ALEA, Lat. Am. J. Probab. Math. Stat. 21, 791–813 (2024)
DOI: 10.30757/ALEA.v21-32

Scaling limits for the random walk penalized by its range in dimension one

Nicolas Bouchot

LPSM, Sorbonne Université, UMR 8001, Campus Pierre et Marie Curie, Boîte courrier 158, 4 Place Jussieu, 75252 Paris Cedex 05, France.
E-mail address: nicolas.bouchot@sorbonne-universite.fr

Abstract. In this article we study a one dimensional model for a polymer in a poor solvent: the random walk on $\mathbb{Z}$ penalized by its range. More precisely, we consider a Gibbs transformation of the law of the simple symmetric random walk by a weight $\exp(-h_n|\mathcal{R}_n|)$, with $|\mathcal{R}_n|$ the number of visited sites and h_n a size-dependent positive parameter. We use gambler's ruin estimates to obtain exact asymptotics for the partition function, that enables us to obtain a precise description of trajectories, in particular scaling limits for the center and the amplitude of the range. A phase transition for the fluctuations around an optimal amplitude is identified at $h_n \asymp n^{1/4}$, inherent to the underlying lattice structure.

1. Introduction of the model and main results

Consider a simple symmetric random walk $(S_k)_{k\geq 0}$ on $\mathbb{Z}^d$, $d \geq 1$, starting from 0, with law denoted $\mathbb{P}$. For $h > 0$, we define the following Gibbs transformation of $\mathbb{P}$, called the *polymer measure*

$$\mathrm{d}\mathbb{P}_{n,h}(S) = \frac{1}{Z_{n,h}} e^{-h|\mathcal{R}_n(S)|} \mathrm{d}\mathbb{P}(S)\,,$$

where $\mathcal{R}_n(S) := \{S_0, \dots, S_n\}$ is the range of the random walk up to time n and $|\cdot|$ is the cardinal measure. The normalizing quantity

$$Z_{n,h} = \mathbb{E}\Big[e^{-h|\mathcal{R}_n(S)|}\Big]$$

is called the *partition function* and is such that $\mathbb{P}_{n,h}$ is a probability measure on the space of trajectories of length n.

The random walk is the "correct" mathematical tool to study polymers, which we briefly explain here. In physics, a polymer is a long chain of n comparatively small molecules called monomers, which can be modeled as a simple path of given length n on $\mathbb{Z}^d$. In this model, the random variable S_i is the location of the i-th monomer and the polymer is the random walk trajectory with law $\mathbb{P}_{n,h}$. To model physical interactions of the polymer with its environment and study typical behavior, the usual tool is to define $\mathbb{P}_{n,h}$ as a Gibbs measure on this set of paths. The reference measure

Received by the editors July 20th, 2022; accepted March 20th, 2024.
2010 *Mathematics Subject Classification.* 60G50, 82B41, 60C05.
Key words and phrases. random walk, directed polymer, confined walk, gambler's ruin.

from which we define the Gibbs transformation is taken to be uniform over these path, meaning the random walk law.

In any dimension $d \geq 1$, the asymptotics for the log-partition function are known since Donsker and Varadhan (1979). These asymptotics strongly suggest that a polymer of length n will typically fold in (and fill up) a ball of radius $\rho n^{\frac{1}{d+2}}$ for some specific constant $\rho = \rho(d,h)$. This has been proved by Bolthausen (1994) in dimension $d = 2$, but only much more recently in dimension $d \geq 3$, by Berestycki and Cerf (2021) and Ding, Fukushima, Sun and Xu (see Ding et al., 2020). More precisely, for $h = 1$ (easily generalized to any $h > 0$), they prove that there exists a positive ρ_d, which only depends on the dimension d, such that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}_{n,1}\Big(\exists x \in \mathbb{R}^d, B\big(x,(1-\varepsilon)\rho_d n^{\frac{1}{d+2}}\big) \cap \mathbb{Z}^d \subset \mathcal{R}_n \subset B\big(x,(1+\varepsilon)\rho_d n^{\frac{1}{d+2}}\big)\Big) = 1\,,$$

where $B(x,r)$ is the d-dimensional Euclidean ball centered at x with radius r.

In dimension $d = 1$, this is much easier since the range is uniquely determined by its two endpoints (and always fills completely the one-dimensional ball). This allows for more explicit calculations using mostly gambler's ruin estimates. In particular, one easily derives that $n^{-1/3}|\mathcal{R}_n|$ converges to $\big(\frac{\pi^2}{h}\big)^{1/3}$ in $\mathbb{P}_{n,h}$-probability.

1.1. *Outline of the paper.* In the current work, we focus only on the case of dimension $d = 1$. Also, we allow the penalization intensity to depend on the length of the polymer, meaning $h = h_n$ now depends on n. We exploit gambler's ruin estimates to their full potential and derive exact asymptotics for the partition function (not only for the log-partition function). Afterwards, we will be able to prove a scaling limit (actually we prove a local limit theorem) for the joint law of the center W_n and the amplitude T_n of the range:

$$T_n := \max_{k\leq n} S_k - \min_{k\leq n} S_k = |\mathcal{R}_n| - 1, \qquad W_n := \frac{T_n}{2} + \min_{k\leq n} S_k = \frac{1}{2}\Big(\max_{k\leq n} S_k + \min_{k\leq n} S_k\Big).$$

For the sake of the exposition, let us consider the case

$$\lim_{n\to\infty} n^{-\gamma} h_n = \hat{h} \in (0,+\infty)\,, \qquad \text{for some } \gamma \in \mathbb{R}\,. \tag{1.1}$$

Some results are already presented in Berger et al. (2022a) which considers a disordered version of the model:

(i) if $\gamma < -\frac{1}{2}$ then $\mathbb{P}_{n,h_n}$ converges to $\mathbb{P}$ in total variation;
(ii) if $\gamma \in (-\frac{1}{2}, 1)$ then $(\frac{n\pi^2}{h_n})^{-1/3} T_n$ converges to 1 in $\mathbb{P}_{n,h_n}$-probability;
(iii) if $\gamma > 1$ then $\mathbb{P}_{n,h_n}$ is concentrated on trajectories visiting only two sites.

Since cases (i) and (iii) are degenerate, we focus on the case $\gamma \in (-\frac{1}{2}, 1)$. In this paper, we give another proof of the convergence $(\frac{n\pi^2}{h_n})^{-1/3} T_n \to 1$ and we additionally identify the fluctuations of $T_n - (\frac{n\pi^2}{h_n})^{1/3}$. We find that a phase transition occurs at $\gamma = \frac{1}{4}$ for the fluctuations:

(i) if $\gamma < \frac{1}{4}$ then the fluctuations, normalized by $(\frac{n}{h_n^4})^{1/6}$, converge to a Gaussian variable;
(ii) if $\gamma > \frac{1}{4}$ then the range penalization is strong enough to collapse the range on $(\frac{n\pi^2}{h_n})^{1/3}$ in the sense that the fluctuations live on a finite set (of cardinality 1, sometimes 2).

We will also prove that $(\frac{n\pi^2}{h_n})^{-1/3} W_n$ converges to a random variable with density $\frac{\pi}{2}\cos(\pi u)\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(u)$ with respect to the Lebesgue measure and is independent of the fluctuations. This type of results appears to be folklore for confined polymers, as the density is the eigenfunction associated with the principal Dirichlet eigenvalue of the Laplacian on $[0,1]$ (see e.g. den Hollander (2009, Ch. 8)), but we are not aware of a proof written in detail (at least for the random walk penalized by its range).

Notations. In the rest of the paper we shall use the standard notations: as $x \to a$, we write $g(x) \sim f(x)$ if $\lim_{x\to a} \frac{g(x)}{f(x)} = 1$, $g(x) = \bar{o}(f(x))$ if $\lim_{x\to a} \frac{g(x)}{f(x)} = 0$, $g(x) = \bar{\mathcal{O}}(f(x))$ if $\limsup_{x\to a} \left|\frac{g(x)}{f(x)}\right| < +\infty$ and $f \asymp g$ if $g(x) = \bar{\mathcal{O}}(f(x))$ and $f(x) = \bar{\mathcal{O}}(g(x))$. When a is not specified, these notations are used with $x \to +\infty$.

We also extensively use the following notation: for $\mathcal{A}$ an event, we denote

$$Z_{n,h_n}(\mathcal{A}) := \mathbb{E}\Big[e^{-h_n|\mathcal{R}_n(S)|}\mathbb{1}_{\{S\in\mathcal{A}\}}\Big],$$

so that in particular $\mathbb{P}_{n,h_n}(\mathcal{A}) = \frac{1}{Z_{n,h_n}} Z_{n,h_n}(\mathcal{A})$.

1.2. *Main results.* The following two theorems summarize our results, the first being the main result regarding the asymptotic behavior of (T_n, W_n) and the second being asymptotics for Z_{n,h_n} that have a use of their own.

We define the following quantities, that will be used throughout the paper:

$$T_n^* = T_n^*(h_n) := \left(\frac{n\pi^2}{h_n}\right)^{1/3}, \qquad a_n = a_n(h_n) := \frac{1}{\sqrt{3}}\left(\frac{n\pi^2}{h_n^4}\right)^{1/6} = \frac{1}{\sqrt{3n\pi^2}}(T_n^*)^2. \tag{1.2}$$

Note that $\lim_{n\to\infty} a_n = +\infty$ if and only if $\lim_{n\to\infty} n^{-1/4}h_n = 0$.

Theorem 1.1. • *Assume that $h_n \geq n^{-1/2}(\log n)^{3/2}$ and $\lim_{n\to\infty} n^{-1/4}h_n = 0$; in other words, $\gamma \in (-\frac{1}{2}, \frac{1}{4})$ in (1.1). Then under $\mathbb{P}_{n,h_n}$, we have the following convergence in distribution*

$$\left(\frac{T_n - T_n^*}{a_n}\,;\,\frac{W_n}{T_n^*}\right) \xrightarrow[n\to+\infty]{(d)} (\mathcal{T}, \mathcal{W}),$$

where the random variables $\mathcal{T}$ and $\mathcal{W}$ are independent with $\mathcal{T} \sim \mathcal{N}(0,1)$ and $\mathcal{W}$ with density given by $\frac{\pi}{2}\cos(\pi u)\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(u)$.

• *Assume that $\lim_{n\to\infty} n^{-1/4}h_n = +\infty$ and $\lim_{n\to\infty} n^{-1}h_n = 0$; in other words, $\gamma \in (\frac{1}{4}, 1)$ in (1.1). Then,*

$$\lim_{n\to\infty} \mathbb{P}_{n,h_n}\big(T_n - \lfloor T_n^* - 2\rfloor \notin \{0,1\}\big) = 0\,.$$

Also, under $\mathbb{P}_{n,h_n}$ we have the convergence in distribution $\frac{W_n}{T_n^*} \xrightarrow{(d)} \mathcal{W}$.

Remark 1.2. The term $a_n = \frac{1}{\sqrt{3n\pi^2}}(T_n^*)^2$ in Theorem 1.1 arises naturally as a Taylor expansion coefficient in the exponential part of the partition function after injecting gambler's ruin formulae, see Section 1.4 below.

The assumption that $h_n \geq n^{-1/2}(\log n)^{3/2}$ is due to technicalities in the proof of Theorem 1.6 below and gambler's ruin formulae.

It should be noted that when $n^{-1/4}h_n \to 0$, we have $a_n \to +\infty$, while we have $a_n \to 0$ if $n^{-1/4}h_n \to \infty$. The condition $h_n n^{-1} \to 0$ ensures that $T_n^* \to +\infty$, meaning the range is still growing with n.

Theorem 1.3. *We have the following exact asymptotics:*

• *Assume that $h_n \geq n^{-1/2}(\log n)^{3/2}$ and $\lim_{n\to\infty} n^{-1/4}h_n = 0$; in other words, $\gamma \in (-\frac{1}{2}, \frac{1}{4})$ in (1.1). Then, as $n \to \infty$,*

$$Z_{n,h_n} = (1 + \bar{o}(1))\frac{16\sqrt{2}}{\sqrt{3\pi}}\Big(\frac{\cosh(h_n) - 1}{h_n}\Big)\sqrt{n}\,\exp\Big(-\frac{3}{2}h_n T_n^*\Big)\,.$$

- *Assume that* $\lim_{n\to\infty} n^{-1/4}h_n = +\infty$ *and* $\lim_{n\to\infty} n^{-1}h_n = 0$*; in other words,* $\gamma \in (\frac{1}{4}, 1)$ *in* (1.1). *Define* $t_n^o = T_n^* - \lfloor T_n^* \rfloor$ *the decimal part of* T_n^**. Then, as* $n \to \infty$*,*

$$Z_{n,h_n} = \frac{16}{\pi^{4/3}}\Big(\frac{\cosh(h_n)-1}{h_n^{1/3}}\Big)n^{1/3}e^{-\frac{3}{2}h_nT_n^*}\sum_{t\in\{0,1\}}\exp\Big(-\Phi_n(t)\frac{n}{(T_n^*)^4}(1+\bar{o}(1))\Big). \tag{1.3}$$

with

$$\Phi_n(t) := \frac{\pi^4}{12} + \frac{3\pi^2}{2}\varsigma_n(t), \quad \varsigma_n(t) := (t-t_n^o)^2 - \frac{2\pi^2}{9}\frac{t-t_n^o}{T_n^*}\mathbb{1}_{\{0,1\}}(t).$$

Note that $\lim_{n\to\infty}\frac{n}{(T_n^*)^4} = +\infty$ *means* $\bar{o}(\frac{n}{(T_n^*)^4})$ *could still diverge.*

Remark 1.4. If we assume that there is a $\delta > 0$ such that for all n large enough we have $t_n^o \leq \frac{1}{2} - \delta$, then $\mathbb{P}_{n,h_n}\big(T_n - \lfloor T_n^* - 2\rfloor = 0\big) \to 1$. If on the other hand $t_n^o \geq \frac{1}{2} + \delta$ we have $\mathbb{P}_{n,h_n}\big(T_n - \lfloor T_n^* - 2\rfloor = 1\big) \to 1$ instead. This is due to the fact that with these assumptions, one of the terms in (1.3) dominates the other.

For the sake of completeness, we add the following result concerning the critical case $\lim_{n\to\infty} n^{-\frac{1}{4}}h_n = \hat{h} \in (0, +\infty)$.

Proposition 1.5. *Suppose that* $\lim_{n\to\infty} n^{-1/4}h_n = \hat{h} \in (0,+\infty)$*, so in particular we have* $\lim_{n\to\infty} a_n = \frac{\pi^{1/3}}{\sqrt{3}\hat{h}^{2/3}} =: a$*. Then, as* $n \to \infty$*, we have*

$$Z_{n,h_n} = (1+\bar{o}(1))\frac{16}{\pi^{4/3}}\Big(\frac{\cosh(h_n)-1}{h_n^{1/3}}\Big)n^{1/3}e^{-\frac{3}{2}h_nT_n^*}\theta_n(a), \quad \textit{with } \theta_n(a) := \sum_{t=-\infty}^{+\infty} e^{-\frac{\varsigma_n(t)}{2a^2}}.$$

Furthermore, for any integers $r \leq s$*, as* $n \to \infty$ *we have*

$$\mathbb{P}_{n,h_n}\Big(r \leq T_n - \lfloor T_n^* - 2\rfloor \leq s\Big) = (1+\bar{o}(1))\frac{1}{\theta_n(a)}\sum_{t=r}^{s} e^{-\frac{\varsigma_n(t)}{2a^2}}.$$

1.3. *Range's endpoints and confinement estimates.* Let us now state some estimates for the probability that the range of a random walk is exactly a given interval. The proof is postponed to Section 4 and follows from gambler's ruin estimates that can be found in Feller (1968, Chap. XIV).

Let x, y be two non-negative integers and denote by $E_x^y(n)$ the following event

$$E_x^y(n) := \big\{\mathcal{R}_n = [\![-x, y]\!]\big\} = \big\{M_n^- = -x\,,\, M_n^+ = y\big\},$$

where we also introduced $M_n^- := \min_{k\leq n} S_k$, $M_n^+ := \max_{k\leq n} S_k$, and used the standard notation $[\![a,b]\!] = [a,b] \cap \mathbb{Z}$. We also define the following function g, that encodes the exponential decay rate of confinement probabilities inside a strip:

$$g(T) := -\log\cos\left(\frac{\pi}{T}\right) = \frac{\pi^2}{2T^2} + \frac{\pi^4}{12T^4} + \bar{O}(T^{-6}) \quad \text{as } T \to \infty. \tag{1.4}$$

The main result used in the rest of the paper is the following. It is based on sharp gambler's ruin estimates, see Lemmas 4.2-4.4 in Section 4.

Theorem 1.6. *For any positive* $T = T(n) \to +\infty$ *with* $n \geq \frac{1}{4}T^2\log T$*, we have:*

$$\lim_{n\to\infty} \sup_{\substack{x,y\in\mathbb{N}\\ x+y=T}} \left|\frac{\mathbb{P}\big(E_x^y(n)\big)}{\Theta_n(x,y)} - 1\right| = 0, \tag{1.5}$$

where we defined the function $\Theta_n(x,y)$ *for* $x+y=T$ *as*

$$\Theta_n(x,y) := \begin{cases} \frac{4}{\pi}\sin\left(\frac{\pi(x+1)}{T}\right)e^{-g(T+2)n} & \text{if } \frac{n}{T^3}\to+\infty; \\ \frac{4}{\pi}(e^{\alpha\pi^2}-1)\left[e^{\alpha\pi^2}\sin\left(\frac{\pi(x+1)}{T}\right)-\sin\left(\frac{\pi x}{T}\right)\right]e^{-g(T)n} & \text{if } \frac{n}{T^3}\to\alpha\in(0,+\infty); \\ \frac{2\pi^3n^2}{T^6}\sin\left(\frac{\pi x}{T}\right)\left[2+\frac{\pi}{T\tan\frac{x\pi}{T}}+\frac{T^2}{n}\frac{1-\frac{2x}{T}}{\pi\tan\frac{x\pi}{T}}\right]e^{-g(T+1)n} & \text{if } \frac{n}{T^3}\to 0\,. \end{cases}$$

Remark 1.7. The condition $n \geq \frac{1}{4}T^2\log T$ is a technicality required to neglect small gambler's ruin contributions. In the proof of Theorem 1.6 we have two terms that appear: a $\bar{\mathcal{O}}(T^{-2})$ and a $\bar{\mathcal{O}}(e^{-\frac{n\pi^2}{T^2}})$. This condition helps getting rid of the second term which is used to get the definitive asymptotics given by $\Theta_n(x,y)$. In Theorem 1.1, the condition $h_n \geq n^{-1/2}(\log n)^{3/2}$ ensures that $\frac{1}{4}(T_n^*)^2\log T_n^* \leq n$ so we can apply Theorem 1.6.
Note that $\frac{1}{4}(T_n^*)^2\log T_n^* \leq n$ also means that in Theorem 1.6, since the range up to time n is typically of size $\sqrt{n}$, the events $E_x^y(n)$ that we consider are rare events for the random walk.

Remark 1.8. For the rest of the paper we will prefer to write $\mathbb{P}\big(E_x^y(n)\big) = (1+\bar{o}(1))\Theta_n(x,y)$ with $\bar{o}(1)$ uniform in x,y and only depending on $T=x+y$, in the sense of (1.5).

We can summarize these results in a more compact way, if we exclude the case where $\frac{x}{T}$ is close to 0 when $\frac{n}{T^3}\to 0$ (more precisely if $x \lesssim \frac{T^3}{n} = \bar{o}(T)$):

$$\mathbb{P}\big(E_x^y(n)\big) = \psi\Big(\frac{n\pi^2}{T^3}\Big)\left[\sin\left(\frac{\pi x}{T}\right)+\bar{o}(1)\right]e^{-g(T+2)n}, \quad \text{with } \psi(r) := \frac{4}{\pi}(1-e^{-r})^2. \tag{1.6}$$

Here, the $\bar{o}(1)$ is uniform in x,y and depends only on $T=x+y$ satisfying $n\geq\frac{1}{4}T^2\log T$, in the spirit of (1.5). To get (1.6), we have used in particular that $g(T+2)n - g(T)n \sim -\frac{2\pi^2 n}{T^3}$ as $T\to\infty$, which converges to $2\alpha\pi^2$ if $\lim_{n\to\infty}\frac{n}{T^3}=\alpha\in[0,+\infty)$.

In what follows, we will always use (1.6) instead of Theorem 1.6 when it is possible. The contribution to the partition function of the trajectories in $E_x^y(n)$ with $x\lesssim\frac{T^3}{n}$ (for which (1.6) is not valid) are examined separately.

Remark 1.9. Whenever $x=0$ (or $y=0$ using symmetry) we have the same Theorem 1.6 applied to $x=0$, except when $\frac{n}{T^3}\to 0$ in which case we take instead

$$\Theta_n(0,T) = \Theta_n(T,0) = \frac{4n\pi}{T^3}\sin\left(\frac{\pi}{T+2}\right)e^{-g(T+1)n}\,. \tag{1.7}$$

This will not be significant starting from Section 2.2 as it only consists of two not-so-peculiar range configurations among the *many* configurations in the partition function. We refer to Section 4.2.4 for the proof of this claim.

Let us stress that one easily deduces from Theorem 1.6 the following statement, leading to the asymptotic independence in Theorem 1.1, as well as the convergence in distribution of $\frac{W_n}{T_n^*}$ to $\mathcal{W}$.

Proposition 1.10. *Let* $(t_n)_{n\geq1}$ *be any sequence of integers such that* $t_n\to+\infty$ *and* $n\geq\frac{1}{4}t_n^2\log t_n$*. Then, conditioning on* $T_n=t_n$*,* $\frac{W_n}{t_n}$ *converges in distribution to* $\mathcal{W}$*.*

More precisely, we have the following local limit convergence: uniformly for w *such that* $2w\in[\![-t_n,t_n]\!]$*, or* $2w\in[\![-(1-\varepsilon)t_n,(1-\varepsilon)t_n]\!]$ *for some* $\varepsilon>0$ *if* $nt_n^{-3}\to0$*, we have*

$$\mathbb{P}\Big(W_n = w\,|\,T_n=t_n\Big) = \frac{\pi}{2t_n}\left[\cos\left(\frac{w\pi}{t_n}\right)+\bar{o}(1)\right] \quad \text{as } n\to\infty\,.$$

Note that this proposition allows us to focus our study on T_n instead of (T_n, W_n).

Proof: For $-\frac{1}{2} \le a \le b \le \frac{1}{2}$, we get thanks to (1.6) that

$$\mathbb{P}\Big(a \le \frac{W_n}{t_n} \le b\,; T_n = t_n\Big) = \sum_{\substack{x+y=t_n \\ 2at_n \le y-x \le 2bt_n}} \mathbb{P}\big(E_x^y(n)\big)$$

$$= \frac{4}{\pi}\Big(1 - e^{-\frac{n\pi^2}{t_n^3}}\Big)^2 e^{-g(t_n+2)n} \sum_{2at_n \le 2w \le 2bt_n} \left[\cos\left(\frac{w\pi}{t_n}\right) + \bar{o}(1)\right],$$

where we have set $w = w(x,y) := \frac{y-x}{2}$. Similarly,

$$\mathbb{P}\big(T_n = t_n\big) = \frac{4}{\pi}\Big(1 - e^{-\frac{n\pi^2}{t_n^3}}\Big)^2 e^{-g(t_n+2)n} \sum_{-t_n \le 2w \le t_n} \left[\cos\left(\frac{w\pi}{t_n}\right) + \bar{o}(1)\right].$$

These sums being Riemann sums, we therefore end up with

$$\mathbb{P}\Big(a \le \frac{W_n}{t_n} \le b \,\Big|\, T_n = t_n\Big) = \frac{\sum_{at_n \le w \le bt_n}\left[\cos(\frac{w\pi}{t_n}) + \bar{o}(1)\right]}{\sum_{-t_n \le 2w \le t_n}\left[\cos(\frac{w\pi}{t_n}) + \bar{o}(1)\right]} \xrightarrow[n\to\infty]{} \frac{\pi}{2}\int_a^b \cos(\pi u)\,\mathrm{d}u\,.$$

Taking $a = b = w/t_n$, the denominator is a Riemman sum and thus

$$\mathbb{P}\Big(W_n = w \,|\, T_n = t_n\Big) = \frac{t_n^{-1}\cos(\frac{w\pi}{t_n}) + \bar{o}(t_n^{-1})}{t_n^{-1}\sum_{-t_n \le 2w \le t_n}\left[\cos(\frac{w\pi}{t_n}) + \bar{o}(1)\right]} = \frac{\pi}{2t_n}\left[\cos\left(\frac{w\pi}{t_n}\right) + \bar{o}(1)\right].$$

When $\frac{n}{T^3} \to 0$, recall that the formula (1.6) fails for $|M_n^-| = \bar{o}(t_n)$. By taking $W_n = w$ with $2w \in [\![-(1-\varepsilon)t_n, (1-\varepsilon)t_n]\!]$ we have $|M_n^-| \ge \varepsilon t_n$ and thus (1.6) can be applied. Thus, with the same proof we get the result and since $\varepsilon > 0$ is arbitrary we get the convergence in distribution. □

1.4. *Some heuristics.* Let us present some heuristics for obtaining the asymptotics of the partition function, and explain how the quantities T_n^* and a_n (recall (1.2)) appear. We can decompose the partition function as

$$Z_{n,h_n} = \sum_{x,y\ge 0} e^{-h_n(T+1)}\mathbb{P}\big(E_x^y(n)\big)\,,$$

where we have set $T = T(x,y) = x+y$. In view of Theorem 1.6, we have $\mathbb{P}\big(E_x^y(n)\big) = u_n(x,y)e^{-g(T)n}$ with $g(T) = (1+\bar{o}(1))\frac{\pi^2}{2T^2}$. Hence, the main contribution to the sum will come from x, y with T that is close to minimizing the function

$$\phi_n(T) := h_n T + \frac{n\pi^2}{2T^2}\,. \tag{1.8}$$

Then, notice that ϕ_n is minimal at $T = T_n^* := \big(\frac{n\pi^2}{h_n}\big)^{1/3}$ (recall (1.2)) and that

$$\phi_n(T_n^*) = \frac{3\pi^{1/3}}{2}n^{1/3}h_n^{2/3} = \frac{3}{2}h_n T_n^*.$$

Let us now factorize $e^{\phi_n(T_n^*)}$ (and e^{h_n}) in the sum above, to get that

$$e^{\frac{3}{2}h_n T_n^*}e^{h_n}Z_{n,h_n} \approx \sum_{x,y\ge 0} u_n(x,y)\exp\big(-(\phi_n(T) - \phi_n(T_n^*))\big)\,.$$

Now, since $\phi_n'(T_n^*) = 0$, we have $\phi_n(T) \approx \phi_n(T_n^*) + \frac{1}{2}(T - T_n^*)^2\phi_n''(T_n^*)$, with $\phi_n''(T_n^*) = \frac{3n\pi^2}{(T_n^*)^4} = \frac{1}{a_n^2}$ (recall (1.2)). In the sum above the main contribution therefore comes from values of T that are such that $\phi_n(T) - \phi_n(T_n^*)$ is at most of order 1, that is with $T - T_n^* = \bar{\mathcal{O}}(a_n)$.

1.5. *Further comments on the results.* Theorem 1.1 states that asymptotically, the polymer behaves as a random walk whose range's size T_n fluctuates around the optimal $T_n^* = \left(\frac{n\pi^2}{h_n}\right)^{1/3}$. If $h_n n^{-1/4} \to 0$ (weak penalization), then the fluctuations are Gaussian at a scale $a_n = \frac{1}{\sqrt{3}}\left(\frac{n}{h_n^4}\right)^{1/6}$. On the other hand, if $h_n n^{-1/4} \to \infty$ (strong penalization), then the fluctuations vanish and T_n is equal to either $\lfloor T_n^* \rfloor - 2$ or $\lfloor T_n^* \rfloor - 1$.

Recall that if $\lim\limits_{n\to\infty} n^{-1/4}h_n = 0$ then $\lim\limits_{n\to\infty} a_n = +\infty$, whereas if $\lim\limits_{n\to\infty} n^{-1/4}h_n = +\infty$ then $\lim\limits_{n\to\infty} a_n = 0$. Theorem 1.1 states that a_n is the scale of the Gaussian fluctuations of T_n around T_n^*, with $a_n \to 0$ corresponding to zero fluctuation (after a slight correction on T_n^*). The condition $\lim\limits_{n\to\infty} n^{-1}h_n = 0$ ensures that $\lim\limits_{n\to\infty} T_n^* = +\infty$ meaning that in both cases, the range grows with n while the fluctuations undergo a phase transition at $h_n \asymp n^{1/4}$.

In both cases, the relative position of the center of the range is asymptotically independent of its size, with distribution given by the density $\frac{\pi}{2}\cos(\pi u)\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(u)$, conjectured or discussed in previous works (see den Hollander (2009, Theorem 8.3) for example) but with no concrete proof (to the best of our knowledge).

1.5.1. *Continuous analogue of the model.* One can easily see the similarities between this polymer model and the study of the Brownian motion penalized by the amplitude of its trajectory. For a Brownian motion β, define $|C_T| := |\{\beta_t : t \le T\}|$ its amplitude at time T (here $|\cdot|$ is the Lebesgue measure). Then, Donsker and Varadhan (1975) proved that

$$\lim_{T\to\infty} \frac{1}{T^{1/3}} \log \mathbb{E}\left[e^{-\nu|C_T|}\right] = -\frac{3}{2}(\nu\pi)^{2/3}.$$

Schmock later expanded on this result in Schmock (1990) and obtained that the associated Gibbs measures $\mathbb{P}_{T,\nu}(\mathrm{d}\omega) = e^{-\nu|C_T|}\mathbb{W}(\mathrm{d}\omega)$ (with $\mathbb{W}$ the Wiener measure) converge weakly to a measure $\mathbb{P}_{\infty,\nu}$ given by

$$\mathbb{P}_{\infty,\nu}(A) = \int_0^{c_\nu} \frac{\pi}{2c_\nu} \sin\left(\frac{\pi u}{c_\nu}\right) P_{u-c_\nu,u}(A)\,\mathrm{d}u\,,$$

with $c_\nu = (\pi^2/\nu)^{1/3}$, where $P_{u-c_\nu,u}$ denotes the path measure of a Brownian taboo process with taboo set $\{u-c_\nu, u\}$. In other words, $\mathbb{P}_{\infty,\nu}$ is a mixture of taboo processes $P_{u-c_\nu,u}$, which correspond to the actual diffusion process conditioned to stay in an interval of length c_ν and upper edge u; additionally, the mixing measure selecting the upper edge u is identical to $\mathcal{W}$ in Theorem 1.1 (if one selects the center of the range rather than the upper edge). This is therefore completely analogous to our Theorem 1.1.

However, because there is no underlying lattice, the continuous case should not display a transition for the fluctuations at $\nu = \nu_T \asymp T^{1/4}$: when $\lim_{T\to\infty} T^{-1/4}\nu_T = +\infty$, fluctuations become $\bar{o}(1)$ but still remain Gaussian after a proper scaling. Let us also stress that in the continuous case, well-known results such as Lévy triple law (see Schilling and Partzsch, 2014, Theorem 6.18) allow for relatively simple computations of the law of the endpoint β_T for a large T conditioning on the range's endpoints — which Theorem 1.6 does not provide in our setting, we only get the position of the starting point relative to the range, see Proposition 1.10. Obtaining a result for the starting and endpoint for our model would require the joint law of (M_n^-, M_n^+, S_n) or a study based on local times of the polymer, which are both beyond the scope of this paper.

1.5.2. *Other related models.* Related models for self-interacting polymers have been studied in the literature these past years. We mention here two of these models and their recent advancements.

First, one can consider a disordered version of the random walk penalized by its range, *i.e.* the case where the penalization by the range is perturbed by a random environment. Take a collection of i.i.d variables $(\omega_z)_{z\in\mathbb{Z}}$ and consider the random polymer measure

$$\mathrm{d}\mathbb{P}^{\omega,\beta}_{n,h}(S) = \frac{1}{Z^{\beta,\omega}_{n,h}} \exp\Big(\sum_{z\in\mathcal{R}_n(S)} \big(\beta\omega_z - h\big)\Big) \mathrm{d}\mathbb{P}(S)\,,$$

in particular $\mathbb{P}_{n,h} = \mathbb{P}^{\omega,0}_{n,h}$. This quenched model was studied in Berger et al. (2022a,b); Huang (2019), for size-dependent parameters h_n and β_n. In dimension $d = 1$, Berger et al. (2022a) finds a wide range of behaviors for the polymer depending on the sign and the growth speed of the parameters h_n, β_n. However, several questions remain open, such as determining the location and fluctuations of the range (in the spirit of Theorem 1.1) in a regime where the range size (properly rescaled) converges to a non-random quantity — we are currently investigating this question Bouchot (2024).

Another related model is the charged polymer, where charges are attached to the different monomers and interact with each other, see den Hollander (2009, Chapter 8) for an overview. Take i.i.d. random variables $(\omega_k)_{k\in\mathbb{N}}$, and consider the following quenched Gibbs measure on random walk trajectories

$$\mathrm{d}\mathbb{P}^{\omega}_{n,\beta}(S) = \frac{1}{Z^{\omega}_{n,\beta}} \exp\Big(-\beta \sum_{1\le i<j\le n} \omega_i\omega_j \mathbb{1}_{\{S_i=S_j\}}\Big) \mathrm{d}\mathbb{P}(S)\,.$$

Some recent papers Berger et al. (2018); Caravenna et al. (2016); Athreya et al. (2019) are dealing with the annealed version of the model, that can be written in the following form

$$\mathrm{d}\mathbb{P}^{\mathrm{ann}}_{n,\beta}(S) = \frac{1}{Z^{\mathrm{ann}}_{n,\beta}} \exp\Big(-\sum_{x\in\mathbb{Z}^d} g_\beta(\ell_n(x))\Big) \mathrm{d}\mathbb{P}(S)\,,$$

where $\ell_n(x) = \sum_{i=1}^n \mathbb{1}_{\{S_i=x\}}$ is the local time at site x and where g_β is a function that depends on β and on the distribution of ω. This model has been shown to undergo a folding/unfolding phase transition, and the case of dimension $d = 1$ has been investigated in remarkable detail in Caravenna et al. (2016). Our model falls in the same class of models: it corresponds to using the function $h\mathbb{1}_{\{\ell_n(x)>0\}}$ instead of the function $g_\beta(\ell_n(x))$; note that our model also displays a folding/unfolding transition when h goes from positive to negative values.

Finally, we mention a link to the annealed polymer model among Bernoulli obstacles. Take a Bernoulli site percolation with parameter p, meaning a collection $\mathcal{O} = \big\{z\in\mathbb{Z}^d, \eta_z = 1\big\}$ where η_z are i.i.d. Bernoulli variables with parameter p, and denote by $\mathcal{P} = \mathcal{B}(p)^{\otimes\mathbb{Z}^d}$ its law on $\mathbb{Z}^d$. Consider the random walk starting at 0 and killed when it first encounters $\mathcal{O}$, see for example Ding and Xu (2019) and its references. The annealed partition function of the corresponding polymer measure is given by

$$\mathbb{E}^{\mathcal{P}} \otimes \mathbf{E}\Big[\mathbb{1}_{\{\mathcal{R}_n\cap\mathcal{O}=\varnothing\}}\Big] = \mathbf{E}\Big[\mathcal{P}\Big(\forall z\in\mathcal{R}_n, \eta_z = 0\Big)\Big] = \mathbf{E}\Big[(1-p)^{|\mathcal{R}_n|}\Big] = \mathbf{E}\Big[e^{|\mathcal{R}_n|\log(1-p)}\Big].$$

Observe that this partition function is exactly Z_{n,h_p} with $h_p = -\log(1-p)$, thus our model can be seen as an annealed version of the random walk among Bernoulli obstacles with common parameter $1 - e^{-h_n}$.

Organization of the rest of the paper. The rest of the paper is organized as follows:

- In Section 2 we focus on the case of a "weak" penalization, that is $\lim_{n\to\infty} n^{-1/4}h_n = 0$: we give local asymptotic estimates for the partition function (Lemma 2.1), from which we deduce the first point of both Theorem 1.3 and Theorem 1.1 (in that order).
- In Section 3 we treat the case of a "strong" penalization, that is $\liminf_{n\to\infty} n^{-1/4}h_n > 0$: we modify the arguments of Section 2 to provide local asymptotic estimates for the partition function (Lemma 2.1). From this, we deduce first the second point of Theorems 1.1-1.3, *i.e.*

in the case $\lim_{n\to\infty} n^{-1/4}h_n = +\infty$, before we turn to the border case of Proposition 1.5, *i.e.* $\lim n^{-1/4}h_n = \hat{h} \in (0,+\infty)$.

- Finally, in Section 4 we derive sharp gambler's ruin estimates (see Lemmas 4.2-4.4) and their consequences for the range of a random walk, that is we prove Theorem 1.6.

2. Weak penalization: the case $\lim_{n\to\infty} n^{-1/4}h_n = 0$

Recall that $T_n^* := \big(\frac{n\pi^2}{h_n}\big)^{1/3}$ and $a_n := \frac{(T_n^*)^2}{\sqrt{3n\pi^2}}$, as well as $Z_{n,h_n}(\mathcal{A}) = \mathbb{E}\Big[e^{-h_n|\mathcal{R}_n(S)|}\mathbb{1}_{\{S\in\mathcal{A}\}}\Big]$.

We start with the case where $h_n \not\to 0$ meaning $n(T_n^*)^{-3} \not\to 0$ to avoid considering the cases where M_n^- is close to 0 or T. The case $n(T_n^*)^{-3} \to 0$ is adressed at the end of the present section and only consists of splitting the partition function as $Z_{n,h_n} = Z_{n,h_n}(\mathcal{A}_n) + Z_{n,h_n}({}^c\mathcal{A}_n)$ with

$$\mathcal{A}_n := \left\{ \frac{T_n}{\sqrt{\log T_n}} \le |M_n^-| \le T_n - \frac{T_n}{\sqrt{\log T_n}}, |T_n - T_n^*| \le \varepsilon_n \right\}.$$

Recall that formula (1.6) is valid for $|M_n^-| \gtrsim (T_n^*)^3/n$ which is at most $4T_n^*/\log T_n^*$ (recall that $n \ge \frac14 (T_n^*)^2 \log T_n^*$). Since on $\mathcal{A}_n$ we have $|M_n^-| \gg T_n^*/\log T_n^*$, we can use (1.6). Thus, with the same techniques as for $h_n \not\to 0$, we can get results on $Z_{n,h_n}(\mathcal{A}_n)$ and $\mathbb{P}_{n,h_n}(\cdot \cap \mathcal{A}_n)$. On the other hand, writing ${}^c\mathcal{A}_n$ for the complement of $\mathcal{A}_n$, we will prove that $Z_{n,h_n}({}^c\mathcal{A}_n)$ is negligible compared to $Z_{n,h_n}(\mathcal{A}_n)$, leading to Theorem 1.3 and 1.1 for this case.

2.1. *Local asymptotics for the partition function.* Our first preliminary result computes the contribution of the partition function from trajectories with a fixed size of the range T_n, with $T_n = T_n^* + \bar{o}(T_n^*)$.

Lemma 2.1. *Assume that $h_n \not\to 0$ and that $\lim_{n\to\infty} n^{-1/4}h_n = 0$. Let $(\varepsilon_n)_{n\ge1}$ be any vanishing sequence. Then, for any $t \in \mathbb{Z}$ and $w \in \frac12\mathbb{Z}$ such that $|t| \le \varepsilon_n T_n^*$ and $w \in [\![-\frac12(\lfloor T_n^*\rfloor + t), \frac12(\lfloor T_n^*\rfloor + t)]\!]$, we have*

$$Z_{n,h_n}\Big(T_n = \lfloor T_n^*\rfloor + t, W_n = w\Big) = \psi_n \times \Big(\cos\Big(\frac{\pi w}{T_n^*}\Big) + \bar{o}(1)\Big) \times e^{-(1+\bar{o}(1))\frac{t^2}{2a_n^2} + \bar{o}(1)}, \tag{2.1}$$

where the $\bar{o}(1)$ only depends on ε_n, and where we have set:

$$\psi_n := \psi(h_n) \exp\Big(-h_n(T_n^*+1) - g(T_n^*+2)n\Big), \qquad \textit{with} \;\; \psi(r) = \frac{4}{\pi}(1-e^{-r})^2. \tag{2.2}$$

Proof: We have

$$Z_{n,h_n}\Big(T_n = \lfloor T_n^*\rfloor + t, W_n = w\Big) = e^{-h_n(x+y+1)}\mathbb{P}\big(E_x^y(n)\big),$$

with $x + y = \lfloor T_n^*\rfloor + t$ and $\frac12(y-x) = w$. Thanks to Theorem 1.6, we can estimate this term. Indeed, for every x, y such that $\lim_{n\to\infty} \frac{x+y}{T_n^*} = 1$, using (1.6) we have

$$\mathbb{P}\big(E_x^y(n)\big) = \psi\Big(\frac{n\pi^2}{(T_n^*)^3}\Big)\Big(\sin\Big(\frac{\pi x}{T_n^*}\Big) + \bar{o}(1)\Big)e^{-g(x+y+2)n}, \tag{2.3}$$

where the $\bar{o}(1)$ is uniform in x, y. Note that by the definition of T_n^* we have $\frac{n\pi^2}{(T_n^*)^3} = h_n$.

Now, here we have that $x = \frac12(\lfloor T_n^*\rfloor + t) - w$, with $\frac{1}{2T_n^*}(\lfloor T_n^*\rfloor + t) \to \frac12$. Hence, we can write $\sin(\frac{\pi x}{T_n^*}) = \cos(\frac{\pi w}{T_n^*}) + \bar{o}(1)$ in (2.3). Recall the definition (1.8) $\phi_n(T) = h_nT + \frac{n\pi^2}{2T^2}$ and write $h_n(T+1) + g(T+2)n = \varphi_n(T) + h_n + \tilde{g}(T)n$, with $\tilde{g}(T) = g(T+2) - \frac{\pi^2}{2T^2}$, to get that

$$\begin{aligned} &Z_{n,h_n}\Big(T_n = \lfloor T_n^*\rfloor + t, W_n = w\Big) \\ &\quad = \psi(h_n)\Big(\cos\Big(\frac{\pi w}{T_n^*}\Big) + \bar{o}(1)\Big)\exp\Big(-\phi_n(\lfloor T_n^*\rfloor + t) - h_n - \tilde{g}(\lfloor T_n^*\rfloor + t)n\Big). \end{aligned} \tag{2.4}$$

We can use that $\phi_n'(T_n^*) = 0$, $\phi_n''(T_n^*) = \frac{3n\pi^2}{(T_n^*)^4} = 1/a_n^2$ and $\phi_n'''(T) = -\frac{12n\pi^2}{T^5}$, as well as Taylor's theorem, to get that for any $\frac{1}{2}T_n^* \leq T \leq 2T_n^*$

$$\left|\varphi_n(T) - \varphi_n(T_n^*) - \frac{(T-T_n^*)^2}{2a_n^2}\right| \leq C\frac{|T-T_n^*|^3 n}{(T_n^*)^5} = C'\frac{|T-T_n^*|^3}{T_n^*\, a_n^2}.$$

Hence, using that $a_n \to \infty$, we get that

$$\phi_n(\lfloor T_n^* \rfloor + t) = \varphi_n(T_n^*) + (1+\bar{o}(1))\frac{t^2}{2a_n^2}. \tag{2.5}$$

We can also perform the same expansion for $\tilde{g}(T) = g(T+2) - \frac{\pi^2}{2T^2}$, for which $\tilde{g}'(T) = \frac{\pi^2}{T^3} - \frac{\pi}{(T+2)^2}\tan(\frac{\pi}{T+2}) = \bar{\mathcal{O}}((T_n^*)^{-4})$:

$$|\tilde{g}(T) - \tilde{g}(T_n^*)| \leq C\frac{|T-T_n^*|}{(T_n^*)^4},$$

so that, inserting $T = \lfloor T_n^* \rfloor + t = T_n^* - t_n^o + t$ (where we recall $t_n^o = T_n^* - \lfloor T_n^* \rfloor$),

$$|\tilde{g}(\lfloor T_n^* \rfloor + t)n - \tilde{g}(T_n^*)n| \leq C\frac{|t-t_n^o|}{a_n^2} = \bar{o}(1)\frac{t^2}{a_n^2} + \bar{o}(1) \tag{2.6}$$

as $n \to \infty$, uniformly in t (consider separately the case $|t| \leq a_n$ and $|t| \geq a_n$).

All together, plugging (2.5)-(2.6) into (2.4), we end up with the desired result, with $\psi_n := \psi(h_n)\exp(-h_n - \phi_n(T_n^*) - \tilde{g}(T_n^*)n)$ which coincides with the definition (2.2) above. □

2.2. *Asymptotics of the partition function.* Lemma 2.1 allows us to obtain the correct behavior for the partition function.

Proof of Theorem 1.3: Assume that $h_n \not\to 0$ and that $\lim_{n\to\infty} n^{-1/4}h_n = 0$, so in particular $a_n \to +\infty$.

Note that by Berger et al. (2022a, Theorem 3.7/Region 5), we have for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}_{n,h_n}\Big(|T_n - T_n^*| > \varepsilon T_n^*\Big) = 0.$$

Therefore, one can find some vanishing sequence $(\varepsilon_n)_{n\geq 0}$ such that we have the asymptotic equivalence $Z_{n,h_n} = (1+\bar{o}(1))Z_{n,h_n}(|T_n - T_n^*| \leq \varepsilon_n T_n^*)$. We therefore only have to estimate that last partition function. We may decompose it as

$$Z_{n,h_n}\Big(|T_n - T_n^*| \leq \varepsilon_n T_n^*\Big) = \sum_{t=-\varepsilon_n T_n^*}^{\varepsilon_n T_n^*} \sum_{-(T_n^*+t)\leq 2w \leq T_n^*+t} Z_{n,h_n}\Big(T_n = \lfloor T_n^* \rfloor + t\,, W_n = w\Big).$$

where for the sake of clarity, we omitted the integer part in the bounds of the sums, to write $-\varepsilon_n T_n^* \leq t \leq \varepsilon_n T_n^*$ instead of $\lceil (1-\varepsilon_n)T_n^* \rceil \leq \lfloor T_n^* \rfloor + t \leq \lfloor (1+\varepsilon_n)T_n^* \rfloor$. Therefore, thanks to Lemma 2.1, we get that

$$Z_{n,h_n}\Big(|T_n - T_n^*| \leq \varepsilon_n T_n^*\Big) = \psi_n \sum_{t=-\varepsilon_n T_n^*}^{\varepsilon_n T_n^*} e^{-(1+\bar{o}(1))\frac{t^2}{2a_n^2} + \bar{o}(1)} \sum_{\substack{-(T_n^*+t)\leq 2w\leq T_n^*+t \\ w\in\frac{1}{2}(\lfloor T_n^*\rfloor + t)+\mathbb{Z}}} \Big(\cos\Big(\frac{\pi w}{T_n^*}\Big) + \bar{o}(1)\Big).$$

Now, as T_n^* goes to $+\infty$, the internal sum is a Riemann sum: we have, uniformly for $\lceil (1-\varepsilon_n)T_n^* \rceil \leq \lfloor T_n^* \rfloor + t \leq \lfloor (1+\varepsilon_n)T_n^* \rfloor$,

$$\lim_{n\to\infty} \frac{1}{T_n^*} \sum_{\substack{-(T_n^*+t)\leq 2w\leq T_n^*+t \\ w\in\frac{1}{2}(\lfloor T_n^*\rfloor + t)+\mathbb{Z}}} \Big(\cos\Big(\frac{\pi w}{T_n^*}\Big) + \bar{o}(1)\Big) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(\pi v)\,\mathrm{d}v = \frac{2}{\pi}.$$

Then, a sum over t remains, which is also a Riemann sum: as $a_n \to +\infty$ and $\varepsilon_n T_n^*/a_n \to +\infty$ ($T_n^*/a_n \geq (cst.)\sqrt{\log n}$ so such sequence (ε_n) exists), we have

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{t=-\varepsilon_n T_n^*}^{\varepsilon_n T_n^*} e^{-(1+\bar{o}(1))\frac{t^2}{2a_n^2}+\bar{o}(1)} = \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \mathrm{d}u = \sqrt{2\pi}\,.$$

Altogether, we have proved that, as $n \to \infty$

$$Z_{n,h_n} \sim \frac{2\sqrt{2}}{\sqrt{\pi}} \psi_n a_n T_n^*\,. \tag{2.7}$$

Recalling the definition of T_n^* and a_n, we have $a_n T_n^* = \frac{\pi}{\sqrt{3}} \frac{\sqrt{n}}{h_n}$. Additionally, recalling the definition (2.2) of ψ_n, and using that $g(T+2) = \frac{\pi^2}{2T^2} - \frac{2\pi^2}{T^3} + \bar{\mathcal{O}}(\frac{1}{T^4})$ as $T \to \infty$, we get that

$$\psi_n = \psi(h_n) \exp\Big(- h_n T_n^* - h_n - \frac{n\pi^2}{2(T_n^*)^2} + \frac{2\pi^2 n}{(T_n^*)^3} + \bar{o}(1) \Big)\,,$$

since $\lim_{n\to\infty} \frac{n}{(T_n^*)^4} = 0$ because $\lim_{n\to\infty} n^{-1/4} h_n = 0$. By the definition of T_n^* we have $\frac{n\pi^2}{(T_n^*)^3} = h_n$, we get that $\psi_n \sim \psi(h_n) e^{h_n} e^{-\frac{3}{2} h_n T_n^*}$. Putting all estimates together and noting that $e^{\alpha}(1-e^{-\alpha})^2 = 2(\cosh(\alpha)-1)$, this concludes the proof. □

Proof of Theorem 1.1: The proof reduces to showing the following Lemma.

Lemma 2.2. *Let $h_n \not\to 0$ be such that $\lim_{n\to\infty} n^{-1/4} h_n = 0$. Then, for any $r < s$, we have*

$$\lim_{n\to\infty} \frac{1}{\psi_n a_n T_n^*} Z_{n,h_n}\Big(r \leq \frac{|T_n - T_n^*|}{a_n} \leq s \Big) = \frac{2}{\pi} \int_r^s e^{-\frac{u^2}{2}} \mathrm{d}u\,,$$

where ψ_n is the sequence that appears in Lemma 2.1.

Indeed, once we have this lemma, in view of the asymptotics (2.7) and Proposition 1.10, we get that for any $r < s$ and any $a < b$,

$$\begin{aligned}\mathbb{P}_{n,h_n}\Big(r \leq \frac{|T_n - T_n^*|}{a_n} \leq s, a \leq \frac{W_n}{T_n^*} \leq b \Big) &= \frac{1}{Z_{n,h_n}} Z_{n,h_n}\Big(r \leq \frac{|T_n - T_n^*|}{a_n} \leq s, a \leq \frac{W_n}{T_n^*} \leq b \Big) \\ &\xrightarrow{n\to\infty} \int_r^s \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \mathrm{d}u \int_a^b \frac{\pi}{2} \cos(\pi v) \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} \mathrm{d}v\,,\end{aligned}$$

which concludes the proof. □

Proof of Lemma 2.2: The proof proceeds as for the proof of Theorem 1.3. We can decompose the partition function as

$$\begin{aligned}&\sum_{t=ra_n}^{sa_n} \sum_{\substack{-\lfloor T_n^*\rfloor - t \leq 2w \leq \lfloor T_n^*\rfloor + t \\ w \in \frac{1}{2}(\lfloor T_n^*\rfloor + t) + \mathbb{Z}}} Z_{n,h_n}\Big(T_n = \lfloor T_n^* \rfloor + t\,, W_n = w \Big) \\ &\quad = \psi_n \sum_{t=ra_n}^{sa_n} e^{-(1+\bar{o}(1))\frac{t^2}{2a_n^2}} \sum_{\substack{-\lfloor T_n^*\rfloor - t \leq 2w \leq \lfloor T_n^*\rfloor + t \\ w \in \frac{1}{2}(\lfloor T_n^*\rfloor + t) + \mathbb{Z}}} \Big(\cos\Big(\frac{\pi w}{T_n^*} \Big) + \bar{o}(1) \Big)\,,\end{aligned}$$

where we have used Lemma 2.1 as above (using that $a_n \to \infty$) and where we omitted the integer parts for the bounds on t, meaning that $\lceil ra_n \rceil \leq t \leq \lfloor sa_n \rfloor$.

Again, as T_n^* goes to $+\infty$, the internal sum is a Riemann sum: we have, uniformly for $\lceil T_n^*+ra_n\rceil \le \lfloor T_n^*\rfloor + t \le \lfloor T_n^* + sa_n\rfloor$ and since $a_n/T_n^* \to 0$,

$$\lim_{n\to\infty}\frac{1}{T_n^*}\sum_{\substack{-\lfloor T_n^*\rfloor - t\le 2w\le \lfloor T_n^*\rfloor + t\\ w\in\frac12(\lfloor T_n^*\rfloor+t)+\mathbb{Z}}}\Big(\cos\Big(\frac{\pi w}{T_n^*}\Big)+\bar{o}(1)\Big) = \int_{-\frac12}^{\frac12}\cos(\pi v)\,\mathrm{d}v = \frac{2}{\pi}\,.$$

Then, the sum over t that remains is also a Riemann sum: as $a_n \to +\infty$, we have

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{t=ra_n}^{sa_n} e^{-(1+\bar{o}(1))\frac{t^2}{2a_n^2}} = \int_r^s e^{-\frac{u^2}{2}}\,\mathrm{d}u\,,$$

which concludes the proof. □

2.3. *Adapting to reduced penalization.* Now assume that $h_n \to 0$ with $h_n \ge n^{-1/2}(\log n)^{3/2}$ and write $\nu_n = (\log T_n^*)^{-1/2}$. We first adapt Lemma 2.1.

Lemma 2.3. *Assume that $h_n \to 0$ and that $h_n \ge n^{-1/2}(\log n)^{3/2}$. Let $(\varepsilon_n)_{n\ge1}$ be any vanishing sequence. Then, for any $t\in\mathbb{Z}$ and $w\in\frac12\mathbb{Z}$ such that $|t|\le\varepsilon_n T_n^*$ and $w\in[\![-(1-\nu_n)\frac12(\lfloor T_n^*\rfloor+t),(1-\nu_n)\frac12(\lfloor T_n^*\rfloor+t)]\!]$, we have*

$$Z_{n,h_n}\Big(T_n=\lfloor T_n^*\rfloor+t, W_n=w\Big) = \psi_n\times\Big(\cos\Big(\frac{\pi w}{T_n^*}\Big)+\bar{o}(1)\Big)\times e^{-(1+\bar{o}(1))\frac{t^2}{2a_n^2}}\,, \tag{2.8}$$

where the $\bar{o}(1)$ only depends on ε_n and we used the same notations as in Lemma 2.1.

Proof: We only use formula (1.6) that is valid since we have $\frac{x}{T} > \nu_n \gg \frac{(T_n^*)^2}{n}$, where x is defined as in the proof of Lemma 2.1 and corresponds to the lower edge of the polymer. The proof then proceeds as previously with a Taylor expansion in the exponential. □

Recall the definition of $\mathcal{A}_n$:

$$\mathcal{A}_n = \Big\{\frac{T_n}{\sqrt{\log T_n}}\le |M_n^-|\le T_n - \frac{T_n}{\sqrt{\log T_n}}, |T_n - T_n^*|\le\varepsilon_n\Big\}\,.$$

Proof of Theorem 1.3 for $h_n\to0$: First we get the asymptotics for $Z_{n,h_n}(\mathcal{A}_n)$ We repeat the proof of Theorem 1.3 for $h_n\not\to 0$, except by restricting the sum to the suitable w given in Lemma 2.3. This means that

$$Z_{n,h_n}(\mathcal{A}_n) = \psi_n\sum_{t=-\varepsilon_nT_n^*}^{\varepsilon_nT_n^*} e^{-(1+\bar{o}(1))\frac{t^2}{2a_n^2}}\sum_{2w=-(1-\nu_n)(\lfloor T_n^*\rfloor+t)}^{(1-\nu_n)(\lfloor T_n^*\rfloor+t)}\Big(\cos\Big(\frac{\pi w}{T_n^*}\Big)+\bar{o}(1)\Big)\,.$$

The sums still are Riemann sums, leading to

$$Z_{n,h_n}(\mathcal{A}_n)\sim\psi_n a_n T_n^*\sqrt{2\pi}\int_{-\frac12+\nu_n}^{\frac12-\nu_n}\cos(\pi v)\mathrm{d}v\,,$$

which leads to the same asymptotics as the one announced in Theorem 1.3.

Now we turn to the term $Z_{n,h_n}({}^c\mathcal{A}_n)$: recall Theorem 1.6 when $n/T^3\to0$ and $x/T\to0$, we have

$$\Theta_n(x,T-x) = \frac{2\pi^3n^2}{T^6}\sin\Big(\frac{\pi x}{T}\Big)\Big[2+\frac{\pi}{T\tan\frac{x\pi}{T}}+\frac{T^2}{n}\frac{1-\frac{2x}{T}}{\tan\frac{x\pi}{T}}\Big]e^{-g(T)n}. \tag{2.9}$$

In the sum appearing in (2.9), the third term behaves as $T^3/xn\to\infty$ when x/T is small, compared to $1/x$ for the second term. Thus, it is dominant and we can get an upper bound $\Theta_n(x,T-x)\le(cst.)\frac{n}{T^4}e^{-g(T)n}$ for all $x=\bar{o}(T)$. By symmetry, we get the same bound for y/T

small meaning that this bound encompasses both $x/T_n^* \le \nu_n T_n^*$ and $x/T_n^* \ge (1-\nu_n)T_n^*$. Thus, consider any (x,t) such that $\{|M_n^-| = x, T_n = t\} \subseteq \mathcal{A}_n^c$, we have

$$Z_{n,h_n}\Big(T_n - T_n^* = t, W_n = \frac{T}{2} - x\Big) \le (cst.)\frac{n}{T^4} e^{-h(T-1)-g(T)n} = (cst.)\frac{n}{T^4} e^{-\frac{3}{2}h_n T_n^* - (1+\bar{o}(1))\frac{t^2}{2a_n^2}} .$$

Therefore,

$$Z_{n,h_n}(\mathcal{A}_n^c) e^{\frac{3}{2}h_n T_n^*} \le \sum_{t=-\varepsilon_n T_n^*}^{\varepsilon_n T_n^*} e^{-(1+\bar{o}(1))\frac{t^2}{2a_n^2}} \times 2 \sum_{0\le x\le \nu_n T_n^*} (cst.)\frac{n}{(T_n^*)^4} \asymp \frac{a_n \nu_n n}{(T_n^*)^3} \asymp \nu_n \frac{(T_n^*)^2}{n} h_n \sqrt{n}.$$

Since $(T_n^*)^2/n$ and ν_n both go to 0, this means that $Z_{n,h_n}(\mathcal{A}_n^c)$ is negligible compared to $Z_{n,h_n}(\mathcal{A}_n) \asymp h_n\sqrt{n} e^{-\frac{3}{2}h_n T_n^*}$ and we get the full Theorem 1.3. □

Theorem 1.1 is adapted with no additional difficulty using Proposition 1.10.

3. Strong penalization and vanishing fluctuations

3.1. *The case* $\liminf_{n\to\infty} n^{-1/4}h_n = +\infty$. See that the case where $a_n \to 0$ is much more restrictive to establish an analog of Lemma 2.1, as $\bar{\mathcal{O}}(a_n^{-2})$ quantities now bring extremely large contributions to the exponential part of Z_{n,h_n} and slight deviations from the optimal size T_n^* will be penalized by a large factor. Indeed, if we are to get a *continuity* from Theorem 1.1 when $\limsup_{n\to\infty} a_n < \infty$, we want to know the exact asymptotic law of fluctuations without renormalization. We define

$$\bar{\phi}_n(T) := h_n(T+1) + \frac{n\pi^2}{2(T+2)^2} \quad \text{and} \quad T_n^o := \arg\min \bar{\phi}_n(T)\,. \tag{3.1}$$

Lemma 3.1. *Assume that* $\lim_{n\to\infty} n^{-1/4}h_n = +\infty$ *and* $\lim_{n\to\infty} n^{-1}h_n = 0$ *and let* $(\varepsilon_n)_{n\ge1}$ *be any vanishing sequence. Then, for any* $t \in \mathbb{Z}\setminus\{0,1\}$ *and* $w \in \frac{1}{2}\mathbb{Z}$ *such that* $|t| \le \varepsilon_n T_n^o$ *and* $w \in [\![-\frac{1}{2}(\lfloor T_n^o\rfloor + t), \frac{1}{2}(\lfloor T_n^o\rfloor + t)]\!]$, *we have*

$$Z_{n,h_n}\Big(T_n = \lfloor T_n^* - 2\rfloor + t, W_n = w\Big) = \bar{\psi}_n \times \Big(\cos\Big(\frac{\pi w}{T_n^*}\Big) + \bar{o}(1)\Big) \times e^{-(1+\bar{o}(1))\frac{(t-t_n^o)^2}{2a_n^2}}\,, \tag{3.2}$$

where $t_n^o := T_n^o - \lfloor T_n^o\rfloor$, $\bar{o}(1)$ *is a vanishing quantity that depends only on* ε_n *and*

$$\bar{\psi}_n := \psi(h_n)\exp\Big(-h_n(T_n^*-1) - g(T_n^*)n\Big)\,, \qquad \text{with} \;\; \psi(\alpha) = \frac{4}{\pi}(1-e^{-\alpha})^2\,. \tag{3.3}$$

When $t\in\{0,1\}$ *we instead have*

$$Z_{n,h_n}\Big(T_n = \lfloor T_n^* - 2\rfloor + t, W_n = w\Big) = \bar{\psi}_n \times \Big(\cos\Big(\frac{\pi w}{T_n^*}\Big) + \bar{o}(1)\Big) \times e^{-\frac{1+\bar{o}(1)}{2a_n^2}\Big[(t-t_n^o)^2 - \frac{2\pi^2}{9}\frac{t-t_n^o}{T_n^*}\Big]}. \tag{3.4}$$

Proof: We can perform the same decomposition as in Lemma 2.1 and setting $T = \lfloor T_n^* - 2\rfloor + t$, we arrive at (analogously to (2.4))

$$\begin{aligned} Z_{n,h_n}\Big(T_n = T, W_n = w\Big) &= \psi(h_n)\Big(\cos\Big(\frac{\pi w}{T}\Big) + \bar{o}(1)\Big) \times e^{-h_n(T+1) - g(T+2)n} \\ &= \psi(h_n)\Big(\cos\Big(\frac{\pi w}{T}\Big) + \bar{o}(1)\Big) \times e^{-\bar{\phi}_n(T) - \bar{g}(T)n}\,, \end{aligned} \tag{3.5}$$

with $\bar{\phi}_n$ defined above in (3.1) and where $\bar{g}(T) := g(T+2) - \frac{\pi^2}{2(T+2)^2}$. One can then easily check that $T_n^o = T_n^* - 2$ and that $\bar{\phi}_n''(T_n^o) = \frac{3n\pi^2}{(T_n^o+2)^4} = a_n^{-2}$ and $\bar{\phi}_n^{(3)}(T_n^o) = -\frac{12n\pi^2}{(T_n^o+2)^5}$, thus

$$\Big|\bar{\phi}_n(T) - \bar{\phi}_n(T_n^o) - \frac{(T-T_n^o)^2}{2a_n^2}\Big| \le \frac{12n\pi^2}{6(T_n^o+2)^5}|T-T_n^o|^3 \le \varepsilon_n \frac{(T-T_n^o)^2}{a_n^2}\,.$$

Furthermore, a series expansion of the first two orders of the Taylor expansion of $\bar{g}(T)$ around T_n^o gives

$$\bar{g}(T) = \bar{g}(T_n^o) - (T - T_n^o)\frac{\pi^4}{3(T_n^*)^5}(1 + \bar{o}(1)) + \frac{5\pi^4(T - T_n^o)^2}{6(T_n^*)^6}(1 + \bar{o}(1)) .$$

Assembling the two previous equations and using $a_n^{-2} = 3\pi^2 n(T_n^*)^{-4}$ leads to

$$\bar{\phi}_n(T) + \bar{g}(T)n = \bar{\phi}_n(T_n^o) + \bar{g}(T_n^o)n + \frac{(T - T_n^o)^2}{2a_n^2}(1 + \bar{o}(1)) - (T - T_n^o)\frac{\pi^2}{9a_n^2 T_n^*}(1 + \bar{o}(1)) .$$

Now, write $T = \lfloor T_n^* - 2 \rfloor + t = T_n^o - t_n^o + t$ (recall that t_n^o is the decimal part of T_n^o), we have

$$\bar{\phi}_n(T) + \bar{g}(T)n = \bar{\phi}_n(T_n^o) + \bar{g}(T_n^o)n + \frac{(t - t_n^o)^2}{2a_n^2}(1 + \bar{o}(1)) - (t - t_n^o)\frac{\pi^2}{9a_n^2 T_n^*}(1 + \bar{o}(1)) .$$

For $t \notin \{0, 1\}$, we have $|T - T_n^o| = |t - t_n^o| > 1$ meaning that $(t - t_n^o)\frac{\pi^2}{9a_n^2 T_n^*} = \bar{o}(1)\frac{(t-t_n^o)^2}{a_n^2}$ which proves (3.2) by injecting in (3.5) and using $\bar{\phi}_n(T_n^o) + n\bar{g}(T_n^o) = h_n(T_n^* - 1) + ng(T_n^*)$.

For $t \in \{0, 1\}$, there is a possibility that $|t - t_n^o|$ gets so small that $T_n^*(t - t_n^o)^2 = \bar{o}(1)(t - t_n^o)$. In this case, we have to keep both terms in $t - t_n^o$ which leads to (3.4). □

Proof of Theorem 1.3: Suppose $\liminf_{n\to\infty} n^{-1/4}h_n = +\infty$ meaning $a_n \to 0$. Apply Lemma 3.1 with the same sequence (ε_n) to write

$$Z_{n,h_n}\Big(|T_n - T_n^o| \le \varepsilon_n T_n^o\Big) = \bar{\psi}_n \sum_{t=-\varepsilon_n T_n^o}^{\varepsilon_n T_n^o} e^{-(1+\bar{o}(1))\frac{\varsigma_n(t)}{2a_n^2}} \sum_{\substack{-(T_n^o+t)\le 2w \le T_n^o+t \\ w\in\frac{1}{2}(\lfloor T_n^o\rfloor+t)+\mathbb{Z}}} \Big(\cos\Big(\frac{\pi w}{T_n^*}\Big) + \bar{o}(1)\Big) , \tag{3.6}$$

where we wrote $\varsigma_n(t) := (t - t_n^o)^2 - \frac{2\pi^2}{9}\frac{t - t_n^o}{T_n^*}\mathbb{1}_{\{t\in\{0,1\}\}}$ and used abusive notations in the sum to mean that $\lceil (1-\varepsilon_n)T_n^* \rceil \le \lfloor T_n^* \rfloor + t \le \lfloor (1+\varepsilon_n)T_n^* \rfloor$. As T_n^* goes to $+\infty$, the sum in w in (3.6) is a Riemann sum and thus, uniformly for $|t| \le \varepsilon_n T_n^o$,

$$\lim_{n\to\infty} \frac{1}{T_n^*} \sum_{\substack{-(T_n^o+t)\le 2w \le T_n^o+t \\ w\in\frac{1}{2}(\lfloor T_n^o\rfloor+t)+\mathbb{Z}}} \Big(\cos\Big(\frac{\pi w}{T_n^*}\Big) + \bar{o}(1)\Big) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(\pi v)\,\mathrm{d}v = \frac{2}{\pi} .$$

This means that injecting in (3.6), we have

$$Z_{n,h_n}\Big(|T_n - T_n^o| \le \varepsilon_n T_n^o\Big) = (1 + \bar{o}(1))\frac{2}{\pi}T_n^*\bar{\psi}_n \sum_{t=-\varepsilon_n T_n^o}^{\varepsilon_n T_n^o} e^{-(1+\bar{o}(1))\frac{\varsigma_n(t)}{2a_n^2}} . \tag{3.7}$$

The largest term in the sum in (3.7) is attained at $t = 0$ or $t = 1$ depending on the value of t_n^o. Thus, write $\hat{\varsigma}_n := \varsigma_n(0) \wedge \varsigma_n(1)$ so that $e^{-\hat{\varsigma}_n/2a_n^2}$ is the largest term of the sum in (3.7). By computing $\varsigma_n(1) - \varsigma_n(0) = 1 - \frac{2\pi^2}{9T_n^*} - 2t_n^o$, we can check that $\hat{\varsigma}_n = \varsigma_n(\mathbb{1}_{\{t_n^o > \tau_n^o\}})$, with $\tau_n^o := \frac{1}{2} - \frac{\pi^2}{9T_n^*}$. We have

$$\sum_{t=-\varepsilon_n T_n^o}^{\varepsilon_n T_n^o} e^{-(1+\bar{o}(1))\frac{\varsigma_n(t)}{2a_n^2}} = \sum_{t\in\{0,1\}} e^{-\frac{1}{2a_n^2}\varsigma_n(t)} + e^{-\frac{\hat{\varsigma}_n}{2a_n^2}} \sum_{\substack{t=-\varepsilon_n T_n^o \\ t\neq 0,1}}^{\varepsilon_n T_n^o} e^{\frac{\hat{\varsigma}_n - (t-t_n^o)^2}{2a_n^2}(1+\bar{o}(1))} . \tag{3.8}$$

To get (1.3) we only need to prove that the sum in the second term of (3.8) goes to 0 as $n \to +\infty$. Indeed, the first sum being of the same order as $e^{-\hat{\varsigma}_n/2a_n^2}$, this would prove that the second term is negligible. To do so, note that for any $t \notin \{0,1\}$,

$$\hat{\varsigma}_n - (t - t_n^o)^2 = \begin{cases} 2tt_n^o - t^2 + \dfrac{2\pi^2}{9T_n^*} t_n^o & \text{if } \hat{\varsigma}_n = \varsigma_n(0)\,; \\ 1 - t^2 + 2t_n^o(t-1) - \dfrac{2\pi^2}{9T_n^*}(1 - t_n^o) & \text{if } \hat{\varsigma}_n = \varsigma_n(1)\,. \end{cases} \tag{3.9}$$

One can easily check that if $\hat{\varsigma}_n = \varsigma_n(1)$, this is always less than $-(t-1)^2$, while if $\hat{\varsigma}_n = \varsigma_n(1)$, this is less than $-\frac{1}{2}t$ for $t \geq 3$, less than $-(t^2+1)$ for $t \leq -2$. Thus, we have

$$\sum_{\substack{t=-\varepsilon_n T_n^o \\ t \notin \{-1,0,1,2\}}}^{\varepsilon_n T_n^o} e^{\frac{\hat{\varsigma}_n - (t-t_n^o)^2}{2a_n^2}} \leq \sum_{t=2}^{+\infty} e^{-\frac{(t-1)^2}{2a_n^2}} + \sum_{t=3}^{+\infty} e^{-\frac{t}{4a_n^2}}\,. \tag{3.10}$$

On the other hand for $t \in \{-1, 2\}$ we work out $\hat{\varsigma}_n - \varsigma_n(t)$:

$$\hat{\varsigma}_n - (-1 - t_n^o)^2 = -1 - 2t_n^o + \frac{2\pi^2}{9T_n^*} t_n^o + (1 - 2t_n^o - \frac{2\pi^2}{9T_n^*}) \mathbb{1}_{\{t_n^o > \tau_n^o\}} \leq -1 + \frac{2\pi^2}{9T_n^*}\,,$$

$$\hat{\varsigma}_n - (2 - t_n^o)^2 = -4 + 4t_n^o + \frac{2\pi^2}{9T_n^*} t_n^o + (1 - 2t_n^o - \frac{2\pi^2}{9T_n^*}) \mathbb{1}_{\{t_n^o > \tau_n^o\}} \leq -1 + \frac{2\pi^2}{9T_n^*}\,,$$

by splitting on whether $t_n^o \leq \tau_n^o$ or not. Combining with (3.10), this shows that

$$\sum_{\substack{t=-\varepsilon_n T_n^o \\ t \neq 0,1}}^{\varepsilon_n T_n^o} e^{\frac{\hat{\varsigma}_n - (t-t_n^o)^2}{2a_n^2}} \leq \sum_{t=2}^{+\infty} e^{-\frac{(t-1)^2}{2a_n^2}} + \sum_{t=3}^{+\infty} e^{-\frac{t}{4a_n^2}} + 2e^{-(\frac{1}{2} - \frac{2\pi^2}{9T_n^*})a_n^{-2}}\,. \tag{3.11}$$

Recall that $a_n \to 0$ as $n \to +\infty$, thus there is a $n_0 \in \mathbb{N}$ such that we have $a_n^2 \leq \frac{1}{2}$ for all $n \geq n_0$, and thus

$$\forall t \geq 2,\ \sup_{n \geq n_0} e^{-\frac{(t-1)^2}{2a_n^2}} \leq e^{-(t-1)^2} \quad \text{as well as} \quad \forall t \geq 3,\ \sup_{n \geq n_0} e^{-\frac{t}{4a_n^2}} \leq e^{-\frac{t}{2}}\,.$$

Therefore, with the use of Lebesgue's dominated convergence, both sums in the right-hand side of (3.11) converge to zero as $n \to +\infty$. Since it is also true for the third term at the right of (3.11), we have

$$\sum_{t=-\varepsilon_n T_n^o}^{\varepsilon_n T_n^o} e^{-(1+\bar{o}(1))\frac{\varsigma_n(t)}{2a_n^2}} = \sum_{t \in \{0,1\}} e^{-\frac{1}{2a_n^2}\varsigma_n(t)} + e^{-\frac{\hat{\varsigma}_n}{2a_n^2}} \bar{o}(1) = (1 + \bar{o}(1)) \sum_{t \in \{0,1\}} e^{-\frac{1}{2a_n^2}\varsigma_n(t)}\,,$$

where for the second equality we used the fact that the sum over $t \in \{0,1\}$ is at least equal to $e^{-\frac{\hat{\varsigma}_n}{2a_n^2}}$.

We are left to get an asymptotic expansion of $g(T_n^*)n$ by expanding g up to order T^{-4}

$$g(T_n^*)n = \frac{n\pi^2}{2(T_n^*)^2} + \frac{n\pi^4}{12(T_n^*)^4}(1 + \bar{o}(1)) = \frac{1}{2}h_n T_n^* + \frac{n\pi^4}{12(T_n^*)^4}(1 + \bar{o}(1))\,,$$

which finally means that

$$Z_{n,h_n} = \frac{2}{\pi} e^{h_n} \psi(h_n)(1 + \bar{o}(1)) \sum_{t \in \{0,1\}} e^{-\frac{\varsigma_n(t)}{2a_n^2}(1+\bar{o}(1))} e^{-\frac{3}{2}h_n T_n^* - \frac{n\pi^4}{12(T_n^*)^4}(1+\bar{o}(1))}\,,$$

and thus proving Theorem 1.3 after using $e^{\alpha}(1 - e^{-\alpha})^2 = 2(\cosh(\alpha) - 1)$, explicitly writing a_n^{-2} and factorizing in the exponential. $\square$

Proof of Theorem 1.1: The above proof of Theorem 1.3 already shows that

$$\mathbb{P}_{n,h_n}\Big(|T_n - \lfloor T_n^* - 2\rfloor| \in \{0,1\}\Big) = \frac{1}{Z_{n,h_n}} Z_{n,h_n}\Big(|T_n - \lfloor T_n^* - 2\rfloor| \in \{0,1\}\Big) \xrightarrow[n\to\infty]{} 1\,.$$

We also see that if $t_n^o \leq \frac{1}{2} - \delta$ for some $\delta > 0$ and all n large enough, $\hat{\varsigma}_n = \varsigma_n(0)$ and $e^{(\hat{\varsigma}_n - \varsigma_n(1))a_n^{-2}} \leq e^{-\delta a_n^{-2}} \to 0$, hence the comment below Theorem 1.3. It is similar in the case where $t_n^o \geq \frac{1}{2} + \delta$. ☐

3.2. *Case* $\lim_{n\to\infty} n^{-1/4} h_n = \hat{h}$*: order one fluctuations.*

Proof of Proposition 1.5: Going back to use Lemma 3.1, we get for any vanishing sequence $(\varepsilon_n)_{n\geq 1}$

$$Z_{n,h_n}\Big(|T_n - T_n^*| \leq \varepsilon_n T_n^*\Big) = (1+\bar{o}(1))\bar{\psi}_n \sum_{t=-\lfloor \varepsilon_n T_n^*\rfloor+2}^{\lfloor \varepsilon_n T_n^*\rfloor+2} e^{-(1+\bar{o}(1))\frac{\varsigma_n(t)}{2a_n^2}} \sum_{\substack{-(T_n^*+t)\leq 2w\leq T_n^*+t \\ w\in\frac{1}{2}(\lfloor T_n^*\rfloor+t)+\mathbb{Z}}} \cos\left(\frac{\pi w}{T_n^*}\right).$$

The internal Riemann sum is dealt with the same method as before, while we can take the limit for a_n in the external sum. Thus we have, as $n \to \infty$

$$Z_{n,h_n}\Big(|T_n - T_n^*| \leq \varepsilon_n T_n^*\Big) \sim \frac{2}{\pi} T_n^* \bar{\psi}_n \sum_{t=-\lfloor \varepsilon_n T_n^*\rfloor}^{\lfloor \varepsilon_n T_n^*\rfloor} e^{-\frac{\varsigma_n(t)}{2a_n^2}}\,,$$

which clearly gives that $Z_{n,h_n} \sim \frac{2}{\pi}\bar{\psi}_n T_n^* \theta_n(a)$, because we already know that taking ε_n going to zero sufficiently slowly we have $\lim_{n\to\infty} \mathbb{P}_{n,h_n}(|T_n - T_n^*| > \varepsilon_n T_n) = 0$.

Moreover, applying again Lemma 3.1, we also get that for any fixed integer $t \in \mathbb{Z}$,

$$Z_{n,h_n}\Big(T_n = \lfloor T_n^* - 2\rfloor + t\Big) \sim \frac{2}{\pi} T_n^* \bar{\psi}_n e^{-\frac{\varsigma_n(t)}{2a_n^2}}\,,$$

using the same calculation as above. This concludes the proof of Proposition 1.5. ☐

4. Range endpoints and gambler's ruin estimates

4.1. *Gambler's ruin estimates.* We consider a band $[0,T]$ with T some positive integer, and choose a starting point $0 \leq z \leq T$. We denote $\tau_0 := \min\{n \geq 0\,, S_n = 0\}$, resp. $\tau_T := \min\{n \geq 0\,, S_n = T\}$, the hitting time of the boundary at 0, resp. at T. We also denote $\tau := \tau_0 \wedge \tau_T$. We recall the formulae of Feller (1968, §14.5) for the ruin problem, in the case of a symmetric walk. We use the notation $n \leftrightarrow z$ if $n - z$ is even and denote by $\mathbb{P}_z$ the law of the simple random walk starting at $z \in \mathbb{Z}$.

Proposition 4.1. *For any* $z \in [\![1, T-1]\!]$ *and* $n > 1$,

$$\mathbb{P}_z(\tau = \tau_0 = n) = \frac{2}{T} \sum_{1\leq k<T/2} \cos^{n-1}\left(\frac{\pi k}{T}\right) \sin\left(\frac{\pi k z}{T}\right) \sin\left(\frac{\pi k}{T}\right) \mathbb{1}_{\{n\leftrightarrow z\}}\,. \tag{4.1}$$

By symmetry, we also have $\mathbb{P}_z(\tau = \tau_T = n) = \mathbb{P}_{T-z}(\tau = \tau_0 = n)$*:*

$$\mathbb{P}_z(\tau = \tau_T = n) = \frac{2}{T} \sum_{1\leq k<T/2} (-1)^{k+1} \cos^{n-1}\left(\frac{\pi k}{T}\right) \sin\left(\frac{\pi k z}{T}\right) \sin\left(\frac{\pi k}{T}\right) \mathbb{1}_{\{n\leftrightarrow T-z\}}\,. \tag{4.2}$$

Note that if $T - z \leftrightarrow n \leftrightarrow z$, if we sum (4.1), (4.2), the terms for even k cancel out and we get

$$\mathbb{P}_z(\tau = n) = \frac{4}{T} \sum_{j=0}^{j_T} \cos^{n-1}\left(\frac{(2j+1)\pi}{T}\right) \sin\left(\frac{(2j+1)\pi z}{T}\right) \sin\left(\frac{(2j+1)\pi}{T}\right).$$

where j_T is the largest integer j such that $2j + 1 < T/2$, which depends on the parity of T.

Let us now give the *sharp* asymptotic behavior of the probabilities (4.1)-(4.2) above. Recall that since we are interested in the case $nT^{-2} \to \infty$, these events $\{\tau = \tau_{0/T} = n\}$ are rare as typically we have $|\mathcal{R}_n| \asymp \sqrt{n}$ and thus $\tau \asymp T^2 \ll n$. Recall the definition (1.4): $g(T) = -\log\cos(\frac{\pi}{T})$. By symmetry, we only deal with the case $z \in [\![0, \frac{T}{2}]\!]$.

Lemma 4.2. *Suppose that $T = T(n) \to \infty$ as $n \to \infty$ and that $\lim_{n\to\infty} \frac{n}{T^2} = +\infty$. Then, we have the following asymptotics: for all $z \in [\![0, \frac{T}{2}]\!]$,*

$$\mathbb{P}_z(\tau = \tau_0 = n) = \big(1 + \bar{\mathcal{O}}(e^{-\frac{\pi^2 n}{T^2}})\big)\frac{2}{T}\sin\Big(\frac{z\pi}{T}\Big)\tan\Big(\frac{\pi}{T}\Big)e^{-g(T)n}\mathbb{1}_{\{n\leftrightarrow z\}}\,, \tag{4.3}$$

$$\mathbb{P}_z(\tau = \tau_T = n) = \big(1 + \bar{\mathcal{O}}(e^{-\frac{\pi^2 n}{T^2}})\big)\frac{2}{T}\sin\Big(\frac{z\pi}{T}\Big)\tan\Big(\frac{\pi}{T}\Big)e^{-g(T)n}\mathbb{1}_{\{n\leftrightarrow T-z\}}\,. \tag{4.4}$$

Here, $\bar{\mathcal{O}}(e^{-\frac{\pi^2 n}{T^2}})$ is uniform in z.

Remark 4.3. We recall that equations such as (4.3) are to be understood in the sense that

$$\exists C>0, \forall T = T(n), \qquad \sup_{\substack{z\in[\![0,\frac{T}{2}]\!]\\ n\leftrightarrow z}} \left| \frac{\mathbb{P}_z(\tau = \tau_0 = n)}{\frac{2}{T}\sin\big(\frac{z\pi}{T}\big)\tan\big(\frac{\pi}{T}\big)e^{-g(T)n}} - 1 \right| \le Ce^{-\frac{\pi^2 n}{T^2}} \quad \text{as } n \to \infty\,.$$

Proof: The proof is inspired by Caravenna and Pétrélis (2009, Appendix B), but we need here a slightly sharper version. In (4.1) and (4.2) we denote $V_0 = V_0(n, T)$ the first term:

$$V_0 = \frac{2}{T}\cos^{n-1}\Big(\frac{\pi}{T}\Big)\sin\Big(\frac{\pi z}{T}\Big)\sin\Big(\frac{\pi}{T}\Big) = \frac{2}{T}\sin\Big(\frac{z\pi}{T}\Big)\tan\Big(\frac{\pi}{T}\Big)e^{-g(T)n}\,.$$

It remains to control the remaining terms. We let

$$V_1 := \frac{2}{T}\sum_{2\le k<T/2}\cos^{n-1}\Big(\frac{\pi k}{T}\Big)\sin\Big(\frac{\pi k z}{T}\Big)\sin\Big(\frac{\pi k}{T}\Big)\,,$$

and we only need to bound V_1/V_0. Using the bounds $\frac{2}{\pi}x \le \sin(x) \le x$ for $x \in [0, \frac{\pi}{2}]$, we get that

$$\frac{V_1}{V_0} \le \frac{\pi^2}{4}\sum_{2\le k<T/2} k^2 \left(\frac{\cos\big(\frac{\pi k}{T}\big)}{\cos\big(\frac{\pi}{T}\big)}\right)^{n-1}.$$

Now, as $\frac{k}{T} \to 0$, we have

$$\frac{\cos\big(\frac{\pi k}{T}\big)}{\cos\big(\frac{\pi}{T}\big)} = 1 - \frac{\pi^2(k^2-1)}{2T^2}\Big(1 + \bar{\mathcal{O}}\Big(\frac{k^2}{T^2}\Big)\Big)\,.$$

Hence, the l.h.s. is bounded by $\exp(-\frac{2\pi^2(k^2-1)}{5T^2})$ provided that $\frac{k}{T} \le \varepsilon$, for some given $\varepsilon \in (0, \frac{1}{2})$. If $\frac{k}{T} \ge \varepsilon$, we can simply bound $\cos\big(\frac{\pi k}{T}\big) \le \cos(\pi\varepsilon) \le e^{-\frac{1}{2}\pi^2\varepsilon^2}$. We therefore get that V_1/V_0 is bounded by a constant times

$$\sum_{2\le k<\varepsilon T} k^2 e^{-\frac{2n\pi^2(k^2-1)}{5T^2}} + \sum_{\varepsilon T\le k<T/2} k^2 e^{-\frac{1}{2}n\pi^2\varepsilon^2}\,.$$

For the first sum, we write that it is

$$\begin{aligned} e^{\frac{2\pi^2 n}{5T^2}}T^3 \times \frac{1}{T}\sum_{2\le k<\varepsilon T}\frac{k^2}{T^2}e^{-\frac{2\pi^2 nk^2}{5T^2}} &\le e^{\frac{2\pi^2 n}{5T^2}}T^3\int_{2/T}^{\infty} x^2 e^{-\frac{2\pi^2}{5}nx^2}\mathrm{d}x \\ &= e^{\frac{2\pi^2 n}{5T^2}}\frac{T^3}{n^{3/2}}\int_{2\sqrt{n}/T}^{\infty} u^2 e^{-\frac{2\pi^2}{5}u^2}\mathrm{d}u \le (cst.)\frac{T^2}{n}\exp\Big(-\frac{6\pi^2 n}{5T^2}\Big)\,, \end{aligned}$$

using that $\int_v^\infty u^2 e^{-\frac{2\pi^2}{5}u^2}\mathrm{d}u \sim (cst.)\, v\, e^{-\frac{2\pi^2}{5}v^2}$ as $v \to \infty$. This term is therefore bounded by a constant times $\exp(-\frac{\pi^2 n}{T^2})$, as n/T^2 goes to infinity.

For the other sum, we bound it by a constant times

$$T^3 \exp\Big(-\frac{1}{2}n\pi^2\varepsilon^2\Big) \le n^{3/2}\exp\Big(-\frac{1}{2}n\pi^2\varepsilon^2\Big) = \bar{o}\Big(\exp\Big(-\frac{\pi^2 n}{T^2}\Big)\Big),$$

as $\frac{n}{T^2} \to +\infty$ and $T \to \infty$. We have therefore shown that V_1/V_0 is bounded by a constant times $\exp(-\frac{\pi^2 n}{T^2})$, which concludes the proof. □

We now obtain an expression for the probability of staying in the band $[0,T]$, without touching the border, during a time $n \gg T^2$.

Lemma 4.4. *Assume that $T = T(n) \to \infty$ and that $\lim_{n\to\infty}\frac{n}{T^2} = +\infty$. Then, we have:*

- *If T is odd,*

$$\mathbb{P}_z(\tau > n) = \frac{2}{T}\sin\Big(\frac{z\pi}{T}\Big)\frac{1}{\tan\big(\frac{\pi}{2T}\big)}e^{-ng(T)}\big(1+\bar{\mathcal{O}}(e^{-\frac{\pi^2 n}{T^2}})\big). \tag{4.5}$$

- *If T is even, letting $a = \mathbb{1}_{\{n\leftrightarrow z\}}$,*

$$\mathbb{P}_z(\tau > n) = \frac{4}{T}\sin\Big(\frac{z\pi}{T}\Big)\frac{\cos^a\big(\frac{\pi}{T}\big)}{\sin\big(\frac{\pi}{T}\big)}e^{-ng(T)}\big(1+\bar{\mathcal{O}}(e^{-\frac{\pi^2 n}{T^2}})\big). \tag{4.6}$$

In particular, with a Taylor expansion, we get that

$$f_n(z,T) := \mathbb{P}_z(\tau > n) = \frac{4}{\pi}\sin\Big(\frac{z\pi}{T}\Big)e^{-ng(T)}\Big[1+\bar{\mathcal{O}}(T^{-2})+\bar{\mathcal{O}}\big(e^{-\frac{\pi^2 n}{T^2}}\big)\Big], \tag{4.7}$$

and note that if $n \ge \frac{1}{4}T^2\log T$ then $e^{-\frac{\pi^2 n}{T^2}} \le T^{-\frac{1}{4}\pi^2} \le T^{-2}$.

Proof: First of all, we write

$$\mathbb{P}_z(\tau > n) = \sum_{k>n}\Big(\mathbb{P}_z(\tau = \tau_0 = k) + \mathbb{P}_z(\tau = \tau_T = k)\Big). \tag{4.8}$$

When T is odd, then in (4.8), for each k in the sum there is only one term which is non-zero: applying Lemma 4.2 to estimate that term, we get

$$\begin{aligned}\mathbb{P}_z(\tau > n) &= \big(1+\bar{\mathcal{O}}(e^{-\frac{\pi^2 n}{T^2}})\big)\frac{2}{T}\sin\Big(\frac{z\pi}{T}\Big)\tan\Big(\frac{\pi}{T}\Big)\sum_{k>n}e^{-g(T)k}\\ &= \big(1+\bar{\mathcal{O}}(e^{-\frac{\pi^2 n}{T^2}})\big)\frac{2}{T}\sin\Big(\frac{z\pi}{T}\Big)\tan\Big(\frac{\pi}{T}\Big)e^{-g(T)n}\frac{\cos\big(\frac{\pi}{T}\big)}{1-\cos\big(\frac{\pi}{T}\big)},\end{aligned}$$

recalling that $e^{-g(T)} = \cos\big(\frac{\pi}{T}\big)$. This gives the desired result since $\frac{\sin(\theta)}{1-\cos(\theta)} = \frac{1}{\tan(\theta/2)}$.

When T is even, notice that in (4.8), either $k \leftrightarrow z$ and then both terms are non-zero or $k \not\leftrightarrow z$ and then both terms are zero. Applying Lemma 4.2, we get

$$\mathbb{P}_z(\tau > n) = \big(1+\bar{\mathcal{O}}(e^{-\frac{\pi^2 n}{T^2}})\big)\frac{4}{T}\sin\Big(\frac{z\pi}{T}\Big)\tan\Big(\frac{\pi}{T}\Big)\sum_{k>n}e^{-g(T)k}\mathbb{1}_{\{k\leftrightarrow z\}}.$$

To deal with the last sum, denote $n^* = n^*(z) := \min\{k > n, k \leftrightarrow z\}$: note that n^* is equal to $n+1+a$ with $a = \mathbb{1}_{\{n\leftrightarrow z\}}$. The indices for which the term is not zero can be written as $k = n^* + 2j$ and thus

$$\sum_{k>n}e^{-g(T)k}\mathbb{1}_{\{k\leftrightarrow z\}} = e^{-n^* g(T)}\sum_{j\ge 0}e^{-2g(T)j} = e^{-ng(T)}\frac{\cos^{1+a}\big(\frac{\pi}{T}\big)}{1-\cos^2\big(\frac{\pi}{T}\big)},$$

recalling that $e^{-g(T)} = \cos\big(\frac{\pi}{T}\big)$. This gives the announced expression. □

4.2. *Range estimates: proof of Theorem 1.6.* Recall the definition of the event $E_x^y(n) = \{M_n^- = -x, M_n^+ = y\}$ for any two positive integers x and y (the case where one equals 0 is dealt with in Section 4.2.4). We use Lemma 4.4 to estimate $\mathbb{P}_0(E_x^y(n))$, *i.e.* to prove Theorem 1.6. From this point onward, we always denote $T := x + y$ and consider that $\frac{n}{T^2} \to +\infty$. Using the spatial invariance of the random walk, we study the probability starting from x to stay in the strip $[0, T]$ *and* to touch both borders before time n. The symmetry of the walk allows us to assume $x \le y$ and $0 < x \le \frac{T}{2}$.

We now write the probability of $E_x^y(n)$ as the following differences

$$\begin{aligned}\mathbb{P}_0\big(E_x^y(n)\big) &= \mathbb{P}_0\big(M_n^+ < y+1, M_n^- = -x\big) - \mathbb{P}_0\big(M_n^+ < y, M_n^- = -x\big)\\ &= \mathbb{P}_0\big(M_n^+ < y+1, M_n^- > -x-1\big) - \mathbb{P}_0\big(M_n^+ < y+1, M_n^- > -x\big)\\ &\quad - \Big[\mathbb{P}_0\big(M_n^+ < y, M_n^- > -x-1\big) - \mathbb{P}_0\big(M_n^+ < y, M_n^- > -x\big)\Big].\end{aligned}$$

Then each of those probabilities is of a strict confinement event with different strips widths and starting points: we get

$$\mathbb{P}_0\big(E_x^y(n)\big) = f_n(x+1, T+2) - f_n(x, T+1) - f_n(x+1, T+1) + f_n(x, T). \tag{4.9}$$

Since for Theorem 1.6 we also assume that T and n satisfy the assumptions of Lemma 4.4, we can use it to estimate each of these terms.

In each of the following sections, we prove one of the formulae for $\Theta_n(x, y)$ given in Theorem 1.6, which we recall here.

With the assumptions of Theorem 1.6/Lemma 4.4 ($nT^{-2} \ge (cst.) \log T \to \infty$), we want to prove that $\mathbb{P}\big(E_x^y(n)\big) = (1 + \bar{o}(1))\Theta_n(x, y)$ with

$$\Theta_n(x,y) := \begin{cases} \dfrac{4}{\pi} \sin\left(\dfrac{\pi(x+1)}{T}\right) e^{-g(T+2)n} & \text{if } \dfrac{n}{T^3} \to +\infty;\\ \dfrac{4}{\pi}(e^{\alpha\pi^2} - 1)\left[e^{\alpha\pi^2} \sin\left(\dfrac{\pi(x+1)}{T}\right) - \sin\left(\dfrac{\pi x}{T}\right)\right] e^{-g(T)n} & \text{if } \dfrac{n}{T^3} \to \alpha \in (0, +\infty);\\ \dfrac{2\pi^3 n^2}{T^6} \sin\left(\dfrac{\pi x}{T}\right)\left[2 + \dfrac{\pi}{T \tan \frac{x\pi}{T}} + \dfrac{T^2}{n} \dfrac{1 - \frac{2x}{T}}{\pi \tan \frac{x\pi}{T}}\right] e^{-g(T+1)n} & \text{if } \dfrac{n}{T^3} \to 0. \end{cases}$$

Remark 4.5. The ratio $\frac{n}{T^3}$ that separates the cases is known to be the relevant quantity when studying such constrained random walk (see Caravenna and Pétrélis (2009)). The main reason is the terms in $e^{-ng(T)}, e^{-ng(T+1)}$ and $e^{-ng(T+2)}$ that appear in the expansion (4.9). Depending on the convergence of $\frac{n}{T^3}$, the ratios between these terms have different convergences, each leading to its own $\Theta_n(x, y)$.

4.2.1. *First case:* $\lim_{n\to\infty} \frac{n}{T^3} = +\infty$*;* In that case, we have

$$\frac{e^{-g(T+1)n}}{e^{-g(T+2)n}} = \left(1 - (1 + \bar{o}(1))\frac{\pi^2}{T^3}\right)^n \xrightarrow{n\to\infty} 0, \quad \frac{e^{-g(T)n}}{e^{-g(T+2)n}} = \left(1 - (1 + \bar{o}(1))\frac{2\pi^2}{T^3}\right)^n \xrightarrow{n\to\infty} 0.$$

Therefore, in view of (4.7), we have that $f_n(x, T+1), f_n(x+1, T+1), f_n(x, T)$ are all negligible compared to $f_n(x+1, T+2)$. Using (4.9), and (4.7), we therefore get that

$$\mathbb{P}_0\big(E_x^y(n)\big) = (1 + \bar{o}(1)) f_n(x+1, T+2) = (1 + \bar{o}(1)) \frac{4}{\pi} \sin\left(\frac{\pi(x+1)}{T+2}\right) e^{-g(T+2)n}, \tag{4.10}$$

where the $\bar{o}(1)$ depends only on T and is uniform in x, and we can use $\sin\left(\frac{\pi(x+1)}{T+2}\right) = (1 + \bar{O}(\frac{1}{T}))\sin\left(\frac{\pi(x+1)}{T}\right)$ uniformly in x to get the desired result.

4.2.2. *Second case:* $\lim_{n\to\infty} \frac{n}{T^3} = \alpha \in (0, +\infty)$*;* Similarly as above, we have

$$\lim_{n\to\infty} \frac{e^{-g(T+2)n}}{e^{-g(T)n}} = e^{2\alpha\pi^2}, \quad \lim_{n\to\infty} \frac{e^{-g(T+1)n}}{e^{-g(T)n}} = e^{\alpha\pi^2}.$$

Therefore, using (4.9) and (4.7), we get that

$$\begin{aligned}&\frac{\pi}{4} e^{g(T)n} \mathbb{P}_0\big(E_x^y(n)\big)\\ &\quad = (1+\bar{o}(1)) \Bigg[\sin\left(\frac{\pi(x+1)}{T}\right) e^{2\alpha\pi^2} + \sin\left(\frac{\pi x}{T}\right) - \left[\sin\left(\frac{\pi(x+1)}{T}\right) + \sin\left(\frac{\pi x}{T}\right)\right] e^{\alpha\pi^2} \Bigg],\end{aligned}$$

where we have used that $\sin(\frac{\pi(x+1)}{T+2}) = (1+\bar{o}(1))\sin(\frac{\pi(x+1)}{T})$ with $\bar{o}(1)$ uniform in x, and similarly with $\sin(\frac{\pi(x+1)}{T+1})$, $\sin(\frac{\pi x}{T+1})$. This gives the announced asymptotics.

Let us stress that, in the case where $x = \bar{\mathcal{O}}(1)$, we get

$$\mathbb{P}_0\big(E_x^y(n)\big) = (1+\bar{o}(1)) \frac{4}{T} (e^{\alpha\pi^2} - 1)(x(e^{\alpha\pi^2} - 1) + e^{\alpha\pi^2}) e^{-g(T)n}. \tag{4.11}$$

If on the other hand we have $x \to \infty$, then we have

$$\mathbb{P}_0\big(E_x^y(n)\big) = (1+\bar{o}(1)) \frac{4}{\pi} (e^{\alpha\pi^2} - 1)^2 \sin\left(\frac{x\pi}{T}\right) e^{-g(T)n}, \tag{4.12}$$

and one can also write $(e^{\alpha\pi^2} - 1)^2 = 4e^{\alpha\pi^2} \cosh^2(\alpha\pi^2)$.

4.2.3. *Last case:* $\lim_{n\to\infty} \frac{n}{T^3} = 0$*;* Recall that $\lim_{n\to\infty} \frac{n}{T^2} = +\infty$ by assumption, in particular $\frac{n}{T^4} = \bar{o}(\frac{n^2}{T^6})$. We then have the following expansions:

$$\frac{e^{-g(T)n}}{e^{-g(T+1)n}} = 1 - \frac{\pi^2 n}{T^3} + (1+\bar{o}(1))\frac{\pi^4 n^2}{2T^6}, \qquad \frac{e^{-g(T+2)n}}{e^{-g(T+1)n}} = 1 + \frac{\pi^2 n}{T^3} + (1+\bar{o}(1))\frac{\pi^4 n^2}{2T^6}.$$

Hence, from (4.9), using (4.7) (with the fact that $n \geq \frac{1}{4}T^2 \log T$ so that $\bar{\mathcal{O}}\big(e^{-\frac{\pi^2 n}{T^2}}\big) = \bar{o}(T^{-2})$) we get

$$\begin{aligned}\frac{\pi}{4}\mathbb{P}_0\big(E_x^y(n)\big)e^{g(T+1)n} = \Bigg\{ &\sin\left(\frac{\pi x}{T}\right)\left[1 - \frac{\pi^2 n}{T^3} + (1+\bar{o}(1))\frac{\pi^4 n^2}{2T^6}\right] \\ &- \left(\sin\left(\frac{\pi x}{T+1}\right) + \sin\left(\frac{\pi(x+1)}{T+1}\right)\right)\left[1 + \bar{\mathcal{O}}(T^{-2})\right] \\ &+ \sin\left(\frac{\pi(x+1)}{T+2}\right)\left[1 + \frac{\pi^2 n}{T^3} + (1+\bar{o}(1))\frac{\pi^4 n^2}{2T^6}\right]\Bigg\}.\end{aligned}$$

Note that we absorbed all terms $\bar{\mathcal{O}}(T^{-2})$ in the $\bar{o}(\frac{n^2}{T^6})$, since $\lim_{n\to\infty} \frac{n}{T^2} = +\infty$. Hence, we get that

$$\frac{\pi}{4}\mathbb{P}_0\big(E_x^y(n)\big)e^{g(T+1)n} = A + B + (1+\bar{o}(1))\frac{\pi^4 n^2}{2T^6}\left[\sin\left(\frac{\pi x}{T}\right) + \sin\left(\frac{\pi(x+1)}{T+2}\right)\right], \tag{4.13}$$

with

$$\begin{aligned}A &= \sin\left(\frac{x\pi}{T}\right) - 2\sin\left(\frac{\pi(x+\frac{1}{2})}{T+1}\right) + \sin\left(\frac{\pi(x+1)}{T+2}\right),\\ B &= \frac{n\pi^2}{T^3}\left[\sin\left(\frac{\pi(x+1)}{T+2}\right) - \sin\left(\frac{\pi x}{T}\right)\right].\end{aligned}$$

Here, for A, we have also used that $\sin(\frac{\pi x}{T+1}) + \sin(\frac{\pi(x+1)}{T+1}) = 2\sin(\frac{\pi(x+\frac{1}{2})}{T+1})\cos(\frac{\pi}{2(T+1)})$ with $\cos(\frac{\pi}{2(T+1)}) = 1 + \bar{\mathcal{O}}(T^{-2})$ and absorbed the $\bar{\mathcal{O}}(T^{-2})$ in the $\bar{o}(\frac{n^2}{T^6})$. Let us now compute both terms A and B.

Term B. Note that setting $v := \frac{T}{2} - x$ we have

$$B = \frac{n\pi^2}{T^3}\left[\cos\left(\frac{\pi v}{T+2}\right) - \cos\left(\frac{\pi v}{T}\right)\right].$$

Using the formula for the difference of cosines, we get that

$$\begin{aligned}\cos\left(\frac{\pi v}{T+2}\right) - \cos\left(\frac{\pi v}{T+1}\right) &= 2\sin\left(\frac{\pi v}{2(T+1)(T+2)}\right)\sin\left(\frac{\pi v}{T+1}\frac{T+\frac{3}{2}}{T+2}\right)\\ &= (1+\bar{o}(1))\frac{\pi v}{T^2}\sin\left(\frac{\pi v}{T}\right).\end{aligned}$$

We end up with

$$B = (1+\bar{o}(1))\frac{n\pi^3 v}{T^5}\sin\left(\frac{\pi v}{T}\right) = (1+\bar{o}(1))\frac{\pi^3 n}{2T^4}\left(1-\frac{2x}{T}\right)\cos\left(\frac{\pi x}{T}\right). \tag{4.14}$$

Term A. As far as A is concerned, notice that setting $v := \frac{T}{2} - x$ we have

$$A = \cos\left(\frac{\pi v}{T}\right) - 2\cos\left(\frac{\pi v}{T+1}\right) + \cos\left(\frac{\pi v}{T+2}\right).$$

Using the formula for the difference of cosines, we get that $A/2$ is equal to

$$\begin{aligned}&-\sin\left(\frac{\pi v}{2T(T+1)}\right)\sin\left(\frac{\pi v}{T}\frac{T+\frac{1}{2}}{T+1}\right) + \sin\left(\frac{\pi v}{2(T+1)(T+2)}\right)\sin\left(\frac{\pi v}{T}\frac{T(T+\frac{3}{2})}{(T+1)(T+2)}\right)\\ &= \frac{\pi v}{2T^2}\left[\sin\left(\frac{\pi v}{T}\frac{T(T+\frac{3}{2})}{(T+1)(T+2)}\right) - \sin\left(\frac{\pi v}{T}\frac{T+\frac{1}{2}}{T+1}\right)\right] + \bar{\mathcal{O}}\left(\frac{v}{T^3}\sin\left(\frac{\pi v}{T}\right)\right),\end{aligned}$$

where we have used that $\sin\left(\frac{\pi v}{2T(T+1)}\right) = \frac{\pi v}{2T^2}(1+\bar{\mathcal{O}}(T^{-1}))$ and similarly for $\sin\left(\frac{\pi v}{2(T+1)(T+2)}\right)$. Using the formula for the difference of sines, we get

$$\begin{aligned}&\sin\left(\frac{\pi v}{T}\frac{T(T+\frac{3}{2})}{(T+1)(T+2)}\right) - \sin\left(\frac{\pi v}{T}\frac{T+\frac{1}{2}}{T+1}\right)\\ &\qquad = -2\sin\left(\frac{\pi v}{2T(T+2)}\right)\cos\left(\frac{\pi v}{T}\left[1+\bar{\mathcal{O}}(T^{-1})\right]\right).\end{aligned}$$

Conclusion. Hence, we end up with $A = \bar{\mathcal{O}}\left(\frac{v^2}{T^4}\cos(\frac{\pi v}{T})\right) + \bar{\mathcal{O}}\left(\frac{v}{T^3}\sin(\frac{\pi v}{T})\right)$. To compare with B, see that

$$\frac{\frac{v^2}{T^4}\cos\left(\frac{\pi v}{T}\right) + \frac{v}{T^3}\sin\left(\frac{\pi v}{T}\right)}{\frac{nv}{T^5}\sin\left(\frac{\pi v}{T}\right)} = \frac{vT}{n\tan\frac{\pi v}{T}} + \frac{T^2}{n}. \tag{4.15}$$

If $\frac{\pi v}{T} \to 0$, we have

$$\frac{vT}{n\tan\frac{\pi v}{T}} + \frac{T^2}{n} = \frac{vT}{n(\frac{\pi v}{T} + \bar{o}(\frac{v}{T}))} + \frac{T^2}{n} = (1+\tfrac{1}{\pi}+\bar{o}(1))\frac{T^2}{n} \xrightarrow[n\to\infty]{} 0.$$

If $\frac{\pi v}{T} \sim \alpha \in (0, \frac{\pi}{2})$ we have

$$\frac{vT}{n\tan\frac{\pi v}{T}} + \frac{T^2}{n} = \frac{(\alpha+\bar{o}(1))T^2}{\pi n(\tan\alpha + \bar{o}(1))} + \frac{T^2}{n} = \left(1 + \frac{\alpha}{\pi\tan\alpha} + \bar{o}(1)\right)\frac{T^2}{n} \xrightarrow[n\to\infty]{} 0.$$

In both cases, the right-hand side of (4.15) goes to zero, which proves that A is negligible compared to B. Therefore,

$$\begin{aligned}\frac{\pi}{4}\mathbb{P}_0\big(E_x^y(n)\big)e^{g(T+1)n} &= (1+\bar{o}(1))\frac{\pi^4 n^2}{2T^6}\Big(\sin\left(\frac{\pi x}{T}\right)+\sin\left(\frac{\pi(x+1)}{T+2}\right)\Big)+(1+\bar{o}(1))B\\ &= (1+\bar{o}(1))\frac{\pi^4 n^2}{2T^6}\sin\left(\frac{\pi x}{T}\right)\Big(2+\frac{\pi}{T\tan\frac{x\pi}{T}}\Big)+(1+\bar{o}(1))B\,,\end{aligned}$$

and plugging in (4.14) yields the result.

4.2.4. *Estimates for a positive random walk.* Note that our estimates need adjustments when $x=0$ or $x=T$, which was commented on just after Theorem 1.6. We announced that the theorem was still true with $x=0$ (or identically with $y=0$), except when $n/T^3\to 0$. This section is devoted to the proof of these claims.

We first go back to correct (4.9) in order to take into account that 0 is the starting point of the walk. We write

$$\begin{aligned}\mathbb{P}_0\big(E_x^y(n)\big) &= \mathbb{P}_0\big(M_n^->-1, M_n^+<y+1\big)-\mathbb{P}_0\big(M_n^->-1, M_n^+<y\big)\\ &= f_n(1,T+2)-f_n(1,T+1)\,.\end{aligned}$$

Note that $y=T$ but we will keep separating the notations y and T. Thus, we have

$$\mathbb{P}_0\big(E_x^y(n)\big) = \frac{4}{\pi}\left[\sin\left(\frac{\pi}{T+2}\right)e^{-g(T+2)n}-\sin\left(\frac{\pi}{T+1}\right)e^{-g(T+1)n}+\bar{\mathcal{O}}(T^{-3})\right]. \tag{4.16}$$

Once again we have different asymptotics depending on the ratio n/T^3 that we rapidly present in the following.

Case $\frac{n}{T^3}\to+\infty$. As previously, $e^{-g(T+2)n}$ is the dominant term and thus

$$\mathbb{P}_0\big(E_x^y(n)\big) = \frac{4}{\pi}(1+\bar{o}(1))\sin\left(\frac{\pi}{T+2}\right)e^{-g(T+2)n}\,, \tag{4.17}$$

and we get the formula of (4.10) applied to $x=0$.

Case $\frac{n}{T^3}\to\alpha$. Factorize by $e^{-g(T)n}$ as in the general case, we thus write

$$\mathbb{P}_0\big(E_x^y(n)\big) = \frac{4}{\pi}\left[\sin\left(\frac{\pi}{T+2}\right)e^{2\alpha\pi^2}-\sin\left(\frac{\pi}{T+1}\right)e^{\alpha\pi^2}+\bar{\mathcal{O}}(T^{-3})\right]e^{-g(T)n}\,.$$

That can be rewritten as

$$\mathbb{P}_0\big(E_x^y(n)\big) = \frac{4}{\pi}(1+\bar{o}(1))e^{\alpha\pi^2}(e^{\alpha\pi^2}-1)\sin\left(\frac{\pi}{T}\right)e^{-g(T)n}\,,$$

which is exactly (4.11) taken at $x=0$.

Case $\frac{n}{T^3}\to 0$. In this case, we again factorize by $e^{-g(T+1)n}$ and write

$$\mathbb{P}_0\big(E_x^y(n)\big) = \frac{4}{\pi}\left[\sin\left(\frac{\pi}{T+2}\right)\Big[1+\frac{n\pi^2}{T^3}+\frac{n^2\pi^4}{2T^6}(1+\bar{o}(1))\Big]-\sin\left(\frac{\pi}{T+1}\right)+\bar{\mathcal{O}}(T^{-3})\right]e^{-g(T+1)n}\,.$$

We are left to compare all the terms in this expression :

$$A=\sin\left(\frac{\pi}{T+2}\right)-\sin\left(\frac{\pi}{T+1}\right)=-2\sin\left(\frac{\pi}{2(T+1)(T+2)}\right)\cos\left(\frac{\pi}{2T}\right)\sim-\frac{\pi}{T^2}\,,$$

$$B=\frac{n\pi^2}{T^3}\sin\left(\frac{\pi}{T+2}\right)\sim\frac{n\pi^3}{T^4},\qquad D=\frac{n^2\pi^4}{T^6}\sin\left(\frac{\pi}{T+2}\right)\sim\frac{n^2\pi^5}{T^7}\,.$$

See that $A\ll B$ and $D\ll B$ using both $\frac{n}{T^2}\to\infty$ and $\frac{n}{T^3}\to 0$, meaning that

$$\mathbb{P}_0\big(E_x^y(n)\big) = \frac{4n\pi}{T^3}(1+\bar{o}(1))\sin\left(\frac{\pi}{T+2}\right)e^{-g(T+1)n}\,.$$

This proves the formula given in (1.7).

Acknowledgements. The author would like to thank his PhD advisors Quentin Berger and Julien Poisat for reviewing this paper and for their continual help, as well as the anonymous reviewers for their comments and for pointing out mistakes in previous versions.

References

Athreya, S., Drewitz, A., and Sun, R. Random walk among mobile/immobile traps: a short review. In *Sojourns in probability theory and statistical physics. III. Interacting particle systems and random walks, a Festschrift for Charles M. Newman*, volume 300 of *Springer Proc. Math. Stat.*, pp. 1–22. Springer, Singapore (2019). MR4044387.

Berestycki, N. and Cerf, R. The random walk penalised by its range in dimensions $d \geqslant 3$. *Ann. H. Lebesgue*, **4**, 1–79 (2021). MR4213155.

Berger, Q., den Hollander, F., and Poisat, J. Annealed scaling for a charged polymer in dimensions two and higher. *J. Phys. A*, **51** (5), 054002, 37 (2018). MR3755571.

Berger, Q., Huang, C.-H., Torri, N., and Wei, R. One-dimensional polymers in random environments: stretching vs. folding. *Electron. J. Probab.*, **27**, Paper No. 162, 45 (2022a). MR4524009.

Berger, Q., Torri, N., and Wei, R. Non-directed polymers in heavy-tail random environment in dimension $d \geq 2$. *Electron. J. Probab.*, **27**, Paper No. 148, 67 (2022b). MR4512391.

Bolthausen, E. Localization of a two-dimensional random walk with an attractive path interaction. *Ann. Probab.*, **22** (2), 875–918 (1994). MR1288136.

Bouchot, N. Scaling limit of a one-dimensional polymer in a repulsive i.i.d. environment. *Electron. J. Probab.*, **29**, no. 61, 1–43 (2024). MR4736268.

Caravenna, F., den Hollander, F., Pétrélis, N., and Poisat, J. Annealed scaling for a charged polymer. *Math. Phys. Anal. Geom.*, **19** (1), Art. 2, 87 (2016). MR3457005.

Caravenna, F. and Pétrélis, N. Depinning of a polymer in a multi-interface medium. *Electron. J. Probab.*, **14**, no. 70, 2038–2067 (2009). MR2550292.

den Hollander, F. *Random polymers*, volume 1974 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (2009). ISBN 978-3-642-00332-5. Lectures from the 37th Probability Summer School held in Saint-Flour, 2007. MR2504175.

Ding, J., Fukushima, R., Sun, R., and Xu, C. Geometry of the random walk range conditioned on survival among Bernoulli obstacles. *Probab. Theory Related Fields*, **177** (1-2), 91–145 (2020). MR4095012.

Ding, J. and Xu, C. Poly-logarithmic localization for random walks among random obstacles. *Ann. Probab.*, **47** (4), 2011–2048 (2019). MR3980914.

Donsker, M. D. and Varadhan, S. R. S. Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.*, **28** (4), 525–565 (1975). MR397901.

Donsker, M. D. and Varadhan, S. R. S. On the number of distinct sites visited by a random walk. *Comm. Pure Appl. Math.*, **32** (6), 721–747 (1979). MR539157.

Feller, W. *An introduction to probability theory and its applications. Vol. I.* John Wiley & Sons, Inc., New York-London-Sydney, third edition (1968). MR228020.

Huang, C.-H. The scaling limits for Wiener sausages in random environments. *ArXiv Mathematics e-prints* (2019). arXiv: 1902.04930.

Schilling, R. L. and Partzsch, L. *Brownian motion. An introduction to stochastic processes.* De Gruyter Graduate. De Gruyter, Berlin, second edition (2014). ISBN 978-3-11-030729-0; 978-3-11-030730-6. MR3234570.

Schmock, U. Convergence of the normalized one-dimensional Wiener sausage path measures to a mixture of Brownian taboo processes. *Stochastics Stochastics Rep.*, **29** (2), 171–183 (1990). MR1041034.