Because $\lim_{x\to-\infty} F(x) = 0$, it will suffice to show that $F'(x) \ge 0$ for all x. Using $f'(x) = -xf(x)$, a computation gives

$$F'(x) = f(x)[(1+x^2)\Phi(x) + xf(x)],$$

so we can just prove that $H(x) \ge 0$ for all x, where $H(x) = (1+x^2)\Phi(x) + xf(x)$. We observe that

$$H'(x) = 2x\Phi(x) + (1+x^2)f(x) + f(x) + xf'(x) = 2(x\Phi(x) + f(x)).$$

and

$$H''(x) = 2[\Phi(x) + xf(x) - xf(x)] = 2\Phi(x) \ge 0.$$

From the first display, we obtain $\lim_{x\to-\infty} H'(x) = 0$, and combining this with the second display, we obtain $H'(x) \ge 0$, for all x. Last, since $\lim_{x\to-\infty} H(x) = 0$, this implies $H(x) \ge 0$ for all x, and so (A.2) follows. □

The next lemma tells us how the quantiles of $\max_i Z_i$ change as we shift the Z_i. Although it is stated for normal variables, it holds for more general distributions.

Lemma A.3. *Let $Z_1, \ldots, Z_n$ be independent normal random variables with distribution functions $F_1, \ldots, F_n$ and densities $f_1, \ldots, f_n$. For $a \in \mathbb{R}$, set*

$$F_{max}(z, a) = \mathbf{P}(\max\{Z_1 + a, Z_2, \ldots, Z_n\} \le z),$$

and for $t \in (0,1)$ define $F_{max}^{-1}(t, a)$ as the unique real such that

$$F_{max}(F_{max}^{-1}(t, a), a) = t. \tag{A.3}$$

Then

$$\frac{\partial}{\partial a} F_{max}^{-1}(t, a) = \left. \frac{\frac{f_1(z-a)}{F_1(z-a)}}{\frac{f_1(z-a)}{F_1(z-a)} + \sum_{i=2}^n \frac{f_i(z)}{F_i(z)}} \right|_{z=F_{max}^{-1}(t,a)}.$$

Proof: Fix $t \in (0,1)$ and define $H : \mathbf{R} \to \mathbf{R}^2$ by $H(a) = (F_{max}^{-1}(t, a), a)$, so that (A.3) becomes $F_{max}(H(a)) = t$. By the chain rule,

$$\left.\frac{\partial}{\partial z} F_{max}(z, a)\right|_{(z,a)=H(a)} \frac{\partial}{\partial a} F_{max}^{-1}(t, a) + \left.\frac{\partial}{\partial a} F_{max}(z, a)\right|_{(z,a)=H(a)} = 0,$$

which implies

$$\frac{\partial}{\partial a} F_{max}^{-1}(t,a) = -\left.\frac{\frac{\partial}{\partial a} F_{max}(z,a)}{\frac{\partial}{\partial z} F_{max}(z,a)}\right|_{(z,a)=H(a)}. \tag{A.4}$$

Using $F_{max}(z,a) = F_1(z-a)\prod_{i=2}^n F_i(z)$, these derivatives are

$$\frac{\partial}{\partial a} F_{max}(z,a) = -f_1(z-a)\prod_{i=2}^n F_i(z),$$

and

$$\frac{\partial}{\partial z} F_{max}(z,a) = F_{max}(z,a)\left(\frac{f_1(z-a)}{F_1(z-a)} + \sum_{i=2}^n \frac{f_i(z)}{F_i(z)}\right).$$

Place these back into (A.4) to complete the proof. □

Our last lemma uses the previous two in its proof, and implies Theorem. A.1 in the case that the Z_i have the same variances but different means. The result is stronger, and is stated in terms of the dispersive order.

Definition A.4. The random variable X, with distribution function F, is said to be less dispersed that the random variable Y, with distribution function G, if

$$F^{-1}(b) - F^{-1}(a) \le G^{-1}(b) - G^{-1}(a) \quad \text{for all } 0 < a \le b < 1.$$

If X is less dispersed than Y, then $\operatorname{Var} X \le \operatorname{Var} Y$ provided that X and Y have finite variances. See, for instance, Shaked and Shanthikumar (2007, Sec. 3.B) for this and more facts on the dispersive order. Moreover, we observe that if X_n is less dispersed than Y_n for all n, then $\{X_n\}_{n=1}^\infty$ has fluctuations of lower order than $\{Y_n\}_{n=1}^\infty$.

The following result on fluctuations of the maximum of shifted normal variables holds for more general distributions. The proof only uses Lemma A.3 and the logconcavity property of normal random variables from Lemma A.2.

Lemma A.5. *Let $X_1,\ldots,X_n$ be i.i.d. normal random variables, let $a_1,\ldots,a_n \in \mathbf{R}$ and let $Z_i = X_i + a_i$, for all $i = 1,\ldots,n$. Then $\max_{1\le i\le n} X_i$ is less dispersed than $\max_{1\le i\le n} Z_i$.*

Proof: We may assume that $a_1 \le a_2 \le \cdots \le a_n$. Write F and f for the common distribution function and density of the X_i, so that the distribution function and density of Z_i is $F(z-a_i)$ and $f(z-a_i)$. Define $F_{max}^{-1}(t,a_1,\ldots,a_n)$ analogously to (A.3): it is the unique real such that

$$\mathbf{P}\left(\max_{i=1,\ldots,n}(a_i+X_i) \le F_{max}^{-1}(t,a_1,\ldots,a_n)\right) = t.$$

We aim to prove that for $t_1 < t_2$, one has

$$F_{max}^{-1}(t_2,a_1,\ldots,a_n) - F_{max}^{-1}(t_1,a_1,\ldots,a_n) \ge F_{max}^{-1}(t_2,a_n,\ldots,a_n) - F_{max}^{-1}(t_1,a_n,\ldots,a_n). \tag{A.5}$$

Putting

$$\psi(a_1,\ldots,a_n) = F_{max}^{-1}(t_2,a_1,\ldots,a_n) - F_{max}^{-1}(t_1,a_1,\ldots,a_n),$$

Lemma A.3 implies that

$$\frac{\partial}{\partial a_k}\psi(a_1,\ldots,a_n) = \left.\frac{\frac{f(z-a_k)}{F(z-a_k)}}{\sum_{i=1}^n \frac{f(z-a_i)}{F(z-a_i)}}\right|_{z=F_{max}^{-1}(t_2,a_1,\ldots,a_n)} - \left.\frac{\frac{f(z-a_k)}{F(z-a_k)}}{\sum_{i=1}^n \frac{f(z-a_i)}{F(z-a_i)}}\right|_{z=F_{max}^{-1}(t_1,a_1,\ldots,a_n)}.$$

We claim that if $a_1 = \cdots = a_k \le a_{k+1} \le \cdots \le a_n$, then

$$\frac{\partial}{\partial a_k}\psi(a_1,\ldots,a_n) \le 0. \tag{A.6}$$

This will imply (A.5) after integrating the derivative of ψ along the concatenation of straight line segments connecting the points $(a_1, a_2 \ldots, a_n)$, $(a_2, a_2, a_3, \ldots, a_n)$, $(a_3, a_3, a_3, a_4, \ldots, a_n)$, $\ldots$, $(a_n, \ldots, a_n)$ in order. To simplify notation, put $z_i = F_{max}^{-1}(t_i, a_1, \ldots, a_n)$, so that $z_2 \geq z_1$, let $g = f/F$, and write

$$\frac{\frac{f(z_2-a_k)}{F(z_2-a_k)}}{\sum_{i=1}^n \frac{f(z_2-a_i)}{F(z_2-a_i)}} = \frac{g(z_2-a_k)}{\sum_{i=1}^n g(z_2-a_i)} = \frac{g(z_1-a_k)\frac{g(z_2-a_k)}{g(z_1-a_k)}}{\sum_{i=1}^n g(z_1-a_i)\frac{g(z_2-a_i)}{g(z_1-a_i)}}. \tag{A.7}$$

By Lemma A.2, g is logconcave, so because $a_i \geq a_k$ for $i = 1, \ldots, n$, we have

$$\begin{aligned}\frac{g(z_2-a_i)}{g(z_1-a_i)} &= \exp\left(\log g(z_2-a_i) - \log g(z_1-a_i)\right) \geq \exp\left(\log g(z_2-a_k) - \log g(z_1-a_k)\right) \\ &= \frac{g(z_2-a_k)}{g(z_1-a_k)},\end{aligned}$$

and (A.7) above becomes

$$\frac{\frac{f(z_2-a_k)}{F(z_2-a_k)}}{\sum_{i=1}^n \frac{f(z_2-a_i)}{F(z_2-a_i)}} \leq \frac{g(z_1-a_k)\frac{g(z_2-a_k)}{g(z_1-a_k)}}{\sum_{i=1}^n \left[g(z_1-a_i)\frac{g(z_2-a_k)}{g(z_1-a_k)}\right]} = \frac{\frac{f(z_1-a_k)}{F(z_1-a_k)}}{\sum_{i=1}^n \frac{f(z_1-a_i)}{F(z_1-a_i)}}.$$

This proves (A.6). □

Given the preparatory lemmas above, we can now prove Theorem A.1.

Proof of Theorem A.1: By Durrett (2019, Ex. 3.2.3), there is an absolute constant $c_{22} > 0$ such that if $W_1, W_2, \ldots$ are i.i.d. normal variables with mean zero and variance σ^2, then there are reals a_n', b_n' such that

$$b_n' - a_n' = \frac{\sigma}{\sqrt{1+\log n}}$$

and both

$$\mathbf{P}\left(\max_{i=1,\ldots,n} W_i \leq a_n'\right) \geq c_{22} \text{ and } \mathbf{P}\left(\max_{i=1,\ldots,n} W_i \geq b_n'\right) \geq c_{22}.$$

We consider $W_i' + V_i$, where $W_i' = W_i + \mathbf{E}Z_i$, and the V_i are independent normal random variables (and independent of (W_i)) with Var $V_i = \sigma_i^2 - \sigma^2$. Then (Z_i) and $(W_i' + V_i)$ have the same distribution. Conditional on $\vec{V} = (V_1, \ldots, V_n)$, the variables $W_i' + V_i$ are independent normals with variance σ^2 and possibly different means $V_i + \mathbf{E}Z_i$. Lemma A.5 therefore gives that they (conditionally) are more dispersed than the sequence $W_1, \ldots, W_n$. Therefore we can find reals $a_n(\vec{V})$ and $b_n(\vec{V})$ such that

$$b_n(\vec{V}) - a_n(\vec{V}) = \frac{\sigma}{\sqrt{1+\log n}}$$

and

$$\mathbf{P}\left(\max_{i=1,\ldots,n} (W_i' + V_i) \geq b_n(\vec{V}) \mid \vec{V}\right) \geq c_{22},$$

$$\mathbf{P}\left(\max_{i=1,\ldots,n} (W_i' + V_i) \leq a_n(\vec{V}) \mid \vec{V}\right) \geq c_{22}.$$

We can choose $a_n(\vec{V})$ and $b_n(\vec{V})$ as Borel measurable functions of the vector $\vec{V}$, for instance by selecting them to be quantiles of the conditional distribution of $\max_i(W_i' + V_i)$ given $\vec{V}$. We now choose reals a_n and b_n depending on the distributions of these $a_n(\vec{V})$ and $b_n(\vec{V})$ as

$$a_n = \sup\left\{x \in \mathbf{R} : \mathbf{P}(a_n(\vec{V}) \leq x) \leq 1/2\right\},$$

$$b_n = a_n + \frac{\sigma}{2\sqrt{1+\log n}}.$$

We observe with this definition that, since the distribution function of $a_n(\vec{V})$ is right-continuous, we have

$$\mathbf{P}(a_n(\vec{V}) \le a_n) \ge 1/2.$$

Furthermore,

$$\mathbf{P}(b_n(\vec{V}) \ge b_n) \ge \mathbf{P}(a_n(\vec{V}) \ge a_n) = 1 - \mathbf{P}(a_n(\vec{V}) < a_n) \ge 1/2.$$

Then

$$\begin{aligned}
\mathbf{P}\left(\max_{i=1,\dots,n}(W_i' + V_i) \le a_n\right) &= \mathbf{E}\left[\mathbf{P}\left(\max_{i=1,\dots,n}(W_i' + V_i) \le a_n \mid \vec{V}\right)\right] \\
&\ge \mathbf{E}\left[\mathbf{P}\left(\max_{i=1,\dots,n}(W_i' + V_i) \le a_n(\vec{V}), a_n(\vec{V}) \le a_n \mid \vec{V}\right)\right] \\
&= \mathbf{E}\left[\mathbf{P}\left(\max_{i=1,\dots,n}(W_i' + V_i) \le a_n(\vec{V}) \mid \vec{V}\right)\mathbf{1}_{\{a_n(\vec{V}) \le a_n\}}\right] \\
&\ge c_{22}\mathbf{P}(a_n(\vec{V}) \le a_n) \\
&\ge \frac{c_{22}}{2},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}\left(\max_{i=1,\dots,n}(W_i' + V_i) \ge b_n\right) &= \mathbf{E}\left[\mathbf{P}\left(\max_{i=1,\dots,n}(W_i' + V_i) \ge b_n \mid \vec{V}\right)\right] \\
&\ge \mathbf{E}\left[\mathbf{P}\left(\max_{i=1,\dots,n}(W_i' + V_i) \ge b_n(\vec{V}), b_n(\vec{V}) \ge b_n \mid \vec{V}\right)\right] \\
&= \mathbf{E}\left[\mathbf{P}\left(\max_{i=1,\dots,n}(W_i' + V_i) \ge b_n(\vec{V}) \mid \vec{V}\right)\mathbf{1}_{\{b_n(\vec{V}) \ge b_n\}}\right] \\
&\ge c_{22}\mathbf{P}(b_n(\vec{V}) \ge b_n) \\
&\ge \frac{c_{22}}{2}.
\end{aligned}$$

Therefore the original claim has been proved with $c_{21} = c_{22}/2$. □

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