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Multitype self-similar growth-fragmentation processes

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Abstract. In this paper, we are interested in multitype self-similar growth-fragmentation processes. More precisely, we investigate a multitype version of the self-similar growth-fragmentation processes introduced by Bertoin, where the type of the particles may also evolve in time according to a Markov chain. This extends the signed case (considered by the first author in a previous work) to the case of finitely many types. Our main result in this direction describes the law of the spine in the multitype setting. In order to do so, we introduce two genealogical martingales, in the same spirit as in the positive case, which allow us not only to obtain the law of the spine but also to study the limit of the empirical measure of fragments. We stress that our arguments only rely on the structure of the underlying Markov additive processes (MAPs), and hence is more general than the treatment of the signed case. Our methods also require new results on exponential functionals for MAPs and a multitype version of the tail estimates in multiplicative cascades which are interesting in their own right.

1. Introduction

Self-similar growth-fragmentation processes first appeared in Bertoin (2017) to describe the evolution of a cloud of atoms which may grow and dislocate in a binary way. More precisely, these atoms are assumed to have a specific one-dimensional trait of interest, which we can think of as its *mass* or *size*. Initially, the cloud starts from one particle (the common ancestor of all future particles) whose size is a positive quantity evolving in time in a Markovian way. This size will have jumps, and at each negative jump $y < 0$, we wish to add to the cloud a new particle whose size at birth will be given by $-y$, at the time when the jump occurs. This creates children of the original ancestor in such a way that the divisions are conservative, that is summing the size of the child and

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the size of the parent just after division exactly gives the size of the parent before division. Then, the newborn particles evolve independently of the parent, and independently of one another, in the same Markovian way as the parent. We proceed similarly creating the offspring of those particles, thereby introducing the grandchildren, great grandchildren, and so on, of the original ancestor.

Such growth-fragmentation models have been given a striking geometric flavour, in the context of random planar maps. This originated from Bertoin et al. (2018b) and Bertoin et al. (2018a), where a remarkable class of self-similar growth-fragmentations shows up in the scaling limit of perimeter processes (see Budd (2016)) in Markovian explorations of Boltzmann planar maps. These growth-fragmentation processes are closely related to stable Lévy processes with stability parameter $\theta \in (\frac{1}{2}, \frac{3}{2}]$. Since then, the same growth-fragmentation processes were directly constructed in the continuum Miller et al. (2022) for $1 < \theta \leq \frac{3}{2}$ by drawing a CLE exploration on a quantum disc. Moreover, the boundary case $\theta = \frac{3}{2}$, corresponding to the random triangulations in Bertoin et al. (2018b), also appears in Le Gall and Riera (2020) as the collection of perimeters obtained when slicing a Brownian disc at heights. The critical Cauchy case $\theta = 1$, in turn, corresponds to slicing a Brownian half-plane excursion at heights, see Aïdékon and Da Silva (2022). This approach was recently extended to $\frac{1}{2} < \theta < 1$ in Da Silva (2023) by considering other half-plane excursions.

Let us point out that in Aïdékon and Da Silva (2022) (and subsequently in Da Silva (2023)), *negative* mass is taken into account in the system, whereas the aforementioned construction of growth-fragmentation processes deals with positive mass only. In those examples, the sign depends on the time orientation of the excursions. In particular, slicing a half-plane Brownian excursion only yields the critical case $\theta = 1$ in Bertoin et al. (2018a) provided one discards the negative cells. On a related note, the driving cell processes in the distinguished family of growth-fragmentations in Bertoin et al. (2018a) also have positive jumps (except for $\theta = \frac{3}{2}$). This has a geometric meaning: Boltzmann planar maps correspond to the gasket of a loop $O(n)$–model, and the positive jumps occur when discovering a loop, which could then be explored. In the continuum, positive jumps also arise in Miller et al. (2022) when hitting a CLE loop for the first time. This prompted Da Silva (2023) to provide a framework for self-similar *signed* growth-fragmentations.

Adding negative mass to the system presents some technical issues. The analysis of the positive case carried out in Bertoin (2017) and Bertoin et al. (2018a) relies heavily on the Lamperti representation, see Lamperti (1972), for positive self-similar Markov processes, allowing for a large toolbox of Lévy techniques. This breaks down if one is willing to deal with signed processes, in other words the effect of introducing a sign is to move from the class of Lévy processes to the one of Markov additive processes, see for instance Chaumont et al. (2013), Kuznetsov et al. (2014), Kyprianou and Pardo (2022) and Pardo and Rivero (2013). One of the aims of this paper is to extend the framework to a general set of types. This has a counterpart in the *pure* fragmentation setting, see for instance Stephenson (2018). In this case, we show that natural martingales arise, in connection to the additive martingales appearing in the context of multitype branching random walks (Section 3). These martingales have the same form as in Bertoin (2017), except that they are weighted by the types. The pair of genealogical martingales arises naturally as intrinsic martingales associated with that of multitype branching random walks. We use a multitype version of Biggins' martingale convergence theorem due to Kyprianou and Rahimzadeh Sani (2001) to deduce that one of the martingales converges to 0 almost surely. The other martingale is uniformly integrable and the tail probability behaviour of its terminal value decreases polynomially. In order to deduce the latter, we extend to the multitype setting the tail estimates in multiplicative cascades due to Jelenković and Olvera-Cravioto (2012). Following the same lines as Bertoin et al. (2018a, Theorem 4.2), our main theorem in the multitype setting (Theorem 4.3) describes the cell system under the change of measures with respect to these martingales (Section 4). We stress that, although the framework developed here includes the signed case which was already treated in Da Silva (2023), our methods are completely different and moreover further results are included which are not treated in the signed case. Indeed, Da Silva (2023) hinges upon a change of driving cell process (which is specific to the signed case) to reduce to the positive case, whereas in this paper we directly work with Markov additive processes (MAPs). Using properties of MAPs, we then arrive at a pair of martingales indexed by continuous time and which are naturally related to the aforementioned genealogical martingales which allow us to describe explicitly the dynamics of multitype growth-fragmentations under the probability measures which are obtained by tilting the initial one with these intrinsic martingales. Such intrinsic martingales are also applied to study the limit of the empirical measure of fragments. Along the way, we provide additional results on exponential functionals for MAPs and the aforementioned multitype version of the tail estimates in multiplicative cascades which are interesting in their own right.

Contents

2. Self-similar Markov processes with types

We start by presenting some shared features of the Markov processes we will be interested in, revolving around the notion of self-similarity. We explain how to deal with *types* for self-similar processes in the case when the set of types is finite. A key ingredient of our analysis is the Lamperti-Kiu representation, which gives a bijection between these self-similar processes and a class of Markov additive processes. We refer to Kyprianou and Pardo (2022) for a detailed treatment of these questions.

Markov additive processes. Let $\mathcal{I}$ be a finite set. We also let $(\xi(t), \Theta(t), t \geq 0)$ be a regular Feller process in $\mathbb{R} \times \mathcal{I}$ with probabilities $\mathtt{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in \mathcal{I}$ and cemetery state $(-\infty, \dagger)$, on $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $(\mathcal{G}_t)_{t\geq 0}$ the natural standard filtration associated with (ξ, Θ). We say that (ξ, Θ) is a *Markov additive process* (MAP for short) if for every bounded measurable $f : \mathbb{R} \times \mathcal{I} \to \mathbb{R}$, $s, t \geq 0$ and $(x, \theta) \in \mathbb{R} \times \mathcal{I}$,

$$\mathtt{E}_{x,\theta}\Big[f(\xi(t+s) - \xi(t), \Theta(t+s))1_{\{t+s<\varsigma\}}\Big|\mathcal{G}_t\Big] = \mathbf{1}_{\{t<\varsigma\}}\mathtt{E}_{0,\Theta(t)}\Big[f(\xi(s), \Theta(s))\mathbf{1}_{\{s<\varsigma\}}\Big],$$

where $\varsigma := \inf\{t > 0,\, \Theta(t) = \dagger\}$. The process Θ is thus a Markov chain on $\mathcal{I}$ and is called the *modulator* of ξ, whereas the latter is called the *ordinator*. The notation

$$\mathtt{P}_\theta(\cdot) := \mathbb{P}(\,\cdot \,|\, \xi(0) = 0 \text{ and } \Theta(0) = \theta) \quad \text{for} \quad \theta \in E,$$

will be in force throughout the paper.

MAPs have received a lot of attention recently, since they have found a prominent role in classical applied probability (see for instance Asmussen (2003) and Ivanovs (2011)) and in the study of self-similar Markov processes (see for instance Chaumont et al. (2013); Kuznetsov et al. (2014); Kyprianou and Pardo (2022)). It is important to note that the concept of MAPs also makes sense when Θ is replaced by a more general Markov process, see for instance Çinlar (75); Kyprianou and Pardo (2022), but this case is out of the scope of this manuscript. For a treatment of the general case in the context of self-similar growth-fragmentation we refer to Da Silva and Pardo (2023).

Informally, one should think of a MAP as a natural extension of a Lévy process in the sense that Θ is an arbitrary well-behaved Markov chain and $((\xi(t), \Theta(t))_{t\geq 0}, \mathtt{P}_{x,\theta})$ is equal in law to $((\xi(t) + x, \Theta(t))_{t\geq 0}, \mathtt{P}_\theta)$. Indeed the ordinate process ξ can be thought of as the concatenation of Lévy processes which depend on the current type in $\mathcal{I}$ given by Θ, as is stated in the following proposition, see Ivanovs (2011), Kuznetsov et al. (2014), Kyprianou and Pardo (2022) or the survey Pardo and Rivero (2013).

Proposition 2.1. *The process (ξ, Θ) is a Markov additive process if and only if there exist independent sequences $(\xi_i^{(n)}, n \geq 0)_{i\in\mathcal{I}}$ and $(U_{i,j}^{(n)}, n \geq 0)_{i,j\in\mathcal{I}}$, all independent of Θ, such that:*

- *for $i \in \mathcal{I}$, $(\xi_i^{(n)}, n \geq 0)$ is a sequence of i.i.d. Lévy processes,*
- *for $i, j \in \mathcal{I}$, $(U_{i,j}^{(n)}, n \geq 0)$ are i.i.d. random variables,*
- *if $(T_n)_{n\geq 0}$ denotes the sequence of jump times of the chain Θ (with the convention $T_0 = 0$), then for all $n \geq 0$,*

$$\xi(t) = \left(\xi(T_n^-) + U^{(n)}_{\Theta(T_n^-),\Theta(T_n)}\right)\mathbf{1}_{\{n\geq 1\}} + \xi^{(n)}_{\Theta(T_n)}(t - T_n), \qquad T_n \leq t < T_{n+1}. \tag{2.1}$$

We now turn to defining the *matrix exponent* of a MAP, which is the analogue of the Laplace exponent in the setting of Lévy processes. Without loss of generality, we assume that $\mathcal{I} = \{1, \ldots, N\}$ where $N \in \mathbb{N}$, and that Θ is irreducible and ergodic. We write $Q = (q_{i,j})_{1\leq i,j\leq N}$ for its intensity matrix, and ρ_i, $i \in \mathcal{I}$, for the exponential time that Θ takes to jump from state i to some other state. Also, we denote for all $i, j \in \mathcal{I}$, all on the same probability space, by ξ_i a Lévy process distributed as the $\xi_i^{(n)}$'s, and by $U_{i,j}$ a random variable distributed as the $U_{i,j}^{(n)}$'s, with the convention $U_{i,i} = 0$ and $U_{i,j} = 0$ if $q_{i,j} = 0$. Finally, we introduce the Laplace exponent ψ_i of ξ_i and the Laplace transform $G_{i,j}(z) := \mathbb{E}\left[\mathrm{e}^{zU_{i,j}}\right]$ of $U_{i,j}$ (this defines a matrix $G(z)$ with entries $G_{i,j}(z)$). Then the matrix exponent F of (ξ, Θ) is defined as

$$F(z) := \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + Q \circ G(z), \tag{2.2}$$

where $\circ$ denotes pointwise multiplication of the entries. Then the following equality holds for all $i, j \in \mathcal{I}$, $z \in \mathbb{C}$, $t \geq 0$, whenever one side of the equality is defined:

$$\mathtt{E}_{0,i}\left[\mathrm{e}^{z\xi(t)}\mathbf{1}_{\{\Theta(t)=j\}}\right] = (\mathrm{e}^{F(z)t})_{i,j}.$$

Let us denote the stationary distribution of Θ by $\pi = (\pi_1, \ldots, \pi_N)$. Given the MAP (ξ, Θ) with probabilities $\mathtt{P}_{x,\theta}, x \in \mathbb{R}, \theta \in E$, we can introduce the dual process; that is the MAP with probabilities $\mathtt{P}^\natural_{x,\theta}, x \in \mathbb{R}, \theta \in E$ whose matrix exponent when it is defined, is given by

$$\mathtt{E}^\natural_{0,i}\left[\mathrm{e}^{z\xi(t)}\mathbf{1}_{\{\Theta(t)=j\}}\right] = (\mathrm{e}^{F^\natural(z)t})_{i,j},$$

where

$$F^\natural(z) := \mathrm{diag}(\psi_1(-z), \ldots, \psi_N(-z)) + Q^\natural \circ G(-z)^{\mathtt{t}},$$

with $A^{\mathtt{t}}$ being the transpose matrix of A and $Q^\natural$ is the intensity matrix of the modulating Markov chain on $\mathcal{I}$ with entries given by

$$q^\natural_{i,j} = \frac{\pi_j}{\pi_i} q_{j,i}, \qquad i,j \in \mathcal{I}. \tag{2.3}$$

Observe that the latter can also be written as $Q^\natural = \Delta_\pi^{-1} Q^{\mathtt{t}} \Delta_\pi$, where $\Delta_\pi = \mathrm{diag}(\pi_1, \dots, \pi_N)$ and hence, when it exists,

$$F^\natural(z) = \Delta_\pi^{-1} F(-z)^{\mathtt{t}} \Delta_\pi, \tag{2.4}$$

showing that

$$\pi_i \mathtt{E}^\natural_{0,i}\left[\mathrm{e}^{z\xi(t)} \mathbf{1}_{\{\Theta(t)=j\}} \right] = \pi_j \mathtt{E}_{0,j}\left[\mathrm{e}^{-z\xi(t)} \mathbf{1}_{\{\Theta(t)=i\}} \right].$$

The previous identity can be understood, at the level of processes, as changing time-directions.

Lemma 2.2. *We have that $(\xi(t-s) - \xi(t), \Theta((t-s)-), 0 \le s \le t)$ under $\mathtt{P}_\pi = \sum_{i=1}^N \pi_i \mathtt{P}_i$ is equal in law to $(\xi(s), \Theta(s), 0 \le s \le t)$ under $\mathtt{P}^\natural_\pi$.*

Spectral properties of MAPs. We also state for future reference the following classical results (see Asmussen (2003); Ivanovs (2011) or the survey Pardo and Rivero (2013)) about the leading eigenvalue of a MAP, often dubbed *Perron-Frobenius* eigenvalue. We consider a MAP (ξ, Θ) on $\mathbb{R} \times \mathcal{I}$ with matrix exponent F.

Proposition 2.3. *Let F denote the matrix exponent of some Markov additive process, and $z \in \mathbb{R}$ such that $F(z)$ is well-defined. Then the matrix $F(z)$ has a real simple eigenvalue $\chi(z)$, which is larger than the real parts of all its other eigenvalues. In addition, $\chi(z)$ is associated to a positive eigenfunction $w(z)$.*

We say that (ξ, Θ) satisfies *Cramér's condition* if there exists $\gamma_0 > 0$ and $\Upsilon \in (0, \gamma_0)$ such that F is defined on $(0, \gamma_0)$ and $\chi(\Upsilon) = 0$. The number Υ is then called a *Cramér number*. The leading eigenvalue enables to identify the following *Wald martingale*, which is a multitype version of the exponential martingale for Lévy processes.

Proposition 2.4. *Fix γ such that $F(\gamma)$ is well-defined. With the notation of Proposition 2.3, let*

$$\mathcal{W}^{(\gamma)}(t) := \frac{w_{\Theta(t)}(\gamma)}{w_{\Theta(0)}(\gamma)} \mathrm{e}^{\gamma \xi(t) - t\chi(\gamma)}, \quad t \ge 0.$$

Then $\mathcal{W}^{(\gamma)}$ is a martingale with respect to $(\mathcal{G}_t)_{t \ge 0}$, and under any initial distribution of $(\xi(0), \Theta(0))$. Moreover, the law of (ξ, Θ) under the corresponding change of measure, that is

$$\left.\frac{\mathrm{d}\mathtt{P}^{(\gamma)}_{0,i}}{\mathrm{d}\mathtt{P}_{0,i}}\right|_{\mathcal{G}_t} = \mathcal{W}^{(\gamma)}(t), \qquad t \ge 0,$$

is that of a Markov additive process with matrix exponent

$$F^{(\gamma)}(z) := \mathrm{diag}(w_i(\gamma), i \in \mathcal{I})^{-1} (F(\gamma + z) - \chi(\gamma)\mathrm{Id}) \mathrm{diag}(w_i(\gamma), i \in \mathcal{I}).$$

In particular, the leading eigenvalue of $F^{(\gamma)}(z)$ is given by $\chi^{(\gamma)}(z) := \chi(\gamma + z) - \chi(\gamma)$.

The following property will also come in useful.

Proposition 2.5. *We take the notation of Proposition 2.3. Let D be an interval of $\mathbb{R}$ on which F is defined. Then the leading eigenvalue χ is smooth and convex on D.*

An important quantity associated to a MAP (ξ,Θ) on $\mathbb{R}\times\mathcal{I}$, particularly in view of the lifetime (2.9) appearing in the next paragraph in relation to the Lamperti-Kiu transform, is the so-called *exponential functional*, namely

$$I(\xi) := \int_0^\infty \mathrm{e}^{\xi(s)}\mathrm{d}s.$$

This quantity has been studied in great detail, first for Lévy processes (see, notably, Behme et al. (2021); Barker and Savov (2021); Bertoin and Yor (2005); Carmona et al. (1997); Kyprianou and Pardo (2022); Pardo and Rivero (2013); Patie and Savov (2018) and references therein), and then more recently for MAPs (see in particular Alili and Woodford (2021); Behme and Sideris (2020); Kuznetsov et al. (2014); Stephenson (2018)). We stress that the study of $I(\xi)$ usually involves the spectral properties of the MAP, and in particular the leading eigenvalue χ. We state for future reference the following result, giving a finiteness criterion for the moments of the exponential functional of Markov additive processes. The case of Lévy processes is also fully understood, see for instance Rivero (2012, Lemma 3).

Proposition 2.6. *Assume that for some $\gamma>0$, F is defined on $[0,\gamma]$, and $\chi(\gamma)<0$. Then $I(\xi)$ has finite moment of order γ, under any initial distribution of (ξ,Θ).*

Proof: We only prove the statement for $\gamma>1$ since the case $\gamma\le 1$ is contained in Kuznetsov et al. (2014, Proposition 3.6) (although Kuznetsov et al. (2014, Proposition 3.6) only states the result for Cramér numbers $\Upsilon<1$, their proof of finiteness of moments carries over to any $\gamma\le 1$ such that $\chi(\gamma)<0$). For the remainder of the proof, we fix $\gamma>1$ and we take $\varepsilon\in(0,\gamma)$. Jensen's inequality provides

$$\begin{aligned}\left(\int_0^\infty \mathrm{e}^{\xi(s)}\mathrm{d}s\right)^\gamma &= \left(\int_0^\infty \mathrm{e}^{\xi(s)}(-\chi(\varepsilon))^{-1}\mathrm{e}^{-s\chi(\varepsilon)}\big(-\chi(\varepsilon)\mathrm{e}^{s\chi(\varepsilon)}\mathrm{d}s\big)\right)^\gamma \\ &\le (-\chi(\varepsilon))^{1-\gamma}\int_0^\infty \mathrm{e}^{\gamma\xi(s)-(\gamma-1)s\chi(\varepsilon)}\mathrm{d}s. \qquad (2.5)\end{aligned}$$

Now write $C_\varepsilon = \max_{i,j\in\mathcal{I}}\frac{w_j(\gamma)}{w_i(\gamma)}\cdot(-\chi(\varepsilon))^{1-\gamma}>0$, and let $i\in\mathcal{I}$. Taking the $\mathtt{P}_{0,i}$–expectation of (2.5), a rough estimate yields

$$\begin{aligned}\mathtt{E}_{0,i}\left[\left(\int_0^\infty \mathrm{e}^{\xi(s)}\mathrm{d}s\right)^\gamma\right] &\le C_\varepsilon\int_0^\infty \mathtt{E}_{0,i}\Big[\mathcal{W}^{(\gamma)}(s)\mathrm{e}^{s\chi(\gamma)}\Big]\mathrm{e}^{-(\gamma-1)s\chi(\varepsilon)}\mathrm{d}s \\ &= C_\varepsilon\int_0^\infty \mathrm{e}^{s(\chi(\gamma)-(\gamma-1)\chi(\varepsilon))}\mathrm{d}s,\end{aligned}$$

by the martingale property of $\mathcal{W}^{(\gamma)}$ in Proposition 2.4. Noting that $\chi(\gamma)<0$ and $\chi(\varepsilon)\to 0$ as $\varepsilon\to 0$, we get that $\chi(\gamma)-(\gamma-1)\chi(\varepsilon)<0$ for small enough ε, which completes the proof. □

Remark 2.7. In particular, if Υ is a Cramér number for (ξ,Θ), then by convexity of χ the exponential functional of ξ has finite moments of order γ for all $\gamma<\Upsilon$. The case when $\Upsilon<1$ already appears in Kuznetsov et al. (2014, Proposition 3.6), but for $\Upsilon>1$ the result does not seem to be contained in the existing literature.

Our next result studies the tail behaviour of $I(\xi)$ under Cramér's condition. The case when $N=2$ has been treated in Alili and Woodford (2021) using direct computations associated with the matrix exponent F. The case of Lévy process was studied in Rivero (2012). Our approach uses the long-term behaviour of MAPs. We recall that $(w_i(z), i\in\mathcal{I})$ is a positive eigenvector associated to the leading eigenvalue $\chi(z)$ of $F(z)$.

Proposition 2.8. *Assume that (ξ,Θ) satisfies Cramér's condition, with Cramér number $\Upsilon > 0$ and that ξ is not concentrated on a lattice. Set for all $i \in \mathcal{I}$,*

$$J_i(\xi) := \int_0^\infty \frac{w_{\Theta(s)}(\Upsilon)}{w_i(\Upsilon)} \mathrm{e}^{\xi(s)} \mathrm{d}s.$$

Then for $i \in \mathcal{I}$, there exists a constant $C_i \geq 0$ such that

$$\mathtt{P}_{0,i}(J_i(\xi) > t) \sim C_i t^{-\Upsilon}, \qquad as \quad t \to \infty.$$

In particular, for all $i \in \mathcal{I}$, there exist nonnegative constants $C_i^{(1)}$ and $C_i^{(2)}$ such that for t large enough,

$$C_i^{(1)} t^{-\Upsilon} \leq \mathtt{P}_{0,i}(I(\xi) > t) \leq C_i^{(2)} t^{-\Upsilon}. \tag{2.6}$$

We believe that under $\mathtt{P}_{0,i}$, the tail probability of $I(\xi)$ should also be equivalent to $C_i t^{-\Upsilon}$ for some constant C_i. Actually we can use the fact that $I(\xi)$ is the solution of a random affine equation and then Theorem 4.1 in Goldie (1991) (see also Kesten (1973)) but for technical reasons our arguments are better suited to deal with $J_i(\xi)$ and only provide the weaker form (2.6).

Proof: We only prove the first statement since (2.6) follows easily from it by bounding $w_{\Theta(t)}$ by the maximal or minimal w_j. In this proof, we shall write w_j instead of $w_j(\Upsilon)$. Fix $i \in \mathcal{I}$ and observe that for all $t \geq 0$,

$$J_i(\xi) = \int_0^t \frac{w_{\Theta(s)}}{w_i} \mathrm{e}^{\xi(s)} \mathrm{d}s + \frac{w_{\Theta(t)}}{w_i} \mathrm{e}^{\xi(t)} \int_t^\infty \frac{w_{\Theta(s)}}{w_{\Theta(t)}} \mathrm{e}^{\xi(s)-\xi(t)} \mathrm{d}s.$$

From the Markov additive property of (ξ,Θ), we deduce that, conditionally on $\Theta(t)$,

$$\int_t^\infty \frac{w_{\Theta(s)}}{w_{\Theta(t)}} \mathrm{e}^{\xi(s)-\xi(t)} \mathrm{d}s,$$

is a copy of $J_{\Theta(t)}(\xi)$, under $\mathtt{P}_{0,\Theta(t)}$, which is further independent of $((\xi_s,\Theta_s), s \leq t)$. In other words, $J_i(\xi)$ is the solution of a multitype random affine equation of the form (5.1). Remark that the matrix m of (5.12) is precisely the matrix exponent F of (ξ,Θ). Hence according to Theorem 5.4 in the appendix, our result will be fulfilled as long as the following conditions are satisfied

- (i) $F(q)$ is well-defined on a domain containing $[\gamma,\Upsilon]$ for some $\gamma \geq 0$,
- (ii) $\chi(\Upsilon) = 0$ and $0 < \mathtt{E}_{0,i}\left[\frac{w_{\Theta(t)}}{w_i}\xi(t)\mathrm{e}^{\Upsilon\xi(t)}\right] < \infty$,
- (iii) $\mathtt{E}_{0,i}[J_i(\xi)^\beta] < \infty$ for all $0 < \beta < \Upsilon$ and $i \in \mathcal{I}$,
- (iv) $\mathtt{E}_{0,i}\left[\left(\int_0^t \frac{w_{\Theta(s)}}{w_i}\mathrm{e}^{\xi_s}\mathrm{d}s\right)^\Upsilon\right] < \infty$ for all $i \in \mathcal{I}$.

Condition (i) and the first claim in condition (ii) follow from Cramér's condition. To see that $\mathtt{E}_{0,i}\left[\frac{w_{\Theta(t)}}{w_i}\xi(t)\mathrm{e}^{\Upsilon\xi(t)}\right] > 0$, we note that the function

$$h : q \mapsto \mathtt{E}_{0,i}\left[\frac{w_{\Theta(t)}}{w_i}\mathrm{e}^{q\xi(t)}\right],$$

is strictly convex and satisfies $h(0) = h(\Upsilon) = 1$, whence $h'(\Upsilon) = \mathtt{E}_{0,i}\left[\frac{w_{\Theta(t)}}{w_i}\xi(t)\mathrm{e}^{\Upsilon\xi(t)}\right] > 0$. Condition (iii) is a consequence of Proposition 2.6 (see also Remark 2.7). It remains to prove (iv), which is equivalent to prove that

$$\mathtt{E}_{0,i}\left[\left(\int_0^t \mathrm{e}^{\xi_s}\mathrm{d}s\right)^\Upsilon\right] < \infty,$$

for all $i \in \mathcal{I}$. Let $i \in \mathcal{I}$. If $\Upsilon \geq 1$, we use Jensen's inequality to get

$$\mathtt{E}_{0,i}\left[\left(\int_0^t \mathrm{e}^{\xi_s}\mathrm{d}s\right)^{\Upsilon}\right] \leq t^{\Upsilon}\mathtt{E}_{0,i}\left[\int_0^t \mathrm{e}^{\Upsilon\xi_s}\frac{\mathrm{d}s}{t}\right] \leq M_i t^{\Upsilon},$$

with $M_i = \max_{j\in\mathcal{I}} \frac{w_i}{w_j}$, since $\mathtt{E}_{0,i}\left[\frac{w_{\Theta(s)}}{w_i}\mathrm{e}^{\Upsilon\xi(s)}\right] = 1$ for all $s \geq 0$. The arguments for $\Upsilon < 1$ are contained in the proof of Kuznetsov et al. (2014, Proposition 3.6), but we repeat them for completeness. In this case, one can bound

$$\mathtt{E}_{0,i}\left[\left(\int_0^t \mathrm{e}^{\xi_s}\mathrm{d}s\right)^{\Upsilon}\right] \leq t^{\Upsilon}\mathtt{E}_{0,i}\left[\sup_{0\leq s\leq t} \mathrm{e}^{\Upsilon\xi_s}\right].$$

Now we take $p > 1$, and bound

$$\mathtt{E}_{0,i}\left[\sup_{0\leq s\leq t} \mathrm{e}^{\Upsilon\xi_s}\right] \leq M_i^{(p)}\mathtt{E}_{0,i}\left[\sup_{0\leq s\leq t}\left(\frac{w_{\Theta(s)}(\Upsilon/p)}{w_i(\Upsilon/p)}\mathrm{e}^{\Upsilon\xi_s/p}\right)^p\right],$$

with $M_i^{(p)} := \max_{j\in\mathcal{I}} \left(\frac{w_i(\Upsilon/p)}{w_j(\Upsilon/p)}\right)^p$ as before. By convexity of χ, $\chi(\Upsilon/p) < 0$, hence

$$\mathtt{E}_{0,i}\left[\sup_{0\leq s\leq t} \mathrm{e}^{\Upsilon\xi_s}\right] \leq M_i^{(p)}\mathtt{E}_{0,i}\left[\sup_{0\leq s\leq t}\left(\mathcal{W}^{(\Upsilon/p)}(s)\right)^p\right],$$

where we recall that $\mathcal{W}^{(\Upsilon/p)}(s) := \frac{w_{\Theta(s)}(\Upsilon/p)}{w_i(\Upsilon/p)}\mathrm{e}^{\Upsilon\xi(s)/p-\chi(\Upsilon/p)s}$, $s \geq 0$, is the Wald martingale associated to Υ/p. We use Doob's maximal inequality:

$$\mathtt{E}_{0,i}\left[\sup_{0\leq s\leq t} \mathrm{e}^{\Upsilon\xi_s}\right] \leq \left(\frac{p}{p-1}\right)^p M_i^{(p)}\mathtt{E}_{0,i}\left[\left(\mathcal{W}^{(\Upsilon/p)}(t)\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p M_i^{(p)} m_i^{(p)}\mathrm{e}^{-p\chi(\Upsilon/p)}\mathtt{E}_{0,i}[\mathrm{e}^{\Upsilon\xi_t}],$$

with $m_i^{(p)} := \max_{j\in\mathcal{I}} \left(\frac{w_j(\Upsilon/p)}{w_i(\Upsilon/p)}\right)^p$. This proves condition (iv) above. Therefore Theorem 5.4 yields the desired estimate. □

We stress that the representation of MAPs given in Proposition 2.1 is quite important in our arguments below. The extension of the aforementioned result, as well as other results regarding the matrix exponent and some related martingales, to the case when $\mathcal{I}$ is countable seems feasible under some general assumptions on the modulator Θ. However, up to our knowledge, this case has not been rigorously treated in the literature, so we will not explore it to keep this paper at a reasonable size.

Self-similar Markov processes with types in $\mathcal{I}$. Similarly to the construction of positive self-similar Markov processes through Lévy processes, it is possible to build a more general family of self-similar Markov processes using MAPs. More precisely, let (ξ, Θ) be a MAP on $\mathbb{R} \times \mathcal{I}$, and fix $\alpha \in \mathbb{R}$. We construct the following process (X, J) with values in $\mathbb{R}_+ \times \mathcal{I}$ with a possible cemetery state ∂ *via* a Lamperti-type procedure. First, introduce

$$\varphi(t) := \inf\left\{s > 0, \int_0^s \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s > t\right\}, \quad t \geq 0. \tag{2.7}$$

Then, for $x > 0$, let

$$X(t) := x\exp(\xi(\varphi(tx^{-\alpha}))) \quad \text{and} \quad J(t) := \Theta(\varphi(tx^{-\alpha})), \quad t \geq 0, \tag{2.8}$$

with the convention that $(X(t), J(t)) = \partial$ when $t \geq \zeta$ where

$$\zeta := x^{\alpha}\int_0^{\infty} \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s. \tag{2.9}$$

We write $\mathbb{P}_{x,i}$, $x > 0$, with $i \in \mathcal{I}$, for the law of (X, J) started from (x, i), and $\mathbb{P}_i = \mathbb{P}_{1,i}$. It is plain from this construction that (X, J) is a Markov process, and that X is a *self-similar* process, that is to say for any $c > 0$ and for all $x > 0$, $i \in \mathcal{I}$,

$$\Big((cX(c^{-\alpha}s), s \geq 0), \mathbb{P}_{x,i}\Big) \stackrel{d}{=} \Big((X(s), s \geq 0), \mathbb{P}_{cx,i}\Big). \tag{2.10}$$

Conversely, if (X, J) is a Markov process in $\mathbb{R}_+ \times \mathcal{I}$, such that X is self-similar with index α in the sense of (2.10), then one can find a MAP such that (2.8) holds, with the time change given in (2.7). This construction is reminiscent of the Lamperti or Lamperti-Kiu representations, see for instance Lamperti (1972); Chaumont et al. (2013); Kuznetsov et al. (2014); Alili et al. (2017) for positive or real-valued self-similar Markov processes respectively. In the latter case, the type J is the sign, see Chaumont et al. (2013); Kuznetsov et al. (2014). We call this process (X, J), or sometimes just X, a *self-similar Markov process with types*.

For our purposes, it is convenient to study self-similar Markov processes (ssMp) with types when the starting point of X tends to the origin. In other words, we would like to find the behaviour of $\mathbb{P}_{x,i}$ as x tends to 0. On the other hand, we see from the scaling property that the question boils down to that of the large times asymptotic behaviour. That is $\mathbb{P}_{x,i}(X_1 \in \cdot, J_1 \in \cdot)$ converges weakly, as $x \to 0$ to a probability measure here denoted as $\eta(\cdot, \cdot)$ if, and only if, for any $y > 0$

$$\mathbb{P}_{y,i}(t^{-1/\alpha}X_t \in \cdot, J_t \in \cdot) \to \eta(\cdot, \cdot) \qquad \text{as} \quad t \to \infty.$$

Let us assume that X is *conservative* that is, that its lifetime ζ is almost surely infinite. Our next result determines the existence of the weak limit of $\mathbb{P}_{x,i}(X_t \in \cdot, J_t \in \cdot)$, as $x \to 0$, under the assumption that ξ is not concentrated on a lattice and $m := \mathtt{E}_{0,\pi}[\xi_1] < \infty$, as a *self-similar entrance law* for the ssMp with types (X, J).

Following Rivero (2016), we say that a family $(\eta_t, t > 0)$ of σ–finite measures on $(0, \infty) \times \mathcal{I}$ is a self-similar entrance law for the semigroup $(P_t^{(X,J)}, t \geq 0)$ of (X, J) if the following two properties are satisfied:

i) the identity between measures

$$\eta_s P_t^{(X,J)} = \eta_{t+s},$$

holds, that is

$$\sum_{i\in\mathcal{I}} \int_{(0,\infty)} \eta_s(\mathrm{d}x, i)\mathbb{E}_{x,i}\left[f(X_t, J_t)\right] = \sum_{i\in\mathcal{I}} \int_{(0,\infty)} \eta_{t+s}(\mathrm{d}x, i) f(x, i),$$

where, for every $i \in \mathcal{I}$, $f(\cdot, i)$ is a positive and measurable function and $s > 0$, $t \geq 0$.

ii) for all $s > 0$,

$$\eta_s f = \eta_1 H_{s^{1/\alpha}} f,$$

where, for every $i \in \mathcal{I}$, $f(\cdot, i)$ is a positive and measurable function, and for $c > 0$, H_c denotes the dilation operator $H_c f(x, i) = f(cx, i)$.

Theorem 2.9. *Assume that X is a conservative ssMp with types and index $\alpha > 0$. Moreover, suppose that the MAP (ξ, Θ) associated with X through the Lamperti transform is such that ξ is not concentrated on a lattice and has positive mean*

$$m = \mathtt{E}_{0,\pi}[\xi_1] < \infty. \tag{2.11}$$

Then as $x \to 0+$, the probability measures $\mathbb{P}_{x,j}$ converge in the sense of finite dimensional marginals towards a probability measure which is determined by the self-similar entrance law $(\eta_t, t \geq 0)$, where, for $j \in \mathcal{I}$ such that $f(\cdot, j)$ is positive and measurable, and $t > 0$,

$$\eta_t f := \frac{1}{\alpha m} \sum_{i\in\mathcal{I}} \pi_i \mathtt{E}^{\natural}_{0,i}\left[\frac{1}{I(\alpha\xi)} f\left(\left(\frac{t}{I(\alpha\xi)}\right)^{1/\alpha}, i\right)\right], \tag{2.12}$$

with

$$I(\alpha\xi) := \int_0^\infty \mathrm{e}^{\alpha\xi(s)}\mathrm{d}s.$$

More precisely, for all measurable sets $A_1, B_1, \ldots, A_n, B_n$ *and all times* $0 < t_1 < \ldots < t_n$*, as* $x \to 0+$*,*

$$\mathbb{P}_{x,j}(X_{t_1} \in A_1, J_{t_1} \in B_1, X_{t_2} \in A_2, J_{t_2} \in B_2, \ldots, X_{t_n} \in A_n, J_{t_n} \in B_n)$$
$$\to \sum_{i\in B_1} \int_{A_1} \eta_{t_1}(dy, i)\mathbb{P}_{y,i}(X_{t_2} \in A_2, J_{t_2} \in B_2, \ldots, X_{t_n} \in A_n, J_{t_n} \in B_n).$$

It is important to note that the self-similar entrance law $(\eta_t, t \geq 0)$ is actually a family of probability measures. Moreover, we stress that the above limit does not depend on the initial type j.

The proof of the previous result follows the same steps as the construction of the law of positive self-similar Markov processes issued from the origin given by Bertoin and Yor (2002) plus Markov renewal theory. We will explain in the Appendix how we can use all these ideas to deduce our result.

We finally emphasise that the above representation (2.12) of the entrance law does not seem to appear anywhere in the literature (even in the monotype case), except in the special setting of a stable process conditioned to avoid the origin, cf. Kuznetsov et al. (2014). A similar claim is nonetheless stated without proof in Theorem 11.16 in Kyprianou and Pardo (2022) for general real self-similar Markov processes, as in this case it can be obtained from copying arguments in the existing literature.

3. Multitype growth-fragmentation processes

In this section, we present an extension of the growth-fragmentation framework Bertoin (2017) to particles with finitely many types in $\mathcal{I}$, where each particle is given a type at birth, which may then evolve according to a Markov chain. We point out that the approach presented here is completely different to the treatment in Da Silva (2023) of the signed case, which relies on a change of Eve cell to go back to the positive setting. We shall describe the martingales appearing in this context, and how they can be found in the roots of *multitype cumulants*.

3.1. *Construction of the multitype growth-fragmentation cell system.* Following section 2, we will consider a *càdlàg* self-similar Markov process with types (X, J). For technical reasons, we further assume that (X, J) is either absorbed at the cemetery state ∂ after a finite time ζ, or that X converges to 0 under $\mathbb{P}_{x,i}$ for all $x \in \mathbb{R}_+^*, i \in \mathcal{I}$. We write $\Delta X(s) := (X(s) - X(s^-))\mathbf{1}_{\{X(s)<X(s^-)\}}$ for the jump of X, when negative.

We now construct a cell system whose building block is the self-similar Markov process with types (X, J). This cell system will start from a single particle whose size and type are given by the process (X, J), that will split in a binary way whenever X has a negative jump. This will create new particles with initial size given by the intensity of the jump, and which will then evolve as (X, J) independently of the mother cell, and independently of one another, conditionally on their sizes at birth. This construction takes the viewpoint presented in Bertoin (2017), but note that in our context we need to clarify the types of the offspring. To this end, we introduce some preliminary notation. Call (ξ, Θ) the MAP underlying (X, J) *via* the Lamperti-Kiu transform (2.8). We assume throughout the paper that for all $i \in \mathcal{I}$, the Lévy measure Λ_i of ξ_i can be decomposed as the sum of Lévy measures

$$\Lambda_i(\mathrm{d}x) := \sum_{k\in\mathcal{I}} \Lambda_i^{(k)}(\mathrm{d}x), \tag{3.1}$$

satisfying the following integrability condition

$$\int_{\mathbb{R}} (1 \wedge |x|^2)\Lambda_i(\mathrm{d}x) < \infty.$$

Likewise, for $i, j \in \mathcal{I}$, we give ourselves a decomposition of the law $\Lambda_{U_{i,j}}$ of $U_{i,j}$ as

$$\Lambda_{U_{i,j}}(\mathrm{d}x) := \sum_{k\in\mathcal{I}} \Lambda^{(k)}_{U_{i,j}}(\mathrm{d}x). \tag{3.2}$$

Equations (3.1) and (3.2) can be interpreted as a thinning of ξ_i and $U_{i,j}$ respectively: the jumps of ξ_i and $U_{i,j}$ should be understood as the result of a superimposition of jumps coming with a type $k \in \mathcal{I}$. Through the Lamperti time change (2.7), we see that any jump $\Delta X(s)$ of X now also comes with some type, that we denote $J_\Delta(s)$.

We may now construct the cell system associated with (X, J) and indexed by the tree $\mathbb{U} := \bigcup_{i\geq 0} \mathbb{N}^i$, with $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 := \{\varnothing\}$ is the label of the *Eve cell.* For $u := (u_1, \dots, u_i) \in \mathbb{U}$, we denote by $|u| = i$, the *generation* of u. In this tree, the offspring of u will be labelled by the lists $(u_1, \dots, u_i, k)$, with $k \in \mathbb{N}$.

We then define the law $\mathcal{P}_{x,i}$, $x > 0$, $i \in \mathcal{I}$, of the cell system $((\mathcal{X}_u(t), \mathcal{J}_u(t)), u \in \mathbb{U})$ driven by X similarly to Bertoin (2017). Let $b_\varnothing = 0$ and take a copy $(\mathcal{X}_\varnothing, \mathcal{J}_\varnothing)$ of (X, J) started from (x, i). We can rank the sequence of positive jumps and times $(x_1, \beta_1), (x_2, \beta_2), \dots$ of $-\mathcal{X}_\varnothing$ by descending lexicographical order of the x_k's (this is possible because in any case X is either absorbed at the cemetery state ∂ or converges to 0). We write $j_1, j_2, \dots$ for the corresponding types. Given this sequence $(x_k, j_k, \beta_k, k \in \mathbb{N})$, we define the first generation $(\mathcal{X}_k, \mathcal{J}_k), k \in \mathbb{N}$, of our cell system as independent processes with respective law $\mathbb{P}_{x_k,j_k}$, and we set $b_k = b_\varnothing + \beta_k$ for the *birth time* of k and ζ_k for its lifetime. Likewise, we define the n-th generation given generations $1, \dots, n-1$. A cell $u' = (u_1, \dots, u_{n-1}) \in \mathbb{N}^{n-1}$ gives birth to the cell $u = (u_1, \dots, u_{n-1}, k)$, with lifetime ζ_u, at time $b_u = b_{u'} + \beta_k$ where β_k is the k-th jump of $\mathcal{X}_{u'}$ (in terms of the same ranking as before). Moreover, conditionally on the jump sizes, types and times of $\mathcal{X}_{u'}$, $(\mathcal{X}_u, \mathcal{J}_u)$ has law $\mathbb{P}_{y,j}$ and is independent of the other daughter cells at generation n, where $-y = \Delta\mathcal{X}_{u'}(\beta_k)$ comes with type j. Note that division events are *conservative* in the sense that the sum of the size of a particle born at time t and of its mother cell at time t exactly equals the size of the mother cell before dislocation.

Although by construction the cells are not labelled chronologically, this uniquely defines the law $\mathcal{P}_{x,i}$ of the cell system driven by (X, J) and started from (x, i). The cell system $((\mathcal{X}_u(t), \mathcal{J}_u(t)), u \in \mathbb{U})$ is meant to describe the evolution of a population of atoms u with size $\mathcal{X}_u(t)$ and type $\mathcal{J}_u(t)$ evolving with its age t and fragmenting in a binary way. We stress that our framework allows for the type $\mathcal{J}_u$ of particle u to vary in time.

Finally, we define the *multitype growth-fragmentation process*

$$\mathbf{X}(t) := \{\{(\mathcal{X}_u(t - b_u), \mathcal{J}_u(t - b_u)),\ u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}\}, \quad t \geq 0,$$

where the double brackets denote multisets: $\mathbf{X}(t)$ is the collection of all the particles alive in the system at time t. We denote by $\mathbf{P}_{x,i}$ the law of $\mathbf{X}$ started from (x, i) and $(\mathcal{F}_t, t \geq 0)$ the natural filtration of $\mathbf{X}$.

Remark 3.1. We emphasise that only the *negative* jumps of X give birth to new cells. One could also be willing to create new particles at the positive jump times, corresponding to cells with negative mass, so that the conservation rule still holds at splittings, similarly as in Da Silva (2023). This will simply result in doubling the number of types of the chain J, by considering the sign itself as a type. Hence we can restrict without loss of generality to considering only positive cells, *i.e.* negative jumps.

We stress that, similarly to Bertoin (2017, Section 3.4), multitype growth-fragmentation processes are closely related to *multitype branching random walks*, as the process $(-\log \mathcal{X}_u(0), \mathcal{J}_u(0))_{u\in\mathbb{U}}$ forms a multitype branching random walk. It is clear from the definition of growth-fragmentation processes that the cell system enjoys a genealogical branching structure. Under mild assumptions, this extends to a temporal branching property. Construct

$$\overline{\mathbf{X}}(t) := \{\{(\mathcal{X}_u(t-b_u), \mathcal{J}_u(t-b_u), |u|),\ u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}\}, \quad t \geq 0,$$

by adjunction of the generations to the growth-fragmentation process; and consider its associated filtration $(\overline{\mathcal{F}}_t, t \geq 0)$. A measurable function $f : \mathbb{R}_+ \to [0, +\infty)$ is called *excessive* for $\mathbf{X}$ if $\inf_{x>a} f(x) > 0$ for all $a > 0$, and for all $(x, i) \in \mathbb{R}_+ \times \mathcal{I}$ and all $t \geq 0$,

$$\mathbf{E}_{x,i}\left[\sum_{(z,j)\in\mathbf{X}(t)} f(z)\right] \leq f(x). \tag{3.3}$$

If such an excessive function exists, then one can rank the elements $(X_1(t), J_1(t)), (X_2(t), J_2(t)), \ldots$ of $\mathbf{X}(t)$ by descending order of their size for any fixed t.

Proposition 3.2. *Assume that* $\mathbf{X}$ *has an excessive function. Then for any* $t \geq 0$*, conditionally on* $\overline{\mathbf{X}}(t) = \{\{(x_i, j_i, n_i)\}\}$*, the process* $(\overline{\mathbf{X}}(t+s), s \geq 0)$ *is independent of* $\overline{\mathcal{F}}_t$ *and distributed as*

$$\bigsqcup_{i\geq 1} \overline{\mathbf{X}}_i(s) \circ \theta_{n_i},$$

where the $\overline{\mathbf{X}}_i, i \geq 1$*, are independent processes distributed as* $\overline{\mathbf{X}}$ *under* $\mathbf{P}_{x_i,j_i}$*,* θ_n *is the shift operator, i.e.* $\{\{(z_i, y_i, k_i), i \geq 1\}\} \circ \theta_n := \{\{(z_i, y_i, k_i + n), i \geq 1\}\}$*, and* $\sqcup$ *denotes union of multisets.*

Proof: We refer to Bertoin (2017, Proposition 2) for a proof of the statement in the classical framework, which is then easily extended to the multitype case. □

3.2. *Martingales in multitype growth-fragmentation processes.* We continue the study of martingales for multitype growth-fragmentation processes initiated in Da Silva (2023) in the signed case. The fact that $(-\log \mathcal{X}_u(0), \mathcal{J}_u(0))_{u\in\mathbb{U}}$ forms a multitype branching random walk provides several tools, including genealogical martingales for the growth-fragmentation cell system. The key feature is the following matrix $m(q)$ indexed by the type set $\mathcal{I}$, with entries

$$m_{i,j}(q) := \mathbb{E}_i\left[\sum_{0<s<\zeta} |\Delta X(s)|^q \mathbf{1}_{\{J_\Delta(s)=j\}}\right], \quad q \in \mathbb{R}. \tag{3.4}$$

This matrix has only nonnegative entries. We make the following two assumptions throughout the paper.

Assumption A : *For* $q \in \mathbb{R}$ *such that* $m(q)$ *has finite entries, the matrix* $m(q)$ *is irreducible.*

In other words, Assumption A means that all the types communicate in the growth-fragmentation cell system (this is not too restrictive, since we could restrict to communication classes otherwise). Since $\mathcal{I}$ is finite, this enables us to consider the Perron-Frobenius eigenvalue $\mathrm{e}^{\lambda(q)}$ and an associated positive eigenvector.

Assumption B : *There exists* $\omega \in \mathbb{R}$ *such that* $\lambda(\omega) = 0$.

Note that, if μ denotes the measure

$$\mu(\mathrm{d}z) := \mathbb{E}_i\left[\sum_{0<s<\zeta} \mathbf{1}_{\{|\Delta X(s)|\in \mathrm{d}z\}} \mathbf{1}_{\{J_\Delta(s)=j\}}\right], \tag{3.5}$$

then for all $q \in \mathbb{R}$, $m_{i,j}(q)$ is the log-Laplace transform of μ with exponent q, and hence is convex. It is a consequence of Kingman's theorem (Kingman, 1961) that if m is finite and invertible in some neighbourhood of ω, then λ is convex on this domain. This implies that λ can only have at most two roots.

We shall give a criterion for Assumption B later on in section 3.3. If $(v_i)_{i\in\mathcal{I}}$ has positive entries and $\omega \geq 0$, we say that $((v_i)_{i\in\mathcal{I}}, \omega)$ is *admissible* for X if $\omega \in D$, $\lambda(\omega) = 0$ and $(v_i)_{i\in\mathcal{I}}$ is an associated eigenvector of $m(\omega)$. In other words, if $(v_i)_{i\in\mathcal{I}}$ has positive entries and $\omega \geq 0$, $((v_i)_{i\in\mathcal{I}}, \omega)$ is admissible for X if and only if

$$\forall i \in \mathcal{I}, \quad \mathbb{E}_i\left[\sum_{0<s<\zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega\right] = v_i.$$

Indeed, by Perron-Frobenius theory, only the leading eigenvalue can be associated to positive eigenvectors. This invariance property extends to a genealogical martingale as follows. Define

$$\mathfrak{G}_n := \sigma((\mathcal{X}_u, \mathcal{J}_u); |u| \leq n),$$

noting that by definition, if $u \in \mathbb{U}$ is such that $|u| = n+1$, then $\mathcal{X}_u(0)$ is $\mathfrak{G}_n$–measurable.

Proposition 3.3. *For all $(x,i) \in \mathbb{R}_+^* \times \mathcal{I}$, the process*

$$\mathcal{M}(n) := \sum_{|u|=n+1} v_{\mathcal{J}_u(0)} \mathcal{X}_u(0)^\omega, \quad n \geq 0,$$

defines a $(\mathfrak{G}_n, n \geq 0)$–martingale under $\mathcal{P}_{x,i}$.

Proof: The process $\mathcal{M}$ is obtained as the genealogical martingale of the multitype branching random walk $(-\log \mathcal{X}_u(0), \mathcal{J}_u(0))_{u\in\mathbb{U}}$, see Da Silva (2023, Theorem 3.3). □

Moreover, the following martingale for X will turn out useful in the next section. In particular, it implies the existence of an excessive function by extending Bertoin (2017, Theorem 1) to the multitype case.

Proposition 3.4. *For all $(x,i) \in \mathbb{R}_+^* \times \mathcal{I}$, under $\mathbb{P}_{x,i}$ the process*

$$M(t) := v_{J(t)} |X(t)|^\omega + \sum_{0<s\leq t\wedge\zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega, \quad t \geq 0,$$

is a uniformly integrable martingale for the filtration $(F_t^X, t \geq 0)$ of X, with terminal value

$$\sum_{0<s<\zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega.$$

Proof: We omit the proof as it essentially follows from Da Silva (2023, Proposition 3.5). □

3.3. *Multitype cumulant functions.* For any sequence $((v_i)_{i\in\mathcal{I}}, \omega)$, define

$$M(t) := v_{J(t)} |X(t)|^\omega + \sum_{0<s\leq t\wedge\zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega, \quad t \geq 0,$$

where we omit the dependence on ω and $(v_i)_{i\in\mathcal{I}}$ in the notation of M for simplicity. Proposition 3.4 states that when the underlying sequence is admissible, M is a martingale under $\mathbb{P}_i$ for all $i \in \mathcal{I}$ (see the signed case Da Silva (2023)). A converse statement also holds, providing a more tractable characterisation of admissibility.

Proposition 3.5. *Let $(v_i)_{i\in\mathcal{I}}$ a vector with positive entries and $\omega \geq 0$. Let H be the first jump time of J. Then $((v_i)_{i\in\mathcal{I}}, \omega)$ is admissible for X if and only if, for all $i \in \mathcal{I}$,*

$$\mathbb{E}_i[M(H)] = v_i.$$

Proof: The implication ($\Rightarrow$) follows easily from the optional stopping theorem applied to the martingale M in Proposition 3.4. Conversely, if we denote $(H_k, k \geq 0)$, the successive jump times of J (with $H_0 = 0$), then for any $i \in \mathcal{I}$, by the Markov property of (X, J) and self-similarity of X,

$$\mathbb{E}_i\left[\sum_{0<s<\zeta} v_{J_\Delta(s)}|\Delta X(s)|^\omega\right] = \sum_{k\geq 0}\mathbb{E}_i\left[\sum_{H_k<s\leq H_{k+1}} v_{J_\Delta(s)}|\Delta X(s)|^\omega\right]$$
$$= \sum_{k\geq 0}\mathbb{E}_i\left[|X(H_k)|^\omega\,\mathbb{E}_{J(H_k)}\left[\sum_{s\leq H} v_{J_\Delta(s)}|\Delta X(s)|^\omega\right]\right].$$

Because we have assumed $\mathbb{E}_j[M(H)] = v_j$ for all $j \in \mathcal{I}$, this is

$$\mathbb{E}_i\left[\sum_{0<s<\zeta} v_{J_\Delta(s)}|\Delta X(s)|^\omega\right] = \sum_{k\geq 0}\mathbb{E}_i\Big[|X(H_k)|^\omega\Big(v_{J(H_k)} - \mathbb{E}_{J(H_k)}\Big[v_{J(H)}|X(H)|^\omega\Big]\Big)\Big].$$

Hence, using again the Markov property and self-similarity of X backwards, we find ourselves with

$$\mathbb{E}_i\left[\sum_{0<s<\zeta} v_{J_\Delta(s)}|\Delta X(s)|^\omega\right] = \sum_{k\geq 0}\mathbb{E}_i\left[v_{J(H_k)}|X(H_k)|^\omega\right] - \sum_{k\geq 0}\mathbb{E}_i\left[v_{J(H_{k+1})}|X(H_{k+1})|^\omega\right],$$

which ultimately cancels out, leaving $\mathbb{E}_i\left[\sum_{0<s<\zeta} v_{J_\Delta(s)}|\Delta X(s)|^\omega\right] = v_i$. □

Next, we identify multitype cumulant functions $\mathcal{K}_i$, $i \in \mathcal{I}$, whose common roots correspond to the admissible exponents ω. To do so, we compute $\mathbb{E}_i[M(H)]$ in terms of the underlying MAP characteristics, for any (not necessarily admissible) sequence $((v_i)_{i\in\mathcal{I}}, \omega)$. The expectation can be written as $\mathbb{E}_i[M(H)] = A + B$, where

$$A := \mathbb{E}_i\left[\sum_{0<s\leq H\wedge\zeta} v_{J_\Delta(s)}|\Delta X(s)|^\omega\right] \quad\text{and}\quad B := \mathbb{E}_i\left[v_{J(H)}|X(H)|^\omega\right].$$

Let us start with the term A. For $s > 0$, we write as in (2.8), $X(\varphi^{-1}(s)) = \mathrm{e}^{\xi(s)}$ and $J(\varphi^{-1}(s)) = \Theta(s)$ under $\mathbb{P}_i$, where φ is the usual time-change (2.7). From this standpoint,

$$A = \mathtt{E}_i\left[\sum_{0<s<\rho_i} v_{\iota_\Delta(s)}\mathrm{e}^{\omega\xi_i(s^-)}\left(1-\mathrm{e}^{\Delta\xi_i(s)}\right)^\omega\right] + \mathtt{E}_i\left[v_{\iota_\Delta(\rho_i)}\mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}\right)^\omega\right], \tag{3.6}$$

where $\iota_\Delta(s)$ stands for the type corresponding to the jump of ξ at time s. By independence and the compensation formula for ξ_i, the first term of (3.6) is

$$\mathtt{E}_i\left[\sum_{s<\rho_i} v_{\iota_\Delta(s)}\mathrm{e}^{\omega\xi_i(s^-)}\left(1-\mathrm{e}^{\Delta\xi_i(s)}\right)^\omega\right]$$
$$= \int_0^\infty \mathrm{d}t\,(-q_{i,i})\mathrm{e}^{q_{i,i}t}\sum_{k\in\mathcal{I}} v_k\mathtt{E}_i\left[\int_0^t \mathrm{d}s\,\mathrm{e}^{\omega\xi_i(s)}\right]\int_{(-\infty,0)}\Lambda_i^{(k)}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega$$
$$= \sum_{k\in\mathcal{I}} v_k\int_{(-\infty,0)}\Lambda_i^{(k)}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega\cdot\frac{1}{\psi_i(\omega)}\int_0^\infty \mathrm{d}t\,(-q_{i,i})\mathrm{e}^{q_{i,i}t}(\mathrm{e}^{\psi_i(\omega)t}-1)$$
$$= -\frac{1}{\psi_i(\omega)+q_{i,i}}\cdot\sum_{k\in\mathcal{I}} v_k\int_{(-\infty,0)}\Lambda_i^{(k)}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega,$$

provided $\psi_i(\omega) + q_{i,i} < 0$ (otherwise the expectation blows up). Now, let $\iota^\star = \Theta(\rho_i)$ be the type to which the Markov chain jumps at time ρ_i. Then $\iota^\star$ is independent of ρ_i, and for all $j \in \mathcal{I} \setminus \{i\}$, $\iota^\star = j$ with probability $-\frac{q_{i,j}}{q_{i,i}}$. By conditioning on ρ_i, we obtain that the second term of (3.6) is

$$\begin{aligned}
&\mathtt{E}_i\left[v_{\iota_\Delta(\rho_i)} \mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\iota^\star}}\right)^\omega\right] \\
&\qquad = \int_0^\infty \mathrm{d}t\,(-q_{i,i})\mathrm{e}^{q_{i,i}t} \sum_{j\in\mathcal{I}\setminus\{i\}} \frac{q_{i,j}}{(-q_{i,i})} \mathtt{E}_i\left[\mathrm{e}^{\omega\xi_i(t)}\right] \sum_{k\in\mathcal{I}} v_k \int_{(-\infty,0)} \Lambda^{(k)}_{U_{i,j}}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega \\
&\qquad = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{(\psi_i(\omega)+q_{i,i})t} \sum_{j\in\mathcal{I}\setminus\{i\}} q_{i,j} \sum_{k\in\mathcal{I}} v_k \int_{(-\infty,0)} \Lambda^{(k)}_{U_{i,j}}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega \\
&\qquad = -\frac{1}{\psi_i(\omega)+q_{i,i}} \sum_{k\in\mathcal{I}} v_k \sum_{j\in\mathcal{I}\setminus\{i\}} q_{i,j} \int_{(-\infty,0)} \Lambda^{(k)}_{U_{i,j}}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega,
\end{aligned}$$

provided again that $\psi_i(\omega) + q_{i,i} < 0$. Therefore, we end up with

$$A = -\frac{1}{\psi_i(\omega)+q_{i,i}} \sum_{k\in\mathcal{I}} v_k \int_{(-\infty,0)} \Pi_{i,k}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega,$$

with $\Pi_{i,k}(\mathrm{d}x) := \Lambda_i^{(k)}(\mathrm{d}x) + \sum_{j\in\mathcal{I}\setminus\{i\}} q_{i,j}\Lambda^{(k)}_{U_{i,j}}(\mathrm{d}x)$.

We now compute

$$B = \mathtt{E}_i\left[v_{\iota^\star}\mathrm{e}^{\omega(\xi_i(\rho_i)+U_{i,\iota^\star})}\right].$$

As before, we condition on ρ_i and decompose over the possible values $j \in \mathcal{I} \setminus \{i\}$ for $\iota^\star$:

$$\begin{aligned}
B &= \sum_{j\in\mathcal{I}\setminus\{i\}} \int_0^\infty \mathrm{d}s\,(-q_{i,i})\mathrm{e}^{q_{i,i}s} \frac{q_{i,j}}{(-q_{i,i})} v_j\, \mathtt{E}_i\left[\mathrm{e}^{\omega(\xi_i(s)+U_{i,j})}\right] \\
&= \sum_{j\in\mathcal{I}\setminus\{i\}} q_{i,j}v_j \int_0^\infty \mathrm{d}s\,\mathrm{e}^{q_{i,i}s}\mathrm{e}^{\psi_i(\omega)s}G_{i,j}(\omega) \\
&= -\frac{1}{\psi_i(\omega)+q_{i,i}} \cdot \sum_{j\in\mathcal{I}\setminus\{i\}} q_{i,j}v_jG_{i,j}(\omega),
\end{aligned}$$

as long as $\psi_i(\omega) + q_{i,i} < 0$. We come to the conclusion that

$$\mathbb{E}_i[M(H)] = -\frac{1}{\psi_i(\omega)+q_{i,i}} \cdot \left(\sum_{k\in\mathcal{I}} v_k \int_{(-\infty,0)} \Pi_{i,k}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega + \sum_{j\in\mathcal{I}\setminus\{i\}} q_{i,j}v_jG_{i,j}(\omega)\right).$$

This is equal to v_i if and only if,

$$\mathcal{K}_i(\omega) := (\psi_i(\omega)+q_{i,i}) + \sum_{k\in\mathcal{I}} \frac{v_k}{v_i} \int_{(-\infty,0)} \Pi_{i,k}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega + \sum_{j\in\mathcal{I}\setminus\{i\}} \frac{v_j}{v_i} q_{i,j}G_{i,j}(\omega) = 0,$$

and, thanks to Proposition 3.5, Assumption A in Section 3.2 boils down to the existence of $\omega \in \mathbb{R}$ and a sequence $(v_i)_{i\in\mathcal{I}}$ of positive numbers such that, for all $i \in \mathcal{I}$, $\mathcal{K}_i(\omega) = 0$. We will call the family $(\mathcal{K}_i, i \in \mathcal{I})$ the *multitype cumulant functions*. We also write

$$\kappa_i(q) := (\psi_i(q)+q_{i,i}) + \int_{(-\infty,0)} \Pi_{i,i}(\mathrm{d}x)(1-\mathrm{e}^x)^q, \quad q \geq 0, \tag{3.7}$$

for the *cumulant* function corresponding to type i, so that for $q \geq 0$,

$$\mathcal{K}_i(q) := \kappa_i(q) + \sum_{j \in \mathcal{I}\setminus\{i\}} \frac{v_j}{v_i} \left(\int_{(-\infty,0)} \Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^q + q_{i,j} G_{i,j}(q) \right). \tag{3.8}$$

4. The spine decomposition of multitype growth-fragmentation processes

4.1. *Description of the spine under the change of measure.*

A change of measure. The martingale $(\mathcal{M}(n), n \geq 0)$ in Proposition 3.3 enables us to introduce a new probability measure $\widehat{\mathcal{P}}_{x,i}$ for $x > 0, i \in \mathcal{I}$. Under this change of measure, the cell system has a spine decomposition that we aim to describe (see Bertoin et al. (2018a, Section 4.1)). The measure $\widehat{\mathcal{P}}_{x,i}$ singles out a particular *leaf* $\mathcal{L} \in \partial\mathbb{U} = \mathbb{N}^{\mathbb{N}}$. On $\mathfrak{G}_n$, for $n \geq 0$, it has Radon-Nikodym derivative $\mathcal{M}(n)$ with respect to $\mathcal{P}_{x,i}$, up to normalisation, *viz.* for all $G_n \in \mathfrak{G}_n$,

$$\widehat{\mathcal{P}}_{x,i}(G_n) := \frac{1}{v_i x^{\omega}} \mathcal{E}_{x,i}\left[\mathcal{M}(n)\mathbf{1}_{G_n}\right].$$

Moreover, conditionally on $\mathfrak{G}_n$, the parent of the particular leaf $\mathcal{L}$ at generation $n+1$ is chosen under $\widehat{\mathcal{P}}_{x,i}$ proportionally to its weight in the martingale $\mathcal{M}(n)$. More precisely, let $\ell(n)$ denote the ancestor of a leaf $\ell \in \partial\mathbb{U}$ at generation n. Then for all $n \geq 0$ and all $u \in \mathbb{U}$ such that $|u| = n+1$,

$$\widehat{\mathcal{P}}_{x,i}\left(\mathcal{L}(n+1) = u \,\middle|\, \mathfrak{G}_n\right) := \frac{v_{\mathcal{J}_u(0)} \mathcal{X}_u(0)^{\omega}}{\mathcal{M}(n)}. \tag{4.1}$$

The consistency of formula (4.1) stems from the martingale property of $(\mathcal{M}(n), n \geq 0)$ and the branching structure of the system, thus defining a unique probability measure by an application of the Kolmogorov extension theorem.

One key player is provided by the so called *tagged cell* or *spine*, which consists in following the evolution of the cell associated with the leaf $\mathcal{L}$. The tagged cell will have the role of a backbone in the *spine decomposition* of the cell system under $\widehat{\mathcal{P}}_{x,i}$. Let $b_\ell = \lim \uparrow b_{\ell(n)}$ for any leaf $\ell \in \partial\mathbb{U}$. Then, we define $\widehat{\mathcal{X}}$ by $(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) := \partial$ if $t \geq b_{\mathcal{L}}$ and

$$\widehat{\mathcal{X}}(t) := \mathcal{X}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}) \quad \text{and} \quad \widehat{\mathcal{J}}(t) := \mathcal{J}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}) \qquad \text{if} \qquad t < b_{\mathcal{L}},$$

where n_t is the unique integer n such that $b_{\mathcal{L}(n)} \leq t < b_{\mathcal{L}(n+1)}$. From the very definition of $\widehat{\mathcal{P}}_{x,i}$, for all nonnegative measurable function f and all $\mathfrak{G}_n$–measurable nonnegative random variable B_n,

$$v_i x^{\omega} \widehat{\mathcal{E}}_{x,i}\Big[f(\mathcal{X}_{\mathcal{L}(n+1)}(0), \mathcal{J}_{\mathcal{L}(n+1)}(0)) B_n\Big] = \mathcal{E}_{x,i}\left[\sum_{|u|=n+1} v_{\mathcal{J}_u(0)} \mathcal{X}_u(0)^{\omega} f(\mathcal{X}_u(0), \mathcal{J}_u(0)) B_n\right].$$

This extends to a temporal identity in the following way. Recall that $\mathbf{X}(t) = \{\{(X_k(t), J_k(t)), k \geq 1\}\}$, for $t \geq 0$, have been enumerated by descending order of the $X_k(t)$.

Proposition 4.1. *For every $t \geq 0$, every nonnegative measurable function f vanishing at ∂, and every $\overline{\mathcal{F}}_t$–measurable nonnegative random variable B_t, we have*

$$v_i x^{\omega} \widehat{\mathcal{E}}_{x,i}\Big[f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) B_t\Big] = \mathcal{E}_{x,i}\left[\sum_{k \geq 1} v_{J_k(t)} X_k(t)^{\omega} f(X_k(t), J_k(t)) B_t\right].$$

Proof: The proof essentially follows from the arguments presented in the proof of Bertoin et al. (2018a, Proposition 4.1). We provide its proof for the sake of completeness.

Let $t \geq 0$. Consider the case when B_t is $\overline{\mathcal{F}}_t \cap \mathfrak{G}_k$–measurable for some $k \in \mathbb{N}$ (the result would then be readily extended by a monotone class argument). Since $f(\partial) = 0$, almost surely,

$$f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) B_t \mathbf{1}_{\{b_{\mathcal{L}(n+1)} > t\}} \underset{n\to\infty}{\longrightarrow} f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) B_t.$$

Therefore, by monotone convergence,

$$\widehat{\mathcal{E}}_{x,i}\Big[f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) B_t\Big] = \lim_{n\to\infty} \widehat{\mathcal{E}}_{x,i}\Big[f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) B_t \mathbf{1}_{\{b_{\mathcal{L}(n+1)} > t\}}\Big].$$

Now, we want to condition on $\mathfrak{G}_n$ and decompose $\mathcal{L}(n+1)$ over the cells at generation $n+1$, provided $n > k$ so that B_t is $\mathfrak{G}_n$–measurable. For u such that $b_u > t$, write $u(t)$ for the most recent ancestor of u at time t. Then

$$\begin{aligned}&\widehat{\mathcal{E}}_{x,i}\left[f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) B_t \mathbf{1}_{\{b_{\mathcal{L}(n+1)} > t\}}\right] \\ &\qquad = \frac{1}{v_i(\omega) x^{\omega}} \mathcal{E}_{x,i}\left[\sum_{|u|=n+1} v_{\mathcal{J}_u(0)} \mathcal{X}_u(0)^{\omega} \mathbf{1}_{\{b_u > t\}} f(\mathcal{X}_{u(t)}(t - b_{u(t)}), \mathcal{J}_{u(t)}(t - b_{u(t)})) B_t\right].\end{aligned}$$

Splitting over the value of $u(t)$ yields

$$\begin{aligned}&\mathcal{E}_{x,i}\left[\sum_{|u|=n+1} v_{\mathcal{J}_u(0)} \mathcal{X}_u(0)^{\omega} \mathbf{1}_{\{b_u > t\}} f(\mathcal{X}_{u(t)}(t - b_{u(t)}), \mathcal{J}_{u(t)}(t - b_{u(t)})) B_t\right] \\ &\qquad = \mathcal{E}_{x,i}\left[\sum_{|u'|\leq n} \sum_{|u|=n+1} v_{\mathcal{J}_u(0)} \mathcal{X}_u(0)^{\omega} \mathbf{1}_{\{b_u > t\}} f(\mathcal{X}_{u'}(t - b_{u'}), \mathcal{J}_{u'}(t - b_{u'})) B_t \mathbf{1}_{\{u(t) = u'\}}\right] \end{aligned} \tag{4.2}$$

and by conditioning on $\overline{\mathcal{F}}_t$ and applying the temporal branching property stated in Proposition 3.2,

$$\begin{aligned}&\mathcal{E}_{x,i}\left[\sum_{|u|=n+1} v_{\mathcal{J}_u(0)} \mathcal{X}_u(0)^{\omega} \mathbf{1}_{\{b_u > t\}} f(\mathcal{X}_{u(t)}(t - b_{u(t)}), \mathcal{J}_{u(t)}(t - b_{u(t)})) B_t\right] \\ &= \mathcal{E}_{x,i}\left[\sum_{|u'|\leq n} f(\mathcal{X}_{u'}(t - b_{u'}), \mathcal{J}_{u'}(t - b_{u'})) B_t \right. \\ &\qquad\qquad \left. \times \mathcal{E}_{\mathcal{X}_{u'}(t-b_{u'}), \mathcal{J}_{u'}(t-b_{u'})}\left[\sum_{|u|=n+1-|u'|} v_{\mathcal{J}_{u'u}(0)} \mathcal{X}_{u'u}(0)^{\omega}\right] \mathbf{1}_{\{b_{u'} \leq t < b_{u'} + \zeta_{u'}\}}\right] \\ &= \mathcal{E}_{x,i}\left[\sum_{|u'|\leq n} f(\mathcal{X}_{u'}(t - b_{u'}), \mathcal{J}_{u'}(t - b_{u'})) B_t \, \mathbf{1}_{\{b_{u'} \leq t < b_{u'} + \zeta_{u'}\}} v_{\mathcal{J}_{u'}(t - b_{u'})} \mathcal{X}_{u'}(t - b_{u'})^{\omega}\right].\end{aligned}$$

Finally, taking $n \to \infty$ and using monotone convergence, we obtain the desired result. □

Remark 4.2. Proposition 4.1 applied with $f := \mathbf{1}_{\{x \neq \partial\}}$ yields that the temporal analogue of $\mathcal{M}(n)$ in Proposition 3.3,

$$\mathcal{M}_t := \sum_{i=1}^{\infty} v_{J_i(t)} X_i(t)^{\omega}, \quad t \geq 0,$$

is a supermartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

The law of the growth-fragmentation under $\widehat{\mathcal{P}}_{x,i}$. We now describe the law of $\widehat{\mathbf{X}}$ under $\widehat{\mathcal{P}}_{x,i}$. Loosely speaking, the tagged cell will serve as a backbone evolving as some explicit self-similar multitype Markov process, to which we attach independent copies of the original growth-fragmentation process. We must first reconstruct the whole cell system from the spine by recording the negative jumps of $\widehat{\mathcal{X}}$, as detailed in Bertoin et al. (2018a, Section 4.1). We will label these by couples (n, j), where $n \geq 0$ is the generation of the tagged cell immediately before the jump, and $j \geq 1$ is the rank (for the usual ranking) of the jump among those of the tagged cell at generation n (including the final jump when the generation changes from n to $n + 1$). To each such (n, j) corresponds a growth-fragmentation $\widehat{\mathbf{X}}_{n,j}$ stemming from the corresponding jump: if the generation does not change during the (n, j)–jump, then we set

$$\widehat{\mathbf{X}}_{n,j}(t) := \{\{(\mathcal{X}_{uw}(t - b_{uw} + b_u), \mathcal{J}_{uw}(t - b_{uw} + b_u)),\ w \in \mathbb{U} \text{ and } b_{uw} \leq t + b_u < b_{uw} + \zeta_{uw}\}\},$$

where u is the label of the cell born at the (n, j)–jump. Otherwise, the (n, j)–jump corresponds to a jump for the generation of the tagged cell and the tagged cell jumps from label u to label uj say, in which case

$$\begin{aligned}\widehat{\mathbf{X}}_{n,j}(t) &:= \{\{(\mathcal{X}_u(t - b_u + b_{uj}), \mathcal{J}_u(t - b_u + b_{uj})),\ b_u \leq t + b_{uj} < b_u + \zeta_u\}\}\\ &\cup \{\{(\mathcal{X}_{uw}(t - b_{uw} + b_{uj}), \mathcal{J}_{uw}(t - b_{uw} + b_{uj})),\ w \notin \mathbb{T}_{uj} \text{ and } b_{uj} \leq b_{uw} \leq t + b_{uj} < b_{uw} + \zeta_{uw}\}\},\end{aligned}$$

where for $v \in \mathbb{U}$, $\mathbb{T}_v := \{vw,\ w \in \mathbb{U}\}$. We agree that $\widehat{\mathbf{X}}_{n,j} := \partial$ when the (n, j)–jump does not exist, and this completely defines $\widehat{\mathbf{X}}_{n,j}$ for all $n \geq 0$ and all $j \geq 1$.

Let $\widehat{F}(q) := (\widehat{F}_{i,j}(q))_{i,j\in\mathcal{I}}$ be the matrix with entries

$$\widehat{F}_{i,j}(q) = \begin{cases} \dfrac{v_j}{v_i}\left(\displaystyle\int_{(-\infty,0)} \Pi_{i,j}(\mathrm{d}x)(1 - \mathrm{e}^x)^{q+\omega} + q_{i,j}G_{i,j}(q + \omega)\right) & \text{if } i \neq j,\\ \kappa_i(\omega + q) & \text{if } i = j.\end{cases} \tag{4.3}$$

Theorem 4.3. *Under $\widehat{\mathcal{P}}_{x,i}$, $(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t), 0 \leq t < b_{\mathcal{L}})$ is a self-similar Markov process with types in $\mathcal{I}$, whose underlying Markov additive process has the matrix exponent $\widehat{F}$ in (4.3). Moreover, conditionally on $(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t))_{0\leq t<b_{\mathcal{L}}}$ and $\widehat{\mathcal{J}}_{n,j}$, the processes $\widehat{\mathbf{X}}_{n,j}$, $n \geq 0$, $j \geq 1$, are independent and each $\widehat{\mathbf{X}}_{n,j}$ has law $\mathbf{P}_{x(n,j),\widehat{\mathcal{J}}_{n,j}}$ where $-x(n, j)$ is the size of the (n, j)–th jump.*

Remark 4.4.
(1) The law of the generation n_t of the spine at time t is not so explicit as in Bertoin et al. (2018a) or Da Silva (2023) in the constant sign case, where it is described by a Poisson process counting some special jumps of $\widehat{\mathcal{X}}$. This is because in the multitype case, one has to keep track of the (random) type of the spine at these special jump times. In fact, for this reason, the random variable $b_{\mathcal{L}(1)}$, which is the first jump time for n_t, may not be exponential.
(2) The proof of Theorem 4.3 goes through determining all three components $\widehat{\psi}_i$, $\widehat{q}_{i,j}$, and $\widehat{G}_{i,j}$ of the MAP in (2.2). This sheds light on the structure of the MAP under (4.3).

We postpone the proof of Theorem 4.3 until Section 4.5, and discuss instead some applications, which are also new for the signed case Da Silva (2023).

4.2. *Martingale exponents in multitype growth-fragmentation processes.* First, we prove that admissible characteristics $(\omega, (v_i, i \geq 1))$ are associated to the roots of the leading eigenvalue of the spine matrix exponent. Recall from Section 3.2 the notation $m(q), q \in \mathbb{R}$, for the matrix with nonnegative entries

$$m_{i,j}(q) := \mathbb{E}_i\left[\sum_{0<s<\zeta} |\Delta X(s)|^q \mathbf{1}_{\{J_\Delta(s)=j\}}\right],$$

and $\mathrm{e}^{\lambda(q)}$ for the Perron-Frobenius eigenvalue of $m(q)$. We will also make use of the spectral properties of Markov additive processes listed in Section 2. In particular, Proposition 2.3 applies to the matrix exponent $\widehat{F}$ of some spine in our growth-fragmentation process (see (4.3)): in this case, we shall denote by $\widehat{\chi}(q)$ the leading eigenvalue.

Proposition 4.5. *Assume that $((v_i)_{i\in\mathcal{I}},\omega)$ is admissible, and let $\widehat{F}$ be the matrix exponent of the associated spine. Then $((v'_i)_{i\in\mathcal{I}},\omega')$ is admissible if and only if $\left(\frac{v'_i}{v_i}\right)_{i\in\mathcal{I}}$ is an eigenvector of $\widehat{F}(\omega'-\omega)$ associated with the eigenvalue 0. In particular, the exponents ω' for which $\lambda(\omega')=0$ are exactly the roots of $\widehat{\chi}(\omega'-\omega)=0$.*

Note that Proposition 4.5 contains (and reproves) the fact that there can be at most two martingale exponents ω, by convexity of the leading eigenvalue (Proposition 2.5).

Proof: Set $\Delta\omega := \omega'-\omega$ and c the vector with entries $c_i := \frac{v'_i}{v_i}$, $i\in\mathcal{I}$. Then for all $i\in\mathcal{I}$,

$$\sum_{j\in\mathcal{I}} \frac{c_j}{c_i}\widehat{F}_{i,j}(\Delta\omega) = \kappa_i(\omega') + \sum_{j\neq i}\frac{v'_j}{v'_i}\left(\int_{(-\infty,0)} \Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^{\omega'} + q_{i,j}G_{i,j}(\omega')\right) = \mathcal{K}_i(\omega'), \qquad (4.4)$$

where the $\mathcal{K}_i$ are the multitype cumulant functions defined in (3.8). This formula proves Proposition 4.5 in both directions. The fact that the admissible exponents correspond to the roots of $\widehat{\chi}(\cdot-\omega)$ is a consequence of Perron-Frobenius theory: if 0 is an eigenvalue of $\widehat{F}(\omega'-\omega)$ associated with a positive eigenvector, then it must be the leading eigenvalue.

□

If ω and ω' are two martingale exponents, one can actually relate the corresponding spines $(\widehat{\mathcal{X}},\widehat{\mathcal{J}})$ and $(\widehat{\mathcal{X}}',\widehat{\mathcal{J}}')$. More precisely, one is obtained from the other one upon tilting the measure by the so-called Wald martingale (see Proposition 2.4) for the underlying Markov additive process. Write $(\widehat{\xi},\widehat{\Theta})$ and $(\widehat{\xi}',\widehat{\Theta}')$ for the Markov additive processes with respective laws $\widehat{\mathtt{P}}$ and $\widehat{\mathtt{P}}'$ appearing in the Lamperti-Kiu representations of $(\widehat{\mathcal{X}},\widehat{\mathcal{J}})$ and $(\widehat{\mathcal{X}}',\widehat{\mathcal{J}}')$ respectively, and let $\widehat{F}$ be the matrix exponent of $(\widehat{\xi},\widehat{\Theta})$. Denote by v and v' the eigenvectors of m corresponding to ω and ω', and write c for the vector with entries $c_i := \frac{v'_i}{v_i}$, $i\in\mathcal{I}$. The proof of Proposition 4.5 gives that the leading eigenvalue of $\widehat{F}(\omega'-\omega)$ is $\widehat{\chi}(\omega'-\omega)=0$ and is associated to the positive eigenvector c. Hence the Wald martingale at $\omega'-\omega$ for $(\widehat{\xi},\widehat{\Theta})$ is

$$\widehat{\mathcal{W}}(t) = \frac{c_{\widehat{\Theta}(t)}}{c_{\widehat{\Theta}(0)}}\mathrm{e}^{(\omega'-\omega)\widehat{\xi}(t)}, \quad t\geq 0.$$

The law of $(\widehat{\xi},\widehat{\Theta})$ under the probability measure biased by $\widehat{\mathcal{W}}$ is also given by Proposition 2.4, and one can check that it coincides with the law of $(\widehat{\xi}',\widehat{\Theta}')$ which is characterised by (4.3) but with (ω',v') instead of (ω,v). In other words, for any nonnegative measurable function f and all $x\in\mathbb{R}, i\in\mathcal{I}, t\in\mathbb{R}_+$,

$$\widehat{\mathtt{E}}'_{x,i}\big[f(\widehat{\xi}'(s),\widehat{\Theta}'(s),s\leq t)\big] = \widehat{\mathtt{E}}_{x,i}\left[f(\widehat{\xi}(s),\widehat{\Theta}(s),s\leq t)\cdot\frac{c_{\widehat{\Theta}(t)}}{c_i}\mathrm{e}^{(\omega'-\omega)\widehat{\xi}(t)}\right]. \qquad (4.5)$$

Alternatively (and perhaps more tellingly), one can apply the many-to-one lemma (Proposition 4.1) to relate the two spines.

4.3. *Two genealogical martingales.* In this subsection, we make the following Cramér-type condition, assuming that there exist two *admissible* exponents ω_- and ω_+ in the interior of the domain where the matrix m is finite, with $0 < \omega_- < \omega_+$. In this case, we will add a $+$ or $-$ superscript (or subscript) to the quantities considered before: for example, we denote by v^+ and v^- the Perron-Frobenius eigenvectors associated to $m(\omega_+)$ and $m(\omega_-)$ respectively. We also assume that $\widehat{\chi}'_-(\omega_+ - \omega_-)$ is finite, where $\widehat{\chi}_-$ is the leading eigenvalue associated to the spine with respect to ω_-. Note that by convexity of $\widehat{\chi}_-$, $\widehat{\chi}_-(0) < 0$ and $\widehat{\chi}'_-(\omega_+ - \omega_-) > 0$. Proposition 2.4 also shows that the condition that $\widehat{\chi}'_-(\omega_+ - \omega_-)$ is finite is the same as assuming that $\widehat{\chi}'_+(\omega_- - \omega_+)$ is finite. Since $\widehat{\chi}_-(0) < 0$, it is known that $\widehat{\xi}^-$ drifts to $-\infty$ (see identity (11.12) in Kyprianou and Pardo (2022)). It is not too hard to see that this entails

$$\mathcal{E}_{1,\pi}\left[\sum_{k\geq 1}\frac{v^-_{\mathcal{J}_k(0)}}{v^-_i}\mathcal{X}_k(0)^{\omega_-}\log|\mathcal{X}_k(0)|\right] = \widehat{\mathcal{E}}_{1,\pi}\Big[\log\widehat{\mathcal{X}}^-(b_{\mathcal{L}(1)})\Big] \in (-\infty, 0).$$

Actually, for some specific point we will need to work under the stronger assumption

$$\text{(H)} \qquad \mathcal{E}_{1,i}\left[\sum_{k\geq 1}\frac{v^-_{\mathcal{J}_k(0)}}{v^-_i}\mathcal{X}_k(0)^{\omega_-}\log|\mathcal{X}_k(0)|\right] \in (-\infty, 0), \quad \text{for all } i \in \mathcal{I}.$$

We will emphasize when we assume (H).

We state for future reference the following lemma.

Lemma 4.6. *For all $\gamma < \omega_+$, the leading eigenvalue of ξ satisfies $\chi(\gamma) < 0$.*

Proof: Fix $\gamma < \omega_+$. We first remark that for $i \in \mathcal{I}$, the multitype cumulant function defined in (3.8),

$$\mathcal{K}_i(q) = \psi_i(q) + \sum_{j\in\mathcal{I}}\frac{v_j}{v_i}\left(\int_{(-\infty,0)}\Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^q + q_{i,j}G_{i,j}(q)\right),$$

is convex. Since ω_+ is a root of $\mathcal{K}_i$, and $\mathcal{K}_i(0) = \psi_i(0) \leq 0$, we infer by convexity that $\mathcal{K}_i$ must be negative on $(0, \omega_+)$. Let $(w_i)_{i\in\mathcal{I}}$ be a positive eigenvector of F associated with $\chi(\gamma)$. Then for all $i \in \mathcal{I}$, using that $\mathcal{K}_i(\gamma) < 0$,

$$\begin{aligned}\chi(\gamma)w_i = w_i(\psi_i(\gamma) + q_{i,i}) + \sum_{j\neq i} w_j q_{i,j}G_{i,j}(\gamma) &< -\sum_{j\neq i}\Big(w_i\frac{v_j}{v_i} - w_j\Big)q_{i,j}G_{i,j}(\gamma) \\ &= -\sum_{j\neq i}\Big(\frac{w_i}{v_i} - \frac{w_j}{v_j}\Big)v_j q_{i,j}G_{i,j}(\gamma).\end{aligned}$$

We now take $i \in \mathcal{I}$ such that $\frac{w_i}{v_i}$ is maximal. This yields $\chi(\gamma)w_i < 0$, and since $w_i > 0$, we get $\chi(\gamma) < 0$. □

Our goal is to carry out in the multitype setting the analysis conducted in Bertoin et al. (2018a) of the two martingales associated to ω_- and ω_+ (see notably Sections 2.3 and 3.3 there). We start by the study of the two genealogical martingales $(\mathcal{M}^+(n), n \geq 0)$ and $(\mathcal{M}^-(n), n \geq 0)$ with associated exponents ω_+ and ω_-, namely

$$\mathcal{M}^+(n) = \sum_{|u|=n+1} v^+_{\mathcal{J}_u(0)}\mathcal{X}_u(0)^{\omega_+}, \quad n \geq 0,$$

and

$$\mathcal{M}^-(n) = \sum_{|u|=n+1} v^-_{\mathcal{J}_u(0)}\mathcal{X}_u(0)^{\omega_-}, \quad n \geq 0.$$

Our arguments are inspired from Bertoin et al. (2018a, Section 2.3), but rely on a multitype version of Biggins' martingale convergence theorem Biggins (1977); Lyons (1997), due to Kyprianou and

Rahimzadeh Sani (2001). Note that both $\mathcal{M}^+$ and $\mathcal{M}^-$ converge almost surely as nonnegative martingales; we now investigate whether the limit is degenerate or not.

Proposition 4.7. *For all $(x,i) \in \mathbb{R}_+^* \times \mathcal{I}$, the martingale $(\mathcal{M}^+(n), n \geq 0)$ converges to 0 almost surely under $\mathcal{P}_{x,i}$.*

Proof: The proof is based on Biggins' martingale convergence theorem for multitype branching random walk (see Kyprianou and Rahimzadeh Sani (2001, Theorem 1)). Recall that, as noted in Section 3.1, the process $(-\log \mathcal{X}_u(0), \mathcal{J}_u(0))_{u\in\mathbb{U}}$ forms a multitype branching random walk. Since $\lambda(\omega_+) = 0$ and $\omega_+ > 0$, the result will follow provided that $\lambda'(\omega_+) > 0$. But since $\omega_+ > \omega_-$, by convexity of λ (see the discussion after identity (3.5)), $\lambda'(\omega_+) > 0$, which concludes the proof. □

Remark 4.8. We observe that we can combine the ideas of Bertoin et al. (2018a, Section 4.2) with Theorem 2.9 to construct the growth-fragmentation under a version of $\widehat{\mathcal{P}}^+_{x,i}$ starting from 0. Indeed, the condition that $\widehat{\chi}'_+(0) > 0$ guarantees that $\widehat{\xi}^+$ drifts to $+\infty$, see for instance the strong law of large numbers for MAPs that appears in identity (11.12) in Kyprianou and Pardo (2022) (see also Theorem 34 in Dereich et al. (2017)). This defines the law of the spine started from 0, and one can then rebuild the tree using the spine as a backbone as in Bertoin et al. (2018a).

In contrast, the martingale $(\mathcal{M}^-(n), n \geq 0)$ satisfies the following properties. We assume that the chain $\widehat{\mathcal{J}} = (\mathcal{J}_{\mathcal{L}(k)}(0), k \geq 0)$ describing the type of the spine at generation k is irreducible and aperiodic, and henceforth has invariant probability measure $(\mu_i)_{i\in\mathcal{I}}$ where $\mu_i = \pi_i q_{i,i}$.

Proposition 4.9. *The following results regarding the martingale $(\mathcal{M}^-(n), n \geq 0)$ hold:*

i) *$\mathcal{M}^-(1) \in L^{\omega_+/\omega_-}$ under any initial distribution;*

ii) *$(\mathcal{M}^-(n), n \geq 0)$ is a uniformly integrable martingale. In particular, its almost sure limit $\mathcal{M}^-(\infty)$ is also an L^1 limit, and hence non degenerate.*

iii) *Assume the ordinate ξ is non-lattice, i.e. does not lie in some $a\mathbb{Z}$, $a > 0$,* and assume that (H) holds. *Then for all $(x,i) \in \mathbb{R}_+ \times \mathcal{I}$,*

$$\mathcal{P}_{x,i}(\mathcal{M}^-(\infty) > t) \underset{t\to\infty}{\sim} Ct^{-\omega_+/\omega_-}, \tag{4.6}$$

for some constant $C > 0$.

Proof: The proof is a generalisation of that of Bertoin et al. (2018a, Lemma 2.3) to the multitype setting, which in particular requires the extension of Jelenković and Olvera-Cravioto (2012) included as an appendix in Section 5.

We first prove part (i) which uses the finiteness criterion for the moments of the exponential functional of Markov additive processes stated as Proposition 2.6. First, we claim that for all $q \geq 0$, for which the matrix $m(q)$ is finite, the process

$$S_t := \sum_{0<s\leq t\wedge\zeta} |\Delta X(s)|^q, \quad t \geq 0, \tag{4.7}$$

has predictable compensator

$$S_t^{(p)} := \int_0^{t\wedge\zeta} \mathrm{d}s X(s)^{q-\alpha} \int_{(-\infty,0)} (1-\mathrm{e}^x)^q \widetilde{\Lambda}_{J(s)}(\mathrm{d}x), \quad t \geq 0, \tag{4.8}$$

where for all $i \in \mathcal{I}$, $\widetilde{\Lambda}_i := \Lambda_i + \sum_{k\in\mathcal{I}} q_{i,k}\Lambda_{U_{i,k}}$. This follows from the same kind of computations as in the determination of the multitype cumulants, see Section 3.3. Note that by the Lamperti-Kiu representation,

$$\int_0^\zeta X(s)^{q-\alpha}\mathrm{d}s = \int_0^\infty \mathrm{e}^{q\xi(s)}\mathrm{d}s,$$

is the exponential functional of ξ. It therefore follows from Lemma 4.6 and Proposition 2.6 (together with a crude bound) that $S^{(p)}$ is bounded in $L^{\omega_+/q}$. Now we take $q=\omega_-$. It remains to prove that $S-S^{(p)}$ is bounded in L^{ω_+/ω_-}: this would entail that S is bounded in L^{ω_+/ω_-}, and hence $\mathcal{M}^-(1)\in L^{\omega_+/\omega_-}$ by another crude bound. Now, the process

$$S_t - S_t^{(p)} = \sum_{0<s\le t\wedge\zeta} |\Delta X(s)|^{\omega_-} - \int_0^{t\wedge\zeta} \mathrm{d}s X(s)^{\omega_- -\alpha} \int_{(-\infty,0)} (1-\mathrm{e}^x)^{\omega_-} \widetilde{\Lambda}_{J(s)}(\mathrm{d}x), \quad t\ge 0,$$

is a purely discontinuous martingale, and so has quadratic variation

$$V_t := \sum_{0<s\le t\wedge\zeta} |\Delta X(s)|^{2\omega_-}, \quad t\ge 0.$$

By the Burkholder-Davis-Gundy inequality, it is enough to prove that, for all $(x,i)\in\mathbb{R}_+\times\mathcal{I}$, $\mathbb{E}_{x,i}[V_\zeta^{\omega_+/(2\omega_-)}]<\infty$. Assume for a moment that $\omega_+/\omega_-<2$. Then for $(x,i)\in\mathbb{R}_+\times\mathcal{I}$,

$$\mathbb{E}_{x,i}[V_\zeta^{\omega_+/(2\omega_-)}] \le \mathbb{E}_{x,i}\Bigg[\sum_{0<s\le\zeta} |\Delta X(s)|^{\omega_+}\Bigg].$$

Setting $C=\min_{i\in\mathcal{I}} v_i^+$, and using the definition of v^+ and ω_+, we end up with the desired inequality:

$$\mathbb{E}_{x,i}[V_\zeta^{\omega_+/(2\omega_-)}] \le C^{-1}\mathbb{E}_{x,i}\Bigg[\sum_{0<s\le\zeta} v^+_{J_\Delta(s)} |\Delta X(s)|^{\omega_+}\Bigg] = C^{-1}.$$

Next, suppose $2\le\omega_+/\omega_-<4$. Then replacing q by $2\omega_-$ in (4.7) and (4.8), the previous arguments show that $S_t^{(p)}$ is bounded in $L^{\omega_+/(2\omega_-)}$, and that $S_t-S_t^{(p)}$ is a purely discontinuous martingale with quadratic variation

$$\sum_{0<s\le t\wedge\zeta} |\Delta X(s)|^{4\omega_-}, \quad t\ge 0.$$

The same inequality as before shows that the latter process is bounded in $L^{\omega_+/(4\omega_-)}$, and we conclude again by the Burkholder-Davis-Gundy inequality that $S_t=\sum_{0<s\le t\wedge\zeta}|\Delta X(s)|^{2\omega_-}$ is bounded in $L^{\omega_+/(2\omega_-)}$. The same arguments hold in full generality when $2^k\le\omega_+/\omega_-<2^{k+1}$, $k\ge 0$, by a simple recursion.

Part (ii) is a consequence of Biggins' martingale convergence theorem for multitype branching random walks due to Kyprianou and Rahimzadeh Sani (2001, Theorem 1). We use the same arguments as in the proof of Proposition 4.7: since $\omega_-<\omega_+$ are two roots of λ, and λ is convex (see the discussion after (3.5)), we infer that $\lambda'(\omega_-)<0$. The $L\log L$ integrability condition is straightforward since $m(\omega_+)$ is finite.

Finally, part (iii) comes from a multitype version of the tail estimates in multiplicative cascades of Jelenković and Olvera-Cravioto (2012). Setting $\widetilde{\mathcal{M}}^{(i)} := \mathcal{M}^-(\infty)/v_i^-$, the branching property entails that $\widetilde{\mathcal{M}}^{(i)}$ satisfies the identity in distribution:

$$\widetilde{\mathcal{M}}^{(i)} \overset{\mathcal{L}}{=} \sum_{k\ge 1} \frac{v^-_{\mathcal{J}_k}}{v_i^-} \Delta_k^{\omega_-} \widetilde{\mathcal{M}}_k,$$

where the Δ_k are the absolute values of the jumps of X, the $\mathcal{J}_k$ are the corresponding types, and the $\widetilde{\mathcal{M}}_k$ form a sequence of independent copies of $\widetilde{\mathcal{M}}^{(j)}$, with initial type $j=\mathcal{J}_k$, which are further independent of the jumps Δ_k. We want to apply Theorem 5.5: this would imply the desired tail asymptotics for $\widetilde{\mathcal{M}}^{(i)}$, and hence for $\mathcal{M}^-(\infty)$. In order to apply Theorem 5.5, the only non-trivial assumption to show is that for $\beta<\omega_+/\omega_-$, $\mathcal{M}^-(\infty)\in L^\beta$. We include this fact as Proposition 5.8 in Appendix 5 to keep the paper at its present pace.

□

The next result produces temporal martingales related to the genealogical martingales defined in Section 3.2 (as explained in Remark 4.2, these processes are always supermartingales). The positive case appears as Bertoin et al. (2018a, Corollary 3.5) (beware that our definition of the self-similarity index α is the opposite of that of Bertoin et al. (2018a)).

Proposition 4.10. *For $\alpha \geq 0$, the process*

$$\mathcal{M}_t^+ := \sum_{i=1}^{\infty} v_{J_i(t)}^+ X_i(t)^{\omega_+}, \quad t \geq 0,$$

is a $\mathbf{P}_{x,i}$–martingale, whereas for $\alpha \leq 0$,

$$\mathcal{M}_t^- := \sum_{i=1}^{\infty} v_{J_i(t)}^- X_i(t)^{\omega_-}, \quad t \geq 0,$$

is a $\mathbf{P}_{x,i}$–martingale.

Proof: The proof is close to Bertoin et al. (2018a, Corollary 3.5) once we translate everything into the Markov additive process framework. We restrict to proving the result for $\alpha \geq 0$ (similar arguments work in the case when $\alpha \leq 0$). By the branching property, the result follows if we prove that $\mathbf{E}_{x,i}[\mathcal{M}_t^+] = v_i^+ x^{\omega_+}$. From the many-to-one lemma (Proposition 4.1), we see that

$$\mathbf{E}_{x,i}[\mathcal{M}_t^+] = v_i^+ x^{\omega_+} \widehat{\mathcal{P}}_{x,i}[\widehat{\mathcal{X}}^+(t) \neq \partial],$$

where $\widehat{\mathcal{X}}^+$ denotes the spine corresponding to the exponent ω_+. Hence $\mathcal{M}^+$ is a martingale if and only if $\widehat{\mathcal{X}}^+$ has infinite lifetime. By convexity of $\widehat{\chi}_+$, we know that $\widehat{\chi}_+'(0) > 0$. This implies that for the underlying Markov additive process $(\widehat{\xi}^+, \widehat{\Theta}^+)$ of $\widehat{\mathcal{X}}^+$, $\widehat{\xi}^+$ drifts to $+\infty$ independently of the initial state (see Asmussen (2003, Corollary XI.2.7) and Ivanovs (2011, Lemma 2.14)). Therefore, the exponential functional

$$I(\alpha\widehat{\xi}^+) := \int_0^\infty \exp(\alpha\widehat{\xi}^+(u))\mathrm{d}u,$$

is infinite since $\alpha \geq 0$, and we conclude by the Lamperti-Kiu construction that $\widehat{\mathcal{X}}^+$ has infinite lifetime. □

Similarly to the positive case (see Bertoin et al. (2018a, Corollary 3.5)), we deduce the long-term behaviour of $\mathbf{E}_{x,i}[\mathcal{M}_t^-]$ as t increases. Such a result will be very useful in the next subsection.

Proposition 4.11. *Assume that the Markov additive processes $(\widehat{\xi}^-, \Theta^-)$ and $(\widehat{\xi}^+, \Theta^+)$ are not concentrated on a lattice. For $\alpha < 0$, there exists some constant $c_i^+, C_i^+ \in [0, \infty)$ such that for t large enough,*

$$v_i^+ x^{\omega_-} c_i^+ t^{\frac{\omega_+ - \omega_-}{\alpha}} \leq \mathbf{E}_{x,i}[\mathcal{M}_t^+] \leq v_i^+ x^{\omega_-} C_i^+ t^{\frac{\omega_+ - \omega_-}{\alpha}}, \tag{4.9}$$

and for $\alpha > 0$, there exists some constant $c_i^-, C_i^- \in [0, \infty)$ such that for t large enough,

$$v_i^- x^{\omega_+} c_i^- t^{-\frac{\omega_+ - \omega_-}{\alpha}} \leq \mathbf{E}_{x,i}[\mathcal{M}_t^-] \leq v_i^- x^{\omega_+} C_i^- t^{-\frac{\omega_+ - \omega_-}{\alpha}}. \tag{4.10}$$

Moreover, for $1 < p < \omega_+/\omega_-$, we have

$$\mathbf{E}_{x,i}\left[\sum_{i=1}^{\infty} \left(v_{J_i(t)}^- X_i(t)^{\omega_-}\right)^p\right] \to 0, \qquad \text{as} \quad t \to \infty. \tag{4.11}$$

Proof: We only prove (4.10) since the behaviour in (4.9) follows from similar arguments. Similarly to the proof of the previous proposition, from the many-to-one lemma (Proposition 4.1), we see that

$$\mathbf{E}_{x,i}[\mathcal{M}_t^-] = v_i^- x^{\omega_-} \widehat{\mathcal{P}}_{x,i}(\widehat{\mathcal{X}}^-(t) \neq \partial),$$

where $\widehat{\mathcal{X}}^-$ denotes the spine corresponding to the exponent ω_-. From the Lamperti-Kiu representation (2.8), we observe

$$\mathbf{E}_{x,i}[\mathcal{M}_t^-] = v_i^- x^{\omega_-} \mathtt{P}_{0,i}\left(I(\alpha\widehat{\xi}^-) > tx^{-\alpha}\right),$$

where

$$I(\alpha\widehat{\xi}^-) := \int_0^\infty e^{\alpha\widehat{\xi}^-(s)}\mathrm{d}s,$$

and $(\widehat{\xi}^-, \widehat{\Theta}^-)$ denotes the underlying Markov additive process with Matrix exponent $\widehat{F}_-$ and leading eigenvalue $\widehat{\chi}_-$. Since $\alpha > 0$ and $\widehat{\chi}'_-(0) < 0$, the exponential functional $I(\alpha\widehat{\xi}^-)$ is a.s. finite. Moreover, since $(\widehat{\xi}^-, \widehat{\Theta}^-)$ satisfies Cramér's condition with $\omega_+ - \omega_-$, we deduce from Proposition 2.8 that the asymptotic behaviour in (4.10) is satisfied.

In order to obtain (4.11), we first observe again from the many-to-one lemma (Proposition 4.1) and the Lamperti-Kiu representation (2.8) that

$$\begin{aligned}
\mathbf{E}_{x,i}\left[\sum_{i=1}^\infty \left(v^-_{J_i(t)} X_i(t)^{\omega_-}\right)^p\right] &= v_i^- x^{\omega_-} \widehat{\mathcal{E}}_{x,i}\left[(v^-_{\widehat{\mathcal{J}}(t)})^{(p-1)} \widehat{\mathcal{X}}(t)^{\omega_-(p-1)} \mathbf{1}_{\{t<\zeta\}}\right] \\
&= v_i^- x^{p\omega_-} \mathtt{E}_{0,i}\left[(v^-_{\widehat{\Theta}(\varphi(tx^{-\alpha}))})^{(p-1)} e^{\omega_-(p-1)\widehat{\xi}^-(\varphi(tx^{-\alpha}))} \mathbf{1}_{\{t<x^\alpha I(\alpha\widehat{\xi}^-)\}}\right] \\
&\le C_{x,v^-,p}\Bigg(\mathtt{E}_{0,i}\left[e^{\omega_-(p-1)\widehat{\xi}^-(\varphi(tx^{-\alpha}))}\mathbf{1}_{\{I(\alpha\widehat{\xi}^-)=\infty\}}\right] \\
&\qquad\qquad + \mathtt{P}_{0,i}\left(t < x^\alpha I(\alpha\widehat{\xi}^-) < \infty\right)\Bigg),
\end{aligned}$$

where $C_{x,v^-,p} = v_i^- x^{p\omega_-} \max_{j\in\mathcal{I}}(v_j^-)^{(p-1)}$. On the one hand, the second term in the right-hand side clearly goes to 0 as t increases. On the other hand, from the change of measure induced by the Wald martingale (see Proposition 2.4) and the fact that φ is a stopping time, it is clear that

$$\mathtt{E}_{0,i}\left[e^{\omega_-(p-1)\widehat{\xi}^-(\varphi(tx^{-\alpha}))}\mathbf{1}_{\{I(\alpha\widehat{\xi}^-)=\infty\}}\right] \le C'\mathbf{E}_{0,i}^{((p-1)\omega_-)}\left[e^{\varphi(tx^{-\alpha})\widehat{\chi}_-(\omega_-(p-1))}\mathbf{1}_{\{I(\alpha\widehat{\xi}^-)=\infty\}}\right],$$

where $C' = \max_{j\in\mathcal{I}} \frac{w_i(\omega_-(p-1))}{w_j(\omega_-(p-1))}$. The right hand side of the above identity clearly goes to 0 since $\widehat{\chi}_-(\omega_-(p-1)) < 0$, as t increases. In particular, this implies that (4.11) holds. □

We conclude this section by the following result on the finiteness of moments of the temporal martingale, which will come in handy in the next section.

Proposition 4.12. *Let $\alpha \le 0$. For all $0 < p < \omega_+/\omega_-$, the martingale $(\mathcal{M}_t^-, t \ge 0)$ is bounded in L^p.*

Proof: The proof follows closely that of Bertoin et al. (2018a, Theorem 3.7). Recall from Section 3.1 the notation $(\overline{\mathcal{F}}_t, t \ge 0)$ for the filtration of the growth-fragmentation process enriched with the generations. Let $x > 0$ and $i \in \mathcal{I}$. By the branching property in Proposition 3.2, for $t \ge 0$,

$$\mathcal{E}_{x,i}\left[\mathcal{M}^-(n)\,\middle|\,\overline{\mathcal{F}}_t\right] \ge \sum_{|u|\le n} \mathbf{1}_{\{b_u\le t\}} v^-_{\mathcal{J}_u(t-b_u)} \mathcal{X}_u(t-b_u)^{\omega_-}.$$

Take $n \to \infty$. The left-hand side converges to $\mathcal{E}_{x,i}[\mathcal{M}^-(\infty)\,|\,\overline{\mathcal{F}}_t]$, whereas the right-hand side converges to $\mathcal{M}_t^-$. Therefore

$$\mathcal{E}_{x,i}\left[\mathcal{M}^-(\infty)\,\middle|\,\overline{\mathcal{F}}_t\right] \ge \mathcal{M}_t^-.$$

Then, by Jensen's inequality, we get

$$\mathbf{E}_{x,i}[|\mathcal{M}_t^-|^p] \le \mathcal{E}_{x,i}[\mathcal{M}^-(\infty)^p],$$

for all $p \in (1, \omega_+/\omega_-)$. This concludes the proof. □

4.4. *Convergence of the empirical measure.* Here, we are interested in the convergence of the empirical measure $\rho^{(\alpha)}$ given by

$$\langle \rho_t^{(\alpha)}, f \rangle := \sum_{i=1}^{\infty} v^-_{J_i(t)} X_i(t)^{\omega_-} f(t^{-1/\alpha} X_i(t), J_i(t)),$$

for $f : (0,\infty) \times \mathcal{I} \to \mathbb{R}$ bounded function and such that $x \mapsto f(x,i)$ is continuous, for all $i \in \mathcal{I}$, and $\alpha < 0$. Throughout this section, we write $\widehat{\xi}$ instead of $\widehat{\xi}^-$, and $(\widehat{\mathcal{X}}, \widehat{\mathcal{J}})$ instead of $(\widehat{\mathcal{X}}^-, \widehat{\mathcal{J}}^-)$. We shall also suppose that the process $\widehat{\xi}$ associated with the tagged cell $\widehat{\mathcal{X}}$ is not concentrated on a lattice. In order to state our result, we introduce the probability measure ρ on $(0,\infty) \times \mathcal{I}$ by

$$\int_{(0,\infty)\times\mathcal{I}} f(y,i)\rho(\mathrm{d}y, \mathrm{d}i) = -\frac{1}{\alpha \left| \widehat{\mathtt{E}}^{\natural}_{0,\pi}[\widehat{\xi}_1] \right|} \sum_{i \in \mathcal{I}} \pi_i \widehat{\mathtt{E}}^{\natural}_{0,i} \left[\frac{1}{I(\alpha\widehat{\xi}\,)} f\left(\frac{1}{I(\alpha\widehat{\xi}\,)^{1/\alpha}}, i \right) \right],$$

where we recall that $\widehat{\mathtt{P}}^{\natural}$ denotes the law of the dual of $\widehat{\xi}$ and

$$I(\alpha\widehat{\xi}\,) := \int_0^\infty e^{\alpha \widehat{\xi}(s)} \mathrm{d}s.$$

We also observe that $\left| \widehat{\mathtt{E}}^{\natural}_{0,\pi}[\widehat{\xi}_1] \right| = -\widehat{\mathtt{E}}_{0,\pi}[\widehat{\xi}_1] = -\widehat{\chi}'_-(0)$. We have the following convergence result for the empirical measure $\rho_t^{(\alpha)}$ (see Dadoun (2017) for the classical growth-fragmentation framework).

Theorem 4.13. *For every $1 < p < \omega_+/\omega_-$ and for every $f : (0,\infty) \times \mathcal{I} \to \mathbb{R}$ bounded function and such that $x \mapsto f(x,i)$ is continuous, for all $i \in \mathcal{I}$,*

$$\lim_{t\to\infty} \langle \rho_t^{(\alpha)}, f \rangle = \mathcal{M}^-(\infty) \int_{(0,\infty)\times\mathcal{I}} f(y,i)\rho(\mathrm{d}y, \mathrm{d}i), \qquad \text{in} \quad L^p.$$

In particular, the random measure $\rho_t^{(\alpha)}$ converges in probability towards $\mathcal{M}^-(\infty)\rho$, as $t \to \infty$, in the space of finite measures on $(0,\infty) \times \mathcal{I}$ endowed with the topology of weak convergence.

In order to deduce the above result, the following lemma is required.

Lemma 4.14. *Under $\widehat{\mathcal{P}}_{1,i}$, the pair $(t^{-1/\alpha}\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t))$ converges in distribution to ρ. Moreover*

$$\int_{(0,\infty)\times\mathcal{I}} y^{-\alpha q} \rho(\mathrm{d}y, \mathrm{d}i) < \infty,$$

for $0 < q < 1 - (\omega_+ - \omega_-)/\alpha$.

Proof: We observe that $(1/\widehat{\mathcal{X}}, \widehat{\mathcal{J}})$ is a self-similar Markov process with types and index $-\alpha$ which is associated with the Markov additive process $(-\widehat{\xi}, \widehat{\Theta})$. Hence, from Theorem 2.9, all we need to verify is that $-\widehat{\xi}$ has finite and strictly positive mean which, in particular, implies that the associated ascending ladder MAP $\hat{H}^+$ has also finite mean, see for instance Theorem 35 in Dereich et al. (2017). The finiteness of the mean of $-\widehat{\xi}$ follows from the identity

$$-\widehat{\mathtt{E}}_{0,\pi}[\widehat{\xi}_1] = -\widehat{\chi}'_-(0) > 0.$$

In order to deduce the existence of moments, we first observe that since $\omega_+ - \omega_-$ is a Cramér number for $\widehat{F}(z)$, $-(\omega_+ - \omega_-)/\alpha$ is a Cramér number for $\widehat{F}(-\alpha z)$. By identity (2.4) and the fact that all positive eigenvectors of $\widehat{F}^{\natural}(\alpha z)$ are associated to the same eigenvalue, we infer that $-(\omega_+ - \omega_-)/\alpha$ is also a Cramér number for $\widehat{F}^{\natural}(\alpha z)$. Thus a straightforward application of Theorem 2.9 and Proposition 2.6 will imply the second part of this Lemma. □

Proof of Theorem 4.13: From the branching property at time t and self-similarity of $\mathbf{X}$, we have under the event $\mathbf{X}(t) = \{\{(x_1, j_1), (x_2, j_2), \cdots\}\}$ that the following identity holds

$$\begin{aligned}\langle \rho^{(\alpha)}_{t+t^2}, f\rangle &= \sum_{i=1}^{\infty} v^-_{J_i(t+t^2)} X_i(t+t^2)^{\omega_-} f\Big((t+t^2)^{-1/\alpha} X_i(t+t^2), J_i(t+t^2)\Big) \\ &= \sum_{i=1}^{\infty} v^-_{j_i} x_i^{\omega_-} \sum_{k=1}^{\infty} \frac{v^-_{J_{i,k}(x_i^{-\alpha}t^2)}}{v^-_{j_i}} X_{i,k}(x_i^{-\alpha}t^2)^{\omega_-} f\Big((t+t^2)^{-1/\alpha} x_i X_{i,k}(x_i^{-\alpha}t^2), J_{i,k}(x_i^{-\alpha}t^2)\Big).\end{aligned}$$

Observe that for each $i \geq 1$, the families $\{(X_{i,k}, J_{i,k}),\ k \geq 1\}$ are independent and have the same law as $\mathbf{X}$ under $\mathbf{P}_{1,j_i}$. For simplicity of exposition, we introduce the random variables

$$\mathcal{N}_i(t) := \sum_{k\geq 1} \frac{v^-_{J_{i,k}(x_i^{-\alpha}t^2)}}{v^-_{j_i}} X_{i,k}(x_i^{-\alpha}t^2)^{\omega_-} f\Big((t+t^2)^{-1/\alpha} x_i X_{i,k}(x_i^{-\alpha}t^2), J_{i,k}(x_i^{-\alpha}t^2)\Big),$$

and observe that $(\mathcal{N}_i(t))_{i\geq 1}$ are independent conditionally on $\{\{(x_1, j_1), (x_2, j_2), \cdots\}\}$. In other words, we may rewrite

$$\langle \rho^{(\alpha)}_{t+t^2}, f\rangle = \sum_{i=1}^{\infty} \lambda_i(t)\mathcal{N}_i(t),$$

where $\lambda_i(t) = v^-_{J_i(t)} X_i(t)^{\omega_-} = v^-_{j_i} x_i^{\omega_-}$. Now, let us introduce

$$\overline{\mathcal{N}}_i := C_1 ||f||_\infty \sup_{t\geq 0} \sum_{k\geq 1} v^-_{J_{i,k}(t)} X_{i,k}(t)^{\omega_-}, \qquad \text{with} \qquad C_1 := \max_{\ell\in\mathcal{I}} \frac{1}{v^-_\ell},$$

and observe $|\mathcal{N}_i(t)| \leq \overline{\mathcal{N}}_i$, for all $t \geq 0$. Moreover, conditionally on $\{\{(x_1, j_1), (x_2, j_2), \cdots\}\}$, by setting $\mathfrak{s}^{(\ell)} := \{k : j_k = \ell\}$ for $\ell \in \mathcal{I}$, we deduce that the random variables $(\overline{\mathcal{N}}_i)_{i\in\mathfrak{s}^{(\ell)}}$ are i.i.d with common distribution

$$\mathcal{N} := C_1 ||f||_\infty \sup_{t\geq 0} \sum_{k\geq 1}^{\infty} v^-_{J_k(t)} X_k(t)^{\omega_-}, \qquad \text{under} \quad \mathbf{P}_{1,\ell}.$$

Additionally, from Proposition 4.12 (recall that $1 < p < \omega_+/\omega_-$), it is clear that $(\overline{\mathcal{N}}_i)_{i\in\mathfrak{s}^{(\ell)}} \in L^p(\mathbf{P}_{1,\ell})$. Similarly, we obtain

$$\sup_{t\geq 0} \mathbf{E}_{1,\ell}\left[\left(\sum_{i=1}^{\infty} v^-_{J_i(t)} X_i(t)^{\omega_-}\right)^p\right] < \infty,$$

and using Proposition 4.11, we deduce

$$\lim_{t\to\infty} \mathbf{E}_{1,\ell}\left[\sum_{i=1}^{\infty} \left(v^-_{J_i(t)} X_i(t)^{\omega_-}\right)^p\right] = 0. \tag{4.12}$$

With these properties at hand, we will deduce the following variation of the law of large numbers for any $\mathbf{k} \in \mathcal{I}$,

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} \lambda_i(t)\Big(\mathcal{N}_i(t) - \mathbf{E}_{1,\mathbf{k}}\Big[\mathcal{N}_i(t)|\mathbf{X}(t)\Big]\Big) = 0, \qquad \text{in} \quad L^p(\mathbf{P}_{1,\mathbf{k}}). \tag{4.13}$$

In order to do so, we follow a similar strategy as in Lemma 1.5 in Bertoin (2006). We take $a > 0$, an arbitrarily large real number and introduce the truncated random variables

$$\mathcal{N}_i(t, a) := \mathbf{1}_{\{|\mathcal{N}_i(t)|<a\}} \mathcal{N}_i(t) \qquad \text{for} \quad i \geq 1.$$

Thus, we deduce the following upper-bound

$$\begin{aligned}\left|\sum_{i=1}^{\infty}\lambda_i(t)\Big(\mathcal{N}_i(t)-\mathbf{E}_{1,\mathbf{k}}\Big[\mathcal{N}_i(t)|\mathbf{X}(t)\Big]\Big)\right| &\le \left|\sum_{i=1}^{\infty}\lambda_i(t)\Big(\mathcal{N}_i(t)-\mathcal{N}_i(t,a)\Big)\right| \\ &+\left|\sum_{i=1}^{\infty}\lambda_i(t)\Big(\mathcal{N}_i(t,a)-\mathbf{E}_{1,\mathbf{k}}\Big[\mathcal{N}_i(t,a)\Big|\mathbf{X}(t)\Big]\Big)\right| \\ &+\left|\sum_{i=1}^{\infty}\lambda_i(t)\mathbf{E}_{1,\mathbf{k}}\Big[\Big(\mathcal{N}_i(t,a)-\mathcal{N}_i(t)\Big)\Big|\mathbf{X}(t)\Big]\right|.\end{aligned} \tag{4.14}$$

We consider the first series in the right-hand side of the above inequality and since

$$\Big|\mathcal{N}_i(t)-\mathcal{N}_i(t,a)\Big| \le \mathbf{1}_{\left\{|\overline{\mathcal{N}}_i|>a\right\}}\overline{\mathcal{N}}_i,$$

we get from Jensen's inequality and the Markov property, the following inequality

$$\begin{aligned}&\mathbf{E}_{1,\mathbf{k}}\left[\left|\sum_{i=1}^{\infty}\lambda_i(t)\Big(\mathcal{N}_i(t)-\mathcal{N}_i(t,a)\Big)\right|^p\right] \\ &\quad\le \mathbf{E}_{1,\mathbf{k}}\left[\left(\sum_{i=1}^{\infty}\lambda_i(t)\right)^p\sum_{i=1}^{\infty}\frac{\lambda_i(t)}{\sum_{k=1}^{\infty}\lambda_k(t)}\mathbf{1}_{\left\{|\overline{\mathcal{N}}_i|>a\right\}}\overline{\mathcal{N}}_i^p\right] \\ &\quad= \mathbf{E}_{1,\mathbf{k}}\left[\left(\sum_{i=1}^{\infty}\lambda_i(t)\right)^p\sum_{i=1}^{\infty}\frac{\lambda_i(t)}{\sum_{k=1}^{\infty}\lambda_k(t)}\mathbf{E}_{1,J_i(t)}\left[\mathbf{1}_{\left\{|\overline{\mathcal{N}}|>a\right\}}\overline{\mathcal{N}}^p\right]\right] \\ &\quad\le \mathbf{E}_{1,\mathbf{k}}\left[\left(\sum_{i=1}^{\infty}v^-_{J_i(t)}X_i(t)^{\omega_-}\right)^p\right]\max_{j\in\mathcal{I}}\mathbf{E}_{1,j}\left[\mathbf{1}_{\left\{|\overline{\mathcal{N}}|>a\right\}}\overline{\mathcal{N}}^p\right],\end{aligned}$$

and the latter quantity converges to 0 as $a\to\infty$, uniformly for $t\ge 0$. The same argument also shows that

$$\lim_{a\to\infty}\sup_{t\ge 0}\mathbf{E}_{1,\mathbf{k}}\left[\left|\sum_{i=1}^{\infty}\lambda_i(t)\mathbf{E}_{1,\mathbf{k}}\Big[\Big(\mathcal{N}_i(t,a)-\mathcal{N}_i(t)\Big)\Big|\mathbf{X}(t)\Big]\right|^p\right]=0.$$

Next, for the second term in the right hand side in (4.14), we observe that conditionally on $\mathbf{X}(t)=\{\{(x_1,j_1),(x_2,j_2),\cdots\}\}$, the random variables

$$\mathcal{N}_i(t,a)-\mathbf{E}_{1,\mathbf{k}}\Big[\mathcal{N}_i(t,a)\Big|\mathbf{X}(t)\Big],$$

are centered, independent and bounded in absolute value by a. Hence, conditionally on $\mathbf{X}(t)$,

$$\sum_{i=1}^{n}\lambda_i(t)\left(\mathcal{N}_i(t,a)-\mathbf{E}_{1,\mathbf{k}}\Big[\mathcal{N}_i(t,a)\Big|\mathbf{X}(t)\Big]\right),\qquad n\ge 1,$$

is a martingale bounded in $L^p(\mathbf{P}_{1,\mathbf{k}})$ and there exists a universal constant C_p such that

$$\mathbf{E}_{1,\mathbf{k}}\left[\left|\sum_{i=1}^{\infty}\lambda_i(t)\left(\mathcal{N}_i(t,a)-\mathbf{E}_{1,\mathbf{k}}\Big[\mathcal{N}_i(t,a)\Big|\mathbf{X}(t)\Big]\right)\right|^p\Big|\mathbf{X}(t)\right]\le C_pa^p\sum_{i=1}^{\infty}\lambda_i(t)^p.$$

Thus from (4.12), the latter quantity converges to 0 as $t\to\infty$ in $L^p(\mathbf{P}_{1,\mathbf{k}})$. Putting all pieces together allows us to deduce (4.13).

In other words, the proof will be completed if we show

$$\lim_{t\to\infty}\sum_{i=1}^{\infty}\lambda_i(t)\mathbf{E}_{1,\mathbf{k}}\Big[\mathcal{N}_i(t)\Big|\mathbf{X}(t)\Big]=\mathcal{M}^-(\infty)\int_{(0,\infty)\times\mathcal{I}}f(y,i)\rho(\mathrm{d}y,\mathrm{d}i),\qquad\text{in}\quad L^p. \tag{4.15}$$

From Proposition 4.1, we get

$$\mathbf{E}_{1,\mathbf{k}}\Big[\mathcal{N}_i(t)\Big|\mathbf{X}(t)\Big] = \widehat{\mathcal{E}}_{1,j_i}\Big[f((1+t^{-1})^{-1/\alpha}t^{-2/\alpha}x_i\widehat{\mathcal{X}}(x_i^{-\alpha}t^2), \widehat{\mathcal{J}}(x_i^{-\alpha}t^2))\Big].$$

From Lemma 4.14, the pair $(s^{-1/\alpha}\widehat{\mathcal{X}}(s), \widehat{\mathcal{J}}(s))$ converges in distribution to ρ, under $\widehat{\mathcal{P}}_{1,j_i}$. On the one hand, it follows that

$$\widehat{\mathcal{E}}_{1,j_i}\Big[f((1+t^{-1})^{-1/\alpha}t^{-2/\alpha}x_i\widehat{\mathcal{X}}(x_i^{-\alpha}t^2), \widehat{\mathcal{J}}(x_i^{-\alpha}t^2))\Big] \xrightarrow[t\to\infty]{} \int_{(0,\infty)\times\mathcal{I}} f(y,i)\rho(\mathrm{d}y,\mathrm{d}i),$$

uniformly in i such that $x_i^{-\alpha}t^2 > \sqrt{t}$, i.e. $x_i > t^{3/(2\alpha)}$. On the other hand, applying Proposition 4.1, we see that

$$\sum_{i=1}^{\infty} v^-_{J_i(t)} X_i(t)^{\omega_-}\mathbf{1}_{\left\{X_i(t)\le t^{3/(2\alpha)}\right\}}, \tag{4.16}$$

has, under $\mathbf{P}_{1,\mathbf{k}}$, mean equals

$$v_{\mathbf{k}}\widehat{\mathcal{P}}_{1,\mathbf{k}}\Big(t^{-1/\alpha}\widehat{\mathcal{X}}(t) < t^{1/(2\alpha)}\Big)$$

which tends to 0 as $t\to\infty$. Since (4.16) is bounded in $L^q(\mathbf{P}_{1,\mathbf{k}})$ for every $p < q < \omega_+/\omega_-$ (Proposition 4.12), by Hölder's inequality it also converges to 0. Putting all pieces together, we deduce that (4.15) holds and thus the first part of the statement.

The second part of the statement is derived using the same arguments as in Dadoun (2017), that is we use a diagonal extraction procedure since the space $\mathcal{C}_c((0,\infty)\times\mathcal{I})$ of continuous function on $(0,\infty)\times\mathcal{I}$ with compact support is separable. In other words there exists an extraction $\sigma:\mathbb{N}\to\mathbb{N}$, such that, almost surely for all $f\in\mathcal{C}_c((0,\infty)\times\mathcal{I})$

$$\langle\rho^{(\alpha)}_{t_{\sigma(n)}}, f\rangle \xrightarrow[n\to\infty]{} \mathcal{M}^-(\infty)\int_{(0,\infty)\times\mathcal{I}} f(y,i)\rho(\mathrm{d}y,\mathrm{d}i),$$

in other words $\rho^{(\alpha)}_{t_{\sigma(n)}}$ converges vaguely to $\mathcal{M}^-(\infty)\rho$, a.s. Since the total mass is conserved, i.e.

$$\langle\rho^{(\alpha)}_{t_{\sigma(n)}}, 1\rangle = \sum_{i=1}^{\infty} v^-_{J_i(t)}X_i(t)^{\omega_-} \xrightarrow[n\to\infty]{} \mathcal{M}^-(\infty) = \langle\mathcal{M}^-(\infty)\rho, 1\rangle, \qquad \text{a.s.},$$

thus the convergence of $\rho^{(\alpha)}_{t_{\sigma(n)}}$ towards $\mathcal{M}^-(\infty)\rho$ is weak which allows us to conclude. □

4.5. *Proof of Theorem 4.3.* It is plain that the spine $(\widehat{\mathcal{X}}, \widehat{\mathcal{J}})$ inherits the Markov property and self-similarity of (X, J), and therefore it can be described in terms of a MAP *via* the representation (2.8). Without loss of generality, we may restrict to the homogeneous case, i.e. when $\alpha = 0$. The result is then easily extended thanks to the Lamperti time-change. We aim at finding the characteristics $(\widehat{\psi}_i, \widehat{q}_{i,j}, \widehat{G}_{i,j})$ of the matrix exponent of this MAP.

Description of the spine. Let $\widehat{H}$ be the first time when the type of the spine changes, and $\widehat{\mathcal{J}}(\widehat{H})$ denote the corresponding type. Fix $q\ge 0$, and $i,j\in\mathcal{I}$.

▷ Determining the Laplace exponent $\widehat{\psi}_i$. This part is similar to the proof of Bertoin et al. (2018a, Theorem 4.2). We denote by $\widehat{\xi}$ the first component of the MAP corresponding to $\widehat{\mathcal{X}}$, that is

$$\widehat{\xi}(s) = \log\widehat{\mathcal{X}}(s), \qquad s\ge 0,$$

and $\widehat{\xi}_k$ the underlying Lévy processes depending on type $k\in\mathcal{I}$. We want to show that the Lévy process $\widehat{\xi}_i$ has Laplace exponent $\widehat{\psi}_i(q) = \kappa_i(q+\omega) - \kappa_i(\omega)$. Notice that a process η with Laplace exponent $\widehat{\psi}_i$ can be written as the superposition $\eta = \eta^{(1)} + \eta^{(2)}$ of independent Lévy processes $\eta^{(1)}$ and $\eta^{(2)}$, with respective Laplace exponents $\psi^{(1)}(q) := \psi_i(q+\omega) - \psi_i(\omega)$ and

$$\psi^{(2)}(q) := \int_{-\infty}^{0} \left((1-\mathrm{e}^x)^{q+\omega} - (1-\mathrm{e}^x)^{\omega}\right) \Pi_{i,i}(\mathrm{d}x).$$

In particular, $\eta^{(2)}$ is a compound Poisson process with Lévy measure $\mathrm{e}^{\omega x}\widetilde{\Pi}_{i,i}(\mathrm{d}x)$, where $\widetilde{\Pi}_{i,i}(\mathrm{d}x)$ is the image measure of $\Pi_{i,i}(\mathrm{d}x)\mathbf{1}_{\{x<0\}}$ through $x \mapsto \log(1-\mathrm{e}^x)$. Let T be the first time when $\eta^{(2)}$ has a jump. The branching property of the cell system and the Markov property of η ensures that the result will hold if we manage to prove that the distribution of $(\widehat{\xi}_i(t), t \le b_{\mathcal{L}(1)})$ is the same as that of $(\eta(t), t \le T)$. Let f, g be two nonnegative measurable functions defined respectively on the space of *càdlàg* trajectories and on $(-\infty, 0)$. Let

$$\Delta\widehat{\xi}(s) = \log \frac{\widehat{\mathcal{X}}(s)}{\widehat{\mathcal{X}}(s-)}, \qquad s \ge 0,$$

then

$$\begin{aligned}
&\widehat{\mathcal{E}}_i\Big[f(\widehat{\xi}(s), s < b_{\mathcal{L}(1)})g(\Delta\widehat{\xi}(b_{\mathcal{L}(1)}))\mathbf{1}_{\{b_{\mathcal{L}(1)}<\widehat{H}\}}\Big] \\
&= \mathtt{E}_i\left[\sum_{t>0} \frac{v_{\iota_\Delta(t)}}{v_i}\mathrm{e}^{\omega\xi(t^-)}\left(1-\mathrm{e}^{\Delta\xi(t)}\right)^{\omega}\mathbf{1}_{\{\iota_\Delta(t)=i\}}\mathbf{1}_{\{t\le\rho_i\}}f(\xi(s), s<t)g(\log(1-\mathrm{e}^{\Delta\xi(t)}))\right] \\
&= \mathtt{E}_i\left[\sum_{0<t<\rho_i} \mathrm{e}^{\omega\xi_i(t^-)}\left(1-\mathrm{e}^{\Delta\xi_i(t)}\right)^{\omega}\mathbf{1}_{\{\iota_\Delta(t)=i\}}f(\xi_i(s), s<t)g(\log(1-\mathrm{e}^{\Delta\xi_i(t)}))\right] \\
&\quad + \mathtt{E}_i\Big[\mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}\right)^{\omega}\mathbf{1}_{\{\iota_\Delta(\rho_i)=i\}}f(\xi_i(s), s<\rho_i)g(\log(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}))\Big].
\end{aligned}$$

The compensation formula for ξ_i entails that the first term is

$$\begin{aligned}
&\mathtt{E}_i\left[\sum_{0<t<\rho_i} \mathrm{e}^{\omega\xi_i(t^-)}\left(1-\mathrm{e}^{\Delta\xi_i(t)}\right)^{\omega}\mathbf{1}_{\{\iota_\Delta(t)=i\}}f(\xi_i(s), s<t)g(\log(1-\mathrm{e}^{\Delta\xi_i(t)}))\right] \\
&\qquad = \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{q_{i,i}t}\mathtt{E}_i\Big[f(\xi_i(s), s<t)\mathrm{e}^{\omega\xi_i(t)}\Big] \int_{-\infty}^{0} g(\log(1-\mathrm{e}^x))(1-\mathrm{e}^x)^{\omega}\Lambda_i^{(i)}(\mathrm{d}x). \qquad (4.17)
\end{aligned}$$

The second term can be computed as follows:

$$\begin{aligned}
&\mathtt{E}_i\Big[\mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}\right)^{\omega}\mathbf{1}_{\{\iota_\Delta(\rho_i)=i\}}f(\xi_i(s), s<\rho_i)g(\log(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}))\Big] \\
&= \int_0^{\infty} \mathrm{d}t\,(-q_{i,i})\mathrm{e}^{q_{i,i}t}\mathtt{E}_i\Big[f(\xi_i(s), s<t)\mathrm{e}^{\omega\xi_i(t)}\Big] \sum_{k\in\mathcal{I}\setminus\{i\}} \frac{q_{i,k}}{(-q_{i,i})} \int_{-\infty}^{0} g(\log(1-\mathrm{e}^x))(1-\mathrm{e}^x)^{\omega}\Lambda_{U_{i,k}}^{(i)}(\mathrm{d}x) \\
&= \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{q_{i,i}t}\mathtt{E}_i\Big[f(\xi_i(s), s<t)\mathrm{e}^{\omega\xi_i(t)}\Big] \sum_{k\in\mathcal{I}\setminus\{i\}} q_{i,k} \int_{-\infty}^{0} g(\log(1-\mathrm{e}^x))(1-\mathrm{e}^x)^{\omega}\Lambda_{U_{i,k}}^{(i)}(\mathrm{d}x). \qquad (4.18)
\end{aligned}$$

Combining (4.17) and (4.18), we finally obtain

$$\begin{aligned}
&\widehat{\mathcal{E}}_i\Big[f(\widehat{\xi}(s), s < b_{\mathcal{L}(1)})g(\Delta\widehat{\xi}(b_{\mathcal{L}(1)}))\mathbf{1}_{\{b_{\mathcal{L}(1)}<\widehat{H}\}}\Big] \\
&\qquad\qquad = \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{q_{i,i}t}\mathtt{E}_i\Big[f(\xi_i(s), s<t)\mathrm{e}^{\omega\xi_i(t)}\Big] \int_{-\infty}^{0} g(y)\mathrm{e}^{\omega y}\widetilde{\Pi}_{i,i}(\mathrm{d}y).
\end{aligned}$$

This proves that $(\widehat{\xi}_i(s), s < b_{\mathcal{L}(1)})$ and $\Delta\widehat{\xi}(b_{\mathcal{L}(1)})$ are independent. The latter is distributed as $-\mathbf{1}_{\{y<0\}}\frac{1}{q_{i,i}+\psi_i(\omega)}\mathrm{e}^{\omega y}\widetilde{\Pi}_{i,i}(\mathrm{d}y)$, which is the law of $\Delta\eta^{(2)}(T)$. The former is ξ_i biased by the exponential martingale, and killed at an independent exponential time with parameter $-q_{i,i}-\psi_i(\omega)$, hence has Laplace exponent $\psi_i(q+\omega)+q_{i,i}$. We retrieve the Laplace exponent of $\eta^{(1)}$ killed at T, and conclude that $(\widehat{\xi}_i(t), t \le b_{\mathcal{L}(1)})$ evolves as $(\eta(t), t \le T)$.

▷ Determining the Laplace transform $\widehat{G}_{i,j}$ of the special jumps. We want to compute

$$\widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\right].$$

We first split over the possible current generations for this special jump to occur:

$$\widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\right] = \sum_{k=0}^{\infty}\underbrace{\widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\cdot\mathbf{1}_{\{b_{\mathcal{L}(k)}<\widehat{H}\le b_{\mathcal{L}(k+1)}\}}\right]}_{:=a_k}. \tag{4.19}$$

For $k \ge 1$, applying successively the Markov property at time $b_{\mathcal{L}(1)}$, the self-similarity of $\widehat{\mathcal{X}}$ and the definition of $\widehat{\mathcal{P}}_i$, we get

$$\begin{aligned} a_k &= \widehat{\mathcal{E}}_i\left[\mathbf{1}_{\widehat{H}>b_{\mathcal{L}(1)}}\cdot\widehat{\mathcal{E}}_{\widehat{\mathcal{X}}(b_{\mathcal{L}(1)}),i}\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\cdot\mathbf{1}_{\{b_{\mathcal{L}(k-1)}<\widehat{H}\le b_{\mathcal{L}(k)}\}}\right]\right] \\ &= \widehat{\mathcal{P}}_i(\widehat{H}>b_{\mathcal{L}(1)})\cdot\widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\cdot\mathbf{1}_{\{b_{\mathcal{L}(k-1)}<\widehat{H}\le b_{\mathcal{L}(k)}\}}\right] \\ &= \mathbb{E}_i\left[\sum_{0<s<\zeta}\frac{v_{J_\Delta(s)}}{v_i}|\Delta X(s)|^{\omega}\mathbf{1}_{\{H\ge s\}}\mathbf{1}_{\{J_\Delta(s)=i\}}\right]\cdot a_{k-1}, \end{aligned}$$

where recall that H denotes the first jump time of J. Therefore $a_k = \mu_{i,i}(\omega)\cdot a_{k-1} = \mu_{i,i}(\omega)^k\cdot a_0$, with

$$\mu_{i,i}(\omega) := \mathbb{E}_i\left[\sum_{0<s<\zeta}|\Delta X(s)|^{\omega}\mathbf{1}_{\{H\ge s\}}\mathbf{1}_{\{J_\Delta(s)=i\}}\right].$$

Then, provided $\mu_{i,i}(\omega) < 1$, identity (4.19) triggers

$$\widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\right] = \frac{a_0}{1-\mu_{i,i}(\omega)}. \tag{4.20}$$

It remains to compute a_0 and $\mu_{i,i}(\omega)$. We begin with the latter:

$$\begin{aligned} \mu_{i,i}(\omega) &= \mathtt{E}_i\left[\sum_{0<s\le\rho_i}\mathrm{e}^{\omega\xi(s^-)}\left(1-\mathrm{e}^{\Delta\xi(s)}\right)^{\omega}\mathbf{1}_{\{\iota_\Delta(s)=i\}}\right] \\ &= \mathtt{E}_i\left[\sum_{0<s<\rho_i}\mathrm{e}^{\omega\xi_i(s^-)}\left(1-\mathrm{e}^{\Delta\xi_i(s)}\right)^{\omega}\mathbf{1}_{\{\iota_\Delta(s)=i\}}\right] + \mathtt{E}_i\left[\mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}\right)^{\omega}\mathbf{1}_{\{\iota_\Delta(\rho_i)=i\}}\right]. \end{aligned}$$

A computation similar to (3.6) gives

$$\mu_{i,i}(\omega) = -\frac{1}{q_{i,i}+\psi_i(\omega)}\int_{-\infty}^{0}\Pi_{i,i}(\mathrm{d}x)(1-\mathrm{e}^{x})^{\omega},$$

provided $\psi_i(\omega)+q_{i,i}<0$. On the other hand, in a_0 the type of the spine can either change because J jumps to j (*i.e.* $\widehat{H}<b_{\mathcal{L}(1)}$), or because one picks a jump of type j at time $b_{\mathcal{L}(1)}$ (*i.e.* $\widehat{H}=b_{\mathcal{L}(1)}$). This writes

$$a_0 = A + B,$$

with

$$A = \widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\cdot\mathbf{1}_{\{\widehat{H}<b_{\mathcal{L}(1)}\}}\right] \quad\text{and}\quad B = \widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\cdot\mathbf{1}_{\{\widehat{H}=b_{\mathcal{L}(1)}\}}\right].$$

Performing the change of measure, we first get

$$A = \mathbb{E}_i\left[\sum_{H<t<\zeta}\frac{v_{J_\Delta(t)}}{v_i}|\Delta X(t)|^\omega\left|\frac{X(H)}{X(H^-)}\right|^q\mathbf{1}_{\{J(H)=j\}}\right],$$

with $J(H)$ being the type to which J first jumps. We now apply the Markov property at time H and self-similarity of X:

$$A = \mathbb{E}_i\left[\left|\frac{X(H)}{X(H^-)}\right|^q\mathbf{1}_{\{J(H)=j\}}|X(H)|^\omega\mathbb{E}_j\left[\sum_{0<t<\zeta}\frac{v_{J_\Delta(t)}}{v_i}|\Delta X(t)|^\omega\right]\right].$$

By admissibility of $((v_i)_{i\in\mathcal{I}},\omega)$, $\mathbb{E}_j\left[\sum_{0<t<\zeta}v_{J_\Delta(t)}|\Delta X(t)|^\omega\right]=v_j$. Hence,

$$\begin{aligned}A &= \frac{v_j}{v_i}\mathbb{E}_i\left[\left|\frac{X(H)}{X(H^-)}\right|^{q+\omega}|X(H^-)|^\omega\mathbf{1}_{\{J(H)=j\}}\right]\\ &= \frac{v_j}{v_i}\mathtt{E}_i\left[\mathrm{e}^{(q+\omega)U_{i,j}}\mathrm{e}^{\omega\xi_i(\rho_i^-)}\mathbf{1}_{\{\Theta(\rho_i)=j\}}\right],\end{aligned}$$

and by independence we obtain

$$\begin{aligned}A &= \frac{v_j}{v_i}\frac{q_{i,j}}{(-q_{i,i})}G_{i,j}(q+\omega)\int_0^\infty \mathrm{d}s\,(-q_{i,i})\mathrm{e}^{q_{i,i}s}\,\mathtt{E}_i[\mathrm{e}^{\omega\xi_i(s)}]\\ &= -\frac{v_j}{v_i}q_{i,j}\frac{G_{i,j}(q+\omega)}{\psi_i(\omega)+q_{i,i}}.\end{aligned}$$

Besides,

$$\begin{aligned}B &= \mathbb{E}_i\left[\sum_{0<t<\zeta}\mathbf{1}_{\{t\le H\}}\mathbf{1}_{\{J_\Delta(t)=j\}}\frac{v_{J_\Delta(t)}}{v_i}|\Delta X(t)|^\omega\left|\frac{\Delta X(t)}{X(t^-)}\right|^q\right]\\ &= \frac{v_j}{v_i}\mathtt{E}_i\left[\sum_{0<t<\rho_i}\mathrm{e}^{\omega\xi_i(t^-)}\left(1-\mathrm{e}^{\Delta\xi_i(t)}\right)^{q+\omega}\mathbf{1}_{\{\iota_\Delta(t)=j\}}\right]+\frac{v_j}{v_i}\mathtt{E}_i\left[\mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}\right)^\omega\mathbf{1}_{\{\iota_\Delta(\rho_i)=j\}}\right].\end{aligned}$$

By the compensation formula as in (3.6), we finally get

$$B = -\frac{v_j}{v_i}\frac{1}{q_{i,i}+\psi_i(\omega)}\int_{-\infty}^0\Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^{q+\omega}.$$

We can now come back to (4.20) and deduce that

$$\widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q\mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\right] = -\frac{\frac{v_j}{v_i}\left(\int_{-\infty}^0\Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^{q+\omega}+q_{i,j}G_{i,j}(q+\omega)\right)}{(\psi_i(\omega)+q_{i,i})+\int_{-\infty}^0\Pi_{i,i}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega}.$$

Recalling (3.7), we are left with

$$\widehat{\mathcal{E}}_i\left[\left|\frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)}\right|^q \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\right] = -\frac{\frac{v_j}{v_i}\left(\int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^{q+\omega} + q_{i,j}G_{i,j}(q+\omega)\right)}{\kappa_i(\omega)}.$$

Note that, because $\mathcal{K}_i(\omega)=0$, we get

$$\kappa_i(\omega) = -\sum_{j\in\mathcal{I}\setminus\{i\}} \frac{v_j}{v_i}\left(\int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x)|\mathrm{e}^x-1|^\omega + q_{i,j}G_{i,j}(\omega)\right),$$

which upon taking $q=0$ already gives $\widehat{q}_{i,j}$ up to a multiplicative constant.

▷ Determining the exponential parameters $\widehat{q}_{i,j}$. Recall that we have assumed homogeneity. Thus, for $q\geq 0$, we wish to compute,

$$\widehat{\mathcal{E}}_i\left[\mathrm{e}^{q\widehat{H}}\mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\right] = \sum_{k=0}^\infty \underbrace{\widehat{\mathcal{E}}_i\left[\mathrm{e}^{q\widehat{H}}\mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\mathbf{1}_{\{b_{\mathcal{L}(k)}<\widehat{H}\leq b_{\mathcal{L}(k+1)}\}}\right]}_{:=a'_k}.$$

Again, using the definition of $\widehat{\mathcal{P}}_i$ and the Markov property just as we did with a_k, we end up with

$$a'_k = ra'_{k-1}, \quad k\geq 1,$$

where

$$\begin{aligned} r &= \mathtt{E}_i\left[\sum_{0<s\leq\rho_i} \mathrm{e}^{qs}\mathrm{e}^{\omega\xi(s^-)}\left(1-\mathrm{e}^{\Delta\xi(s)}\right)^\omega \mathbf{1}_{\{\iota_\Delta(s)=i\}}\right] \\ &= \mathtt{E}_i\left[\sum_{0<s<\rho_i} \mathrm{e}^{qs}\mathrm{e}^{\omega\xi_i(s^-)}\left(1-\mathrm{e}^{\Delta\xi_i(s)}\right)^\omega \mathbf{1}_{\{\iota_\Delta(s)=i\}}\right] + \mathtt{E}_i\left[\mathrm{e}^{q\rho_i}\mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}\right)^\omega \mathbf{1}_{\{\iota_\Delta(\rho_i)=i\}}\right]. \end{aligned} \tag{4.21}$$

Then, we use the compensation formula and we obtain that the first term is

$$\begin{aligned} \mathtt{E}_i\left[\sum_{0<s<\rho_i} \mathrm{e}^{qs}\mathrm{e}^{\omega\xi_i(s^-)}\left(1-\mathrm{e}^{\Delta\xi_i(s)}\right)^\omega \mathbf{1}_{\{\iota_\Delta(s)=i\}}\right] &= \int_0^\infty \mathrm{d}s\,\mathrm{e}^{(q+q_{i,i})s}\mathrm{e}^{\psi_i(\omega)s}\int_{-\infty}^0 \Lambda_i^{(i)}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega \\ &= -\frac{1}{q+q_{i,i}+\psi_i(\omega)}\int_{-\infty}^0 \Lambda_i^{(i)}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega. \end{aligned} \tag{4.22}$$

By independence, the second term of (4.21) is

$$\begin{aligned} &\mathtt{E}_i\left[\mathrm{e}^{q\rho_i}\mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}\right)^\omega \mathbf{1}_{\{\iota_\Delta(\zeta)=i\}}\right] \\ &\qquad = \sum_{k\in\mathcal{I}\setminus\{i\}} \frac{q_{i,k}}{(-q_{i,i})}\int_0^\infty \mathrm{d}s\,(-q_{i,i})\mathrm{e}^{(q+q_{i,i})s}\mathrm{e}^{\psi_i(\omega)s}\int_{-\infty}^0 \Lambda_{U_{i,k}}^{(i)}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega \\ &\qquad = -\frac{1}{q+q_{i,i}+\psi_i(\omega)}\sum_{k\in\mathcal{I}\setminus\{i\}} q_{i,k}\int_{-\infty}^0 \Lambda_{U_{i,k}}^{(i)}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega. \end{aligned} \tag{4.23}$$

Thanks to (4.22) and (4.23), equation (4.21) boils down to

$$r = -\frac{1}{q+q_{i,i}+\psi_i(\omega)}\int_{-\infty}^0 \Pi_{i,i}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega.$$

On the other hand,

$$a_0' = \widehat{\mathcal{E}}_i\left[\mathrm{e}^{q\widehat{H}}\mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\mathbf{1}_{\{\widehat{H}\le b_{\mathcal{L}(1)}\}}\right],$$

and we may split the indicator over $\{\widehat{H} < b_{\mathcal{L}(1)}\}$ and $\{\widehat{H} = b_{\mathcal{L}(1)}\}$. We therefore get $a_0' = A' + B'$, where

$$A' = \mathbb{E}_i\left[\sum_{0<t<\zeta} \mathrm{e}^{qH}\mathbf{1}_{\{J(H)=j\}}\mathbf{1}_{\{H<t\}}\frac{v_{J_\Delta(t)}}{v_i}|\Delta X(t)|^\omega\right],$$

and

$$B' = \mathtt{E}_i\left[\sum_{0<t\le\rho_i} \mathrm{e}^{qt}\frac{v_j}{v_i}\mathrm{e}^{\omega\xi(t^-)}\left(1-\mathrm{e}^{\Delta\xi(t)}\right)^\omega\mathbf{1}_{\{\iota_\Delta(t)=j\}}\right].$$

First of all, B' can be rewritten as follows

$$B' = \frac{v_j}{v_i}\mathtt{E}_i\left[\sum_{0<t<\rho_i}\mathrm{e}^{qt}\mathrm{e}^{\omega\xi(t^-)}\left(1-\mathrm{e}^{\Delta\xi(t)}\right)^\omega\mathbf{1}_{\{\iota_\Delta(t)=j\}}\right] + \frac{v_j}{v_i}\mathtt{E}_i\left[\mathrm{e}^{q\rho_i}\mathrm{e}^{\omega\xi_i(\rho_i^-)}\left(1-\mathrm{e}^{U_{i,\Theta(\rho_i)}}\right)^\omega\mathbf{1}_{\{\iota_\Delta(\rho_i)=j\}}\right].$$

Continuing along the lines of (4.22), (4.23), we eventually get to

$$B' = -\frac{v_j}{v_i}\frac{1}{q+q_{i,i}+\psi_i(\omega)}\int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega.$$

Moreover, by using the Markov property at time H, self-similarity of X, and by admissibility of $((v_i)_{i\in\mathcal{I}},\omega)$, we have

$$\begin{aligned}
A' &= \mathbb{E}_i\left[\mathrm{e}^{qH}\mathbf{1}_{\{J(H)=j\}}\mathbb{E}_{X(H),j}\left[\sum_{0<t<\zeta}\frac{v_{J_\Delta(t)}}{v_i}|\Delta X(t)|^\omega\right]\right] \\
&= \mathbb{E}_i\left[\mathrm{e}^{qH}\mathbf{1}_{\{J(H)=j\}}|X(H)|^\omega\mathbb{E}_j\left[\sum_{0<t<\zeta}\frac{v_{J_\Delta(t)}}{v_i}|\Delta X(t)|^\omega\right]\right] \\
&= \frac{v_j}{v_i}\mathbb{E}_i\left[\mathrm{e}^{qH}\mathbf{1}_{\{J(H)=j\}}|X(H)|^\omega\right].
\end{aligned}$$

Now, on the event that $J(H) = j$, we have $X(H) = \mathrm{e}^{\xi_i(\rho_i^-)+U_{i,j}}$ under $\mathbb{P}_i$. This entails

$$A' = -\frac{v_j}{v_i}\frac{q_{i,j}}{q_{i,i}}\mathtt{E}_i[\mathrm{e}^{(q+\psi_i(\omega))\rho_i}]G_{i,j}(\omega) = -\frac{v_j}{v_i}\frac{q_{i,j}G_{i,j}(\omega)}{q+q_{i,i}+\psi_i(\omega)}.$$

Therefore,

$$a_0 = -\frac{\frac{v_j}{v_i}q_{i,j}G_{i,j}(\omega) + \frac{v_j}{v_i}\int_{-\infty}^0\Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega}{q+q_{i,i}+\psi_i(\omega)}.$$

We finally conclude that

$$\widehat{\mathcal{E}}_i\left[\mathrm{e}^{q\widehat{H}}\mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}}\right] = \frac{a_0}{1-r} = -\frac{\frac{v_j}{v_i}q_{i,j}G_{i,j}(\omega) + \frac{v_j}{v_i}\int_{-\infty}^0\Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega}{q+q_{i,i}+\psi_i(\omega)+\int_{-\infty}^0\Pi_{i,i}(\mathrm{d}x)(1-\mathrm{e}^x)^\omega}.$$

This shows that, for all $i,j\in\mathcal{I}$, with $j\ne i$, the jump time of the chain $\widehat{\mathcal{J}}$ from state i to state j is an exponential random variable, with parameter

$$\widehat{q}_{i,j} = \frac{v_j}{v_i}\left(q_{i,j}G_{i,j}(\omega) + \int_{-\infty}^0\Pi_{i,j}(\mathrm{d}x)|1-\mathrm{e}^x|^\omega\right).$$

▷ THE MATRIX EXPONENT. The previous calculations determine $\widehat{F}(q) = (\widehat{F}_{i,j}(q))_{i,j\in\mathcal{I}}$, the matrix exponent of the spine as the matrix with entries:

$$\forall i \in \mathcal{I}, \quad \widehat{F}_{i,i}(q) = \kappa_i(\omega + q)$$

and

$$\forall i,j \in \mathcal{I}, i \neq j, \quad \widehat{F}_{i,j}(q) = \frac{v_j}{v_i}\left(\int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x)(1-\mathrm{e}^x)^{q+\omega} + q_{i,j}G_{i,j}(q+\omega)\right).$$

Proof of the second assertion. We finally prove the second assertion of Theorem 4.3 directly in the general setting, by mimicking Da Silva (2023). We shall only prove the statement for the first generation (this is then easily extended using the branching property). Let f_1, f_2 be nonnegative measurable functionals respectively on the space of *càdlàg* trajectories and sequences of types in $\mathcal{I}$, and g_k, $k \geq 1$, be nonnegative measurable functionals on the space of multiset–valued paths. For $t > 0$, denote by $(\mathfrak{X}_k(t), j \geq 1)$ the sequence consisting of the value of $\mathcal{X}_\varnothing(t)$, and all those jumps of $\mathcal{X}_\varnothing$ that happened strictly before time t, ranked in descending order of their absolute value, and write $(\mathfrak{j}_k(t), k \geq 1)$ for the corresponding types. Our goal is to show that

$$\widehat{\mathcal{E}}_{x,i}\left[f_1(\mathcal{X}_\varnothing(s), 0 \leq s \leq b_{\mathcal{L}(1)}) f_2(\mathfrak{j}_k(b_{\mathcal{L}(1)}), k \geq 1) \prod_{k\geq 1} g_k(\widehat{\mathbf{X}}_{0,k})\right]$$
$$= \widehat{\mathcal{E}}_{x,i}\left[f_1(\mathcal{X}_\varnothing(s), 0 \leq s \leq b_{\mathcal{L}(1)}) f_2(\mathfrak{j}_k(b_{\mathcal{L}(1)}), k \geq 1) \prod_{k\geq 1} \mathbf{E}_{\mathfrak{X}_k(b_{\mathcal{L}(1)}),\mathfrak{j}_k(b_{\mathcal{L}(1)})}\left[g_k(\mathbf{X})\right]\right].$$

But,

$$\widehat{\mathcal{E}}_{x,i}\left[f_1(\mathcal{X}_\varnothing(s), 0 \leq s \leq b_{\mathcal{L}(1)}) f_2(\mathfrak{j}_k(b_{\mathcal{L}(1)}), k \geq 1) \prod_{k\geq 1} g_k(\widehat{\mathbf{X}}_{0,k})\right]$$
$$= \mathcal{E}_{x,i}\left[\sum_{t>0} \frac{v_{J_\Delta(t)}}{v_i} |\Delta \mathcal{X}_\varnothing(t)|^\omega f_1(\mathcal{X}_\varnothing(s), 0 \leq s \leq t) f_2(\mathfrak{j}_k(t), k \geq 1) \prod_{k\geq 1} g_k(\widehat{\mathbf{X}}_{0,k})\right],$$

and the definition of the $\widehat{\mathbf{X}}_{0,k}$ together with the branching property under $\mathcal{P}_{x,i}$ give

$$\widehat{\mathcal{E}}_{x,i}\left[f_1(\mathcal{X}_\varnothing(s), 0 \leq s \leq b_{\mathcal{L}(1)}) f_2(\mathfrak{j}_k(b_{\mathcal{L}(1)}), k \geq 1) \prod_{k\geq 1} g_k(\widehat{\mathbf{X}}_{0,k})\right]$$
$$= \mathcal{E}_{x,i}\left[\sum_{t>0} \frac{v_{J_\Delta(t)}}{v_i} |\Delta \mathcal{X}_\varnothing(t)|^\omega f_1(\mathcal{X}_\varnothing(s), 0 \leq s \leq t) f_2(\mathfrak{j}_k(t), k \geq 1) \prod_{k\geq 1} \mathbf{E}_{\mathfrak{X}_k(t),\mathfrak{j}_k(t)}\left[g_k(\mathbf{X})\right]\right].$$

Applying the change of measure backwards, we get the desired identity. Therefore Theorem 4.3 is proved.

5. Appendix: implicit renewal theory on multitype branching trees

In the case when there is no type, the study of some variants of $\mathcal{M}^-(\infty)$ has arised in a variety of contexts and dates back to Mandelbrot (1974) and then Kahane and Peyrière (1976). In Liu (2000), the author presents a unified account on these *multiplicative cascades*, through a distributional equation.

This appendix is concerned with a multitype version of Goldie's implicit renewal theorem Goldie (1991), with applications to two random equations. The first set of equations that we consider in Section 5.2 is:

$$R \overset{\mathcal{L}}{=} \frac{v_J}{v_i} A R^{(J)} + B, \tag{5.1}$$

under $\mathbb{P}_i$ for all $i \in \mathcal{I}$, where $(v_j, j \in \mathcal{I})$ is a (deterministic) positive vector, A and B are positive random variables, J is a random variable in $\mathcal{I}$, $(R^{(j)}, j \in \mathcal{I})$ denotes a random vector, independent of (A, B, J), whose entries are distributed according to R under $\mathbb{P}_j, j \in \mathcal{I}$. The second equation of interest (Section 5.3) is the *multitype* linear homogeneous recursion or smoothing transform

$$R \overset{\mathcal{L}}{=} \sum_{k=1}^{\infty} \frac{v_{J_k}}{v_i} C_k R_k^{(J_k)}, \tag{5.2}$$

under $\mathbb{P}_i$ for all $i \in \mathcal{I}$, where $(v_j, j \in \mathcal{I})$ is a (deterministic) positive vector, $(C_k, k \geq 1)$ is a nonnegative random vector, $(J_k, k \geq 1)$ is a sequence of types in $\mathcal{I}$, $(R_k^{(j)}, k \geq 1)$ are nonnegative i.i.d. variables with the same law as R, under $\mathbb{P}_j$, for each $j \in \mathcal{I}$, independent of $(J_k, C_k, k \geq 1)$. We denote by Q_i the law of R under $\mathbb{P}_i$. Our goal is to generalise to some extent the results of Jelenković and Olvera-Cravioto (2012). We follow closely their approach. Although the statements we give are often tailored for our purposes, we believe that this approach could be extended in many different ways: random number of offspring, removing the moment condition in Theorem 5.5 in the spirit of Jelenković and Olvera-Cravioto (2012, Theorem 4.2), treating the $\alpha \leq 1$ case in Theorem 5.5, etc.

We construct the following multitype branching tree indexed by $\mathbb{U}$, similarly to the growth-fragmentation tree. Denote by P_i, $i \in \mathcal{I}$, the law of a vector $(J_k, C_k, k \geq 1)$ which will stand for the types and displacements of the offspring in the tree conditionally on the parent type being i. Let $i \in \mathcal{I}$, and construct the law $\mathcal{P}_i$ of the multitype tree as follows. The root $\varnothing$ has type i and marks $M_\varnothing := (J_k, C_k, k \geq 1)$ distributed according to P_i. Independently of each another, each child k of $\varnothing$ is then assigned the type J_k and the displacement C_k, and comes itself with marks $M_k := (J_{(k,l)}, C_{(k,l)}, l \geq 1)$ which, conditionally on $M_\varnothing$, is a sample of P_{J_k}. More generally, any $u \in \mathbb{U}$ with $|u| = n+1$ will have a type J_u and a displacement C_u from its parent, and is assigned marks M_u which are distributed as P_{J_u} conditionally on $\mathfrak{G}_n := \sigma(M_v, |v| \leq n)$. To each $u \in \mathbb{U}$, we also associate its *position* X_u defined by recursion by

$$X_\varnothing = 1, \quad \text{and} \quad X_{(u,k)} = C_{(u,k)} \cdot X_u, \tag{5.3}$$

for all $u \in \mathbb{U}, k \in \mathbb{N}$. We will also need independent copies $R_u^{(J_u)}$ with law Q_{J_u} (we will sometimes omit the superscript J_u, writing $R_u = R_u^{(J_u)}$ for ease of notation). This completes the construction of the multitype branching tree $\mathcal{T}$. We see equation (5.2) as a fixed point equation on the genealogies of this tree.

5.1. *The implicit renewal theorem on $\mathcal{T}$.* We describe and prove a version of Goldie's implicit renewal theorem Goldie (1991) on multitype branching trees. Our statement is largely inspired by Jelenković and Olvera-Cravioto (2012, Theorem 3.1), from which our proof is a simple adaptation. A key role will be played by the analogue of the matrix exponent in (3.4), defined here as the matrix with entries

$$m_{i,j}(q) = E_i \left[\sum_{k \geq 1} C_k^q \mathbf{1}_{\{J_k = j\}} \right]. \tag{5.4}$$

Again, by Perron-Frobenius theory, whenever $m(q)$ is finite and irreducible (note that irreducibility does not depend on q), we can define its leading eigenvalue to be $\mathrm{e}^{\lambda(q)}$. We shall then write $(v_i(q), i \in \mathcal{I})$ for an associated positive eigenvector (we will often omit the dependence on q).

Theorem 5.1. *Under $\mathbb{P}_i$, for any $i \in \mathcal{I}$, let $(J_k, C_k, k \geq 1)$ be distributed as P_i, $(R_k^{(j)}, k \geq 1)$ be an independent sequence of nonnegative i.i.d. variables with law Q_j for each $j \in \mathcal{I}$, and R be a random variable with law Q_i. Assume that $P_i(C_k > 0) > 0$ and that the measure $P_i(\log C_k \in \mathrm{d}y, C_k > 0)$ is non-lattice for some $k \geq 1$. We assume that there exist $0 \leq \gamma < \alpha$ such that*

(i) $m(q)$ *is finite and irreducible for* q *in a domain containing* $[\gamma, \alpha]$,
(ii) $\lambda(\alpha) = 0$ *and if* $(v_j, j \in \mathcal{I})$ *denotes a positive eigenvector associated to* $\lambda(\alpha) = 0$,

$$0 < E_i \left[\sum_{k \geq 1} \frac{v_{J_k}}{v_i} C_k^\alpha \log C_k \right] < \infty,$$

(iii) $\mathbb{E}_j[R^\beta] < \infty$, *for all* $0 < \beta < \alpha$ *and all* $j \in \mathcal{I}$.

If, for every $i \in \mathcal{I}$,

$$\int_0^\infty \left| \mathbb{P}_i(R > t) - \mathbb{E}_i \left[\sum_{k \geq 1} \mathbf{1}_{\{\frac{v_{J_k}}{v_i} C_k R_k^{(J_k)} > t\}} \right] \right| t^{\alpha - 1} \mathrm{d}t < \infty, \tag{5.5}$$

then for all $i \in \mathcal{I}$, *there exists a constant* $a_i \geq 0$ *such that*

$$\mathbb{P}_i(R > t) \underset{t \to \infty}{\sim} a_i \cdot t^{-\alpha}.$$

The proof of Theorem 5.1 relies on the following two lemmas. For a measure μ on $\mathcal{I} \times \mathbb{R}$ and a function $f : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$, we define (whenever we can) the convolution $f * \mu$ as

$$\forall x \in \mathbb{R}, \quad f * \mu(x) := \int_{\mathcal{I} \times \mathbb{R}} f(j, x - y) \mu(\mathrm{d}j, \mathrm{d}y).$$

Similarly, for two measure-valued mappings $\mu : i \in \mathcal{I} \mapsto \mu_i(\mathrm{d}j, \mathrm{d}y)$ and $\nu : i \in \mathcal{I} \mapsto \nu_i(\mathrm{d}j, \mathrm{d}y)$ on $\mathcal{I} \times \mathbb{R}$, let $\mu * \nu_i$, $i \in \mathcal{I}$, be the measure on $\mathcal{I} \times \mathbb{R}$:

$$(\mu * \nu)_i(\mathrm{d}j, \mathrm{d}x) := \int_{\mathcal{I} \times \mathbb{R}} \mu_i(\mathrm{d}J, \mathrm{dY}) \nu_J(\mathrm{d}j, \mathrm{d}x - Y). \tag{5.6}$$

We will mostly restrict to the case when $\nu = \mu$, and write $(\mu_i)^{*n}$ for the measure $(\mu * \cdots * \mu)_i$ (n times). This corresponds simply to the distribution of a multitype random walk $S_n := \sum_{i=1}^n Y_i$, where the distribution of Y_i at each step depends on the type in $\mathcal{I}$.

Lemma 5.2. *Recall the notation regarding the tree* $\mathcal{T}$, *and write* $Y_u = \log X_u$. *Let* $\alpha > 0$ *and for* $m \in \mathbb{N}$ *and* $i \in \mathcal{I}$, *define the measure on* $\mathcal{I} \times \mathbb{R}$

$$\mu_m^{(i)}(\mathrm{d}j, \mathrm{d}y) := \mathbb{E}_i \left[\sum_{|u|=m} \frac{v_j}{v_i} \mathrm{e}^{\alpha y} \mathbf{1}_{\{J_u = j, Y_u \in \mathrm{d}y\}} \right], \tag{5.7}$$

and $\eta^{(i)} = \mu_1^{(i)}$. *Assume that* $P_i(C_k > 0) > 0$ *and that the measure* $P_i(\log C_k \in \mathrm{d}y, C_k > 0)$ *is non-lattice for some* $k \geq 1$. *Moreover, suppose that assumption (ii) of Theorem 5.1 holds. Then, for all* $i \in \mathcal{I}$, $\eta^{(i)}$ *is a non-lattice probability measure with positive mean, and for all* $m \in \mathbb{N}$, $\mu_m^{(i)} = \left(\eta^{(i)}\right)^{*m}$ *is the* m *times convolution of* $\eta^{(i)}$ *in the sense of* (5.6).

Proof: The fact that $\eta^{(i)}$ is a non-lattice probability measure is clear from our assumptions. Furthermore, by assumption (ii), $\eta^{(i)}$ has positive mean

$$\mathbb{E}_i \left[\sum_{k \geq 1} \frac{v_{J_k}}{v_i} C_k^\alpha \log C_k \right] > 0.$$

Finally, by conditioning on $\mathfrak{G}_m$,

$$\begin{aligned}\mu_{m+1}^{(i)}(\mathrm{d}j,\mathrm{d}y) &= \mathbb{E}_i\left[\sum_{|u|=m}\frac{v_{J_u}}{v_i}\mathrm{e}^{\alpha Y_u}\sum_{k\geq 1}\frac{v_j}{v_{J_u}}\mathrm{e}^{\alpha(y-Y_u)}\mathbf{1}_{\{J_{uk}=j,\log C_k\in\mathrm{d}y-Y_u\}}\right]\\ &= \mathbb{E}_i\left[\sum_{|u|=m}\frac{v_{J_u}}{v_i}\mathrm{e}^{\alpha Y_u}\eta^{(J_u)}(\mathrm{d}j,\mathrm{d}y-Y_u)\right]\\ &= \mu_m^{(i)}*\eta^{(i)}(\mathrm{d}j,\mathrm{d}y).\end{aligned}$$

The last statement of Lemma 5.2 follows. □

Recall the notation X_u and R_u from (5.3).

Lemma 5.3. *Assume that assumptions (i), (ii) and (iii) from Theorem 5.1 hold, and denote by $(v_j, j\in\mathcal{I})$ a positive eigenvector associated to $\lambda(\alpha)=0$. For all $i\in\mathcal{I}$, $n\in\mathbb{N}$ and $t\in\mathbb{R}$, set*

$$\delta_n^{(i)}(t)=\mathrm{e}^{\alpha t}\mathbb{E}_i\left[\sum_{|u|=n}\mathbf{1}_{\{\frac{v_{J_u}}{v_i}X_uR_u>\mathrm{e}^t\}}\right],$$

and let

$$\tilde{\delta}_n^{(i)}(t):=\int_{-\infty}^t\mathrm{e}^{-(t-s)}\delta_n^{(i)}(s)\mathrm{d}s,\quad t\geq\mathbb{R},$$

be its Laplace transform. Then for all $i\in\mathcal{I}$ and $t\in\mathbb{R}$, $\tilde{\delta}_n^{(i)}(t)\to 0$ as $n\to\infty$.

Proof: We notice the following facts from assumptions (i) and (ii): for all $i\in\mathcal{I}$,

$$\mathbb{E}_i\left[\sum_{k\geq 1}\frac{v_k}{v_i}C_k^{\alpha}\right]=1\qquad\text{and}\qquad\mathbb{E}_i\left[\sum_{k\geq 1}\frac{v_k}{v_i}C_k^{\gamma}\right]<\infty.$$

Moreover, as a byproduct of Lemma 5.2, the measure $\nu^{(i)}$ has positive mean

$$\mathbb{E}_i\left[\sum_{k\geq 1}\frac{v_{J_k}}{v_i}C_k^{\alpha}\log C_k\right]>0.$$

By convexity of $\beta\mapsto\mathbb{E}_i\Big[\sum_{k\geq 1}\frac{v_k}{v_i}C_k^{\beta}\Big]$, these three facts entail that there exists $\beta\in(0,\alpha)$ such that, for all $i\in\mathcal{I}$, $\mathbb{E}_i\Big[\sum_{k\geq 1}\frac{v_k}{v_i}C_k^{\beta}\Big]<1$. Call $K:=\max_{i\in\mathcal{I}}\mathbb{E}_i\Big[\sum_{k\geq 1}\frac{v_k}{v_i}C_k^{\beta}\Big]<1$. Now observe the following: for $n\in\mathbb{N}$ and $t\in\mathbb{R}$,

$$\begin{aligned}\tilde{\delta}_n^{(i)}(t) &= \int_{-\infty}^t\mathbb{E}_i\left[\sum_{|u|=n}\mathbf{1}_{\{\frac{v_{J_u}}{v_i}X_uR_u>\mathrm{e}^s\}}\right]\mathrm{e}^{-(t-s)}\mathrm{e}^{\alpha s}\mathrm{d}s\\ &\leq \int_{-\infty}^t\mathbb{E}_i\left[\sum_{|u|=n}\mathbf{1}_{\{\frac{v_{J_u}}{v_i}X_uR_u>\mathrm{e}^s\}}\right]\mathrm{e}^{\alpha s}\mathrm{d}s\\ &\leq \mathrm{e}^{(\alpha-\beta)t}\mathbb{E}_i\left[\sum_{|u|=n}\int_{-\infty}^t\mathbf{1}_{\{\frac{v_{J_u}}{v_i}X_uR_u>\mathrm{e}^s\}}\mathrm{e}^{\beta s}\mathrm{d}s\right]\\ &\leq \frac{\mathrm{e}^{(\alpha-\beta)t}}{\beta}\mathbb{E}_i\left[\sum_{|u|=n}\left(\frac{v_{J_u}}{v_i}X_uR_u\right)^{\beta}\right],\end{aligned}$$

by direct integration. Set $c_1 := \max_{i,j\in\mathcal{I}} \left(\frac{v_j}{v_i}\right)^{\beta-1} > 0$. By assumption (iii), $c_2 := \max_{i\in\mathcal{I}} \mathbb{E}_i[R^\beta] < \infty$, so by conditioning on the type J_u of u and using independence, we end up with

$$\tilde{\delta}_n^{(i)}(t) \leq \frac{c_1 c_2 \mathrm{e}^{(\alpha-\beta)t}}{\beta} \cdot \mathbb{E}_i \left[\sum_{|u|=n} \frac{v_{J_u}}{v_i} X_u^\beta \right]. \tag{5.8}$$

The expectation in (5.8) can now be split using the branching property:

$$\mathbb{E}_i \left[\sum_{|u|=n} \frac{v_{J_u}}{v_i} X_u^\beta \right] = \mathbb{E}_i \left[\sum_{|u|=n-1} \frac{v_{J_u}}{v_i} X_u^\beta \cdot \mathbb{E}_{J_u} \left[\sum_{k\geq 1} \frac{v_k}{v_{J_u}} C_k^\beta \right] \right] \leq K \cdot \mathbb{E}_i \left[\sum_{|u|=n-1} \frac{v_{J_u}}{v_i} X_u^\beta \right],$$

by definition of K. Iterating this argument yields that $\tilde{\delta}_n^{(i)}(t) \to 0$ (actually faster than K^n) as $n \to \infty$. □

Proof of Theorem 5.1: We write $X_u = \mathrm{e}^{Y_u}$, $u \in \mathbb{U}$. Let $i \in \mathcal{I}$. In view of applying renewal theory, considering telescoping sums over the tree $\mathcal{T}$ provides, for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}_i(R > \mathrm{e}^t) \\
&= \sum_{m=0}^{n-1} \mathbb{E}_i \left[\sum_{|u|=m} \mathbf{1}_{\{\frac{v_{J_u}}{v_i} X_u R_u > \mathrm{e}^t\}} - \sum_{|w|=m+1} \mathbf{1}_{\{\frac{v_{J_w}}{v_i} X_w R_w > \mathrm{e}^t\}} \right] + \mathbb{E}_i \left[\sum_{|u|=n} \mathbf{1}_{\{\frac{v_{J_u}}{v_i} X_u R_u > \mathrm{e}^t\}} \right] \\
&= \sum_{m=0}^{n-1} \mathbb{E}_i \left[\sum_{|u|=m} \left(\mathbf{1}_{\{\frac{v_{J_u}}{v_i} R_u > \mathrm{e}^{t-Y_u}\}} - \sum_{k\geq 1} \mathbf{1}_{\{\frac{v_{J(u,k)}}{v_i} C_{(u,k)} R_{(u,k)} > \mathrm{e}^{t-Y_u}\}} \right) \right] + \mathbb{E}_i \left[\sum_{|u|=n} \mathbf{1}_{\{\frac{v_{J_u}}{v_i} X_u R_u > \mathrm{e}^t\}} \right].
\end{aligned} \tag{5.9}$$

Conditioning on $\mathfrak{G}_m$, $0 \leq m \leq n-1$, each node u in the first term corresponds to

$$\frac{v_{J_u}}{v_i} \mathrm{e}^{\alpha(Y_u - t)} g(J_u, t - Y_u),$$

where

$$g(j, y) = \frac{v_i}{v_j} \mathrm{e}^{\alpha y} \left(\mathbb{P}_j \left(R > \frac{v_i}{v_j} \mathrm{e}^y \right) - \mathbb{E}_j \left[\sum_{k\geq 1} \mathbf{1}_{\{C_k R_k^{(J_k)} > \frac{v_i}{v_{J_k}} \mathrm{e}^y\}} \right] \right).$$

Set also

$$\delta_n^{(i)}(t) = \mathrm{e}^{\alpha t} \mathbb{E}_i \left[\sum_{|u|=n} \mathbf{1}_{\{\frac{v_{J_u}}{v_i} X_u R_u > \mathrm{e}^t\}} \right].$$

Hence, multiplying (5.9) by $\mathrm{e}^{\alpha t}$, we get

$$\mathrm{e}^{\alpha t} \mathbb{P}_i(R > \mathrm{e}^t) = \sum_{m=0}^{n-1} \mathbb{E}_i \left[\sum_{|u|=m} \frac{v_{J_u}}{v_i} \mathrm{e}^{\alpha Y_u} g(J_u, t - Y_u) \right] + \delta_n^{(i)}(t).$$

Recall (5.7) and set $\nu_n^{(i)} := \sum_{m=0}^n \mu_m^{(i)}$, so that the previous equation becomes

$$\mathrm{e}^{\alpha t} \mathbb{P}_i(R > \mathrm{e}^t) = g * \nu_{n-1}^{(i)}(t) + \delta_n^{(i)}(t). \tag{5.10}$$

Now let $r^{(i)}(t) := \mathrm{e}^{\alpha t} \mathbb{P}_i(R > \mathrm{e}^t)$, for $t \in \mathbb{R}$, and define the operator $f \mapsto \tilde{f}$ by

$$\tilde{f}(t) := \int_{-\infty}^t \mathrm{e}^{-(t-s)} f(s) \mathrm{d}s, \quad t \geq \mathbb{R}.$$

Applying this operator to (5.10), we obtain for all $t \in \mathbb{R}$,

$$\tilde{r}^{(i)}(t) = \tilde{g} * \nu_{n-1}^{(i)}(t) + \tilde{\delta}_n^{(i)}(t). \tag{5.11}$$

We want to take $n \to \infty$ in the previous equality. Thanks to Lemma 5.3, the second term of (5.11) vanishes when $n \to \infty$. We now deal with the second term using Lemma 5.2. Indeed, the latter provides that the measure $\eta^{(i)} := \mu_1^{(i)}$ is a non-lattice probability measure with positive mean, and that $\nu_n^{(i)} = \sum_{m=0}^n \left(\eta^{(i)}\right)^{*m}$. Let

$$\nu^{(i)} = \sum_{m=0}^{\infty} \left(\eta^{(i)}\right)^{*m},$$

its renewal measure. By Kesten's theorem Kesten (1973) (see also Athreya et al. (1978)), for every function $G : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$ which is measurable, bounded, continuous in the second variable, and directly Riemann integrable, the integral $G * \nu^{(i)}$ is finite. Now, by our assumption (5.5), $g(j, \cdot) \in L^1$ for all $j \in \mathcal{I}$. Hence, by Goldie (1991, Lemma 9.2), $|\tilde{g}|$ is directly Riemann integrable, and therefore $|\tilde{g}| * \nu^{(i)}$ is finite. By Fubini's theorem, we get that $\tilde{g} * \nu_{n-1}^{(i)}(t) \to \tilde{g} * \nu^{(i)}(t)$ for all $t \in \mathbb{R}$.

We come to the conclusion that as $n \to \infty$, (5.11) boils down to $\tilde{r}^{(i)} = \tilde{g} * \nu^{(i)}$. Now take $t \to \infty$. By Kesten's renewal theorem Kesten (1973),

$$\tilde{r}^{(i)}(t) = \mathrm{e}^{-t} \int_0^{\mathrm{e}^t} s^\alpha \mathbb{P}_i(R > s) \mathrm{d}s \underset{t \to \infty}{\longrightarrow} a_i,$$

where a_i is an (explicit) nonnegative constant. It remains to apply Goldie (1991, Lemma 9.3) to obtain

$$\mathbb{P}_i(R > t) \underset{t \to \infty}{\sim} a_i \cdot t^{-\alpha},$$

which is Theorem 5.1. □

5.2. *Tail behaviour of the random affine equation.* We present a first application of Theorem 5.1, where we study the solutions of (5.1).

Theorem 5.4. *Under $\mathbb{P}_i$, for any $i \in \mathcal{I}$, let (A, B, J) be distributed as P_i, $(R^{(j)}, j \in \mathcal{I})$ be a sequence of nonnegative variables with law Q_j for each $j \in \mathcal{I}$, independent of (A, B, J), and R be a random variable with law Q_i. Assume that $P_i(A > 0) > 0$ and that the measure $P_i(\log A \in \mathrm{d}y, A > 0)$ is non-lattice. Similarly to (5.4), for $q \geq 0$, let $m(q)$ be the matrix*

$$m_{i,j}(q) = E_i\left[A^q \cdot \mathbf{1}_{\{J=j\}}\right], \quad i, j \in \mathcal{I}, \tag{5.12}$$

and $\lambda(q)$ its leading eigenvalue. We assume that there exist $\alpha > 0$ and $0 \leq \gamma < \alpha$ such that

- (i) *The matrix $m(q)$ defined by (5.12) is finite and irreducible for q in a domain containing $[\gamma, \alpha]$,*
- (ii) *$\lambda(\alpha) = 0$ and if $(v_j, j \in \mathcal{I})$ denotes a positive eigenvector associated to $\lambda(\alpha) = 0$,*

$$0 < E_i\left[\frac{v_J}{v_i} A^\alpha \log A\right] < \infty,$$

- (iii) *$\mathbb{E}_j[R^\beta] < \infty$ for all $0 < \beta < \alpha$ and all $j \in \mathcal{I}$,*
- (iv) *$E_i[A^\alpha] < \infty$ and $E_i[B^\alpha] < \infty$ for all $i \in \mathcal{I}$.*

Assume that (5.1) holds with the same notation. Then for all $i \in \mathcal{I}$, there exists a constant $b_i \geq 0$ such that

$$\mathbb{P}_i(R > t) \underset{t \to \infty}{\sim} b_i \cdot t^{-\alpha}.$$

Proof: We need to show that, under the assumptions in Theorem 5.4, (5.5) holds. Then Theorem 5.1 would imply the desired assymptotics. In our setting, (5.5) rephrases as

$$\mathbb{E}_i\left[R^\alpha - \left(\frac{v_J}{v_i}AR^{(J)}\right)^\alpha\right] < \infty.$$

As $E_i[B^\alpha] < \infty$ by assumption, this is equivalent to

$$\mathbb{E}_i\left[R^\alpha - \left(\frac{v_J}{v_i}AR^{(J)}\right)^\alpha - B^\alpha\right] < \infty.$$

Since R solves (5.1), we can take $R = (v_J/v_i)AR^{(J)} + B$ under $\mathbb{P}_i$. Write $\alpha = p\gamma$, with $p := \lceil \alpha \rceil$, and $\gamma \in [0,1]$. Then expanding the first sum, and using the inequality $(x+y)^\gamma \le x^\gamma + y^\gamma$ for $x, y \ge 0$, we get

$$\mathbb{E}_i\left[R^\alpha - \left(\frac{v_J}{v_i}AR^{(J)}\right)^\alpha - B^\alpha\right] \le \mathbb{E}_i\left[\left(\sum_{k=1}^{p-1}\binom{p}{k}\left(\frac{v_J}{v_i}AR^{(J)}\right)^k B^{p-k}\right)^\gamma\right].$$

We now condition on (A, B, J). By Jensen's inequality ($\gamma \le 1$), and the independence of $(R_j, j \in \mathcal{I})$ and (A, B, J),

$$\mathbb{E}_i\left[R^\alpha - \left(\frac{v_J}{v_i}AR^{(J)}\right)^\alpha - B^\alpha\right] \le \mathbb{E}_i\left[\left(\sum_{k=1}^{p-1}\binom{p}{k}\left(\frac{v_J}{v_i}A\right)^k \mathbb{E}_J[R^k]B^{p-k}\right)^\gamma\right].$$

Using the monotonicity of $k \mapsto \mathbb{E}_j[R^k]^{1/k}$, we obtain for all $1 \le k \le p-1$,

$$\mathbb{E}_J[R^k] \le \mathbb{E}_J[R^{p-1}]^{k/(p-1)} \le M^p \qquad \text{a.s.},$$

with $M = \max_{j\in\mathcal{I}}(1 \vee \mathbb{E}_j[R^{p-1}]^{1/(p-1)})$ (note that $M < \infty$ by assumption (iii)). Therefore

$$\mathbb{E}_i\left[R^\alpha - \left(\frac{v_J}{v_i}AR^{(J)}\right)^\alpha - B^\alpha\right] \le M^\alpha \mathbb{E}_i\left[\left(\sum_{k=1}^{p-1}\binom{p}{k}\left(\frac{v_J}{v_i}A\right)^k B^{p-k}\right)^\gamma\right].$$

Finally, we can factorise the sum, and we get

$$\begin{aligned}\mathbb{E}_i\left[R^\alpha - \left(\frac{v_J}{v_i}AR^{(J)}\right)^\alpha - B^\alpha\right] &\le M^\alpha \mathbb{E}_i\left[\left(\left(\frac{v_J}{v_i}A + B\right)^p - \left(\frac{v_J}{v_i}A\right)^p - B^p\right)^\gamma\right] \\ &\le M^\alpha \mathbb{E}_i\left[\left(\frac{v_J}{v_i}A + B\right)^\alpha\right].\end{aligned}$$

Thus we conclude using (iv) that

$$\mathbb{E}_i\left[R^\alpha - \left(\frac{v_J}{v_i}AR^{(J)}\right)^\alpha - B^\alpha\right] < \infty.$$

This proves (5.5), and the renewal theorem (Theorem 5.1) provides a constant $b_i \ge 0$ such that

$$\mathbb{P}_i(R > t) \underset{t\to\infty}{\sim} b_i \cdot t^{-\alpha}.$$

□

5.3. *Tail behaviour of the multitype smoothing transform.* We now turn to describing the tail asymptotics of the solutions of (5.2). The next result is tailored for our purposes, but we believe that one could replace condition (iii) by a condition on $(J_k, C_k, k \ge 1)$, in the spirit of Jelenković and Olvera-Cravioto (2012, Theorem 4.1 or 4.2), and that their proofs should extend to our setup. We here decided to give a minimal statement for our purposes. Again, we follow Jelenković and Olvera-Cravioto (2012) closely.

Theorem 5.5. *Under $\mathbb{P}_i$, for any $i \in \mathcal{I}$, let $(J_k, C_k, k \geq 1)$ be distributed as P_i, $(R_k^{(j)}, k \geq 1)$ be an independent sequence of nonnegative i.i.d. variables with law Q_j for each $j \in \mathcal{I}$, and R be a random variable with law Q_i. Assume that $P_i(C_k > 0) > 0$ and that the measure $P_i(\log C_k \in \mathrm{d}y, C_k > 0)$ is non-lattice for some $k \geq 1$. We assume that there exist $\alpha > 1$ and $0 \leq \gamma < \alpha$ such that*

(i) *The matrix $m(q)$ defined by (5.4) is finite and irreducible for q in a domain containing $[\gamma, \alpha]$,*

(ii) *$\lambda(\alpha) = 0$ and if $(v_j, j \in \mathcal{I})$ denotes a positive eigenvector associated to $\lambda(\alpha) = 0$,*

$$0 < E_i \left[\sum_{k \geq 1} \frac{v_{J_k}}{v_i} C_k^\alpha \log C_k \right] < \infty,$$

(iii) *$\mathbb{E}_j[R^\beta] < \infty$ for all $0 < \beta < \alpha$ and all $j \in \mathcal{I}$,*

(iv) *$\mathbb{E}_j \left[\left(\sum_{k \geq 1} C_k \right)^\alpha \right] < \infty$ for all $j \in \mathcal{I}$.*

Assume that (5.2) holds with the same notation. Then for all $i \in \mathcal{I}$, there exists a constant $a_i \geq 0$ such that

$$\mathbb{P}_i(R > t) \underset{t \to \infty}{\sim} a_i \cdot t^{-\alpha}.$$

Our proof relies on the following two technical lemmas.

Lemma 5.6. *Under $\mathbb{P}_i$, for any $i \in \mathcal{I}$, let $(J_k, C_k, k \geq 1)$ be distributed as P_i, $(R_k^{(j)}, k \geq 1)$ be an independent sequence of i.i.d. variables with law Q_j for each $j \in \mathcal{I}$, and R be a random variable with law Q_i. For $\alpha > 0$, assume that $\sum_{k \geq 1} \left(C_k R_k^{(J_k)} \right)^\alpha < \infty$ a.s., and that $\mathbb{E}_i[R^\beta] < \infty$ for all $i \in \mathcal{I}$ and $0 < \beta < \alpha$. Moreover, assume that for some $\varepsilon \in (0,1)$, $\mathbb{E}_i\big[\big(\sum_{k \geq 1} C_k^{\alpha/(1+\varepsilon)}\big)^{1+\varepsilon}\big] < \infty$ for all $i \in \mathcal{I}$. Then for all $i \in \mathcal{I}$,*

$$\int_0^\infty \left(\mathbb{E}_i \left[\sum_{k \geq 1} \mathbf{1}_{\{C_k R_k^{(J_k)} > t\}} \right] - \mathbb{P}_i \left(\sup_{k \geq 1} \left(C_k R_k^{(J_k)} \right) > t \right) \right) t^{\alpha - 1} \mathrm{d}t$$
$$= \mathbb{E}_i \left[\sum_{k \geq 1} \left(C_k R_k^{(J_k)} \right)^\alpha - \left(\sup_{k \geq 1} C_k R_k^{(J_k)} \right)^\alpha \right] < \infty.$$

Proof: The integrand is of course positive, so that the integral makes sense. Moreover, the equality is well-known, and stems from a simple application of Fubini's theorem. The point is to show that this integral is finite. We condition the integrand on $(J_k, C_k, k \geq 1)$:

$$\mathbb{E}_i \left[\sum_{k \geq 1} \mathbf{1}_{\{C_k R_k^{(J_k)} > t\}} - \mathbf{1}_{\{\sup_{k \geq 1} \left(C_k R_k^{(J_k)} \right) > t\}} \right] =$$
$$\mathbb{E}_i \left[\mathbb{E}_i \left[\sum_{k \geq 1} \mathbf{1}_{\{C_k R_k^{(J_k)} > t\}} \,\middle|\, (J_k, C_k, k \geq 1) \right] - 1 + \mathbb{E}_i \left[\mathbf{1}_{\{\sup_{k \geq 1} \left(C_k R_k^{(J_k)} \right) \leq t\}} \,\middle|\, (J_k, C_k, k \geq 1) \right] \right]. \tag{5.13}$$

By independence, the last term in the expectation of (5.13) is

$$\mathbb{E}_i \left[\mathbf{1}_{\{\sup_{k \geq 1} \left(C_k R_k^{(J_k)} \right) \leq t\}} \,\middle|\, (J_k, C_k, k \geq 1) \right] = \prod_{k \geq 1} Q_{J_k} \left(R \leq \frac{t}{C_k} \,\middle|\, (C_k, k \geq 1) \right) = \prod_{k \geq 1} (1 - F_{J_k}(t/C_k)),$$

where $F_j(r) := Q_j(R > r)$ for $j \in \mathcal{I}$ and $r \in \mathbb{R}$. The simple inequality $1 - x \leq \mathrm{e}^{-x}$, $x \geq 0$, yields

$$\mathbb{E}_i \left[\mathbf{1}_{\{\sup_{k \geq 1} \left(C_k R_k^{(J_k)} \right) \leq t\}} \,\middle|\, (J_k, C_k, k \geq 1) \right] \leq \mathrm{e}^{-\sum_{k \geq 1} F_{J_k}(t/C_k)}.$$

Hence the right-hand side of (5.13) is less than $\mathbb{E}_i\big[g\big(\sum_{k\geq 1} F_{J_k}(t/C_k)\big)\big]$, where $g : x \mapsto x-1+\mathrm{e}^{-x}$. We now use that g is non-decreasing on $\mathbb{R}_+$. For future purposes, let $\delta = \alpha\varepsilon/(1+\varepsilon)$, and $\beta = \alpha - \delta$. Then by Markov's inequality,

$$\sum_{k\geq 1} F_{J_k}(t/C_k) \leq \sum_{k\geq 1} t^{-\beta} C_k^{\beta} \mathbb{E}_{J_k}[R^{\beta}] \leq M t^{-\beta} \sum_{k\geq 1} C_k^{\beta},$$

where $M := \max_{j\in\mathcal{I}} \mathbb{E}_j[R^{\beta}] < \infty$ by assumption. Define $S_\beta := \sum_{k\geq 1} C_k^{\beta}$. By monotonicity of g,

$$\mathbb{E}_i\left[g\bigg(\sum_{k\geq 1} F_{J_k}(t/C_k)\bigg)\right] \leq \mathbb{E}_i\left[g\big(Mt^{-\beta}S_\beta\big)\right].$$

In total, integrating (5.13) gives

$$\int_0^\infty \mathbb{E}_i\left[\sum_{k\geq 1}\mathbf{1}_{\{C_k R_k^{(J_k)}>t\}} - \mathbf{1}_{\{\sup_{k\geq 1}\left(C_k R_k^{(J_k)}\right)>t\}}\right] t^{\alpha-1}\mathrm{d}t \leq \int_0^\infty \mathbb{E}_i\left[g\big(Mt^{-\beta}S_\beta\big)\right] t^{\alpha-1}\mathrm{d}t,$$

and by the change of variables $s := Mt^{-\beta}S_\beta$, we get

$$\int_0^\infty \mathbb{E}_i\left[\sum_{k\geq 1}\mathbf{1}_{\{C_k R_k^{(J_k)}>t\}} - \mathbf{1}_{\{\sup_{k\geq 1}\left(C_k R_k^{(J_k)}\right)>t\}}\right] t^{\alpha-1}\mathrm{d}t \leq \frac{1}{\beta} M^{\alpha/\beta}\mathbb{E}_i\left[S_\beta^{\alpha/\beta}\right]\int_0^\infty g(s)s^{-\alpha/\beta-1}\mathrm{d}s.$$

Plainly, since $\alpha/\beta = 1+\varepsilon \in (1,2)$, the above integral is finite. The expectation is also finite by assumption, and the result follows. □

Lemma 5.7. *Under $\mathbb{P}_i$, for any $i\in\mathcal{I}$, let $(J_k, C_k, k\geq 1)$ be distributed as P_i, and $(R_k^{(j)}, k\geq 1)$ be an independent sequence of i.i.d. variables with law Q_j for each $j\in\mathcal{I}$. For $\alpha > 1$, assume that $\sum_{k\geq 1} C_k R_k^{(J_k)} < \infty$ a.s., that $Q_i[R^\beta] < \infty$, and that $\mathbb{E}_i\big[\big(\sum_{k\geq 1} C_k\big)^\alpha\big] < \infty$ for all $i\in\mathcal{I}$ and $0<\beta<\alpha$. Then for all $i\in\mathcal{I}$,*

$$\mathbb{E}_i\left[\bigg(\sum_{k\geq 1} C_k R_k^{(J_k)}\bigg)^\alpha - \sum_{k\geq 1}\Big(C_k R_k^{(J_k)}\Big)^\alpha\right] < \infty.$$

Proof: Define $p := \lceil \alpha \rceil$, and $\gamma := \alpha/p$. We prove the following inequality:

$$\mathbb{E}_i\left[\bigg(\sum_{k\geq 1} C_k R_k^{(J_k)}\bigg)^\alpha - \sum_{k\geq 1}\Big(C_k R_k^{(J_k)}\Big)^\alpha\right] \leq M^\alpha \mathbb{E}_i\left[\bigg(\sum_{k\geq 1} C_k\bigg)^\alpha\right], \tag{5.14}$$

where $M := \max_{j\in\mathcal{I}} \mathbb{E}_j\big[R^{p-1}\big]^{1/(p-1)}$. The claim is straigthforward with our assumptions once (5.14) is established. For $k\in\mathbb{N}$, write

$$A_p := \Big\{\mathbf{q} = (q_k)_{k\in\mathbb{N}},\ 0\leq q_k\leq p-1 \text{ and } \sum_{k\in\mathbb{N}} q_k = p\Big\}. \tag{5.15}$$

A sequence $\mathbf{q}$ in A_p has only finitely many non-zero entries $r_1,\dots,r_k$, and for ease of notation we define $\binom{p}{\mathbf{q}}$ as the corresponding multinomial coefficient $\binom{p}{r_1,\dots,r_k}$, and the product $\prod_{k\geq 1} a_k^{q_k}$ of the terms of a nonnegative sequence $(a_k)_{k\geq 1}$ in a similar way. Then, expanding the first sum, and using that $\big(\sum_{k\geq 1} x_k\big)^\gamma \leq \sum_{k\geq 1} x_k^\gamma$, for $0<\gamma\leq 1$ and $x_k\geq 0$,

$$\mathbb{E}_i\left[\bigg(\sum_{k\geq 1} C_k R_k^{(J_k)}\bigg)^\alpha - \sum_{k\geq 1}\Big(C_k R_k^{(J_k)}\Big)^\alpha\right] \leq \mathbb{E}_i\left[\bigg(\sum_{\mathbf{q}\in A_p}\binom{p}{\mathbf{q}}\prod_{k\geq 1}(C_k R_k^{(J_k)})^{q_k}\bigg)^\gamma\right].$$

We now condition on $(J_k, C_k, k \geq 1)$ and use Jensen's inequality with $\gamma \leq 1$ to obtain

$$\mathbb{E}_i\left[\left(\sum_{k\geq 1} C_k R_k^{(J_k)}\right)^\alpha - \sum_{k\geq 1}\left(C_k R_k^{(J_k)}\right)^\alpha\right] \leq \mathbb{E}_i\left[\left(\sum_{\mathbf{q}\in A_p}\binom{p}{\mathbf{q}}\prod_{k\geq 1} C_k^{q_k}\mathbb{E}_{J_k}\left[R^{q_k}\right]\right)^\gamma\right]. \tag{5.16}$$

Fix $\mathbf{q} \in A_p$. By monotonicity of $q \mapsto \mathbb{E}_i[X^q]^{1/q}$, we can upper-bound the product of expectations:

$$\prod_{k\geq 1}\mathbb{E}_{J_k}\left[R^{q_k}\right] \leq \prod_{k\geq 1}\mathbb{E}_{J_k}\left[R^{p-1}\right]^{q_k/(p-1)} \leq M^p,$$

where $M := \max_{j\in\mathcal{I}}\mathbb{E}_j\left[R^{p-1}\right]^{1/(p-1)}$. Plugging this inequality into (5.16) provides

$$\mathbb{E}_i\left[\left(\sum_{k\geq 1} C_k R_k^{(J_k)}\right)^\alpha - \sum_{k\geq 1}\left(C_k R_k^{(J_k)}\right)^\alpha\right] \leq M^\alpha\mathbb{E}_i\left[\left(\sum_{\mathbf{q}\in A_p}\binom{p}{\mathbf{q}}\prod_{k\geq 1} C_k^{q_k}\right)^\gamma\right].$$

Finally, we can factorise everything back to obtain

$$\begin{aligned}\mathbb{E}_i\left[\left(\sum_{k\geq 1} C_k R_k^{(J_k)}\right)^\alpha - \sum_{k\geq 1}\left(C_k R_k^{(J_k)}\right)^\alpha\right] &\leq M^\alpha\mathbb{E}_i\left[\left(\left(\sum_{k\geq 1} C_k\right)^p - \sum_{k\geq 1} C_k^p\right)^\gamma\right] \\ &\leq M^\alpha\mathbb{E}_i\left[\left(\sum_{k\geq 1} C_k\right)^\alpha\right],\end{aligned}$$

which is (5.14). □

Proof of Theorem 5.5: The theorem is an immediate consequence of Theorem 5.1 once we prove (5.5). To do so, first write, for $i \in \mathcal{I}$ and $t \geq 0$,

$$\left|\mathbb{P}_i(R>t) - \mathbb{E}_i\left[\sum_{k\geq 1}\mathbf{1}_{\left\{\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}>t\right\}}\right]\right| \leq \left|\mathbb{P}_i(R>t) - \mathbb{P}_i\left(\sup_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)>t\right)\right| \tag{5.17}$$

$$+\left|\mathbb{E}_i\left[\sum_{k\geq 1}\mathbf{1}_{\left\{\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}>t\right\}}\right] - \mathbb{P}_i\left(\sup_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)>t\right)\right|. \tag{5.18}$$

Both differences in (5.17) and (5.18) are nonnegative, so that we can remove the absolute values.

We first deal with the second term using Lemma 5.6. Since $\alpha > 1$, we can choose $\varepsilon > 0$ small enough so that $\alpha/(1+\varepsilon) > 1$. Now for all $i \in \mathcal{I}$,

$$\mathbb{E}_i\left[\left(\sum_{k\geq 1}\left(C_k R_k^{(J_k)}\right)^{\alpha/(1+\varepsilon)}\right)^{1+\varepsilon}\right] \leq \mathbb{E}_i\left[\left(\sum_{k\geq 1} C_k R_k^{(J_k)}\right)^\alpha\right] < \infty,$$

and we conclude by an application of Lemma 5.6 that

$$\int_0^\infty\left(\mathbb{E}_i\left[\sum_{k\geq 1}\mathbf{1}_{\left\{\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}>t\right\}}\right] - \mathbb{P}_i\left(\sup_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)>t\right)\right)t^{\alpha-1}\mathrm{d}t < \infty. \tag{5.19}$$

As for the first term (5.17), Fubini's theorem implies

$$\int_0^\infty\left(\mathbb{P}_i(R>t) - \mathbb{P}_i\left(\sup_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)>t\right)\right)t^{\alpha-1}\mathrm{d}t = \mathbb{E}_i\left[R^\alpha - \sup_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)^\alpha\right].$$

We then again split

$$\mathbb{E}_i\left[R^\alpha - \sup_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)^\alpha\right] = \mathbb{E}_i\left[R^\alpha - \sum_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)^\alpha\right] + \mathbb{E}_i\left[\sum_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)^\alpha - \sup_{k\geq 1}\left(\frac{v_{J_k}}{v_i}C_k R_k^{(J_k)}\right)^\alpha\right].$$

Notice that both terms are positive, so that the splitting makes sense. The last term is actually (5.19), and so is finite. For the first one, we use Lemma 5.7. This concludes the proof of (5.5), and hence Theorem 5.5 is proved. □

5.4. *Moments of* $\mathcal{M}^-(\infty)$. In this section, we prove that the limit $\mathcal{M}^-(\infty)$ of the martingale $(\mathcal{M}^-(n), n \geq 0)$ is in L^β for all $0 < \beta < \omega_+/\omega_-$. This allows to leverage Theorem 5.5 to prove the last item of Proposition 4.9 (the other assumptions being either clearly fulfilled or already proved). We mostly rely on Liu (2000).

Proposition 5.8. *The variable* $\mathcal{M}^-(\infty)$ *is in* L^β *for all* $0 < \beta < \omega_+/\omega_-$.

We divide the proof of Proposition 5.8 into several steps, following Liu (2000). First, we call attention to a multitype version of a random linear recursion, for which we provide conditions for the solutions to be in L^β. Then, we prove that (a variant of) $\mathcal{M}^-(\infty)$ satisfies such a recursion and that all the conditions are fulfilled, thus concluding the proof of Proposition 5.8.

The multitype linear recursion. We start by setting the framework for the multitype linear recursion. Consider, for each $i \in \mathcal{I}$, two positive random variables $A^{(i)}$ and $B^{(i)}$ whose distributions depend on the type i. Let $J^{(i)}, i \in \mathcal{I}$, be random variables taking values in $\mathcal{I}$. We are interested in the solutions X to the following random equation in law:

$$\forall i \in \mathcal{I}, \quad X^{(i)} \overset{\mathcal{L}}{=} A^{(i)} X^{(J^{(i)})} + B^{(i)}, \tag{5.20}$$

where the $X^{(i)}, i \in \mathcal{I}$, are independent of the $A^{(i)}, B^{(i)}, J^{(i)}, i \in \mathcal{I}$. It will be convenient to consider, for $i \in \mathcal{I}$, i.i.d. sequences $(A_k^{(i)}, k \geq 1)$, $(B_k^{(i)}, k \geq 1)$, $(X_k^{(i)}, k \geq 1)$ and $(J_k^{(i)}, k \geq 1)$ with respective laws $A^{(i)}$, $B^{(i)}$, $X^{(i)}$ and $J^{(i)}$ under measure P_i. Moreover, with a slight abuse of notation we set, under P_i, $J_0 = i$ and by recursion $J_{k+1} = J_{k+1}^{(J_k)}$. We assume that the chain $J = (J_k, k \geq 0)$ is irreducible and aperiodic, and write π for its invariant probability measure. Denote P_π the measure $\sum_{i\in\mathcal{I}} \pi(i) P_i$. The following lemma is a multitype version of Grincevičjus (1974, Theorem 1).

Lemma 5.9. *Suppose that for all* $i \in \mathcal{I}$, $P_i(A^{(i)} = 0) = 0$ *and that for some positive vector* $(c_i, i \in \mathcal{I})$,

$$-\infty < E_\pi\left[\log\left(\frac{c_{J_1}}{c_{J_0}} A^{(J_0)}\right)\right] < 0, \quad \textit{and} \quad E_i\left[\log^+ B^{(i)}\right] < \infty.$$

Then the recursion (5.20) *has a unique solution, which is* P_i*–almost surely given by*

$$X^{(i)} = \sum_{k\geq 0} A_1^{(i)} A_2^{(J_1)} \cdots A_k^{(J_{k-1})} B_{k+1}^{(J_k)}.$$

Proof: It is sufficient to prove that the series $\sum_k A_1^{(i)} A_2^{(J_1)} \cdots A_k^{(J_{k-1})} B_{k+1}^{(J_k)}$ is convergent (the claim then follows by iterating (5.20)). We mimic the proof in Grincevičjus (1974). Write $C_k^{(i)} := \log^+ B_k^{(i)}$. Since

$$\sum_{k\geq 1} P_i(C_k^{(i)} \geq Ak) \leq \frac{1}{A}\sum_{k\geq 1}\int_{A(k-1)}^{Ak} P_i(C_1^{(i)} \geq x)\mathrm{d}x = \frac{1}{A}\int_0^\infty P_i(C_1^{(i)} \geq x)\mathrm{d}x = E_i[C_1^{(i)}] < \infty,$$

by Borel-Cantelli lemma, $\limsup_{k\to\infty} C_k^{(i)}/k < A$, P_i–a.s. Taking $A \to 0$, we deduce that $C_k^{(i)}/k \to 0$, P_i–a.s. This being true for all $i \in \mathcal{I}$, we have $C_k^{(J_{k-1})}/k \to 0$, P_i–a.s. Now for $k \geq 0$,

$$\left(A_1^{(i)} A_2^{(J_1)} \cdots A_k^{(J_{k-1})} B_{k+1}^{(J_k)}\right)^{1/k} = \left(\frac{c_i}{c_{J_k}}\right)^{1/k} \left(\frac{c_{J_1}}{c_i} A_1^{(i)} \cdot \frac{c_{J_2}}{c_{J_1}} A_2^{(J_1)} \cdots \frac{c_{J_k}}{c_{J_{k-1}}} A_k^{(J_{k-1})} B_{k+1}^{(J_k)}\right)^{1/k}$$

$$\leq C^{1/k} \exp\left(C_{k+1}^{(J_k)}/k\right) \cdot \exp\left(\frac{1}{k}\sum_{l=1}^{k} \log\left(\frac{c_{J_l}}{c_{J_{l-1}}} A_l^{(J_{l-1})}\right)\right),$$

where $C = \max_{j\in\mathcal{I}} \frac{c_i}{c_j}$. By the previous point and the ergodic theorem of Lalley (1984), P_i–almost surely,

$$\limsup_{k\to\infty} \left(A_1^{(i)} A_2^{(J_1)} \cdots A_k^{(J_{k-1})} B_{k+1}^{(J_k)}\right)^{1/k} \leq \exp\left(E_\pi\left[\log\left(\frac{c_{J_1}}{c_{J_0}} A^{(J_0)}\right)\right]\right) < 1.$$

We conclude the proof by an application of Cauchy's criterion on series convergence. □

We now use Lemma 5.9 to provide conditions for the solutions of (5.20) to be in some L^β (see Liu (2000, Lemma 3.2)).

Proposition 5.10. *In addition to the assumptions of Lemma 5.9, suppose that for some $\beta > 0$ and some positive vector $(c_i, i \in \mathcal{I})$, $E_i[(\frac{c_{J_1}}{c_i}|A^{(i)}|)^\beta] < 1$ and $E_i[|B^{(i)}|^\beta] < \infty$ for all $i \in \mathcal{I}$. Then the solution of (5.20) satisfies $E_i[|X^{(i)}|^\beta] < \infty$, for all $i \in \mathcal{I}$.*

Proof: Denote by $\widetilde{A}_k^{(j)} := \frac{c_{J_k}}{c_j} A_k^{(j)}$. Using Lemma 5.9, one can write $X^{(i)}$ as

$$X^{(i)} = \sum_{k\geq 0} \frac{c_i}{c_{J_k}} \widetilde{A}_1^{(i)} \widetilde{A}_2^{(J_1)} \cdots \widetilde{A}_k^{(J_{k-1})} B_{k+1}^{(J_k)}.$$

Assume $\beta \geq 1$ for the moment. Then by Minkowski's inequality,

$$||X^{(i)}||_\beta \leq C \sum_{k\geq 0} ||\widetilde{A}_1^{(i)} \widetilde{A}_2^{(J_1)} \cdots \widetilde{A}_k^{(J_{k-1})} B_{k+1}^{(J_k)}||_\beta,$$

where $||Y||_\beta = E_i[|Y|^\beta]^{1/\beta}$ and $C = \max_{j\in\mathcal{I}} \frac{c_i}{c_j}$. We now condition on the types $J_1, \ldots, J_k$ and use conditional independence to get:

$$||X^{(i)}||_\beta \leq C \sum_{k\geq 0} E\left[E_i\left[|\widetilde{A}_1^{(i)}|^\beta\right] E_{J_1}\left[|\widetilde{A}_1^{(J_1)}|^\beta\right] \cdots E_{J_{k-1}}\left[|\widetilde{A}_1^{(J_{k-1})}|^\beta\right] E_{J_k}\left[B_1^\beta\right]\right]^{1/\beta}.$$

Set $a := \max_{j\in\mathcal{I}} E_j\left[|\widetilde{A}_1^{(j)}|^\beta\right]^{1/\beta} < 1$ and $b := \max_{j\in\mathcal{I}} E_j\left[B_1^\beta\right]^{1/\beta}$. Then

$$||X^{(i)}||_\beta \leq Cb \sum_{k\geq 0} a^k = \frac{Cb}{1-a} < \infty.$$

For $\beta < 1$, one uses the inequality

$$||X^{(i)}||_\beta^\beta \leq C \sum_{k\geq 0} ||\widetilde{A}_1^{(i)} \widetilde{A}_2^{(J_1)} \cdots \widetilde{A}_k^{(J_{k-1})} B_{k+1}^{(J_k)}||_\beta^\beta,$$

and concludes in the same way. □

Moments of $\mathcal{M}^-(\infty)$ through equation (5.20). Next, we prove that such an equation as (5.20) arises naturally when considering $\widetilde{\mathcal{M}}^-(\infty) := \mathcal{M}^-(\infty)/v_{\widehat{\mathcal{J}}(0)}$. Recall the notation from the change of measure in Section 4.1, and let $\widehat{\mathcal{J}}_k := \widehat{\mathcal{J}}(b_{\mathcal{L}(k)})$ and $\widehat{\mathcal{X}}_k := \widehat{\mathcal{X}}(b_{\mathcal{L}(k)})$. For all $u \in \mathbb{U}$, denote by M_u the value of the limit $\widetilde{\mathcal{M}}^-(\infty)$ on the tree re-rooted at u.

Proposition 5.11. *Set* $\mathcal{A}_k^{(i)} := \frac{v_{\mathcal{J}_k(0)}}{v_i}|\mathcal{X}_k(0)|^{\omega_-}, k \geq 1,$ *and*

$$A^{(i)} := \mathcal{A}_{\mathcal{L}(1)}^{(i)}, \quad \text{and} \quad B^{(i)} := \sum_{k \neq \mathcal{L}(1)} \mathcal{A}_k^{(i)} M_k.$$

Then $\widetilde{\mathcal{M}}^-(\infty)$ *solves the family of equations, for all* $i \in \mathcal{I}$ *under* $\widehat{\mathcal{P}}_{1,i}^-$,

$$\widetilde{\mathcal{M}}^-(\infty) \stackrel{\mathcal{L}}{=} A^{(i)} M_{\mathcal{L}(1)} + B^{(i)}. \tag{5.21}$$

Moreover, the joint law is described by the following formula: for all nonnegative measurable functions f, g, h,

$$\widehat{\mathcal{E}}_{1,i}^- \left[f(A^{(i)}) g(B^{(i)}) h(M_{\mathcal{L}(1)}) \right] = \mathcal{E}_{1,i} \left[\sum_{k \geq 1} \mathcal{A}_k^{(i)} f(\mathcal{A}_k^{(i)}) g\left(\sum_{l \neq k} \mathcal{A}_l^{(i)} M_l \right) \mathcal{E}_{1,\mathcal{J}_k(0)} \left[h(M_k) M_k \right] \right]. \tag{5.22}$$

In particular, under $\widehat{\mathcal{P}}_{1,i}^-$ *the* $M_{\mathcal{L}(1)}$ *is independent of* $(A^{(i)}, B^{(i)})$ *and conditionally on* $\widehat{\mathcal{J}}_1$, *is distributed as* $\widetilde{\mathcal{M}}^-(\infty)$ *under* $\widehat{\mathcal{P}}_{1,\widehat{\mathcal{J}}_1}^-$.

Proof: We note that under $\widehat{\mathcal{P}}_{1,i}^-$, $\widetilde{\mathcal{M}}^-(\infty)$ satisfies $\widetilde{\mathcal{M}}^-(\infty) = \sum_{k \geq 1} \mathcal{A}_k^{(i)} M_k$ by the branching property. The fact that $\widetilde{\mathcal{M}}^-(\infty)$ solves (5.21) is then obvious by decomposing over the spine and the other cells. For the second statement, by definition of $\widehat{\mathcal{P}}_{1,i}^-$, we have

$$\widehat{\mathcal{E}}_{1,i}^- \left[f(A^{(i)}) g(B^{(i)}) h(M_{\mathcal{L}(1)}) \right] = \mathcal{E}_{1,i} \left[\sum_{k \geq 1} \mathcal{A}_k^{(i)} f(\mathcal{A}_k^{(i)}) g\left(\sum_{l \neq k} \mathcal{A}_l^{(i)} M_l \right) h(M_k) M_k \right].$$

Using the independence of the $M_k, k \geq 1$, we retrieve (5.22). □

Given that (5.21) holds, for all $i \in \mathcal{I}$, it will be convenient to denote by $M^{(i)}$ a random variable with the same law as $\widetilde{\mathcal{M}}^-(\infty)$ under $\widehat{\mathcal{P}}_{1,i}^-$, defined on the same probability space and independent of all other quantities. We are then interested in the set of equations

$$M^{(i)} \stackrel{\mathcal{L}}{=} A^{(i)} M^{(\widehat{\mathcal{J}}_1)} + B^{(i)},$$

for all $i \in \mathcal{I}$. This recasts (5.21) in the framework of (5.20). In order to apply Proposition 5.10, we will need the following moment bounds on $B^{(i)}$.

Lemma 5.12. *We have the following upper-bounds on* $\widehat{\mathcal{E}}_{1,i}[|B^{(i)}|^p], p > 0$:

- *If* $p \leq 1$, *then* $\widehat{\mathcal{E}}_{1,i}[|B^{(i)}|^p] \leq \mathcal{E}_{1,i}\left[\left(\sum_{k \geq 1} \mathcal{A}_k^{(i)}\right)^{p+1}\right]$;
- *If* $p > 1$, *then* $\widehat{\mathcal{E}}_{1,i}[|B^{(i)}|^p] \leq \mathcal{E}_{1,i}\left[\left(\sum_{k \geq 1} \mathcal{A}_k^{(i)}\right)^{p+1}\right] \cdot \max_{j \in \mathcal{I}} \mathcal{E}_{1,j}[|M^{(j)}|^p]$.

Furthermore, if $(w_i, i \in \mathcal{I})$ *is a positive vector and we set* $c_i := \frac{w_i^{1/p}}{v_i^{1+1/p}}$ *for* $p > 0$, *then*

$$\widehat{\mathcal{E}}_{1,i} \left[\left(\frac{c_{\widehat{\mathcal{J}}_1}}{c_i} A^{(i)} \right)^p \right] = \mathcal{E}_{1,i} \left[\sum_{k \geq 1} \frac{w_{\mathcal{J}_k(0)}}{w_i} |\mathcal{X}_k(0)|^{(p+1)\omega_-} \right].$$

The proof is adapted from Liu (2000, Lemma 4.2).

Proof: We use (5.22) with $f = h = 1$ and $g : x \mapsto x^p$:

$$\widehat{\mathcal{E}}_{1,i}^- \left[|B^{(i)}|^p \right] = \mathcal{E}_{1,i} \left[\sum_{k \geq 1} \mathcal{A}_k^{(i)} \left(\sum_{l \neq k} \mathcal{A}_l^{(i)} M_l \right)^p \right].$$

Assume $p \le 1$. By the conditional Jensen inequality for $p \le 1$,

$$\widehat{\mathcal{E}}^{-}_{1,i}\left[|B^{(i)}|^p\right] \le \mathcal{E}_{1,i}\left[\sum_{k\ge 1}\mathcal{A}^{(i)}_k \mathcal{E}_{1,i}\left[\sum_{l\neq k}\mathcal{A}^{(i)}_l M_l \,\middle|\, \mathfrak{G}_1\right]^p\right]$$
$$\le \mathcal{E}_{1,i}\left[\sum_{k\ge 1}\mathcal{A}^{(i)}_k\left(\sum_{l\neq k}\mathcal{A}^{(i)}_l\right)^p\right],$$

since $\mathcal{E}_{1,i}[M_l \,|\, \mathfrak{G}_1] = \mathcal{E}_{1,i}\left[\mathcal{E}_{1,\mathcal{J}_l(0)}\left[\widetilde{\mathcal{M}}^{-}(\infty)\right]\right] = 1$ for all l (recall from Proposition 4.9(ii) that $\mathcal{M}(\infty)$ is a limit in L^1). Hence

$$\widehat{\mathcal{E}}^{-}_{1,i}\left[|B^{(i)}|^p\right] \le \mathcal{E}_{1,i}\left[\left(\sum_{k\ge 1}\mathcal{A}^{(i)}_k\right)^{p+1}\right].$$

Now assume $p > 1$. We use Jensen's inequality on the sum:

$$\left(\sum_{k\ge 1} a_k z_k\right)^p \le \sum_{k\ge 1} a_k z_k^p \quad \text{if } \sum_{k\ge 1} a_k = 1.$$

This gives

$$\widehat{\mathcal{E}}^{-}_{1,i}\left[|B^{(i)}|^p\right] \le \mathcal{E}_{1,i}\left[\sum_{k\ge 1}\mathcal{A}^{(i)}_k\left(\sum_{l\neq k}\mathcal{A}^{(i)}_l\right)^{p-1}\cdot\sum_{l\neq k}\mathcal{A}^{(i)}_l M_l^p\right]$$
$$\le \mathcal{E}_{1,i}\left[\left(\sum_{k\ge 1}\mathcal{A}^{(i)}_k\right)^{p}\cdot\sum_{l\neq k}\mathcal{A}^{(i)}_l \mathcal{E}_{1,\mathcal{J}_l(0)}\left[M_l^p\right]\right]$$
$$\le \mathcal{E}_{1,i}\left[\left(\sum_{k\ge 1}\mathcal{A}^{(i)}_k\right)^{p+1}\right]\cdot \max_{j\in\mathcal{I}}\mathcal{E}_{1,j}\left[|M^{(j)}|^p\right].$$

The second claim about $A^{(i)}$ is straightforward. □

Proof of Proposition 5.8. First of all, notice that the recursion (5.21) satisfies the assumptions of Lemma 5.9. Indeed, we assumed that for all $i \in \mathcal{I}$,

$$\widehat{\mathcal{E}}_{1,i}\left[\log\left(\frac{v_i}{v_{\widehat{\mathcal{J}}_1}}A^{(i)}\right)\right] = \omega_{-}\mathcal{E}_{1,i}\left[\sum_{k\ge 1}\frac{v_{\mathcal{J}_k(0)}}{v_i}|\mathcal{X}_k(0)|^{\omega_-}\log|\mathcal{X}_k(0)|\right] \in (-\infty, 0),$$

which is a stronger requirement. Then, we use formula (5.22) to see that

$$\widehat{\mathcal{E}}_{1,i}\left[|M^{(i)}|^p\right] = \widehat{\mathcal{E}}_{1,i}\left[\mathcal{E}_{1,\widehat{\mathcal{J}}_1}\left[|M^{(i)}|^{p+1}\right]\right]. \tag{5.23}$$

Now from Proposition 5.10 and Lemma 5.12, we see that the conditions

$$\mathcal{E}_{1,i}\left[|M^{(i)}|^p\right] < \infty, \quad \mathcal{E}_{1,i}\left[\left(\sum_{k\ge 1}\mathcal{A}^{(i)}_k\right)^{p+1}\right] < \infty \quad \text{and} \quad \mathcal{E}_{1,i}\left[\sum_{k\ge 1}\frac{w_{\mathcal{J}_k(0)}}{w_i}|\mathcal{X}_k(0)|^{(p+1)\omega_-}\right] < 1, \tag{5.24}$$

for all $i \in \mathcal{I}$, for some positive vector w and $p > 0$, imply that $\widehat{\mathcal{E}}_{1,i}[|M^{(i)}|^p] < \infty$ and hence by (5.23), $\mathcal{E}_{1,i}[|M^{(i)}|^{p+1}] < \infty$ for all $i \in \mathcal{I}$.

Assume that $\omega_+/\omega_- \in (1,2]$, and consider $p < \omega_+/\omega_- - 1$. We now check the three assumptions in (5.24). Since $p \le 1$ and $\mathcal{E}_{1,i}[|M^{(i)}|] < \infty$, we have $\mathcal{E}_{1,i}[|M^{(i)}|^p] < \infty$. In addition, since $p+1 < \omega_+/\omega_-$, $\mathcal{E}_{1,i}\left[\left(\sum_{k\ge 1}\mathcal{A}_k^{(i)}\right)^{p+1}\right] < \infty$ by the first claim of Proposition 4.9. Finally, we know that $\omega_- < (p+1)\omega_- < \omega_+$, and that $\lambda(\omega_-) = \lambda(\omega_+) = 0$. By convexity of λ, $\lambda((p+1)\omega_-) < 0$. If we choose w to be a positive eigenvector of $m((p+1)\omega_-)$ associated with the leading eigenvalue, we therefore get

$$\mathcal{E}_{1,i}\left[\sum_{k\ge 1}\frac{w_{\mathcal{J}_k(0)}}{w_i}|\mathcal{X}_k(0)|^{(p+1)\omega_-}\right] < 1.$$

This altogether triggers $\mathcal{E}_{1,i}[|M^{(i)}|^{p+1}] < \infty$ for all $i \in \mathcal{I}$, and hence Proposition 5.8 follows.

The general claim follows by recursion. For example, if $\omega_+/\omega_- \in (2,3]$, we first leverage the previous arguments for $p=1$ and obtain that $\mathcal{E}_{1,i}[|M^{(i)}|^2] < \infty$. Taking any $\beta < \omega_+/\omega_- - 1$, the very same arguments provide $\mathcal{E}_{1,i}[|M^{(i)}|^{\beta+1}] < \infty$ (note that the first step for $p=1$ is here used to deduce that $\mathcal{E}_{1,i}[|M^{(i)}|^{\beta}] < \infty$). This concludes the proof of Proposition 5.8.

5.5. *Proof of Theorem 2.9.* Our arguments rely heavily on the seminal work of Bertoin and Yor (2002) where the entrance law of positive self-similar Markov processes is determined. Before we prove Theorem 2.9, let us first discuss a duality property for self-similar Markov processes with types and establish some nice properties of their associated resolvent operators.

The dual of self-similar Markov processes with types (X,J) is defined as follows. First, we set

$$\varphi^\natural(t) := \inf\left\{s>0,\ \int_0^s \exp(-\alpha\xi(u))\mathrm{d}u > t\right\}, \quad t \ge 0,$$

and then we define for $x>0$ the pair $(X^\natural, J^\natural)$ where

$$X^\natural(t) := x\exp(-\xi(\varphi^\natural(tx^{-\alpha}))) \quad \text{and} \quad J^\natural(t) := \Theta^\natural(\varphi^\natural(tx^{-\alpha})), \quad t \ge 0,$$

where $\Theta^\natural$ is a Markov chain with intensity matrix $Q^\natural$ defined in (2.3) and with the convention that $(X^\natural(t), J^\natural(t)) = \partial$ when $t \ge \zeta^\natural$ where

$$\zeta^\natural := x^\alpha \int_0^\infty \exp(-\alpha\xi(u))\mathrm{d}u.$$

For $x>0$ and $j \in \mathcal{I}$, we denote by $\mathbb{P}^\natural_{x,j}$ for the distribution of the pair $(X^\natural, J^\natural)$.

Recall that for $q \ge 0$ and a measurable $f : \mathbb{R}_+ \times \mathcal{I} \to \mathbb{R}_+$, the resolvent operators are given by

$$V^{(q)}f(x,j) = \mathbb{E}_{x,j}\left[\int_0^\infty e^{-qt} f(X(t),J(t))\mathrm{d}t\right] \quad \text{and} \quad V^{\natural,(q)}f(x,j) = \mathbb{E}^\natural_{x,j}\left[\int_0^\zeta e^{-qt} f(X(t),J(t))\mathrm{d}t\right]. \tag{5.25}$$

From Lemma 2.2, we deduce the following result.

Proposition 5.13. *For every $q \ge 0$ and every measurable functions $f, g : \mathbb{R}\times\mathcal{I} \to \mathbb{R}$, we have*

$$\sum_{j\in\mathcal{I}}\int_0^\infty f(x,j)V^{(q)}g(x,j)\mu(\mathrm{d}x,j) = \sum_{k\in\mathcal{I}}\int_0^\infty g(x,k)V^{\natural,(q)}f(x,k)\mu(\mathrm{d}x,k), \tag{5.26}$$

where

$$\mu(\mathrm{d}x,j) = x^{\alpha-1}\pi_j\mathrm{d}x, \qquad \text{for} \quad x>0 \text{ and } j\in\mathcal{I}.$$

Proof: Using the Lamperti-type transform of (X,J) in (2.8), we first observe

$$V^{(q)}g(x,j) = \mathtt{E}_j\left[\int_0^\infty e^{-qx^\alpha I_u(\xi)} g(xe^{\xi(u)},\Theta(u))x^\alpha e^{\alpha\xi(u)}\mathrm{d}u\right],$$

where

$$I_u(\xi) = \int_0^u \exp(\alpha\xi(s))\mathrm{d}s.$$

Thus, it is clear

$$\begin{aligned}\sum_{j\in\mathcal{I}} \int_0^\infty f(x,j)V^{(q)}g(x,j)\mu(\mathrm{d}x,j) \\ &= \sum_{j\in\mathcal{I}} \pi_j \mathtt{E}_j\left[\int_0^\infty \int_0^\infty x^{\alpha-1} f(x,j)e^{-qx^\alpha I_u(\xi)} g(xe^{\xi(u)},\Theta(u))x^\alpha e^{\alpha\xi(u)}\mathrm{d}u\mathrm{d}x\right] \\ &= \sum_{j\in\mathcal{I}} \pi_j \int_0^\infty y^{\alpha-1}\int_0^\infty \mathtt{E}_j\left[g(y,\Theta(u))e^{-qy^\alpha e^{-\alpha\xi(u)}I_u(\xi)} f(ye^{-\xi(u)},j)y^\alpha e^{-\alpha\xi(u)}\mathrm{d}u\right]\mathrm{d}y.\end{aligned}$$

From Lemma 2.2, we deduce

$$\begin{aligned}\sum_{j\in\mathcal{I}} \pi_j \mathtt{E}_j\left[g(y,\Theta'(0))e^{-qy^\alpha I'_u(\xi)} f(ye^{\xi'(u)},\Theta'(u))y^\alpha e^{\alpha\xi'(u)}\right] \\ = \sum_{k\in\mathcal{I}} \pi_k g(y,k)\mathtt{E}^\natural_k\left[e^{-qy^\alpha I_u(\xi)} f(ye^{\xi(u)},\Theta(u))y^\alpha e^{\alpha\xi(u)}\right],\end{aligned}$$

where $\xi'(s) = \xi(u-s) - \xi(u)$ and $\Theta'(s) = \Theta(u-s)$. Therefore

$$\begin{aligned}\sum_{j\in\mathcal{I}} \int_0^\infty f(x,j)V^{(q)}g(x,j)\mu(\mathrm{d}x,j) \\ = \sum_{k\in\mathcal{I}} \pi_k \int_0^\infty y^{\alpha-1}g(y,k)\int_0^\infty \mathtt{E}^\natural_k\left[e^{-qy^\alpha I_u(\xi)} f(ye^{\xi(u)},\Theta(u))y^\alpha e^{\alpha\xi(u)}\right]\mathrm{d}u\mathrm{d}y.\end{aligned}$$

Finally, using the Lamperti-type transform of (X,J), under $\mathbb{P}^\natural_{y,k}$, we observe

$$\begin{aligned}\sum_{k\in\mathcal{I}} \int_0^\infty g(y,k)V^{\natural,(q)}f(y,k)\mu(\mathrm{d}y,k) \\ = \sum_{k\in\mathcal{I}} \pi_k \int_0^\infty y^{\alpha-1}g(y,k)\int_0^\infty \mathtt{E}^\natural_k\left[e^{-qy^\alpha I_u(\xi)} f(ye^{\xi(u)},\Theta(u))y^\alpha e^{\alpha\xi(u)}\right]\mathrm{d}u\mathrm{d}y.\end{aligned}$$

Hence putting all the pieces together, we observe that (5.26) holds. □

The key to the proof of Theorem 2.9 is a consequence of the Markov additive renewal theorem for the potential measure of the MAP (ξ,Θ). There is a relatively wide body of literature concerning Markov additive renewal theory, see for instance Alsmeyer (1994, 2014); Kesten (1974); Lalley (1984). Although the literature mostly deals with the case of discrete-time, one can nonetheless identify the following renewal-type theorem for the potential measure $U_{ij}(\mathrm{d}x)$, defined as

$$U_{ij}(\mathrm{d}x) = \mathtt{E}_i\left[\int_0^\infty \mathbf{1}_{\{\xi(t)\in\mathrm{d}x,\Theta(t)=j\}}\mathrm{d}t\right], \qquad x\in\mathbb{R}.$$

To give a precise statement, let us recall that a function $g:\mathbb{R}\to\mathbb{R}$ is said to be directly Riemann integrable if

$$\lim_{n\to\infty} \underline{g_n}(y) = \lim_{n\to\infty} \overline{g_n}(y) = g(y) \qquad \text{is integrable,}$$

where for every integer n, $\underline{g_n}$ (respectively $\overline{g_n}$) denotes the largest (respectively, the smallest) function with $\underline{g_n} \le g$ (respectively $g \le \overline{g_n}$) which is constant on the intervals $[k2^{-n},(k+1)2^{-n})$, for every $k\in\mathbb{Z}$. We may now state the following

Theorem 5.14. *Assume that ξ is not concentrated on a lattice and has positive mean*

$$0 < m = \mathtt{E}_{0,\pi}[\xi(1)] < \infty.$$

Then the following hold

(i) *For all $i, j \in \mathcal{I}$,*

$$\lim_{x\to\infty} \frac{Ui,j([0,x])}{x} = \frac{\pi_j}{m}.$$

(ii) *For $x > 0$ and $i, j \in \mathcal{E}$, the weak limit of the overshoot $(\xi(\tau_a^+) - a, \Theta(\tau_a^+))$ exists as a goes to ∞, where $\tau_z^+ = \inf\{t : \xi(t) \geq z\}$. More precisely*

$$\nu(\mathrm{d}x, j) := w - \lim_{a\to\infty} \mathtt{P}_i\Big(\xi(\tau_a^+) - a \in \mathrm{d}x, \Theta(\tau_a^+) = j\Big),$$

where $w - \lim$ denotes weak limit of probability measures.

(iii) *Let us assume that for each $j \in \mathcal{I}$, $g(\cdot, j)$ is direct Riemann integrable, then*

$$\lim_{z\to\infty} \sum_{j\in\mathcal{I}} \int_{\mathbb{R}} g(y - z, j) U_{i,j}(\mathrm{d}y) = \frac{1}{m} \sum_{j\in\mathcal{I}} \pi_j \int_{\mathbb{R}} g(x, j)\mathrm{d}x.$$

Part (i) and (ii) is the continuous-time analogue of the Markov additive renewal theorem in Lalley (1984). Both follow from Theorem 28 in Dereich et al. (2017). Part (iii) follows from similar arguments as in Alsmeyer (2014).

With the previous result in hand, we may now state the following result.

Lemma 5.15. *Let $f : (0,\infty) \times \mathcal{I} \to \mathbb{R}$ be such that for each $j \in \mathcal{I}$, $y \mapsto e^{\alpha y} f(e^y, j)$ is directly Riemann integrable. Then*

$$\lim_{x\to 0+} V^{(0)} f(x, i) = \frac{1}{m} \sum_{j\in\mathcal{I}} \pi_j \int_0^\infty f(y, j) y^{\alpha-1} \mathrm{d}y$$

Proof: Observe, from the Lamperti-type transform of (X, J) that the potential $V^{(0)}$ can be expressed in terms of the potential operator of (ξ, Θ) as follows

$$\begin{aligned} V^{(0)} f(x, i) &= \mathtt{E}_i \left[\int_0^\infty f(e^{\xi(t)+\log x}, \Theta(t)) e^{\alpha(\xi(t)+\log x)} \mathrm{d}t \right] \\ &= \sum_{j\in\mathcal{I}} \int_{\mathbb{R}} f(e^{y+\log x}, j) e^{\alpha(y+\log x)} U_{i,j}(\mathrm{d}y). \end{aligned}$$

Hence from Theorem 5.14 part (iii), we deduce

$$\lim_{x\to 0+} V^{(0)} f(x, i) = \frac{1}{m} \sum_{j\in\mathcal{I}} \pi_j \int_{-\infty}^\infty f(e^y, j) e^{\alpha y} \mathrm{d}y = \frac{1}{m} \sum_{j\in\mathcal{I}} \pi_j \int_0^\infty f(z, j) z^{\alpha-1} \mathrm{d}z,$$

as expected. □

Next, we study regularity properties of the resolvent $V^{(q)}$.

Lemma 5.16. *For each $i \in \mathcal{I}$, we assume that $f(\cdot, i) : \mathbb{R}_+ \to \mathbb{R}$ is a bounded continuous function. Then for every $q > 0$, $V^{(q)} f(\cdot, i)$ is continuous and bounded on $(0,\infty)$. Moreover, if $f(\cdot, i)$ has compact support, then the function $y \mapsto e^{\alpha y} V^{(q)} f(e^y, i)$ is directly Riemann integrable on $\mathbb{R}$.*

Proof: Since the MAP (ξ, Θ) is a Feller process with càdlàg paths then, for each $i \in \mathcal{I}$ the map $x \mapsto \mathbb{E}_{x,i}[f(X(t), J(t))] = \mathtt{E}_i \left[f(xe^{\xi(\varphi(tx^{-\alpha})}), \Theta(\varphi(tx^{-\alpha}))) \right]$ is continuous and bounded on $(0,\infty)$, for each $i \in \mathcal{I}$, and in particular the same holds for $x \mapsto V^{(q)} f(x, i)$, for $q > 0$.

For the direct Riemann integrability property, and without loss of generality, we assume that $|f(x,i)| \le \mathbf{1}_{[0,1]\times\mathcal{I}}(x,i)$. We also introduce the step function

$$g(x,i) := \inf_{y\in[k-2,k-1]} e^{\alpha y} V^{(q)} \mathbf{1}_{[0,1]\times\mathcal{I}}(e^y,i) \qquad \text{for every } x \in [k,k+1) \text{ and } k \in \mathbb{Z}.$$

and observe from Proposition 5.13 that

$$\begin{aligned}\int_{-\infty}^{\infty} g(x,i)\mathrm{d}x &\le \int_{-\infty}^{\infty} e^{\alpha y} V^{(q)} \mathbf{1}_{[0,1]\times\mathcal{I}}(e^y,i)\mathrm{d}y \\ &\le C\sum_{j\in\mathcal{I}} \pi_j \int_0^{\infty} V^{(q)}\mathbf{1}_{[0,1]\times\mathcal{I}}(x,i)x^{\alpha-1}\mathrm{d}x \\ &= C\sum_{k\in\mathcal{I}} \pi_k \int_0^1 V^{\natural,(q)}\mathbf{1}_{[0,\infty)\times\mathcal{I}}(y,k)y^{\alpha-1}\mathrm{d}y < \infty,\end{aligned}$$

where $C = \max_{j\in\mathcal{I}} \frac{1}{\pi_j}$. Next, for $y \le x$, we observe from the scaling property that

$$\begin{aligned}|V^{(q)} f(x,i)| &\le V^{(q)}\mathbf{1}_{[0,1]\times\mathcal{I}}(x,i) \\ &\le \int_0^{\infty} e^{-qt}\mathbb{P}_{x,i}(X(t)\le 1)\mathrm{d}t \\ &= \int_0^{\infty} e^{-qt}\mathbb{P}_{1,i}(X(tx^{-\alpha})\le 1/x)\mathrm{d}t \\ &\le \int_0^{\infty} e^{-qt}\mathbb{P}_{1,i}(X(tx^{-\alpha})\le 1/y)\mathrm{d}t \\ &= \int_0^{\infty} e^{-qt}\mathbb{P}_{y,i}(X(t(x/y)^{-\alpha})\le 1)\mathrm{d}t \\ &= \left(\frac{x}{y}\right)^{\alpha} V^{(q)}\mathbf{1}_{[0,1]\times\mathcal{I}}(y,i).\end{aligned}$$

In other words, we get $e^{\alpha x}|V^{(q)} f(e^x,i)| \le e^{4\alpha} g(x,i)$, for every $x \in \mathbb{R}$. Since the function $x \mapsto e^{\alpha x} V^{(q)} f(e^x,i)$ is continuous, the direct Riemann integrability follows. □

Finally, we present a tightness property for the distribution of self-similar Markov processes with types.

Lemma 5.17. *Let T be a random time with an exponential distribution that is independent of the self-similar Markov process with types (X,J). Then the familly of probability measures on $[0,\infty)\times\mathcal{I}$, $\{\mathbb{P}_{x,j}((X(T),J(T))\in\cdot), 0<x\le 1, j\in\mathcal{I}\}$ is tight.*

Proof: Let $\overline{X}$ be the supremum process of X, that is

$$\overline{X}(t) = \sup_{u\le t} X(u), \qquad t \ge 0,$$

and let $\sigma^+(\cdot) = \inf\{t\ge 0 : \overline{X}(t) > \cdot\}$ be its right-continuous inverse. For $x\in(0,1]$ and $k\ge 1$, it is clear that

$$\mathbb{P}_{x,i}\Big(\overline{X}(T)\le k^3\Big) \le \mathbb{P}_{x,i}(X(T)\le k^3).$$

On the other hand, using the Markov property (in the third line),

$$\begin{aligned}\mathbb{P}_{x,i}\Big(\overline{X}(T) \le k^3\Big) &\ge \mathbb{P}_{x,i}\Big(\overline{X}(\sigma^+(k)+T) \le k^3\Big) \\ &\ge \mathbb{P}_{x,i}\Big(\overline{X}(\sigma^+(k)+T) \le k^3, X(\sigma^+(k)) \le k^2\Big) \\ &= \sum_{j\in\mathcal{I}} \int_{[k,k^2]} \mathbb{P}_{x,i}(X(\sigma^+(k)) \in \mathrm{d}y, J(\sigma^+(k)) = j) \\ &\qquad\qquad \times \mathbb{P}_{y,j}\Big(\overline{X}(T) \le k^3\Big) \\ &\ge \mathbb{P}_{x,i}(X(\sigma^+(k)) \le k^2) \inf_{y\in[k,k^2], j\in\mathcal{I}} \mathbb{P}_{y,j}\Big(\overline{X}(T) \le k^3\Big).\end{aligned}$$

From the scaling property, for every $y \in [k, k^2]$,

$$\begin{aligned}\mathbb{P}_{y,j}\Big(\overline{X}(T) \le k^3\Big) &= \mathbb{P}_{1,j}\Big(\overline{X}(Ty^{-\alpha}) \le k^3/y\Big) \\ &\ge \mathbb{P}_{1,j}\Big(\overline{X}(T) \le k\Big),\end{aligned}$$

which is close to 1, uniformly for $y \in [k, k^2]$, by taking k large enough.

Moreover, recalling the notation τ_a^+ in Theorem 5.14, from the Lamperti-type transform and Theorem 5.14 we may deduce that

$$\begin{aligned}\mathbb{P}_{x,i}(X(\sigma^+(k)) \le k^2) &= \sum_{j\in\mathcal{I}} \mathbb{P}_{x,i}(X_{\sigma^+(k)} \le k^2, J_{\sigma^+(k)} = j) \\ &= \sum_{j\in\mathcal{I}} \mathsf{P}_i\Big(\xi(\tau^+_{\log(k/x)}) \le \log(k^2/x), \Theta(\tau^+_{\log(k/x)}) = j\Big) \xrightarrow[k\to\infty]{} 1.\end{aligned}$$

Putting all pieces together, allow us to deduce our claim, with a choice of compact given by $[0, k^3] \times \mathcal{I}$. □

We are now ready to prove Theorem 2.9.

Proof of Theorem 2.9: For each $i \in \mathcal{I}$, we let $f(\cdot, i) : [0, \infty) \to \mathbb{R}$ be a continuous function with compact support and $q > 0$. It is clear that $y \mapsto e^{\alpha y} f(e^y, i)$ is directly Riemann integrable and from Lemma 5.16, $y \mapsto e^{\alpha y} V^{(q)} f(e^y, i)$ is also directly Riemann integrable. Thus from Lemma 5.15, we have

$$\lim_{x\to 0+} V^{(0)} f(x,i) = \frac{1}{m} \sum_{j\in\mathcal{I}} \pi_j \int_0^\infty f(y,j) y^{\alpha-1} \mathrm{d}y$$

and

$$\lim_{x\to 0+} V^{(0)} V^{(q)} f(x,i) = \frac{1}{m} \sum_{j\in\mathcal{I}} \pi_j \int_0^\infty V^{(q)} f(y,j) y^{\alpha-1} \mathrm{d}y.$$

From the resolvent equation $V^{(q)}f(x,i) = V^{(0)}f(x,i) - qV^{(0)}V^{(q)}f(x,i)$, see for instance identity (2.7) in Chapter 1 in Ethier and Kurtz (1986), Proposition 5.13 and identity (2.9), we deduce

$$\begin{aligned}
\lim_{x\to 0+} V^{(q)}f(x,i) &= \frac{1}{m}\sum_{j\in\mathcal{I}}\pi_j\int_0^\infty \Big(f(y,j) - qV^{(q)}f(y,j)\Big)y^{\alpha-1}\mathrm{d}y \\
&= \frac{1}{m}\sum_{j\in\mathcal{I}}\pi_j\int_0^\infty f(y,j)\Big(1 - qV^{\natural,(q)}\mathbf{1}(y,j)\Big)y^{\alpha-1}\mathrm{d}y \\
&= \frac{1}{m}\sum_{j\in\mathcal{I}}\pi_j\int_0^\infty f(y,j)\mathbb{E}^\natural_{y,j}[e^{-q\zeta}]y^{\alpha-1}\mathrm{d}y \\
&= \frac{1}{m}\sum_{j\in\mathcal{I}}\pi_j\int_0^\infty f(y,j)\mathtt{E}^\natural_j\Big[e^{-qy^\alpha I(\alpha\xi)}\Big]y^{\alpha-1}\mathrm{d}y.
\end{aligned}$$

Here we used the definition of $\overline{V}^{(q)}$ (see (5.25)) in the third line. Using Lemma 5.17, we observe that

$$\alpha m = \sum_{j\in\mathcal{I}}\pi_j\mathtt{E}^\natural_j\left[\frac{1}{I(\alpha\xi)}\right],$$

and that for every continuous $f(\cdot,i) : [0,\infty) \to \mathbb{R}$ with compact support,

$$\begin{aligned}
\lim_{x\to 0+} V^{(q)}f(x,i) &= \frac{1}{m}\sum_{j\in\mathcal{I}}\pi_j\int_0^\infty f(y,j)\mathtt{E}^\natural_j\Big[e^{-qy^\alpha I(\alpha\xi)}\Big]y^{\alpha-1}\mathrm{d}y \\
&= \frac{1}{m}\sum_{j\in\mathcal{I}}\pi_j\int_0^\infty \mathtt{E}^\natural_j\left[f\left(\left(\frac{u}{I(\alpha\xi)}\right)^{1/\alpha}, j\right)\frac{1}{I(\alpha\xi)}\right]e^{-qu}\mathrm{d}u.
\end{aligned}$$

This completes the proof. □

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Index of notation

$\mathcal{I}$	Finite set of types
(ξ, Θ)	Markov additive process (MAP) in $\mathbb{R} \times \mathcal{I}$
$\mathtt{P}_{\mathtt{x},\theta}$	Law of (ξ, Θ) started from (x, θ)
$(\xi_i, U_{i,j}, q_{i,j})$	Characteristics of the MAP
$F(z)$	Matrix exponent of the MAP (ξ, Θ)
$\chi(z)$	Leading eigenvalue of $F(z)$
$\natural$	Denotes quantities associated with the dual MAP
(X, J)	Self-similar Markov process with types
$\varphi(t)$	Lamperti time-change
$I(\xi)$	Exponential functional
$(\eta_t, t > 0)$	Entrance law of (X, J)
$\Lambda_i^{(k)}$	Lévy measure of ξ_i associated to type k
$\Lambda_{U_{i,j}}^{(k)}$	Jump measure of $U_{i,j}$ associated to type k
$J_\Delta(s)$	Type corresponding to the jump at type s
$\mathcal{P}_{x,i}$	Law of the growth-fragmentation cell system $(\mathcal{X}_u, \mathcal{J}_u, u \in \mathbb{U})$
$(\mathcal{X}_u, \mathcal{J}_u)$	Mass and type of particle indexed by u
b_u, ζ_u	Birth and lifetime of particle indexed by u
$\mathbf{P}_{x,i}$	Law of the growth-fragmentation process $\mathbf{X}$
$m(q) := (m_{i,j}(q))_{i,j\in\mathcal{I}}$	Cumulant matrix of jumps
$((v_i)_{i\in\mathcal{I}}, \omega)$	Admissible sequence
$\mathcal{M}(n)$	Genealogical martingale
$\widehat{\mathcal{P}}_{x,i}$	Change of measures induced by $\mathcal{M}$
$\mathcal{L}$	Tagged cell or spine
$(\widehat{\mathcal{X}}, \widehat{\mathcal{J}})$	Mass and type of the tagged cell
$\mathcal{M}^-(\infty)$	Limit of the martingale $(\mathcal{M}^-(n), n \geq 0)$
$\rho^{(\alpha)}$	Empirical measure
$\mathcal{K}_i(q)$	Multitype cumulant function corresponding to type i
C_k, R_k, J_k	Random variables involved in the smoothing transform
C_u, R_u, J_u	Tree version of C_k, R_k, J_k
A, B	Random variables involved in the random affine equation
$A^{(i)}, B^{(i)}, J^{(i)}$	Random variables involved in the multitype linear recursion
$(\pi_k, k \in \mathcal{I})$	Invariant law of J
$V^{(q)}$	Resolvent operators

References

Aïdékon, E. and Da Silva, W. Growth-fragmentation process embedded in a planar Brownian excursion. *Probab. Theory Related Fields*, **183** (1-2), 125–166 (2022). MR4421172.

Alili, L., Chaumont, L., Graczyk, P., and Żak, T. Inversion, duality and Doob h-transforms for self-similar Markov processes. *Electron. J. Probab.*, **22**, Paper No. 20, 18 (2017). MR3622890.

Alili, L. and Woodford, D. On the finiteness and tails of perpetuities under a Lamperti-Kiu MAP. *J. Appl. Probab.*, **58** (4), 1086–1113 (2021). MR4342595.

Alsmeyer, G. On the Markov renewal theorem. *Stochastic Process. Appl.*, **50** (1), 37–56 (1994). MR1262329.

Alsmeyer, G. Quasistochastic matrices and Markov renewal theory. *J. Appl. Probab.*, **51A** (Celebrating 50 Years of The Applied Probability Trust), 359–376 (2014). MR3317369.

Asmussen, S. *Applied probability and queues*, volume 51 of *Applications of Mathematics. Stochastic Modelling and Applied Probability.* Springer-Verlag, New York, second edition (2003). ISBN 0-387-00211-1. MR1978607.

Athreya, K. B., McDonald, D., and Ney, P. Limit theorems for semi-Markov processes and renewal theory for Markov chains. *Ann. Probab.*, **6** (5), 788–797 (1978). MR503952.

Barker, A. and Savov, M. Bivariate Bernstein-gamma functions and moments of exponential functionals of subordinators. *Stochastic Process. Appl.*, **131**, 454–497 (2021). MR4165648.

Behme, A., Lindner, A., and Reker, J. On the law of killed exponential functionals. *Electron. J. Probab.*, **26**, Paper No. 60, 35 (2021). MR4254802.

Behme, A. and Sideris, A. Exponential functionals of Markov additive processes. *Electron. J. Probab.*, **25**, Paper No. 37, 25 (2020). MR4089787.

Bertoin, J. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2006). ISBN 978-0-521-86728-3; 0-521-86728-2. MR2253162.

Bertoin, J. Markovian growth-fragmentation processes. *Bernoulli*, **23** (2), 1082–1101 (2017). MR3606760.

Bertoin, J., Budd, T., Curien, N., and Kortchemski, I. Martingales in self-similar growth-fragmentations and their connections with random planar maps. *Probab. Theory Related Fields*, **172** (3-4), 663–724 (2018a). MR3877545.

Bertoin, J., Curien, N., and Kortchemski, I. Random planar maps and growth-fragmentations. *Ann. Probab.*, **46** (1), 207–260 (2018b). MR3758730.

Bertoin, J. and Yor, M. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, **17** (4), 389–400 (2002). MR1918243.

Bertoin, J. and Yor, M. Exponential functionals of Lévy processes. *Probab. Surv.*, **2**, 191–212 (2005). MR2178044.

Biggins, J. D. Martingale convergence in the branching random walk. *J. Appl. Probability*, **14** (1), 25–37 (1977). MR433619.

Budd, T. The peeling process of infinite Boltzmann planar maps. *Electron. J. Combin.*, **23** (1), Paper 1.28, 37 (2016). MR3484733.

Carmona, P., Petit, F., and Yor, M. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Bibl. Rev. Mat. Iberoamericana, pp. 73–130. Rev. Mat. Iberoamericana, Madrid (1997). MR1648657.

Çinlar, E. Lévy systems of Markov additive processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **31**, 175–185 (75). MR370788.

Chaumont, L., Pantí, H., and Rivero, V. The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli*, **19** (5B), 2494–2523 (2013). MR3160562.

Da Silva, W. Self-similar signed growth-fragmentations. *Electron. J. Probab.*, **28**, Paper No. 49, 45 (2023). MR4567459.

Da Silva, W. and Pardo, J. C. Spatial growth-fragmentations and excursions from hyperplanes. *ArXiv Mathematics e-prints* (2023). arXiv: 2309.08459.

Dadoun, B. Asymptotics of self-similar growth-fragmentation processes. *Electron. J. Probab.*, **22**, Paper No. 27, 30 (2017). MR3629871.

Dereich, S., Döring, L., and Kyprianou, A. E. Real self-similar processes started from the origin. *Ann. Probab.*, **45** (3), 1952–2003 (2017). MR3650419.

Ethier, S. N. and Kurtz, T. G. *Markov processes. Characterization and convergence.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York (1986). ISBN 0-471-08186-8. MR838085.

Goldie, C. M. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, **1** (1), 126–166 (1991). MR1097468.

Grincevičjus, A. K. The continuity of the distribution of a certain sum of dependent variables that is connected with independent walks on lines. *Teor. Verojatnost. i Primenen.*, **19**, 163–168 (1974). MR345178.

Ivanovs, J. *One-sided Markov additive processes and related exit problems.* Ph.D. thesis, Universiteit van Amsterdam (2011). Available at https://hdl.handle.net/11245/1.360305.

Jelenković, P. R. and Olvera-Cravioto, M. Implicit renewal theory and power tails on trees. *Adv. in Appl. Probab.*, **44** (2), 528–561 (2012). MR2977407.

Kahane, J.-P. and Peyrière, J. Sur certaines martingales de Benoit Mandelbrot. *Advances in Math.*, **22** (2), 131–145 (1976). MR431355.

Kesten, H. Random difference equations and renewal theory for products of random matrices. *Acta Math.*, **131**, 207–248 (1973). MR440724.

Kesten, H. Renewal theory for functionals of a Markov chain with general state space. *Ann. Probability*, **2**, 355–386 (1974). MR365740.

Kingman, J. F. C. A convexity property of positive matrices. *Quart. J. Math. Oxford Ser. (2)*, **12**, 283–284 (1961). MR138632.

Kuznetsov, A., Kyprianou, A. E., Pardo, J. C., and Watson, A. R. The hitting time of zero for a stable process. *Electron. J. Probab.*, **19**, no. 30, 26 (2014). MR3183574.

Kyprianou, A. E. and Pardo, J. C. *Stable Lévy processes via Lamperti-type representations*, volume 7 of *Institute of Mathematical Statistics (IMS) Monographs.* Cambridge University Press, Cambridge (2022). ISBN 978-1-108-48029-1. MR4692990.

Kyprianou, A. E. and Rahimzadeh Sani, A. Martingale convergence and the functional equation in the multi-type branching random walk. *Bernoulli*, **7** (4), 593–604 (2001). MR1849369.

Lalley, S. P. Conditional Markov renewal theory. I. Finite and denumerable state space. *Ann. Probab.*, **12** (4), 1113–1148 (1984). MR757772.

Lamperti, J. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **22**, 205–225 (1972). MR307358.

Le Gall, J.-F. and Riera, A. Growth-fragmentation processes in Brownian motion indexed by the Brownian tree. *Ann. Probab.*, **48** (4), 1742–1784 (2020). MR4124524.

Liu, Q. On generalized multiplicative cascades. *Stochastic Process. Appl.*, **86** (2), 263–286 (2000). MR1741808.

Lyons, R. A simple path to Biggins' martingale convergence for branching random walk. In *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *IMA Vol. Math. Appl.*, pp. 217–221. Springer, New York (1997). MR1601749.

Mandelbrot, B. Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire. *C. R. Acad. Sci. Paris Sér. A*, **278**, 289–292 (1974). MR431351.

Miller, J., Sheffield, S., and Werner, W. Simple conformal loop ensembles on Liouville quantum gravity. *Ann. Probab.*, **50** (3), 905–949 (2022). MR4413208.

Pardo, J. C. and Rivero, V. Self-similar Markov processes. *Bol. Soc. Mat. Mexicana (3)*, **19** (2), 201–235 (2013). MR3183994.

Patie, P. and Savov, M. Bernstein-gamma functions and exponential functionals of Lévy processes. *Electron. J. Probab.*, **23**, Paper No. 75, 101 (2018). MR3835481.

Rivero, V. Tail asymptotics for exponential functionals of Lévy processes: the convolution equivalent case. *Ann. Inst. Henri Poincaré Probab. Stat.*, **48** (4), 1081–1102 (2012). MR3052404.

Rivero, V. Entrance laws for positive self-similar Markov processes. In *Mathematical Congress of the Americas*, volume 656 of *Contemp. Math.*, pp. 119–140. Amer. Math. Soc., Providence, RI (2016). MR3457599.

Stephenson, R. On the exponential functional of Markov additive processes, and applications to multi-type self-similar fragmentation processes and trees. *ALEA Lat. Am. J. Probab. Math. Stat.*, **15** (2), 1257–1292 (2018). MR3867206.