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First passage percolation for weakly correlated fields

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Abstract. Let T be a random ergodic pseudometric over $\mathbb{R}^d$. This setting generalizes classical *first passage percolation* over $\mathbb{Z}^d$. We provide simple conditions on T (decay of instant one-arms and quasi-independence) that ensure the positivity of its time constants, that is almost surely, the pseudo-distance given by T from the origin is asymptotically a norm. This theorem applies in particular to Voronoi percolation and smooth Gaussian fields with weak positive correlations.

Contents

1. Introduction 345
2. Random pseudometrics 348
2.1. Statements of the main results 348
2.2. Positivity of the time constant 350
2.3. Vanishing of the time constant 355
2.4. The shape theorem 355
3. Applications 358
3.1. Voronoi FPP 359
3.2. Gaussian FPP 362
Acknowledgements 365
References 366

1. Introduction

First passage percolation (FPP) was first introduced by Hammersley and Welsh (1965), see Auffinger et al. (2017) for an introduction to the subject. Let $(\mathbb{Z}^d, \mathbb{E}^d)$ be the hypercubic lattice, ν be a probability law on $\mathbb{R}_+$, and $\sigma_\nu : \mathbb{E}^d \to \mathbb{R}_+$ be such that every edge $e \in \mathbb{E}^d$ is endowed with an independent time $\sigma_\nu(e) \in \mathbb{R}_+$ following the law ν. For any two vertices x, y in $\mathbb{Z}^d$, a *path* between

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x and y is a continuous path from x to y made of edges. Then, the random time or first passage time between x and y is defined by:

$$T(x,y) := \inf_{\gamma \text{ path } x\to y} \sum_{e\in\gamma} \sigma_\nu(e). \tag{1.1}$$

We have hence endowed $\mathbb{Z}^d$ with a random pseudometric. It is not necessarily a metric since T can vanish even if the points are different. For any probability measure ν on $\mathbb{R}_+$, define the following conditions:

(a) (Finite expectation) $\mathbb{E}\min(\sigma_\nu(1),\cdots,\sigma_\nu(2d)) < \infty$
(b) (Finite moment) $\mathbb{E}[\min(\sigma_\nu(1),\cdots,\sigma_\nu(2d))^d] < \infty$,

where the $\sigma_\nu(i)$'s are i.i.d random variables with law ν. Recall that a $\mathbb{Q}$-*semi-norm* over $\mathbb{R}^d$ is a map $\mu : \mathbb{R}^d \to \mathbb{R}_+$ satisfying

$$\forall(\lambda,x)\in\mathbb{Q}\times\mathbb{R}^d, \mu(\lambda x) = |\lambda|\mu(x),$$

and $\forall(x,y)\in(\mathbb{R}^d)^2,\ \mu(x+y)\le\mu(x)+\mu(y)$. The first main result in this domain is a consequence of the ergodicity of the model:

Theorem 1.1. *(Hammersley and Welsh, 1965) Let ν be a probability measure over $\mathbb{R}_+$ satisfying condition (a). Then, there exists a $\mathbb{Q}$-semi-norm μ_ν such that for any $w\in\mathbb{Z}^d$,*

$$\lim_{n\to+\infty}\frac{1}{n}T(0,nw) = \mu_\nu(w) \quad \textit{almost surely and } L^1. \tag{1.2}$$

For Bernoulli percolation, $p\in[0,1]$ is fixed, and any edge is given independently a number σ_p, 0 with probability p and 1 with probability $1-p$, that is $\nu = p\delta_0 + (1-p)\delta_1$. Let $p_c(d)$ be the critical threshold for Bernoulli bond percolation on $\mathbb{Z}^d$, that is

$$p_c(d) = \sup\{p\in[0,1],\ \text{there is no infinite component of } \{\sigma_p = 0\} \text{ a.s.}\}.$$

It is well known Grimmett (1999) that for any $d\ge 2$, $p_c(d)\in]0,1[$, and that $p_c(2)=1/2$. The second FPP result, namely Theorem 1.2, provides a precise link between the local law ν and the global behaviour of the time constant μ_ν.

Theorem 1.2. *(Kesten, 1986) Let ν be a probability measure over $\mathbb{R}_+$ satisfying condition (a). Then,*

$$\mu_\nu \textit{ is a norm } \Leftrightarrow \nu(\{0\}) < p_c(d).$$

Notice that for Bernoulli percolation, the condition is equivalent to $p < p_c(d)$. For subcritical laws, a natural question is to study the geometry of the large balls defined by the pseudometric T. For this define:

$$\forall t\ge 0,\ B_t := \{x\in\mathbb{Z}^d, T(x,0)\le t\} + [-1/2,1/2]^d$$

the family of balls in $\mathbb{R}^d$ defined by the pseudometric T. In 1981, J. T. Cox and R. Durrett proved the following geometric result:

Theorem 1.3. *(Cox and Durrett (1981) for $d=2$ and Kesten (1986) for $d\ge 2$) Let ν be a probability measure over $\mathbb{R}_+$ satisfying condition (b) and T be defined by (1.1).*

(1) If $\nu(\{0\})\ge p_c(d)$, then for any $M>0$,

$$\mathbb{P}[M\mathbb{B}\subset \frac{1}{t}B_t \textit{ for } t \textit{ large enough }] = 1,$$

where $\mathbb{B}$ denotes the unit standard open ball in $\mathbb{R}^d$.

(2) If $\nu(\{0\}) < p_c(d)$, there exists a deterministic compact convex set $K\subset\mathbb{R}^d$ with non-empty interior, such that for any positive ϵ,

$$\mathbb{P}\left[(1-\epsilon)K \subset \frac{1}{t}B_t \subset (1+\epsilon)K \textit{ for all } t \textit{ large enough}\right] = 1. \tag{1.3}$$

In Ziesche (2016), a wide generalization of the classical FPP was proposed: general random ergodic pseudometrics $T : (\mathbb{R}^d)^2 \to \mathbb{R}_+$ over the whole affine space $\mathbb{R}^d$. In this continuous setting we can also define the family of time constants $(\mu(v))_{v\in\mathbb{R}^d}$, under mild conditions, see Theorem 2.1. In this paper we prove a general theorem, see Theorem 2.2, which asserts that under two simply stated main conditions, the time constants associated with T are positive. More precisely, if T is ergodic, satisfies a polynomial decay (for a large enough degree depending only on the dimension d) of correlations, see condition 6, and if the probability that the origin and a large sphere are at vanishing T-distance decreases polynomially fast (with degree depending only on d), see condition 5, then the time constants of T are positive. We also prove a shape theorem, see Theorem 2.5. One fundamental tool for the proof of the positivity of the time constant in the case of classical percolation is the so-called van den Berg-Kesten (BK) inequality. This inequality no longer holds for dependent models. In this paper, we explain how to bypass this crucial tool, if the correlations are weak. For this, we provide a renormalization scheme which holds in a very general way, see Theorem 2.2.

Quite surprinsingly, Theorem 2.2 applies to all the known natural sorts of FPP, discrete or continuous, that is classical, Boolean (but with a non-optimal degree for the polynomial tail for radii) or Riemannian FPP, with the notable exception of the Gaussian free field Ding and Goswami (2019), where the correlations are too strong for the present setting. Moreover, we provide two new applications in two very natural settings: Voronoi and Gaussian FPP. Historically, the first natural generalization of the classical FPP on $\mathbb{Z}^d$ has been provided by random measurable *colourings* $\sigma : \mathbb{R}^d \to \{0, 1\}$ (see e.g Gouéré and Théret (2017), Ziesche (2016)). Here, the associated first passage time $T(x, y)$ is the least integral of σ over the piecewise C^1 paths between two points x, y of $\mathbb{R}^d$, see (3.1). *Voronoi percolation* is defined in the following way. First, a Poisson process in $\mathbb{R}^d$ of intensity one provides a locally finite random set X of points in $\mathbb{R}^d$. Then X induces a partition of the space into *Voronoi cells* defined by the points which are closest to a particular point in X. Now for a given $p \in [0, 1]$, all the points in a given random cell are given a common number σ_p, 0 or 1, with respective probability p and $1 - p$, as in Bernoulli percolation, and this is done independently over the cells. It is classical that this model undergoes a phase transition for the infinite components of $\{\sigma_p = 0\}$. Recently, new results about the associated percolation and criticality properties have been proved, see Theorems 3.2, 3.3 and 3.4. We prove in this paper, using the aforementioned results and Theorem 2.2, that the same phase transition occurs for the associated FPP, see Theorem 3.5.

Also very recently, another class of continuous percolation model was reborn, *Gaussian percolation*, that is connectivity properties associated with the sign of a stationnary smooth Gaussian field over $\mathbb{R}^d$. Common features with Bernoulli percolation have been revealed some years ago for planar fields with positive and strongly decorrelating fields, see Theorems 3.14 and 3.15, the latter providing a phase transition for the levels of the random field. More precisely, for $p \in \mathbb{R}$ and a random real centered Gaussian field f over $\mathbb{R}^2$, let σ_p be the colouring equal to 0 if $f + p \leq 0$ and 1 if $f + p > 0$. Then, almost surely $\{\sigma_p = 0\}$ has an infinite component if and only if $p < 0$. In this planar context, for the same conditions on the correlations, we apply Theorem 2.2 to prove that the FPP model associated with σ_p undergoes the same phase transition, see Theorem 3.16. All this applies to the Bargmann-Fock model defined by (3.10), which is a field of particular interest due to its connections with complex geometry (see Beffara and Gayet (2017)).

This paper is a shortened version of the preprint Dewan and Gayet (2020). In the latter applications are described in detail (in particular, classical, Boolean and Riemannian FPP). Since the real novelty our work lies, firstly in the general proof of the positivity of the constant, and secondly in the applications to Voronoi and Gaussian FPP, we prefered to restrict ourselves to these sections in this published version.

We finish this introduction with some open questions.

- One main conjecture for discrete FPP is the universality of the fluctuations of $T(0,x) - \mu(x) = o(x)$. It is conjectured Auffinger et al. (2017, §3.1) that

$$\text{var } T(0,x) \sim_{\|x\|\to\infty} \|x\|^{2/3}$$

on $\mathbb{R}^2$, where the symbol $\sim$ has various interpretations. Does the previous estimate hold for Gaussian fields, for instance the Bargmann-Fock field? Note that in our continuous setting, since we don't work on a lattice, the issues associated with lattice rigidity don't arise. However, one of the main problems in our context is the infinite dependency, an issue which does not arise in classical Bernoulli percolation.
- Another conjecture is related to the deviations of geodesics for the pseudometric from the straight line, for instance the maximal distance between these two kinds of geodesics. It is conjectured that this distance should be of order $\|x\|^\gamma$ for a certain exponent $\gamma < 1$, see Auffinger et al. (2017, §4.2). It is very natural to conjecture that this should be the case for Gaussian fields as well.
- The proof of Theorem 2.2 involves a combinatorial bound, which must be fought by, among others, the asymptotic independence given by condition 6. In the Gaussian case, this independence is provided by the fast decay of the correlation function. If said function decreasestoo slowly, the combinatorics win and we cannot get any upper bound.

2. Random pseudometrics

2.1. *Statements of the main results.* Let $T : (\mathbb{R}^d)^2 \to \mathbb{R}_+$ be a random pseudometric, that is T is a random map, almost surely satisfying the axioms of a metric except the non-degeneracy. We further suppose that T is *geodesic* in the following sense: there exists a function $\tilde{T}$ defined over piecewise $\mathcal{C}^1$ paths $\gamma : [0,1] \mapsto \mathbb{R}^d$ such that for any $x, y \in (\mathbb{R}^d)^2$,

$$T(x,y) = \inf_{\gamma \text{ from } x \text{ to } y} \tilde{T}(\gamma).$$

We will always assume that T is measurable with respect to the Σ-algebra of $\tilde{T}$. In the two main applications $\tilde{T}$ is the integral of a random function, see (3.1).

For every $v \in \mathbb{R}^d$, τ_v denotes the translation associated with v. The translations of $\mathbb{R}^d$ act on the set $\mathcal{T}(\mathbb{R}^d)$ of pseudometrics over $\mathbb{R}^d$ by

$$\tau_v(T)(x,y) = T(v+x, v+y). \tag{2.1}$$

The action τ_v is said to be *ergodic* if the law of the pseudometric T is invariant under the action τ_v, and if for any event E, E invariant under τ_v implies that E has measure 0 or 1.

The following two first assumptions are used for the existence of μ, and the third one is secondary.

(1) (Ergodicity) T is ergodic under the action of the translations of $\mathbb{R}^d$.
(2) (Finite moment) For any $x \in \mathbb{R}^d$, $\mathbb{E}(T(0,x))$ is finite.
(3) (Isotropy) The measure of T is invariant under the action of the orthogonal group of $\mathbb{R}^d$.

The following is a standard consequence of Kingman's subadditive ergodic theorem.

Theorem 2.1. *Let T be a random pseudometric satisfying conditions 1 and 2. Then, there exists a $\mathbb{Q}$-semi-norm $\mu : \mathbb{R}^d \to \mathbb{R}_+$ such that*

$$\forall v \in \mathbb{R}^d, \lim_{n\to+\infty} \frac{1}{n} T(0,nv) = \mu(v) \quad \textit{almost surely and } L^1. \tag{2.2}$$

If T satisfies the further condition 3 then μ is constant over $\mathbb{S}^{d-1}$.

Note that a semi-norm over $\mathbb{R}^d$ is always continuous. For the main theorem we need further notations. The set $\mathcal{T}$ of pseudometrics over $\mathbb{R}^d$ is equipped with the natural partial order $\leq$. An event E in $\mathcal{T}$ is said to be *increasing* if

$$\varphi \in E \text{ and } \varphi \leq \psi \Rightarrow \psi \in E.$$

An event is *decreasing* if $\varphi \in E$ and $\varphi \geq \psi \Rightarrow \psi \in E$. For any pair of subsets $A, B \subset \mathbb{R}^d$, let $\mathcal{A}^-$ and $\mathcal{B}^-$ be the set of decreasing events in $\mathcal{T}$ depending only on the values of $\tilde{T}$ for paths contained in A and B respectively. For any positive reals Q, S, let

$$\mathrm{Ind}^-(Q,S) := \sup_{\substack{A,B\subset\mathbb{R}^d, \mathrm{Diam}\, A\leq 2S, \mathrm{Diam}\, B\leq 2S \\ \mathrm{dist}(A,B)>Q, E_A\in\mathcal{A}^-, E_B\in\mathcal{B}^-}} \left|\mathbb{P}[E_A \cap E_B] - \mathbb{P}[E_A]\mathbb{P}[E_B]\right|. \tag{2.3}$$

For any $0 < r < R$, let

$$A_{r,R} = B(0,R) \setminus B(0,r) \subset \mathbb{R}^d,\ A_R = A_{1,R} \tag{2.4}$$

$$\text{and } T(A_{r,R}) = \inf_{x\in S(0,r), y\in S(0,R)} T(x,y), \tag{2.5}$$

where $B(x,r)$ (resp. $S(x,r)$) is the Euclidean ball (resp. sphere) of center x and radius r. The following assumptions are needed for the positivity of μ (Theorem 2.2)

(4) (Shell measurability) For any $0 < r < R$, $T(A_{r,R})$ is measurable with respect to the $\Sigma-$algebra of the random pseudometric T.

(5) (Decay of instant one-arms) There exist $R_0 > 0$, $\eta > (d-1)/4$ such that

$$\forall R \geq R_0,\ \mathbb{P}\left[T(A_R) = 0\right] \leq \frac{1}{R^{d-1+\eta}}.$$

(6) (Quasi-independence) There exist constants $\alpha > 1$, $Q_0 > 0$ such that for any $Q \geq Q_0$,

$$\mathrm{Ind}^-(Q, Q^\alpha) \leq Q^{-19(d-1)},$$

where Ind^- is defined by (2.3).

The main result of this paper is the following:

Theorem 2.2. *Let T be a random pseudometric over $\mathbb{R}^d$ satisfying conditions 1, 2, 4, 5 and 6. Then μ is a norm, that is $\mu > 0$.*

Remark 2.3.
- We emphasize that this theorem is general, and does not deal with the particularities of the model. This is the reason why we can apply it to such different models as Gaussian fields, Voronoi percolation, Boolean percolation or smooth random metrics.
- Condition 5 is one of the two crucial assumptions needed for our main Theorem 2.2. This fact is intuitive: if the random time across a spherical shell is too small, then it is believable that the time constant will drop to zero.
- Condition 6 asserts that the restrictions of the random pseudometric over two disjoint boxes are weakly correlated. We must allow the size of the boxes to increase polynomially with their distance. This kind of measure of dependency was used in Beffara and Gayet (2017) for topological events related to Gaussian fields. Because of this small asymptotic dependence, in the Gaussian application, we will need fields with polynomially fast decorrelation. Notice that this condition enables us to deal with infinite correlations and to have an alternative to the van den Berg-Kesten (BK) inequality, which is a crucial tool for percolation in independent settings. Also, condition 6 could be weakened in considering only events which are finite intersections of events of the type $\{T(A_R) < \delta\}$, see the proof of Proposition 2.9.

The following assumptions are needed for the vanishing of μ (Theorem 2.4).

(7) (Instant crossings of large rescaled spherical shells)

$$\limsup_{R\to\infty} \mathbb{P}[T(A_{R,2R}) = 0] > 0.$$

(8) There exists a positive $C > 0$ such that almost surely T is C-Lipschitz for the Euclidean metric.

In Gouéré and Théret (2017) it is proved the following:

Theorem 2.4. *(Gouéré and Théret, 2017, §2) Let T be a random pseudometric over $\mathbb{R}^d$ satisfying conditions 1, 3, 4, 7 and 8. Then $\mu = 0$.*

The aforementioned article Gouéré and Théret (2017) is written for Boolean percolation, but the proof holds in our context. We explain it in §2.3. Theorem 2.2 is extended into the shape theorem, the exact counterpart of Theorem 1.3. For this, for any $t \geq 0$ let

$$\begin{aligned} B_t &= \{x \in \mathbb{R}^d, T(0,x) \leq t\} \\ \text{and } K &= \left\{x \in \mathbb{R}^d, \mu(x) \leq 1\right\}, \end{aligned} \tag{2.6}$$

where μ is defined by (2.2).

Theorem 2.5. *Let T be a random pseudometric over $\mathbb{R}^d$ satisfying 1, 2 and 8.*

(1) If $\mu = 0$ then for any positive M,

$$\mathbb{P}\Big[M\mathbb{B} \subset \frac{1}{t}B_t \text{ for all } t \text{ large enough}\Big] = 1,$$

where $\mathbb{B}$ is the unit Euclidean ball.

(2) If μ is a norm then the subset K defined by (2.6) is a convex compact subset of $\mathbb{R}^d$ with non-empty interior. Moreover, for any positive ϵ,

$$\mathbb{P}\Big[(1-\epsilon)K \subset \frac{1}{t}B_t \subset (1+\epsilon)K \text{ for all } t \text{ large enough}\Big] = 1. \tag{2.7}$$

If T further satisfies condition 3, then $K = \frac{1}{\mu(1)}\mathbb{B}$, where $\mathbb{B} \subset \mathbb{R}^d$ denotes the unit ball and $\mu(1)$ denotes $\mu(v)$ for any vector v of norm 1.

Corollary 2.6. *Let T be a random pseudometric over $\mathbb{R}^d$ satisfying conditions 1, 2, 4 and 6. Assume also that $\mu = 0$, where μ is the pseudo-norm defined by Theorem 2.1. Then*

$$\forall \eta > \frac{d-1}{4}, \ \limsup_{R\to\infty} R^{d-1+\eta}\mathbb{P}[T(A_R) = 0] > 0,$$

where $T(A_R)$ denotes the first passage time between the two spheres $S(0,1)$ and $S(0,R)$, see (2.4) and (2.5).

Under an assumption of exponential decay of correlations, one can get the result of Corollary 2.6 for all $\eta > 0$ instead of $\eta > (d-1)/4$, see Dewan and Gayet (2020).

2.2. *Positivity of the time constant.* Theorem 2.2, which asserts that μ is a norm if the ergodic pseudometric T satisfies condition 5 and 6, is a consequence of the following Proposition 2.7. Recall that for any $M > 1$, A_M denotes the spherical shell centered at 0 of inner radius 1 and outer radius M, see (2.4), and $T(A_M)$ denotes the minimal time of a path from the interior sphere to the outside of A_M, see (2.5).

Proposition 2.7. *Let $T : (\mathbb{R}^d)^2 \to \mathbb{R}_+$ be a random pseudometric satisfying conditions 1, 4, 5 for $\eta > (d-1)/4$ and 6. Then, there exists an unbounded positive increasing sequence $(M_n)_n$ and a positive number c such that*

$$\forall n \in \mathbb{N}, \ \mathbb{P}\left[\frac{T(A_{M_n})}{M_n} < c\right] \leq \frac{1}{M_n^{d-1+\eta}}.$$

Remark 2.8. Note that a large deviation result would suffice to get exponential decay. It is possible that the wide applicability range of Theorem 2.2 is a consequence of the leniency of this result. Moreover, it yields Corollary 2.6, which is the best general result known for percolation.

Given this proposition, we can prove the main Theorem 2.2.

Proof of Theorem 2.2: Let $v \in \mathbb{S}^{d-1}$ and $(M_n)_n$ be the sequence given by Theorem 2.7. By Theorem 2.1 there exists a constant $\mu(v) \geq 0$ such that

$$\frac{1}{\lfloor M_n \rfloor + 1} T(0, (\lfloor M_n \rfloor + 1)v) \xrightarrow[n\to\infty]{a.s.} \mu(v).$$

Since for any n, $T(A_{M_n}) \leq T(0, (\lfloor M_n \rfloor + 1)v)$, the limit and Proposition 2.7 imply that $\mu(v) \geq c$ and thus $\mu(v) > 0$. □

Proposition 2.7 will be proved by induction over scales. However we will need to renormalize the constant c, see Corollary 2.10 below. To this end, we begin by proving the following Proposition 2.9 which compares the crossing time probabilities of two spherical shells with different exterior radii.

Proposition 2.9. *Let T be a random pseudometric over $\mathbb{R}^d$ satisfying assumption 1 and 4. Then, for any $1 \leq Q < R$ and $S \geq 100R^2/Q$, for any positive constant δ,*

$$\mathbb{P}\left[\frac{T(A_S)}{S} < \frac{\delta}{1+\frac{Q}{R}}\right] \leq \left(c_d S^{d-1}\frac{R}{Q}\right)^n \left(\mathbb{P}\left[\frac{T(A_R)}{R} < \delta\right]^n + n\,\mathrm{Ind}^-(Q,S)\right), \tag{2.8}$$

where $c_d > 0$ is a constant depending only on the dimension d, where $n = \lfloor N\frac{Q}{3R+3Q}\rfloor$ with $N = \lfloor\frac{S-1}{2R+Q}\rfloor$, and where Ind^- is defined by (2.3).

Proof: Let

$$N = \left\lfloor \frac{S-1}{2R+Q} \right\rfloor.$$

Then, there exist $\{B_1, \cdots, B_N\}$ a set of disjoint spherical shells, centered at 0, included in A_S, of increasing radii, of width $2R$, such that the interior sphere of B_1 is the unit sphere, and separated by a sequence $(C_1, \cdots, C_N)$ of spherical shells centered on 0, of width Q and of increasing radii, see Figure 2.1.

For any $j \in \{1, \cdots, N\}$, we consider a minimal set of k_j translates of A_R inside B_j, such that the closure of the union of their interior disks contains the middle sphere of B_j, that is $S(0, 1 + (j-1)(2R+Q) + R)$. These conditions ensure that any continuous crossing of B_j crosses at least one of the k_j copies of A_R inside B_j at least twice. It is true that there exists $c_d > 0$ depending only on the dimension d, such that

$$\forall 1 \leq j \leq N,\ k_j \leq c_d S^{d-1}. \tag{2.9}$$

Let γ be a path across the shell A_S. By the previous remark, γ necessarily crosses one copy of A_R in each B_j, once to enter the interior ball, and then once more to leave it. It thus crosses at least N such shells, each of them twice. Any path thus crosses a sequence $(a_1, \cdots, a_N)$, the first copies of A_R it crosses twice in each B_j, see Figure 2.1. We therefore have

$$T(A_S) \geq \inf_{\substack{a_1,\cdots,a_n \text{ copies of } A_R\\ \text{on disjoint shells } B_j}} \sum_{j=1}^{N} 2T(a_j). \tag{2.10}$$

Now, for any $j \in \{1, \cdots, N\}$, consider the corresponding event:

$$E_j := \left\{\frac{T(a_j)}{R} < \delta(1+\frac{Q}{R})\right\}.$$

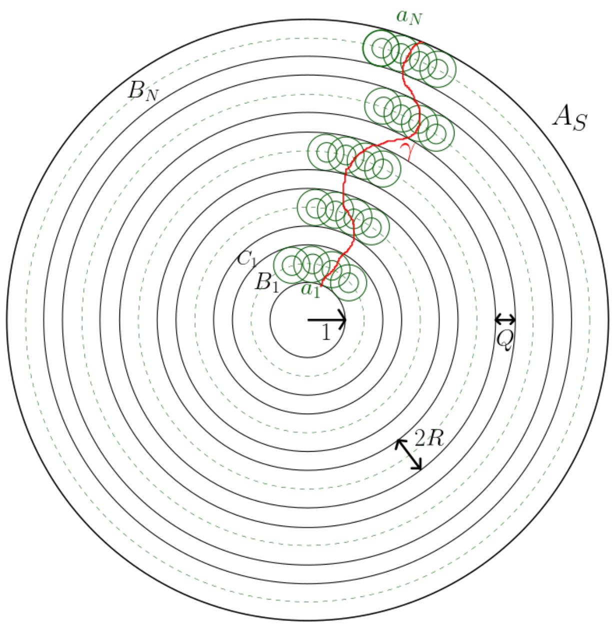

FIGURE 2.1. A path γ going across A_S crosses a certain number of copies of A_R.

When the event $\left\{\frac{T(A_S)}{S} < \delta\right\}$ occurs, at least

$$n = \lfloor N \frac{Q}{3R+3Q} \rfloor$$

events of the form E_j occur. Indeed, otherwise we would have by (2.10)

$$\begin{aligned}\frac{T(A_S)}{S} &\geq 2\frac{R}{S}(N - N\frac{Q}{3Q+3R})\delta(1+\frac{Q}{R}) \\ &\geq 2\frac{R}{S}\left(\frac{S-1}{2R+Q} - 1\right)\frac{2Q+3R}{3Q+3R}\frac{R+Q}{R}\delta \\ &= \frac{2}{3}\frac{S-1-2R-Q}{S}\frac{3R+2Q}{2R+Q}\delta \\ &\geq \left(1 - \frac{4R}{S}\right)\left(1+\frac{Q}{9R}\right)\delta \\ &\geq \delta,\end{aligned}$$

where in the last step we have used that $S \geq 100R^2/Q$. Assume from now on that $n \geq 1$. Note that if $n = 0$ then (2.8) is trivially true. Using (2.9),

$$\mathbb{P}\left(\frac{T(A_S)}{S} < \delta\right) \leq \binom{N}{n}(c_d S^{d-1})^n \sup_{\substack{a_1,\cdots,a_n \text{ copies of } A_R \\ \text{on disjoint shells } B_j}} \mathbb{P}\left[\bigcap_{j=1}^{n} E_j\right].$$

Indeed, there are $\binom{N}{n}$ ways to choose the n shells $(B_{j_1}, \cdots, B_{j_n})$ where $E_{j_1}, \cdots, E_{j_n}$ occur, and for any $i = 1, \cdots, n$, there are at most $c_d S^{d-1}$ choices for the small shell a_{j_i}. Now, given such a deterministic sequence $a_1, \cdots, a_n$, since by definition the distance between any two of the shells B_j is at least Q, the distance between any two of the a_j's has the same lower bound. By definition of

Ind$^-$, using the fact that a finite intersection of E_j's is a decreasing event, for all $S > R > Q \geq 1$,

$$\forall i \in \{1, \cdots, n\},\ \mathbb{P}\left[E_i \cap \bigcap_{j=i+1}^{n} E_j\right] \leq \mathbb{P}[E_i]\mathbb{P}\left[\bigcap_{j=i+1}^{n} E_j\right] + \mathrm{Ind}^-(Q, S).$$

By an immediate induction, this implies

$$\mathbb{P}\left[\bigcap_{j=1}^{n} E_j\right] \leq (\mathbb{P}[E_1])^n + n\,\mathrm{Ind}^-(Q, S).$$

By the classical inequality

$$\forall 1 \leq n \leq N, \binom{N}{n} \leq \left(\frac{eN}{n}\right)^n$$

and the definition of n, the combinatorial term satisfies

$$\binom{N}{n}(c_d S^{d-1})^n \leq \left(\frac{6c_d e S^{d-1} R}{Q}\right)^n.$$

Replacing δ with $\delta(1 + Q/R)^{-1}$, we obtain the result. □

In the next Corollary 2.10, Proposition 2.9 is applied to a sequence of growing scales, threatening the inductive renormalized constant δ to drop to zero. However, the sequence is chosen so that the infinite product of the renormalization factors converges to a positive constant.

Corollary 2.10. *Let $\eta > 0$ and T be a random pseudometric satisfying assumptions 1, 4 and 6. Let $(d-1)/4 < \eta < d-1$. Then there exists $R_0, \epsilon > 0$, such that for any positive constant δ and any $R \geq R_0$,*

$$\mathbb{P}\left[\frac{T(A_R)}{R} < \delta\right] \leq \frac{1}{R^{d-1+\eta}} \Rightarrow \mathbb{P}\left[\frac{T(A_{100R^{1+\epsilon}})}{100R^{1+\epsilon}} < \frac{\delta}{1+R^{-\epsilon}}\right] \leq \frac{1}{(100R^{1+\epsilon})^{d-1+\eta}}. \tag{2.11}$$

Proof: For any $0 < \epsilon < 1$ and $R \geq 1$, let $(Q, R, S) = (R^{1-\epsilon}, R, 100R^{1+\epsilon})$ so that, in the notations of Proposition 2.9, for all $R \geq 1$,

$$N = \left\lfloor \frac{100R^\epsilon - R^{-1}}{2 + R^{-\epsilon}} \right\rfloor, \quad n = \left\lfloor \frac{R^{-\epsilon}}{3 + 3R^{-\epsilon}} N \right\rfloor = \left\lfloor \frac{100}{6} \frac{1 - [R^{-1-\epsilon} + R^{-\epsilon} q_N]/100}{(1 + R^{-\epsilon})(1 + R^{-\epsilon}/2)} \right\rfloor,$$

where $0 \leq q_N \leq 2 + R^{-\epsilon}$ (q_N is $2 + R^{-\epsilon}$ times the fractional part of N). Thus, for all $R \geq 1$, for any $0 < \epsilon < 1$,

$$5 \leq n \leq 17 \quad \text{and} \quad \left(c_d S^{d-1} \frac{R}{Q}\right)^n \leq (c_d (100R^{1+\epsilon})^{d-1} R^\epsilon)^n \leq c_d' R^{n[(d-1)(1+\epsilon)+\epsilon]}. \tag{2.12}$$

Now, since $\eta > (d-1)/4$ and $n \geq 5$, we have

$$n(d-1+\eta) > n(d-1) + d - 1 + \eta.$$

Further, since $n \leq 17$ and $\eta < d-1$, we have

$$19(d-1) > n(d-1) + d - 1 + \eta.$$

We are thus able to define a small $\epsilon > 0$ (small enough so that $\alpha := \frac{1+\epsilon}{1-\epsilon}$ is smaller than α_0 from condition 6, and uniformly in n, hence in $R \geq 1$) such that

$$\begin{aligned} n(d-1+\eta) &> n[(d-1)(1+\epsilon)+\epsilon] + (1+\epsilon)(d-1+\eta) \\ \text{and } 19(d-1)(1-\epsilon) &> n[(d-1)(1+\epsilon)+\epsilon] + (1+\epsilon)(d-1+\eta). \end{aligned} \tag{2.13}$$

Now, by Proposition 2.9 and condition 6, and recalling (2.12), there exists $R_1(\epsilon) > 0$ such that for any $R \geq R_1$, for any $\delta > 0$, if the left-hand side of (2.11) holds, then

$$\mathbb{P}\left[\frac{T(A_{100R^{1+\epsilon}})}{100R^{1+\epsilon}} < \frac{\delta}{1+R^{-\epsilon}}\right] \leq c'_d R^{n[(d-1)(1+\epsilon)+\epsilon]}\big(R^{-(d-1+\eta)n} + 17R^{-19(d-1)(1-\epsilon)}\big). \tag{2.14}$$

And by (2.13), there exists $R_2(\epsilon) \geq R_1(\epsilon)$ such that for all $R \geq R_2$,

$$c'_d R^{n[(d-1)(1+\epsilon)+\epsilon]} R^{-(d-1+\eta)n} \leq \frac{1}{2}\frac{1}{(100R^{1+\epsilon})^{d-1+\eta}}$$

and

$$17c'_d R^{n[(d-1)(1+\epsilon)+\epsilon]} R^{-19(d-1)(1-\epsilon)} \leq \frac{1}{2}\frac{1}{(100R^{1+\epsilon})^{d-1+\eta}}.$$

Hence, the right-hand side of (2.14) is bounded above by $(100R^{1+\epsilon})^{-(d-1+\eta)}$. □

To implement the implication (2.11), we need to find a scale where the left-hand side holds. This is done by the following lemma:

Lemma 2.11. *Let T be a random pseudometric over $\mathbb{R}^d$ satisfying conditions 1, 4 and 5 for some $R_0 > 0$ and $\eta > (d-1)/4$. Then, there exists $M_0 > 0$ and $\eta' > (d-1)/4$ such that*

$$\forall M \geq M_0,\ \exists c_M,\ \forall c \leq c_M, \mathbb{P}\left[\frac{T(A_M)}{M} \leq c\right] \leq \frac{1}{M^{d-1+\eta'}}.$$

Proof: Let $\eta' := (d-1)/4 + [\eta - (d-1)/4]/2$. By condition 5, there exists $M_0 > 0$ such that for all $M \geq M_0$,

$$\mathbb{P}\big[T(A_M) = 0\big] \leq \frac{1}{2M^{d-1+\eta'}}.$$

Since for a real-valued random variable X, the function $x \mapsto \mathbb{P}(X \leq x)$ is right continuous, we obtain the result. □

We can now prove Proposition 2.7.

Proof of Proposition 2.7: By condition 5 and Lemma 2.11, there exist $R_0 \geq 1$ and $\eta > (d-1)/4$ such that

$$\forall M \geq R_0,\ \exists c_M,\ \mathbb{P}\left[\frac{T(A_M)}{M} \leq c_M\right] \leq \frac{1}{M^{d-1+\eta}}. \tag{2.15}$$

Moreover, by Corollary 2.10 there exists $R_1 \geq R_0$ and $\epsilon > 0$ such that such that for any $R \geq R_1$ and any $\delta > 0$, the implication (2.11) holds. Let

$$\delta := c_{R_1}$$

be defined and given by (2.15) and define $M_0 = R_1$ and for any integer $k \geq 1$:

$$M_k := 100M_{k-1}^{1+\epsilon},$$

i.e for any integer k,

$$M_k = 100^{-1/\epsilon}(100^{1/\epsilon}R_1)^{(1+\epsilon)^k},$$

Then by an immediate induction and Corollary 2.10,

$$\forall k \geq 1,\ \mathbb{P}\left[\frac{T(A_{M_k})}{M_k} \leq \delta \prod_{j=0}^{k-1}(1+M_j^{-\epsilon})^{-1}\right] \leq \frac{1}{M_k^{d-1+\eta}}.$$

Now note that $M_k^{-\epsilon} = 100(100^{1/\epsilon}R_1)^{-(1+\epsilon)^k\epsilon}$, so that the product $\prod_{j=0}^{\infty}(1+M_j^{-\epsilon})^{-1}$ converges to a constant $\gamma > 0$ (recalling $R_1 \geq 1$). Hence, we then obtain

$$\forall k \geq 1, \mathbb{P}\left[\frac{T(A_{M_k})}{M_k} \leq \delta\gamma\right] \leq \frac{1}{M_k^{d-1+\eta}}, \tag{2.16}$$

which implies the result. □

We finish this paragraph with the proof of the estimate for the one-arm decay.

Proof of Corollary 2.6: If the conclusion does not hold, then T satisfies condition 5, so that by Theorem 2.2, $\mu > 0$, which is a contradiction. □

2.3. *Vanishing of the time constant.* We explain why Theorem 2.4 proved in a Boolean setting extends to ours.

Proof of Theorem 2.4: Under conditions 3 (isotropy) and 8 (Lipschitz), the convergence given by Theorem 2.1 is uniform in all directions. This is proved by Gouéré and Théret (2017, Theorem 1.1) for the Boolean setting, but the proof given by Gouéré and Théret (2017, §B) only uses the Lipschitz property of T and isotropy. Now, in Gouéré and Théret (2017, §2), the authors proved

$$\mu \text{ is a norm } \Rightarrow \mathbb{P}[\mathrm{Cross}_0(A_{R,2R})] \to_R 0, \tag{2.17}$$

where any $0 < r < R$,

$$\mathrm{Cross}_0(A_{r,R}) = \{\exists \text{ a } C^0 \text{ path included in } \{\sigma = 0\} \text{ crossing } A_{r,R}\ \}$$

and $\sigma : \mathbb{R}^d \to \mathbb{R}$ denotes the indicator function of the union of the random balls. However, their proof gives the stronger

$$\mu \text{ is a norm } \Rightarrow \mathbb{P}[T(A_{R,2R}) = 0] \to_R 0. \tag{2.18}$$

Now, under isotropy, μ is either a norm or vanishes, so that the contrapositive of (2.18) gives the result. □

Example 2.12. Condition 2.17 is indeed weaker than condition 2.18 in our general setting. Here is a deterministic example (provided by a referee) where $T(A_{R,2R}) = 0$ but $\mathrm{Cross}_0(A_{R,2R})$ does not occur: let $\sigma : \mathbb{R}^2 \to \mathbb{R}_+$ be such that if $(x, y) \in \mathbb{R}^2$ with $y \neq 0$, then $\sigma(x, y)$ is the positive angle between (x, y) and the horizontal axis, and $\sigma(x, y) = 1$ if $y = 0$. Let T be defined by the infimum of the integral of σ over C^1 paths between two points, see (3.1). Then T satisfies the aformentioned properties.

2.4. *The shape theorem.* We set out to prove Theorem 2.5 (shape theorem). Firstly, let us define a particular event:

- Let T be a random pseudometric over $\mathbb{R}^d$ satisfying conditions 1 and 2. Denote by E the event
$$E := \left\{\forall b \in \mathbb{Q}^d,\ \frac{1}{n}T(0, nb) \to_{n\to+\infty} \mu(b)\right\}, \tag{2.19}$$
where μ is the time constant defined by Theorem 2.1.

Note that by Theorem 2.1, E holds almost surely. For both cases of Theorem 2.5, $\mu = 0$ or $\mu > 0$, we will use the same compactness lemma:

Lemma 2.13. *Let T be a random pseudometric satisfying conditions 1, 2 and 8. Assume E is satisfied, and let $(z_n)_n$ be a sequence in $\mathbb{R}^d$ such that $\|z_n\| \to_n \infty$. Then, there exist a subsequence $(y_n)_n$ of $(z_n)_n$ and $a \in \mathbb{S}^{d-1}$ such that*

$$\frac{y_n}{\|y_n\|} \to_n a \text{ and } \frac{1}{\|y_n\|}T(0, y_n) \to_n \mu(a). \tag{2.20}$$

Proof: Since μ is a semi-norm, it is C_μ-Lipschitz for a constant C_μ, and by condition 8 there exists $C_T > 0$ such that T is C_T-Lipschitz. By compactness, we can assume that there exist a subsequence $(y_n)_n$ of $(z_n)_n$ and $a \in \mathbb{S}^{d-1}$, such that

$$\frac{y_n}{\|y_n\|} \to_{n\to\infty} a. \tag{2.21}$$

Let $\eta > 0$ and $b = b(\eta) \in \mathbb{Q}^d$ be such that

$$\|a - b\| < \frac{\eta}{9\max(C_T, C_\mu)}. \tag{2.22}$$

Let N be so large that

$$\forall n \geq N, \ \left\| \frac{y_n}{\|y_n\|} - a \right\| < \frac{\eta}{3\max(C_\mu, C_T)}. \tag{2.23}$$

Since μ is C_μ-Lipschitz, (2.23) implies that

$$\forall n \geq N, \ \big|\mu(y_n) - \|y_n\|\mu(a)\big| < \frac{\eta}{3}\|y_n\|. \tag{2.24}$$

Since E given by (2.19) holds, there exists $N_\eta \geq N$, such that

$$\forall n \geq N_\eta, \ \left| \frac{T(0, \|y_n\|b)}{\|y_n\|} - \mu(b) \right| < \frac{\eta}{9}.$$

Moreover by (2.22) and since T is C_T-Lipschitz,

$$\forall n \in \mathbb{N}, \ \left| \frac{T(0, \|y_n\|a)}{\|y_n\|} - \frac{T(0, \|y_n\|b)}{\|y_n\|} \right| < \frac{\eta}{9},$$

so that we have for all $n \geq N_\eta$, using again (2.22) and that μ is C_μ-Lipschitz for the last term,

$$\begin{aligned} \left| \frac{T(0, \|y_n\|a)}{\|y_n\|} - \mu(a) \right| \quad &\leq \quad \left| \frac{T(0, \|y_n\|a)}{\|y_n\|} - \frac{T(0, \|y_n\|b)}{\|y_n\|} \right| + \left| \frac{T(0, \|y_n\|b)}{\|y_n\|} - \mu(b) \right| \\ &\quad + |\mu(b) - \mu(a)| < \frac{\eta}{3}. \end{aligned} \tag{2.25}$$

Now, for all $n \geq N_\eta$:

$$\begin{aligned} \Big|T(0, y_n) - \mu(y_n)\Big| \quad &\leq \quad \Big|T(0, y_n) - T(0, \|y_n\|a)\Big| + \Big|T(0, \|y_n\|a) - \mu(\|y_n\|a)\Big| \\ &\quad + \Big|\mu(\|y_n\|a) - \mu(y_n)\Big|. \end{aligned}$$

Since T is C_T-Lipschitz and by (2.23), for any $n \geq N$ the first term is upper bounded by $\frac{\eta}{3}\|y_n\|$. By (2.25), for any $n \geq N_\eta$ the second term is bounded by $\frac{\eta}{3}\|y_n\|$. By (2.24) the third term is less than $\frac{\eta}{3}\|y_n\|$ for all $n \geq N$. We deduce that

$$\forall n \geq N_\eta, \ |T(0, y_n) - \mu(y_n)| < \eta\|y_n\|.$$

Hence, we have proved that

$$\frac{1}{\|y_n\|}T(0, y_n) - \mu(\frac{y_n}{\|y_n\|}) \to_n 0, \tag{2.26}$$

which implies by continuity of μ and (2.21) that

$$\frac{1}{\|y_n\|}T(0, y_n) \to_n \mu(a). \tag{2.27}$$

□

Proof of Theorem 2.5: First, the compact K defined by (2.6) is convex. Indeed, since μ is a semi-norm, for any $x, y \in \mathbb{R}^d$ and $t \in [0, 1]$,

$$\mu\left(tx + (1-t)y\right) \leq \mu\left(tx\right) + \mu\left((1-t)y\right) = t\mu(x) + (1-t)\mu(y).$$

For the rest of the proof, we begin with general implications. Firstly,

$$\forall \epsilon, t > 0, \forall x \in \mathbb{R}^d, \ x \in \frac{1}{t}B_t \setminus (1+\epsilon)K \quad \Rightarrow \quad \mu(tx) - T(0, tx) > \epsilon t \tag{2.28}$$

$$\text{and } x \in (1-\epsilon)K \setminus \frac{1}{t}B_t \quad \Rightarrow \quad T(0, tx) - \mu(tx) > \epsilon t. \tag{2.29}$$

Moreover, since μ is a semi-norm, it is C_μ-Lipschitz for a constant C_μ, so that

$$\forall \epsilon, t > 0, \forall x \in \mathbb{R}^d,\ x \in \frac{1}{t}B_t \setminus (1+\epsilon)K \quad \Rightarrow \quad \|x\| \geq \frac{1+\epsilon}{C_\mu}. \tag{2.30}$$

Besides under condition 8 there exists $C_T > 0$ such that T is C_T-Lipschitz, so that

$$\forall \epsilon, t > 0, \forall x \in \mathbb{R}^d,\ x \in (1-\epsilon)K \setminus \frac{1}{t}B_t \quad \Rightarrow \quad \|x\| \geq \frac{1}{C_T}. \tag{2.31}$$

Lastly, if μ is a norm,

$$\forall \epsilon \in]0,1[, t > 0, \forall x \in \mathbb{R}^d,\ x \in (1-\epsilon)K \quad \Rightarrow \quad \|x\| \leq \frac{1-\epsilon}{\mu(\frac{x}{\|x\|})}. \tag{2.32}$$

We now prove the second assertion of Theorem 2.5. Let E be the event defined by (2.19). By Theorem 2.1 (existence of μ), $\mathbb{P}(E) = 1$. For any $\epsilon \in]0,1[$, define

$$I_\epsilon := \left\{(1-\epsilon)K \subset \frac{1}{t}B_t \subset (1+\epsilon)K \text{ for all } t \text{ large enough}\right\}.$$

It is enough to prove that

$$E \subset I_\epsilon.$$

Assume on the contrary that there exists $\epsilon > 0$ such that E occurs but not I_ϵ. By (2.28) and (2.29), it implies that there exists a sequence $(t_n)_n$ of positive reals such that

$$t_n \to_n +\infty, \tag{2.33}$$

and a sequence $(x_n)_n \in (\mathbb{R}^d)^{\mathbb{N}}$, such that

$$\forall n \in \mathbb{N},\ x_n \in \left(\frac{B_{t_n}}{t_n} \setminus (1+\epsilon)K\right) \cup \left((1-\epsilon)K \setminus \frac{B_{t_n}}{t_n}\right) \tag{2.34}$$

and thus

$$|T(0, x_n t_n) - \mu(t_n x_n)| \geq \epsilon t_n. \tag{2.35}$$

For any integer n, let

$$z_n := t_n x_n.$$

Note that for any n, $z_n \neq 0$ and by (2.33), (2.34), (2.30) and (2.31),

$$\|z_n\| \to_n +\infty.$$

By Lemma 2.13, there exist $a \in \mathbb{R}^d$ and a subsequence $(y_n)_n$ of $(z_n)_n$ such that

$$\frac{y_n}{\|y_n\|} \to_n a \text{ and } \frac{1}{\|y_n\|}T(0, y_n) \to_n \mu(a).$$

Since μ is a norm, there exists N', such that for $n \geq N'$,

$$\frac{1}{\|y_n\|}T(0, y_n) > \frac{1}{2}\mu(a). \tag{2.36}$$

Fix $n \in \mathbb{N}$. If $x_n \in \frac{1}{t_n}B_{t_n} \setminus (1+\epsilon)K$, then $T(0, y_n) \leq t_n$ so that by (2.36),

$$\|x_n\| \leq \frac{2}{\mu(a)}.$$

If on the contrary $x_n \in (1-\epsilon)K \setminus \frac{1}{t}B_t$, by (2.32)

$$\|x_n\| \leq \frac{1-\epsilon}{\inf_{\mathbb{S}^{d-1}} \mu}.$$

In all cases, we see that $(x_n)_n$ is bounded so that by (2.20) and the continuity of μ at a,

$$\frac{1}{t_n}T(0, y_n) - \mu(\frac{y_n}{t_n}) \to_n 0,$$

which contradicts (3.20) and proves the second assertion of Theorem 2.5.

We prove now the first assertion of the theorem, again by contradiction. Assume $\mu = 0$, that E is satisfied and that there exist $M > 0$ and a sequence $(t_n)_n$ diverging to infinity, such that $\forall n,\ \frac{1}{t_n}B(t_n)$ does not contain $M\mathbb{B}$. Hence, there exists $(x_n) \in (M\mathbb{B})^{\mathbb{N}}$ such that

$$\forall n,\ T(0, x_n t_n) > t_n. \tag{2.37}$$

As before, let $(z_n)_n := (t_n x_n)_n$. Then again $\|z_n\| \to_n +\infty$. By Lemma 2.13, there exists a subsequence $(z_n)_n$ of $(y_n)_n$ such that

$$\frac{1}{\|y_n\|}T(0, y_n) \to_n 0.$$

Because again $(x_n)_n$ is bounded, this implies that $\frac{1}{t_n}T(0, y_n) \to_n 0$, which contradicts (2.37). □

3. Applications

We present the main two (new) applications of Theorem 2.2 to Voronoi and Gaussian percolations. The general setting is the following. Let $\sigma : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function over $\mathbb{R}^d$ with non-negative values. It induces a pseudometric T defined by:

$$\forall (x, y) \in (\mathbb{R}^d)^2,\ T(x, y) := \inf_{\substack{\gamma \text{ piecewise } \mathcal{C}^1 \\ \text{path } x \to y}} \int_\gamma \sigma. \tag{3.1}$$

Here, if $\sigma : [0, 1] \to \mathbb{R}^d$, then

$$\int_\gamma \sigma = \int_0^1 \sigma(\gamma(t))\|\gamma'(t)\|dt.$$

Recall that T has possibly infinite values and is not a distance in general. As a particular but very natural case, a *colouring* σ has values in $\{0, 1\}$. In this case, we travel over $\{\sigma = 1\}$ with speed one and with infinite speed over $\{\sigma = 0\}$. The following lemma transfers some properties of σ to its assocated pseudometric.

Lemma 3.1. *Let $\sigma : \mathbb{R}^d \to \mathbb{R}_+$ be a random function, such that almost surely, for any segment $I \subset \mathbb{R}^d$, $\sigma_{|I}$ is a regulated function. Then, the associated pseudometric defined by (3.1) is measurable with respect to the Σ-algebra of σ. Furthermore, it satisfies condition 4 (shell measurability). If σ is ergodic, so is T (condition 1) and if σ is bounded, then T satisfies conditions 2 (finite moment) and 8 (Lipschitz).*

Proof: The proof of the first assertion is similar to and easier than the second one. The ergodicity of T is an immediate consequence of the ergodicity of σ and the first assertion. Let us prove the last condition. For any finite set of points $x_1, ..., x_n$, we denote by $\gamma_{x_1,...,x_n}$ the piecewise affine path starting at x_1, ending at x_n, going through $x_2, ..., x_{n-1}$ in order and following the straight line in between two consecutive x_i's with speed 1. Then,

$$T(A_{r,R}) = \inf_{n\in\mathbb{N}} \inf_{\substack{x_2,...,x_{n-1} \in \mathbb{Q}^d \cap A_{r,R} \\ x_1 \in \mathbb{Q}^d \cap \mathbb{B}_r, x_n \in \mathbb{Q}^d \setminus \mathbb{B}_R}} \int_{\gamma_{x_1,...,x_n}} \sigma.$$

Since any infimum of a sequence of measurable maps is measurable, it suffices to show that for a fixed segment $I \subset A_{r,R}$,

$$\sigma \mapsto \int_I \sigma$$

is Σ-measurable. By hypothesis, $\sigma_{|I}$ is the uniform limit over the segment of simple functions $(f_n)_n$, such that for any n, f_n has finite values in $\sigma(I) \subset \sigma(A_{r,r})$. Hence, the integral is Σ-measurable. In conlusion, $T(A_{r,R})$ is Σ-measurable. The last assertion is immediate. □

We will need the following definition, for densities defined on $\mathbb{R}^2$: for any rectangle $R \subset \mathbb{R}^2$,

$$\mathrm{Cross}_0(R) = \left\{\exists \text{ a } C^0 \text{ path included in } \{\sigma = 0\} \text{ crossing } R \text{ horizontally}\right\}. \tag{3.2}$$

The event $\mathrm{Cross}_1(R)$ is defined in a similar way. Recall that for any $0 < r < R$,

$$\mathrm{Cross}_0(A_{r,R}) = \left\{\exists \text{ a } C^0 \text{ path included in } \{\sigma = 0\} \text{ crossing } A_{r,R} \right\}. \tag{3.3}$$

3.1. *Voronoi FPP.* The first natural and new example for density FPP is Voronoi percolation. Let X be a Poisson process over $\mathbb{R}^d$ with intensity 1. Recall that X is a random subset of points, locally finite, such that for any Borel subset $A \subset \mathbb{R}^d$, the probability that $X \cap A$ has exactly k points equals

$$\frac{(\mathrm{Vol}\, A)^k}{k!} \exp(-\mathrm{Vol}\, A).$$

Moreover, for two disjoint subsets A and B, $X_{|A}$ is independent of $X_{|B}$. To X we can associate the so-called *Voronoi tiling*: any point x of X has a cell $V_x \subset \mathbb{R}^d$ defined by the points in $\mathbb{R}^d$ which are closer to x than any other point of X. Then, we colour any cell in black (value 1) with probability $1 - p$ or in white (value 0) with probability p. The boundaries of two cells with different colour are coloured white. This provides a random colouring

$$\sigma_p : \mathbb{R}^d \to \{0, 1\}.$$

Let $p_c(d) \in [0, 1]$ be defined by

$$p_c(d) = \inf\{p, \text{there exists an infinite white component a.s.}\}. \tag{3.4}$$

Note that we have flipped the classical definition of the colouring, in order to fit the general setting. It is classical (Bollobás and Riordan, 2006b, pp. 270–272) that for any $d \geq 2$, $p_c(d) \in]0, 1[$. In 2006, B. Bollobás and O. Riordan proved:

Theorem 3.2. *(Bollobás and Riordan, 2006a, Theorems 1.1 and 1.2) For Voronoi percolation, $p_c(2) = 1/2$.*

Then V. Tassion proved that at criticality, planar Voronoi percolation σ_0 satisfies a Russo-Seymour-Welsh type theorem:

Theorem 3.3. *(Tassion, 2016, Theorem 3) If $p = p_c(2) = 1/2$,*
(1) for any rectangle $R \subset \mathbb{R}^2$, $\liminf_{n \to \infty} \mathbb{P}[\mathrm{Cross}_0(nR)] > 0$;
(2) there exist $C, \alpha > 0$, such that $\forall R \geq 1, \mathbb{P}[\mathrm{Cross}_0(A_R)] \leq \frac{C}{R^\alpha}$.

More recently, H. Duminil-Copin, A. Raoufi and V. Tassion proved the following result:

Theorem 3.4. *(Duminil-Copin et al., 2019, Theorem 1) For any $p \in [0, 1]$, let σ_p be the Voronoi percolation model defined above. For $p < p_c$, there exist $c > 0$ and $R_0 > 0$, such that*

$$\forall R \geq R_0,\ \mathbb{P}[\mathrm{Cross}_0(A_R)] \leq \exp(-cR).$$

For $d = 2$, it was already proved by Bollobás and Riordan (2006a, Theorem 1.2).

As a first application of our general theorems, we obtain the following.

Theorem 3.5. *For any integer $d \geq 2$ and $p \in [0, 1]$, let σ_p be the Voronoi percolation model defined above, T be its associated pseudometric and μ_p be its time constant. Then,*

$$p < p_c(d) \Rightarrow \mu_p > 0 \text{ and } \mu_p > 0 \Rightarrow p \leq p_c(d).$$

Moreover, Theorem 2.5 (shape theorem) applies, and the convex K is a Euclidean ball.

Remark 3.6. (1) In fact, when $d = 2$, Theorem 3.2 and Theorem 3.3, one can prove further that

$$\mu_p > 0 \Leftrightarrow p < \frac{1}{2}.$$

(2) There exist other models of FPP for Voronoi tesselations, see Howard and Newman (1997) and Pimentel (2006). The first one always gives positive times, and the second one is associated with the graph given by the tesselation, hence is topological.

Corollary 3.7. *Let $\sigma_{1/2} : \mathbb{R}^2 \to \{0,1\}$ be the planar critical Voronoi percolation model defined above. Then,*

$$\forall \eta > 0,\ \limsup_{R\to\infty} R^{1+\eta}\mathbb{P}[\mathrm{Cross}_0(A_R)] > 0.$$

The rest of this section is devoted to the proof of Theorem 3.5. For condition 1 and condition 6, we will need the following lemmas.

Proposition 3.8. *(Tassion, 2016) Let $p \in]0,1[$ and σ_p be the associated Voronoi percolation over $\mathbb{R}^d$. Then, there exist constants $c, M_0 > 0$ such that for all $M \geq M_0$ and A_1, A_2 two compact subsets of $\mathbb{R}^d$, both of diameter less than M and at a distance $\geq M$ from each other, for all events E_1, E_2 depending respectively on the colour over A_1, A_2 respectively, we have:*

$$\left|\mathbb{P}\left[E_1 \cap E_2\right] - \mathbb{P}[E_1]\mathbb{P}[E_2]\right| \leq e^{-cM^d}.$$

In particular the pseudometric T associated to σ_p satisfies condition 6 (quasi-independence).

The proof of this proposition can be extracted from the proof of Lemma 1.1 of Tassion (2016). For sake of clarity, we give here a proof of it. It is a consequence of the following lemma:

Lemma 3.9. *(Tassion, 2016) Let X be a Poisson process over $\mathbb{R}^d$ with intensity 1, and for $x \in X$, denote by V_x the Voronoi cell based on x. Then there exist $c > 0$ and $M_0 > 0$ such that the following holds. For any open bounded subset $A \subset \mathbb{R}^d$ with diameter less than $M \geq M_0$, let $E(A,M)$ be the event*

$$E(A,M) := \{A \subset \bigcup_{x\in X\cap(A+B(0,M))} V_x\}. \tag{3.5}$$

Then, $\mathbb{P}[E(A,M)] \geq 1 - \exp(-cM^d)$.

In other terms, with exponentially high probability the Voronoi cells intersecting A do not go too far off of A.

Proof of Lemma 3.9: There exists $C > 0$, such that for any $M > 0$ and A as in the lemma, A can be covered by at most C balls of radius M. With probability at least $1 - C\exp\left(-(\mathrm{Vol}\,\mathbb{B})^d M^d\right)$, there exists at least one point of the Poisson process in every ball. Consequently, with the same probability, any point of A is M-close to a point of the Poisson process. □

Proof of Proposition 3.8: By Lemma 3.9, with probability at least $1-2e^{-c(M/2)^d}$, the event $E(A, M/2)\cap E(B, M/2)$ happens, where $E(A,M)$ is defined by (3.5). Since the distance beween A and B is larger than M, this implies the result. □

We could not find in the literature the proof that Voronoi percolation is ergodic under the actions of translations, hence the following proposition:

Proposition 3.10. *For any $p \in \mathbb{R}$, the translations over $\mathbb{R}^d$ are ergodic for the Voronoi percolation σ_p.*

Proof: Let $\epsilon > 0$ and A be a translation-invariant event. Since A is measurable, there exist a compact subset $S \subset \mathbb{R}^d$ and an event A_S depending only on the value of σ_p on S such that

$$\mathbb{P}(A\Delta A_S) \leq \epsilon. \tag{3.6}$$

Let $c, M_0 > \mathrm{Diam}\, S$ be given by Lemma 3.9 such that

$$\forall R \geq M_0,\ \mathbb{P}[E(S,R)] \geq 1 - \exp(-cR^d) \geq 1 - \epsilon,$$

where $E(S, R)$ is defined by (3.5). Let

$$v = (4M_0, 0, \cdots, 0) \in \mathbb{R}^d.$$

Then conditioned on an event of probability at least $1 - \epsilon$, A_S is independent of $\tau_v A_S$, so that

$$|\mathbb{P}(A_S \cap \tau_v A_S) - \mathbb{P}(A_S)^2| \leq \epsilon. \quad (3.7)$$

Since A is invariant under τ_v, $\mathbb{P}(A \cap \tau_v A) = \mathbb{P}(A)$. Now

$$\begin{aligned} \mathbb{P}\Big[(A_S \cap \tau_v A_S) \triangle A\Big] &\leq \mathbb{P}(A_S \triangle A) + \mathbb{P}(\tau_v A_S \triangle A) \\ &\leq \mathbb{P}(A_S \triangle A) + \mathbb{P}(A_S \triangle \tau_{-v} A). \end{aligned}$$

But $\tau_{-v} A = A$. Thus,

$$\mathbb{P}\Big[(A_S \cap \tau_v A_S) \triangle A\Big] \leq 2\mathbb{P}(A_S \triangle A) \leq 2\epsilon.$$

Therefore, $|\mathbb{P}(A_S \cap \tau_v A_S) - \mathbb{P}(A)| \leq 2\epsilon$. Hence by (3.7) we get

$$|\mathbb{P}(A_S)^2 - \mathbb{P}(A)| \leq 3\epsilon.$$

Now using (3.6),

$$|\mathbb{P}(A)^2 - \mathbb{P}(A)| \leq 3\epsilon + |\mathbb{P}(A_S)^2 - \mathbb{P}(A)^2| \leq 5\epsilon.$$

Consequently, $\mathbb{P}(A) \in \{0, 1\}$. □

Proposition 3.11. *For any integer $d \geq 2$ and $p \in [0, 1]$, let σ_p be the Voronoi percolation model defined above. Then, the associated pseudometric T is measurable with respect to the $\Sigma-$algebra of σ_p, T satisfies conditions 1, 2 and 4, and the associated time constant μ_p defined by (1.2) is well-defined.*

Proof: By Proposition 3.10, σ_p is ergodic. The colour of Voronoi percolation is constant on each tile, and a tile τ is semi-algebraic, that is, τ is a finite union of subsets defined by a finite number of algebraic inequalities, see Bochnak et al. (1998, Definition 2.1.4). This implies that that the restriction of σ_p to any segment is the indicator function of a semi-algebraic subset. Since by Bochnak et al. (1998, Theorem 2.4.5), any such subset has a finite number of connected components, the restriction is regulated over I. By Lemma 3.1, this implies that T is measurable with respect to the Σ-algebra of σ_p, is ergodic and satisfies condition 4. Since $\sigma_p = 0$ or 1, T satisfies condition 2. Hence, by Theorem 2.1, μ_p is well-defined and isotropic. □

We carry on with an intermediate result for the proof of Theorem 3.5.

Proposition 3.12. *Let $\sigma_p : \mathbb{R}^d \to \{0, 1\}$ be the Voronoi percolation with parameter $p \in]0, 1[$. If $p < p_c(d)$, then the associated pseudometric T satisfies condition 5.*

Proof: Let $R > 1$ and assume that $T(A_R) = 0$. Then, there doesn't exist any black circuit in A_R. Indeed, if there were such a circuit C, any deterministic crossing path would cross C. However, with probability one, the union of Voronoi cell touching the path has positive width, so that its time T would be non vanishing. Hence, there is a white crossing, that is, the event $\mathrm{Cross}_0(A_R)$ occurs. Now, if $p < p_c(d)$, Theorem 3.4 implies that the probability of this event is exponentialy small with R. Hence, σ_p satisfies condition 5. □

Proof of Theorem 3.5: By Proposition 3.8, the pseudometric T associated to σ_p satisfies condition 6 (quasi-independence). Let $p < p_c(d)$. By Proposition 3.12, T satisfies condition 5. By Theorem 2.2, $\mu_p > 0$. For $p > p_c(d)$, by the definition (3.4) of $p_c(d)$, almost surely there is an infinite connected component of $\{\sigma_p = 0\}$, so that condition 7 (white crossing of large spherical shells) holds, which implies that $\mu_p = 0$ by Theorem 2.4. □

3.2. *Gaussian FPP.* Continous Gaussian fields are very natural object in probability. Gaussian percolation, which can be defined by the connectivity features of the associated nodal domains, that is the subset of points where the function is positive, has recently become a very active research area. Let

$$f : \mathbb{R}^2 \to \mathbb{R}$$

be any centered planar Gaussian field. To this field we associate a family $(\sigma_p)_{p\in\mathbb{R}}$ of colouring functions over $\mathbb{R}^p$ defined by:

$$\forall p \in \mathbb{R},\ \sigma_p := \frac{1}{2}\left(1 + \operatorname{sign}(f + p)\right), \tag{3.8}$$

where the sign of 0 is considered to be -1. This choice will have no influence if f satisfies condition 12, see Proposition 3.19. Recall that f is entirely determined by its covariance kernel:

$$\forall (x, y) \in (\mathbb{R}^2)^2,\ e(x, y) := \mathbb{E}\left(f(x)f(y)\right).$$

In the sequel, the centered Gaussian field f will satisfy the following conditions:

(9) (Stationarity) the covariance is stationary, that is invariant under translations, so that there exists $\kappa : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\forall x, y \in \mathbb{R}^2,\ e(x, y) = \kappa(x - y). \tag{3.9}$$

(10) (Normalization) $\kappa(0) = 1$.
(11) (Symmetries) κ is symmetric under both reflection across the x-axis, and rotation by $\pi/2$ about the origin.
(12) (Regularity) κ is C^3.
(13) (Positive correlations) $\kappa \geq 0$.
(14) (Decay of correlations)
  (a) (weak) $\kappa(x) \to_{\|x\|\to\infty} 0$.
  (b) (strong) There exist two positive constants C, β such that for every multi-index α with $|\alpha| \leq 3$,
$$\forall x \in \mathbb{R}^2,\ |\partial^\alpha \kappa(x)| \leq C\|x\|^{-\beta}.$$
(15) (Isotropy) κ depends only on the distance between two points.

Example 3.13. The full list of conditions from 9 to 15 are satisfied by the *Bargmann-Fock field*. This field arises naturally from random complex and real algebraic geometry as explained in Beffara and Gayet (2017). It is given by the non-negative correlation function:

$$e(x, y) = \exp\left(-\frac{1}{2}\|x - y\|^2\right).$$

Equivalently, we can explicitly write it as the following random field f:

$$f(x) = \exp\left(-\frac{1}{2}\|x\|^2\right) \sum_{i,j\in\mathbb{N}} a_{i,j} \frac{x_1^i x_2^j}{\sqrt{i!j!}}, \tag{3.10}$$

where the $a_{i,j}$'s are i.i.d centered Gaussians of variance 1.

In Beffara and Gayet (2017), V. Beffara and the second author of this work proved a Russo-Seymour-Welsh theorem for the nodal domains $\{f > 0\}$:

Theorem 3.14. *(Beffara and Gayet (2017), Beliaev and Muirhead (2018), Rivera and Vanneuville (2019) and Muirhead and Vanneuville (2020)) Let f be a Gaussian field on $\mathbb{R}^2$ satisfying the assumptions given above, where the exponent from assumption 14b verifies $\beta > 2$. Let σ_0 be the associated colour function defined by (3.8) for $p = 0$. Then,*

(1) for any rectangle $R \subset \mathbb{R}^2$, $\liminf_{n\to\infty} \mathbb{P}[\mathrm{Cross}_0(nR)] > 0$;
(2) there exist $C, \alpha > 0$, such that $\forall R \geq 1,\ \mathbb{P}[\mathrm{Cross}_0(A_R)] \leq \frac{C}{R^\alpha}$.

The second assertion implies that there is no infinite component of $\{f > 0\}$, a negative result which was already in Alexander (1996), with a different (sketched) proof. Secondly, Rivera and Vanneuville (2020) proved that for the Bargmann-Fock field (3.10) below the value $p = 0$ is critical:

Theorem 3.15. *(?, Muirhead and Vanneuville (2020), Rivera (2021), Garban and Vanneuville (2020)) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Gaussian field satisfying the assumptions given above, where the exponent from assumption 14b verifies $\beta > 2$.*

(1) If $p \geq 0$, then a.s. there is no unbounded connected component of $\{\sigma_p = 0\}$.
(2) If $p < 0$, then a.s. there is a unique unbounded connected component of $\{\sigma_p = 0\}$.

Theorem 3.16. *Let f be a centered Gaussian field over $\mathbb{R}^2$ satisfying the above assumptions, where the exponent from assumption 14b verifies $\beta > 21$. Let $(\sigma_p)_{p\in\mathbb{R}}$ be the associated family of colour functions given by (3.8), and $(T_p)_p$ the associated pseudometric defined by (3.1). Then,*

(1) the time constants $(\mu_p)_p$ are well-defined ;
(2) the conclusions of Theorem 2.5 (shape theorem) hold.
(3) $\mu_p > 0 \Leftrightarrow p > 0$.

Corollary 3.17. *Let f be a centered Gaussian field over $\mathbb{R}^2$ and satisfying and satisfying the above assumptions. For $p = 0$, that is the colouring function is σ_0, then*

$$\forall \eta > 5/4,\ \limsup_{R\to\infty} R^\eta \mathbb{P}[\mathrm{Cross}_0(A_R)] > 0.$$

The rest of this section is devoted to the proof of Theorem 3.16. We begin by recalling two important classical regularity results. The first one concerns analytic regularity:

Proposition 3.18. *(Nazarov and Sodin, 2016, §A.3) Let $k \in \mathbb{N}^*$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a Gaussian field with covariance e, such that e can be differentiated at least k times in x and k times in y, and that these derivatives are continuous. Then, almost surely f is C^{k-1}.*

The second one concerns the geometric regularity of the vanishing locus of the field:

Proposition 3.19. *(Adler and Taylor, 2007, Lemma 12.11.12) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Gaussian field, almost surely C^1. Then, almost surely f vanishes transversally. In particular, $\{f = 0\}$ is empty or has codimension 1.*

By condition 12, f is almost surely C^1, so that by Proposition 3.19, the vanishing locus has a vanishing Lebesgue measure, so that the previous choice has no influence on the value of the random pseudometric T for σ_p defined by (3.1).

Proposition 3.20. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a centered Gaussian field satisfying the conditions above, $(\sigma_p)_{p\in\mathbb{R}}$ be the colouring defined by (3.8) and $(T_p)_p$ their associated pseudometrics. Then, T_p is measurable with respect to the Σ-algebra of f, satisfies conditions 1, 2, 3 and 4. In particular, the constants μ_p are well-defined.*

Proof: Since f is continuous almost surely by Proposition 3.18, it is continuous over any segment $I \subset \mathbb{R}^2$. Consequently, σ_p is regulated over I. By Lemma 3.1, this implies that T is measurable with respect to the Σ-algebra of σ_p, hence with respect to that of f as well, and that T satisfies condition 4. By Adler (2010, Theorem 6.5.4), any stationnary centered Gaussian field which is almost surely continuous, and whose correlation function converges to zero at infinity, is ergodic. All these conditions are true by hypothesis. This implies that σ_p is also ergodic, so that T satisfies condition 1. Since $\sigma_p = 0$ or 1, T satisfies condition 2. The isotropy condition is a consequence of the isotropy of κ. Hence, by Theorem 2.1, μ_p is well-defined and isotropic. □

Fields with positive correlations enjoy a very important property, namely the FKG inequality:

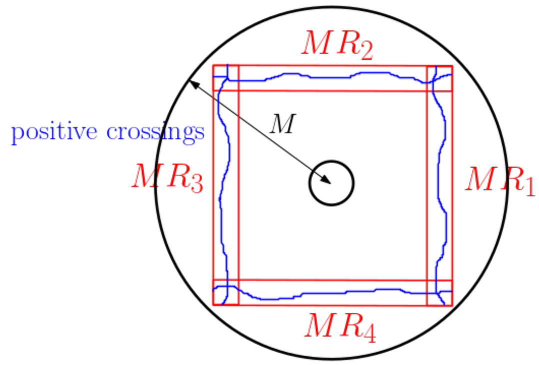

FIGURE 3.2. Positive crossings of the four rectangles implies no negative crossing of the shell

Theorem 3.21. *(Pitt (1982); Rivera and Vanneuville (2019, Lemma A.12)) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Gaussian field satisfying conditions 11, 12 and 13 (positive correlations). Then for any $p \in \mathbb{R}$, σ_p defined by (3.8) satisfie the following Fortuin-Kasteleyn-Ginibre inequality for crossings: for any positive crossing events E_1 and E_2 of the form $\mathrm{Cross}_1(R)$ (see 3.2),*

$$\mathbb{P}[E_1 \cap E_2] \geq \mathbb{P}[E_1]\mathbb{P}[E_2].$$

In dimension 2, when $p = 0$, Theorem 3.14 asserts that both probabilities of $\mathrm{Cross}_0(nR)$ and $\mathrm{Cross}_1(nR)$ are uniformly lower bounded by a positive constant when n goes to infinity. When $p \neq 0$, this situation changes drastically:

Theorem 3.22. *(?; Muirhead and Vanneuville (2020); see also Rivera (2021)) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a planar Gaussian field satisfying the assumptions above. For any $p \in \mathbb{R}$, let σ_p be the associated random planar colouring defined by (3.8), and $R \subset \mathbb{R}^2$ be a rectangle. Then*

$$p > 0 \Leftrightarrow \exists c > 0, M_0 > 0, \forall M \geq M_0,\ \mathbb{P}\Big[\mathrm{Cross}_0(MR)\Big] \leq e^{-cM}.$$

We state now a simple corollary of Theorem 3.22 which will be used for the proof of Theorem 3.16, and which relies only on the FKG condition.

Corollary 3.23. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a planar Gaussian field satisfying the above assumptions, and let $p > 0$. Then, there exist positive constants c, M_0 such that*

$$\forall M \geq M_0,\ \mathbb{P}\left[\mathrm{Cross}_0(A_M)\right] \leq e^{-cM}.$$

In particular, σ_p satisfies condition 5.

Proof: Consider four fixed horizontal or vertical rectangles $(R_i)_{i=1,\cdots,4}$ inside $A_2 = A_{1,2}$, and such that their open union contains a closed circuit inside A_2 around its center, see Figure 3.2. Note that for all $M \geq 2$ the union of the four copies $MR_1, \cdots MR_4$ lies in A_M. By Theorem 3.22, there exist $M_0 > 0$ and $c > 0$ such that

$$\forall i \in \{1, \cdots, 4\},\ \forall M \geq M_0,\ \mathbb{P}\Big[\mathrm{Cross}_1(MR_j)\Big] \geq 1 - e^{-cM}.$$

Here we used the symmetry of the law under rotation of right angle and by translations. We also used that a lengthwise positive crossing is the complement event of there being a widthwise negative crossing. Since a positive circuit inside the union of the four rectangles prevents any negative crossing of the annulus, Theorem 3.21 (FKG) implies that

$$\forall M \geq M_0,\ \mathbb{P}(\mathrm{Cross}_0(A_M)^{\mathsf{c}}) \geq \mathbb{P}\left[\bigcap_{i=1}^{4} \mathrm{Cross}_1(R_i)\right] \geq (1 - e^{-cM})^4.$$

Thus there exist $M_1 \geq M_0$ and $c' > 0$ such that

$$\forall M \geq M_1,\ \mathbb{P}(\text{Cross}_0(A_M)) \leq e^{-c'M}.$$

Now, let $R > 1$ and assume that $T(A_R) = 0$. Then, there is no black circuit in A_R. Indeed, if there were one, by Proposition 3.18 and then Proposition 3.19, almost surely the positive region $\{\sigma = 1\}$ would be a d-submanifold with smooth boundary, hence any crossing of A_R would cross an open set in $\{\sigma = 1\}$, hence could not have vanishing time. Hence, T satisfies condition 5. □

In order to ensure that the sign of σ_p satisfies the quasi-independence condition 6, we will use the following theorem.

Theorem 3.24. *(Muirhead and Vanneuville, 2020, Theorem 4.2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a planar Gaussian field satisfying the above conditions (14b for a certain $\beta > 0$). Then, there exist $c, R_0 > 0$ such that for any $R \geq R_0$, $r \geq 1$ and $t \geq \log R$, for any pair of compact sets A and B of diameters bounded above by R with $\text{dist}(A, B) \geq r$, for any events E_1, E_2 which are both increasing or both decreasing events, and depending only of the field f over A and B respectively, we have:*

$$\Big|\mathbb{P}(E_1 \cap E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2)\Big| \leq cRtr^{1-\beta} + ce^{-ct^2}.$$

Corollary 3.25. *Let f be satisfying the conditions of Theorem 3.24 for some $\beta > 1$. Then for any $1 < \alpha_0 < \beta - 1$ and any $0 < \beta' < \beta - 1 - \alpha_0$, and any $p \in \mathbb{R}$, σ_p defined by (3.8) satisfies the inequality of condition 6 (quasi-independence) with exponent β' instead of* $19(d-1)$.

Proof: Firstly, notice that a decreasing event E depending only on the value of σ_p over some subset $A \subset \mathbb{R}^d$ is also decreasing for f. Hence, by the definition of Ind^- given by (2.3) and Theorem 3.24 taking $R = S$, $r = Q$ and $t = \log S$, there exists $Q_0 \geq 0$, such that

$$\forall Q \geq 1,\ \text{Ind}^-(Q, S) \leq SQQ^{-\beta} \log S + cS^{-c \log S}.$$

So that if $S = Q^\alpha$, $\alpha < \alpha_0$,

$$\text{Ind}^-(Q, S) \leq \alpha Q^{1+\alpha} Q^{-\beta} \log Q + cQ^{-c\alpha^2 \log Q} \leq Q^{-\beta'}$$

for Q large enough, since $\beta' < \beta - 1 - \alpha_0$. So that condition 6 is verified for β'. □

Proof of Theorem 3.16: First, let us suppose $p > 0$ and that f satisfies assumption 14b for some $\beta > 21$. By Corollary 3.25, if we let $\alpha_0 = (\beta - 19)/2$, the associated pseudometric satisfies a polynomial decay of correlations with exponent $\beta' = (\beta + 19)/2 - 1 > 20 - 1 = 19$, hence it satisfies assumption 6. Besides, by Corollary 3.23, T satisfies condition 5. By Propoposition 3.20, T satisfies all the other conditions needed for Theorem 2.2. In conclusion, $\mu_p > 0$. If $p = 0$, by Theorem 3.14, for any rectangle R, $\liminf_{n\to\infty} \mathbb{P}[\text{Cross}_0(nR)] > 0$. This implies that $\liminf_{n\to\infty} \mathbb{P}[\text{Cross}_0(A_{n,2n})] > 0$, so that

$$\liminf_{n\to\infty} \mathbb{P}[T(A_{n,2n}) = 0] > 0.$$

In other terms, T satisfies condition 7. Lastly, by Lemma 3.1, T is Lipschitz, hence satisfies condition 8. By Theorem 2.4, this implies that $\mu_p = 0$. For $p < 0$, since the white crossing probabilities decrease with p, we obtain the same conclusion. □

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