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Arrivals are universal in coalescing ballistic annihilation

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Abstract. Coalescing ballistic annihilation is an interacting particle system intended to model features of certain chemical reactions. Particles are placed with independent and identically distributed spacings on the real line and begin moving with velocities sampled from $-1, 0$, and 1. Collisions result in either coalescence or mutual annihilation. For a variety of symmetric coalescing rules, we prove that the law of the index of the first particle to arrive at the origin does not depend on the law for spacings between particles.

1. Introduction

We study a universality property of the coalescing ballistic annihilation process from Benitez et al. (2023). These dynamics were introduced and studied by physicists at the end of the 20th century Carnevale et al. (1990); Ermakov et al. (1998); Blythe et al. (2000). There has been a recent revival of interest from mathematicians Haslegrave et al. (2021); Junge and Lyu (2022); Burdinski et al. (2019); Sidoravicius and Tournier (2017); Dygert et al. (2019). The motivation for the dynamics comes from diffusion-limited annihilating systems Bramson and Lebowitz (1991) inspired by natural phenomena such as thermal variation Toussaint and Wilczek (1983), turbulent flows Hill (1976), and porous media Raje and Kapoor (2000).

The process, though simple to define, has complex combinatorial and probabilistic structures. As is commonly done, we consider the *one-sided* version of ballistic annihilation. The initial conditions have a particle $\bullet_k$ at $x_k \in (0,\infty)$ for $k \geq 1$ with interdistances $x_{k+1} - x_k$ that are independent and identically distributed with nonnegative, continuous probability distribution μ. Each particle is independently assigned a velocity from $-1, 0, 1$, and we designate the event describing the velocity of $\bullet_k$ with $\overleftarrow{\bullet}_k, \dot{\bullet}_k$, and $\vec{\bullet}_k$, respectively. We assume that $\mathbf{P}(\dot{\bullet}_k) = p$ and $\mathbf{P}(\overleftarrow{\bullet}_k) = (1-p)/2 = \mathbf{P}(\vec{\bullet}_k)$

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for a fixed parameter $p \in (0,1)$. At time 0, each particle begins moving at its assigned velocity across $\mathbb{R}$.

We denote the event that two particles $\bullet_j$ and $\bullet_k$ meet at the same time and location by $\bullet_j \longleftrightarrow \bullet_k$. Upon meeting the particles react. While there are many possible reactions, it was discovered in Benitez et al. (2023) that some are more amenable to analysis. We restrict our attention to *three-parameter coalescing ballistic annihilation* (TCBA) systems. TCBA allows for moving particles to spontaneously survive collisions (equivalently, to generate a new moving particle), or to generate a $\dot{\bullet}$-particle. Fix parameters $0 \leq a, b, c \leq 1$ with $a + b \leq 1$. Let $[\bullet \longleftrightarrow \bullet \implies \Theta,\ \theta]$ denote a collision that generates $\Theta \in \{\dot{\bullet}, \vec{\bullet}, \overleftarrow{\bullet}, \varnothing\}$ independently with probability θ. The reaction rules are:

$$\vec{\bullet} \longleftrightarrow \overleftarrow{\bullet} \implies \begin{cases} \overleftarrow{\bullet}, & a/2 \\ \vec{\bullet}, & a/2 \\ \dot{\bullet}, & b \\ \varnothing, & 1-(a+b) \end{cases} \qquad \begin{aligned} \dot{\bullet} \longleftrightarrow \overleftarrow{\bullet} &\implies \begin{cases} \overleftarrow{\bullet}, & c \\ \varnothing, & 1-c \end{cases} \\ \vec{\bullet} \longleftrightarrow \dot{\bullet} &\implies \begin{cases} \vec{\bullet}, & c \\ \varnothing, & 1-c \end{cases}. \end{aligned} \tag{1.1}$$

We will refer to the special case $a = b = c = 0$ with only mutual annihilation as *simple ballistic annihilation.*

Denote the event that the site x is visited by the particle started at x_k by $x \longleftrightarrow \overleftarrow{\bullet}_k$. Let $A = \min\{k \colon 0 \longleftrightarrow \overleftarrow{\bullet}_k\}$ be the index of the first particle to reach the origin. It was proven in Haslegrave et al. (2021, Theorem 2) that the law of A does not depend on the spacing distribution μ for simple ballistic annihilation. We generalize this to TCBA.

Theorem 1.1. *The law of A does not depend on μ for TCBA and $\mathbf{E}[t^A]$ satisfies the recursion at* (2.20).

One of the main quantities of interest in ballistic annihilation is $q := \mathbf{P}(A < \infty)$ and the phase transition $p_c = \sup\{p \colon q = 1\}$. It is proven in Haslegrave et al. (2021) for simple ballistic annihilation and in Benitez et al. (2023) for TCBA that q and p_c do not depend on μ. In Haslegrave et al. (2021), the authors further discovered that the *skyline* of collision shapes for $p > p_c$ does not depend on μ. In Haslegrave and Tournier (2021), additional spacing universality properties were observed for simple ballistic annihilation as well as for an asymmetric version introduced in Junge and Lyu (2022). Broutin and Marckert proved that the related bullet process with finitely many particles has a universal law governing the number of surviving particles that does not depend on the velocity or spacing laws Broutin and Marckert (2020). Junge, Miguel, Reeves, and Sanchéz proved a non-universality result for ballistic annihilation with superimposed clusters with a random number of blockades Junge et al. (2023). Namely, the critical value depends on the variance of the cluster size.

Ballistic annihilation dynamics are notoriously sensitive to perturbation; changing the velocity of a single particle can have cascading effects. This feature makes the coalescing version significantly more complex. Our interest in establishing Theorem 1.1 comes from a desire to understand the limits of techniques successfully applied to simple ballistic annihilation. Theorem 1.1 marks a step in this direction and suggests that TCBA may share other universality properties with simple ballistic annihilation. An additional feature of Theorem 1.1 is the implicit recursion of the generating function of A. A special case of this recursion was utilized in Haslegrave et al. (2021, Theorem 3) to describe the rate of decay of $\mathbf{P}(A > n)$. Our more general formula at (2.20) is a first step towards describing the right tail of the distribution of A in TCBA.

The method of proving Theorem 1.1 is similar to what was done in Haslegrave et al. (2021). The idea is to prove by induction that the coefficients of the generating function $\mathbf{E}[t^A]$ do not depend on μ. Coalescence makes the details more involved and requires additional considerations. For example, we must distinguish between strong (the visiting arrow will destroy the next arrow it meets) and weak (the next arrow met by the visiting arrow would survive) visits. This brings in a second generating function related to the index of the first strong visit to 0.

A natural extension of this work would be to prove that the analogue of A is universal with respect to particle spacings in *four-parameter ballistic annihilation* (FCBA) that includes the extra reactions $(\vec{\bullet} \text{ — } \dot{\bullet}) \to \dot{\bullet}$ and $(\dot{\bullet} \text{ — } \overleftarrow{\bullet}) \to \dot{\bullet}$ in which blockades survive collisions with some probability β. Recently, Affeld, Dean, Junge, Liu, Panish, and Reeves overcame a difficulty observed in Benitez et al. (2023) to compute p_c in FCBA Affeld et al. (2024). It may be possible to apply their technique to prove universality of A in FCBA. However, when computing p_c, they relied on events with an arbitrary number of particles. The approach we use to prove universality of A often restricts to systems with a fixed number of particles. To quote Affeld et al. (2024), "Fixing the number of particles poses a serious difficulty when making various distance comparisons, because it introduces an extra constraint to already delicate calculations." We leave generalizing our result to FCBA as an open problem.

2. Proof of Theorem 1.1

We will write $\hat{\bullet}$ to denote a stationary particle generated from a $\vec{\bullet} \text{ — } \overleftarrow{\bullet}$ collision. If the collision involved $\vec{\bullet}_j$ and $\overleftarrow{\bullet}_k$, then we write $\hat{\bullet}_{j,k}$ for the stationary particle now inhabiting $(x_j + x_k)/2$.

For positive integers j and k, we define the collision events

$$
\begin{aligned}
\dot{\bullet}_j \longleftrightarrow \overleftarrow{\bullet}_k &:= (\dot{\bullet}_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\dot{\bullet}_j \text{ — } \overleftarrow{\bullet}_k \implies \varnothing\} \\
\dot{\bullet}_j \longleftarrow \overleftarrow{\bullet}_k &:= (\dot{\bullet}_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\dot{\bullet}_j \text{ — } \overleftarrow{\bullet}_k \implies \overleftarrow{\bullet}_k\} \\
\vec{\bullet}_j \longleftrightarrow \overleftarrow{\bullet}_k &:= (\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k \implies \varnothing\} \\
\vec{\bullet}_j \longleftarrow \overleftarrow{\bullet}_k &:= (\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k \implies \overleftarrow{\bullet}_k\} \\
\vec{\bullet}_j \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_k &:= (\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k \implies \hat{\bullet}_{j,k}\} \\
\vec{\bullet}_j \leftharpoonup \overleftarrow{\bullet}_k &:= (\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_j \implies \text{any but } \vec{\bullet}_j\} \\
\vec{\bullet}_j \leftharpoondown \overleftarrow{\bullet}_k &:= (\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\vec{\bullet}_j \text{ — } \overleftarrow{\bullet}_k \implies \vec{\bullet}_j\}.
\end{aligned}
$$

Specify generic collision events by:

$$
\begin{aligned}
\bullet_j \text{ — } \overleftarrow{\bullet} &:= \{\text{there exists } k \text{ with } \{\bullet_j \text{ — } \overleftarrow{\bullet}_k\}\} \\
\vec{\bullet}_j \text{ — } \dot{\bullet} &:= \{\text{there exists } k \text{ with } \{\vec{\bullet}_j \text{ — } \dot{\bullet}_k\}\} \\
\vec{\bullet}_j \text{ — } \hat{\bullet} &:= \{\text{there exist } k \text{ and } \ell \text{ with } \{\vec{\bullet}_j \text{ — } \hat{\bullet}_{k,\ell}\}\},
\end{aligned}
$$

where — can be replaced by any admissible collision event.

To determine how reactions occur, we assign to each $\vec{\bullet}$-particle a stack of independent instructions for $\vec{\bullet} \text{ — } \dot{\bullet}$ reaction types with probabilities as at (1.1). When $\vec{\bullet}_j$ collides with a blockade the smallest index unused instruction is used to determine the reaction type. We assign to each $\overleftarrow{\bullet}$-particle two independent stacks of reaction instructions distributed as at (1.1) to determine the outcomes of $\vec{\bullet} \text{ — } \overleftarrow{\bullet}$ and $\dot{\bullet} \text{ — } \overleftarrow{\bullet}$ collisions. This construction ensures that the reaction type of the next collision can be read off from the instructions before it occurs. Thus, the following visiting events are well-defined.

$$
\begin{aligned}
x_j \overset{\cdot}{\longleftrightarrow} \bullet_k &:= (x_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\overleftarrow{\bullet}_k \text{ mutually annihilates in its next } \dot{\bullet} \text{ collision}\} \\
x_j \longleftarrow \overleftarrow{\bullet}_k &:= (x_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\overleftarrow{\bullet}_k \text{ survives its next } \dot{\bullet} \text{ collision}\} \\
x_j \leftharpoonup \overleftarrow{\bullet}_k &:= (x_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\text{the next } \vec{\bullet}\text{-particle } \overleftarrow{\bullet}_k \text{ meets is annihilated}\} \\
x_j \leftharpoondown \overleftarrow{\bullet}_k &:= (x_j \text{ — } \overleftarrow{\bullet}_k) \wedge \{\text{the next } \vec{\bullet}\text{-particle } \overleftarrow{\bullet}_k \text{ meets survives}\}.
\end{aligned}
$$

Given an interval I and an event B, we write B_I for the event in TCBA restricted to only the particles initially in I. We will use various forms of renewal that occur in TCBA. These come from the fact that the particles behind a moving particle cannot influence events involving the moving particle. For example, $\mathbf{P}((x_\ell \text{ — } \overleftarrow{\bullet}) \mid (\dot{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet}_\ell)) = \mathbf{P}(x_\ell \text{ — } \overleftarrow{\bullet}) = \mathbf{P}(0 \text{ — } \overleftarrow{\bullet})$.

We will call a visit to x_j by a left-moving particle where the left-moving particle will destroy the next right-moving particle it meets after having visited x_j (as in $\{x_j \leftharpoonup \overleftarrow{\bullet}\}$) a *strong* visit. On the other hand, we refer to *weak* visits to x_j as those by a left-moving particle whose next collision with a right-moving particle after visiting x_j will result in the right-moving particle surviving. Note that even if no such arrow-arrow collision occurs, we can sample the outcome of such an event in advance at the moment $\overleftarrow{\bullet}$ visits x_j. Thus, each visit to x_j by a left-arrow must be either strong or weak and is decidable at the instant the visit occurs.

We designate first visits to a given site with the following notation:

$$x_j \overset{1}{-\!\!-} \overleftarrow{\bullet}_k := \{\overleftarrow{\bullet}_k \text{ is the first left-moving particle to reach } x_j\}.$$

It will be necessary to refine the notion of a first visit into the following events:

$$\begin{aligned}
x_j \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_k &:= \{\overleftarrow{\bullet}_k \text{ is the first left-moving particle to strongly visit } x_j\} \\
x_j \overset{1}{\leftharpoondown} \overleftarrow{\bullet}_k &:= \{\overleftarrow{\bullet}_k \text{ is the first left-moving particle to weakly visit } x_j\} \\
x_j \overset{1}{\longleftrightarrow} \overleftarrow{\bullet}_k &:= (x_j \overset{1}{-\!\!-} \overleftarrow{\bullet}_k) \wedge (x_j \overset{\cdot}{\longleftrightarrow} \overleftarrow{\bullet}_k) \\
x_j \overset{1}{\longleftarrow} \overleftarrow{\bullet}_k &:= (x_j \overset{1}{-\!\!-} \overleftarrow{\bullet}_k) \wedge (x_j \longleftarrow \overleftarrow{\bullet}_k).
\end{aligned}$$

We define the index of the first particle to strongly visit 0 by

$$A^* := \min\{k \colon 0 \leftharpoonup \overleftarrow{\bullet}_k\}.$$

Let $p_n := \mathbf{P}(A = n)$ and $p_n^* := \mathbf{P}(A^* = n)$.

We start by describing a relationship between p_n and p_n^*. Note that both quantities are 0 for $n = 0$, and that $p_1 = (1 - p)/2$.

Lemma 2.1. *For $n \geq 1$ it holds that*

$$p_n^* = \sum_{w=0}^{n-1} (\tfrac{a}{2})^w (1 - \tfrac{a}{2}) \sum_{0=\ell_0<\ell_1<\cdots<\ell_w<\ell_{w+1}=n} \prod_{i=1}^{w+1} p_{\ell_i - \ell_{i-1}}. \tag{2.1}$$

Proof: We can decompose $\{A^* = n\}$ in terms of the number of weak visits w to 0 that precede the first strong visit from $\overleftarrow{\bullet}_n$:

$$p_n^* = \sum_{w=0}^{n-1} \sum_{0=\ell_0<\ell_1<\cdots<\ell_w<n} \mathbf{P}\left((0 \leftharpoondown \overleftarrow{\bullet}_{\ell_1}) \wedge \cdots (0 \leftharpoondown \overleftarrow{\bullet}_{\ell_w}) \wedge (0 \leftharpoonup \overleftarrow{\bullet}_n)\right)$$

This is the same as having $w + 1$ visits to 0 where the first w visits are weak and the last is strong and each visit occurs independently. Formally,

$$p_n^* = \sum_{w=0}^{n-1} (\tfrac{a}{2})^w (1 - \tfrac{a}{2}) \sum_{0=\ell_0<\ell_1<\cdots<\ell_w<n} p_{\ell_1} p_{\ell_2 - \ell_1} \cdots p_{\ell_w - \ell_{w-1}} p_{n - \ell_w},$$

which is equivalent to the expression in (2.1). ☐

Our main tool is the following decomposition result for $\mathbf{P}(A = n)$.

Proposition 2.2. *For $n \geq 2$ it holds that*

$$p_n = \alpha_n + \dot{\beta}_n + \hat{\beta}_n + \gamma_n + \hat{\gamma}_n + \overleftarrow{\gamma}_n \tag{2.2}$$

with

$$\alpha_n := \mathbf{P}[(A=n)\wedge \dot{\bullet}_1] = cpp_{n-1} + (1-c)p \sum_{1<k<n} p_{k-1}p_{n-k} \tag{2.3}$$

$$\dot{\beta}_n := \mathbf{P}[(A=n)\wedge (\vec{\bullet}_1 \longleftrightarrow \dot{\bullet})] = \tfrac{1-c}{2} p \sum_{1<k<n} p_{k-1}p_{n-k} \tag{2.4}$$

$$\hat{\beta}_n := \mathbf{P}[(A=n)\wedge (\vec{\bullet}_1 \longleftrightarrow \hat{\bullet})] = \tfrac{1-c}{2} \sum_{1<k<\ell<n} \hat{\delta}_{\ell-k+1}p_{k-1}p_{n-\ell} \tag{2.5}$$

$$\gamma_n := \mathbf{P}[(A=n)\wedge (\vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet})] = \sum_{1<k<n} \delta_k p_{n-k} \tag{2.6}$$

$$\begin{aligned}\hat{\gamma}_n &:= \mathbf{P}[(A=n)\wedge (\vec{\bullet}_1 \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet})] \\ &= c\sum_{1<k<n} \hat{\delta}_k p_{n-k} + (1-c)\sum_{1<k<\ell<n} \hat{\delta}_k p_{\ell-k}p_{n-\ell}\end{aligned} \tag{2.7}$$

$$\overleftarrow{\gamma}_n := \mathbf{P}[(A=n)\wedge (\vec{\bullet}_1 \longleftarrow \overleftarrow{\bullet})] = \tfrac{a}{2}\bar{\delta}_n, \tag{2.8}$$

and

$$\begin{aligned}\delta_n^* &:= \mathbf{P}(\vec{\bullet}_1 \leftharpoondown \overleftarrow{\bullet}_n) \\ &= \tfrac{1-p}{2}p^*_{n-1} - \sum_{1<k<n} (\dot{\beta}_k + \hat{\beta}_k)p^*_{n-k} \\ &\quad - \tfrac{1}{2}(1-\tfrac{a}{2})(1-c)c \sum_{1<k<n} p_{k-1}p_{n-k} \\ &\quad - \tfrac{1}{2}(1-\tfrac{a}{2})(1-c)c \sum_{1<k<\ell<n} \hat{\delta}_{\ell-k+1}p_{k-1}p_{n-\ell}.\end{aligned} \tag{2.9}$$

Also let

$$\bar{\delta}_n := \mathbf{P}(\vec{\bullet}_1 \;\text{—}\; \overleftarrow{\bullet}_n) = \frac{\delta_n^*}{1-\frac{a}{2}} \tag{2.10}$$

$$\delta_n := \mathbf{P}(\vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet}_n) = (1-(a+b))\bar{\delta}_n \tag{2.11}$$

$$\hat{\delta}_n := \mathbf{P}(\vec{\bullet}_1 \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_n) = b\bar{\delta}_n. \tag{2.12}$$

Proof of (2.2)*:* This is a partitioning of the event $\{A=n\}$ based on the velocity of $\bullet_1$. We use the fact (ensured by symmetry) that $\vec{\bullet}_1$ is almost surely annihilated as observed in Benitez et al. (2023). □

Proof of (2.3)*:* Conditional on $\dot{\bullet}_1$, there are precisely two manners in which $A=n$. One is that $\overleftarrow{\bullet}_n$ is the first left-moving particle to reach x_1 and the reaction $\dot{\bullet} - \overleftarrow{\bullet} \implies \overleftarrow{\bullet}$ occurs. This occurs with probability

$$\mathbf{P}(\dot{\bullet}_1)\mathbf{P}(\dot{\bullet} \;\text{—}\; \overleftarrow{\bullet} \implies \overleftarrow{\bullet})p_{n-1} = pcp_{n-1}. \tag{2.13}$$

The other manner in which $A=n$ may occur conditional on $\dot{\bullet}_1$, is if there is some $1<k<n$ such that $\bullet_k$ is the first particle to reach x_1 from the right and a $[\dot{\bullet} \;\text{—}\; \overleftarrow{\bullet} \implies \varnothing]$ reaction occurs. Then $\overleftarrow{\bullet}_n$ is the first to reach x_k from the right. This second part happens with probability $\mathbf{P}(A=k-1)\mathbf{P}(A=n-k) = p_{k-1}p_{n-k}$. So, for each k we acquire the probability

$$\mathbf{P}(\dot{\bullet}_1)\mathbf{P}(\dot{\bullet} \;\text{—}\; \overleftarrow{\bullet} \implies \varnothing)p_{k-1}p_{n-k} = p(1-c)p_{k-1}p_{n-k}.$$

Summing over k and combining with (2.13) gives (2.3). □

Proof of (2.4)*:* The event $\{(A=n)\wedge(\vec{\bullet}_1 \longleftrightarrow \dot{\bullet}_k)\}$ occurs if and only if the following hold:

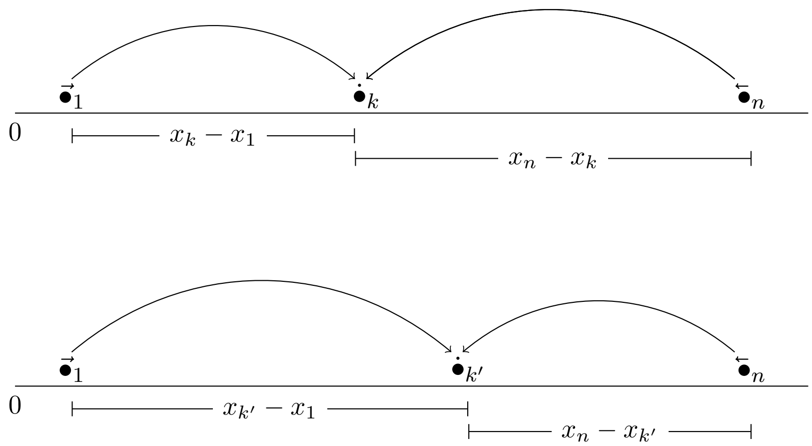

FIGURE 2.1. A configuration $\omega \in \{(A = n) \wedge (\vec{\bullet}_1 \longleftrightarrow \dot{\bullet}_k)\}$ (top) and its reversal (bottom). Particles between $\bullet_1$ and $\bullet_k$ and $\bullet_k$ and $\bullet_n$ are not shown. The arced arrows indicate that the particle is the first to reach that site among all particles started under the arc. Summing over all k and k' gives complementary events. This allows us to bypass any computations involving the interdistances.

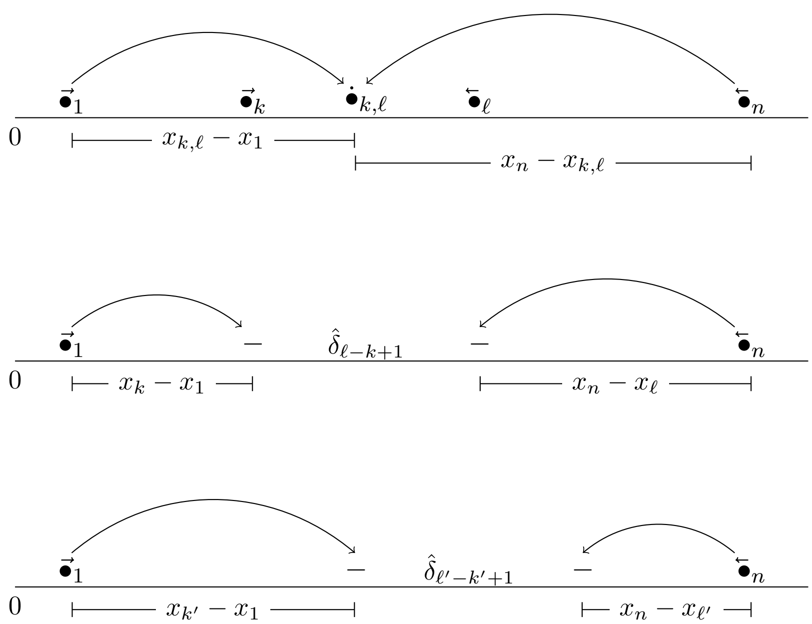

FIGURE 2.2. A configuration $\omega \in \{(A = n) \wedge (\vec{\bullet}_1 \longleftrightarrow \dot{\bullet}_{k,\ell})\}$ (top). The middle figure shows an equivalent formulation conditional on $\vec{\bullet}_k \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_\ell$. The bottom figure shows the configuration after reversing the particles in $[x_1, x_n]$.

- $\dot{\bullet}_k$ occurs.
- The first particle to reach x_k from the left is $\vec{\bullet}_1$, which mutually annihilates with $\dot{\bullet}_k$.
- The first particle to reach x_k from the right is $\overleftarrow{\bullet}_n$.
- And, $x_k - x_1 < x_n - x_k$.

We can then write

$$\dot{\beta}_n = \sum_{1<k<n} \mathbf{P}\big((\dot{\bullet}_k) \wedge (\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_k)_{(0,x_k)} \wedge (x_k \overset{1}{\text{---}} \overleftarrow{\bullet}_n)_{(x_k,\infty)} \\ \wedge (x_k - x_1 < x_n - x_k)\big).$$

Given a configuration $\omega \in \{(A = n) \wedge (\vec{\bullet}_1 \longleftrightarrow \dot{\bullet}_k)\}$ of particle locations and velocities, define $\mathrm{rev}_n(\omega)$ to be the reversed configuration. The particle at $x \in [x_1, x_n]$ corresponds to the particle in $\mathrm{rev}_n(\omega)$ with position $x_1 + (x_n - x)$ and is moving in the opposite direction. The symmetry of the parameters ensures that reversing the configuration preserves the probability: $\mathbf{P}(\omega) = \mathbf{P}(\mathrm{rev}_n(\omega))$. Since reversing maps the index k to $k' = n + 1 - k$, we may also write

$$\dot{\beta}_n := \sum_{1<k'<n} \mathbf{P}\big((\dot{\bullet}_{k'}) \wedge (\vec{\bullet}_1 \overset{1}{\text{---}} x_{k'})_{(0,x_{k'})} \wedge (x_{k'} \overset{1}{\longleftrightarrow} \overleftarrow{\bullet}_n)_{(x_{k'},\infty)} \\ \wedge (x_{k'} - x_1 > x_n - x_{k'})\big). \tag{2.14}$$

Since reactions are determined independently, we can swap the reaction types in (2.14) so that the probabilities

$$\mathbf{P}\Big((\dot{\bullet}_{k'}) \wedge (\vec{\bullet}_1 \overset{1}{\text{---}} x_{k'})_{(0,x_{k'})} \wedge (x_{k'} \overset{1}{\longleftrightarrow} \overleftarrow{\bullet}_n)_{(x_{k'},\infty)} \wedge (x_{k'} - x_1 > x_n - x_{k'})\Big)$$

and

$$\mathbf{P}\Big((\dot{\bullet}_{k'}) \wedge (\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_{k'})_{(0,x_{k'})} \wedge (x_{k'} \overset{1}{\text{---}} \overleftarrow{\bullet}_n)_{(x_{k'},\infty)} \wedge (x_{k'} - x_1 > x_n - x_{k'})\Big)$$

are equal.

Summing the two formulas for $\dot{\beta}_n$ and combining terms with the same index partitions the comparison between $x_k - x_1$ and $x_n - x_k$. See Figure 2.1. Note that this depends crucially on the continuity of μ, which ensures that we need not worry about events like $\{x_k - x_1 = x_n - x_k\}$. Thus,

$$\begin{aligned} 2\dot{\beta}_n &= \sum_{1<k<n} \mathbf{P}((\dot{\bullet}_k) \wedge (\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_k)_{(0,x_k)} \wedge (x_k \overset{1}{\text{---}} \overleftarrow{\bullet}_n)_{(x_k,\infty)}) \\ &= \sum_{1<k<n} p(1-c)p_{k-1}p_{n-k}. \end{aligned}$$

At the second step we apply independence. Dividing by 2 gives the claimed formula. □

Proof of (2.5)*:* The event $\{(A = n) \wedge (\vec{\bullet}_1 \longleftrightarrow \hat{\bullet}_{k,\ell})\}$ occurs if and only if the following hold:

- $\hat{\bullet}_{k,\ell}$ is generated from $\vec{\bullet}_k \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_\ell$ at $x_{k,\ell} = (x_k + x_\ell)/2$ for some $1 < k < \ell < n$.
- The first particle to reach $x_{k,\ell}$ from the left of x_k is $\vec{\bullet}_1$, which mutually annihilates with $\dot{\bullet}_{k,\ell}$.
- The first particle to reach $x_{k,\ell}$ from the right of x_ℓ is $\overleftarrow{\bullet}_n$.
- And, $x_{k,\ell} - x_1 < x_n - x_{k,\ell}$.

Thus,

$$\hat{\beta}_n = \sum_{1<k<\ell<n} \mathbf{P}\big((\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_k)_{(0,x_k)} \wedge (\vec{\bullet}_k \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_\ell)_{[x_k,x_\ell]} \\ \wedge (x_\ell \overset{1}{\text{---}} \overleftarrow{\bullet}_n)_{(x_\ell,\infty)} \wedge (x_{k,\ell} - x_1 < x_n - x_{k,\ell})\big).$$

Let $G_{k,\ell} = (\vec{\bullet}_k \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_\ell)_{[x_k,x_\ell]}$ so that $\mathbf{P}(G_{k,\ell}) = \hat{\delta}_{\ell-k+1}$. Conditioning gives

$$\hat{\beta}_n = \sum_{1<k<\ell<n} \hat{\delta}_{\ell-k+1}\mathbf{P}\big((\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_k)_{(0,x_k)} \wedge (x_\ell \overset{1}{\text{---}} \overleftarrow{\bullet}_n)_{(x_\ell,\infty)} \\ \wedge (x_{k,\ell} - x_1 < x_n - x_{k,\ell}) \mid G_{k,\ell}\big).$$

Since moving particles have unit speed, we have $x_{k,\ell} - x_1 < x_n - x_{k,\ell}$ if and only if $x_k - x_1 < x_n - x_\ell$. Using this observation and the fact that the events $(\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_k)_{(0,x_k)}$ and $(x_\ell \overset{1}{\text{——}} \overleftarrow{\bullet}_n)_{(x_\ell,\infty)}$ are independent of $G_{k,\ell}$ yields

$$\hat{\beta}_n = \sum_{1<k<\ell<n} \hat{\delta}_{\ell-k+1} \mathbf{P}\Big((\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_k)_{[x_1,x_k)} \wedge (x_\ell \overset{1}{\text{——}} \overleftarrow{\bullet}_n)_{(x_\ell,\infty)} \wedge (x_k - x_1 < x_n - x_\ell)\Big). \tag{2.15}$$

By reversing the configuration of particles in $[x_1, x_n]$ as with the proof of (2.4) (illustrated in Figure 2.2), we may also write

$$\hat{\beta}_n = \sum_{1<k'<\ell'<n} \hat{\delta}_{\ell'-k'+1} \mathbf{P}\Big((\vec{\bullet}_1 \overset{1}{\text{——}} x_{k'})_{[x_1,x_{k'})} \wedge (x_{\ell'} \overset{1}{\longleftrightarrow} \overleftarrow{\bullet}_n)_{(x_{\ell'},\infty)} \wedge (x_{k'} - x_1 > x_n - x_{\ell'})\Big). \tag{2.16}$$

Since reactions are determined independently, we can swap the reaction types in (2.16) as we did in the proof of (2.4). We may then rewrite (2.16) as

$$\hat{\beta}_n = \sum_{1<k'<\ell'<n} \hat{\delta}_{\ell'-k'+1} \mathbf{P}\Big((\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_{k'})_{[x_1,x_{k'})} \wedge (x_{\ell'} \overset{1}{\text{——}} \overleftarrow{\bullet}_n)_{(x_{\ell'},\infty)} \wedge (x_{k'} - x_1 > x_n - x_{\ell'})\Big). \tag{2.17}$$

Summing the two formulations of $\hat{\beta}$ at (2.15) and (2.17) removes the interval comparisons. Thus,

$$\begin{aligned} 2\hat{\beta}_n &= \sum_{1<k<\ell<n} \hat{\delta}_{\ell-k+1} \mathbf{P}\Big((\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_k)_{[x_1,x_k)} \wedge (x_\ell \overset{1}{\text{——}} \overleftarrow{\bullet}_n)_{(x_\ell,\infty)}\Big) \\ &= (1-c) \sum_{1<k<\ell<n} \hat{\delta}_{\ell-k+1} p_{k-1} p_{n-\ell}. \end{aligned}$$

Dividing by 2 gives (2.5). □

Proof of (2.6)*:* Using the definition of δ_n, it is straightforward to see that

$$\gamma_n = \sum_{1<k<n} \mathbf{P}(\vec{\bullet}_1 \longleftrightarrow \overleftarrow{\bullet}_k) \mathbf{P}(x_k \overset{1}{\text{——}} \overleftarrow{\bullet}_n) = \sum_{1<k<n} \delta_k p_{n-k}$$

which gives (2.6). □

Proof of (2.7)*:* The event in the probability at (2.7) may occur in two ways. One, there exists a $1 < k < n$ such that: $(\vec{\bullet}_1 \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_k) \wedge (\hat{\bullet}_{1,k} \longleftarrow \overleftarrow{\bullet}_n)$ occurs. Each such event is equivalent to $(\vec{\bullet}_1 \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_k) \wedge (x_k \longleftarrow \overleftarrow{\bullet}_n)$, which has probability $\hat{\delta}_k c p_{n-k}$. The other manner in which the event in the probability at (2.7) may occur is if for $1 < k < \ell < n$ we have

$$(\vec{\bullet}_1 \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_k) \wedge (x_{1,k} \overset{1}{\longleftrightarrow} \overleftarrow{\bullet}_\ell) \wedge (x_\ell \overset{1}{\text{——}} \overleftarrow{\bullet}_n)).$$

Conditional independence ensures that this event has probability $\hat{\delta}_k(1-c)p_{\ell-k+1}p_{n-\ell}$ as claimed in the second part of (2.7). □

Proof of (2.8)*:* The formula for $\overleftarrow{\gamma}_n$ is the simple observation that the event in question occurs if and only if $\{\vec{\bullet}_1 \longleftarrow \overleftarrow{\bullet}_n\}$, which has the claimed probability. □

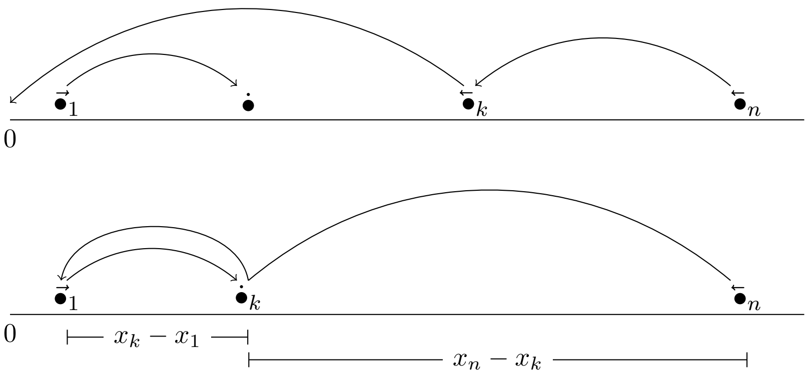

FIGURE 2.3. The top diagram shows a configuration in $\vec{\bullet}_1 \wedge (x_1 \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_n)_{(x_1,x_n]}$ for which $\vec{\bullet}_1 \mathrel{-\!\!-} \overleftarrow{\bullet}_n$ fails to occur. Arrows indicate that the particle from the tail of the arrow is the first to visit the location at the head of the arrow. The bottom diagram shows another type of configuration in which this may occur. Note that $\overleftarrow{\bullet}_n$ survives the indicated $\dot{\bullet} \mathrel{-\!\!-} \overleftarrow{\bullet}_n$ collision.

Proof of (2.9), (2.10), (2.11), *and* (2.12): The main work is proving (2.9). The other three formulas follow immediately by specifying the reaction. Towards (2.9), let

$$G = \vec{\bullet}_1 \wedge (x_1 \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_n)_{(x_1,\infty)}.$$

We can easily compute $\mathbf{P}(G) = \frac{1-p}{2} p^*_{n-1}$. We further claim that

$$(\vec{\bullet}_1 \leftharpoonup \overleftarrow{\bullet}_n) = G \setminus [\dot{B}_1 \cup \dot{B}_2 \cup \hat{B}_1 \cup \hat{B}_2] \tag{2.18}$$

with

$$\dot{B}_1 = \bigcup_{1<k<n} (\vec{\bullet}_1 \longleftrightarrow \dot{\bullet})_{[x_1,\infty)} \wedge (x_1 \overset{1}{\longleftrightarrow} \overleftarrow{\bullet}_k)_{[x_1,x_k]} \wedge (x_k \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_n)_{(x_k,\infty)},$$

$$\dot{B}_2 = \bigcup_{1<k<n} (\vec{\bullet}_1 \longleftrightarrow \dot{\bullet}_k)_{[x_1,\infty)} \wedge (x_k \overset{1}{\longleftarrow} \overleftarrow{\bullet}_n)_{(x_k,\infty)} \wedge (x_k \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_n)_{(x_k,\infty)}$$
$$\wedge (x_k - x_1 < x_n - x_k),$$

$$\hat{B}_1 = \bigcup_{1<k<n} (\vec{\bullet}_1 \longleftrightarrow \hat{\bullet})_{[x_1,\infty)} \wedge (x_1 \overset{1}{\longleftrightarrow} \overleftarrow{\bullet}_k)_{[x_1,x_k]} \wedge (x_k \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_n)_{(x_k,\infty)},$$

and

$$\hat{B}_2 = \bigcup_{1<k<\ell<n} (\vec{\bullet}_1 \overset{1}{\longleftrightarrow} x_k)_{[x_1,x_k)} \wedge (\vec{\bullet}_k \overset{\hat{\bullet}}{\longleftrightarrow} \overleftarrow{\bullet}_\ell)_{[x_k,x_\ell]} \wedge (x_\ell \overset{1}{\longleftarrow} \overleftarrow{\bullet}_n)_{(x_\ell,\infty)}$$
$$\wedge (x_\ell \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_n)_{(x_\ell,\infty)} \wedge (x_k - x_1 < x_n - x_\ell).$$

To see why (2.18) holds, first note that G is necessary for $\vec{\bullet}_1 \leftharpoonup \overleftarrow{\bullet}_n$. Next, we claim that $\dot{B}_1 \cup \dot{B}_2 \cup \hat{B}_1 \cup \hat{B}_2$ contains precisely the configurations in G for which $\vec{\bullet}_1$ does not collide with $\overleftarrow{\bullet}_n$. Indeed, $\vec{\bullet}_1$ cannot collide and be destroyed by a smaller index $\overleftarrow{\bullet}$-particle, since otherwise, that smaller index $\overleftarrow{\bullet}$-particle would strongly visit x_1 before $\overleftarrow{\bullet}_n$ in the process restricted to $(x_1, x_n]$. So, the configurations from G for which $\vec{\bullet}_1$ does not collide with $\overleftarrow{\bullet}_n$ must have $\vec{\bullet}_1$ mutually annihilating with a blockade. The events in $\dot{B}_1 \cup \dot{B}_2$ describe the configurations for which $\vec{\bullet}_1 \longleftrightarrow \dot{\bullet}$ and $x_1 \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_n$. See Figure 2.3. The configurations in $\hat{B}_1 \cup \hat{B}_2$ describe the configurations for which $\vec{\bullet}_1 \longleftrightarrow \hat{\bullet}$ and $x_1 \overset{1}{\leftharpoonup} \overleftarrow{\bullet}_n$.

Using independence and the definition of $\dot\beta_k$, it is easily seen that

$$\mathbf{P}(\dot B_1) = \sum_{1<k<n} \dot\beta_k p^*_{n-k}.$$

A similar reversal argument as in the proof of (2.4) yields

$$\mathbf{P}(\dot B_2) = (1-\tfrac{a}{2}) \sum_{1<k<n} \tfrac{1}{2}(1-c)cp_{k-1}p_{n-k}.$$

Similarly,

$$\mathbf{P}(\hat B_1) = \sum_{1<k<n} \hat\beta_k p^*_{n-k},$$

and a reversal argument like the one used to obtain (2.5) yields

$$\mathbf{P}(\hat B_2) = (1-\tfrac{a}{2}) \sum_{1<k<\ell<n} \tfrac{1}{2}(1-c)c\hat\delta_{\ell-k+1}p_{k-1}p_{n-\ell}.$$

Since all the individual events in $\dot B_1 \cup \dot B_2 \cup \hat B_1 \cup \hat B_2$ are disjoint, we subtract these equations from (2.18) to obtain (2.10).

□

Before we prove Theorem 1.1, we define the generating functions for $0 \le t \le 1$

$$f(t) = \mathbf{E}[t^A] = \sum_{n=0}^{\infty} p_n t^n \qquad \text{and} \qquad f^*(t) = \mathbf{E}[t^{A^*}] = \sum_{n=0}^{\infty} p^*_n t^n.$$

Note that $p_0 = 0$, so the sums could begin at $n = 1$. These are related by the following formula.

Lemma 2.3. *For all $0 \le t \le 1$ it holds that*

$$f^*(t) = \frac{(1-\frac{a}{2})f(t)}{1-\frac{a}{2}f(t)}.$$

Proof: Recall from (2.1) that

$$p^*_n = \sum_{w=0}^{n-1} (\tfrac{a}{2})^w (1-\tfrac{a}{2}) \sum_{0=\ell_0<\ell_1<\cdots<\ell_w<\ell_{w+1}=n} \prod_{i=1}^{w+1} p_{\ell_i-\ell_{i-1}}.$$

Since each weak visit occurs independently with probability $a/2$ and the index spacings between successive visits are independent with the same distribution as A, we may view this as computing the generating function for a sum of $N \sim \text{Geometric}(1-\frac{a}{2})$ independent copies of A. This classically gives, $f^*(t) = f_N(f(t))$, with $f_N(x) = (1-\frac{a}{2})x/(1-\frac{a}{2}x)$ the generating function of N.

Plugging in the formula for p^*_n and exchanging the order of summation via Fubini's theorem gives

$$\begin{aligned} f^*(t) &= \sum_{n=0}^{\infty} p^*_n t^n \\ &= \sum_{w=0}^{\infty} (\tfrac{a}{2})^w (1-\tfrac{a}{2}) \sum_{n=0}^{\infty} \sum_{0=\ell_0<\ell_1<\cdots<\ell_w<\ell_{w+1}=n} \prod_{i=1}^{w+1} p_{\ell_i-\ell_{i-1}} t^{\ell_i-\ell_{i-1}} \\ &= \sum_{w=0}^{\infty} (\tfrac{a}{2})^w (1-\tfrac{a}{2}) f(t)^{w+1}. \end{aligned}$$

This is a geometric series whose closed form is the claimed formula for $f^*(t)$. □

Proof of Theorem 1.1: For $n = 1$, it is immediate that $p_1 = \mathbf{P}(\overleftarrow{\bullet}_1) = (1-p)/2$ and the quantities in (2.3)–(2.12) are all equal to 0. Hence, none depend on μ. Given $n \geq 2$, it follows from Proposition 2.2 that the quantities in (2.3)–(2.12) can be expressed solely in terms of quantities with index strictly less than n. Thus, we may proceed by induction to infer that these quantities do not depend on μ for all n. It follows then from (2.2) that p_n does not depend on μ. Since $f(t) = \mathbf{E}[t^A] = \sum_{n\geq 1} p_n t^n$ uniquely determines the distribution of A, we obtain the first part of Theorem 1.1.

The implicit recursion for f is obtained by summing the generating functions corresponding to both sides of (2.2) and then applying the equations (2.3)–(2.12). This gives

$$
\begin{aligned}
A(t) &:= \sum_{n\geq 0} \alpha_n t^n = cptf(t) + (1-c)ptf(t)^2 \\
B(t) &:= \sum_{n\geq 0} (\dot{\beta}_n + \hat{\beta}_n)t^n = \frac{1-c}{2}ptf(t)^2 + \frac{1-c}{2}\hat{D}(t)f(t)^2 \\
C(t) &:= \sum_{n\geq 0} (\gamma_n + \hat{\gamma}_n + \overleftarrow{\gamma}_n)t^n \\
&= D(t)f(t) + c\hat{D}(t)f(t) + (1-c)\hat{D}(t)f(t)^2 + \frac{a}{2}\bar{D}(t) \\
D^*(t) &:= \sum_{n\geq 0} \delta_n^* t^n \\
&= \frac{1-p}{2}tf^*(t) - B(t)f^*(t) - \tfrac{1}{2}(1-\tfrac{a}{2})(1-c)c(t+\hat{D}(t))f(t)^2 \\
\bar{D}(t) &:= \sum_{n\geq 0} \bar{\delta}_n t^n = \tfrac{1}{1-\frac{a}{2}} D^*(t) \\
D(t) &:= \sum_{n\geq 0} \delta_n t^n = (1-(a+b))\bar{D}(t) \\
\hat{D}(t) &:= \sum_{n\geq 0} \hat{\delta}_n t^n = b\bar{D}(t).
\end{aligned}
$$

We may apply Lemma 2.3 to write the $f^*(t)$ terms in $D^*(t)$ in terms of $f(t)$. As an example of the calculations that lead to the formulas for A, B, C, D, and $\hat{D}$, we provide the derivation for summing the $\hat{\beta}_n$. First, we apply the formula at (2.5) to write

$$\sum_{n=0}^{\infty} \hat{\beta}_n t^n = \tfrac{1-c}{2} \sum_{n=0}^{\infty} \sum_{1<k<\ell<n} \hat{\delta}_{\ell-k+1} p_{k-1} p_{n-\ell} t^n. \tag{2.19}$$

Expanding and rearranging the sums, then applying Fubini's theorem gives (2.19) is equal to

$$\tfrac{1-c}{2} \sum_{k=0}^{\infty} p_k t^k \sum_{\ell=0}^{\infty} \hat{\delta}_\ell t^\ell \sum_{n=0}^{\infty} p_n t^n = \tfrac{1-c}{2}\hat{D}(t)f(t)^2.$$

The other derivations are similar.

We have thus established that

$$f(t) = p_0 + p_1 t + A(t) + B(t) + C(t) \tag{2.20}$$

with $p_0 = 0$ and $p_1 = (1-p)/2$. The formula is implicit since the formula for $\bar{D}(t)$ is also a recursive equation and may not necessarily have a solution. For the purpose of illustration, here is the recursion for the special case (proven in Haslegrave et al. (2021, Theorem 2)) of simple ballistic annihilation $a = b = c = 0$,

$$f(t) = -\frac{1}{2}ptf(t)^4 + ptf(t)^2 + \frac{1}{2}tf(t)^2 + \frac{1-p}{2}t.$$

□

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