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The variance of the graph distance in the infinite cluster of percolation is sublinear

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Abstract. We consider the standard model of i.i.d. bond percolation on $\mathbb{Z}^d$ of parameter p. When $p > p_c$ where p_c denotes the critical parameter, there exists almost surely a unique infinite cluster $\mathcal{C}_\infty$. Using the recent techniques of Cerf and Dembin (2022), we prove that the variance of the graph distance in $\mathcal{C}_\infty$ between two points of $\mathcal{C}_\infty$ is sublinear. This result extends the works of Benjamini et al. (2003), Benaïm and Rossignol (2008) and Damron et al. (2015) for the study of the variance of passage times in first passage percolation without moment conditions on the edge-weight distribution.

1. Introduction

Bernoulli percolation. The model of Bernoulli percolation is formally defined as follows. Let $\mathbb{E}^d$ be the set of all pairs of nearest neighbours in $\mathbb{Z}^d$. We consider i.i.d. Bernoulli random variables $(B_e)_{e\in\mathbb{E}^d}$ of parameter $p \in [0,1]$. If $B_e = 1$, then the edge e is called **open**; otherwise, the edge is called **closed**. Let $\mathcal{G}_p$ be the graph of the open edges:

$$\mathcal{G}_p := (\mathbb{Z}^d, \{e \in \mathbb{E}^d : B_e = 1\}) .$$

The graph $\mathcal{G}_p$ is called the **percolation graph**. A path is said to be **open** if the path consists only of open edges. This model exhibits a phase transition. Indeed, when $d \geq 2$, there exists a critical parameter $p_c \in (0,1)$ such that for $p > p_c$ (**supercritical regime**), there almost surely exists a unique infinite connected component $\mathcal{C}_\infty$ in $\mathcal{G}_p$. In contrast, for $p < p_c$ (**subcritical regime**), there are no infinite open clusters. We refer to Grimmett (1989) for general backgrounds and known results on Bernoulli percolation. We denote by $\mathcal{D}^{\mathcal{C}_\infty}$ the graph distance in the cluster $\mathcal{C}_\infty$ that is

$$\forall x, y \in \mathbb{Z}^d \qquad \mathcal{D}^{\mathcal{C}_\infty}(x,y) := \inf\Big\{ |r| : r \text{ is a path from } x \text{ to } y \text{ in } \mathcal{C}_\infty \Big\} \tag{1.1}$$

where $|r|$ denotes the number of edges in the path r and we use the convention that $\inf \emptyset = +\infty$. In particular, if x and y are not connected in $\mathcal{C}_\infty$, then we have $\mathcal{D}^{\mathcal{C}_\infty}(x,y) = \infty$. To deal with the fact that $\mathcal{D}^{\mathcal{C}_\infty}(x,y)$ is infinite with positive probability, we will use the technique of Cerf and Théret

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(2016) and introduce regularized points. For x in $\mathbb{Z}^d$, we define $\widetilde{x}$ to be the closest point in $\mathcal{C}_\infty$ for the ℓ_2-norm to x with a deterministic rule to break ties. The advantage of defining regularized points is that for any $x, y \in \mathbb{Z}^d$, $\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{x}, \widetilde{y}) < \infty$ almost surely. A geodesic between $\widetilde{x}$ and $\widetilde{y}$ is a path achieving the infimum in $\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{x}, \widetilde{y})$. Note that the geodesics are not necessarily unique. **From now on, if there are several possible geodesics, we will choose one according to a deterministic rule to break ties**. When we refer to the geodesic, we will refer to the geodesic chosen according to this deterministic rule.

First passage percolation. The model of first passage percolation may be seen as a generalization of the model of percolation. Let G be a distribution on $\mathbb{R}_+ \cup \{+\infty\}$. To each edge $e \in \mathbb{E}^d$, we assign a random variable t_e such that the family $(t_e, e \in \mathbb{E}^d)$ is independent and identically distributed with distribution G. The random variable t_e may be interpreted as the time needed to cross the edge e. We define a random pseudo-metric T on this graph: for any pair of vertices x, $y \in \mathbb{Z}^d$, the random variable $T(x, y)$ is the shortest time to go from x to y, *i.e.*,

$$T(x,y) := \inf \left\{ \sum_{e \in r} t_e \; : \; r \text{ is a path joining } x \text{ to } y \right\} .$$

Note that for the distribution

$$G_p := p\delta_1 + (1-p)\delta_\infty, \quad p > p_c \tag{1.2}$$

the travel time $T(x, y)$ for the law G_p coincides with the graph distance between x and y in $\mathcal{G}_p$ where the edges with infinite passage time correspond to the closed edges. Thanks to classical tools used in first passage percolation, in particular the subadditive ergodic theorem, we can study $\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0}, \widetilde{nx})$. In particular, Cerf and Théret proved in Cerf and Théret (2016) a law of large numbers for the regularized graph distance: there exists a deterministic function $\mu_p : \mathbb{Z}^d \to [0, +\infty)$ such that

$$\forall x \in \mathbb{Z}^d \qquad \lim_{n\to\infty} \frac{\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0}, \widetilde{nx})}{n} = \mu_p(x) \qquad \text{a.s. and in } L^1.$$

The function μ_p is the so-called time constant. This was first proved by Garet–Marchand without introducing the regularized points in Garet and Marchand (2004).

Fluctuations of the travel time. The question of the fluctuations of $T(0, x)$ for general distributions G is a very central question. It has been conjectured by physicists that the variance $\mathrm{Var}(T(0, x))$ should scale as $\|x\|_1^\alpha$ for some constant $\alpha < 1$ depending on the dimension. In particular, in dimension 2, it is conjectured that the model belongs to the KPZ universality class that was introduced by Kardar et al. (1986) in 1986, and that $\alpha = 2/3$. However, beyond some related integrable models, the results obtained in this direction are still very modest. The first upper-bound on the variance was obtained by Kesten (1993). He proved that there exists a constant C such that for all $x \in \mathbb{Z}^d$

$$\mathrm{Var}(T(0,x)) \leq C\|x\|_1$$

under some integrability condition on the distribution G. In their seminal paper Benjamini et al. (2003), Benjamini–Kalai–Schramm proved that the fluctuations are sublinear there exists a constant C such that for all $x \in \mathbb{Z}^d$, $\|x\| \geq 2$

$$\mathrm{Var}(T(0,x)) \leq C \frac{\|x\|_1}{\log \|x\|}$$

for the case of a distribution G that takes only two values. The results of Benjamini et al. (2003) were later extended to continuous distributions that satisfied a modified logarithmic Sobolev inequality by Benaïm and Rossignol (2008) and to more general distributions under moment conditions using a Bernoulli encoding by Damron et al. (2015). In this paper, we aim to extend these results to the case of the distribution G_p defined in (1.2). Namely, we are interested in the variance of the graph distance $\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx})$. The main obstacle to overcome is that the distribution G_p has no finite moment. We obtain that the variance of the graph distance in $\mathcal{C}_\infty$ is sublinear.

Theorem 1.1. *Let $p>p_c$. There exists a positive constant C_0 depending on p and d such that*

$$\forall x\in\mathbb{Z}^d\setminus\{0\}\quad \forall n\geq 1\qquad \mathrm{Var}(\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))\leq C_0\frac{n}{\log n}\,. \tag{1.3}$$

Let us first explain the proof strategy used in Benjamini et al. (2003). To prove that the variance of $T(0,nx)$ is sublinear, Benjamini–Kalai–Schramm use an inequality of Talagrand (1994) related to hypercontractivity. They need to study the influence of the edges that is the expected impact on the passage time $T(0,nx)$ of changing the value of a given edge. To get the logarithm in the denominator using Talagrand's inequality, they need that almost all the edges have a small influence. In this context, the influence of an edge e is related to the probability that the geodesic γ between 0 and nx goes through the edge e. Since there exists no result controlling the probability that the geodesic goes through a given edge, the authors use a trick to circumvent this issue. They randomize the starting point of the geodesic in such a way that the new random variable has a variance that is still close to the original one and such that all the edges of the lattice have a small influence. This trick of randomizing the starting point was later replaced by Damron–Hanson–Sosoe by a geometric average in Damron et al. (2015).

In Benjamini et al. (2003); Benaïm and Rossignol (2008); Damron et al. (2015), moment conditions on the distribution are needed. A first reason why a moment condition is needed, is that without it, we have $\mathrm{Var}(T(x,y))=+\infty$. Note that even if we don't have a good moment condition, this problem may be solved by the use of regularized points. This is exactly in the same spirit as the use of regularized points for the study of the graph distance to ensure that the graph distance is finite and has good moment properties. But, the main reason why moment conditions are needed, is that it enables to obtain a good control on the impact of resampling an edge. When the distribution is bounded by M, resampling an edge on the geodesic cannot affect the passage time by more than M. We can easily upperbound the maximum impact of changing the value of an edge. However, in the context of the graph distance in the infinite cluster of percolation, closing one edge on the geodesic can greatly increase the graph distance and we cannot get a good uniform control of the impact of closing an edge, This is the main issue to extend the previous results to the distribution G_p that can take infinite value. To solve this issue, we use here the recent technology developed by Cerf and Dembin (2022). Before stating the key ingredient of the proof, let us introduce some definitions. Let $p>p_c$. Let $\mathcal{C}_p^e$ be the infinite connected component of $\mathcal{C}_\infty\setminus\{e\}$, it is almost surely unique. For $x\in\mathbb{Z}^d$, denote by $\widetilde{x}^e$ the closest point to x in $\mathcal{C}_\infty^e$. Let us denote by $\mathcal{R}_e$ the following event

$$\mathcal{R}_e:=\{\widetilde{0}=\widetilde{0}^e,\widetilde{nx}=\widetilde{nx}^e\}\,. \tag{1.4}$$

The event $\mathcal{R}_e$ is the event that the regularized points are unchanged when closing the edge e. We will need the following theorem that is the key result to prove the main theorem.

Theorem 1.2. *Let $p>p_c$. There exists a positive constant c_0 depending on p and d such that for any $x\in\mathbb{Z}^d$ and $n\geq 1$, let γ be the geodesic from $\widetilde{0}$ to $\widetilde{nx}$, we have*

$$\mathbb{E}\left[\sum_{e\in\gamma}(\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(\widetilde{0},\widetilde{nx})-\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))^2\mathbf{1}_{\mathcal{R}_e}\right]\leq c_0 n\,.$$

Roughly speaking, this result says that on average, closing an edge on the geodesic modifies the graph distance by at most a constant. This theorem is a consequence of the work of Cerf and Dembin (2022). This theorem together with the Efron–Stein inequality leads to an upper-bound on the variance of order n which is already a new result in the context of the graph distance in the infinite cluster (with some additional technical details due to the use of regularized points). To prove that the variance is sublinear we will use the geometric averaging trick and concentration inequalities used by Damron et al. (2015). This geometric average will ensure that every edge in the lattice has a small influence. Once the key result Theorem 1.2 is proved, the remaining of the proof uses the concentration inequalities in the same way as Damron et al. (2015) with some additional technical difficulties due to the fact that we use regularized points.

Remark 1.3. We believe that our proof strategy together with the Bernoulli encoding used by Damron–Hanson–Sosoe in Damron et al. (2015) can also work for any distribution G on $\mathbb{R}_+\cup\{+\infty\}$ such that $G(\{+\infty\})<1-p_c$.

In Section 2, we present some standard facts about supercritical percolation and we present the concentration inequalities we will use. In Section 3, we prove Theorems 1.1 and 1.2.

2. Background

2.1. *Background on percolation.* We will need the following standard facts about percolation. For $x\in\mathbb{Z}^d$, let $\mathcal{C}(x)$ be the p-open cluster of x. We denote by $\|\cdot\|_2$ the ℓ_2 norm. We have the following theorem that controls the probability of having a large and finite open cluster.

Theorem 2.1 (Theorems 8.18 and 8.19 in Grimmett (1989)). *Let $p>p_c$. There exist positive constants A_1 and A_2 such that*

$$\forall n\geq 1 \qquad \mathbb{P}(0\notin\mathcal{C}_\infty,\mathcal{C}(0)\cap\partial\Lambda_n\neq\emptyset)\leq A_1\exp(-A_2 n)$$

where $\Lambda_n:=[-n,n]^d\cap\mathbb{Z}^d$ and $\partial\Lambda_n:=\{y\notin\Lambda_n:\exists x\in\Lambda_n,\{x,y\}\in\mathbb{E}^d\}$.

The following theorem controls the probability of having a big hole in the infinite cluster.

Theorem 2.2 (Theorem 7 in Grimmett (1989)). *Let $p>p_c$. There exist positive constants A_3 and A_4 such that*

$$\forall n\geq 1 \qquad \mathbb{P}(\mathcal{C}_\infty\cap\Lambda_n=\emptyset)\leq A_3\exp(-A_4 n)\,.$$

The following theorem gives a control on the graph distance.

Theorem 2.3 (Antal and Pisztora (1996)). *Let $p>p_c$. There exist positive constants β, A_5 and A_6 such that*

$$\forall x,y\in\mathbb{Z}^d \quad \forall m\geq\beta\|x-y\|_2 \qquad \mathbb{P}(m\leq\mathcal{D}^{\mathcal{C}_\infty}(x,y)<\infty)\leq A_5\exp(-A_6 m)\,.$$

We will need in what follows the two following estimates that are consequences of Theorem 2.3.

Lemma 2.4. *Let $p>p_c$. Let $k\geq 2$. There exists a constant $\kappa>0$ (depending on k) such that*

$$\forall x,y\in\mathbb{Z}^d \qquad \mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{x},\widetilde{y})^k]\leq\kappa\|x-y\|_2^k\,.$$

The following lemma controls the expected intersection of the geodesic with a box.

Lemma 2.5. *Let $p>p_c$. There exists $\alpha>0$ such that for any geodesic γ between two points in $\mathcal{C}_\infty$, we have*

$$\forall z\in\mathbb{Z}^d \quad \forall m\geq 1 \qquad \mathbb{E}|\gamma\cap(z+\Lambda_m)|\leq\alpha m\,.$$

Let us now prove these two lemmas.

Proof of Lemma 2.4: Set $l=\|\widetilde{x}-x\|_2+\|\widetilde{y}-y\|_2$. Let $m\geq 2\beta\|x-y\|_2$, we have using Theorems 2.2 and 2.3

$$\begin{aligned}\mathbb{P}(\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{x},\widetilde{y})\geq m)&\leq\mathbb{P}\left(l\geq\frac{m}{4d\beta}\right)+\mathbb{P}(\exists w\in(x+\Lambda_{\frac{m}{4d\beta}})\,\exists z\in(y+\Lambda_{\frac{m}{4d\beta}}):m\leq\mathcal{D}^{\mathcal{C}_\infty}(w,z)<\infty)\\&\leq 2A_3\exp\left(-A_4\frac{m}{8d\beta}\right)+2m^{2d}A_5\exp(-A_6 m)\end{aligned}$$

where we use that for any $w \in (x+\Lambda_{m/(4d\beta)})$ and $z \in (y+\Lambda_{m/(4d\beta)})$

$$\beta\|w-z\|_2 \le \frac{m}{2} + \beta\|x-y\|_2 \le m\,.$$

Hence, it yields that for $k \ge 2$,

$$\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{x},\widetilde{y})^k] \le (2\beta\|x-y\|_2)^k + \sum_{j\ge 2\beta\|x-y\|_2} j^k \mathbb{P}(\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{x},\widetilde{y}) \ge j)$$

and the result follows. □

Proof of Lemma 2.5: Let γ be a geodesic between two points in $\mathcal{C}_\infty$. Let $z \in \mathbb{Z}^d$ and $m \ge 1$. Let us assume $\gamma \cap (z+\Lambda_m) \neq \emptyset$, otherwise the result follows trivially. Let us denote by w and y the first and last intersection of γ with $z+\Lambda_m$. The portion of γ between w and y is a geodesic inside $\mathcal{C}_\infty$, its length is equal to $\mathcal{D}^{\mathcal{C}_\infty}(w,y)$. Let us denote by $\mathcal{E}$ the following event

$$\mathcal{E} := \left\{\forall u,v \in (z+\Lambda_m)\cap\mathcal{C}_\infty : \mathcal{D}^{\mathcal{C}_\infty}(u,v) \le 2d\beta m\right\}\,.$$

Thanks to Theorem 2.3, we have

$$\mathbb{P}(\mathcal{E}^c) \le (2m)^{2d} A_5 \exp(-A_6 m)\,.$$

Hence, we have

$$\begin{aligned}\mathbb{E}[|\gamma\cap(z+\Lambda_m)|] &\le \mathbb{E}[|\gamma\cap(z+\Lambda_m)|\mathbf{1}_{\mathcal{E}}] + \mathbb{E}[|\gamma\cap(z+\Lambda_m)|\mathbf{1}_{\mathcal{E}^c}]\\ &\le 2d\beta m + (2m)^{3d}A_5\exp(-A_6 m)\,.\end{aligned}$$

The result follows. □

2.2. *Concentration inequalities.* Let f be a real-valued function on $\{0,1\}^{\mathbb{E}^d}$ in L^2. Let us enumerate the edges of the lattice $\{e_1,e_2,\dots\}$, we can write the following martingale decomposition

$$f - \mathbb{E}f = \sum_{k=1}^{\infty} \mathbb{E}(f|\mathcal{F}_k) - \mathbb{E}(f|\mathcal{F}_{k-1})$$

where $\mathcal{F}_k$ is the σ-algebra generated by the first k edge weights $t_{e_1},\dots,t_{e_k}$ and $\mathcal{F}_0$ is the trivial σ-algebra. Set

$$V_k = \mathbb{E}(f|\mathcal{F}_k) - \mathbb{E}(f|\mathcal{F}_{k-1})\,.$$

We have the following inequality that is proved in Damron et al. (2015) using an inequality proved by Falik and Samorodnitsky (2007). Note that all the content of this section was already present in the work of Benaïm and Rossignol (2008).

Lemma 2.6 (Lemma 3.3. in Damron et al. (2015))**.**

$$\mathrm{Var}(f)\log\left(\frac{\mathrm{Var}(f)}{\sum_{k=1}^{\infty}\mathbb{E}(|V_k|)^2}\right) \le \sum_{k=1}^{\infty}\mathrm{Ent}(V_k^2) \tag{2.1}$$

where Ent *denotes the entropy.*

We recall that the entropy of a function $f \ge 0$, $f \neq 0$ on $\{0,1\}^{\mathbb{E}^d}$ given a distribution π on $\{0,1\}^{\mathbb{E}^d}$ is defined as follows

$$\mathrm{Ent}_\pi(f) := \mathbb{E}_\pi\left[f\log\frac{f}{\mathbb{E}_\pi(f)}\right]\,.$$

Let $p \in (0,1)$. We will omit π when π is $\otimes_{e\in\mathbb{E}^d}\mathrm{Ber}(p)$ where $\mathrm{Ber}(p)$ denotes the Bernoulli distribution of parameter p. For $e \in \mathbb{E}^d$, we define σ_e^1 (respectively σ_e^0 the function that for $t \in \{0,1\}^{\mathbb{E}^d}$ changes t_e into 1 (respectively into 0). Set

$$\Delta_e f := f\circ\sigma_e^0 - f\circ\sigma_e^1\,.$$

Remark 2.7. The edges with value 0 will correspond to the closed edges and the edges with value 1 to the open edges. For f a non-increasing functions on $\{0,1\}^{\mathbb{E}^d}$ we have $\Delta_e f \geq 0$.

We will need the following lemma to control the entropy.

Lemma 2.8. *Let f be such that $\mathbb{E}[f^2] < \infty$. There exists a constant $C > 0$ depending on p such that*

$$\sum_{k=1}^{\infty} \mathrm{Ent}(V_k^2) \leq C \sum_{k=1}^{\infty} \mathbb{E}[(\Delta_{e_k} f)^2]\,. \tag{2.2}$$

For short, we will write Δ_k instead of Δ_{e_k}. The proof of this lemma follows the same idea as in Lemma 6.3 in Damron et al. (2015) but in a simpler context. Note that the proof of this lemma can also be deduced from the proofs in Benaïm and Rossignol (2008). For the sake of completeness, we include the proof of this lemma here. To prove this lemma we need the following lemma.

Lemma 2.9 (Bernoulli log-Sobolev inequalities)**.** *Let $p \in (0,1)$. There exists a positive constant C depending on p such that for any function $g \geq 0$ on $\{0,1\}$*

$$\mathrm{Ent}_{\mathrm{Ber}(p)}[g^2] \leq C(g(0) - g(1))^2\,.$$

We will also need the following theorem.

Theorem 2.10 (Tensorization of the entropy, Theorem 2.3 in Damron et al. (2015))**.** *Let $p \in (0,1)$. Let f be a non-negative L^2 random variable on $\{0,1\}^{\mathbb{E}^d}$. Let $(t_e)_{e\in\mathbb{E}^d}$ be a family of i.i.d. Bernoulli random variables of parameter p and denote π the distribution of the family. For $t \in \{0,1\}^{\mathbb{E}^d}$, denote by $\pi_k(t)$ be the distribution with respect to the k^{th} coordinate, all the other coordinates remain fix. We have*

$$\mathrm{Ent}_\pi(f) \leq \sum_{k=1}^{\infty} \mathbb{E}_\pi[\mathrm{Ent}_{\pi_k}(f)]\,.$$

We have now all the ingredients to prove Lemma 2.8.

Proof of Lemma 2.8: Let $k \geq 1$. The random variable V_k only depends on $t_{e_1}, \dots, t_{e_k}$. Using Theorem 2.10 and Lemma 2.9, we have

$$\mathrm{Ent}_\pi(V_k^2) \leq \sum_{j=1}^{k} \mathbb{E}_\pi[\mathrm{Ent}_{\pi_j}(V_k^2)] \leq C \sum_{j=1}^{k} \mathbb{E}_\pi[(\Delta_j V_k)^2]$$

It follows that

$$\mathbb{E}[(\Delta_j V_k)^2] = \begin{cases} \mathbb{E}[\mathbb{E}[(\Delta_k f)|\mathcal{F}_k]^2] & \text{if } j = k \\ \mathbb{E}[(\mathbb{E}[\Delta_j f|\mathcal{F}_k] - \mathbb{E}[\Delta_j f|\mathcal{F}_{k-1}])^2] & \text{if } j < k\,. \end{cases}$$

Hence

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{j=1}^{k} \mathbb{E}[(\Delta_j V_k)^2] &= \sum_{j=1}^{\infty}\left(\mathbb{E}[\mathbb{E}[(\Delta_j f)|\mathcal{F}_j]^2] + \sum_{k\geq j+1} \mathbb{E}[(\mathbb{E}[\Delta_j f|\mathcal{F}_k] - \mathbb{E}[\Delta_j f|\mathcal{F}_{k-1}])^2]\right)\\ &= \sum_{j=1}^{\infty} \lim_{N\to\infty} \mathbb{E}[\mathbb{E}[\Delta_j f|\mathcal{F}_N]^2] = \sum_{j=1}^{\infty} \mathbb{E}[(\Delta_j f)^2]\end{aligned}$$

where we used the orthogonality of the martingale increments in L^2 and the convergence of closed martingales. The result follows. □

3. Proofs

3.1. *Proof of Theorem 1.1.* Let $p > p_c$. Let $(B_e)_{e\in\mathbb{E}^d}$ be an i.i.d. family of Bernoulli random variables of parameter p. Let $n \geq 1$ and $x \in \mathbb{Z}^d$. Set m be the largest integer such that $m \leq n^{1/4}$ and

$$f((B_e)_{e\in\mathbb{E}^d}) := \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m} \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{z}, \widetilde{nx+z}) \tag{3.1}$$

where edges that have value 1 correspond to edges that are open. The function f is a geometric average of the graph distance, the interest of considering such a function is that it is simpler to prove that all the edges have a small influence. We can prove that the variance of f is close to the original variance we aim to estimate. This is the purpose of the following lemma.

Lemma 3.1. *We have for n large enough*

$$\operatorname{Var} \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0}, \widetilde{nx}) \leq 2\operatorname{Var}(f) + n^{3/4}\,.$$

Let $e \in \mathbb{E}^d$. We recall that $\mathcal{C}_\infty^e$ is the infinite connected component of $\mathcal{C}_\infty \setminus \{e\}$. For $z \in \mathbb{Z}^d$, denote by $\widetilde{z}^e$ the closest point to z in $\mathcal{C}_\infty^e$. For short, set

$$\ell(e) := \mathcal{D}^{\mathcal{C}_\infty^e}(\widetilde{0}^e, \widetilde{nx}^e) - \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0}, \widetilde{nx})\,. \tag{3.2}$$

We will need the two following lemmas that give an upper-bound on $\ell(e)$. Recall that $\mathcal{R}_e$ was defined in (1.4).

Lemma 3.2. *Let $k \geq 1$. There exists κ_1 (depending on k) such that for any $x \in \mathbb{Z}^d$, $n \geq 1$ and $e \in \mathbb{E}^d$*

$$\mathbb{E}[\ell(e)^k \mathbf{1}_{e\in\gamma}\mathbf{1}_{\mathcal{R}_e}] \leq \kappa_1$$

where γ is the geodesic between $\widetilde{0}$ and $\widetilde{nx}$ in $\mathcal{C}_\infty$.

The following lemma upperbounds the total influence of the edges that change the regularized points.

Lemma 3.3. *There exists $C > 0$ such that for any $x \in \mathbb{Z}^d$, $n \geq 1$*

$$\sum_{k=1}^{\infty} \mathbb{E}[\ell(e_k)^2 \mathbf{1}_{\mathcal{R}_{e_k}^c}] \leq C \log^d n\,.$$

Lemma 3.4. *We have for n large enough*

$$\sum_{k=1}^{\infty} \mathbb{E}(|V_k|)^2 \leq n^{15/16}.$$

Before proving all these lemmas, let us show how togteher with Theorem 1.2 they imply Theorem 1.1.

Proof of Theorem 1.1: If $\operatorname{Var}(f) \leq n^{31/32}$, then thanks to Lemma 3.1, the result follows. Otherwise, we have thanks to Lemma 2.6

$$\operatorname{Var}(f)\log\left(\frac{n^{31/32}}{\sum_{k=1}^{\infty}\mathbb{E}(|V_k|)^2}\right) \leq \sum_{k=1}^{\infty}\operatorname{Ent}(V_k^2)$$

and using Lemma 3.4 we get

$$\operatorname{Var}(f) \leq \frac{32}{\log n}\sum_{k=1}^{\infty}\operatorname{Ent}(V_k^2). \tag{3.3}$$

Thanks to Lemma 2.4, we get $\mathbb{E}(f^2)<\infty$. Finally using Lemma 2.8, we get

$$\sum_{k=1}^{\infty}\mathrm{Ent}(V_k^2)\leq C\sum_{k=1}^{\infty}\mathbb{E}((\Delta_{e_k}f)^2)\leq C\sum_{k=1}^{\infty}\frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}\mathbb{E}((\Delta_{e_k}\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{z},\widetilde{nx+z}))^2)$$

where we use the Cauchy-Schwarz in the second inequality. Using the invariance by translation in distribution, it follows that

$$\sum_{k=1}^{\infty}\mathrm{Ent}(V_k^2)\leq C\frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}\sum_{k=1}^{\infty}\mathbb{E}((\Delta_{e_k}\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{z},\widetilde{nx+z}))^2)=C\sum_{k=1}^{\infty}\mathbb{E}((\Delta_{e_k}\tau)^2)$$

where for short we write $\tau=\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx})$. Let $e\in\mathbb{E}^d$. Note that $\Delta_e\tau$ is independent of B_e, it follows that

$$\mathbb{E}((\Delta_e\tau)^2)=\frac{1}{p}\mathbb{E}((\Delta_e\tau)^2\mathbf{1}_{B_e=1})\,.$$

Let us denote by γ the geodesic between $\widetilde{0}$ and $\widetilde{nx}$. We recall that $\mathcal{R}_e$ was defined in (1.4) as the event where closing the edge e does not modify the regularized points. Let us assume that we are on the event $\mathcal{R}_e$ and that e is originally open and outside the geodesic γ, then closing e has no impact on the geodesic and $\Delta_e\tau=0$. It yields that

$$(\Delta_e\tau)^2\mathbf{1}_{B_e=1}\mathbf{1}_{\mathcal{R}_e}=(\Delta_e\tau)^2\mathbf{1}_{\mathcal{R}_e}\mathbf{1}_{e\in\gamma}\mathbf{1}_{B_e=1}\,.$$

Besides, we have

$$\Delta_e\tau\mathbf{1}_{B_e=1}=\ell(e)\mathbf{1}_{B_e=1}\,.$$

Hence, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\mathrm{Ent}(V_k^2)&\leq\frac{C}{p}\sum_{k=1}^{\infty}\mathbb{E}((\Delta_{e_k}\tau)^2\mathbf{1}_{\mathcal{R}_{e_k}}\mathbf{1}_{e_k\in\gamma})+\mathbb{E}((\Delta_{e_k}\tau)^2\mathbf{1}_{\mathcal{R}_{e_k}^c}\mathbf{1}_{B_{e_k}=1})\\&\leq\frac{C}{p}\mathbb{E}\left[\sum_{e\in\gamma}(\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(\widetilde{0},\widetilde{nx})-\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))^2\mathbf{1}_{\mathcal{R}_e}\right]+\frac{C}{p}\sum_{k=1}^{\infty}\mathbb{E}(\ell(e_k)^2\mathbf{1}_{\mathcal{R}_{e_k}^c})\,.\end{aligned}$$

Thanks to Theorem 1.2 and Lemma 3.3, we have for n large enough

$$\sum_{k=1}^{\infty}\mathrm{Ent}(V_k^2)\leq 2c_0\frac{C}{p_c}n\,.$$

Using inequality (3.3) and Lemma 3.1, the result follows.

□

Let us now prove the lemmas.

Proof of Lemma 3.1: We have

$$|\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx})-\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{z},\widetilde{z+nx})|\leq\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{z})+\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{nx},\widetilde{z+nx})\,.$$

Hence,

$$|f-\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx})|\leq\frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}(\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{z})+\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{nx},\widetilde{z+nx}))\tag{3.4}$$

Recall that f was defined in (3.1). It is easy to check using the invariance by translation in distribution that

$$\mathbb{E}(f)=\mathbb{E}(\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))\,.$$

We recall that $m \le n^{1/4}$. It follows that

$$\begin{aligned}\mathrm{Var}(\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx})) &= \mathbb{E}((\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}) - \mathbb{E}(f))^2) \le 2\mathbb{E}((f - \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))^2) + 2\,\mathrm{Var}(f)\\ &\le \frac{4}{|\Lambda_m|}\sum_{z\in\Lambda_m}\mathbb{E}(\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{z})^2) + 2\,\mathrm{Var}(f)\\ &\le 2\,\mathrm{Var}(f) + 4d^2\kappa m^2 \le 2\,\mathrm{Var}(f) + n^{3/4}\end{aligned}$$

where in the second inequality we use the Cauchy Schwarz inequality together with (3.4) and in the second to last inequality, we used Lemma 2.4. The result follows. □

Theorem 1.1 trivially holds for $p = 1$. We here work for $p \in (p_c, 1)$.

Proof of Lemma 3.2: Let $e = \{w, z\} \in \mathbb{E}^d$ and $k \ge 1$. We have

$$\begin{aligned}\mathbb{E}[\ell(e)^k \mathbf{1}_{\mathcal{R}_e}\mathbf{1}_{e\in\gamma}] &\le \mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(w,z)^k \mathbf{1}_{w,z\in\mathcal{C}_\infty^e}]\\ &= \frac{1}{1-p}\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(w,z)^k \mathbf{1}_{w,z\in\mathcal{C}_\infty^e}\mathbf{1}_{B_e=0}]\\ &\le \frac{1}{1-p}\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(w,z)^k \mathbf{1}_{w,z\in\mathcal{C}_\infty}] \le \frac{\kappa}{1-p}\end{aligned}$$

where we use that $\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(w,z)^k \mathbf{1}_{w,z\in\mathcal{C}_\infty^e}$ is independent of B_e. In the last inequality, we used Lemma 2.4.

□

Proof of Lemma 3.3: Let $e = \{w, z\} \in \mathbb{E}^d$. Let $\mathcal{C}_\infty^e$ be the infinite connected component of $\mathcal{C}_\infty \setminus \{e\}$. For $y \in \mathbb{Z}^d$, denote by $\mathcal{C}^e(y)$ the connected component of y in the graph $\mathcal{G}_p \setminus \{e\}$. We will need the following estimate

$$\begin{aligned}\mathbb{E}[\ell(e)^4] &= \mathbb{E}[(\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(\widetilde{0}^e,\widetilde{nx}^e) - \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))^4]\\ &\le 8(\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(\widetilde{0}^e,\widetilde{nx}^e)^4] + \mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx})^4])\\ &\le 8\frac{2-p}{1-p}\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))^4] \le C_2 n^4\end{aligned} \tag{3.5}$$

where we use Lemma 2.4 in the last inequality and C_2 is a constant depending on p. Set

$$l := \|w\|_\infty\,.$$

We have

$$\mathbb{E}[\ell(e)^2\mathbf{1}_{\mathcal{R}_e^c}] \le \mathbb{E}[\ell(e)^2\mathbf{1}_{\widetilde{0}\neq\widetilde{0}^e}] + \mathbb{E}[\ell(e)^2\mathbf{1}_{\widetilde{nx}\neq\widetilde{nx}^e,\widetilde{0}=\widetilde{0}^e}].$$

Let us upper-bound the probability of the event $\{\widetilde{0} \neq \widetilde{0}^e\}$. Note that if $\widetilde{0} \neq \widetilde{0}^e$ then $\widetilde{0}$ is in a finite cluster in $\mathcal{C}_\infty^e$ and is connected to one of the endpoint of e. Hence, either $\mathcal{C}^e(w)$ is finite and contains $\widetilde{0}$ or $\mathcal{C}^e(z)$ is finite and contains $\widetilde{0}$. Hence,

$$\begin{aligned}\mathbb{P}(\widetilde{0}\neq\widetilde{0}^e) &\le \mathbb{P}(\widetilde{0}\notin\Lambda_{l/2}) + \mathbb{P}(\widetilde{0}\in\Lambda_{l/2},\widetilde{0}\neq\widetilde{0}^e)\\ &\le \mathbb{P}(\mathcal{C}_\infty\cap\Lambda_{l/2}=\emptyset) + \mathbb{P}\left(\frac{l}{2}\le|\mathcal{C}^e(z)|<\infty\right) + \mathbb{P}\left(\frac{l}{2}\le|\mathcal{C}^e(w)|<\infty\right)\\ &\le A_3\exp\left(-A_4\frac{l}{2}\right) + \frac{2A_1}{1-p}\exp\left(-A_2\frac{l}{2}\right)\end{aligned}$$

where we used Theorems 2.2 and 2.1. It follows that using the previous inequality and inequality (3.5)

$$\mathbb{E}[\ell(e)^2\mathbf{1}_{\widetilde{0}\neq\widetilde{0}^e}] \le \sqrt{\mathbb{E}[\ell(e)^4]\mathbb{P}(\widetilde{0}\neq\widetilde{0}^e)} \le Cn^2\exp(-c\|w\|_\infty)$$

for some constants $C, c > 0$ depending on p and d. Similarly we have

$$\mathbb{E}[\ell(e)^2 \mathbf{1}_{\widetilde{nx}\neq\widetilde{nx}^e}] \leq Cn^2 \exp(-c\|nx - w\|_\infty)\,.$$

Note that these bounds are not good enough when w is close to one of the endpoint. In that case, we use a different upperbound. On the event $\{\widetilde{0} = \widetilde{0}^e, \widetilde{nx} \neq \widetilde{nx}^e\}$, we must have $e \in \gamma$ otherwise it would imply that $\widetilde{0}$ is connected to $\widetilde{nx}$ in $\mathcal{C}^e_\infty$ and contradict $\widetilde{nx} \neq \widetilde{nx}^e$. Besides, we have either $\widetilde{nx}^e \in \mathcal{C}^e(w)$ and $\widetilde{0} \in \mathcal{C}^e(w)$ or $\widetilde{nx}^e \in \mathcal{C}^e(z)$ and $\widetilde{0} \in \mathcal{C}^e(z)$. Let us consider the first case. We have

$$\mathcal{D}^{\mathcal{C}^e_\infty}(\widetilde{0}, \widetilde{nx}^e) \leq \mathcal{D}^{\mathcal{C}^e_\infty}(\widetilde{0}, w) + \mathcal{D}^{\mathcal{C}^e_\infty}(w, \widetilde{nx}^e) = \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0}, w) + \mathcal{D}^{\mathcal{C}^e_\infty}(w, \widetilde{nx}^e)$$

and

$$\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0}, \widetilde{nx}^e) = \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0}, w) + \mathcal{D}^{\mathcal{C}_\infty}(w, \widetilde{nx}^e)\,.$$

Finally, we have

$$\begin{aligned}&\ell(e)^2 \mathbf{1}_{\widetilde{0}=\widetilde{0}^e} \mathbf{1}_{\widetilde{nx}\neq\widetilde{nx}^e}\\ &\qquad\leq (\mathcal{D}^{\mathcal{C}^e_\infty}(w, \widetilde{nx}^e) - \mathcal{D}^{\mathcal{C}_\infty}(w, \widetilde{nx}))^2 \mathbf{1}_{\widetilde{nx}^e\in\mathcal{C}^e(w)} + (\mathcal{D}^{\mathcal{C}^e_\infty}(z, \widetilde{nx}^e) - \mathcal{D}^{\mathcal{C}_\infty}(z, \widetilde{nx}))^2 \mathbf{1}_{\widetilde{nx}^e\in\mathcal{C}^e(z)}\,.\end{aligned}$$

Using Lemma 2.4 and similar arguments as in the proof of Lemma 3.2, it yields that

$$\begin{aligned}&\mathbb{E}[\ell(e)^2 \mathbf{1}_{\widetilde{0}=\widetilde{0}^e} \mathbf{1}_{\widetilde{nx}\neq\widetilde{nx}^e}]\\ &\quad\leq 2\mathbb{E}[\mathcal{D}^{\mathcal{C}^e_\infty}(w, \widetilde{nx}^e)^2 \mathbf{1}_{\widetilde{nx}^e\in\mathcal{C}^e(w)}] + 2\mathbb{E}[\mathcal{D}^{\mathcal{C}^e_\infty}(z, \widetilde{nx}^e)^2 \mathbf{1}_{\widetilde{nx}^e\in\mathcal{C}^e(z)}]\\ &\quad+ 2\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(w, \widetilde{nx})^2 \mathbf{1}_{\widetilde{nx}^e\in\mathcal{C}^e(w)}] + 2\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(z, \widetilde{nx})^2 \mathbf{1}_{\widetilde{nx}^e\in\mathcal{C}^e(z)}]\\ &\quad\leq 2\mathbb{E}[\mathcal{D}^{\mathcal{C}^e_\infty}(\widetilde{w}^e, \widetilde{nx}^e)^2] + 2\mathbb{E}[\mathcal{D}^{\mathcal{C}^e_\infty}(\widetilde{z}^e, \widetilde{nx}^e)^2] + 2\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{w}, \widetilde{nx})^2] + 2\mathbb{E}[\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{z}, \widetilde{nx})^2]\\ &\quad\leq \frac{8\kappa_1}{1-p}\,.\end{aligned}$$

As a result, we have

$$\begin{aligned}\mathbb{E}[\ell(e)^2 \mathbf{1}_{\mathcal{R}^c_e}] &\leq \mathbb{E}[\ell(e)^2 \mathbf{1}_{\widetilde{0}\neq\widetilde{0}^e}] + \mathbb{E}[\ell(e)^2 \mathbf{1}_{\widetilde{0}=\widetilde{0}^e} \mathbf{1}_{\widetilde{nx}\neq\widetilde{nx}^e}]\\ &\leq Cn^2 \exp(-c\|w\|_\infty) + \min\left(\frac{8\kappa_1}{1-p}, Cn^2 \exp(-c\|nx - w\|)_\infty\right)\,.\end{aligned}$$

and similarly,

$$\mathbb{E}[\ell(e)^2 \mathbf{1}_{\mathcal{R}^c_e}] \leq Cn^2 \exp(-c\|nx - w\|_\infty) + \min\left(\frac{8\kappa_1}{1-p}, Cn^2 \exp(-c\|w\|)_\infty\right)\,.$$

Finally, we have

$$\sum_{e\in\mathbb{E}^d} \mathbb{E}[\ell(e)^2 \mathbf{1}_{\mathcal{R}^c_e}] \leq \sum_{e\in(\Lambda_{C_0 \log n}\cup(nx+\Lambda_{C_0 \log n}))} \frac{16\kappa_1}{1-p} + \sum_{j\geq C_0 \log n} c_d j^{d-1} C \exp(-cj)$$

where c_d is a constant depending only on d. It follows that for n large enough, we have

$$\sum_{e\in\mathbb{E}^d} \mathbb{E}[\ell(e)^2 \mathbf{1}_{\mathcal{R}^c_e}] = O(\log^d n)\,.$$

The result follows. □

Proof of Lemma 3.4: For short write $\tau_z = \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{z}, \widetilde{z + nx})$. Note that $\mathbb{E}[\tau_z|\mathcal{F}_{k-1}] = p\mathbb{E}[\tau_z \circ \sigma^1_k|\mathcal{F}_k] + (1-p)\mathbb{E}[\tau_z \circ \sigma^0_k|\mathcal{F}_k]$ and $\mathbb{E}[\tau_z|\mathcal{F}_{k-1}] = B_{e_k}\mathbb{E}[\tau_z \circ \sigma^1_k|\mathcal{F}_k] + (1 - B_{e_k})\mathbb{E}[\tau_z \circ \sigma^0_k|\mathcal{F}_k]$. It follows that

$$\mathbb{E}|\mathbb{E}(\tau_z|\mathcal{F}_k) - \mathbb{E}(\tau_z|\mathcal{F}_{k-1})| = \mathbb{E}|(B_{e_k} - p)\mathbb{E}[\Delta_{e_k}\tau_z|\mathcal{F}_k]| = 2p(1-p)\mathbb{E}[\Delta_{e_k}\tau_z].$$

It yields

$$\begin{aligned}\mathbb{E}[|V_k|] &\le \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}\mathbb{E}|\mathbb{E}(\tau_z|\mathcal{F}_k)-\mathbb{E}(\tau_z|\mathcal{F}_{k-1})|\\ &= \frac{2p(1-p)}{|\Lambda_m|}\sum_{z\in\Lambda_m}\mathbb{E}|\Delta_{e_k}\tau_z|\\ &\le \frac{1}{|\Lambda_m|}\sum_{z\in\Lambda_m}\mathbb{E}|\Delta_{e_k}\tau_z| \qquad (3.6)\\ &= \frac{1}{p|\Lambda_m|}\sum_{z\in\Lambda_m}\mathbb{E}[|\Delta_{e_k-z}\tau_0|\mathbf{1}_{B_{e_k-z}=1}]\\ &= \frac{1}{p|\Lambda_m|}\sum_{z\in\Lambda_m}(\mathbb{E}[|\ell(e_k-z)|\mathbf{1}_{\mathcal{R}^c_{e_k-z}}]+\mathbb{E}[\ell(e_k-z)\mathbf{1}_{\mathcal{R}_{e_k-z}}\mathbf{1}_{e_k-z\in\gamma}]\end{aligned}$$

where we used similar arguments to the ones in the proof of Theorem 1.1, in particular that $\Delta_e\tau_z$ is independent of B_e. Recall that γ is the geodesic between $\widetilde{0}$ and $\widetilde{nx}$ Using Lemma 3.3, we get

$$\sum_{z\in\Lambda_m}\mathbb{E}[|\ell(e_k-z)|\mathbf{1}_{\mathcal{R}^c_{e_k-z}}]\le\sum_{k=1}^{\infty}\mathbb{E}[\ell(e_k)^2\mathbf{1}_{\mathcal{R}^c_{e_k}}]\le C\log^d n\,. \qquad (3.7)$$

Using the Cauchy-Schwarz inequality and Lemma 3.2, we have

$$\begin{aligned}\sum_{z\in\Lambda_m}\mathbb{E}[\ell(e_k-z)\mathbf{1}_{\mathcal{R}_{e_k-z}}\mathbf{1}_{e_k-z\in\gamma}] &\le \left(\sum_{z\in\Lambda_m}\mathbb{E}[\ell(e_k-z)^2\mathbf{1}_{\mathcal{R}_{e_k-z}}\mathbf{1}_{e_k-z\in\gamma}]\right)^{1/2}\left(\sum_{z\in\Lambda_m}\mathbb{E}[\mathbf{1}_{e_k-z\in\gamma}]\right)^{1/2}\\ &\le \sqrt{\kappa_1|\Lambda_m|}\sqrt{\mathbb{E}|\gamma\cap(\Lambda_m+e_k)|}\,.\end{aligned} \qquad (3.8)$$

Using Lemma 2.5 and the inequalities (3.6), (3.7) and (3.8), it follows that for n large enough

$$\mathbb{E}[|V_k|]\le\frac{2^d}{p_c}\sqrt{\alpha\kappa_1}\,m^{(1-d)/2}\,.$$

Besides, using Theorem 1.2, Lemma 3.3 and inequality (3.6), we have for n large enough

$$\sum_{k=1}^{\infty}\mathbb{E}[|V_k|]\le\frac{1}{p}\mathbb{E}\left[\sum_{e\in\gamma}(\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(\widetilde{0},\widetilde{nx})-\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))^2\mathbf{1}_{\mathcal{R}_e}\right]+\frac{1}{p}\sum_{k=1}^{\infty}\mathbb{E}[|\ell(e_k)|\mathbf{1}_{\mathcal{R}^c_{e_k}}]\le\frac{2}{p_c}c_0 n\,.$$

Finally, combining the two previous inequalities we get for some constant C for n large enough

$$\sum_{k=1}^{\infty}\mathbb{E}[|V_k|]^2\le C\,m^{(1-d)/2}\sum_{k=1}^{\infty}\mathbb{E}[|V_k|]\le\frac{4}{p}Cc_0 n\,m^{(1-d)/2}\le n^{15/16}$$

where we use that $m^{(d-1)/2}\ge n^{1/8}/2$ (we recall that m is the largest integer such that $m\le n^{1/4}$). ☐

3.2. *Proof of Theorem 1.2.* In this section, we will use results of Cerf and Dembin (2022). The following Theorem is deduced from the results in Cerf and Dembin (2022) Set $\mathrm{N}=\lfloor n^{1/3d}\rfloor$.

Theorem 3.5. *Let $p>p_c$. There exit positive constants C_1, c_2, κ such that the following holds. Let $x\in\mathbb{Z}^d$ and $n\ge 1$. Let γ be the geodesic between $\widetilde{0}$ and $\widetilde{nx}$. Set $\overline{\gamma}=\gamma\setminus((\Lambda_\mathrm{N}+\widetilde{0})\cup(\Lambda_\mathrm{N}+\widetilde{nx})$. There exists a random family $(\mathfrak{c}_e)_{e\in\overline{\gamma}}$ of positive real numbers such that*

$$\forall e\in\overline{\gamma}\qquad \ell(e)\mathbf{1}_{\mathcal{R}_e}=(\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(\widetilde{0},\widetilde{nx})-\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))\mathbf{1}_{\mathcal{R}_e}\le\mathfrak{c}_e \qquad (3.9)$$

and

$$\mathbb{P}\left(\sum_{e\in\overline{\gamma}} \mathfrak{c}_e^2 \geq \kappa n\right) \leq C_1 \exp(-c_2 n^{1/(6d^2+1)}). \tag{3.10}$$

Sketch of the proof of Theorem 3.5: Proposition 3.5 in Cerf and Dembin (2022) states that for any path we can associate to each edge in $\overline{\gamma}$ a shell. A shell is a $*$-connected set of boxes with good connectivity properties surrounding the edge. We refer to Cerf and Dembin (2022) for a precise definition of the shells. Thanks to the good connectivity of the boxes in the shell, it enables us to build a bypass of the edge e in a neighborhood of the shell of well-controlled length. In particular, Proposition 3.6 enables to bound the size of this bypass of the edge e by $\kappa_0|\,\mathrm{shell}(e)|$. Knowing that there exists a bypass of length at most $\kappa_0|\,\mathrm{shell}(e)|$ enables us to upper-bound the number of extra edges we need to join $\widetilde{0}$ and $\widetilde{nx}$ when we close the edge e. Finally, Proposition 4.5 enables us to have with very high probability a good control on the average size of the shells built in Proposition 3.5. Let $\varepsilon > 0$ small enough depending on p and such that $p-\varepsilon > p_c$. Applying Proposition 3.5 in Cerf and Dembin (2022) to $p-\varepsilon, p$ and γ, there exists a random variable $N_{M(\gamma)}$ associated to γ and a family $(\mathrm{shell}(e), e\in\widetilde{\gamma})$ where $\widetilde{\gamma} = \gamma \setminus ((\Lambda_{4N_{M(\gamma)}} + \widetilde{0}) \cup (\Lambda_{4N_{M(\gamma)}} + \widetilde{nx})$.

Remark 3.6. Note that here we will build $(p-\varepsilon)$ bypass, so we implicitly work here with a coupling of the bond percolation of parameter $p-\varepsilon$ and p in such a way that a $(p-\varepsilon)$-open edge is also p-open. Actually, the proof of Proposition 3.5 still holds true when $p = q$ and the definition of a good box becomes simpler.

Thanks to Proposition 3.6 in Cerf and Dembin (2022)

$$\forall e \in \overline{\gamma} \qquad \ell(e)\mathbf{1}_{\mathcal{R}_e} = (\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(\widetilde{0},\widetilde{nx}) - \mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))\mathbf{1}_{\mathcal{R}_e} \leq \kappa_0|\,\mathrm{shell}(e)| \tag{3.11}$$

where κ_0 is a constant depending on p, d and ε. Thanks to the control on the size of the family $(|\,\mathrm{shell}(e)|, e\in\overline{\gamma})$ in Proposition 3.5 and the Proposition 4.5 in Cerf and Dembin (2022), there exist positive constants C_1, c_2 and κ such that

$$\mathbb{P}(\mathcal{E}_0^c) \leq C_1 \exp(-c_2 n^{1/(6d^2+1)})\,. \tag{3.12}$$

where

$$\mathcal{E}_0 := \left\{ N_M(\gamma) \leq n^{1/3d},\ \sum_{e\in\overline{\gamma}} (|\,\mathrm{shell}(e)|)^2 \leq \kappa n \right\}.$$

We conclude by setting for all $e\in\overline{\gamma}\cap\widetilde{\gamma}$, $\mathfrak{c}_e = \kappa_0|\,\mathrm{shell}(e)|$ and for $e\in\overline{\gamma}\setminus\widetilde{\gamma}$, $\mathfrak{c}_e = \infty$. The result follows. □

Proof of Theorem 1.1: Let $(\mathfrak{c}_e)_{e\in\overline{\gamma}}$ be the family defined in Theorem 3.5. Define

$$\mathcal{E}_0 := \left\{\sum_{e\in\overline{\gamma}} \mathfrak{c}_e^2 \geq \kappa n\right\}.$$

Using Theorem 3.5, it follows

$$\begin{aligned}\mathbb{E}\left[\sum_{e\in\gamma} \ell(e)^2\mathbf{1}_{\mathcal{R}_e}\right] &\leq \mathbb{E}\left[\sum_{e\in\gamma} \ell(e)^2\mathbf{1}_{\mathcal{R}_e}\mathbf{1}_{\mathcal{E}_0}\right] + \mathbb{E}\left[\sum_{e\in\gamma} \ell(e)^2\mathbf{1}_{\mathcal{R}_e}\mathbf{1}_{\mathcal{E}_0^c}\right]\\ &\leq \kappa n + \mathbb{E}\left[\sum_{e\in\gamma\cap(\Lambda_{\mathrm{N}}\cup(nx+\Lambda_{\mathrm{N}}))} \ell(e)^2\mathbf{1}_{\mathcal{R}_e}\right] + \sum_{e\in\mathbb{E}^d} \mathbb{E}\left[\ell(e)^4\mathbf{1}_{\mathcal{R}_e}\mathbf{1}_{e\in\gamma}\right]^{1/2}\sqrt{\mathbb{P}(\mathcal{E}_0^c)}\end{aligned}$$

where we use the Cauchy-Schwarz inequality in the last inequality. Besides, we have by the Cauchy-Schwarz inequality and Lemma 3.2 that

$$\sum_{e\in\mathbb{E}^d}\mathbb{E}\left[\ell(e)^4\mathbf{1}_{\mathcal{R}_e}\mathbf{1}_{e\in\gamma}\right]^{1/2}\leq\sum_{e\in\mathbb{E}^d}\mathbb{E}[\ell(e)^8\mathbf{1}_{\mathcal{R}_e}\mathbf{1}_{e\in\gamma}]^{1/4}\mathbb{P}(e\in\gamma)^{1/4}\leq\kappa_1^{1/4}\sum_{e\in\mathbb{E}^d}\mathbb{P}(e\in\gamma)^{1/4}\,.$$

It is easy to check that the right hand side is at most polynomial in n using for instance Theorem 2.3. Using (3.12), it follows that the following quantity goes to 0 when n goes to infinity

$$\sum_{e\in\mathbb{E}^d}\mathbb{E}\left[\ell(e)^4\mathbf{1}_{\mathcal{R}_e}\mathbf{1}_{e\in\gamma}\right]^{1/2}\sqrt{\mathbb{P}(\mathcal{E}_0^c)}\,.$$

Thanks to Lemma 3.2, we have

$$\mathbb{E}\left[\sum_{e\in\gamma\cap(\Lambda_{\mathrm{N}}\cup(\Lambda_{\mathrm{N}}+nx))}\ell(e)^2\mathbf{1}_{\mathcal{R}_e}\right]=\sum_{e\in(\Lambda_{\mathrm{N}}\cup(\Lambda_{\mathrm{N}}+nx))}\mathbb{E}[\ell(e)^2\mathbf{1}_{e\in\gamma}\mathbf{1}_{\mathcal{R}_e}]\leq 2\kappa_1(2N)^d\leq 2^{d+1}\kappa_1 n^{1/3}.$$

Combining the previous inequalities, we get for n large enough

$$\mathbb{E}\left[\sum_{e\in\gamma}(\mathcal{D}^{\mathcal{C}_\infty\setminus\{e\}}(\widetilde{0},\widetilde{nx})-\mathcal{D}^{\mathcal{C}_\infty}(\widetilde{0},\widetilde{nx}))^2\mathbf{1}_{\mathcal{R}_e}\right]\leq 2\kappa n\,.$$

The result follows.

□

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