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Functional Gaussian approximations on Hilbert-Poisson spaces

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Abstract. We develop a functional Stein-Malliavin method in a non-diffusive Poissonian setting, thus obtaining a) quantitative central limit theorems for approximation of arbitrary non-degenerate Gaussian random elements taking values in a separable Hilbert space and b) fourth moment bounds for approximating sequences with finite chaos expansion. Our results rely on an infinite-dimensional version of Stein's method of exchangeable pairs combined with the so-called Gamma calculus. Two applications are included: Brownian approximation of Poisson processes in Besov-Liouville spaces and a functional limit theorem for an edge-counting statistic of a random geometric graph.

1. Introduction

The now classical Stein-Malliavin method, a combination of Stein's method with Malliavin calculus, has been very successful in deriving quantitative central limit theorems for non-linear approximation. Since its inception by Nourdin and Peccati in 2013 (see Nourdin and Peccati (2009)), it has formed a vivid community which developed the theory further and applied it to numerous situations. An excellent exposition of the basic method is available in the monograph Nourdin and Peccati (2012) , while I. Nourdin keeps a rather exhaustive and continuously updated list of references on the webpage https://sites.google.com/site/malliavinstein. From a theoretical point of view, one of the main remaining challenges is an adaptation of the method to the infinite-dimensional setting, with quantitative approximation of Gaussian processes as main application. For random elements taking values in a Hilbert space, and in a diffusive context, this has recently been achieved by Bourguin and Campese (2020). In this work, we provide the natural analogue in

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the non-diffusive context of Poisson spaces. More specifically, let X be a square-integrable measurable transformation of a Poisson process and Z be a Gaussian process, both taking values in some separable Hilbert space K. Informally, our main results (Theorems 3.1 and 3.7 on page 526) provide bounds on a probabilistic distance between X and Z (metrizing convergence in law) in terms of the first four strong moments of X or alternatively in terms of so called contractions. From these bounds, one can directly deduce quantitative and functional central limit theorems for convergence towards a Gaussian process, as well as an infinite-dimensional version of the Fourth Moment Theorem, which says that for a sequence of K-valued multiple Poisson-integrals, convergence of the second and fourth moments implies convergence towards a Gaussian process.

It is noteworthy to observe that while the analogous diffusive statements in Bourguin and Campese (2020) look similar to our non-diffusive ones, their proofs are rather different, for the same reason as in the finite-dimensional case: No chain rule is available in the non-diffusive case, which renders the usual integration by parts argument unfeasible. Instead, one can construct an appropriate exchangeable pair and then apply a Taylor argument in order to control the term resulting from an application of Stein's method. Compared to the finite-dimensional setting, several technical issues arise which require the use of Hilbert-space techniques. A commonality with the diffusive statements is, however, that our main results subsume all known finite-dimensional Malliavin-Stein bounds in a Poissonian context as special cases (see Remark 3.2 on page 526 for details).

In order to illustrate our results, we provide two applications: The first one concerns the classical approximation of a Brownian motion by a normalized Poisson process with growing intensity λ. A natural class of Hilbert spaces accommodating the sample paths of both processes are the so-called Besov-Liouville spaces. In Coutin and Decreusefond (2013), the authors showed that convergence takes place at rate $\lambda^{-1/2}$ (as in the classical one-dimensional case). To prove this, they first transferred both processes isometrically $\ell^2(\mathbb{N})$ and then had to go through rather tedious calculations. In contrast to this, our bounds yield the same result in just a few lines, and no isometry is necessary. As a second application we illustrate, using an edge counting statistic of a random graph, how known one-dimensional central limit theorem can be made functional with very little additional effort.

Besides the already mentioned reference Bourguin and Campese (2020), the work Coutin and Decreusefond (2013) is also concerned with quantitative functional approximation in a Malliavin-Stein context. As already mentioned, the authors use a different approach which crucially depends on isometrically mapping all random elements to $\ell^2(\mathbb{N})$. In applications, the need to explicitly evaluate such an isometry can be seen as a drawback. Also, our setting seems to be more general and does not rely on ad-hoc arguments depending on the Gaussian process at hand. Other related references proving functional central limit theorems using Malliavin-Stein techniques are Kasprzak (2017, 2020); Döbler and Kasprzak (2021); Döbler et al. (2022).

The rest of this paper is organized as follows. In Section 2 we introduce the necessary preliminaries, followed by the main results in Section 3. The proofs are given in Section 4 which is followed by the two aforementioned applications in Section 5. An appendix contains several technical lemmas required for the proofs.

2. Preliminaries

2.1. *Probability on Hilbert spaces.*

Let K be a real separable Hilbert space, $\mathcal{B}(K)$ the Borel σ-algebra of K and $(\Omega, \mathcal{F}, P)$ a complete probability space. A K-valued random variable X is a measurable map from $(\Omega, \mathcal{F})$ to $(K, \mathcal{B}(K))$. Such random variables are characterized by the property that for any continuous linear functional $\phi \in K^*$, the function $\phi(X) : \Omega \to \mathbb{R}$ is a real-valued random variable. As usual, the distribution or law of X is the push-forward probability measure $P \circ X^{-1}$ on $(K, \mathcal{B}(K))$. The set of all K-valued random variables is a a vector space over the field of real numbers. If the Lebesgue integral $\mathbb{E}[\|X\|_K] = \int_\Omega \|X\|_K \, dP$ exists and is finite, then the Bochner integral $\int_\Omega X dP$ exists in K and is called the expectation of X. Slightly abusing notation, we denote this integral by $\mathbb{E}[X]$ as well, and it can always inferred from the context whether $\mathbb{E}[\cdot]$ refers to Lebesgue or Bochner integration with respect to P. For $p \geq 1$, $L^p(\Omega, P)$ denotes the Banach space of all equivalence classes (under almost sure equality) of K-valued random variables X with finite p-th moment, i.e., such that

$$\|X\|_{L^p(\Omega,P)} = \mathbb{E}\big[\|X\|_K^p\big]^{1/p} < \infty.$$

Note that for all $X \in L^p(\Omega, P)$, the Bochner integral $\mathbb{E}[X]$ exists. In the case $X \in L^2(\Omega, P)$, the covariance operator $S : K \to K$ of X is defined by

$$Su = \mathbb{E}[\langle X, u\rangle_K X].$$

S is a positive, self-adjoint trace-class operator that verifies the identity

$$\operatorname{Tr} S = \mathbb{E}\Big[\|X\|_K^2\Big].$$

We denote by $S_1(K)$ the Banach space of all trace-class operators on K, equipped with norm $\|T\|_{S_1(K)} = \operatorname{Tr}|T|$, where $|T| = \sqrt{TT^*}$ and T^* denotes the adjoint of T. The subspace of Hilbert-Schmidt operators on K is denoted by $\mathrm{HS}(K)$, its inner product and norm by $\langle \cdot, \cdot\rangle_{\mathrm{HS}(K)}, \|\cdot\|_{\mathrm{HS}(K)}$ respectively. Recall that

$$\|\cdot\|_{\mathrm{op}} \leq \|\cdot\|_{\mathrm{HS}(K)} \leq \|\cdot\|_{S_1(K)},$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm.

2.2. *Gaussian measures and Stein's method.*

In this section, we introduce Gaussian measures, the associated abstract Wiener spaces and Stein characterization of Gaussian measures. The theory will be presented within a general Banach space setting. Keep in mind that at the end of this section and beyond that, we will assume any target Gaussian measure under consideration is defined on a Hilbert space such as K above. Standard references for Gaussian measures and abstract Wiener spaces are the monographs Bogachev (1998); Kuo (1975), while Stein's method for Gaussian measures has been developed by Shih (2011) (see also Barbour (1990), an earlier work for the special case of Brownian motion).

2.2.1. *Abstract Wiener spaces.* Let H be a real separable Hilbert space equipped with inner product $\langle \cdot, \cdot\rangle_H$ and $\|\cdot\|$ be a norm on H weaker than $\|\cdot\|_H$. Denote B the Banach space obtained via completion of H with respect to $\|\cdot\|$ and i the canonical embedding of H into B. The triple (i, H, B) defines an abstract Wiener space and has first been introduced by Gross (1967a). We identify B^* as a dense subspace of H^* under the adjoint i^* of i, so that we have the continuous embeddings $B^* \subseteq H \subseteq B$, where, as usual, H is identified with its dual H^*. All of this can be summarized via the diagram

$$B^* \xrightarrow{i^*} H^* = H \xrightarrow{i} B.$$

The abstract Wiener measure p on B is characterized as the Borel measure on B satisfying

$$\int_B \exp\Big(i\,\langle x, \eta\rangle_{B,B^*}\Big)p(dx) = \exp\left(-\frac{\|\eta\|_H^2}{2}\right),$$

for any $\eta \in B^*$.

2.2.2. *Gaussian measures.* Let B be a separable Banach space, with $\mathcal{B}(B)$ its Borel σ-algebra. A Gaussian measure μ is a probability measure on $(B, \mathcal{B}(B))$ such that every non-zero linear functional $x \in B^*$, considered as a (real-valued) random variable on $(B, \mathcal{B}(B), \mu)$, has a Gaussian distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Such a Gaussian measure is called centered and/or non-degenerate, if these properties hold for the distributions of every non-zero $x \in B^*$.

We can see that every abstract Wiener measure is a Gaussian measure, and conversely, for every Gaussian measure μ on B, there exists a Hilbert space H such that (i, H, B) forms an abstract Wiener space. The space H is known as the Cameron Martin space.

2.2.3. *Stein characterization of Gaussian measures.* Let B be a real separable Banach space with norm $\|\cdot\|$. Let Z be a B-valued random variable which induces a centered Gaussian measure μ_Z on B and let (i, H, B) be the associated abstract Wiener space. By $\{P_t : t \geq 0\}$ we denote the Ornstein-Uhlenbeck semi-group of Z. It has the Mehler representation

$$P_t f(x) = \int_B f\Big(e^{-t}x + \sqrt{1-e^{-2t}}y\Big)\mu_Z(dy),$$

provided such an integral exists. In Shih (2011, Theorem 3.1), Shih proved the following Stein lemma for abstract Wiener measures.

Theorem 2.1. *Let X be a B-valued random variable with distribution μ_X.*

i) If B is finite-dimensional, then $\mu_X = \mu_Z$ if and only if

$$\mathbb{E}\Big[\langle X, \nabla f(X)\rangle_{B,B^*} - \Delta_G f(X)\Big] = 0 \tag{2.1}$$

for any twice-differentiable function f on B such that $\mathbb{E}\Big[\big\|\nabla^2 f(Z)\big\|_{S_1(H)}\Big] < \infty$.

ii) If B is infinite-dimensional, then $\mu_X = \mu_Z$ if and only if (2.1) holds for every twice H-differentiable function f on B such that $\nabla f(x) \in B^*$ for every $x \in B$, $\mathbb{E}\Big[\big\|\nabla^2 f(Z)\big\|_{S_1(H)}\Big] < \infty$ and $\mathbb{E}\Big[\|\nabla f(Z)\|^2_{B^*}\Big] < \infty$.

The notion of H-derivative which is also known as Fréchet derivative along H and appears in Theorem 2.1 was introduced by Gross (1967b), and we briefly recall it here for the sake of self-containedness. A function $f : U \to W$ from an open set U of B into a Banach space W is said to be H-differentiable at $x \in U$ if the map $\phi(h) = f(x+h), h \in H$, regarded as a function defined in a neighborhood of the origin of H is Fréchet-differentiable at 0. The H-derivative of f at x in the direction $h \in H$ is denoted by $\langle \nabla f(x), h\rangle_H$. The k-th order H-derivatives of f at x can then be constructed inductively and are denoted by $\nabla^k f(x)$, provided they exist. If f is scalar-valued, $\nabla f(x) \in H^* \simeq H$ and $\nabla^2 f(x)$ is a bounded linear operator from H to H^* for every $x \in U$. The notation $\left\langle \nabla^2 f(x)h, k\right\rangle_H$ or $\nabla^2 f(x)(h,k)$ will stand for the action of the linear form $\nabla^2 f(x)(h, \cdot)$ on k.

If $\nabla^2 f(x)$ is a trace-class operator on H, the Gross Laplacian $\Delta_G f(x)$ of f at x is defined as $\Delta_G f(x) = \mathrm{Tr}_H(\nabla^2 f(x))$.

2.2.4. *Stein's equation.* In view of Theorem 2.1, the associated Stein equation is given by

$$\langle x, \nabla g(x)\rangle_{B,B^*} - \Delta_G g(x) = h(x) - \mathbb{E}[h(Z)]$$

for $x \in B$, where h belongs to a suitable class of test functions. Recall from earlier that we assume K to be a separable Hilbert space. From this point forward, we will let $B = K$ and assume our test functions belong to $C_b^3(K)$, the class of real-valued functions on K that have bounded Fréchet derivatives up to order three. This space is equipped with the norm

$$\|h\|_{C_b^3(K)} = \sup_{j=1,2,3}\sup_{x\in K}\left\|D^j h(x)\right\|_{K^{\otimes j}}.$$

Using standard semigroup techniques, the first two authors of this work showed in Bourguin and Campese (2020) that there is a solution $g_h(x)$ for every test function $h(x)$ and that $g_h \in C_b^3(K)$ when $h \in C_b^3(K)$. Specifically, Bourguin and Campese (2020, Lemma 2.4) provides the estimates

$$\sup_{x \in K} \left\| D^j g_h(x) \right\|_{K^{\otimes j}} \leq \frac{1}{j} \left\| h \right\|_{C_b^j(K)} \tag{2.2}$$

and

$$\left\| g_h \right\|_{C_b^3(K)} \leq \left\| h \right\|_{C_b^3(K)}.$$

Thus, using the probability distance

$$d_3(X_1, X_2) = \sup_{\substack{h \in C_b^3(K) \\ \|h\|_{C_b^3(K)} \leq 1}} \left| \mathbb{E}[h(X_1) - h(X_2)] \right|,$$

Stein's equation implies that

$$d_3(X, Z) = \sup_{\substack{h \in C_b^3(K) \\ \|h\|_{C_b^3(K)} \leq 1}} \left| \mathbb{E}[\Delta_G g_h(X) - \langle X, Dg_h(X) \rangle_K] \right|.$$

Remark 2.2. To obtain fourth moment estimate on separable Hilbert spaces, we choose to work with a rather strong notion of probability distance that is $d_3(\cdot,\cdot)$. In fact, proofs of our main results (see Section 4) only require bounds on second and third derivatives of g_h. Then per (2.2), the results in Section 3 still hold true if $d_3(\cdot,\cdot)$ is replaced by the probability distance

$$\widetilde{d}(X_1, X_2) = \sup_{h \in \mathcal{A}} \left| \mathbb{E}[h(X_1) - h(X_2)] \right|$$

such that $\mathcal{A} = \{h : K \to \mathbb{R} \text{ and } \sup_{x \in K} \left\| D^2 h(x) \right\|_{K^{\otimes 2}} \vee \sup_{x \in K} \left\| D^3 h(x) \right\|_{K^{\otimes 3}} \leq 1\}$. Alternatively, one can refer to Shih (2011, Theorem 4.9v) which states that if h is a Lipschitz function on K then $\sup_{x \in K} \left\| D^2 g_h(x) \right\|_{\mathrm{HS}(K)} \leq \|h\|_{\mathrm{Lip}}$. Based on this fact, one can replace $d_3(\cdot,\cdot)$ in the results in Section 3 with

$$\bar{d}(X_1, X_2) = \sup_{h \in \mathcal{B}} \left| \mathbb{E}[h(X_1) - h(X_2)] \right|$$

such that $\mathcal{B} = \{h : K \to \mathbb{R} \text{ and } \sup_{x \in K} \|Dh(x)\|_K \vee \sup_{x \in K} \left\| D^3 h(x) \right\|_{K^{\otimes 3}} \leq 1\}$.

A more interesting question here is whether one can obtain the results in Section 3 with Wasserstein distance, that is

$$d_{\mathrm{Wass}}(X_1, X_2) = \sup_{\|h\|_{\mathrm{Lip}(K)} \leq 1} \left| \mathbb{E}[h(X_1) - h(X_2)] \right|.$$

During the revision of this paper, it was pointed out to us that the recent reference Fang and Koike (2022) contains a four moment estimate in d_{Wass} for $\mathbb{R}^d$-valued Poisson functionals. We expect that the technique in the aforementioned reference can be adapted to the infinite-dimensional setting to produce bounds in the Wasserstein distance.

2.3. *Dirichlet structure.*

This section contains an overview of Dirichlet structures, which is the framework we will be working within alongside Stein's method. We start by recalling the definition and properties of a Dirichlet structure on $L^2(\Omega;\mathbb{R})$ (full details can be found in the monographs Bakry et al. (2014); Bouleau and Hirsch (1991)) before focusing on an extension to $L^2(\Omega;K)$. Given a probability space $(\Omega,\mathcal{F},P)$, a Dirichlet structure $(\mathbb{D},\mathcal{E})$ on $L^2(\Omega;\mathbb{R})$ with the associated carré du champ operator Γ consists of a Dirichlet domain $\mathbb{D}$, which is a dense subset of $L^2(\Omega;\mathbb{R})$ and a carré du champ operator $\Gamma : \mathbb{D} \times \mathbb{D} \to L^1(\Omega,\mathbb{R})$ characterized by the following properties.

- Γ is bilinear, symmetric ($\Gamma(F,G) = \Gamma(G,F)$) and positive ($\Gamma(F,F) \geq 0$).

- the induced positive linear form $F \to \mathcal{E}(F,F)$, where $\mathcal{E}(F,G) = \frac{1}{2}\mathbb{E}[\Gamma(F,G)]$, is closed in $L^2(\Omega;\mathbb{R})$, i.e., $\mathbb{D}$ is complete when equipped with the norm

$$\|\cdot\|_{\mathbb{D}}^2 = \|\cdot\|_{L^2(\Omega;\mathbb{R})}^2 + \mathcal{E}(\cdot).$$

Remark 2.3. We do not assume that Γ satisfies the so-called diffusion property – see Bakry et al. (2014, Definition 3.1.3) – as opposed to what is being done in Bourguin and Campese (2020).

Here and in the following, $\mathbb{E}[\cdot]$ denotes the expectation on $(\Omega,\mathcal{F})$ with respect to P. The linear form $\mathcal{E}$ is known as a Dirichlet form and for brevity we write $\mathcal{E}(F)$ for $\mathcal{E}(F,F)$. Every Dirichlet form gives rise to a strongly continuous semigroup $\{P_t\}_{t\geq 0}$ on $L^2(\Omega;\mathbb{R})$ and an associated symmetric Markov generator $-L$, defined on a dense subset $\mathrm{dom}(-L) \subseteq \mathbb{D}$. We will assume that $\ker(-L)$ only consists of constants. There are two important relations between Γ and L, the first being the integration by part formula

$$\mathbb{E}[\Gamma(F,G)] = -\mathbb{E}[FLG] = -\mathbb{E}[GLF],$$

which is valid for $F,G \in \mathbb{D}$. The second relation is

$$\Gamma(F,G) = \frac{1}{2}(L(FG) - GLF - FLG),$$

which holds for all $F,G \in \mathrm{dom}(L)$ such that $FG \in \mathrm{dom}(L)$. If $-L$ is diagonalizable with spectrum $\mathbb{N}_0$ (the set of natural numbers plus 0) and F_q is an eigenfunction corresponding to the eigenvalue q, then $-LF_q = qF_q$. The pseudo-inverse L^{-1} is defined by $-L^{-1}F_q = \frac{1}{q}F_q$ when $q \neq 0$ and 0 otherwise. The definition of $-L$ and $-L^{-1}$ for a general $F = \sum_{q\in\mathbb{N}_0} F_q$ follows naturally via linearity. Alternatively, L can be defined as the generator of the heat semigroup $\{P_t\}_{t\geq 0}$ (on $\mathrm{dom}(L)$) which satisfies

$$\partial_t P_t = LP_t = P_t L.$$

Next we present what is meant by a Dirichlet structure on $L^2(\Omega;K)$. Let us adopt the notations $\widetilde{\mathbb{D}}, \widetilde{\Gamma}, \widetilde{L}, \widetilde{P}_t$ for the Dirichlet domain, Dirichlet form, carré du champ operator, generator and semigroup associated with elements in $L^2(\Omega;\mathbb{R})$. Meanwhile, $\mathbb{D}, \Gamma, L, P_t$ are reserved for the counterpart objects associated with elements in $L^2(\Omega;K)$. Given a separable Hilbert space K, one has that $L^2(\Omega;K)$ is isomorphic to $L^2(\Omega;\mathbb{R}) \otimes K$. The Dirichlet structure on $L^2(\Omega;\mathbb{R})$ can therefore be extended to $L^2(\Omega;K)$ via a tensorization procedure. Let $\mathbb{N}_0$ be the spectrum of $-\widetilde{L}$ and $\{k_i\}_{i\in\mathbb{N}}$ an orthonormal basis of K. $\mathcal{A}$ will be the set of all functions X taking the form

$$X = \sum_{q,i\in I} F_{q,i} \otimes k_i$$

such that $I \subseteq \mathbb{N}^2$ is a finite set and $F_{q,i} \in \ker\left(-\widetilde{L} + qI\right)$. Assuming another element $Y = \sum_{p,j\in J} G_{p,j} \otimes k_j$ in $\mathcal{A}$, we can define $L, \Gamma, P_t, \mathcal{E}$ for $t \geq 0$ via

$$\begin{cases} LX = L\sum_{q,i\in I} F_{q,i} \otimes k_i = \sum_{q,i\in I}\left(\widetilde{L}F_{q,i}\right) \otimes k_i \\ P_t X = P_t \sum_{q,i\in I} F_{q,i} \otimes k_i = \sum_{q,i\in I}\left(\widetilde{P}_t F_{q,i}\right) \otimes k_i \\ \Gamma(X,Y) = \frac{1}{2}\sum_{q,i\in I}\sum_{p,j\in J} \widetilde{\Gamma}(F_{q,i}, G_{p,j}) \otimes (k_i \otimes k_j + k_j \otimes k_i) \end{cases}$$

and

$$\mathcal{E}(X,Y) = \mathbb{E}[\mathrm{Tr}\,\Gamma(X,Y)].$$

In the last line, we identify $\Gamma(X, Y)$ as an element of $L^2(\Omega; \mathbb{R}) \otimes K \otimes K \simeq L^2(\Omega, \mathcal{L}(K, K))$ via the action

$$\Gamma(X, Y)u = \frac{1}{2} \sum_{q,i \in I} \sum_{p,j \in J} \widetilde{\Gamma}(F_{q,i}, G_{p,j}) \otimes \left(\langle k_i, u \rangle_K \otimes k_j + \langle k_j, u \rangle_K \otimes k_i \right).$$

Remark 2.4. The definitions above are independent of the choice of basis of K. We include here a brief explanation for the operator L for the sake of completeness. First, the definition of L is equivalent to

$$LX = \sum_{i \in N} \left(\widetilde{L} \langle X, k_i \rangle_K \right) k_i.$$

Let $\{e_j\}_{j \in \mathbb{N}}$ be another orthonormal basis of K and define

$$L_0 X = \sum_{j \in \mathbb{N}} \left(\widetilde{L} \langle X, e_j \rangle_K \right) e_j.$$

Then by using $e_j = \sum_{n \in \mathbb{N}} \langle e_j, k_n \rangle_K k_n$ such that $\sum_{n \in \mathbb{N}} \langle e_j, k_n \rangle_K^2 = 1$, and also the identity $\langle u, v \rangle_K = \sum_{i \in \mathbb{N}} \langle u, k_i \rangle_K \langle v, k_i \rangle_K$, one can deduce that $L_0 X = LX$.

Since $\mathcal{A}$ is clearly dense in $L^2(\Omega; K)$, the operators above can be extended to appropriate domains in $L^2(\Omega; K)$. This has been verified in Bourguin and Campese (2020, Proposition 2.5 and Theorem 2.6) (excluding the diffusion identity), which we restate below for the reader's convenience.

Proposition 2.5 (Proposition 2.5 in Bourguin and Campese (2020))**.** *The operators L L^{-1}, $\mathcal{E}$ and Γ can be extended to* $\mathrm{dom}(L)$, $\mathrm{dom}(L^{-1})$ *and* $\mathrm{dom}(\Gamma) = \mathrm{dom}(\mathcal{E}) = \mathbb{D} \times \mathbb{D}$, *respectively, given by*

$$\mathrm{dom}(L) = \left\{ X \in L^2(\Omega; K) : \sum_{q \in \mathbb{N}_0} q^2 \widetilde{J}_q \left(\|X\|_K^2 \right) < \infty \right\},$$

$\mathrm{dom}(L^{-1}) = L^2(\Omega; K)$ *and*

$$\mathbb{D} = \left\{ X \in L^2(\Omega; K) : \sum_{q \in \mathbb{N}_0} q \widetilde{J}_q \left(\|X\|_K^2 \right) < \infty \right\},$$

where $\widetilde{J}_q(\cdot)$ denotes the projection onto $\ker\left(\widetilde{L} + qI \right) \subseteq L^2(\Omega; \mathbb{R})$. *In particular, one has*

$$\mathcal{A} \subseteq \mathrm{dom}(L) \subseteq \mathbb{D} \subseteq \mathrm{dom}(L^{-1}) = L^2(\Omega; K),$$

and all inclusions are dense.

Theorem 2.6 (Theorem 2.6 in Bourguin and Campese (2020))**.** *For a Dirichlet structure $(\mathbb{D}, \Gamma)$ on $L^2(\Omega; K)$, the following is true.*

(i) Γ is bilinear, almost surely positive, symmetric and self-adjoint with respect to $\langle \cdot, \cdot \rangle_K$.

(ii) The Dirichlet domain $\mathbb{D}$ equipped with the norm

$$\|X\|_{\mathbb{D}}^2 = \|X\|_{L^2(\Omega; K)} + \|\Gamma(X, X)\|_{L^1(\Omega; S_1)}$$

is complete, so that Γ is closed.

(iii) The generator $-L$ acting on $L^2(\Omega; K)$ is positive, symmetric, densely defined and has the same spectrum as $-\widetilde{L}$.

(iv) There is a compact pseudo-inverse L^{-1} of L such that

$$LL^{-1}X = X - \mathbb{E}[X]$$

for all $X \in L^2(\Omega; K)$, where the expression on the right is a Bochner integral.

(v) The integration by parts formula

$$\mathbb{E}[\operatorname{Tr}\Gamma(X,Y)] = -\mathbb{E}[\langle LX, Y\rangle_K] = -\mathbb{E}[\langle X, LY\rangle_K]$$

is satisfied for all $X, Y \in \operatorname{dom}(-L)$.

(vi) The generators $\Gamma, L, \widetilde{L}$ *are related via*

$$\operatorname{Tr}\Gamma(X,Y) = \frac{1}{2}\Big(\widetilde{L}\,\langle X, Y\rangle_K - \langle LX, Y\rangle_K - \langle X, LY\rangle_K\Big) \tag{2.3}$$

for all $X, Y \in \operatorname{dom}(-L)$.

(vii) The identity

$$\langle \Gamma(X,Y)u, v\rangle_K = \frac{1}{2}\Big(\widetilde{\Gamma}(\langle X, u\rangle_K, \langle Y, v\rangle_K) + \widetilde{\Gamma}(\langle Y, u\rangle_K, \langle X, v\rangle_K)\Big),$$

is valid for all $X, Y \in \mathbb{D}$ *and* $u, v \in K$.

2.4. *Analysis on Poisson space.*

So far we have been working with a general probability space. In this section we will get more specific and describe the Poisson space on which most of our objects of interest are defined. We direct the reader to the references Last and Penrose (2018); Nualart and Nualart (2018) for an extensive treatment of this topic. Let $(\mathcal{Z}, \mathscr{L}, \mu)$ be a measure space such that μ is σ-finite. A Poisson random measure η on $(\mathcal{Z}, \mathscr{L})$ with control measure μ is a family of distributions defined on some probability space $(\Omega, \mathcal{F}, P)$ that satisfies

- $\eta(B)$ is a Poisson distribution on Ω with mean $\mu(B)$,
- for every $m \in \mathbb{N}$ and all pairwise disjoint sets $B_1, \dots, B_m \in \mathcal{Z}$, the random variables $\eta(B_1), \dots, \eta(B_m)$ are independent.

If such a Poisson random measure exists, the associated probability space $(\Omega, \mathcal{F}, P)$ is called a Poisson space. Next, let $\widehat{\eta}$ be the compensated Poisson random measure, that is $\widehat{\eta}(B) = \eta(B) - \mu(B)$, whenever $\mu(B)$ is finite. Denote $L^2_s(\mu^q)$ the set of all symmetric functions in $L^2(\mu^q)$. For $f \in L^2_s(\mu^q)$, $I^\eta_q(f)$ denotes a multiple (Wiener-Itô) integral of order q. Unless we are simultaneously dealing with two different Poisson random measures, $I_q(\cdot)$ will be understood as an integral with respect to $\widehat{\eta}$. Multiple integrals have the following isometry property: for any integers $q, p \geq 1$,

$$\mathbb{E}[I_q(f)I_p(g)] = \mathbb{1}_{\{q=p\}}q!\langle \tilde{f}, \tilde{g}\rangle_{L^2(\mu^q)},$$

where $\tilde{f}$ denotes the symmetrization of f, and we recall that $I_q(f) = I_q(\tilde{f})$. The contraction of two kernels $f \in L^2_s(\mu^q)$ and $g \in L^2_s(\mu^p)$, denoted by $f \star_r^l g$ for $0 \leq l \leq r \leq q \wedge p$, is obtained by identifying r variables and then integrating l of those:

$$\begin{aligned} &f \star_r^l g(y_1, \dots, y_{r-l}, y_{r-l+1}, \dots, y_{q-l}, z_1, \dots, z_{p-r}) \\ &= \int_{\mathcal{Z}^l} f(x_1, \dots, x_l, y_1, \dots, y_{r-l}, y_{r-l+1}, \dots, y_{q-l}) g(x_1, \dots, x_l, y_1, \dots, y_{r-l}, z_1, \dots, z_{p-r}) \\ &\qquad\qquad d\mu(x_1, \dots, x_l) \end{aligned}$$

provided the integral exists in $L^2(\mu^{q+p-r-l})$. Contractions are central objects for analysis on Poisson space as they appear in the product formula for multiple integrals. There are two ways of stating this product formula on Poisson space: Last (2016, Proposition 6.1) and Döbler and Peccati (2018, Lemma 2.4), each having different assumptions. We will state both below.

Lemma 2.7 (Proposition 6.1 in Last (2016))**.** *Let* $f \in L^2_s(\mu^q), g \in L^2_s(\mu^p)$ *and assume that* $f \star_r^l g \in L^2(\mu^{q+p-r-l})$. *Then,*

$$I_q(f)I_p(g) = \sum_{r=0}^{q\wedge p} r!\binom{q}{r}\binom{p}{r}\sum_{l=0}^{r}\binom{r}{l} I_{q+p-r-l}(f \star_r^l g). \tag{2.4}$$

Lemma 2.8 (Lemma 2.4 in Döbler and Peccati (2018)). *Let $f \in L^2_s(\mu^q), g \in L^2_s(\mu^p)$ and assume that $F = I_q(f), G = I_p(g) \in L^4(P)$. Then*

$$FG = \sum_{k=1}^{q\wedge p-1} \widetilde{J}_k(FG) + I_{q+p}(f\widetilde{\otimes}g).$$

The collection of all multiple integrals of order q form the so-called Poisson chaos of order q in $L^2(\Omega;\mathbb{R})$, which is denoted by $\mathcal{H}_q$. We have the orthogonal decomposition

$$L^2(\Omega,\mathcal{F},P) = \bigoplus_{q=1}^{\infty} \mathcal{H}_q.$$

Similarly as what we did for Dirichlet structures, we define $\mathcal{H}_q(K)$ (K-valued Poisson chaos of order q) as the closure of $\mathcal{H}_q \otimes K$ in $L^2(\Omega, K)$. Then,

$$L^2(\Omega;K) = \bigoplus_{q=1}^{\infty} \mathcal{H}_q(K).$$

Consequently, every $X \in L^2(\Omega, K)$ can be decomposed as

$$X = \sum_{q\in\mathbb{N}_0} F_q = \sum_{\substack{i\in\mathbb{N},\\ q\in\mathbb{N}_0}} \langle F_q, k_i\rangle_K k_i = \sum_{\substack{i\in\mathbb{N},\\ q\in\mathbb{N}_0}} F_{q,i}k_i,$$

where $F_q \in \mathcal{H}_q(K)$, $F_{q,i} \in \mathcal{H}_q$ with $F_{q,i} = I_q(f_{q,i})$ for some $f_{q,i} \in L^2_s(\mu^q)$.

2.5. *An exchangeable pair on Poisson space.* Another tool that we employ alongside Stein's method is an exchangeable pair on the Poisson space. Construction of this crucial exchangeable pair is done in Döbler et al. (2018) (see also Zheng (2019); Nourdin and Zheng (2019) for analogous construction on Rademacher and Gaussian spaces).

Per Last and Penrose (2018, Corollary 3.7), the Poisson random measure η on $(\mathcal{Z},\mathscr{L},\mu)$ equals in distribution to the proper Poisson point process

$$\eta = \sum_{n=1}^{\kappa} \delta_{X_n},$$

such that X_n and κ are random elements in respectively $\mathcal{Z}$ and $\mathbb{N}\cup\{0,\infty\}$.

Let Q be the standard exponential distribution. Assume that $\{Y_n\}_{n\in\mathbb{N}}$ is an i.i.d. family with distribution Q and is independent of (κ, X_n). Based on Last and Penrose (2018, Theorem 5.6), the marked point process $\xi = \sum_{n=1}^{\kappa} \delta_{(X_n,Y_n)}$ is a Poisson point process with control $\mu\times Q$. Then the e^{-t}-thinning of η is defined as

$$\eta_{e^{-t}} = \xi(A\times[t,\infty)]),$$

which is obtained by removing point X_n of η with probability $1-e^{-t}$ such that the thinning decisions are independent among different points.

Now let f be a measurable function $\mathbb{N}\to\mathbb{R}$ such that $\mathbb{E}[f(\eta)] < \infty$. $\hat{\eta}_t$ is another Poisson point process with control $\left(1-e^{-t}\right)\mu(\cdot)$ and is independent from $(\eta,\eta_{e^{-t}})$. Based on Last and Penrose (2018, Chapter 20), the semi-group $\{P_t\}_{t\geq 0}$ associated with η admits the representation

$$P_t f(\eta) = \mathbb{E}\left[f(\eta_{e^{-t}} + \hat{\eta}_t) \mid \eta\right]$$

for $t \geq 0$. This representation is also known as Mehler's formula on the Poisson space.

The paper Döbler et al. (2018) contains the following important result regarding the process $\eta^t = \eta_{e^{-t}} + \hat{\eta}_t$.

Lemma 2.9 (Lemma 3.1 in Döbler et al. (2018)). *For each $t \geq 0$, $\left(\eta, \eta^t\right)$ is an exchangeable pair of Poisson random measures.*

In fact, the construction of the specific exchangeable pair above is just one example of how one can in general construct exchangeable pairs using reversible Markov chain (see for example Ross (2009)). Moreover, based on the above result, for any kernel $g \in L^2_s(\mu^p)$, the pair $\left(I_p^\eta(g), I_p^{\eta^t}(g)\right)$ is exchangeable, and further relations between $I_p^\eta(g)$ and $I_p^{\eta^t}(g)$ are stated in Lemma 5.4.

3. Statement of main results

In what follows, let K be a separable Hilbert space with orthonormal basis $\{k_i\}_{i\in\mathbb{N}}$, and let X denote a K-valued centered random variable in $L^2(\Omega; K)$ with finite chaos decomposition

$$X = \sum_{1\leq q\leq N} F_q, \tag{3.1}$$

where each F_q belongs to the q-th K-valued Poisson chaos. Furthermore, assume that X has covariance operator S, which in turn decomposes as

$$S = \sum_{1\leq q\leq N} S_q,$$

where, for each $1 \leq q \leq N$, S_q is the covariance operator of F_q. Finally, we will denote by $f_{q,i} \in \mathfrak{H}^{\otimes q}$ the kernel of $F_{q,i} = \langle F_q, k_i\rangle_K = I_q(f_{q,i})$.

Our first main result provides a quantitative bound on the distance between the law of X and a centered K-valued Gaussian random variable Z in terms of the first four moments of X.

Theorem 3.1. *Assume X is a K-valued random variable as described above such that for every $1 \leq q \leq N$, F_q has finite fourth moment, i.e., $\mathbb{E}\left[\|F_q\|_K^4\right] < \infty$. Then, letting Z be a centered Gaussian random variable on K with covariance operator S', the following estimate holds*

$$\begin{aligned} d_3(X,Z) &\leq \frac{1}{2}\left\|S - S'\right\|_{\mathrm{HS}} \\ &\quad + \left(\frac{2N-1}{4} + \frac{N}{2}\sqrt{(4N-3)\max_{1\leq p\leq N}\mathbb{E}\left[\|F_p\|_K^2\right]}\right)\sum_{1\leq q\leq N}\sqrt{\|F_q\|_K^4 - \mathbb{E}\left[\|F_q\|_K^2\right]^2 - 2\left\|S_q\right\|_{\mathrm{HS}}^2} \\ &\quad + \frac{2N-1}{4}\sum_{1\leq p\neq q\leq N}\sqrt{\mathbb{E}\left[\|F_p\|_K^2\|F_q\|_K^2\right] - \mathbb{E}\left[\|F_p\|_K^2\right]\mathbb{E}\left[\|F_q\|_K^2\right]}. \end{aligned}$$

Remark 3.2. Theorem 3.1 is an infinite-dimensional version of the fourth moment theorems on the Poisson space obtained in Döbler et al. (2018, Theorem 1.2, Theorem 1.7) and Döbler and Peccati (2018, Theorem 1.3). In particular, the aforementioned results are special cases of Theorem 3.1 obtained by setting $K = \mathbb{R}^d$ for a positive integer d.

Remark 3.3. Theorem 3.1 can be viewed as a Poissonian counterpart of Bourguin and Campese (2020, Theorem 3.10) in the context of a non-diffusive chaos structure. The fact that we are working with a non-diffusive structure (where no chain rule is available for the Gamma calculus introduced in Section 2) forces us to use different techniques in order to obtain the above quantitative bounds than the ones used in Bourguin and Campese (2020), making these results comparable in nature, but very different in their methodologies of proof.

Remark 3.4. We explain here why we assume in Theorem 3.1 (and consequently in all other theoretical results of this paper, which follow from this theorem) that for every $1 \leq q \leq N$, $\mathbb{E}\left[\|F_q\|_K^4\right] < \infty$. In the proof of Theorem 3.1 in Section 4, we will apply at several locations an important estimate that is Lemma 5.5. This lemma is originally established in Döbler and Peccati (2018); Döbler et al. (2018) and is obtained via the product formula (2.4), which requires that $F_{q,i} = \langle F_q, k_i \rangle_K$ has finite fourth moment. This requirement is satisfied in the event $\mathbb{E}\left[\|F_q\|_K^4\right] < \infty$, since

$$\mathbb{E}\left[\|F_q\|_K^4\right] = \mathbb{E}\left[\left\|\sum_{i\in\mathbb{N}} F_{q,i}k_i\right\|_K^4\right] = \sum_{i,j\in\mathbb{N}} \mathbb{E}\left[F_{q,i}^2 F_{q,j}^2\right] \geq \sum_{i\in\mathbb{N}} \mathbb{E}\left[F_{q,i}^4\right].$$

Remark 3.5. Let us mention some helpful criteria for the purpose of verifying finite fourth moment of $F = I_q(f)$ for some $f \in L_s^2(\mu^q) \otimes K$. Based on Döbler and Peccati (2018, Remark 1.2b), a sufficient condition is that $\|f\|_K$ is bounded and the support of $\|f\|_K$ is contained in a rectangle of the type $C \times C \ldots \times C$ such that $\mu(C) < \infty$, where we recall μ is the control of our Poisson random measure. This detail can be easily verified via the product formula (2.4). Poisson multiple integrals with this type of kernels include most U-statistics that are relevant to geometric applications (see Lachièze-Rey and Peccati (2013a,b); Reitzner and Schulte (2013) and the references therein).

Another way to verify that F has finite fourth moment is via restricted hypercontractivity on the Poisson space recently discovered in Nourdin et al. (2020). Specifically, under technical assumptions in their Theorem 1.4, one can bound the fourth moment of F by its variance.

Whenever X belongs to a single chaos, we can reformulate Theorem 3.1 in a more compact form:

Corollary 3.6 (Quantitative Fourth Moment Theorem)**.** *Let the notation and setup of Theorem 3.1 prevail. When X belongs to a single chaos, i.e., $X \in \mathcal{H}_q(K)$ for some $q \geq 1$, one has*

$$\begin{aligned} d_3(X, Z) \leq& \frac{1}{2}\left\|S - S'\right\|_{\mathrm{HS}} \\ &+ \left(\frac{2q-1}{4} + \frac{q}{2}\sqrt{(4q-3)\mathbb{E}\left[\|X\|_K^2\right]}\right)\sqrt{\|X\|_K^4 - \mathbb{E}\left[\|X\|_K^2\right]^2 - 2\|S\|_{\mathrm{HS}}^2}. \end{aligned}$$

As d_3 metrizes convergence in law, the above corollary in particular shows that within a single non-diffusive chaos, convergence of the second and fourth strong moments implies convergence towards a (Hilbert-valued) Gaussian.

A particularly useful formulation of the above moment bounds for applications uses contraction operators acting on the kernels of the multiple integrals appearing in the chaos decomposition representation of X given in (3.1). Contractions, which are the analytic quantities defined in Section 2, allow for much simpler computation compared to dealing directly with the first four moments. Some examples of previous works that use contraction norms to obtain quantitative limit theorem for Poisson random variables include Lachièze-Rey and Peccati (2013a,b); Reitzner and Schulte (2013).

Our second main result is the following contraction bound.

Theorem 3.7. *Let the notation and setup of Theorem 3.1 prevail. Moreover, let $\mathfrak{H} = L^2(\mathcal{Z}, \mu)$ where $\mathcal{Z}$ is the σ-finite measure space described in Subsection 2.4. Then it holds that*

$$\begin{aligned} d_3(X, Z) \leq& \frac{1}{2}\left\|S - S'\right\|_{\mathrm{HS}} \\ &+ \left(\frac{2N-1}{4} + \frac{N}{2}\sqrt{(4N-3)\max_{1\leq p\leq N}\mathbb{E}\left[\|F_p\|_K^2\right]}\right)\beta_1 + \frac{2N-1}{4}\beta_2. \end{aligned}$$

where

$$\beta_1 = \sum_{1\le q\le N}\left(\sum_{r=1}^{q-1} b_{q,q}(r)\left\|f_q \star_r^r f_q\right\|^2_{\mathfrak{H}^{\otimes(2q-2r)}\otimes K^{\otimes 2}} + \sum_{(r,s,l,m)\in I} c_{q,q,l,m}(r,s)\left\|f_q \widetilde{\star}_r^l f_q\right\|_{\mathfrak{H}^{\otimes(2q-r-l)}\otimes K^{\otimes 2}}\left\|f_q \widetilde{\star}_s^m f_q\right\|_{\mathfrak{H}^{\otimes(2q-m-s)}\otimes K^{\otimes 2}}\right)^{1/2}$$

and

$$\beta_2 = \sum_{1\le p\ne q\le N}\left(a_{p,q}(p\wedge q)\left\|f_q \star_{q\wedge p}^{q\wedge p} f_p\right\|^2_{\mathfrak{H}^{\otimes|q-p|}\otimes K^{\otimes 2}} + \sum_{r=1}^{q\wedge p-1} b_{p,q}(r)\left\|f_q \star_r^r f_p\right\|^2_{\mathfrak{H}^{\otimes(q+p-2r)}\otimes K^{\otimes 2}} + \sum_{(r,s,l,m)\in I} c_{p,q,l,m}(r,s)\left\|f_q \widetilde{\star}_r^l f_p\right\|_{\mathfrak{H}^{\otimes(q+p-r-l)}\otimes K^{\otimes 2}}\left\|f_q \widetilde{\star}_s^m f_p\right\|_{\mathfrak{H}^{\otimes(q+p-m-s)}\otimes K^{\otimes 2}}\right)^{1/2}.$$

Here, the combinatorial coefficients are given by

$$\begin{cases} a_{p,q}(r) = p!q!\binom{q}{r}\binom{p}{r} + r!^2\binom{q}{r}^2\binom{p}{r}^2|p-q|! \\ b_{p,q}(r) = p!q!\binom{q}{r}\binom{p}{r} \\ c_{p,q,l,m}(r,s) = r!s!\binom{q}{r}\binom{q}{s}\binom{p}{r}\binom{p}{s}\binom{r}{l}\binom{s}{m}(p+q-r-l)! \end{cases},$$

and the index set I is defined by

$$I = \{(r,s,l,m)\in\mathbb{N}^4 \colon 0\le r,s\le q\wedge p,\ 0\le l\le r,\ 0\le m\le s,\ r+l=s+m,\ (r,s,l,m)\notin\{(0,0,0,0),(q\wedge p, q\wedge p, q\wedge p, q\wedge p)\}\}.$$

Example 3.8. If X is a sum of elements of the first two chaoses, i.e., $X = I_1(f_1) + I_2(f_2)$, Theorem 3.7 requires the contraction norms $\left\|f_1 \star_1^1 f_2\right\|_{\mathfrak{H}\otimes K^{\otimes 2}}$, $\left\|f_2 \star_1^1 f_2\right\|_{\mathfrak{H}^{\otimes 2}\otimes K^{\otimes 2}}$, $\left\|f_1 \star_1^0 f_2\right\|_{\mathfrak{H}^{\otimes 2}\otimes K^{\otimes 2}}$, $\left\|f_1 \star_1^0 f_2\right\|_{\mathfrak{H}^{\otimes 2}\otimes K^{\otimes 2}}$, $\left\|f_2 \star_2^0 f_2\right\|_{\mathfrak{H}^{\otimes 2}\otimes K^{\otimes 2}}$, $\left\|f_2 \star_2^1 f_2\right\|_{\mathfrak{H}\otimes K^{\otimes 2}}$ and $\left\|f_1 \star_1^0 f_1\right\|_{\mathfrak{H}\otimes K^{\otimes 2}}$ to converge to 0 to get convergence towards a Gaussian law.

Example 3.9. Let μ be a σ-finite measure on some measure space. By setting $K=\mathbb{R}, \mathfrak{H} = L^2(\mu)$ and $X = I_p(f)$ for some $p \ge 2$ in Theorem 3.7, we get a result comparable to Peccati et al. (2010, Theorem 5.1) and Peccati and Taqqu (2008, Theorem 2). For instance, whenever $X = I_2(f)$, Theorem 3.7 and Peccati et al. (2010, Example 5.2) both state that normal convergence happens if $\left\|f \star_1^1 f\right\|_{L^2(\mu^2)}$, $\|f\|_{L^4(\mu^2)}$ and $\left\|f \star_2^1 f\right\|_{L^2(\mu)}$ converge to 0, keeping in mind that $\|f\|^2_{L^4(\mu^2)} = \left\|f \star_2^0 f\right\|_{L^2(\mu^2)}$, and $\left\|f \star_1^0 f\right\|_{L^2(\mu^3)} = \left\|f \star_2^1 f\right\|_{L^2(\mu)}$.

Another example is Peccati et al. (2010, Example 5.3), which states that $X = I_3(g)$ converges to a Gaussian distribution if $\|g\|^2_{L^4(\mu^3)}$, $\left\|g \star_1^1 g\right\|_{L^2(\mu^4)}$, $\left\|g \star_2^1 g\right\|_{L^2(\mu^3)}$, $\left\|g \star_3^1 g\right\|_{L^2(\mu^2)}$ and $\left\|g \star_3^2 g\right\|_{L^2(\mu)}$ all converge to 0, which is the same condition suggested in Theorem 3.7.

Further, we would like to mention Eichelsbacher and Thäle (2014); Lachièze-Rey and Peccati (2013a,b) which also offer contraction bounds for normal approximation on the Poisson space.

4. Proof of main results

We begin with the proof of Theorem 3.1 which uses the method of exchangeable pairs developed in Section 2.

4.1. *Proof of Theorem 3.1.* Let G be a Gaussian random variable on K with the same covariance operator as X, i.e., G has covariance operator S. Similarly to Bourguin and Campese (2020, Corrolary 3.3), it holds that

$$d_3(G,Z) \leq \frac{1}{2}\left\|S - S'\right\|_{\mathrm{HS}}.$$

Therefore, it suffices to derive the desired moment bound for $d_3(X,G)$ which yields the first item in Theorem 3.1 as

$$d_3(X,Z) \leq d_3(X,G) + d_3(G,Z).$$

In Subsection 2.5, we constructed an exchangeable pair of the form (F_q, F_q^t) based on an element of a fixed K-valued chaos F_q, where q denotes the order of the Poisson chaos. Recall that X has the chaos decomposition (3.1). It follows that, for any $t \geq 0$, if we define X^t as

$$X^t = \sum_{q=1}^{N} F_q^t,$$

then the pair

$$(X, X^t) = \left(\sum_{q=1}^{N} F_q, \sum_{q=1}^{N} F_q^t\right)$$

is also exchangeable. Since $\langle x-y, Dg(x)+Dg(y)\rangle_K$ is an anti-symmetric expression, the exchangeability implies

$$\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle \sum_{q=1}^{N}\frac{1}{q}\big(F_q^t - F_q\big), Dg(X^t) + Dg(X)\right\rangle_K\right] = 0.$$

Furthermore, applying Taylor's theorem yields

$$\begin{aligned}
0 &= \lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle \sum_{q=1}^{N}\frac{1}{q}\big(F_q^t - F_q\big), Dg(X^t) + Dg(X)\right\rangle_K\right] \\
&= \lim_{t\to 0}\mathbb{E}\left[\frac{1}{2t}\left\langle \sum_{q=1}^{N}\frac{1}{q}\big(F_q^t - F_q\big), Dg(X^t) - Dg(X)\right\rangle_K + \frac{1}{t}\left\langle \sum_{q=1}^{N}\frac{1}{q}\big(F_q^t - F_q\big), Dg(X)\right\rangle_K\right] \\
&= \lim_{t\to 0}\mathbb{E}\left[\frac{1}{2t}\left\langle \sum_{q=1}^{N}\frac{1}{q}\big(F_q^t - F_q\big), D^2g(X)(X^t - X) + r\right\rangle_K + \frac{1}{t}\left\langle \sum_{q=1}^{N}\frac{1}{q}\big(F_q^t - F_q\big), Dg(X)\right\rangle_K\right].
\end{aligned}$$

Here r denotes the remainder term for which $\|r\|_K \leq \frac{1}{2}\left\|D^3g(\xi)(X^t - X)^2\right\|_K$, and ξ is in the open ball centered at X with radius $\|X_t - X\|_K$.

Now set

$$R(t) = \mathbb{E}\left[\frac{1}{2t}\left\langle \sum_{q=1}^{N}\frac{1}{q}\big(F_q^t - F_q\big), r\right\rangle_K\right].$$

Note that $\mathbb{E}[\Delta_G g(X)] = \sum_{1\leq q\leq N} \mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X) S_q\big)\big]$. Combined with part (a) and (c) of Lemma 5.7 and keeping in mind $F_q = \sum_{i\in\mathbb{N}} F_{q,i} k_i$, this leads to

$$\begin{aligned}
0 &= \sum_{1\leq q\leq N} \mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X)\Gamma\big(F_q, -L^{-1}F_q\big)\big)\big] \\
&\quad + \sum_{1\leq p\neq q\leq N}\sum_{i,j\in\mathbb{N}} \mathbb{E}\Big[\Big\langle k_i, D^2 g(X)\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i}, F_{q,j}\Big)k_j\Big\rangle_K\Big] - \mathbb{E}[\langle X, Dg(X)\rangle_K] + \lim_{t\to 0} R(t) \\
&= \mathbb{E}[\Delta_G g(X)] - \mathbb{E}[\langle X, Dg(X)\rangle_K] + \sum_{1\leq q\leq N} \mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X)\big(\Gamma\big(F_q, -L^{-1}F_q\big) - S_q\big)\big)\big] \\
&\quad + \sum_{1\leq p\neq q\leq N}\sum_{i,j\in\mathbb{N}} \mathbb{E}\Big[\Big\langle k_i, D^2 g(X)\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i}, F_{q,j}\Big)k_j\Big\rangle_K\Big] + \lim_{t\to 0} R(t).
\end{aligned}$$

The above equation and the Stein equation introduced in Section 2 imply

$$\begin{aligned}
d_3(X,G) &= \sup_{h\in C_b^3(K)} \mathbb{E}[|\Delta_G g(X) - \langle X, Dg(X)\rangle_K|] \\
&\leq \sup_{h\in C_b^3(K)} \Bigg\{ \sum_{1\leq q\leq N} \big|\mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X)\big(\Gamma\big(F_q, -L^{-1}F_q\big) - S_q\big)\big)\big]\big| \\
&\qquad + \sum_{1\leq p\neq q\leq N} \left| \sum_{i,j\in\mathbb{N}} \mathbb{E}\Big[\Big\langle k_i, D^2 g(X)\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i}, F_{q,j}\Big)k_j\Big\rangle_K\Big]\right| + \left|\lim_{t\to 0} R(t)\right| \Bigg\}.
\end{aligned} \tag{4.1}$$

For the first term on the right side of (4.1), it holds that

$$\begin{aligned}
&\sum_{1\leq q\leq N} \big|\mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X)\big(\Gamma\big(F_q, -L^{-1}F_q\big) - S_q\big)\big)\big]\big| \\
&\qquad \leq \sum_{1\leq q\leq N} \big\|D^2 g(X)\big\|_{L^2(\Omega;\mathrm{HS}(K))} \left\| \frac{1}{q}\Gamma(F_q, F_q) - S_q \right\|_{L^2(\Omega;\mathrm{HS}(K))} \\
&\qquad \leq \sum_{1\leq q\leq N} \frac{1}{2q} \sqrt{\sum_{i,j\in\mathbb{N}} \mathrm{Var}(\Gamma(F_{q,i}, F_{q,j}))} \\
&\qquad \leq \sum_{1\leq q\leq N} \frac{2q-1}{4q} \sqrt{\sum_{i,j\in\mathbb{N}} \mathbb{E}\Big[F_{q,i}^2 F_{q,j}^2\Big] - \mathbb{E}\Big[F_{q,i}^2\Big]\mathbb{E}\Big[F_{q,j}^2\Big] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2} \\
&\qquad = \sum_{1\leq q\leq N} \frac{2q-1}{4q} \sqrt{\mathbb{E}\Big[\|F_q\|_K^4\Big] - \mathbb{E}\Big[\|F_q\|_K^2\Big]^2 - 2\,\|S_q\|_{\mathrm{HS}}^2}.
\end{aligned}$$

In particular, we have used the fact that $\big\|D^2 g(x)\big\|_{K^{\otimes 2}} = \big\|D^2 g(x)\big\|_{\mathrm{HS}(K)}$ and Bourguin and Campese (2020, Lemma 2.4) to get the third line above. The fourth line is a consequence of Lemma 5.5. Finally, the identity $\langle Sf, g\rangle_K = \mathbb{E}[\langle X, f\rangle_K \langle X, g\rangle_K]$ allows us to get the term $\|S_q\|_{\mathrm{HS}}$ in the last line.

Now we study the second term on the right side of (4.1). Application of Bourguin and Campese (2020, Lemma 2.4) and Lemma 5.5 gives

$$\begin{aligned}
&\sum_{1\le p\ne q\le N}\left|\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[\Big\langle k_i, D^2g(X)\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i},F_{q,j}\Big)k_j\Big\rangle_K\Big]\right|\\
&\qquad\le \sum_{1\le p\ne q\le N}\mathbb{E}\left[\sqrt{\sum_{i,j\in\mathbb{N}}\langle k_i, D^2g(X)k_j\rangle_K}\sqrt{\sum_{i,j\in\mathbb{N}}\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i},F_{q,j}\Big)^2}\right]\\
&\qquad\le \sum_{1\le p\ne q\le N}\sqrt{\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[\langle k_i, D^2g(X)k_j\rangle_K^2\Big]}\sqrt{\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i},F_{q,j}\Big)^2\Big]}\\
&\qquad\le \sum_{1\le p\ne q\le N}\frac{p+q-1}{2p}\left\|D^2g(X)\right\|_{L^2(\Omega;\mathrm{HS}(K))}\sqrt{\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[F_{p,i}^2F_{q,j}^2\Big]-\mathbb{E}\Big[F_{p,i}^2\Big]\mathbb{E}\Big[F_{q,j}^2\Big]}\\
&\qquad\le \sum_{1\le p\ne q\le N}\frac{p+q-1}{4p}\sqrt{\mathbb{E}\Big[\|F_p\|_K^2\,\|F_q\|_K^2\Big]-\mathbb{E}\Big[\|F_p\|_K^2\Big]\mathbb{E}\Big[\|F_q\|_K^2\Big]}.
\end{aligned}$$

As the last step, we apply Lemma 5.8 to the remainder term in (4.1).

$$\begin{aligned}
\left|\lim_{t\to 0}R(t)\right| &\le \lim_{t\to 0}\frac{1}{4t}\left\|D^3g\right\|_{\mathrm{op}}\mathbb{E}\left[\left\|\sum_{q=1}^{N}\frac{1}{q}\big(F_q^t-F_q\big)\right\|_K\left\|X^t-X\right\|_K^2\right]\\
&\le \frac{N}{2}\sqrt{\max_{1\le p\le N}\mathbb{E}\Big[\|F_p\|_K^2\Big]}\sum_{1\le q\le N}\sqrt{4q-3}\sqrt{\|F_q\|_K^4-\mathbb{E}\Big[\|F_q\|_K^2\Big]^2-2\left\|S_q\right\|_{\mathrm{HS}}^2}.
\end{aligned}$$

We can hence deduce from (4.1) the inequality

$$\begin{aligned}
d_3(X,G)\le &\sum_{1\le q\le N}\frac{2q-1}{4q}\sqrt{\mathbb{E}\Big[\|F_q\|_K^4\Big]-\mathbb{E}\Big[\|F_q\|_K^2\Big]^2-2\left\|S_q\right\|_{\mathrm{HS}}^2}\\
&+\sum_{1\le p\ne q\le N}\frac{p+q-1}{4p}\sqrt{\mathbb{E}\Big[\|F_p\|_K^2\,\|F_q\|_K^2\Big]-\mathbb{E}\Big[\|F_p\|_K^2\Big]\mathbb{E}\Big[\|F_q\|_K^2\Big]}\\
&+\frac{N}{2}\sqrt{\max_{1\le p\le N}\mathbb{E}\Big[\|F_p\|_K^2\Big]}\sum_{1\le q\le N}\sqrt{4q-3}\sqrt{\|F_q\|_K^4-\mathbb{E}\Big[\|F_q\|_K^2\Big]^2-2\left\|S_q\right\|_{\mathrm{HS}}^2}. \qquad (4.2)
\end{aligned}$$

This combined with

$$\begin{aligned}
\frac{2q-1}{4q}\vee\frac{p+q-1}{4p}&\le\frac{2N-1}{4} &&\text{for } 1\le p,q\le N,\\
4q-3&\le 4N-3 &&\text{for } 1\le q\le N,
\end{aligned}$$

yields

$$\begin{aligned}
&d_3(X,G)\\
&\le\left(\frac{2N-1}{4}+\frac{N}{2}\sqrt{(4N-3)\max_{1\le p\le N}\mathbb{E}\Big[\|F_p\|_K^2\Big]}\right)\sum_{1\le q\le N}\sqrt{\|F_q\|_K^4-\mathbb{E}\Big[\|F_q\|_K^2\Big]^2-2\left\|S_q\right\|_{\mathrm{HS}}^2}\\
&\qquad+\frac{2N-1}{4}\sum_{1\le p\ne q\le N}\sqrt{\mathbb{E}\Big[\|F_p\|_K^2\,\|F_q\|_K^2\Big]-\mathbb{E}\Big[\|F_p\|_K^2\Big]\mathbb{E}\Big[\|F_q\|_K^2\Big]}.
\end{aligned}$$

□

We now turn to the proof of Theorem 3.7, which makes use of the second estimate in Theorem 3.1.

4.2. *Proof of Theorem 3.7.* The strategy here consists of making use of the product formula (2.4) for Poisson multiple integrals in order to represent the quantities

$$\mathbb{E}\Big[\|F_q\|_K^4\Big] - \mathbb{E}\Big[\|F_q\|_K^2\Big]^2 - 2\,\|S_q\|_{\mathrm{HS}}^2$$

and

$$\mathbb{E}\Big[\|F_p\|_K^2\,\|F_q\|_K^2\Big] - \mathbb{E}\Big[\|F_p\|_K^2\Big]\mathbb{E}\Big[\|F_q\|_K^2\Big], \quad p \neq q$$

which appear in the second estimate of Theorem 3.1 in term of contraction norms. We begin by noting that

$$\mathbb{E}\Big[\|F_q\|_K^4\Big] - \mathbb{E}\Big[\|F_q\|_K^2\Big]^2 - 2\,\|S_q\|_{\mathrm{HS}}^2 = \sum_{i,j\in\mathbb{N}} \Big(\mathbb{E}\big[F_{q,i}^2 F_{q,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{q,j}^2\big] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2\Big)$$

and for $p \neq q$,

$$\mathbb{E}\Big[\|F_p\|_K^2\,\|F_q\|_K^2\Big] - \mathbb{E}\Big[\|F_p\|_K^2\Big]\mathbb{E}\Big[\|F_q\|_K^2\Big] = \sum_{i,j\in\mathbb{N}} \left(\mathbb{E}\big[F_{q,i}^2 F_{p,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{p,j}^2\big]\right).$$

An application of the product formula (2.4) for Poisson multiple integrals yields

$$F_{q,i}F_{p,j} = \sum_{r=0}^{q\wedge p} r!\binom{q}{r}\binom{p}{r}\sum_{l=0}^{r}\binom{r}{l} I_{q+p-r-l}\Big(f_{q,i}\widetilde{\star}_r^l f_{p,j}\Big).$$

Now by the orthogonality of Poisson chaos of different orders, one has

$$\mathbb{E}\big[F_{q,i}^2 F_{p,j}^2\big] = \sum_{r,s=0}^{q\wedge p} \sum_{\substack{0\le l\le r\\ 0\le m\le s\\ r+l=s+m}} c_{p,q,l,m}(r,s)\left\langle f_{q,i}\widetilde{\star}_r^l f_{p,j}, f_{q,i}\widetilde{\star}_s^m f_{p,j}\right\rangle_{\mathfrak{H}^{\otimes(q+p-r-l)}}, \tag{4.3}$$

where the coefficient $c_{p,q,l,m}(r,s)$ is given by

$$c_{p,q,l,m}(r,s) = r!s!\binom{q}{r}\binom{q}{s}\binom{p}{r}\binom{p}{s}\binom{r}{l}\binom{s}{m}(p+q-r-l)!.$$

Let us define the index set I as

$$\begin{aligned} I = \Big\{&(r,s,l,m)\in\mathbb{N}^4 \colon 0\le r,s\le q\wedge p,\ 0\le l\le r,\ 0\le m\le s,\\ &r+l=s+m,\ (r,s,l,m)\notin\{(0,0,0,0),(q\wedge p,q\wedge p,q\wedge p,q\wedge p)\}\Big\}. \end{aligned}$$

Then, using Lemma 5.9, equation (4.3) can be rewritten as

$$\begin{aligned} \mathbb{E}\big[F_{q,i}^2 F_{p,j}^2\big] =& q!p!\,\|f_{q,i}\|_{\mathfrak{H}^{\otimes q}}^2\,\|f_{p,j}\|_{\mathfrak{H}^{\otimes q}}^2 + 2q!^2\,\langle f_{q,i}, f_{q,j}\rangle_{\mathfrak{H}^{\otimes q}}^2\,\mathbb{1}_{\{q=p\}}\\ &+ a_{p,q}(p\wedge q)\left\|f_{q,i}\star_{q\wedge p}^{q\wedge p} f_{p,j}\right\|_{\mathfrak{H}^{\otimes|q-p|}}^2 \mathbb{1}_{\{q\neq p\}} + \sum_{r=1}^{q\wedge p-1} b_{p,q}(r)\,\|f_{q,i}\star_r^r f_{p,j}\|_{\mathfrak{H}^{\otimes(q+p-2r)}}^2\\ &+ \sum_{(r,s,l,m)\in I} c_{p,q,l,m}(r,s)\left\langle f_{q,i}\widetilde{\star}_r^l f_{p,j}, f_{q,i}\widetilde{\star}_s^m f_{p,j}\right\rangle_{\mathfrak{H}^{\otimes(q+p-r-l)}}, \end{aligned}$$

where the combinatorial coefficients $a_{p,q}(r)$ and $b_{p,q}(r)$ are given by

$$\begin{cases} a_{p,q}(r) = p!q!\binom{q}{r}\binom{p}{r} + r!^2\binom{q}{r}^2\binom{p}{r}^2 |p-q|! \\ b_{p,q}(r) = p!q!\binom{q}{r}\binom{p}{r} \end{cases} .$$

It follows that

$$\sum_{i,j\in\mathbb{N}} \Big(\mathbb{E}\big[F_{q,i}^2 F_{q,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{q,j}^2\big] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2\Big) = \sum_{i,j\in\mathbb{N}}\sum_{r=1}^{q-1} b_{q,q}(r)\,\|f_{q,i}\star_r^r f_{q,j}\|^2_{\mathfrak{H}^{\otimes(2q-2r)}}$$
$$+ \sum_{\substack{i,j\in\mathbb{N}\\(r,s,l,m)\in I}} c_{q,q,l,m}(r,s)\left\langle f_{q,i}\widetilde{\star}_r^l f_{q,j}, f_{q,i}\widetilde{\star}_s^m f_{q,j}\right\rangle_{\mathfrak{H}^{\otimes(2q-r-l)}};$$

and for $p \neq q$

$$\sum_{i,j\in\mathbb{N}} \big(\mathbb{E}\big[F_{q,i}^2 F_{p,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{p,j}^2\big]\big) = \sum_{i,j\in\mathbb{N}} a_{p,q}(p\wedge q)\,\big\|f_{q,i}\star_{q\wedge p}^{q\wedge p} f_{p,j}\big\|^2_{\mathfrak{H}^{\otimes|q-p|}}$$
$$+ \sum_{i,j\in\mathbb{N}}\sum_{r=1}^{q\wedge p-1} b_{p,q}(r)\,\|f_{q,i}\star_r^r f_{p,j}\|^2_{\mathfrak{H}^{\otimes(q+p-2r)}}$$
$$+ \sum_{\substack{i,j\in\mathbb{N}\\(r,s,l,m)\in I}} c_{p,q,l,m}(r,s)\left\langle f_{q,i}\widetilde{\star}_r^l f_{p,j}, f_{q,i}\widetilde{\star}_s^m f_{p,j}\right\rangle_{\mathfrak{H}^{\otimes(q+p-r-l)}}.$$

Finally observe that

$$\left\|f_q\star_r^l f_p\right\|^2_{\mathfrak{H}^{\otimes(q+p-r-l)}\otimes K^{\otimes 2}} = \sum_{i,j\in\mathbb{N}} \left\|\langle f_q, k_i\rangle_K \star_r^l \langle f_p, k_j\rangle_K\right\|^2_{\mathfrak{H}^{\otimes(q+p-r-l)}} = \sum_{i,j\in\mathbb{N}} \left\|f_{q,i}\star_r^l f_{p,j}\right\|^2_{\mathfrak{H}^{\otimes(q+p-r-l)}},$$

Therefore, we can sum over $i, j\in\mathbb{N}$ and apply Cauchy Schwarz's inequality to get

$$\mathbb{E}\Big[\|F_q\|_K^4\Big] - \mathbb{E}\Big[\|F_q\|_K^2\Big]^2 - 2\,\|S_q\|^2_{\mathrm{HS}} \le \sum_{r=1}^{q-1} b_{q,q}(r)\,\|f_q\star_r^r f_q\|^2_{\mathfrak{H}^{\otimes(2q-2r)}\otimes K^{\otimes 2}}$$
$$+ \sum_{(r,s,l,m)\in I} c_{q,q,l,m}(r,s)\left\|f_q\widetilde{\star}_r^l f_q\right\|_{\mathfrak{H}^{\otimes(2q-r-l)}\otimes K^{\otimes 2}} \|f_q\widetilde{\star}_s^m f_q\|_{\mathfrak{H}^{\otimes(2q-m-s)}\otimes K^{\otimes 2}};$$

and

$$\mathbb{E}\Big[\|F_p\|_K^2\,\|F_q\|_K^2\Big] - \mathbb{E}\Big[\|F_p\|_K^2\Big]\mathbb{E}\Big[\|F_q\|_K^2\Big]$$
$$\le a_{p,q}(p\wedge q)\,\big\|f_q\star_{q\wedge p}^{q\wedge p} f_p\big\|^2_{\mathfrak{H}^{\otimes|q-p|}\otimes K^{\otimes 2}} + \sum_{r=1}^{q\wedge p-1} b_{p,q}(r)\,\|f_q\star_r^r f_p\|^2_{\mathfrak{H}^{\otimes(q+p-2r)}\otimes K^{\otimes 2}}$$
$$+ \sum_{(r,s,l,m)\in I} c_{p,q,l,m}(r,s)\left\|f_q\widetilde{\star}_r^l f_p\right\|_{\mathfrak{H}^{\otimes(q+p-r-l)}\otimes K^{\otimes 2}} \|f_q\widetilde{\star}_s^m f_p\|_{\mathfrak{H}^{\otimes(q+p-m-s)}\otimes K^{\otimes 2}}.$$

This combined with the moment estimate in Theorem 3.1 concludes our proof of contraction bound. □

5. Applications

5.1. *Brownian approximation of a Poisson process in Besov-Liouville spaces.*

5.1.1. *A brief overview of Besov-Liouville spaces.* For an extensive account on the current topic, we invite readers to view Samko et al. (1993). For $f \in L^p([0,1], ds)$ and $\beta > 0$, we define the left and right fractional integrals respectively as

$$\left(I^{\beta}_{0+} f\right)(s) = \frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} f(r) dr$$

and

$$\left(I^{\beta}_{1-} f\right)(s) = \frac{1}{\Gamma(\beta)} \int_s^1 (r-s)^{\beta-1} f(r) dr.$$

This allows us to define the Besov-Liouville spaces

$$\mathcal{I}^{+}_{\beta,p} = \left\{ I^{\beta}_{0+} \widehat{f}, \ \widehat{f} \in L^p([0,1]) \right\},$$

which are Banach spaces when equipped with the norm $\|f\|_{\mathcal{I}^{+}_{\beta,p}} = \left\| \widehat{f} \right\|_{L^p([0,1])}$. The Besov-Liouville spaces $\mathcal{I}^{-}_{\beta,p}$ are defined accordingly with the right fractional integrals. When $\beta p < 1$, the spaces $\mathcal{I}^{+}_{\beta,p}$ and $\mathcal{I}^{-}_{\beta,p}$ are canonically isomorphic and therefore will both be denoted by $\mathcal{I}_{\beta,p}$.

Remark 5.1. As pointed out in Coutin and Decreusefond (2013), $\mathcal{I}_{\beta,2}$ for $\beta < 1/2$ is an appropriate class of Besov-Liouville spaces for the functional approximation of a Poisson process by a Brownian motion since they are Hilbert spaces containing both the sample paths of the Poisson process and the Brownian motion.

Similarly to the left and right fractional integrals, one can define left and right fractional derivatives as

$$\left(D^{\beta}_{0+} f\right)(s) = \frac{1}{\Gamma(1-\beta)} \frac{d}{ds} \int_0^s (s-r)^{-\beta} f(r) dr$$
$$\left(D^{\beta}_{1-} f\right)(s) = \frac{1}{\Gamma(1-\beta)} \frac{d}{ds} \int_s^1 (r-s)^{-\beta} f(r) dr$$

As the name suggests, D^{β}_{0+} is the inverse of I^{β}_{0+} (see Samko et al. (1993, Theorem 2.4)). Two examples for the action of this operator that will be useful later are

$$\left(D^{\beta}_{0+} \operatorname{Id}\right)(r) = \frac{r^{-\beta+1}}{(-\beta+1)\Gamma(-\beta+1)} \quad \text{and} \quad \left(D^{\beta}_{0+} 1_{[a,\infty)}\right)(r) = \frac{(r-a)_+^{-\beta}}{\Gamma(-\beta+1)}, \tag{5.1}$$

where Id denotes the identity function. Let us also mention a few important facts about fractional integrals and derivatives. Given $0 < \beta < 1$ and $1 < p < 1/\beta$, I^{β}_{0+} is a bounded operator from $L^p([0,1])$ to $L^q([0,1])$ with $q = p(1-\beta p)^{-1}$. Moreover, for $\beta > 0$ and $p \geq 1$, I^{β}_{0+} is bounded from $L^p([0,1])$ into itself (see for instance Samko et al. (1993, Equation (2.72))). Next, fractional derivatives are the inverses of fractional integrals, in the sense that

$$\left(D^{\beta}_{0+} I^{\beta}_{0+} f\right)(s) = f(s)$$

for $f \in L^1([0,1])$. Furthermore, fractional integrals enjoy the semigroup property (see Samko et al. (1993, Theorem 2.5)), that is

$$\left(I^{\alpha}_{0+} I^{\beta}_{0+} f\right)(s) = \left(I^{\alpha+\beta}_{0+} f\right)(s)$$

as long as $\beta > 0$, $\alpha + \beta > 0$ and $f \in L^1([0,1])$.

5.1.2. *A functional central limit theorem.* We consider a Poisson process $N_\lambda(t)$ with intensity λ. It is well known (see for instance Nualart and Nualart (2018, Example 9.1.3)) that it can be represented as

$$N_\lambda(t) = \sum_{n\in\mathbb{N}} 1_{[T_n,\infty)}(t), \tag{5.2}$$

where $T_n = \sum_{i=1}^n \alpha_i$ and $\{\alpha_i : i \in \mathbb{N}\}$ are independent exponentially distributed random variables with parameter λ, i.e., $\alpha_i \sim \mathrm{Exp}(\lambda)$ for all $i \in \mathbb{N}$. This implies that T_n is Gamma distributed with shape n and rate λ, i.e., $T_n \sim \mathrm{Gamma}(n,\lambda)$. As pointed out in Coutin and Decreusefond (2013), $N_\lambda(t)$ maps into $\mathcal{I}_{\beta,2}$ for $\beta < 1/2$.

For any $t \in [0,1]$, define

$$X_\lambda(t) = \frac{N_\lambda(t) - \lambda t}{\sqrt{\lambda}}$$

and let Z be a Brownian motion on $\mathcal{I}_{\beta,2}$, that is a $\mathcal{I}_{\beta,2}$-valued Gaussian random variable with covariance operator

$$S' = I^\beta_{0^+} I^{1-\beta}_{0^+} I^{1-\beta}_{1^-} D^\beta_{0^+}, \tag{5.3}$$

where the expression of the covariance operator was derived in Coutin and Decreusefond (2013). We are now ready to state the main result of this application, namely the Brownian approximation of a Poisson process in $\mathcal{I}_{\beta,2}$.

Theorem 5.2. *On a Besov-Liouville space $\mathcal{I}_{\beta,2}$ with $\beta < 1/2$, the distributions of X_λ and Z are asymptotically close as $\lambda \to \infty$. Their closeness can be quantified by*

$$d_3(X_\lambda, Z) \lesssim \frac{1}{\sqrt{\lambda}}.$$

Proof: $X_\lambda(t)$ can be represented as a Poisson multiple integral of order one. Let $\mathfrak{H} = L^2(\mathbb{R}^+, \lambda dx)$ be the underlying Hilbert space to the compensated Poisson process $N_\lambda(t) - \lambda t$. Furthermore, let $f(t) = \frac{1}{\sqrt{\lambda}} 1_{[0,t]} \in \mathfrak{H}$. We can hence write

$$X_\lambda(t) = I_1(f(t)).$$

Theorem 3.7 then provides us with the estimate

$$d_3(X_\lambda, Z) \lesssim \left\| f \star_1^0 f \right\|^2_{\mathfrak{H}\otimes K^{\otimes 2}} + \left\| S_\lambda - S' \right\|_{\mathrm{HS}(K)}, \tag{5.4}$$

where S_λ denotes the covariance operator of X_λ and where $K = \mathcal{I}_{\beta,2}$. We begin by computing the contraction norm appearing above. We have

$$(f \star_1^0 f)(x) = \frac{1}{\lambda} 1_{[0,t]}(x) 1_{[0,s]}(x) = 1_{[x,\infty)}(t) 1_{[x,\infty)}(s),$$

so that

$$\begin{aligned} \left\| f \star_1^0 f \right\|^2_{\mathfrak{H}\otimes K^{\otimes 2}} &= \frac{1}{\lambda^2} \int_0^1 \int_0^1 \int_0^1 \left(\left(D^\beta_{0^+} 1_{[x,\infty)} \right)(t) \left(D^\beta_{0^+} 1_{[x,\infty)} \right)(s) \right)^2 \lambda dx ds dt \\ &= \frac{1}{\lambda \Gamma(-\beta+1)^4} \int_0^1 \int_0^1 (t-x)_+^{-2\beta} (s-x)_+^{-2\beta} ds dt \lesssim \frac{1}{\lambda}, \end{aligned}$$

where the last inequality simply comes from the fact that $\int_0^1 \int_0^1 (t-x)_+^{-2\beta} (s-x)_+^{-2\beta} ds dt$ is finite.

Regarding the remaining term, namely $\|S_\lambda - S'\|_{\mathrm{HS}(K)}$, we apply Lemma 5.10 and Lemma 5.12. This yields

$$\begin{aligned}&\left\|S_\lambda - S'\right\|^2_{\mathrm{HS}(K)}\\ &= \left\|\mathbb{E}\left[\left(D^\beta_{0+}X_\lambda\right)(r)\left(D^\beta_{0+}X_\lambda\right)(s)\right] - \mathbb{E}\left[\left(D^\beta_{0+}Z\right)(r)\left(D^\beta_{0+}Z\right)(s)\right]\right\|^2_{L^2([0,1]^{\otimes 2})} = 0,\end{aligned}$$

which concludes the proof. □

5.2. *Edge counting in random graphs.* In Lachièze-Rey and Peccati (2013a), the authors studied Gaussian fluctuations of real-valued U-statistics related to graphs generated by Poisson point processes. We will apply Theorem 3.7 to obtain a functional version of their results in all three regimes mentioned in Lachièze-Rey and Peccati (2013a, Example 4.13). Recall from Subsection 2.5 the definition of a proper Poisson point process

$$\eta_\lambda = \sum_{i=1}^{\mathrm{Po}(\lambda)} \delta_{Y_i},$$

where $\mathrm{Po}(\lambda)$ is a Poisson distribution on $\mathbb{R}$, while $\{Y_i\}_{i\in\mathbb{N}}$ is an i.i.d. sequence of $\mathbb{R}^d$-valued random variables distributed as ℓ and independent from $\mathrm{Po}(\lambda)$. Let W be a symmetric and bounded set in $\mathbb{R}^d$. For simplicity and illustration purposes, let us assume ℓ is the uniform measure on W. The control measure of η_λ is therefore

$$\mu_\lambda(\cdot) = \lambda\ell(\cdot).$$

Let G be a graph generated by η_λ, so that G has the vertex set $\{Y_1,\ldots,Y_{\mathrm{Po}(\lambda)}\}$. The set W will serve as our original window in which we monitor the edges of G, and let $H_\lambda \subseteq \mathbb{R}^{2d}$ be a symmetric set which will serve as our original edge set. For $0\le t\le 1$, define

$$\begin{cases} W_t = t^{\frac{1}{2d}}W\\ H_{\lambda,t} = t^{\frac{1}{2d}}H_\lambda\\ \widehat{W}_t = \{x-y : x,y\in W_t\}\\ \overline{H}_{\lambda,t} = \{x-y : x,y\in H_{\lambda,t}\}\end{cases}.$$

We will assume that any edge, written in pairs (x,y), belongs to $H_{\lambda,t}$ if and only if $x-y\in\overline{H}_{\lambda,t}$. For example, this property holds for a disk graph with base edge set $\overline{H}_\lambda = B(0,r_\lambda)$, an open ball of radius r_λ at the origin. We note that compared to the setup in Lachièze-Rey and Peccati (2013a), our window and edge set are not static but evolve with time.

We are interested in a Poissonized U-statistics of the form

$$F_\lambda(t) = \sum_{\substack{(x,y)\in\eta^2_\lambda\\ x\neq y}} 1_{H_{\lambda,t}\cap W_t^2}(x,y) = \sum_{1=i_1<i_2}^{\mathrm{Po}(\lambda)} 1_{H_{\lambda,t}\cap W_t^2}(Y_{i_1},Y_{i_2})$$

which counts edges of G that belong to the set $\overline{H}_{\lambda,t}$ and lie inside the window W_t at time t. It is clear from the hypothesis that $\{F_\lambda(t)\}_{t\in[0,1]}$ as a process belongs to $K = L^2([0,1])$. As proved in Reitzner and Schulte (2013), our U-statistic has a finite chaos expansion given by

$$F_\lambda(t) = \mathbb{E}[F_\lambda(t)] + I_1(f_1(t)) + I_2(f_2(t)),$$

where the (functional) kernels $f_1(t)$ and $f_2(t)$ are given by

$$\begin{cases} f_1(t) = 2\displaystyle\int_{\mathbb{R}^d} 1_{H_{\lambda,t}\cap W_t^2}(x,y)\lambda dy\\ f_2(t) = 1_{H_{\lambda,t}\cap W_t^2}(x,y)\end{cases}.$$

Let $\bar{F}_\lambda(t)$ denote the centered and normalized version of $F_\lambda(t)$ given by

$$\bar{F}_\lambda(t) = \frac{F_\lambda(t) - \mathbb{E}[F_\lambda(t)]}{\sigma} = I_1\left(g_1(t)\right) + I_2\left(g_2(t)\right),$$

where $\sigma^2 = \mathrm{Var}(F_\lambda(1))$, $g_1(t) = \frac{f_1(t)}{\sigma}$ and $g_2(t) = \frac{f_2(t)}{\sigma}$. For convenience, we will also write ℓ_t for $\ell(W_t)$ and $\psi_{\lambda,t}$ for $\ell\Big(\overline{H}_{\lambda,t} \cap \widehat{W}_t\Big)$. Using the scaling properties of the Lebesgue measure, we can write

$$\ell_t = \sqrt{t}\ell_1 \quad \text{and} \quad \psi_{\lambda,t} = \sqrt{t}\psi_{\lambda,1}.$$

We can actually compute σ^2 explicitly, using the orthogonality of Wiener chaos of different orders and the isometry property of Poisson multiple integrals. This yields

$$\begin{aligned}\sigma^2 =& \|f_1(1)\|^2_{L^2(\mu_\lambda)} + \|f_2(1)\|^2_{L^2\left(\mu^2_\lambda\right)} \\ =& 4\lambda^3 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} 1_{W_1}(x) 1_{\overline{H}_{\lambda,1}\cap\widehat{W}_1}(y-x)d(y-x) \right)^2 dx + \int_{\mathbb{R}^{2d}} 1_{H_{\lambda,1}\cap W_1^2}(x,y)\lambda^2 dxdy \\ =& 4\ell_1\lambda^3\psi^2_{\lambda,1} + \ell_1\lambda^2\psi_{\lambda,1}.\end{aligned}$$

Based on the above expression for σ^2, we can consider three different regimes (similarly to what was done in Lachièze-Rey and Peccati (2013a)), namely

- Regime 1: $\lambda\psi_{\lambda,1} \to \infty$ as $\lambda \to \infty$;
- Regime 2: $\lambda\psi_{\lambda,1} \to 1$ for $c > 0$ as $\lambda \to \infty$;
- Regime 3: $\lambda\psi_{\lambda,1} \to 0$ and $\lambda\sqrt{\psi_{\lambda,1}} \to \infty$ as $\lambda \to \infty$.

Within Regime 1, σ^2 is dominated by $\|f_1(1)\|^2_{L^2(\mu_\lambda)}$ for large values of λ, which implies

$$\sigma^2 \asymp 4\ell_1\lambda^3\psi^2_{\lambda,1},$$

whereas in Regime 2, we get

$$\sigma^2 \asymp 4\ell_1\lambda^3\psi^2_{\lambda,1} \asymp \ell_1\lambda^2\psi_{\lambda,1},$$

and finally in Regime 3, it holds that

$$\sigma^2 \asymp \ell_1\lambda^2\psi_{\lambda,1}.$$

We are now ready to present the application of our results to edge counting in random graphs.

Theorem 5.3. *As $\lambda \to \infty$, $\bar{F}_\lambda(t)$ converges in $K = L^2([0,1])$ to a K-valued Gaussian random variable Z with covariance function $\phi(s,t) = \mathbb{E}[Z(s)Z(t)]$. More specifically,*

- *In Regime 1, $\phi(t,s) = \sqrt{ts(t \wedge s)}$ and*

$$d_3\big(\bar{F}_\lambda, Z\big) \lesssim \lambda^{-\frac{1}{2}} + \frac{1}{\lambda\psi_{\lambda,1}};$$

- *In Regime 2, $\phi(t,s) = \frac{4\sqrt{ts(t\wedge s)}+t\wedge s}{5}$ and*

$$d_3\big(\bar{F}_\lambda, Z\big) \lesssim \lambda^{-\frac{1}{2}} + |\lambda\psi_{\lambda,1} - 1|;$$

- *In Regime 3, $\phi(t,s) = t \wedge s$ which implies that Z is a Brownian motion, and*

$$d_3\big(\bar{F}_\lambda, Z\big) \lesssim \lambda^{-1}\psi^{-1/2}_{\lambda,1} + \lambda\psi_{\lambda,1}.$$

Proof: In order to make use of Theorem 3.7, we will need to evaluate contraction norms, but also the Hilbert-Schmidt norm of the difference between the covariance operators, i.e., $\|S - S'\|_{\mathrm{HS}}$. Let us start with this term before we turn to the contraction norms themselves. As before, S_λ and

S' denotes the covariance operator of $\bar{F}_\lambda$ and Z respectively. Based on Hsing and Eubank (2015, Theorem 7.4.3) and how Hilbert-Schmidt norms are defined for integral operators, we can use

$$\begin{aligned}\left\|S_\lambda - S'\right\|_{\mathrm{HS(K)}} &= \left\|\mathbb{E}\left[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\right] - \mathbb{E}[Z(t)Z(s)]\right\|_{L^2([0,1]^{\otimes 2})} \\ &\leq \left\|\mathbb{E}\left[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\right] - \mathbb{E}[Z(t)Z(s)]\right\|_{\infty}.\end{aligned}$$

Our task is hence to compute $\mathbb{E}\left[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\right]$. We have

$$\langle f_1(t), f_1(s)\rangle_{L^2(\mu_\lambda)} = 4\lambda^3\psi_{\lambda,t}\psi_{\lambda,s}\ell_{t\wedge s} = \sqrt{ts(t\wedge s)}4\ell_1\lambda^3\psi_{\lambda,1}^2$$

and

$$\langle f_2(t), f_2(s)\rangle_{L^2(\mu_\lambda^{\otimes 2})} = \lambda^2\psi_{\lambda,t\wedge s}\ell_{t\wedge s} = (t\wedge s)\ell_1\lambda^2\psi_{\lambda,1},$$

so that

$$\begin{aligned}\mathbb{E}\left[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\right] =& \frac{\langle f_1(t), f_1(s)\rangle_{L^2(\mu_\lambda)} + \langle f_2(t), f_2(s)\rangle_{L^2(\mu_\lambda^2)}}{\sigma^2} \\ =& \frac{\sqrt{ts(t\wedge s)}4\lambda\psi_{\lambda,1} + t\wedge s}{4\lambda\psi_{\lambda,1}+1}.\end{aligned}$$

At this step, we need to differentiate our analysis depending on what regime we are in.

Regime 1: We assume here that $\lambda\psi_{\lambda,1} \to \infty$. The limiting covariance operator S' then has covariance function $\phi(t,s) = \sqrt{ts(t\wedge s)}$. We can use the fact that for $a \ll A, b \ll B$,

$$\left|\frac{A+a}{B+b} - \frac{A}{B}\right| \lesssim \left|\frac{a}{B}\right| + \left|\frac{b}{B}\right|$$

in order to deduce that

$$\left\|S_\lambda - S'\right\|_{\mathrm{HS(K)}} \leq \sup_{1\leq s,t\leq M}\left|\mathbb{E}\left[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\right] - \phi(t,s)\right| \lesssim \frac{1}{\lambda\psi_{\lambda,1}}. \tag{5.5}$$

Regime 2: Here, $\lambda\psi_{\lambda,1} \to 1$, so that the limiting covariance function is given by $\phi(t,s) = \frac{4\sqrt{ts(t\wedge s)}+t\wedge s}{5}$. Moreover,

$$\begin{aligned}\left\|S_\lambda - S'\right\|_{\mathrm{HS(K)}} &\leq \sup_{1\leq s,t\leq M}\left|\mathbb{E}\left[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\right] - \phi(t,s)\right| \\ &= \sup_{1\leq s,t\leq M}\left|\frac{4\sqrt{ts(t\wedge s)}\lambda\psi_{\lambda,1} + t\wedge s}{4\lambda\psi_{\lambda,1}+1} - \frac{4\sqrt{ts(t\wedge s)} + t\wedge s}{5}\right| \lesssim \left|\lambda\psi_{\lambda,1} - 1\right|. \end{aligned}\tag{5.6}$$

Regime 3: The fact that $\lambda\psi_{\lambda,1} \to 0$ implies in this case that the limiting covariance function is given by $\phi(t,s) = t\wedge s$, and we hence have

$$\left\|S_\lambda - S'\right\|_{\mathrm{HS(K)}} \lesssim \frac{\lambda^3\psi_{\lambda,1}^2}{\lambda^2\psi_{\lambda,1}} \asymp \lambda\psi_{\lambda,1}. \tag{5.7}$$

We now turn to the second part of the bound appearing in Theorem 3.7, namely the contraction norms. We need to evaluate the norms of $g_1(t)\star_1^0 g_1(t)$, $g_1(t)\star_1^0 g_2(t)$, $g_1(t)\star_1^1 g_2(t)$, $g_2(t)\star_1^0 g_2(t)$, $g_2(t)\star_2^0 g_2(t)$ and $g_2(t)\star_1^1 g_2(t)$. The calculations we need to perform are very similar to the ones appearing in the proof of Lachièze-Rey and Peccati (2013a, Theorem 4.7), hence we will not provide full details and proceed straight to the result. Let us still include two examples of these calculations (the cases of the contractions $g_1(t)\star_1^0 g_1(t)$ and $g_2(t)\star_1^1 g_2(t)$) for the reader's convenience and for the sake of staying self-contained. Recall that W_t, $H_{\lambda,t}$ are symmetric sets, W_t (respectively $H_{\lambda,t}$)

is contained in $W_{t'}$ (respectively $H_{\lambda,t'}$) for $t \le t'$, and that $\psi_{\lambda,t} = \sqrt{t}\psi_{\lambda,1}$, while $\ell_t = \sqrt{t}\ell_1 < \infty$. We can then write

$$\begin{aligned}
&\left\|f_1(t) \star_1^0 f_1(s)\right\|^2_{L^2(\mu_\lambda)\otimes K^{\otimes 2}} \\
&\quad= \left\| \int_{\mathbb{R}^d} \left(4 \int_{Z^2} 1_{H_{\lambda,t}\cap W_t^{\otimes 2}}(x,y) 1_{H_{\lambda,s}\cap W_s^{\otimes 2}}(x,u)\lambda dy \lambda du \right)^2 \lambda dx \right\|_{K^{\otimes 2}} \\
&\quad\le 16\lambda^5 \left\| \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{2d}} 1_{H_{\lambda,s\vee t}\cap W^{\otimes 2}_{s\vee t}}(x,y) 1_{H_{\lambda,s\vee t}\cap W^{\otimes 2}_{s\vee t}}(x,u) dy du \right)^2 dx \right\|_{K^{\otimes 2}} \\
&\quad\asymp \lambda^5 \left\| \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{2d}} 1_{W_{s\vee t}}(x) 1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(y-x) 1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(u-x) d(y-x) d(u-x) \right)^2 dx \right\|_{K^{\otimes 2}} \\
&\quad\asymp \lambda^5 \left\| \ell_{s\vee t} \psi^4_{\lambda,s\vee t} \right\|_{K^{\otimes 2}} \asymp \lambda^5 \psi^4_{\lambda,1}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|f_2(t) \star_1^1 f_2(s)\right\|^2_{L^2(\mu_\lambda^{\otimes 2})\otimes K^{\otimes 2}} \\
&\le \left\| \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^d} 1_{H_{\lambda,s\vee t}\cap W^{\otimes 2}_{s\vee t}}(x,y) 1_{H_{\lambda,s\vee t}\cap W^{\otimes 2}_{s\vee t}}(x,u) \lambda dx \right)^2 \lambda^2 dy du \right\|_{K^{\otimes 2}} \\
&= \lambda^4 \left\| \int_{\mathbb{R}^{4d}} 1_{H_{\lambda,s\vee t}\cap W^{\otimes 2}_{s\vee t}}(x,y) 1_{H_{\lambda,s\vee t}\cap W^{\otimes 2}_{s\vee t}}(x,u) 1_{H_{\lambda,s\vee t}\cap W^{\otimes 2}_{s\vee t}}(v,y) 1_{H_{\lambda,s\vee t}\cap W^{\otimes 2}_{s\vee t}}(v,u) dx dy du dv \right\|_{K^{\otimes 2}} \\
&\le \lambda^4 \Bigg\| \int_{\mathbb{R}^{4d}} 1_{W_{s\vee t}}(x) 1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(y-x) 1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(u-x) 1_{W_{s\vee t}}(x) 1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(y-v) \\
&\qquad\qquad\qquad\qquad\qquad\qquad dx d(y-x) d(u-v) d(v-y) \Bigg\|_{K^{\otimes 2}} \\
&\quad\asymp \lambda^4 \left\| \ell_{s\vee t} \psi^3_{\lambda,s\vee t} \right\|_{K^{\otimes 2}} \asymp \lambda^4 \psi^3_{\lambda,1}.
\end{aligned}$$

For the remaining contractions, performing similar calculations yields $\left\|f_1(t) \star_1^0 f_2(t)\right\|^2_{L^2(\mu_\lambda^{\otimes 2})\otimes K^{\otimes 2}} \lesssim \lambda^4 \psi^3_{\lambda,1}$, $\left\|f_2(t) \star_1^0 f_2(t)\right\|^2_{L^2(\mu_\lambda^{\otimes 3})\otimes K^{\otimes 2}} \lesssim \lambda^3 \psi^2_{\lambda,1}$, $\left\|f_2(t) \star_2^0 f_2(t)\right\|^2_{L^2(\mu_\lambda^{\otimes 2})\otimes K^{\otimes 2}} \lesssim \lambda^2 \psi_{\lambda,1}$, and finally $\left\|f_1(t) \star_1^1 f_2(t)\right\|^2_{L^2(\mu_\lambda)\otimes K^{\otimes 2}} \lesssim \lambda^5 \psi^4_{\lambda,1}$. We split the remainder of the proof into three cases corresponding to the three possible regimes.

Regime 1: Here, $\lambda\psi_{\lambda,1} \to \infty$ as $\lambda \to \infty$, and since $\sigma^2 \asymp \lambda^3 \psi^2_{\lambda,1}$, we have $\left\|g_1(t) \star_1^0 g_1(t)\right\|^2_{L^2(\mu_\lambda)\otimes K^{\otimes 2}} \lesssim \lambda^{-1}$, $\left\|g_1(t) \star_1^0 g_2(t)\right\|^2_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}} \lesssim \lambda^{-2}\psi^{-1}_{\lambda,1}$, $\left\|g_2(t) \star_1^0 g_2(t)\right\|^2_{L^2(\mu_\lambda^3)\otimes K^{\otimes 2}} \lesssim \lambda^{-3}\psi^{-2}_{\lambda,1}$, $\left\|g_2(t) \star_2^0 g_2(t)\right\|^2_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}} \lesssim \lambda^{-4}\psi^{-3}_{\lambda,1}$, $\left\|g_2(t) \star_1^1 g_2(t)\right\|^2_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}} \lesssim \lambda^{-2}\psi^{-1}_{\lambda,1}$ and lastly $\left\|g_1(t) \star_1^1 g_2(t)\right\|^2_{L^2(\mu_\lambda)\otimes K^{\otimes 2}} \lesssim \lambda^{-1}$. Note that all the above estimates are asymptotically bounded from above by λ^{-1}, and using (5.5), the estimate in Theorem 3.7 yields

$$d_3\big(\bar{F}_\lambda, Z\big) \lesssim \lambda^{-\frac{1}{2}} + \frac{1}{\lambda\psi_{\lambda,1}}.$$

Regime 2: As in this case, we have $\lambda\psi_{\lambda,1} \to 1$ as $\lambda \to \infty$, we get $\sigma^2 \asymp \lambda^3\psi^2_{\lambda,1} \asymp \lambda^2\psi_{\lambda,1}$. Therefore, we can reuse the computations from Regime 1 combined with (5.6) to get

$$d_3\big(\bar{F}_\lambda, Z\big) \lesssim \lambda^{-\frac{1}{2}} + \left|\lambda\psi_{\lambda,1} - 1\right|.$$

Regime 3: In this regime, $\lambda\psi_{\lambda,1} \to 0$ and $\lambda\sqrt{\psi_{\lambda,1}} \to \infty$ as $\lambda \to \infty$, so that $\sigma^2 \asymp \lambda^2\psi_{\lambda,1}$. This allows us to deduce that $\left\|g_1(t) \star_1^0 g_1(t)\right\|^2_{L^2(\mu_\lambda)\otimes K^{\otimes 2}} \lesssim \lambda\psi^2_{\lambda,1}$, $\left\|g_1(t) \star_1^0 g_2(t)\right\|^2_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}} \lesssim \psi_{\lambda,1}$, $\left\|g_2(t) \star_1^0 g_2(t)\right\|^2_{L^2(\mu_\lambda^3)\otimes K^{\otimes 2}} \lesssim \lambda^{-1}$, $\left\|g_2(t) \star_2^0 g_2(t)\right\|^2_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}} \lesssim \lambda^{-2}\psi^{-1}_{\lambda,1}$, $\left\|g_2(t) \star_1^1 g_2(t)\right\|^2_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}} \lesssim \psi_{\lambda,1}$ and $\left\|g_1(t) \star_1^1 g_2(t)\right\|^2_{L^2(\mu_\lambda)\otimes K^{\otimes 2}} \lesssim \lambda\psi^2_{\lambda,1}$. Since $\lambda^{-2} \ll \psi_{\lambda,1} \ll \lambda^{-1}$, all terms listed are asymptotically bounded by $\lambda^{-2}\psi^{-1}_{\lambda,1}$. Combining this fact with (5.7) yields

$$d_3\big(\bar{F}_\lambda, Z\big) \lesssim \lambda^{-1}\psi^{-1/2}_{\lambda,1} + \lambda\psi_{\lambda,1},$$

which concludes the proof. □

Appendix

This section gathers ancillary lemmas used in the proofs of our main results as well as in the different applications presented in this paper.

5.3. *Lemmas related to the proofs of Theorems 3.1 and 3.7.* Our first two lemma contain important results from Döbler et al. (2018) which we restate here for reader's convenience.

Lemma 5.4. *Let $p, q \geq 1$ be integers, and let $F_q = I^\eta_q(f_q), G_p = I^\eta_p(g_p)$ and $F^t_q = I^{\eta^t}_q(f_q), G^t_p = I^{\eta^t}_p(g_p)$ be real-valued Poisson multiple integrals as constructed in Section 2. Assume further that $\mathbb{E}\big[F^4_q\big], \mathbb{E}\big[G^4_p\big] < \infty$. Then, the following limits hold almost surely.*

(a) $\lim_{t\to 0} \frac{1}{t}\mathbb{E}\big[F^t_q - F_q|\eta\big] = -qF$

(b) $\lim_{t\to 0} \frac{1}{t}\mathbb{E}\big[(F^t_q - F_q)(G^t_p - G_p)|\eta\big] = 2\widetilde{\Gamma}(F_q, G_p)$

(c) $\lim_{t\to 0} \frac{1}{t}\mathbb{E}\big[F^t_q(G^t_p - G_p)|\eta\big] = 2\widetilde{\Gamma}(F_q, G_p) - pF_qG_p$

(d) $\lim_{t\to 0} \frac{1}{t}\mathbb{E}\big[(F^t_q - F_q)^4\big] = -4q\mathbb{E}\big[F^4_q\big] + 12\mathbb{E}\Big[F^2_q\widetilde{\Gamma}(F_q, F_q)\Big]$.

Proof: The proof of part (a), (b) and (d) are in Döbler et al. (2018, Proposition 3.2). Part (c) is a consequence of (a) and (b). □

The following is an estimate from Döbler et al. (2018, Lemma 2.2), which improves upon a similar result in Döbler and Peccati (2018, Lemma 3.1).

Lemma 5.5. *Let $p, q \geq 1$ be integers, and let $F_q = I^\eta_q(f_q)$ and $F^t_q = I^{\eta^t}_q(f_q)$ be real-valued Poisson multiple integrals as constructed in Section 2. Assume further that $\mathbb{E}\big[F^4_q\big], \mathbb{E}\big[G^4_p\big] < \infty$. Then, we have*

$$\mathbb{E}\Big[\widetilde{\Gamma}(F_q, G_p)^2\Big] \leq \left(\frac{p+q-1}{2}\right)^2\Big(\mathbb{E}\big[F^2_qG^2_p\big] - \mathbb{E}\big[F^2_q\big]\mathbb{E}\big[G^2_p\big] - 2\mathbb{E}[F_qG_p]^2\Big).$$

Our next lemma states a more general version of Lemma 5.4, part (d).

Lemma 5.6. *Let (X, X^t) be an exchangeable pair such that $X = \sum_{q\in\mathbb{N}} I^\eta_q(x_q)$ and $X^t = \sum_{q\in\mathbb{N}} I^{\eta^t}_q(x_q)$. Let the pairs (Y, Y^t),(U, U^t) and (V, V^t) be defined in the same way. Assume further that $\mathbb{E}\big[X^4\big]$, $\mathbb{E}\big[Y^4\big]$, $\mathbb{E}\big[U^4\big]$ and $\mathbb{E}\big[V^4\big]$ are finite. Then, one has*

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\big[(X^t - X)(Y_t - Y)(U^t - U)(V^t - V)\big] =& 4\mathbb{E}\Big[\widetilde{\Gamma}(X, Y)UV + \widetilde{\Gamma}(X, V)YU \\ &+ \widetilde{\Gamma}(X, U)YV + \widetilde{L}XYUV\Big].\end{aligned}$$

Proof: This limit is a consequence of exchangeability and Lemma 5.4. Indeed, denoting

$$M_t = \frac{1}{t}\mathbb{E}\big[(X^t - X)(Y_t - Y)(U^t - U)(V^t - V)\big],$$

we can write

$$\begin{aligned}
\lim_{t\to\infty} M_t =& 2\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\big[XYUV - X^tYUV - XY^tUV - XYU^tV - XYUV^t\big] \\
&+ 2\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\big[X^tY^tUV + X^tYU^tV + X^tYUV^t\big] \\
=& 2\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\big[-\big(X^t - X\big)YUV - X\big(Y^t - Y\big)UV - XY\big(U^t - U\big)V - XYU\big(V^t - V\big)\big] \\
&+ 2\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\big[\big(X^tY^t - XY\big)UV + \big(X^tU^t - XU\big)YV + \big(X^tV^t - XV\big)YU\big] \\
=& 2\mathbb{E}\big[-\widetilde{L}XYUV - X\widetilde{L}YUV - XY\widetilde{L}UV - XYU\widetilde{L}V + \widetilde{L}(XY)UV + \widetilde{L}(XU)YV \\
&+ \widetilde{L}(XV)YU\big] \\
=& 4\mathbb{E}\Big[\widetilde{\Gamma}(X,Y)UV + \widetilde{\Gamma}(X,V)YU + \widetilde{\Gamma}(X,U)YV + \widetilde{L}XYUV\Big].
\end{aligned}$$

□

The upcoming lemma is a version of Lemma 5.4 in the setting of Hilbert-valued random variables.

Lemma 5.7. *Let* $X = \sum_{1\le q\le N} F_q$*, where* $F_q \in \mathcal{H}^q(K)$ *with covariance operator* S_q *and* $\mathbb{E}\Big[\|F_q\|_K^4\Big] < \infty$*. It holds that*

(a) $\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\big\langle F_q^t - F_q, Dg(X)\big\rangle_K\Big] = -q\mathbb{E}\big[\langle F_q, Dg(X)\rangle_K\big]$.

(b) $\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\big\|F_q^t - F_q\big\|_K^2\Big] = 2q\mathbb{E}\Big[\|F_q\|_K^2\Big]$.

(c) $\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\Big[\Big\langle \frac{1}{q}\big(F_q^t - F_q\big), D^2g(X)(F_p^t - F_p)\Big\rangle_K\Big]$
$= \frac{1}{q}\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[\widetilde{\Gamma}(F_{q,i}, F_{p,j})\left\langle k_i, D^2g(X)k_j\right\rangle_K\Big]$.

(d) $\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\big\|F_q^t - F_q\big\|_K^4\Big] = 4\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[F_{q,i}^2\Big(\widetilde{\Gamma}(F_{q,j}, F_{q,j}) - q\mathbb{E}[F_{q,j}]\Big)\Big]$
$+ 8\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[F_{q,i}F_{q,j}\Big(\widetilde{\Gamma}(F_{q,i}, F_{q,j}) - q\mathbb{E}[F_{q,i}F_{q,j}]\Big)\Big]$
$- 4q\sum_{i,j\in\mathbb{N}}\Big(\mathbb{E}\Big[F_{q,i}^2F_{q,j}^2\Big] - \mathbb{E}\Big[F_{q,i}^2\Big]\mathbb{E}\Big[F_{q,j}^2\Big] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2\Big)$.

In particular, when $q = p$ *then part (c) becomes*

$$\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle\frac{1}{q}\big(F_q^t - F_q\big), D^2g(X)(F_q^t - F_q)\right\rangle_K\right] = \mathrm{Tr}_K\big(D^2g(X)\Gamma\big(F_q, -L^{-1}F_q\big)\big).$$

Proof: Part (a) follows from

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\big\langle F_q^t - F_q, Dg(X)\big\rangle_K\Big] &= \lim_{t\to 0}\frac{1}{t}\sum_{i\in\mathbb{N}}\mathbb{E}\Big[\big\langle\big(F_{q,i}^t - F_{q,i}\big)k_i, Dg(X)\big\rangle_K\Big] \\
&= \sum_{i\in\mathbb{N}}\mathbb{E}\left[\lim_{t\to 0}\frac{1}{t}\mathbb{E}\big[F_{q,i}^t - F_{q,i}|\eta\big]\langle k_i, Dg(X)\rangle_K\right] \\
&= -q\sum_{i\in\mathbb{N}}\mathbb{E}[F_{q,i}\langle k_i, Dg(X)\rangle_K] \\
&= -q\mathbb{E}\big[\langle F_q, Dg(X)\rangle_K\big].
\end{aligned}$$

We need to justify the exchange of expected value and limit in the second line above. This will be done via the dominated convergence theorem, noting that other parts of this proof which are presented below will also require similar arguments. It is sufficient to consider only $0 \leq t \leq 1$. The function $g(t) = e^{-qt}$ is Lipschitz continuous on $[0,1]$. Moreover, $\mathbb{E}\left[F^t_{q,i}|\eta\right] = P_t F_{q,i} = e^{-qt} F_{q,i}$ per Section 2.5 and $\sup_x \|Dg(x)\|_K < \infty$ per Bourguin and Campese (2020, Lemma 2.4). Hence,

$$\left|\frac{1}{t}\mathbb{E}\big[F^t_{q,i} - F_{q,i}|\eta\big]\langle k_i, Dg(X)\rangle_K\right| = \left|\frac{e^{-qt}-1}{t} F_{q,i}\,\langle k_i, Dg(X)\rangle_K\right| \leq C\,|F_{q,i}|$$

for some positive constant C. Then we can apply the dominated convergence theorem.

Part (b) is a result of

$$\begin{aligned}\mathbb{E}\left[\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\big\|F^t_q - F_q\big\|^2_K|\eta\Big]\right] =& \mathbb{E}\left[\sum_{i\in\mathbb{N}}\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\big(F^t_{q,i} - F_{q,i}\big)^2|\eta\Big]\right]\\ =&2\sum_{i\in\mathbb{N}}\mathbb{E}\Big[\widetilde{\Gamma}(F_{q,i}, F_{q,i})\Big]\\ =&2q\mathbb{E}\Big[\|F_q\|^2_K\Big].\end{aligned}$$

For part (c), we can write

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle\frac{1}{q}\big(F^t_q - F_q\big), D^2 g(X)(F^t_p - F_p)\right\rangle_K\right]\\ &\qquad= \lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle\sum_{i\in\mathbb{N}}\frac{1}{q}\big(F^t_{q,i} - F_{q,i}\big)k_i, D^2 g(X)\sum_{j\in\mathbb{N}}\big(F^t_{p,j} - F_{p,j}\big)k_j\right\rangle_K\right]\\ &\qquad= \frac{1}{q}\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\big[\big(F^t_{q,i} - F_{q,i}\big)\big(F^t_{p,j} - F_{p,j}\big)|\eta\big]\left\langle k_i, D^2 g(X)k_j\right\rangle_K\right]\\ &\qquad= \frac{1}{q}\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[\widetilde{\Gamma}(F_{q,i}, F_{p,j})\left\langle k_i, D^2 g(X)k_j\right\rangle_K\Big].\end{aligned}$$

Using the above expression in the case $q = p$, along with the fact that

$$\begin{aligned}\Gamma(F_q, F_q)k_j = \Gamma\left(\sum_{i\in\mathbb{N}}F_{q,i}k_i, \sum_{m\in\mathbb{N}}F_{q,m}k_m\right)k_j &= \sum_{i,m\in\mathbb{N}}\Gamma(F_{q,i}k_i, F_{q,m}k_m)k_j\\ &= \sum_{i,m\in\mathbb{N}}\frac{1}{2}\widetilde{\Gamma}(F_{q,i}, F_{q,m})(k_i\otimes k_m + k_m\otimes k_i)k_j\\ &= \sum_{i\in\mathbb{N}}\widetilde{\Gamma}(F_{q,i}, F_{q,j})k_i\end{aligned}$$

yields

$$\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle\frac{1}{q}\big(F^t_q - F_q\big), D^2 g(X)(F^t_q - F_q)\right\rangle_K\right] = \mathrm{Tr}_K\big(D^2 g(X)\Gamma\big(F_q, -L^{-1}F_q\big)\big).$$

For part (d), the exchangeability of $\left(F_q, F_q^t\right)$ and Lemma 5.6 imply

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|F_q^t-F_q\right\|_K^4\right] =&\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|\sum_{i\in\mathbb{N}}\left(F_{q,i}^t-F_{q,i}\right)k_i\right\|_K^4\right]\\
=&\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\sum_{i,j\in\mathbb{N}}\left(F_{q,i}^t-F_{q,i}\right)^2\left(F_{q,j}^t-F_{q,j}\right)^2\right]\\
=&4\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[F_{q,i}^2\left(\widetilde{\Gamma}(F_{q,j},F_{q,j})-q\mathbb{E}\left[F_{q,j}^2\right]\right)\right]\\
&+8\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[F_{q,i}F_{q,j}\left(\widetilde{\Gamma}(F_{q,i},F_{q,j})-q\mathbb{E}[F_{q,i}F_{q,j}]\right)\right]\\
&-4q\sum_{i,j\in\mathbb{N}}\left(\mathbb{E}\left[F_{q,i}^2F_{q,j}^2\right]-\mathbb{E}\left[F_{q,i}^2\right]\mathbb{E}\left[F_{q,j}^2\right]-2\mathbb{E}[F_{q,i}F_{q,j}]^2\right).
\end{aligned}$$

□

Via Lemma 5.7, we will establish a fourth moment bound that will help with the remainder term in the proof of Theorem 3.1.

Lemma 5.8. *Let $X=\sum_{1\le q\le N}F_q$, where $F_q\in\mathcal{H}^q(K)$ with covariance operator S_q and $\mathbb{E}\left[\|F_q\|_K^4\right]<\infty$. It holds that*

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|\sum_{q=1}^N\frac{1}{q}\left(F_q^t-F_q\right)\right\|_K\left\|X^t-X\right\|_K^2\right]\\
&\le 2N\sqrt{\max_{1\le p\le N}\mathbb{E}\left[\|F_p\|_K^2\right]}\sum_{1\le q\le N}\sqrt{4q-3}\sqrt{\|F_q\|_K^4-\mathbb{E}\left[\|F_q\|_K^2\right]^2-2\left\|S_q\right\|_{\mathrm{HS}}^2}.
\end{aligned}$$

Proof: We have

$$\begin{aligned}
\mathbb{E}\left[\left\|\sum_{q=1}^N\frac{1}{q}\left(F_q^t-F_q\right)\right\|_K\left\|X^t-X\right\|_K^2\right]&\le\mathbb{E}\left[\left(\sum_{1\le q\le N}\frac{1}{q}\left\|F_q^t-F_q\right\|_K\right)\left(\sum_{p=1}^N\left\|F_p^t-F_p\right\|_K\right)^2\right]\\
&\le N\sum_{1\le p,q\le N}\frac{1}{q}\mathbb{E}\left[\left\|F_q^t-F_q\right\|_K\left\|F_p^t-F_p\right\|_K^2\right]\\
&\le N\sum_{1\le p,q\le N}\frac{1}{q}\sqrt{\mathbb{E}\left[\left\|F_q^t-F_q\right\|_K^2\right]}\sqrt{\mathbb{E}\left[\left\|F_p^t-F_p\right\|_K^4\right]}.
\end{aligned}$$

The first line is due to chaos decomposition of X, X^t and triangle inequality of $\|\cdot\|_K$. The second line is a consequence of $\left(\sum_{p=1}^N y_p\right)^2\le N\sum_{i=1}^N y_p^2$. The last line is due to Hölder's inequality.

Next, part (b) of Lemma 5.7 implies that

$$
\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|\sum_{q=1}^{N}\frac{1}{q}\left(F_q^t-F_q\right)\right\|_K \left\|X^t-X\right\|_K^2\right] \\
&= N\sum_{1\le p,q\le N}\frac{1}{q}\sqrt{\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|F_q^t-F_q\right\|_K^2\right]}\sqrt{\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|F_p^t-F_p\right\|_K^4\right]} \\
&\le N\sqrt{2\max_{1\le q\le N}\mathbb{E}\left[\left\|F_q\right\|_K^2\right]}\sum_{1\le q\le N}\sqrt{\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|F_q^t-F_q\right\|_K^4\right]}. \qquad (5.8)
\end{aligned}
$$

Let us study the remaining limit on the right hand side. Lemma 5.7 says

$$
\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|F_q^t-F_q\right\|_K^4\right] =&4\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[F_{q,i}^2\left(\widetilde{\Gamma}(F_{q,j},F_{q,j})-q\mathbb{E}\left[F_{q,j}^2\right]\right)\right] \\
&+8\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[F_{q,i}F_{q,j}\left(\widetilde{\Gamma}(F_{q,i},F_{q,j})-q\mathbb{E}[F_{q,i}F_{q,j}]\right)\right] \\
&-4q\sum_{i,j\in\mathbb{N}}\left(\mathbb{E}\left[F_{q,i}^2F_{q,j}^2\right]-\mathbb{E}\left[F_{q,i}^2\right]\mathbb{E}\left[F_{q,j}^2\right]-2\mathbb{E}[F_{q,i}F_{q,j}]^2\right). \qquad (5.9)
\end{aligned}
$$

We will treat each term of (5.9) separately. For the first term of (5.9), our proof will use an argument similar to the proof of Döbler et al. (2018, Lemma 2.2) or Döbler and Peccati (2018, Lemma 3.1). First, observe that if k is a fixed positive integer and J_k denotes the projection into the k-th Poisson chaos, then

$$
\mathbb{E}\left[J_k\left(\|F_p\|_K^2\right)^2\right]=\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[J_k\left(F_{q,i}^2\right)J_k\left(F_{q,j}^2\right)\right].
$$

In particular, the expansion in Döbler et al. (2018, Lemma 5.1) yields

$$
\begin{aligned}
\mathbb{E}\left[J_{2q}\left(\|F_q\|_K^{\otimes 2}\right)^2\right] &= \sum_{i,j\in\mathbb{N}}(2q)!\left\langle f_{q,i}\widetilde{\otimes}f_{q,i},f_{q,j}\widetilde{\otimes}f_{q,j}\right\rangle_{\mathfrak{H}^{\otimes 2q}} \\
&=\sum_{i,j\in\mathbb{N}}\left(2\mathbb{E}[F_{q,i}F_{q,j}]^2+\sum_{r=1}^{q-1}q!^2\binom{q}{r}^2\left\langle f_{q,i}\star_r^r f_{q,j},f_{q,j}\star_r^r f_{q,i}\right\rangle_{\mathfrak{H}^{\otimes 2q-2r}}\right).
\end{aligned}
$$

Thus, the first term of (5.9) can be bounded via

$$
\begin{aligned}
\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[F_{q,i}^2\left(\widetilde{\Gamma}(F_{q,j},F_{q,j})-q\mathbb{E}\left[F_{q,j}^2\right]\right)\right] \le&\frac{1}{2}\sum_{i,j\in\mathbb{N}}\sum_{k=1}^{2q-1}(2p-k)\mathbb{E}\left[J_k\left(F_{q,i}^2\right)J_k\left(F_{q,j}^2\right)\right] \\
=&\frac{1}{2}\sum_{k=1}^{2q-1}(2p-k)\mathbb{E}\left[J_k\left(\|F_q\|_K^2\right)^2\right] \\
\le&\frac{2q-1}{2}\sum_{k=1}^{2q-1}\mathbb{E}\left[J_k\left(\|F_q\|_K^2\right)^2\right] \\
=&\frac{2q-1}{2}\left(\|F_q\|_K^4-\mathbb{E}\left[\|F_q\|_K^2\right]^2-2\left\|S_q\right\|_{\mathrm{HS}}^2\right) \\
&-\frac{2q-1}{2}\sum_{i,j\in\mathbb{N}}\sum_{r=1}^{q-1}q!^2\binom{q}{r}^2\left\langle f_{q,i}\star_r^r f_{q,j},f_{q,j}\star_r^r f_{q,i}\right\rangle_{\mathfrak{H}^{\otimes 2q-2r}}.
\end{aligned}
$$

The second term of (5.9) will receive a similar treatment. Based on Döbler et al. (2018, Lemma 5.1), we have

$$\begin{aligned}\mathbb{E}[J_{2q}(F_{q,i}F_{q,j})] &= \sum_{i,j\in\mathbb{N}} (2q)!\,\left\|f_{q,i}\widetilde{\otimes} f_{q,j}\right\|^2_{\mathfrak{H}^{\otimes 2q}} \\ &= \sum_{i,j\in\mathbb{N}}\left(\mathbb{E}[F_{q,i}F_{q,j}]^2 + \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{q,j}^2\big] + \sum_{r=1}^{q} q!^2\binom{q}{r}^2 \|f_{q,i}\star_r^r f_{q,j}\|^2_{\mathfrak{H}^{\otimes 2q-2r}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[F_{q,i}F_{q,j}\Big(\widetilde{\Gamma}(F_{q,i},F_{q,j}) - q\mathbb{E}[F_{q,i}F_{q,j}]\Big)\Big] &= \frac{1}{2}\sum_{i,j\in\mathbb{N}}\sum_{k=1}^{2q-1}(2q-k)\mathbb{E}\Big[J_k(F_{q,i}F_{q,j})^2\Big] \\ &\le \frac{2q-1}{2}\sum_{i,j\in\mathbb{N}}\sum_{k=1}^{2q-1}\mathbb{E}\Big[J_k(F_{q,i}F_{q,j})^2\Big] \\ &= \frac{2q-1}{2}\Big(\mathbb{E}\big[F_{q,i}^2F_{q,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{q,j}^2\big] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2\Big) \\ &\quad - \frac{2q-1}{2}\sum_{i,j\in\mathbb{N}}\sum_{r=1}^{q-1} q!^2\binom{q}{r}^2\|f_{q,i}\star_r^r f_{q,j}\|^2_{\mathfrak{H}^{\otimes 2q-2r}} \\ &\le \frac{2q-1}{2}\Big(\|F_q\|_K^4 - \mathbb{E}\Big[\|F_q\|_K^2\Big]^2 - 2\,\|S_q\|^2_{\mathrm{HS}}\Big) \\ &\quad - \frac{2q-1}{4}\sum_{i,j\in\mathbb{N}}\sum_{r=1}^{q-1} q!^2\binom{q}{r}^2\|f_{q,i}\star_r^r f_{q,j}\|^2_{\mathfrak{H}^{\otimes 2q-2r}}\,.\end{aligned}$$

In addition, based on that fact that

$$\begin{aligned}&\sum_{i,j\in\mathbb{N}}\Big(2\,\|f_{q,i}\star_r^r f_{q,j}\|^2_{\mathfrak{H}^{\otimes 2q-2r}} + 2\,\langle f_{q,i}\star_r^r f_{q,j}, f_{q,j}\star_r^r f_{q,i}\rangle_{\mathfrak{H}^{\otimes 2q-2r}}\Big) \\ &\quad= \sum_{i,j\in\mathbb{N}}\Big(\|f_{q,i}\star_r^r f_{q,j}\|^2_{\mathfrak{H}^{\otimes 2q-2r}} + 2\,\langle f_{q,i}\star_r^r f_{q,j}, f_{q,j}\star_r^r f_{q,i}\rangle_{\mathfrak{H}^{\otimes 2q-2r}} + \|f_{q,j}\star_r^r f_{q,i}\|^2_{\mathfrak{H}^{\otimes 2q-2r}}\Big) \\ &\quad= \sum_{i,j\in\mathbb{N}}\Big(\|f_{q,i}\star_r^r f_{q,j} + f_{q,j}\star_r^r f_{q,i}\|^2_{\mathfrak{H}^{\otimes 2q-2r}}\Big) \\ &\quad\ge 0,\end{aligned}$$

we deduce from (5.9) that

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\|F_q^t - F_q\|_K^4\Big] \le (8q-6)\Big(\|F_q\|_K^4 - \mathbb{E}\Big[\|F_q\|_K^2\Big]^2 - 2\,\|S_q\|^2_{\mathrm{HS}}\Big).$$

Combining this with (5.8), we arrive at the fourth moment bound in the statement of this lemma. □

The result below is an adaptation to our setting of a classical combinatorial identity appearing in Peccati and Taqqu (2011, Proof of Proposition 11.2.2).

Lemma 5.9. *The quantity* $\left\| f_{q,i} \widetilde{\star}_0^0 f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes q+p}}$ *appearing in Equation (4.3) can be written in terms of norms of non-symmetrized contractions as*

$$\begin{aligned}(q+p)! \left\| f_{q,i} \widetilde{\star}_0^0 f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes q+p}} =& q!p! \left\| f_{q,i} \right\|^2_{\mathfrak{H}^{\otimes q}} \left\| f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes p}} + q!^2 \left\langle f_{q,i}, f_{q,j} \right\rangle^2_{\mathfrak{H}^{\otimes q}} \mathbb{1}_{\{q=p\}} \\ &+ q!p! \binom{q}{q \wedge p} \binom{p}{q \wedge p} \left\| f_{q,i} \star^{q\wedge p}_{q \wedge p} f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes |q-p|}} \mathbb{1}_{\{q \neq p\}} \\ &+ \sum_{r=1}^{q\wedge p-1} q!p! \binom{q}{r} \binom{p}{r} \left\| f_{q,i} \star^r_r f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes q+p-2r}}.\end{aligned}$$

Proof: The procedure in Peccati and Taqqu (2011, Proof of Proposition 11.2.2) will be slightly modified to fit our situation. Let $\mathfrak{S}_{q+p}$ be the sets of all permutations of $(q+p)$ elements and assume $\pi, \rho \in \mathfrak{S}_{q+p}$. When the intersection set $\{\pi(1), \dots, \pi(q)\} \cap \{\rho(q+1), \dots, \rho(q+p)\}$ contains r element, this will be denoted by $\pi \overset{r}{\sim} \rho$. Since $\mathfrak{H} = L^2(\mathcal{Z}, \mu)$, we have that

$$\begin{aligned}\left\| f_{q,i} \widetilde{\star}_0^0 f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes q+p}} &= \left\| f_{q,i} \widetilde{\otimes} f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes q+p}} \\ &= \frac{1}{(q+p)!^2} \sum_{\pi, \rho \in \mathfrak{S}_{q+p}} \int_{\mathcal{Z}^{q+p}} f_{q,i}\big(z_{\pi(1)}, \dots, z_{\pi(q)}\big) f_{p,j}\big(z_{\pi(q+1)}, \dots, z_{\pi(q+p)}\big) \\ &\qquad f_{q,i}\big(z_{\rho(1)}, \dots, z_{\rho(q)}\big) f_{p,j}\big(z_{\rho(q+1)}, \dots, z_{\rho(q+p)}\big) \mu(dz_1 \dots dz_{q+p}) \\ &= \frac{1}{(q+p)!^2} \sum_{\pi \in \mathfrak{S}_{q+p}} \left(\sum_{r=1}^{q \wedge p-1} \sum_{\pi \overset{r}{\sim} \rho} A_{1,r} + \sum_{\pi \overset{0}{\sim} \rho} A_2 + \sum_{\pi \overset{q\wedge p}{\sim} \rho} A_3 \right). \end{aligned} \tag{5.10}$$

For the second sum in (5.10), $\pi \overset{0}{\sim} \rho$ is equivalent to

$$\begin{cases} \{\pi(1), \dots, \pi(q)\} \cap \{\rho(1), \dots, \rho(q)\} = \{\pi(1), \dots, \pi(q)\} \\ \{\pi(q+1), \dots, \pi(q+p)\} \cap \{\rho(q+1), \dots, \rho(q+p)\} = \{\pi(q+1), \dots, \pi(q+p)\} \end{cases},$$

which implies that

$$\begin{aligned}A_2 = &\int_{\mathcal{Z}^{q\wedge p}} \left(\int_{\mathcal{Z}^{q\wedge p}} f_{q,i}\big(z_{\pi(1)}, \dots, z_{\pi(q)}\big) f_{q,i}\big(z_{\pi(1)}, \dots, z_{\pi(q)}\big) \right) \\ &\big(f_{p,j}\big(z_{\pi(q+1)}, \dots, z_{\pi(q+p)}\big) f_{p,j}\big(z_{\pi(q+1)}, \dots, z_{\pi(q+p)}\big)\big) \mu(dz_1 \dots dz_{q+p}) = \left\| f_{q,i} \right\|^2_{\mathfrak{H}^{\otimes q}} \left\| f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes p}}.\end{aligned}$$

Furthermore, observe that for a fixed element $\pi \in \mathfrak{S}_{q+p}$, there are $q!$ ways to permute $\{1, \dots, q\}$ and $p!$ ways to permute $\{q+1, \dots, q+p\}$. Since $f_{q,i}$ and $f_{p,j}$ are symmetric functions, we have

$$\sum_{\pi \overset{0}{\sim} \rho} A_2 = q!p! \left\| f_{q,i} \right\|^2_{\mathfrak{H}^{\otimes q}} \left\| f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes p}}.$$

For the third sum in (5.10), there are two cases to consider. If $q = p$ then $\pi \overset{q}{\sim} \rho$ means

$$\begin{cases} \{\pi(1), \dots, \pi(q)\} \cap \{\rho(q+1), \dots, \rho(2q)\} = \{\pi(1), \dots, \pi(q)\} \\ \{\pi(q+1), \dots, \pi(2q)\} \cap \{\rho(1), \dots, \rho(q)\} = \{\pi(q+1), \dots, \pi(2q)\} \end{cases},$$

which implies that

$$\begin{aligned}
A_3 &= \int_{\mathcal{Z}^q} \left(\int_{Z^q} f_{q,i}\big(z_{\pi(1)}, \dots, z_{\pi(q)}\big) f_{q,j}\big(z_{\pi(1)}, \dots, z_{\pi(q)}\big) \right) \\
&\qquad\qquad f_{q,i}\big(z_{\pi(q+1)}, \dots, z_{\pi(2q)}\big) f_{q,j}\big(z_{\pi(q+1)}, \dots, z_{\pi(2q)}\big) \mu(dz_1 \dots dz_{2q}) \\
&= \langle f_{q,i}, f_{q,j} \rangle^2_{\mathfrak{H}^{\otimes q}} \mathbb{1}_{\{q=p\}},
\end{aligned}$$

and there are $q!^2$ copies like the one above. On the other hand for $q \neq p$,

$$\begin{aligned}
A_3 &= \int_{\mathcal{Z}^{|q-p|}} \left(\int_{\mathcal{Z}^{q\wedge p}} f_{q,i}\big(z_{\pi(1)}, \dots, z_{\pi(q)}\big) f_{p,j}\big(z_{\rho(q+1)}, \dots, z_{\rho(q+p)}\big) \right) \\
&\qquad\qquad \left(\int_{\mathcal{Z}^{q\wedge p}} f_{q,i}\big(z_{\rho(1)}, \dots, z_{\rho(q)}\big) f_{p,j}\big(z_{\pi(q+1)}, \dots, z_{\pi(q+p)}\big) \right) \mu(dz_1 \dots dz_{q+p}) \\
&= \int_{\mathcal{Z}^{|q-p|}} \big(f_{q,i} \star^{q\wedge p}_{q\wedge p} f_{p,j}\big)^2 \mu(dz_1 \dots dz_{|q-p|}) \\
&= \big\| f_{q,i} \star^{q\wedge p}_{q\wedge p} f_{p,j} \big\|^2_{\mathfrak{H}^{\otimes |q-p|}} .
\end{aligned}$$

Given a fixed π such that $\pi \overset{q\wedge p}{\sim} \rho$ and $q \neq p$, there is a total of $\binom{q}{q\wedge p}\binom{p}{q\wedge p}$ ways of choosing $q \wedge p$ elements in $\{\pi(1), \dots, \pi(q)\} \cap \{\rho(q+1), \dots, \rho(q+p)\}$ and $q \wedge p$ elements in $\{\pi(q+1), \dots, \pi(q+p)\} \cap \{\rho(1), \dots, \rho(q)\}$. In addition, there are $q!p!$ ways to organize $\{\rho(1), \dots, \rho(q)\}$ and $\{\rho(q+1), \dots, \rho(q+p)\}$. Therefore, combining the case $q = p$ and $q \neq p$ gives us

$$\sum_{\pi \overset{q\wedge p}{\sim} \rho} A_3 = q!^2 \langle f_{q,i}, f_{q,j} \rangle^2_{\mathfrak{H}^{\otimes q}} \mathbb{1}_{\{q=p\}} + q!p! \binom{q}{q\wedge p} \binom{p}{q\wedge p} \big\| f_{q,i} \star^{q\wedge p}_{q\wedge p} f_{p,j} \big\|^2_{\mathfrak{H}^{\otimes |q-p|}} \mathbb{1}_{\{q\neq p\}}.$$

We now turn to the first sum on the right side of (5.10), that is when $\pi \overset{r}{\sim} \rho$ for $1 \leq r \leq q \wedge p - 1$. We can write

$$\begin{aligned}
A_{1,r} &= \int_{\mathcal{Z}^{q+p-2r}} \left(\int_{Z^r} f_{q,i}\big(z_{\pi(1)}, \dots, z_{\pi(q)}\big) f_{p,j}\big(z_{\rho(q+1)}, \dots, z_{\rho(q+p)}\big) \right) \\
&\qquad\qquad \left(\int_{\mathcal{Z}^r} f_{q,i}\big(z_{\rho(1)}, \dots, z_{\rho(q)}\big) f_{p,j}\big(z_{\pi(q+1)}, \dots, z_{\pi(q+p)}\big) \right) \mu(dz_1 \dots dz_{q+p}) \\
&= \int_{\mathcal{Z}^{q+p-2r}} \big(f_{q,i} \star^r_r f_{p,j}(z_1, \dots, z_{q+p-2r})\big)^2 \mu(dz_1 \dots dz_{q+p-2r}) \\
&= \| f_{q,i} \star^r_r f_{p,j} \|^2_{\mathfrak{H}^{\otimes q+p-2r}} .
\end{aligned}$$

There are $\binom{q}{r}\binom{p}{r}$ ways to choose r elements in $\{\pi(1), \dots, \pi(q)\} \cap \{\rho(q+1), \dots, \rho(q+p)\}$ and r elements in $\{\pi(q+1), \dots, \pi(q+p)\} \cap \{\rho(1), \dots, \rho(q)\}$. Furthermore, there are $q!p!$ ways to organize $\{\rho(1), \dots, \rho(q)\}$ and $\{\rho(q+1), \dots, \rho(q+p)\}$. This yields

$$\sum_{r=1}^{q\wedge p-1} \sum_{\pi \overset{r}{\sim} \rho} A_{1,r} = \sum_{r=1}^{q\wedge p-1} q!p! \binom{q}{r} \binom{p}{r} \| f_{q,i} \star^r_r f_{p,j} \|^2_{\mathfrak{H}^{\otimes q+p-2r}} .$$

Thus, we can expand (5.10) as

$$\left\| f_{q,i} \widetilde{\star}_0^0 f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes q+p}} = \frac{(q+p)!}{(q+p)!^2} \Bigg(q!p! \left\| f_{q,i} \right\|^2_{\mathfrak{H}^{\otimes q}} \left\| f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes p}} + q!^2 \left\langle f_{q,i}, f_{q,j} \right\rangle^2_{\mathfrak{H}^{\otimes q}} \mathbb{1}_{\{q=p\}}$$
$$+ q!p! \binom{q}{q \wedge p} \binom{p}{q \wedge p} \left\| f_{q,i} \star_{q\wedge p}^{q \wedge p} f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes |q-p|}} \mathbb{1}_{\{q \neq p\}}$$
$$+ \sum_{r=1}^{q\wedge p-1} q!p! \binom{q}{r} \binom{p}{r} \left\| f_{q,i} \star_r^r f_{p,j} \right\|^2_{\mathfrak{H}^{\otimes q+p-2r}} \Bigg),$$

which is the desired statement. □

5.4. *Lemmas related to the proof of Theorem 5.2.* Our first lemma expresses the Hilbert-Schmidt norm in a Besov-Liouville space as an norm in $L^2\left([0,1]^{\otimes 2}\right)$.

Lemma 5.10. *Let $K = \mathcal{I}_{\beta,2}$ and S be the covariance operator of a random variable $X \in L^2(\Omega) \otimes K$. Let $f \in K$, then*

$$\left(D^\beta_{0+} S f\right)(s) = \int_0^1 \mathbb{E}\left[\left(D^\beta_{0+} X\right)(r)\left(D^\beta_{0+} X\right)(s)\right]\left(D^\beta_{0+} f\right)(r) dr \tag{5.11}$$

is in $L^2([0,1])$. This leads to

$$\|S\|_{\mathrm{HS}(K)} = \left\| \mathbb{E}\left[\left(D^\beta_{0+} X\right)(r)\left(D^\beta_{0+} X\right)(s)\right] \right\|_{L^2([0,1]^2)}. \tag{5.12}$$

Proof: Let $f, g \in K$. Applying Fubini's theorem to $\langle Sf, g\rangle_K = \mathbb{E}[\langle f, X\rangle_K \langle g, X\rangle_K]$ and rearranging terms yields

$$\int_0^1 \left(D^\beta_{0+} S f\right)(s) \left(D^\beta_{0+} g\right)(s) ds$$
$$= \int_0^1 \left(\int_0^1 \mathbb{E}\left[\left(D^\beta_{0+} X\right)(r)\left(D^\beta_{0+} X\right)(s)\right]\left(D^\beta_{0+} f\right)(r) dr \right) \left(D^\beta_{0+} g\right)(s) ds,$$

which is equivalent to

$$\int_0^1 \left(\left(D^\beta_{0+} S f\right)(s) - \int_0^1 \mathbb{E}\left[\left(D^\beta_{0+} X\right)(r)\left(D^\beta_{0+} X\right)(s)\right]\left(D^\beta_{0+} f\right)(r) dr \right) \left(D^\beta_{0+} g\right)(s) ds = 0. \tag{5.13}$$

Let $\{g_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{I}_{\beta,2}$. Due to the isometry between $\mathcal{I}_{\beta,2}$ and $L^2([0,1])$, the set $\left\{D^\beta_{0+} g_n\right\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2([0,1])$. Then, Equation (5.13) implies (5.11).

To prove (5.12), let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2([0,1])$. Then, $\{e_m \otimes e_n\}_{m,n\in\mathbb{N}}$ is an orthonormal basis of $L^2([0,1]^{\otimes 2})$. Also, $\left\{I^\beta_{0+} e_n\right\}_{n\in\mathbb{N}}$ is a basis of K. Now observe that, using (5.11), we can write

$$\begin{aligned}
\left\langle I^\beta_{0+} e_m, S I^\beta_{0+} e_n \right\rangle_K &= \int_0^1 e_m(s) \left(D^\beta_{0+} S I^\beta_{0+} e_n\right)(s) ds \\
&= \int_0^1 e_m(s) \left(\int_0^1 \mathbb{E}\left[\left(D^\beta_{0+} X\right)(r)\left(D^\beta_{0+} X\right)(s)\right] e_n(r) dr \right) ds \\
&= \int_0^1 \int_0^1 \mathbb{E}\left[\left(D^\beta_{0+} X\right)(r)\left(D^\beta_{0+} X\right)(s)\right] e_m(s) e_n(r) dr ds,
\end{aligned}$$

which leads to

$$\|S\|^2_{\mathrm{HS}(K)} = \sum_{m,n\in\mathbb{N}} \left\langle I^\beta_{0+} e_m, S I^\beta_{0+} e_n \right\rangle_K^2 = \left\| \mathbb{E}\left[\left(D^\beta_{0+} X\right)(r)\left(D^\beta_{0+} X\right)(s)\right] \right\|^2_{L^2([0,1]^{\otimes 2})},$$

where the first equality comes from the identity $\|T\|^2_{\mathrm{HS}(K)} = \sum_{m,n\in\mathbb{N}} \langle k_m, Tk_n\rangle^2_K$ for an operator $T \in \mathrm{HS}(K)$ and an orthonormal basis $\{k_n\}_{n\in\mathbb{N}}$ of K. □

Remark 5.11. Let ζ be a $L^2([0,1])$-valued random variable with covariance operator T. Then the second statement in Lemma 5.10 is comparable to the identity

$$\|T\|_{\mathrm{HS}(L^2([0,1]))} = \|\mathbb{E}[\zeta(r)\zeta(s)]\|_{L^2([0,1]^{\otimes 2})}$$

whenever $T \in \mathrm{HS}\left(L^2\left([0,1]\right)\right)$.

The following lemma is helpful to compute the Hilbert-Schmidt norms of the Poisson process and Brownian motion appearing in Subsection 5.1.

Lemma 5.12. *Let the setting of Subsection 5.1 prevail, where X_λ and Z denoting a Poisson process and a Brownian motion in $\mathcal{I}_{\beta,2}$, respectively. Then, one has*

$$\begin{aligned}\mathbb{E}\Big[\Big(D^\beta_{0+}Z\Big)(r)\Big(D^\beta_{0+}Z\Big)(s)\Big] &= \mathbb{E}\Big[\Big(D^\beta_{0+}X_\lambda\Big)(r)\Big(D^\beta_{0+}X_\lambda\Big)(s)\Big] \\ &= \frac{1}{\Gamma(-\beta+1)^2}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}dx.\end{aligned}$$

Proof: According to Coutin and Decreusefond (2013, Section 3.1), the covariance operator of our Brownian motion is $S' = I^\beta_{0+}I^{1-\beta}_{0+}I^{1-\beta}_{1-}D^\beta_{0+}$. Substituting this into Equation (5.11), we get

$$\Big(D^\beta_{0+}I^\beta_{0+}I^{1-\beta}_{0+}I^{1-\beta}_{1-}D^\beta_{0+}f\Big)(s) = \int_0^1 \mathbb{E}\Big[\Big(D^\beta_{0+}Z\Big)(r)\Big(D^\beta_{0+}Z\Big)(s)\Big]\Big(D^\beta_{0+}f\Big)(r)dr. \tag{5.14}$$

For the left-hand side, note that $f \in \mathcal{I}_{\beta,2}$ implies that $D^\beta_{0+}f \in L^2 \subseteq L^1$, so that $I^{1-\beta}_{0+}I^{1-\beta}_{1-}D^\beta_{0+}f \in L^1$. Thus, $D^\beta_{0+}I^\beta_{0+} = I$ by Samko et al. (1993, Theorem 2.4). Continuing with the left-hand side, we first write out $I^{1-\beta}_{0+}$ using its definition and then perform an integration by part, which yields

$$\begin{aligned}\Big(I^{1-\beta}_{0+}I^{1-\beta}_{1-}D^\beta_{0+}f\Big)(s) =& \frac{1}{\Gamma(1-\beta)}\int_0^1 1_{[0,s]}(r)(s-r)^{-\beta}\Big(I^{1-\beta}_{1-}D^\beta_{0+}f\Big)(r)dr \\ =& \frac{1}{\Gamma(1-\beta)}\int_0^1 I^{1-\beta}_{0+}\Big(1_{[0,s]}(\cdot)(s-\cdot)^{-\beta}\Big)(r)\Big(D^\beta_{0+}f\Big)(r)dr\end{aligned}$$

In particular, the integration by part is valid since Samko et al. (1993, Equation (2.20)) is satisfied for $p = q = 2$ and $0 < \beta < 1/2$. Equation (5.14) then becomes

$$\int_0^1\left(\frac{1}{\Gamma(1-\beta)}I^{1-\beta}_{0+}\Big(1_{[0,s]}(\cdot)(s-\cdot)^{-\beta}\Big)(r) - \mathbb{E}\Big[\Big(D^\beta_{0+}Z\Big)(r)\Big(D^\beta_{0+}Z\Big)(s)\Big]\right)\Big(D^\beta_{0+}f\Big)(r)dr = 0.$$

Now, using a basis argument like the one in the proof of Lemma 5.10 yields

$$\begin{aligned}\mathbb{E}\Big[\Big(D^\beta_{0+}Z\Big)(r)\Big(D^\beta_{0+}Z\Big)(s)\Big] &= \frac{1}{\Gamma(1-\beta)}I^{1-\beta}_{0+}\Big(1_{[0,s]}(\cdot)(s-\cdot)^{-\beta}\Big)(r) \\ &= \frac{1}{\Gamma(1-\beta)^2}\int_0^r (r-x)^{-\beta}(s-x)^{-\beta}1_{[0,s]}(x)dx \\ &= \frac{1}{\Gamma(1-\beta)^2}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}dx.\end{aligned}$$

Next, we will compute $\mathbb{E}\Big[\Big(D^\beta_{0+}X_\lambda\Big)(r)\Big(D^\beta_{0+}X_\lambda\Big)(s)\Big]$. Recall the representation of X_λ given at (5.2). In order to use this representation, we need the joint density of (T_n, T_m). By definition, $T_{m\wedge n}$

and $T_{m\vee n} - T_{m\wedge n}$ are independent and distributed as $\Gamma(m\wedge n,\lambda)$ and $\Gamma(|m-n|,\lambda)$, respectively. Their joint density is hence given by

$$f_{T_{m\wedge n},T_{m\vee n}-T_{m\wedge n}}(x,y) = \frac{\lambda^{m\vee n}}{\Gamma(n\vee m)\Gamma(|m-n|)} x^{n\wedge m-1} y^{|m-n|-1} e^{-\lambda(x+y)}.$$

Since $T_{m\vee n} = T_{m\wedge n} + (T_{m\vee n} - T_{m\wedge n})$, we can write, using a simple change of variable,

$$f_{T_{m\wedge n},T_{m\vee n}}(x,y) = \frac{\lambda^{m\vee n}}{\Gamma(n\wedge m)\Gamma(|m-n|)} x^{n\wedge m-1} (y-x)^{|m-n|-1} e^{-\lambda y} \mathbb{1}_{\{x<y\}}. \tag{5.15}$$

We are now ready to compute $\mathbb{E}\Big[\Big(D^{\beta}_{0^+}X_\lambda\Big)(r)\Big(D^{\beta}_{0^+}X_\lambda\Big)(s)\Big]$. We have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(D^{\beta}_{0^+}X_\lambda\Big)(r)\Big(D^{\beta}_{0^+}X_\lambda\Big)(s)\Big] \\
&\quad= \frac{1}{\lambda\Gamma(-\beta+1)^2}\Bigg(\sum_{n,m\in\mathbb{N}} \mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big] - \frac{\lambda s^{-\beta+1}}{-\beta+1}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(r-T_n)_+^{-\beta}\Big] \\
&\qquad - \frac{\lambda r^{-\beta+1}}{-\beta+1}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(s-T_m)_+^{-\beta}\Big] + \frac{\lambda^2}{(-\beta+1)^2}s^{-\beta+1}r^{-\beta+1}\Bigg) \\
&\quad= \frac{1}{\lambda\Gamma(-\beta+1)^2}\Bigg(\sum_{n\in\mathbb{N}} \mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_n)_+^{-\beta}\Big] + \sum_{n\in\mathbb{N}}\sum_{m\neq n}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big] \\
&\qquad - \frac{\lambda}{-\beta+1}s^{-\beta+1}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(r-T_n)_+^{-\beta}\Big] - \frac{\lambda}{-\beta+1}r^{-\beta+1}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(s-T_m)_+^{-\beta}\Big] \\
&\qquad + \frac{\lambda^2}{(-\beta+1)^2}s^{-\beta+1}r^{-\beta+1}\Bigg).
\end{aligned} \tag{5.16}$$

The first sum on the right side (consisting of all diagonal terms when $m=n$) simplifies as

$$\begin{aligned}
&\frac{1}{\lambda\Gamma(-\beta+1)^2}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(t-T_n)_+^{-\beta}(s-T_n)_+^{-\beta}\Big] \\
&\quad= \frac{1}{\lambda\Gamma(-\beta+1)^2}\sum_{n\in\mathbb{N}}\int_0^\infty (r-x)_+^{-\beta}(s-x)_+^{-\beta}\frac{\lambda^n}{\Gamma(n)}x^{n-1}e^{-\lambda x}dx \\
&\quad= \frac{1}{\Gamma(-\beta+1)^2}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}e^{-\lambda x}\Bigg(\sum_{n\in\mathbb{N}}\frac{(\lambda x)^{n-1}}{(n-1)!}\Bigg)dx \\
&\quad= \frac{1}{\Gamma(-\beta+1)^2}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}dx.
\end{aligned}$$

Next, we consider the second sum on the right side of (5.16). The joint density of (T_n, T_m) given in (5.15) enables us to write

$$\begin{aligned}
&\sum_{n\in\mathbb{N}}\sum_{m\neq n}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big]\\
&=\sum_{n\in\mathbb{N}}\sum_{m=n+1}^{\infty}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big]+\sum_{m\in\mathbb{N}}\sum_{m+1}^{\infty}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big]\\
&=\lambda^2\int_0^{r\wedge s}\int_0^{s}(r-x)^{-\beta}(s-y)^{-\beta}e^{-\lambda y}\sum_{n\in\mathbb{N}}\sum_{m=n+1}^{\infty}\frac{(\lambda x)^{n-1}(\lambda y-\lambda x)^{m-n-1}}{\Gamma(n)\Gamma(m-n)}dydx\\
&+\lambda^2\int_0^{r\wedge s}\int_0^{r}(s-x)^{-\beta}(r-y)^{-\beta}e^{-\lambda y}\sum_{m\in\mathbb{N}}\sum_{n=m+1}^{\infty}\frac{(\lambda x)^{m-1}(\lambda y-\lambda x)^{n-m-1}}{\Gamma(m)\Gamma(n-m)}dydx
\end{aligned}$$

By letting $k=m-n$, it is easy to see that

$$\sum_{n\in\mathbb{N}}\sum_{m=n+1}^{\infty}\frac{(\lambda x)^{n-1}(\lambda y-\lambda x)^{m-n-1}}{\Gamma(n)\Gamma(m-n)}=e^{\lambda y},$$

and hence

$$\begin{aligned}
&\sum_{n\in\mathbb{N}}\sum_{m\neq n}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big]\\
&=\lambda^2\int_0^{r\wedge s}\int_0^{s}(r-x)^{-\beta}(s-y)^{-\beta}dydx+\lambda^2\int_0^{r\wedge s}\int_0^{r}(s-x)^{-\beta}(r-y)^{-\beta}dydx\\
&=\frac{\lambda^2}{-\beta+1}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}(s+t-2x)dx\\
&=\frac{\lambda^2}{(-\beta+1)^2}(s-x)^{-\beta+1}(r-x)^{-\beta+1}\Big|_0^{r\wedge s}\\
&=\frac{\lambda^2}{(-\beta+1)^2}\Big(s^{-\beta+1}r^{-\beta+1}-(s-r\wedge s)^{-\beta+1}(r-r\wedge s)^{-\beta+1}\Big)\\
&=\frac{\lambda^2}{(-\beta+1)^2}s^{-\beta+1}r^{-\beta+1}.
\end{aligned}$$

For the remaining sums in (5.16), observe that

$$\sum_{m\in\mathbb{N}}\mathbb{E}\Big[(s-T_m)_+^{-\beta}\Big]=\frac{\lambda}{-\beta+1}s^{-\beta+1}.$$

Substituting the previous calculations into (5.16) yields

$$\mathbb{E}\Big[\Big(D_{0+}^{\beta}X_\lambda\Big)(r)\Big(D_{0+}^{\beta}X_\lambda\Big)(s)\Big]=\frac{1}{\Gamma(-\beta+1)^2}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}dx.$$

□

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