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A measure-valued stochastic model for vector-borne viruses

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Abstract. In this work we propose a measure-valued stochastic process representing the dynamics of a virus population, structured by phenotypic traits and geographical space, and where viruses are transported between spatial locations by vectors. We first show how to use this model to infer results on the probability of extinction of the virus population. Later, by combining various scalings on population sizes, speed of diffusion of vectors, and other relevant model parameters, we show the emergence of two systems of integro-differential equations (IDEs) as macroscopic descriptions of the system. Under the existence of densities at time zero, we also show the propagation of this property for later times, and derive the strong formulation of the limiting systems of IDEs. These strong formulations, in a sense, correspond to spatial Lotka-Volterra competition models with mutation and vector-borne dispersal.

1. Introduction

The selective pressure imposed on pathogens during the infection of a new type of host is one of the well known mechanisms that drive their evolution. For plant pathogens, plant genes conferring major or partial resistance to these pathogens are then valuable natural resources which in a context of a plant of agricultural interest may be used in a way that maximizes the preservation of their efficiency in conjunction with the gain of crop productivity they can ensure. To gain some insights on the possible optimized deployment strategies, the study of adaptation of pathogens to their hosts during repeated epidemic events is then of great importance. The understanding of such evolutionary processes is at the heart of evolutionary epidemiology research, see for example the review Restif (2009). In the evolutionary epidemiology literature, most models deal with either the adaptation process with simplified epidemic process, as described in Crow and Kimura (1970), or conversely focus on the epidemic process forgetting a proper description of the evolution processes at play. One

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of the main reasons comes from the modelling tools used to describe these two phenomena. Indeed, selection mutation population models are often used to describe evolutionary phenomena, often overlooking the potential spatial structuring of such epidemiological events. Conversely, epidemics are typically modeled using Susceptible-Exposed-Infectious-Removed (SEIR) type models, wherein the host serves as a surrogate for characterizing the epidemic dynamics, but these models often neglect the description of the evolving pathogen population.

Modelling approaches trying to reconcile these two points of view have been recently introduced Day and Proulx (2004); Day and Gandon (2007); Ohtsuki and Sasaki (2006); Lo Iacono et al. (2012); Djidjou-Demasse et al. (2017); Fabre et al. (2022) where the epidemic approach considering the host as a proxy has been adapted in order to take into account a partial description of the population that evolves and its possible adaptation by mutation. However, whereas this approach seems well suited for pathogens that behave like spores, it does not seems flexible enough to describe viral populations that disseminate in a field through vectors. In addition, they do not take into account stochastic effects due to low population densities at the beginning of the epidemic.

In the spirit of Day and Proulx (2004); Day and Gandon (2007); Ohtsuki and Sasaki (2006); Lo Iacono et al. (2012); Djidjou-Demasse et al. (2017); Fabre et al. (2022), we propose and analyse here new modelling tools that integrate on the same time scale the epidemic events and the adaptation processes. However, unlike the approach proposed in Ohtsuki and Sasaki (2006); Lo Iacono et al. (2012); Djidjou-Demasse et al. (2017) that uses the hosts as a proxy, we fully describe the pathogen population with all its possible interactions making the assumption that the host is an environmental variable for the pathogen population. Our aim here is then to construct a realistic stochastic representation of the main processes of adaptation involved during an epidemic event, and to obtain various large population asymptotic limits depending on the choice of the scaling parameter considered. The stochastic nature of our model gives room to incorporate demographic stochasticity, which is an important feature to consider, since early stages affect later stages of the epidemic. Moreover, our model permits to study both quantitative and qualitative trait scenarios.

Some of the results we present here are in the spirit of the so-called hydrodynamic limits from the theory of interacting particle systems, for physics see for example De Masi and Presutti (1991); Kipnis and Landim (1999); Seppäläinen (2008), and for biology Fournier and Méléard (2004); Champagnat and Méléard (2007); Champagnat et al. (2008); Bansaye and Méléard (2015). In particular, we use the methodology introduced in Fournier and Méléard (2004), and in subsequent works (Champagnat and Méléard (2007); Champagnat et al. (2008); Bansaye and Méléard (2015)), to build a measure-valued stochastic model representing the dynamics of a virus population structured by time, space, and phenotypic traits, where viruses are subject to vector-borne dispersal, and can only reproduce in plants.

In the derivation of the scaling limits, our particular choice of re-scaling is important since it allows us to consider different scenarios that depend on the relation between the orders of magnitude of the total number of viruses and vectors at time zero. In particular, there are fewer vectors than viruses, in order to obtain a sensible limit, it becomes necessary to accelerate the diffusion of vectors. As the reader will see in Theorem 1.11 below, this acceleration has the consequence that for each $t > 0$ the population of vectors is given by the solution of an elliptic system which depends only on the current population of viruses at time t. We can interpret this as saying that the population of vectors is at an equilibrium that only changes through the time evolution of the virus population.

The viral epidemic model. Let us consider three populations, denoted by $\nu_v(t), \nu_c(t)$ and $\nu_u(t)$, representing the viral population on plants, the population of vectors that are charged with a virus, and the free vectors population, respectively. As in Fournier and Méléard (2004); Champagnat and Méléard (2007); Champagnat et al. (2008); Bansaye and Méléard (2015), let us first introduce, for a Polish space X, the set $\mathcal{M}_p(X)$ denoting the space of finite point measures on X. We define these populations by means of point measures on the adequate Polish spaces as follows. First, we consider the viral population $\nu_v(t)$ on plants. We represent this population by a point measure over the space $E \times \mathcal{X}$, where E is a finite set of positions in a bounded smooth, at least C^3, domain $\mathcal{D} \subset \mathbb{R}^d$ and $\mathcal{X} \subseteq \mathbb{R}^n$ is a compact set. The set E corresponds to the locations of the various host plants, and $\mathcal{X}$ corresponds to the phenotypic trait space of viruses. Thus, for $t \geq 0$,

$$\nu_v(t) = \sum_{i=1}^{N_v(t)} \delta_{x_i(t), z_i(t)}, \tag{1.1}$$

where $N_v(t) := \langle \nu_v(t), 1 \rangle$ is the total size of the virus population currently hosted by plants, and

$$\{(x_1(t), z_1(t)) \ldots, (x_{N_v(t)}(t), z_{N_v(t)}(t))\} \in E^{N_v(t)} \times \mathcal{X}^{N_v(t)}$$

correspond respectively to an arbitrary ordering of the positions and traits of the viruses.

Similarly, the uncharged vector population is represented by a finite point measure on $\mathcal{D}$, $\nu_u(t)$, taking the form

$$\nu_u(t) = \sum_{i=1}^{N_u(t)} \delta_{Y_i^u(t)},$$

where $N_u(t)$ is the number of uncharged vectors and $(Y_1^u(t), \ldots, Y_{N_u(t)}^u(t))$ are their positions at time $t \geq 0$. Finally, the population of vectors carrying viruses is represented by a measure $\nu_c(t) \in \mathcal{M}_p(\mathcal{D} \times \mathcal{X})$, of the form

$$\nu_c(t) = \sum_{i=1}^{N_c(t)} \delta_{Y_i^c(t), z_i(t)},$$

where, for $i \in \{1, \ldots, N_c(t)\}$, $Y_i^c(t)$ is the position of the vector and $z_i(t)$ is the trait of the unique virus it is carrying.

Figure 1.1 below, shows an example of the type of environment we have in mind. We have plotted plants in different colors to convey the idea that incorporating spatial dependency of some of the rates in this framework allows us to consider different varieties of plants. Notice that it is also possible to consider non-homogeneous (spatial) distributions of plants.

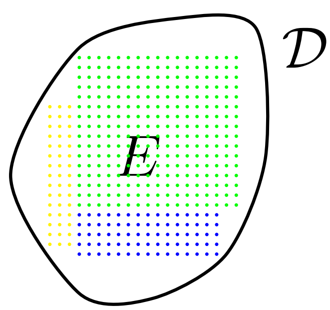

FIGURE 1.1. The spatial domain $\mathcal{D}$ represents the space in which vectors can diffuse (with normal reflection at the boundary). The colored dots represent the set E, i.e. the location of plants

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In addition to the definitions of the point measures $\nu_v(t), \nu_c(t)$ and $\nu_u(t)$, let us now define a specific set of biological rules that determine the way these measures evolve over time. To reflect the adaptation process during an epidemic event, we will assume that viruses undergo reproduction, death, mutation and transportation by vectors and that vectors move in $\mathcal{D}$ according to a reflected Itô diffusion and that they can load and unload viruses on plants. This translates into the following processes :

Reproduction: viruses on plants reproduce asexually at rate $b(x,z)$, where $x \in E$ denotes the position of the hosting plant and $z \in \mathcal{X} \subset \mathbb{R}^n$ the phenotypic trait of the virus. Given a reproductive event, with probability $1-\mu$ the new virus inherits its parent trait, and with probability μ undergoes mutation. In the latter case, the trait z' of the new individual is drawn from a probability distribution $\bar{m}(dz')$ with density $m(z,z')$ with respect to Lebesgue measure on $\mathcal{X}$.

Death: viruses on plants die naturally at rate $d(z) > 0$, and due to competition at rate $cN_x(t)$, where $N_x(t)$ denotes the number of viruses in a plant at position $x \in E$, and c is a positive constant. Likewise, viruses carried by a vector die at rate $\gamma(z)$, where $z \in \mathcal{X}$ is the phenotype of the virus.

Diffusion: vectors, free or charged with virus, diffuse in the spatial domain $\mathcal{D} \subset \mathbb{R}^d$ according to a non-degenerate (see Fournier and Printems (2010)) Itô diffusion with normal reflection at the boundary of the domain $\partial\mathcal{D}$, i.e. the diffusions have infinitesimal generator

$$\mathcal{L}^u \phi_u(y) := a^u(y) \cdot \nabla \phi_u(y) + \frac{\sigma^u(y)^2}{2} \Delta \phi_u(y), \tag{1.2}$$

$$\mathcal{L}^c \phi_c(w,e) := a^c(w) \cdot \nabla_w \phi_c(w,e) + \frac{\sigma^c(w)^2}{2} \Delta_w \phi_c(w,e), \tag{1.3}$$

where $\phi_u : \bar{\mathcal{D}} \to \mathbb{R}$ and $\phi_c : \bar{\mathcal{D}} \times \mathcal{X} \to \mathbb{R}$ are elements of their corresponding domains:

$$D(\mathcal{L}^u) = \left\{ \phi \in \mathcal{C}^2(\mathcal{D}) : \nabla \phi(y) \cdot \vec{n}(y) = 0, \quad \forall y \in \partial D \right\},$$

$$D(\mathcal{L}^c) = \left\{ \phi : \mathcal{D} \times \mathcal{X} \to \mathbb{R} : \forall e \in \mathcal{X}; \phi(\cdot, e) \in \mathcal{C}^2(\mathcal{D}) \text{ and } \nabla \phi(y,e) \cdot \vec{n}(y) = 0, \quad \forall y \in \partial D \right\}$$

where $\nabla_w \phi(w,e)$ denotes the operator ∇ acting on the first variable (similarly for $\Delta_w \phi(w,e)$), and $\vec{n}(y)$ denotes the inward normal at $y \in \partial D$.

Charging of vectors: a vector at position $y \in \mathcal{D}$ *successfully* bites a plant located at $x \in E$, and gets charged with a virus at rate $\beta(t,y,x,N_x(t),z)$, where $N_x(t)$ is the total number of viruses at x, and $z \in \mathcal{X}$ is the phenotype of the virus being taken. In accordance with biological studies (see Moury et al. (2007); Gutiérrez et al. (2012)) which suggest that a vector carries and effectively transmits a small number of viruses, we assume that a vector can carry at most one virus at the same time.

Un-loading of vectors: a charged vector at position $y \in \mathcal{D}$ *successfully* discharges viruses, to a plant located at $x \in E$, at rate $\eta(t,y,x,z)$, where $z \in \mathcal{X}$ is the phenotype of the virus being unloaded.

Furthermore, we also make the following, biologically consistent, additional assumptions:

Assumption 1. We assume that the coefficients $\sigma^\alpha, a^\alpha, \alpha \in \{u,c\}$, are Lipschitz continuous, $\sigma^\alpha > 0$, that the competition parameter c is strictly positive, and that there exist $(\bar{b}, \bar{d}, \bar{\gamma}, \bar{\beta}, \bar{\eta}) \in \mathbb{R}_+^5$, such that for all $t \geq 0$ we have:

$$\sup_{(x,z)\in E\times\mathcal{X}} b(x,z) = \bar{b} < \infty, \quad \sup_{z\in\mathcal{X}} d(z) = \bar{d} < \infty, \quad \sup_{u\in\mathcal{X}} \gamma(u) = \bar{\gamma} < \infty,$$

$$\sup_{r\in\mathbb{R}_+} \sup_{(y,z)\in\mathcal{D}\times\mathcal{X}} \int_E \beta(t,y,x,r,z)\, \zeta_E(dx) = \bar{\beta} < \infty, \qquad \sup_{(y,z)\in\mathcal{D}\times\mathcal{X}} \int_E \eta(t,y,x,z)\, \zeta_E(dx) = \bar{\eta} < \infty,$$

where ζ_E denotes the counting measure on E. Moreover, for all $(t,y,x,z) \in \mathbb{R}_+ \times \mathcal{D} \times E \times \mathcal{X}$, we assume $\beta(t,y,x,\cdot,z) : \mathbb{R}_+ \to \mathbb{R}_+$ to be Lipschitz continuous with constant L_β independent of t.

Our framework allows to incorporate the spatial extent of the plant. We can assume for example the existence of some function $\beta_e : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{X} \to \mathbb{R}_+$, and $r_p > 0$ such that:

$$\beta(t,y,x,N_x(t),z) = \begin{cases} \beta_e(t,|x-y|,N_x(t),z) & \text{if } |x-y| \leq r_p, \\ 0 & \text{otherwise,} \end{cases}$$

for all $t \geq 0$. Analogous assumptions can be imposed over the model parameter η.

Infinitesimal generator. We are interested in the dynamics of the process

$$\{\boldsymbol{\nu}_t : t \geq 0\} = \{(\nu_v(t), \nu_u(t), \nu_c(t)) : t \geq 0\}$$

taking values in the space of measures $\mathcal{M}_p := \mathcal{M}_p(E \times \mathcal{X}) \times \mathcal{M}_p(\mathcal{D}) \times \mathcal{M}_p(\mathcal{D} \times \mathcal{X})$. Let us introduce the set of cylindrical functions that generates the set of bounded and measurable functions from $\mathcal{M}_p$ to $\mathbb{R}$, necessary to describe the generator of the process.

Definition 1.1. We call an admissible triplet a set of bounded and measurable functions $\{\phi_v, \phi_u, \phi_c\}$ such that: $\phi_u : \mathcal{D} \to \mathbb{R}$ is in the domain of the generator $\mathcal{L}^u$, $\phi_c : \mathcal{D} \times \mathcal{X} \to \mathbb{R}$ is in the domain of the generator $\mathcal{L}^c$. Moreover, we denote by $\Phi(\mathcal{M}_p)$ the set of all admissible triplets of this form.

Remark 1.2. For $\boldsymbol{\phi} = \{\phi_v, \phi_u, \phi_c\} \in \Phi(\mathcal{M}_p)$, we denote by $\mathcal{L}^\alpha \boldsymbol{\phi}$ the following triplets:

$$\mathcal{L}^u \boldsymbol{\phi} = \{0, \mathcal{L}^u \phi_u, 0\}, \text{ and } \mathcal{L}^c \boldsymbol{\phi} = \{0, 0, \mathcal{L}^c \phi_c\},$$

where $\mathcal{L}^u$ and L^c denote the generators given in (1.2)-(1.3) acting on the variables y and w respectively, and $\alpha \in \{u, c\}$.

We now define the relevant set of cylindrical functions.

Definition 1.3. The class of cylindrical functions $\mathcal{F}_C$ on $\mathcal{M}_p$ is given by functions $F_{\boldsymbol{\phi}} : \mathcal{M}_p \to \mathbb{R}$, of the form:

$$F_{\boldsymbol{\phi}}(\boldsymbol{\nu}) := F(\langle \boldsymbol{\phi}, \boldsymbol{\nu} \rangle) := F(\langle \nu_v, \phi_v \rangle, \langle \nu_u, \phi_u \rangle, \langle \nu_c, \phi_c \rangle)$$

where $F \in \mathcal{C}^\infty(\mathbb{R}^3; \mathbb{R})$, and $\boldsymbol{\phi} = \{\phi_v, \phi_u, \phi_c\}$ is an admissible triplet, and for $\alpha \in \{v, u, c\}$ we abused notation by using:

$$\langle \nu_\alpha, \phi_\alpha \rangle := \int_{\mathcal{V}_\alpha} \phi_\alpha(x)\, \nu_\alpha(dx),$$

with $\mathcal{V}_p := E \times \mathcal{X}, \mathcal{V}_u := \mathcal{D}$, and $\mathcal{V}_c := \mathcal{D} \times \mathcal{X}$.

Remark 1.4. By Remark 1.1 in Champagnat and Méléard (2007) we know that the spaces of $\mathcal{C}^2$ functions vanishing at the boundary of $\mathcal{D}$, and $\mathcal{V}_c$, $\mathcal{C}_0^2(\mathcal{D})$, and $\mathcal{C}_0^{2,0}(\mathcal{V}_c)$ respectively, are dense in $\mathcal{C}(\mathcal{D})$ and $\mathcal{C}(\mathcal{V}_c)$, respectively, for the uniform topology.

As we did before, and whenever possible in the future, we will be consistent with our notation. Whenever we use the functions $\boldsymbol{\phi} = \{\phi_v, \phi_u, \phi_c\}$ we will refer to functions satisfying the conditions of this section.

The infinitesimal generator $\mathcal{L}$, that corresponds to the dynamics described above, can be written as the sum of a jump part, denoted by $\mathcal{L}_1$, and a diffusive part that we denote by $\mathcal{L}_2$. We further split the jump part of the generator as the sum of operators, acting on cylindrical functions $F_{\boldsymbol{\phi}} \in \mathcal{F}_C$, dealing with every type of jump event. The operator concerning demographics of viruses in plants is given as follows:

$$\begin{aligned}
\mathcal{L}^{dem} F_{\boldsymbol{\phi}}(\boldsymbol{\nu}) := (1-\mu) \int_{\mathcal{V}_p} & b(x,z)\, [F_{\boldsymbol{\phi}}((\nu_v + \delta_{x,z}, \nu_u, \nu_c)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})]\, \nu_v(dx, dz) \\
& + \mu \int_{\mathcal{V}_p} b(x,z) \int_{\mathcal{X}} m(z,e)\ [F_{\boldsymbol{\phi}}((\nu_v + \delta_{x,e}, \nu_u, \nu_c)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})]\ \bar{m}(de)\, \nu_v(dx, dz) \\
& + \int_{\mathcal{V}_p} (d(z) + c\langle \nu_v^x, 1 \rangle)\ [F_{\boldsymbol{\phi}}((\nu_v - \delta_{x,z}, \nu_u, \nu_c)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})]\, \nu_v(dx, dz).
\end{aligned}$$

where for $x \in E$, the measure ν_v^x denotes the restriction of ν_v to $\{x\} \times \mathcal{X}$.

The loading and unloading of viruses on a vector is described in terms of the following operators:

$$\mathcal{L}^{load}F_{\boldsymbol{\phi}}(\boldsymbol{\nu}) := \mathcal{L}^{load}F_{\boldsymbol{\phi}}((\nu_v,\nu_u,\nu_c)) = \int_{\mathcal{V}_p}\int_D \beta(t,y,x,N_x(t),z)\ [F_{\boldsymbol{\phi}}((\nu_v-\delta_{x,z},\nu_u-\delta_y,\nu_c+\delta_{y,z}))-F_{\boldsymbol{\phi}}(\boldsymbol{\nu})]\ \nu_u(dy)\,\nu_v(dx,dz),$$

and

$$\mathcal{L}^{unload}F_{\boldsymbol{\phi}}(\boldsymbol{\nu}) := \mathcal{L}^{unload}F_{\boldsymbol{\phi}}((\nu_v,\nu_u,\nu_c)) = \int_{\mathcal{V}_c}\int_E \eta(t,w,x,e)\ [F_{\boldsymbol{\phi}}((\nu_v+\delta_{x,e},\nu_u+\delta_w,\nu_c-\delta_{w,e}))-F_{\boldsymbol{\phi}}(\boldsymbol{\nu})]\ \zeta_E(dx)\,\nu_c(dw,de).$$

The final type of jump event, the death of viruses on vectors, is given by:

$$\mathcal{L}^{los}F_{\boldsymbol{\phi}}(\boldsymbol{\nu}) := \mathcal{L}^{los}F_{\boldsymbol{\phi}}((\nu_v,\nu_u,\nu_c)) = \int_{\mathcal{V}_c}\gamma(e)\ [F_{\boldsymbol{\phi}}((\nu_v,\nu_u+\delta_w,\nu_c-\delta_{w,e}))-F_{\boldsymbol{\phi}}(\boldsymbol{\nu})]\,\nu_c(dw,de).$$

Summing up we have that the jump part is given by:

$$\mathcal{L}_1 := \mathcal{L}^{dem}+\mathcal{L}^{load}+\mathcal{L}^{unload}+\mathcal{L}^{los}.$$

For the diffusive part we need an admissible triplet $\boldsymbol{\phi}=\{\phi_v,\phi_u,\phi_c\}$ to be as in Definition 1.1. For functions $F_{\boldsymbol{\phi}}\in\mathcal{F}_C$ the diffusive part $\mathcal{L}_2$ can be obtained from Itô's formula. The $\mathcal{L}_2$ operator is given by:

$$\begin{aligned}
&\mathcal{L}_2F_{\boldsymbol{\phi}}(\boldsymbol{\nu})\\
&:= \left(\int_{\mathcal{D}}\mathcal{L}^u_y\phi_u(y)\nu_u(dy)\right)(\partial_uF)_{\boldsymbol{\phi}}(\boldsymbol{\nu})+\left(\int_{\mathcal{V}_c}\mathcal{L}^c\phi_c(w,e)\nu_c(dw,de)\right)(\partial_cF)_{\boldsymbol{\phi}}(\boldsymbol{\nu})\\
&+\left(\int_{\mathcal{D}}\frac{(\sigma^u(y))^2}{2}|\nabla_y\phi_u(y)|^2\nu_u(dy)\right)(\partial_u^2F)_{\boldsymbol{\phi}}(\boldsymbol{\nu})\\
&+\left(\int_{\mathcal{V}_c}\frac{(\sigma^c(w))^2}{2}|\nabla_w\phi_c(w,e)|^2\nu_c(dw,de)\right)(\partial_c^2F)_{\boldsymbol{\phi}}(\boldsymbol{\nu})
\end{aligned}\tag{1.4}$$

where for $\alpha\in\{u,c\}$ we denoted by $\partial_\alpha F_{\boldsymbol{\phi}}$ and $\partial^2_\alpha F_{\boldsymbol{\phi}}$, the first and second partial derivatives with respect to the first coordinate for $\alpha=u$, and second coordinate for $\alpha=c$.

After introducing all the operators we have:

$$\mathcal{L}F_{\boldsymbol{\phi}}(\boldsymbol{\nu}) = \mathcal{L}_1F_{\boldsymbol{\phi}}(\boldsymbol{\nu})+\mathcal{L}_2F_{\boldsymbol{\phi}}(\boldsymbol{\nu}).\tag{1.5}$$

The description of our processes in terms of its infinitesimal generator is rather formal. We refer the reader to the Appendix A of this work, where in the vein of Fournier and Méléard (2004), we provide a rigorous definition on path-space, and a proof of the well-definedness of the processes corresponding to the generator $\mathcal{L}$.

Remark 1.5. Notice that the dynamics described above leaves invariant the total number of vectors.

Existence and uniqueness. We have just described the time-evolution of our measure-valued process in terms of the infinitesimal generator $\mathcal{L}$ given by (1.5). Our first result is to rigorously prove that under Assumption 1 the process is well defined.

Theorem 1.6. *Let $\boldsymbol{\nu}_0=(\nu_v(0),\nu_u(0),\nu_c(0))$ be such that for any $p\ge 2$*

$$\mathbb{E}\left[\langle\boldsymbol{\nu}_0,\mathbf{1}\rangle^p\right]<\infty,\tag{1.6}$$

where

$$\langle\boldsymbol{\nu}_0,\mathbf{1}\rangle := \int_{E\times\mathcal{X}}\nu_v(0)(dx,dz)+\int_{\mathcal{D}}\nu_u(0)(dy)+\int_{\mathcal{D}\times\mathcal{X}}\nu_c(0)(dy,dz).$$

Then, under Assumption 1, there exists a unique solution to the martingale problem associated to $(\mathcal{L}, D(L))$*, which we denote by* $(\boldsymbol{\nu}_t)_{t\geq 0} = (\nu_v(t), \nu_u(t), \nu_c(t) : t \geq 0)$*. Moreover, the process satisfies:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \langle \boldsymbol{\nu}_t, \mathbf{1}\rangle^p\right] < \infty. \tag{1.7}$$

In particular, we also have that the process ν_t *is well-defined.*

For the proof we refer to the Appendix, where we also show that the dynamics of the process $(\boldsymbol{\nu}_t)_{t\geq 0}$ indeed corresponds to the one described by the infinitesimal generator $\mathcal{L}$.

Population-level descriptions as observables. The framework of this paper allows us to directly recover some quantities of interest as observables of the system. For example, the evolution of the total virus population can be recovered by integrating the constant function 1 with respect to the measures ν_v and ν_c:

$$P_v(t) := \langle \nu_v(t), 1\rangle + \langle \nu_c(t), 1\rangle. \tag{1.8}$$

In a similar way, we can also recover the total number of vectors charged with a virus and those free from viruses:

$$N_u(t) := \langle \nu_u(t), 1\rangle, \text{ and } N_c(t) := \langle \nu_c(t), 1\rangle.$$

Moreover, we show how this description can be used to extend known results about extinction to the case of vector-borne dispersal.

Theorem 1.7. *Suppose that the initial set of measures* $(\nu_v(0), \nu_u(0), \nu_c(0))$ *is such that for some* $p \geq 2$ *we have*

$$\mathbb{E}\left[\langle \boldsymbol{\nu}(0), \mathbf{1}\rangle^p\right] < \infty.$$

Under Assumption 1, we further assume that

$$\inf_{z\in\mathcal{X}} d(z) := \hat{d} > 0. \tag{1.9}$$

Then the total-virus population process $P_v(t)$ *given by* (1.8) *goes extinct almost surely.*

We refer to Section 2 for details on the proof of this theorem.

Deterministic limits. Our main results concern the derivation of large population-level deterministic descriptions of our system. By introducing a scaling parameter K, we derive these descriptions in the spirit of a law of large numbers for our processes. In this setting, the scaling parameter K has the biological interpretation of imposing a carrying capacity on the system. We incorporate this idea by letting the competition parameter c depend on K as follows:

$$c_K = \frac{c}{K}.$$

Moreover, we consider two scenarios: one in which the populations of viruses and vectors are of the same order, and one in which the population of vectors is of smaller order than that of viruses. We model these two scenarios by introducing a new parameter $\lambda \in (0, 1]$, which scales the population of viruses and vectors, at time zero, as follows:

$$\nu_v{}^{(K)}(0) = \frac{1}{K}\nu_v(0, K), \quad \nu_u{}^{(K)}(0) = \frac{1}{K^\lambda}\nu_u(0, K), \text{ and } \nu_c{}^{(K)}(0) = \frac{1}{K^\lambda}\nu_c(0, K).$$

where for each $\alpha \in \{v, u, c\}$ and $K \in \mathbb{N}$, the random measure $\nu_\alpha(0, K)$ is an element of $\mathcal{M}_p(\mathcal{V}_\alpha)$. An important ingredient needed for our results is the assumption that at time zero the sequences $\{\nu_\alpha^{(K)}(0)\}_{K\in\mathbb{N}}$ converge, as $K \to \infty$, for $\alpha \in \{v, u, c\}$. The convergence of these sequences implies that for the case $\lambda \in (0, 1)$, the total number of vectors is of smaller order than the total population of viruses. This *lack of vectors* suggests the need to let the rest of the parameters depend on K as well. In particular, roughly speaking, we compensate the lack of vectors by letting the processes $\nu_u(t,K)$ and $\nu_c(t,K)$ (i.e. the time evolution of the processes with initial condition $\nu_u(0,K)$ and $\nu_c(0,K)$) evolve with diffusive generator given by:

$$\mathcal{L}_K^{\alpha,\text{accel}} = K^{1-\lambda}\mathcal{L}^{\alpha}, \tag{1.10}$$

where for $\alpha \in \{u,c\}$, $\mathcal{L}^{\alpha}$ denotes the infinitesimal generator of the Itô diffusion driving the movement of the α-class of vectors. We refer to Section 3 for a full expression of the scaled generator. Furthermore, the explicit way in which the scaled parameters η_K, β_K and γ_K depend on K comes from the idea of making the distance travelled by a vector between loading and unloading of order one. Additionally, we wish the number of virus deaths (on vectors) to be of the same order one. This means:

$$K^{1+\lambda}\beta_K = K^{\lambda}\eta_K = K^{\lambda}\gamma_K = O(K)$$

or equivalently

$$\eta_K = O(K^{1-\lambda}), \quad \gamma_K = O(K^{1-\lambda}), \text{ and } \beta_K = O(K^{-\lambda}).$$

We formalize these ideas with the following assumption:

Assumption 2. There exist a Lipschitz continuous function on its x variable $\beta : \mathbb{R}_+ \times \mathcal{D} \times E \times \mathbb{R}_+ \times \mathcal{X} \to \mathbb{R}$, continuous functions $\eta : \mathbb{R}_+ \times \mathcal{D} \times E \times \mathcal{X} \to \mathbb{R}$ and $\gamma : \mathcal{X} \to \mathbb{R}_+$, and a positive constant c such that:

$$\beta_K(t,y,x,Kn,z) = K^{-\lambda}\beta(t,y,x,n,z), \quad c_K = \frac{c}{K},$$
$$\eta_K(t,y,x,z) = K^{1-\lambda}\eta(t,y,x,z), \quad \gamma_K(z) = K^{1-\lambda}\gamma(z),$$

for all $t \in \mathbb{R}_+$. Moreover, we assume the rest of the parameters to be independent of the scaling parameter K, and all parameters together satisfy Assumption 1.

Under the above assumptions, we normalize our processes as follows:

$$\nu_v{}^{(K)}(t) = \frac{1}{K}\nu_v(t,K), \quad \nu_u{}^{(K)}(t) = \frac{1}{K^{\lambda}}\nu_u(t,K), \text{ and } \nu_c{}^{(K)}(t) = \frac{1}{K^{\lambda}}\nu_c(t,K), \tag{1.11}$$

where for $\alpha \in \{u,c\}$, the process $\nu_\alpha(t,K)$ has dynamics with diffusive generator given by $\mathcal{L}_K^{\alpha,\text{accel}}$ as in (1.10).

In the two regimes described above (i.e. if $\lambda = 1$ or $\lambda \in (0,1)$) we show that, as the parameter K tends to infinity, the normalized triplet of measure-valued processes $(\nu_v{}^{(K)}(t), \nu_u{}^{(K)}(t), \nu_c{}^{(K)}(t))$ converges in path-space to a deterministic limiting triplet $(\xi_v(t), \xi_u(t), \xi_c(t))$ characterized as the solution of a system of non-local integro-differential equations. We refer the reader to Theorem 3.3 and Theorem 3.6 in Section 3, which make rigorous this convergence in a general setting and describe the precise assumptions needed at time zero. In the following paragraphs, we present two simpler versions of Theorem 3.3 and Theorem 3.6 where in particular the deaths of viruses on vectors are neglected, i.e. in both cases we assumed $\gamma = 0$.

First regime. For simplicity let us assume that at time zero the sequences of finite point measures $\{\nu_v{}^{(K)}(0)\}_{K\in\mathbb{N}}$, $\{\nu_u{}^{(K)}(0)\}_{K\in\mathbb{N}}$ and $\{\nu_c{}^{(K)}(0)\}_{K\in\mathbb{N}}$ converge to deterministic limiting measures $\xi_v(0)$, $\xi_u(0)$ and $\xi_c(0)$, respectively. Moreover, we assume that these limiting measures are absolutely continuous with respect to the relevant Haar measures in their respective state spaces (i.e., the counting measure $\zeta_E(dx)$ on E, the Lebesgue measure on $\mathcal{D}$, etc...). Let us denote the densities of the limiting measures $\xi_v(0)$, $\xi_u(0)$ and $\xi_c(0)$, by $g_v(x,z), g_u(y)$, and $g_c(y,z)$ respectively. In Theorem 3.5, we show that for reversible diffusions we have the propagation of absolutely continuity for later times. Let us denote by $g_v(t,x,z), g_u(t,y)$, and $g_c(t,y,z)$ respectively, these densities for later times $t > 0$.

For the sake of clarity let us assume that vectors diffuse in space according to reflected Brownian motion (i.e. the generator obtained by setting $a^\alpha = 0$ and $\sigma^\alpha = 1$ in the generators given by (1.2)). Then a consequence of Theorem 3.3 is the following result which resembles the so-called Lotka-Volterra (L-V) competition system in the presence of diffusion by vectors.

Theorem 1.8. *Let $\lambda = 1$, $\gamma = 0$, and let $\boldsymbol{\nu}_0^{(K)} = (\nu_v{}^{(K)}(0), \nu_u{}^{(K)}(0), \nu_c{}^{(K)}(0))$ be such that:*

$$\sup_{K\in\mathbb{N}} \mathbb{E}\left[\left(\langle \nu_v{}^{(K)}(0), 1\rangle + \langle \nu_u{}^{(K)}(0), 1\rangle + \langle \nu_c{}^{(K)}(0), 1\rangle\right)^3\right] < \infty.$$

Then, the measure-valued process $\{\boldsymbol{\nu}^{(K)}(t) : t \geq 0\}$ converges, as $K \to \infty$, to a deterministic process $\xi_t = (\xi_v(t), \xi_u(t), \xi_c(t))$ with densities (g_v, g_u, g_c) solving the following system:

$$\begin{cases} \dfrac{\partial}{\partial t} g_v(t,x,z) = (1-\mu) b(x,z)\, g_v(t,x,z) + \mu \displaystyle\int_{\mathcal{X}} b(x,z') m(z',z)\, g_v(t,x,z')\, dz' \\ \qquad - \left[d(z) + c\left(\displaystyle\int_{\mathcal{X}} g_v(t,x,z') dz' \right)\right] g_v(t,x,z) \\ \qquad - \left[\displaystyle\int_{\mathcal{D}} \beta\left(y, x, \int_{\mathcal{X}} g_v(t,x,z') dz', z\right) g_u(t,y)\, dy \right] g_v(t,x,z) + \displaystyle\int_{\mathcal{D}} \eta(y,x,z)\, g_c(t,y,z)\, dy, \\ \dfrac{\partial}{\partial t} g_u(t,y) = \Delta g_u(t,y) - \left[\displaystyle\int_{E\times\mathcal{X}} \beta\left(y, x, \int_{\mathcal{X}} g_v(t,x,z') dz', z\right) g_v(t,x,z)\, \zeta_E(dx)\, dz \right] g_u(t,y) \\ \qquad + \displaystyle\int_{E\times\mathcal{X}} \eta(y,x,z)\, g_c(t,y,z)\, \zeta_E(dx)\, dz, \\ \dfrac{\partial}{\partial t} g_c(t,y,z) = \Delta_y g_c(t,y,z) + \left[\displaystyle\int_{E} \beta\left(y, x, \int_{\mathcal{X}} g_v(t,x,z') dz', z\right) g_v(t,x,z)\, \zeta_E(dx) \right] g_u(t,y) \\ \qquad - \left(\displaystyle\int_{E} \eta(y,x,z)\, \zeta_E(dx) \right) g_c(t,y,z), \end{cases}$$

with initial data and boundary conditions given by:

$$\begin{aligned} &g_v(0,x,z) = g_v(x,z), \quad g_u(0,y) = g_u(y), \ \text{and} \quad g_c(0,y,z) = g_c(y,z) \\ &\nabla g_u(t,y) \cdot \vec{n}(y) = 0 \quad \text{and} \quad \nabla g_c(t,y,z) \cdot \vec{n}(y) = 0 \end{aligned}$$

for all $y \in \partial\mathcal{D}$.

Remark 1.9. Notice that this system extends the L-V competition model with mutation (see for example Raoul (2011); Calsina and Cuadrado (2005, 2007)) by incorporating two non-local additional terms representing vector-borne dispersal. Moreover, the system also keeps track of the uncharged and charged vector populations simultaneously with that of the viruses. Finally, this representation allows us to consider a continuous trait space $\mathcal{X} \subset \mathbb{R}^n$. This structure can be used for example to study the adaptation of viruses to resistant genes of plants Fabre et al. (2009, 2012).

Local persistence for the first regime. As a small application of Theorem 1.8, in this section we give sufficient conditions for the local persistence of the virus population of a specific virus trait on a given plant, i.e., conditions that guarantee that:

$$\liminf_{t\to\infty} g_v(t,x,z) > 0,$$

for $x \in E$, and $z \in X$.

Let us introduce the following quantity:

$$R(x,z) = (1-\mu) b(x,z) - d(z) - \int_{\mathcal{D}} \beta(y,x,z)\, dy$$

we then have the following:

Proposition 1.10. *Let $g_v(t,\cdot,\cdot)$ be given as in Theorem 1.8. Assume that, for some fixed $x \in E$, and $z \in X$ we have:*

$$g_v(0,x,z) > 0, \quad R(x,z) > 0. \tag{1.12}$$

Then the virus population of trait z locally persists at x.

We refer to 3.3 for the proof of this result.

Second regime. For the second regime, i.e. $\lambda \in (0,1)$, under the same assumptions on the sequence of initial measures, we make the additional assumption:

Assumption 3. Assume that for all $\nu_v \in \mathcal{M}_p(\mathcal{V}_p)$, and all $t \geq 0$, the function

$$\bar{\beta}(t,y) = \int_{\mathcal{V}_p} \beta(t,y,x,\langle \nu_v^x, 1\rangle, z)\nu_v(dx,dz) \tag{1.13}$$

is in $\mathcal{C}^2(\mathcal{D})$. Moreover, assume the existence of a constant $\beta_0 > 0$ such that

$$\bar{\beta}(t,y) \geq \beta_0 > 0, \tag{1.14}$$

uniformly in y. Finally, assume $\gamma = 0$.

The following result is a consequence of Theorem 3.6. It shows how the speeding up of the diffusion of vectors has the effect of changing the limiting evolution of the two populations of vectors to an equilibrium state.

Theorem 1.11. *Let $\lambda \in (0,1)$, $\sigma^\alpha = 1$, $a^\alpha = 0$, and $\boldsymbol{\nu}_0^{(K)} = ({\nu_v}^{(K)}(0), {\nu_u}^{(K)}(0), {\nu_c}^{(K)}(0))$ be such that:*

$$\sup_{K\in\mathbb{N}} \mathbb{E}\left[\left(\langle {\nu_v}^{(K)}(0), 1\rangle + \langle {\nu_u}^{(K)}(0), 1\rangle + \langle {\nu_c}^{(K)}(0), 1\rangle\right)^3\right] < \infty.$$

Then, under Assumption 3, the measure-valued process $\{\boldsymbol{\nu}^{(K)}(t) : t \geq 0\}$ converges, as $K \to \infty$, to a deterministic process $\xi_t = (\xi_v(t), \xi_u(t), \xi_c(t))$ with densities (g_v, g_u, g_c) solving the following system:

$$\begin{cases} \dfrac{\partial}{\partial t} g_v(t,x,z) = (1-\mu)b(x,z)\, g_v(t,x,z) + \mu \displaystyle\int_{\mathcal{X}} b(x,z')m(z',z)\, g_v(t,x,z')\, dz' \\ \qquad - \left[d(z) + c\left(\displaystyle\int_{\mathcal{X}} g_v(t,x,z')dz'\right)\right] g_v(t,x,z) \\ \qquad - \left[\displaystyle\int_{\mathcal{D}} \beta\left(y,x,\int_{\mathcal{X}} g_v(t,x,z')dz', z\right) g_u(t,y)\, dy\right] g_v(t,x,z) + \left(\displaystyle\int_{\mathcal{D}} \eta(y,x,z)\, g_c(t,y,z)\, dy\right), \\ 0 = \Delta g_u(t,y) - \left[\displaystyle\int_{E\times\mathcal{X}} \beta\left(y,x,\int_{\mathcal{X}} g_v(t,x,z')dz', z\right) g_v(t,x,z)\, \zeta_E(dx)\, dz\right] g_u(t,y) \\ \qquad + \displaystyle\int_{E\times\mathcal{X}} \eta(y,x)\, g_c(t,y,z)\, \zeta_E(dx)\, dz, \\ 0 = \Delta_y g_c(t,y,z) + \left[\displaystyle\int_E \beta\left(y,x,\int_{\mathcal{X}} g_v(t,x,z')dz', z\right) g_v(t,x,z)\, \zeta_E(dx)\right] g_u(t,y) \\ \qquad - \left(\displaystyle\int_E \eta(y,x)\, \zeta_E(dx)\right) g_c(t,y,z), \end{cases}$$

with the same boundary conditions as in Theorem 1.8.

Remark 1.12. Notice that this result in particular is only valid for $\mathcal{L}^u = \mathcal{L}^c = \Delta$. The extension to general reflecting diffusions is not possible with our techniques. The main difficulty comes from the lack of applicability of our proof on existence and uniqueness to the general setting.

Perspectives on extension to a continuous set of plants. As a final remark, we want to mention that in Theorem 1.8 and Theorem 1.11 we have deliberately used the notation:

$$\int_E f(x)\,\zeta_E(dx),$$

instead of the perhaps more natural:

$$\sum_{x\in E} f(x).$$

Our intention with this choice was to suggest in particular the possible extension of our results to the case in which the set of plants becomes increasingly large together with the rescaling parameter K. To be more precise, when the locations of plants form an increasing sequence of lattices E_K approximating the whole plantation space $\mathcal{D}$ (and the measures ζ_{E_K} approximating the Lebesgue measure dx) as $K \to \infty$. We suspect that Theorems 3.3 and 3.6 can be extended to that setting. This extension would require the use of additional techniques, in particular a type of embedding, common to the theory of scaling limits of particle systems (i.e., as in De Masi and Presutti (1991); Kipnis and Landim (1999)) and has been postponed for future work.

Organization of the paper. The rest of our paper is organized as follows. In Section 2 we provide the proof of Theorem 1.7. In Section 3 we state our main theorems, namely Theorem 3.3 and Theorem 3.6, together with the details on the derivation of Theorem 1.8, and Theorem 1.11, from Theorem 3.3 and Theorem 3.6, respectively. Section 3.3 contains the details of the derivation of Proposition 1.10. In Section 4 we prove our main theorems using the standard compactness-uniqueness approach, in particular Section 4.1 deals with the details of the proof of Theorem 3.3, while Section 4.2 deals with the proof of Theorem 3.6 with the help of Kurtz' averaging principle for slow-fast systems Kurtz (1992). Finally, in the Appendix we include the rigorous definition of the processes on path-space. There, we also include the derivation of some standard martingale properties that are used in our proofs. Finally, in part B of the Appendix, we include results related to existence and uniqueness for the averaged dynamics of the vector populations. We also include a comprehensive list of all notations used in this paper, in the last part of the Appendix, Section C.

2. Extinction probabilities

We are interested in the probability of extinction of the virus population, i.e., the probability

$$\mathbb{P}_{\nu(0)}(\exists s > 0, P_v(s) = 0),$$

where the process is initially started from the measure $\boldsymbol{\nu}(0) = (\nu_v(0), \nu_u(0), \nu_c(0))$.

We first make the following remarks:

Remark 2.1. Notice that the random variable $P_v(t)$ does not make distinction about the different phenotypes. Moreover, it also takes into account the sub-population of viruses residing on vectors. This is needed to properly study extinction since it avoids the possibility of re-emergence of the virus population.

Remark 2.2. The random process $\{P_v(t)\}_{t\geq 0}$ is integer valued, in fact it is $(\mathbb{N}\cup\{0\})$-valued. Moreover, zero is an absorbing state, i.e, if for some $s \geq 0$, $P_v(s) = 0$, then $P_v(t) = 0$ for all $t \geq s$.

Remark 2.3. Notice that by Remark 1.5 we have:

$$N_u(t) + N_c(t) =: V_0$$

for all $t \geq 0$.

Before proving Theorem 1.7, we need the following lemma:

Lemma 2.4. *Let $\nu_v \in \mathcal{M}_p(\mathcal{V}_p)$, then we have:*

$$\int_{\mathcal{V}_p} \left[\int_{\mathcal{X}} 1 \cdot {\nu_v}^x(t)(dz') \right] \nu_v(t)(dx,dz) \geq \frac{1}{|E|} N_v(t)^2, \tag{2.1}$$

where $|E|$ denotes the cardinality of the finite set E, and ${\nu_v}^x$ denotes the restriction of the measure ν_v to the set $\{x\}$.

Proof: For $\nu_v \in \mathcal{M}_p(\mathcal{V}_p)$, let us denote by $\bar{\nu}_v \in \mathcal{M}_p(E)$ the following measure:

$$\bar{\nu}_v(dx) = \int_{\mathcal{X}} 1 \, \nu_v(dx,dz).$$

Notice then that we can rewrite the LHS of (2.1) as follows:

$$\begin{aligned} \int_{E\times\mathcal{X}} \left[\int_{\mathcal{X}} 1 \cdot \nu_v^x(t)(dz') \right] \nu_v(t)(dx,dz) &= \int_E \int_E \mathbb{1}_{\{x\}}(x') \cdot \bar{\nu}_v(t)(dx')\, \bar{\nu}_v(t)(dx) \\ &= \sum_{i=1}^{N_v(t)} \sum_{j=1}^{N_v(t)} \mathbb{1}_{\{x_i\}}(x_j). \end{aligned} \tag{2.2}$$

Notice that the last sum counts the sum of the squares of the total number of viruses per plant in each of the plants in the plantation E. We conclude the proof of the lemma by using the fact that the set E is finite, and the fundamental inequality:

$$n \sum_{i=1}^n a_i^2 \geq \left(\sum_{i=1}^n a_i \right)^2$$

for $a_i \geq 0$. □

We can now proceed to the proof of Theorem 1.7.

Proof: Let us first claim:

$$\sup_{t\geq 0} \mathbb{E}\left[P_v(t)\right] < \infty. \tag{2.3}$$

To see that this is the case, let us define $f(t)$ as follows:

$$f(t) := \mathbb{E}\left[P_v(t)\right] = \mathbb{E}\left(\langle \nu_v(t), 1\rangle + \langle \nu_c(t), 1\rangle \right).$$

By Proposition A.8 we have

$$\begin{aligned} f(t) = f(0) + \int_0^t \mathbb{E}\left[\int_{\mathcal{V}_p} b(x,z)\nu_v(s)(dx,dz) - \int_{\mathcal{V}_p} (d(z) + c\langle \nu_v^x, 1\rangle)\, \nu_v(s)(dx,dz) \right] ds, \\ - \int_0^t \mathbb{E}\left[\int_{\mathcal{V}_c} \gamma(y)\, \nu_c(s)(dy,dz) \right] ds \end{aligned}$$

and as a consequence we obtain the differentiability of f. Moreover, using $\gamma \geq 0$, we also have:

$$f'(t) \leq (\bar{b} - \hat{d}) f(t) - c \cdot \mathbb{E}\left[\int_{\mathcal{V}_p} \langle \nu_v^x(t), 1\rangle \nu_v(t)(dx,dz) \right].$$

where $\bar{b}$ is given in Assumption 1, and $\hat{d} := \inf_{z\in\mathcal{X}} d(z)$.

By Jensen's inequality we have:

$$\begin{aligned} f(t)^2 &\leq \mathbb{E}\left[N_v(t)^2 + 2N_c(t)N_v(t) + N_c(t)^2 \right] \\ &\leq \mathbb{E}\left[N_v(t)^2 + 2V_0\big(N_v(t) + N_c(t) \big) \right] \end{aligned}$$

which can be rewritten as:

$$\mathbb{E}\left[N_v(t)^2\right] \geq f(t)^2 - 2V_0 f(t),$$

This, together with Lemma 2.4, implies

$$f'(t) \leq \left((\bar{b} - \hat{d}) + \frac{2cV_0}{|E|}\right) f(t) - \frac{c}{|E|} f(t)^2.$$

From the observation that the function:

$$y(x) = \left((\bar{b} - \hat{d}) + \frac{2cV_0}{|E|}\right) x - \frac{c}{|E|} x^2$$

is negative for any x such that

$$x \geq x_0 := |E| \frac{(\bar{b} - \hat{d})}{c} + 2V_0,$$

we deduce that

$$f(t) \leq f(0) \vee x_0$$

for all $t \geq 0$. This implies (2.3).

Now we claim that:

$$\lim_{t \to \infty} P_v(t) \in \{0, \infty\}.$$

Using the fact that $P_v(t)$ is $(\mathbb{N} \cup \{0\})$-valued, it is enough to check that for any $M \in \mathbb{N}$ we have

$$\mathbb{P}\left[\liminf_{t \to \infty} P_v(t) = M\right] = 0.$$

Assume that it is not the case, and that $\liminf_{t \to \infty} P_v(t) = M$. By definition, this implies that $P_v(t)$ reaches the value M infinitely often, but the value $M - 1$ only a finite number of times. However, this is almost surely impossible since every time that the process P_v is at state M, the probability of going to state $(M - 1)$ is bounded from below by:

$$\frac{\hat{d}M}{\bar{b}M + \bar{d}M + cM^2 \bar{\gamma} V_0} > 0,$$

where we recall that $\hat{d} > 0$. We only have to use the fact that $\{0\}$ is an absorbing state to deduce that the limit exists and

$$\lim_{t \to \infty} P_v(t) \in \{0, \infty\}.$$

To conclude that a.s.

$$\lim_{t \to \infty} P_v(t) = 0,$$

it is enough to show that

$$\mathbb{E}\left[\lim_{t \to \infty} P_v(t)\right] < \infty.$$

This property is a consequence of Fatou's lemma, expression (2.3), and the following reasoning:

$$\mathbb{E}\left[\lim_{t \to \infty} P_v(t)\right] = \mathbb{E}\left[\liminf_{t \to \infty} P_v(t)\right] \leq \liminf_{t \to \infty} \mathbb{E}\left[P_v(t)\right] \leq \sup_{t \geq 0} \mathbb{E}\left[P_v(t)\right] < \infty.$$

□

3. IDE formulation

Let us introduce some additional notation needed to introduce the IDE formulations of Theorem 1.8 and Theorem 1.11.

Rescaled processes. Let us denote by $\Lambda_K(t)$ the measure-valued process of the form:

$$\Lambda_K(t) = ({\nu_v}^{(K)}(t), {\nu_u}^{(K)}(t), {\nu_c}^{(K)}(t)), \tag{3.1}$$

where ${\nu_v}^{(K)}(t), {\nu_u}^{(K)}(t)$, and ${\nu_c}^{(K)}(t)$ are given as in (1.11). We consider $\Lambda_K(t)$ as a process taking values in the product space of measures $\mathcal{M}_p$ given by:

$$\mathcal{M}_p := \mathcal{M}_p(\mathcal{V}_p) \times \mathcal{M}_p(\mathcal{D}) \times \mathcal{M}_p(\mathcal{V}_c).$$

Moreover we introduced the additional notation:

$$\mathcal{M}_F := \mathcal{M}_F(\mathcal{V}_p) \times \mathcal{M}_F(\mathcal{D}) \times \mathcal{M}_F(\mathcal{V}_c),$$

where for any Polish space X, $\mathcal{M}_F(X)$ denotes the space of finite measures on X. Notice that with this notation we have $\mathcal{M}_p \subset \mathcal{M}_F$.

Under Assumption 2, the process $\Lambda_K(t)$ has an infinitesimal generator, that we denote by $\mathcal{L}^{(K)}$, with jump part $\mathcal{L}_1^{(K)}$ given by:

$$\begin{aligned}
&\mathcal{L}_1^{(K)} F_{\boldsymbol{\phi}}(\boldsymbol{\nu}) = K(1-\mu) \int_{\mathcal{V}_p} b(x,z) \left[F_{\boldsymbol{\phi}}((\nu_v + \tfrac{1}{K}\delta_{x,z}, \nu_u, \nu_c)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})\right] \nu_v(dx,dz) \\
&+ K\,\mu \int_{\mathcal{V}_p} b(x,z) \int_{\mathcal{X}} m(z,e) \left[F_{\boldsymbol{\phi}}((\nu_v + \tfrac{1}{K}\delta_{x,e}, \nu_u, \nu_c)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})\right] \nu_v(dx,dz)\,de \\
&+ K \int_{\mathcal{V}_p} (d(z) + c\langle \nu_v^x, 1\rangle) \left[F_{\boldsymbol{\phi}}((\nu_v - \tfrac{1}{K}\delta_{x,z}, \nu_u, \nu_c)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})\right] \nu_v(dx,dz) \\
&+ K \int_{\mathcal{V}_p} \int_D \beta(t,y,x,N_x(t),z) \left[F_{\boldsymbol{\phi}}((\nu_v - \tfrac{1}{K}\delta_{x,z}, \nu_u - \tfrac{1}{K^\lambda}\delta_y, \nu_c + \tfrac{1}{K^\lambda}\delta_{y,z})) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})\right] \nu_v(dx,dz)\,\nu_u(dy) \\
&+ K \int_{\mathcal{V}_c} \int_E \eta(t,w,x,e) \left[F_{\boldsymbol{\phi}}((\nu_v + \tfrac{1}{K}\delta_{x,e}, \nu_u + \tfrac{1}{K^\lambda}\delta_w, \nu_c - \tfrac{1}{K^\lambda}\delta_{w,e})) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})\right] \nu_c(dw,de)\,dx \\
&+ K \int_{\mathcal{V}_c} \gamma(e) \left[F_{\boldsymbol{\phi}}((\nu_v, \nu_u + \tfrac{1}{K^\lambda}\delta_w, \nu_c - \tfrac{1}{K^\lambda}\delta_{w,e})) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu})\right] \nu_c(dw,de),
\end{aligned}$$

and a diffusive part:

$$\begin{aligned}
&\mathcal{L}_2^{(K)} F_{\boldsymbol{\phi}}(\boldsymbol{\nu}) := K^{1-\lambda} \left(\int_{\mathcal{D}} \mathcal{L}_y^u \phi_u(y) \nu_u(dy)\right) (\partial_u F)_{\boldsymbol{\phi}}(\boldsymbol{\nu}) \\
&+ K^{1-2\lambda} \left(\int_{\mathcal{D}} \frac{(\sigma^u(y))^2}{2} |\nabla_y \phi_u(y)|^2 \nu_u(dy)\right) (\partial_u^2 F)_{\boldsymbol{\phi}}(\boldsymbol{\nu}) + K^{1-\lambda} \left(\int_{\mathcal{V}_c} \mathcal{L}^c \phi_c(w,e) \nu_c(dw,de)\right) (\partial_c F)_{\boldsymbol{\phi}}(\boldsymbol{\nu}) \\
&+ K^{1-2\lambda} \left(\int_{\mathcal{V}_c} \frac{(\sigma^c(w))^2}{2} |\nabla_w \phi_c(w,e)|^2 \nu_c(dw,de)\right) (\partial_c^2 F)_{\boldsymbol{\phi}}(\boldsymbol{\nu}).
\end{aligned}$$

where the extra factors $K^{-\lambda}$ in front of the second derivative terms come from Itô's lemma.

Population at time zero. At time zero we consider a sequence of initial measures $\{\Lambda_K(0)\}_{\{k\geq 1\}} \in \mathcal{M}_p$ of the form:

$$\Lambda_K(0) = ({\nu_v}^{(K)}(0), {\nu_u}^{(K)}(0), {\nu_c}^{(K)}(0)).$$

We assume that this measure satisfies an estimate like the one needed in Proposition A.5, and that a law of large numbers is satisfied by the sequence $\{\Lambda_K(0)\}_{\{k\geq 1\}}$. More precisely we make the following assumption:

Assumption 4. The sequence of measures $\{\Lambda_K(0)\}_{\{K\geq 1\}} \in \mathcal{M}_p \subset \mathcal{M}_F$ converges in law and for the weak topology of $\mathcal{M}_F$ to a deterministic finite measure $\xi_0 = (\xi_v(0), \xi_u(0), \xi_c(0)) \in \mathcal{M}_F$. Moreover we have the following estimate:

$$\sup_{K\in\mathbb{N}} \left(\mathbb{E}\left[\langle \nu_\alpha^K(0), 1\rangle^3\right]\right) < \infty,$$

for all $\alpha \in \{v, u, c\}$.

Remark 3.1. Notice that the limiting measure at time zero is allowed to belong to the whole space $\mathcal{M}_F$. This is with the intention to allow for the existence of densities, with respect to the relevant Haar measure, at time zero.

The IDE formulation, given by Theorem 3.3 below, can be thought of as a law of large numbers, for the limit of the sequence of measures $\{\Lambda_K(t) : t \geq 0\}_{K\geq 1}$, seen as taking values in the path space $\mathbb{D}([0,T], \mathcal{M}_F)$, for all $T > 0$. Obtaining such a description requires to work with martingales associated to a Markov process and to control their quadratic variation. We will split the analysis of the martingales into the two cases already described in Section 1.

Remark 3.2. It is possible to find combinations of scaling of parameters, different from those of Assumption 2, such that the speeding of diffusion by vectors becomes unnecessary to derive a limiting description for the case $\lambda < 1$. However, for such a combination, the limiting evolution equation for the population of viruses decouples from that of the vectors (i.e. viruses do not see the effect of vectors). This time scale is biologically irrelevant in the context of vector-borne dynamics.

3.1. *IDE formulation: first case.* We have the following theorem:

Theorem 3.3. *Let $\lambda = 1$. Consider the sequence of measure-valued processes $\{\Lambda_K\}_{K\geq 1}$ given by (3.1)-(1.11). Suppose Assumptions 2-4 are satisfied. Then, for all $T > 0$, the sequence of processes $\{\Lambda_K\}_{K\geq 1}$ converges in law in $\mathbb{D}([0,T], \mathcal{M}_F)$ to a deterministic continuous function $\xi(t) = (\xi_v(t), \xi_u(t), \xi_c(t))$ belonging to the path space $\mathbb{C}([0,T], \mathcal{M}_F)$, and solving the following integro-differential equations:*

$$\begin{aligned}\langle \xi_v(t), \phi_v\rangle = \langle \xi_v(0), \phi_v\rangle &+ \int_0^t \int_{\mathcal{V}_p} b(x,z)\phi_v(x,z)\, \xi_v(s)(dx,dz)\, ds \\ &+ \mu \int_0^t \int_{\mathcal{V}_p} b(x,z) \left[\int_{\mathcal{X}} m(z,e)\phi_v(x,e)\, de - \phi_v(x,z)\right] \xi_v(s)(dx,dz)\, ds \\ &- \int_0^t \int_{\mathcal{V}_p} \left[d(z) + c\langle \xi_v(s)^x, 1\rangle\right] \phi_v(x,z)\xi_v(s)(dx,dz)ds \\ &- \int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v(s)^x, 1\rangle, z)\phi_v(x,z)\, \xi_v(s)(dx,dz)\, \xi_u(s)(dy)\, ds \\ &+ \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x,z)\phi_v(x,z)\, \xi_c(s)(dy,dz)\, \zeta_E(dx)\, ds, \end{aligned} \tag{3.2}$$

$$\begin{aligned}\langle \xi_u(t), \phi_u\rangle = \langle \xi_u(0),\phi_u\rangle &+ \int_0^t \int_{\mathcal{D}} \mathcal{L}^u\phi_u(y)\, \xi_u(s)(dy)\, ds \\ &- \int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v(s)^x, 1\rangle, z)\phi_u(y)\xi_v(s)(dx,dz)\, \xi_u(s)(dy)\, ds \\ &+ \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x,z)\phi_u(y)\xi_c(s)(dy,dz)\, \zeta_E(dx)\, ds \\ &+ \int_0^t \int_{\mathcal{V}_c} \gamma(z)\phi_u(y)\, \xi_c(s)(dy,dz)ds, \end{aligned} \tag{3.3}$$

$$\begin{aligned}\langle \xi_c(t), \phi_c\rangle = \langle \xi_c(0),\phi_c\rangle &+ \int_0^t \int_{\mathcal{V}_c} \mathcal{L}^c\phi_c(y,z)\, \xi_c(s)(dy,dz)\, ds \\ &+ \int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v(s)^x, 1\rangle, z)\phi_c(y,z)\xi_v(s)(dx,dz)\, \xi_u(s)(dy)\, ds \\ &- \int_0^t \int_{\mathcal{V}_p\times E} \eta(s,y,x,z)\phi_c(y,z)\xi_c(s)(dy,dz)\, \zeta_E(dx)\, ds \\ &- \int_0^t \int_{\mathcal{V}_c} \gamma(z)\phi_c(y,z)\, \xi_c(s)(dy,dz)\, ds, \end{aligned} \tag{3.4}$$

for all ϕ_v bounded and measurable, $\phi_u \in \mathcal{C}_b^2(\mathcal{D}) \cap \mathcal{D}(\mathcal{L}^u)$, and $\phi_c \in \mathcal{C}_b^2(\mathcal{I}) \cap D(\mathcal{L}^c)$.

We postpone the proof of Theorem 3.3 to Section 4 for the generalities and Section 4.1 for the particulars.

Remark 3.4. For $\alpha \in \{v,u\}$, using the functions $\phi_\alpha = 1$ and neglecting the negative terms, we can obtain that there exists a positive constant C such that

$$\max_\alpha (\langle \xi_\alpha(t), 1\rangle) \le \max_\alpha (\langle \xi_\alpha(0), 1\rangle) + tV_0 + C\int_0^t \max_\alpha (\langle \xi_\alpha(s), 1\rangle)\, ds.$$

By Gronwall's lemma we then conclude:

$$\sup_{t\in[0,T]} \max_\alpha (\langle \xi_\alpha(t), 1\rangle) \le \left(\max_\alpha (\langle \xi_\alpha(0), 1\rangle) + TV_0\right) e^{CT},$$

which by the conservation of mass in vectors, the fact that the sup is bounded can also be extended to the case $\alpha = c$.

Theorem 1.8 is the strong form of Theorem 3.3. In order to verify that we can indeed obtain such a strong formulation, we need to verify that if we assume that at time zero the triplet $(\xi_v(t), \xi_u(t), \xi_c(t))$ has densities with respect to their respective Haar measures, indeed we have the propagation of this property for later times.

3.2. *Propagation of absolute continuity.*

Theorem 3.5. *Let us assume that for $\alpha \in \{u,c\}$ the diffusive generator $\mathcal{L}^\alpha$ is self-adjoint. Let us also assume that for $\alpha \in \{v,u,c\}$, at time zero each of the limiting measures $\xi_\alpha(0)$ admits a density of the following form:*

$$\xi_v(0)(dx,dz) = g_v(x,z)\,\zeta_E(dx)\,dz, \quad \xi_u(0)(dy,dz) = g_u(y)\,dy\,dz, \text{ and } \xi_c(0)(dy,dz) = g_c(y,z)\,dy\,dz,$$

where ζ_E denotes the counting measure on E. Consider the solution $\xi(t) = (\xi_v(t), \xi_u(t), \xi_c(t))$ of (3.2)-(3.4) with initial condition $(\xi_v(0), \xi_u(0), \xi_c(0))$. Then, under the assumptions of Theorem 3.3, for each time $t > 0$, and each $\alpha \in \{v,u,c\}$, there exist functions $g_\alpha(t,\cdot) : \mathcal{V}_\alpha \to \mathbb{R}$ such that:

$$\xi_v(t)(dx,dz) = g_v(t,x,z)\,dz, \quad \xi_u(t)(dy,dz) = g_u(t,y)\,dy\,dz, \text{ and } \xi_c(t)(dy,dz) = g_c(t,y,z)\,dy\,dz.$$

In other words, we have propagation of absolute continuity.

Proof: Since the space E is finite, and ζ_E is the counting measure on E, any ζ_E-null subset of E is empty. Hence we only analyse null sets of $\mathcal{X}$ and $\mathcal{D}$. Let us consider a set $A \subset \mathcal{X}$ of Lebesgue measure zero. We want to show that both measures $\xi_v^{\mathcal{X}}(t) := \int_E \xi_v(t)(dx, \cdot)$ and $\xi_c^{\mathcal{X}}(t) := \int_{\mathcal{D}} \xi_c(dy, \cdot)$ assign zero mass to A. We want to take advantage of the fact that both measures are non-negative to only deal with the measure:

$$\nu^{\mathcal{X}}(t) = \xi_v^{\mathcal{X}}(t) + \xi_c^{\mathcal{X}}(t).$$

By Theorem 3.3, and the absolute continuity at time zero, we have

$$\begin{aligned}\nu^{\mathcal{X}}(t)(A) = &\int_0^t \int_{\mathcal{V}_p} b(x,z)\mathbb{1}_{\{A\}}(z)\, \xi_v(s)(dx, dz)\, ds \\ &- \int_0^t \int_{\mathcal{V}_p} [d(z) + c\langle \xi_v(s)^x, 1\rangle]\; \mathbb{1}_{\{A\}}(z)\xi_v(s)(dx, dz)ds - \int_0^t \int_{\mathcal{V}_c} \gamma(z)\mathbb{1}_{\{A\}}(z)\, \xi_c(s)(dy, dz)\, ds.\end{aligned} \tag{3.5}$$

Let the function $M(t)$ be given by

$$M(t) := \nu^{\mathcal{X}}(t)(A).$$

Notice that $M(0) = 0$, and by (3.5) $M(t)$ is differentiable.

Moreover, by Assumption 1 and the non-negativity of $\xi_c^{\mathcal{X}}(t)$, we have:

$$\begin{aligned}\frac{d}{dt}M(t) \le \bar{b}M(t) &- \int_{\mathcal{V}_p} [d(z) + c\langle \xi_v(t)^x, 1\rangle]\; \mathbb{1}_{\{A\}}(z)\xi_v(t)(dx, dz) \\ &- \int_{\mathcal{V}_c} \gamma(z)\mathbb{1}_{\{A\}}(z)\, \xi_c(t)(dy, dz),\end{aligned}$$

which implies $\frac{d}{dt}M(t) \le \bar{b}M(t)$, and as a consequence $M(t) = 0$.

We now show absolute continuity of $\xi_u(t)$ with respect to Lebesgue measure on $\mathcal{D}$. For simplicity of exposition let us assume that $\mathcal{L}^u = \mathcal{L}^c$ (denoted by $\mathcal{L}$), and consider a measurable set $D \subset \mathcal{D}$ of null Lebesgue measure. Here we also take advantage of the positivity of both measures $\xi_u(t)$ and $\xi_c^{\mathcal{D}}(t) := \int_{\mathcal{X}} \xi_c(t)(\cdot, dz)$, to only deal with the measure:

$$\nu^{\mathcal{D}}(t) = \xi_u(t) + \xi_c^{\mathcal{D}}(t)$$

We would like to plug $\mathbb{1}_{\{D\}}$ and obtain an expression for $\nu^{\mathcal{D}}(t)(D)$, but the indicator function $\mathbb{1}_{\{D\}}$ is not an element of $D(\mathcal{L})$. However we can define the following function:

$$\phi(s, y) = P_{\mathcal{L}}(t - s)\mathbb{1}_{\{D\}}(y) \quad \forall y \in \mathcal{D},$$

for a fixed $t \in [0, T]$. By construction we have that $\phi(s, y)$ is a solution of the boundary problem:

$$\begin{aligned}&\partial_s \phi(s, y) + \mathcal{L}\phi(s, y) = 0 \quad \text{ on } [0, T] \times \mathcal{D} \\ &\partial_n \phi(s, y) = 0 \quad \text{ on } [0, T] \times \partial\mathcal{D}, \\ &\lim_{s \to t} \phi(s, y) = \mathbb{1}_{\{D\}}(y) \quad \text{ on } \mathcal{D}.\end{aligned}$$

Notice that we can also re-write equations (3.2)-(3.4) in their weak time-space formulation. In particular for the last two equations we obtain:

$$\begin{aligned}\langle \xi_u(t), \phi_u(t,\cdot)\rangle &= \langle \xi_u(0), \phi_u(0,\cdot)\rangle + \int_0^t \int_{\mathcal{D}} \left(\mathcal{L}^u \phi_u(s,y) + \partial_s \phi_u(s,y)\right) \xi_u(s)(dy)\, ds \\ &- \int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v^x(s), 1\rangle, z)\phi_u(s,y)\xi_v(s)(dx,dz)\, \xi_u(s)(dy)\, ds \\ &+ \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x,z)\phi_u(s,y)\xi_c(s)(dy,dz)\, \zeta_E(dx)\, ds + \int_0^t \int_{\mathcal{V}_c} \gamma(z)\phi_u(s,y)\, \xi_c(s)(dy,dz) ds, \qquad (3.6)\end{aligned}$$

$$\begin{aligned}\langle \xi_c(t), \phi_c(t,\cdot)\rangle &= \langle \xi_c(0), \phi_c(0,\cdot)\rangle + \int_0^t \int_{\mathcal{V}_c} \left(\mathcal{L}^c \phi_c(s,y,z) + \partial_s \phi_c(s,y,z)\right) \xi_c(s)(dy,dz)\, ds \\ &+ \int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v^x(s), 1\rangle, z)\phi_c(s,y,z)\xi_v(s)(dx,dz)\, \xi_u(s)(dy)\, ds \\ &- \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x,z)\phi_c(s,y,z)\xi_c(s)(dy,dz)\, \zeta_E(dx)\, ds - \int_0^t \int_{\mathcal{V}_c} \gamma(z)\phi_c(s,y,z)\, \xi_c(s)(dy,dz)\, ds. \qquad (3.7)\end{aligned}$$

Summing (3.6) and (3.7), and using that $\mathcal{L}^u = \mathcal{L}^c = \mathcal{L}$, gives:

$$\begin{aligned}\langle \nu^{\mathcal{D}}(t), \mathbb{1}_{\{D\}}\rangle &= \langle \nu^{\mathcal{D}}(t), \phi(0,\cdot)\rangle \\ &= \int_{\mathcal{D}} g_u(y) P_{\mathcal{L}}(t)\mathbb{1}_{\{D\}}(y)\, dy + \int_{\mathcal{D}} \left(\int_{\mathcal{X}} g_c(y,z)\, dz\right) P_{\mathcal{L}}(t)\mathbb{1}_{\{D\}}(y)\, dy = 0, \qquad (3.8)\end{aligned}$$

where in the last equality we used the reversibility of $\mathcal{L}$.

Finally we check that the measure $\xi_c(t)$ is absolutely continuous with respect to the product of the Lebesgue measures on $\mathcal{D}$ and $\mathcal{X}$. Let us denote by D_c a Lebesgue null-subset of $\mathcal{D}\times\mathcal{X}$. As in the previous case, we define the following function:

$$\phi(s,y,z) = P_{\mathcal{L}_c}(t-s)\mathbb{1}_{\{D_c\}}(y,z) \quad \forall (y,z) \in \mathcal{D}\times\mathcal{X},$$

for a fixed $t \in [0,T]$ which solves its corresponding boundary problem. We then conclude by noticing that in this case, by the non-negativity of ξ_c we have:

$$\begin{aligned}\xi_c(t)(D_c) = \int_{\mathcal{D}\times\mathcal{X}} & g_c(y,z)\, P_{\mathcal{L}_c}(t)\mathbb{1}_{\{D_c\}}(y,z)\, dy\, dz \\ &- \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x,z)\phi_c(s,y,z)\xi_c(s)(dy,dz)\, \zeta_E(dx)\, ds \\ &- \int_0^t \int_{\mathcal{V}_c} \gamma(z)\phi_c(s,y,z)\, \xi_c(s)(dy,dz)\, ds \le 0. \qquad (3.9)\end{aligned}$$

□

3.3. *Proof of Proposition 1.10.* Let us start by giving the linearization of the system given in Theorem 1.8 around the equilibrium point (g_v^*, g_u^*, g_c^*) with originally all vectors being free of viruses:

$$g_v^*(x,z) = 0, \quad g_u^*(y) = 1, \text{ and } g_c^*(y,z) = 0$$

for all $x \in E, y \in \mathcal{D}$, and $z \in \mathcal{X}$.

Let $h = (h_v, h_u, h_c)$ be a perturbation around the equilibrium point (g_v^*, g_u^*, g_c^*). To conclude local persistence it is enough to show that the perturbation h_v is monotonically increasing with time locally in x and z. For this particular set of equilibrium the linearization takes the form:

$$\begin{aligned}\frac{\partial}{\partial t}h_v(t,x,z) &= \left((1-\mu)b(x,z) - d(z) - \int_{\mathcal{D}} \beta(y,x,z)\,dy\right) h_v(t,x,z) \\ &\quad + \mu\int_{\mathcal{X}} b(x,z')m(z',z)\,h_v(t,x,z')\,dz' + \left(\int_{\mathcal{D}} \eta(y,x,z)\,h_c(t,y,z)\,dy\right) \end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{\partial}{\partial t}h_u(t,y) &= \Delta h_u(t,y) - \left(\int_{E\times\mathcal{X}} \beta(y,x,z)\,h_v(t,x,z)\,\zeta_E(dx)\,dz\right) \\ &\quad + \int_{E\times\mathcal{X}} \eta(y,x,z)\,h_c(t,y,z)\,\zeta_E(dx)\,dz \end{aligned} \tag{3.11}$$

$$\begin{aligned}\frac{\partial}{\partial t}h_c(t,y,z) &= \Delta_y h_c(t,y,z) + \left(\int_E \beta(y,x,z)\,h_v(t,x,z)\,\zeta_E(dx)\right) \\ &\quad - \left(\int_E \eta(y,x,z)\,\zeta_E(dx)\right) h_c(t,y,z) \end{aligned} \tag{3.12}$$

with the same boundary conditions as before.

By Theorem 2.2 in Sato and Ueno (1965) we have that

$$\begin{aligned} h_c(t,y,z) &= \int_{\mathcal{D}} p(t,y,y';z)h_c(0,y,z)\,dy' \\ &\quad + \int_0^t \int_{\mathcal{D}} p(s,y,y';z)\left(\int_E \beta(y',x,z)\,h_v(t-s,x,z)\,\zeta_E(dx)\right) dy'\,ds \end{aligned}$$

where for every $z \in \mathcal{X}$, $p(t,y,y';z)$ is solution to:

$$\frac{\partial}{\partial t}p(t,y,y';z) = \Delta_y p(t,y,y';z) - \left(\int_E \eta(y,x,z)\,\zeta_E(dx)\right) p(t,y,y';z)$$

and the same boundary conditions as before.

Assuming $h_c(0,y,z) = 0$, reduces (3.10) to:

$$\begin{aligned}\frac{\partial}{\partial t}h_v(t,x,z) &= \left((1-\mu)b(x,z) - d(z) - \int_{\mathcal{D}} \beta(y,x,z)\,dy\right) h_v(t,x,z) \\ &\quad + \mu\int_{\mathcal{X}} b(x,z')m(z',z)\,h_v(t,x,z')\,dz' \\ &\quad + \int_{\mathcal{D}} \eta(y,x,z)\left(\int_E\int_0^t \left(\int_{\mathcal{D}} p(s,y,y';z)\beta(y',x,z)\,dy'\right) h_v(t-s,x',z)\,ds\,\zeta_E(dx')\right) dy. \end{aligned} \tag{3.13}$$

The positivity of the RHS of (3.13), which is a consequence of (1.12), concludes the proof.

3.4. *IDE formulation: second case.* Suppose Assumptions 2, 3, and 4 are satisfied. Then we have the following:

Theorem 3.6. *Let $\lambda \in (0,1)$ and assume $\mathcal{L}^u = \mathcal{L}^c = \Delta$. Consider the sequence of processes $\left\{{\nu_v}^{(K)}(t) : t \geq 0\right\}_{K\geq 1}$ given by (3.1). Assume that the function η does not depend on the trait z, and that for $\alpha \in \{u,c\}$, at time zero each of the limiting measures $\xi_\alpha(0)$ admits a density of the following form $\xi_u(0)(dy,dz) = g_u(y)\,dy\,dz$ and $\xi_c(0)(dy,dz) = g_c(y,z)\,dy\,dz$. Then for all $T > 0$, the* *sequence* $\left\{{\nu_v}^{(K)}(t) : t \geq 0\right\}_{K\geq 1}$ *converges in law in* $\mathbb{D}([0,T], \mathcal{M}_F(\mathcal{V}_p))$ *to a deterministic continuous function* ξ_v *belonging to the space* $\mathbb{C}([0,T], \mathcal{M}_F(\mathcal{V}_p))$*, and solving the IDE:*

$$\begin{aligned}
\langle \xi_v(t), \phi_v\rangle &= \langle \xi_v(0), \phi_v\rangle + \int_0^t \int_{\mathcal{V}_p} b(x,z)\phi_v(x,z)\, \xi_v(s)(dx,dz)\, ds \\
&+ \mu \int_0^t \int_{\mathcal{V}_p} b(x,z) \left[\int_{\mathcal{X}} m(z,e)\phi_v(x,e)\, de - \phi_v(x,z)\right] \xi_v(s)(dx,dz)\, ds \\
&- \int_0^t \int_{\mathcal{V}_p} \left[d(z) + c\langle \xi_v(s)^x, 1\rangle\right] \phi_v(x,z)\xi_v(s)(dx,dz)ds \\
&- \int_0^t \int_{\mathcal{V}_p} \left(\int_{\mathcal{M}_F(\mathcal{D})} \int_{\mathcal{D}} \beta(s,y,x,\langle \xi_v(s)^x, 1\rangle, z)\, \Pi^B_{\xi_v(s)}(\xi_u(dy) \times \mathcal{M}_F(\mathcal{V}_c))\right) \phi_v(x,z)\, \xi_v(s)(dx,dz)\, ds \\
&+ \int_0^t \int_E \left(\int_{\mathcal{M}_F(\mathcal{V}_c)} \int_{\mathcal{V}_c} \eta(s,y,x)\, \Pi^B_{\xi_v(s)}(\mathcal{M}_F(\mathcal{D}) \times \xi_c(dy,dz))\right) \phi_v(x,z)\, \zeta_E(dx)\, ds,
\end{aligned} \tag{3.14}$$

for all ϕ_v *bounded and measurable, and where for each* $\nu_v \in \mathcal{M}_F(\mathcal{V}_p)$ *the measure* $\Pi^B_{\nu_v}$ *is the unique stationary measure of the Markov process with generator* $B_{\nu_v} : D(B_{\nu_v}) \to \mathcal{C}(\mathcal{D}\times\mathcal{I})$ *given by:*

$$\begin{aligned}
B_{\nu_v} F_{\phi_u,\phi_c}(\nu_u,\nu_c) &= \partial_u F_{\phi_u,\phi_c}(\nu_u,\nu_c) \int_{\mathcal{D}} \Delta\phi_u(y)\, \nu_u(dy) + \partial_c F_{\phi_u,\phi_c}(\nu_u,\nu_c) \int_{\mathcal{V}_c} \Delta_w \phi_c(w,e)\, \nu_c(dw,de) \\
&- \partial_u F_{\phi_u,\phi_c}(\nu_u,\nu_c) \int_{\mathcal{V}_p\times\mathcal{D}} \beta(t,y,x,N_x(t),z)\phi_u(y)\nu_v(dx,dz)\, \nu_u(dy) \\
&+ \partial_c F_{\phi_u,\phi_c}(\nu_u,\nu_c) \int_{\mathcal{V}_p\times\mathcal{D}} \beta(t,y,x,N_x(t),z)\phi_c(y,z)\nu_v(dx,dz)\, \nu_u(dy) \\
&+ \partial_u F_{\phi_u,\phi_c}(\nu_u,\nu_c) \int_{\mathcal{V}_c\times E} \eta(t,y,x)\phi_u(y)\nu_c(dy,dz)\, \zeta_E(dx) \\
&- \partial_c F_{\phi_u,\phi_c}(\nu_u,\nu_c) \int_{\mathcal{V}_c\times E} \eta(t,y,x)\phi_c(y,z)\nu_c(dy,dz)\, \zeta_E(dx) \\
&+ \partial_u F_{\phi_u,\phi_c}(\nu_u,\nu_c) \int_{\mathcal{V}_c} \gamma(z)\phi_u(y)\nu_c(dy,dz) - \partial_c F_{\phi_u,\phi_c}(\nu_u,\nu_c) \int_{\mathcal{V}_c} \gamma(z)\phi_c(y,z)\nu_c(dy,dz),
\end{aligned} \tag{3.15}$$

where for all $\phi_u \in \mathcal{C}^2(\mathcal{D}) \cap \mathcal{D}(\Delta)$*, and* $\phi_c \in \mathcal{C}^2(\mathcal{V}_c) \cap D(\Delta_w)$*, the function* F_{ϕ_u,ϕ_c} *is defined in an analogous way to Definition 1.3:*

$$F_{\phi_u,\phi_c}(\nu_u,\nu_c) := F(\langle \nu_u, \phi_u\rangle, \langle \nu_c, \phi_c\rangle),$$

where $F \in \mathcal{C}^2(\mathbb{R}^2;\mathbb{R})$*.*

Remark 3.7. Notice that the presence of only first order derivatives in the defining expression of the generator B_{ν_v} implies that it corresponds to deterministic dynamics, and as a consequence the measure $\Pi^B_{\nu_v}$ is a Dirac mass on the set $\mathcal{M}_F(\mathcal{D}) \times \mathcal{M}_F(\mathcal{V}_c)$.

3.5. *Absolute continuity and stationarity of the measure* $\Pi^B_{\nu_v}$. Theorem 3.6 seems rather abstract due to the apparent lack of information about the measure $\Pi^B_{\nu_v}$. However, under Assumption 3 we can find an explicit expression for the measure $\Pi^B_{\nu_v}$. The proof of Theorem 3.6 (via Proposition 4.4) requires that the measure $\Pi^B_{\nu_v}$ satisfy

$$\int_{\mathcal{M}_F(\mathcal{D})\times\mathcal{M}_F(\mathcal{V}_c)} B_{\nu_v} F_{\phi_u,\phi_c}(\nu_u,\nu_c)\, \Pi^B_{\nu_v}(d\nu_u, d\nu_c) = 0 \tag{3.16}$$

for all cylindrical function $F_{\phi_u,\phi_c} : \mathcal{M}_F(\mathcal{D}) \times \mathcal{M}_F(\mathcal{V}_c) \to \mathbb{R}$.

Equation (3.16), and the precise form (3.15), imply that the support of the measure $\Pi^B_{\nu_v}$ is inside the set of all measures ν_u, ν_c such that:

$$\int_{\mathcal{D}} \Delta\phi_u(y)\, \nu_u(dy) - \int_{\mathcal{V}_p\times\mathcal{D}} \beta(t,y,x,N_x(t),z)\phi_u(y)\nu_v(dx,dz)\, \nu_u(dy) + \int_{\mathcal{V}_c\times E} \eta(t,y,x)\phi_u(y)\nu_c(dy,dz)\, \zeta_E(dx) = 0, \tag{3.17}$$

and

$$\int_{\mathcal{V}_c} \Delta_w\phi_c(w,e)\, \nu_c(dw,de) + \int_{\mathcal{V}_p\times\mathcal{D}} \beta(t,y,x,N_x(t),z)\phi_c(y,z)\nu_v(dx,dz)\, \nu_u(dy) - \int_{\mathcal{V}_c\times E} \eta(t,y,x)\phi_c(y,z)\nu_c(dy,dz)\, \zeta_E(dx) = 0, \tag{3.18}$$

for all $\phi_u \in \mathcal{C}^2(\mathcal{D}) \cap \mathcal{D}(\Delta)$, and $\phi_c \in \mathcal{C}^2(\mathcal{V}_c) \cap D(\Delta_w)$.

Remark 3.8. Recall that the total number of vectors $V_0 \neq 0$ is a conserved quantity. This is also true for the limit and can be seen directly from the system (3.17)-(3.18). This fact discards the trivial solution, $(\xi_u, \xi_c) = (0,0)$, as a potential solution for the system.

In order to apply Proposition 4.4, we want to use that the system (3.17)-(3.18) has a unique solution $(\xi_u, \xi_c) \in \mathcal{M}_F(\mathcal{D}) \times \mathcal{M}_F(\mathcal{V}_c)$ up to multiplicative factors. In order to see that this is the case we proceed in two steps. First we argue that the strong formulation of it has a unique solution. Second, we compare this unique solution to a generic (measure-valued) solution in total variation. We refer to Appendix B for details.

From uniqueness of solutions of the system (3.17)-(3.18), we have that

$$\text{Supp } \Pi^B_{\nu_v} \subseteq \{(\xi_u, \xi_c)\}.$$

The fact that $\Pi^B_{\nu_v}$ is a probability measure implies that it is of the form $\Pi^B_{\nu_v} = \delta_{\xi_u,\xi_c}$, where $\xi_u \in \mathcal{M}_F(\mathcal{D})$, and $\xi_c \in \mathcal{M}_F(\mathcal{V}_c)$ satisfy (3.17)-(3.18).

4. Proofs of Theorem 3.3 and Theorem 3.6

For the proofs of Theorem 3.3 and Theorem 3.6 we use the standard compactness-uniqueness approach. The proof of both theorems uses the same type of techniques with the exception of the part dealing with the characterization of limit points. The characterization of limit points for Theorem 3.6 is interesting in its own. It is an application of an averaging principle due to T. Kurtz and given originally in Kurtz (1992).

Uniqueness of mild-solutions for Theorem 3.3. Here we will combine the arguments for uniqueness used in the proof of Theorem 5.3 from Fournier and Méléard (2004), and in the proof of Theorem 4.2 in Champagnat and Méléard (2007). Let us first assume that $(\xi_v(t), \xi_u(t), \xi_c(t))_{t\geq 0}$, and $(\bar{\xi}_v(t), \bar{\xi}_u(t), \bar{\xi}_c(t))_{t\geq 0}$ are solutions of (3.2)-(3.4). We want to show that for all $\alpha \in \{v,u,c\}$ we have:

$$\left\|\xi_\alpha - \bar{\xi}_\alpha\right\|_\alpha = 0,$$

where for ν^1_α and $\nu^2_\alpha \in \mathcal{M}_F(D_\alpha)$ their variation norm is given by:

$$\left\|\nu^1_\alpha - \nu^2_\alpha\right\|_\alpha = \sup_{\substack{\phi_\alpha \in L^\infty(\mathcal{V}_\alpha) \\ \|\phi_\alpha\|_\infty \leq 1}} |\langle \nu^1_\alpha - \nu^2_\alpha, \phi_\alpha\rangle|.$$

where $\mathcal{V}_u = \mathcal{D}$.

Let us first deal with the case $\alpha = v$. Let ϕ_v be such that $\|\phi_v\|_\infty \leq 1$, by (3.2) we have:

$$
\begin{aligned}
\big|\langle \xi_v(t) - \bar{\xi}_v(t), \phi_v\rangle\big| &\leq (1-\mu)\int_0^t \left|\int_{\mathcal{V}_p} b(x,z)\phi_v(x,z)\ \big[\xi_v(s)(dx,dz) - \bar{\xi}_v(s)(dx,dz)\big]\right| ds \\
&+ \mu \int_0^t \left|\int_{\mathcal{V}_p} b(x,z)\int_{\mathcal{X}} m(z,z')\phi_v(x,z')\,dz'\ \big[\xi_v(s)(dx,dz) - \bar{\xi}_v(s)(dx,dz)\big]\right| ds \\
&+ \int_0^t \left|\int_{\mathcal{V}_p} d(z)\phi_v(x,z)\,\big[\xi_v(s)(dx,dz) - \bar{\xi}_v(s)(dx,dz)\big]\right| ds \\
&+ c\int_0^t \left|\int_{\mathcal{V}_p} \langle {\xi_v}^x(s), 1\rangle\,\phi_v(x,z)\,\big[\xi_v(s)(dx,dz) - \bar{\xi}_v(s)(dx,dz)\big]\right| ds \\
&+ c\int_0^t \left|\int_{\mathcal{V}_p} \big[\langle \bar{\xi}_v^x(s), 1\rangle - \langle {\xi_v}^x(s), 1\rangle\big]\,\phi_v(x,z)\,\bar{\xi}_v(s)(dx,dz)\right| ds \\
&+ \int_0^t \left|\int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v^x(s),1\rangle,z)\,\xi_u(s)(dy)\,\phi_v(x,z)\ \big[\xi_v(s)(dx,dz) - \bar{\xi}_v(s)(dx,dz)\big]\right| ds \\
&+ \int_0^t \left|\int_{\mathcal{V}_p\times\mathcal{D}} \big[\beta(s,y,x,\langle \xi_v^x(s),1\rangle,z) - \beta(s,y,x,\langle \bar{\xi}_v^x(s),1\rangle,z)\big]\ \xi_u(s)(dy)\,\phi_v(x,z)\,\bar{\xi}_v(s)(dx,dz)\right| ds \\
&+ \int_0^t \left|\int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \bar{\xi}_v^x(s),1\rangle,z)\ \big[\xi_u(s)(dy) - \bar{\xi}_u(s)(dy)\big]\ \phi_v(x,z)\,\bar{\xi}_v(s)(dx,dz)\right| ds \\
&+ \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x,z)\phi_v(x,z)\,\zeta_E(dx)\ \big[\xi_c(s)(dy,dz) - \bar{\xi}_c(s)(dy,dz)\big]\ ds. \qquad (4.1)
\end{aligned}
$$

Notice that the linear terms in the RHS of (4.1) can be bounded from above using $\|\phi_v\|_\infty \leq 1$, and Assumption 1. For example the first term can be bounded from above as

$$
\int_0^t \left|\int_{\mathcal{V}_p} b(x,z)\phi_v(x,z)\ \big[\xi_v(s)(dx,dz) - \bar{\xi}_v(s)(dx,dz)\big]\right| ds \leq \bar{b}\int_0^t \sup_{\phi\leq 1}|\langle \xi_v(s) - \bar{\xi}_v(s), \phi\rangle|\,ds.
$$

Similar bounds can be derived for the other linear terms. The main difficulty comes from the non-linear terms. Let us first deal with the terms coming from competition among viruses, i.e. the integrals with the competition constant c in front. First notice that we have

$$
\begin{aligned}
\big|\langle \bar{\xi}_v^x(s), 1\rangle - \langle {\xi_v}^x(s), 1\rangle\big| = \left|\int_{\mathcal{X}} 1\,\big[\xi_v^x(s)(dz) - \bar{\xi}_v^x(s)(dz)\big]\right| &\leq \left|\int_{E\times\mathcal{X}} \mathbb{1}_{\{x\}}(x')\ \big(\xi_v(s) - \bar{\xi}_v(s)\big)(dx',dz)\right| \\
&\leq \sup_{\phi\leq 1}|\langle \xi_v(s) - \bar{\xi}_v(s), \phi\rangle|,
\end{aligned}
$$

and by Remark 3.4:

$$
\begin{aligned}
&\int_0^t \left|\int_{\mathcal{V}_p} \big[\langle \bar{\xi}_v^x(s), 1\rangle - \langle {\xi_v}^x(s), 1\rangle\big]\,\phi_v(x,z)\,\bar{\xi}_v(s)(dx,dz)\right| ds \\
&\qquad \leq e^{Ct}\max_\alpha\left(\langle \xi_\alpha(0), 1\rangle\right)\int_0^t \sup_{\phi\leq 1}|\langle \xi_v(s) - \bar{\xi}_v(s), \phi\rangle|\,ds.
\end{aligned}
$$

Additionally, also by Remark 3.4 we have:

$$\int_0^t \left| \int_{\mathcal{V}_p} \langle {\xi_v}^x(s), 1\rangle \, \phi_v(x,z) \left[\xi_v(s)(dx,dz) - \bar{\xi}_v(s)(dx,dz)\right] \right| ds$$

$$\leq e^{Ct} \max_\alpha \left(\langle \xi_\alpha(0), 1\rangle\right) \int_0^t \sup_{\phi\leq 1} |\langle \xi_v(s) - \bar{\xi}_v(s), \phi\rangle| \, ds.$$

Now we deal with the non-linear terms representing interactions with free vectors. First, by Assumption 1, we have:

$$\int_{\mathcal{D}} \beta(s,y,x,\langle \xi_v^x(s), 1\rangle, z) \, \left[\xi_u(s)(dy) - \bar{\xi}_u(s)(dy)\right] \leq \bar{\beta} \sup_{\phi\leq 1} |\langle \xi_u(s) - \bar{\xi}_u(s), \phi\rangle|,$$

and by Remark (3.4) we deduce:

$$\int_0^t \left| \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v^x(s), 1\rangle, z) \, \left[\xi_u(s)(dy) - \bar{\xi}_u(s)(dy)\right] \, \phi_v(x,z) \, \xi_v(s)(dx,dz) \right| ds$$

$$\leq \bar{\beta} \, e^{Ct} \max_\alpha \left(\langle \xi_\alpha(0), 1\rangle\right) \int_0^t \sup_{\phi\leq 1} |\langle \xi_u(s) - \bar{\xi}_u(s), \phi\rangle| \, ds.$$

Analogously we also obtain:

$$\int_0^t \left| \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \bar{\xi}_v^x(s), 1\rangle, z) \, \left[\xi_u(s)(dy) - \bar{\xi}_u(s)(dy)\right] \, \phi_v(x,z) \, \bar{\xi}_v(s)(dx,dz) \right| ds$$

$$\leq \bar{\beta} \, e^{Ct} \max_\alpha \left(\langle \xi_\alpha(0), 1\rangle\right) \int_0^t \sup_{\phi\leq 1} |\langle \xi_u(s) - \bar{\xi}_u(s), \phi\rangle| \, ds.$$

The Lipschitz continuity of β also implies the upper bound:

$$\int_0^t \left| \int_{\mathcal{V}_p\times\mathcal{D}} \left[\beta(s,y,\langle \xi_v^x(s), 1\rangle, z) - \beta(s,y,x,\langle \bar{\xi}_v^x(s), 1\rangle, z)\right] \, \xi_u(s)(dy) \, \phi_v(x,z) \, \bar{\xi}_v(s)(dx,dz) \right| ds$$

$$\leq \bar{\beta} \, e^{Ct} \left[\max_\alpha \left(\langle \xi_\alpha(0), 1\rangle\right)\right]^2 \int_0^t \sup_{\phi\leq 1} |\langle \xi_u(s) - \bar{\xi}_u(s), \phi\rangle| \, ds.$$

Altogether implies:

$$\left|\langle \xi_v(t) - \bar{\xi}_v(t), \phi_v\rangle\right| \leq C \left(\int_0^t \sum_\alpha \sup_{\phi\leq 1} |\langle \xi_\alpha(s) - \bar{\xi}_\alpha(s), \phi\rangle| \, ds. \right) \tag{4.2}$$

For $\alpha \in \{u, c\}$, we cannot proceed in the same way to bound

$$\left|\langle \xi_\alpha(t) - \bar{\xi}_\alpha(t), \phi_\alpha\rangle\right|.$$

The reason for the additional difficulty is that for a generic measurable ϕ_α, the action of the generator $\mathcal{L}^\alpha$ is not necessarily well-defined. We will avoid this issue by using an approach similar to the one given in the proof of Theorem 4.2 in Champagnat and Méléard (2007).

For $\alpha \in \{u, c\}$, we consider the semi-groups $P_\alpha(t)$ corresponding to the diffusion process with generator $\mathcal{L}^\alpha$. Let us fix a function $\phi_\alpha \in L^\infty(\mathcal{V}_\alpha)$ with $\|\phi_\alpha\| \leq 1$, take $t \in [0,T]$, and define the following function:

$$\phi_\alpha(s, y_\alpha) = P_\alpha(t-s)\phi_\alpha(y_\alpha) \quad \forall y_\alpha \in \mathcal{V}_\alpha.$$

By construction $\phi_\alpha(s, y_\alpha)$ is a solution of the boundary problem:

$$\begin{aligned} \partial_s \phi_\alpha(s, y_\alpha) + \mathcal{L}^\alpha \phi_\alpha(s, y_\alpha) = 0 \quad &\text{on } [0,T] \times \mathcal{V}_\alpha \\ \nabla \phi_\alpha \cdot \bar{n}(s, y_\alpha) = 0 \quad &\text{on } [0,T] \times \partial\mathcal{V}_\alpha \\ \lim_{s \to t} \phi_\alpha(s, y_\alpha) = \phi_\alpha(y_\alpha) \quad &\text{on } \mathcal{V}_\alpha \end{aligned}$$

where, to save some space, we have made a slight abuse of notation by using $\partial\mathcal{V}_c = \partial\mathcal{D} \times \mathcal{X}$.

Using the weak time-space formulation given in (3.6) and 3.7, we obtain:

$$\begin{aligned} \langle \xi_u(t), \phi_u \rangle = \langle \xi_u(0), P_u(t)\phi_u \rangle &+ \int_0^t \int_{\mathcal{V}_c} \gamma(z) P_u(t-s)\phi_u(y)\, \xi_c(s)(dy, dz) ds \\ &+ \int_0^t \int_{\mathcal{V}_c \times E} \eta(s, y, x, z) P_u(t-s)\phi_u(y)) \xi_c(s)(dy, dz)\, \zeta_E(dx)\, ds \\ &- \int_0^t \int_{\mathcal{V}_p \times \mathcal{D}} \beta(s, y, x, \langle \xi_v^x(s), 1 \rangle, z) P_u(t-s)\phi_u(y) \xi_v(s)(dx, dz)\, \xi_u(s)(dy)\, ds, \end{aligned}$$

and

$$\begin{aligned} \langle \xi_c(t), \phi_c \rangle = \langle \xi_c(0), P_c(t)\phi_c \rangle &- \int_0^t \int_{\mathcal{V}_c \times E} \eta(s, y, x, z) P_c(t-s)\phi_c(y, z) \xi_c(s)(dy, dz)\, \zeta_E(dx)\, ds \\ &+ \int_0^t \int_{\mathcal{V}_p \times \mathcal{D}} \beta(s, y, x, \langle \xi_v^x(s), 1 \rangle, z) P_c(t-s)\phi_c(y, z) \xi_v(s)(dx, dz)\, \xi_u(s)(dy)\, ds \\ &- \int_0^t \int_{\mathcal{V}_c} \gamma(z) P_c(t-s)\phi_c(y, z)\, \xi_c(s)(dy, dz)\, ds. \end{aligned}$$

Analogous estimates to those used for the case $\alpha = v$, give

$$\left|\langle \xi_u(t) - \bar{\xi}_u(t), \phi_u \rangle\right| \leq C \left(\int_0^t \sum_\alpha \sup_{\phi_\alpha \leq 1} |\langle \xi_\alpha(s) - \bar{\xi}_\alpha(s), \phi_\alpha \rangle|\, ds \right)$$

and

$$\left|\langle \xi_c(t) - \bar{\xi}_c(t), \phi_c \rangle\right| \leq C \left(\int_0^t \sum_\alpha \sup_{\phi_\alpha \leq 1} |\langle \xi_\alpha(s) - \bar{\xi}_\alpha(s), \phi_\alpha \rangle|\, ds \right)$$

where we have picked the biggest constant C of all cases. Taking first the sup on the LHS of each case, and then summation gives the inequality:

$$\sum_\alpha \sup_{\phi_\alpha \leq 1} \left|\langle \xi_\alpha(t) - \bar{\xi}_\alpha(t), \phi_\alpha \rangle\right| \leq C \left(\int_0^t \sum_\alpha \sup_{\phi_\alpha \leq 1} |\langle \xi_\alpha(s) - \bar{\xi}_\alpha(s), \phi_\alpha \rangle|\, ds \right),$$

and by Gronwall's lemma we conclude uniqueness of mild solutions.

Uniform estimates. Let us first remark that, under Assumption 4, we have the following estimate:

$$\sup_{K \in \mathbb{N}} \mathbb{E} \left[\sup_{t \in [0,T]} \langle \nu_\alpha^{(K)}(t), 1 \rangle^3 \right] < +\infty, \tag{4.3}$$

for every $\alpha \in \{v, u, c\}$.

To see that this is the case, notice that by replicating the procedure of the proof of Proposition A.5, with $p = 3$, we can obtain an analogous expression to (A.6), namely

$$\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_n^{(K)}]}\langle\nu_\alpha^{(K)}(t),1\rangle^3\right]\leq Ce^{BT}$$

where the constants B and C are independent of K and n and τ_n are defined as in (A.5). Just as in Proposition A.5 we conclude that $\tau_n^{(K)}\to\infty$, as $K\to\infty$ for some well chosen n, and obtain (4.3) as a consequence of the RHS not depending on K.

Remark 4.1. As a consequence of estimate (4.3) we have that for every $\alpha\in\{v,u,c\}$

$$\sup_{K\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\langle\nu_\alpha^{(K)}(t),\phi_\alpha\rangle^3\right]<\infty \tag{4.4}$$

for all ϕ_α as in Definition 1.1.

Tightness. The goal of this section is to show that, for each $\alpha\in\{v,u,c\}$, the sequence of laws Q_α^K of the processes $\nu_\alpha^{(K)}$ are uniformly tight in $\mathbb{P}(\mathcal{D}([0,T],\mathcal{M}_F^\alpha))$. Here we specialize to the case $\lambda=1$. The other case is simpler for the vector populations, and for the virus population it is virtually done in the same way as for $\lambda=1$. Let us first introduce the following lemma:

Lemma 4.2. *Under Assumption 2 we have the following càdlàg martingales; for any admissible triplets* $\{\phi_v,\phi_u,\phi_c\}$.

(1) *For the virus population:*

$$\begin{aligned}
M_K^{v,\phi_v}(t) &= \langle{\nu_v}^{(K)}(t),\phi_v\rangle-\langle{\nu_v}^{(K)}(0),\phi_v\rangle-\int_0^t\int_{\mathcal{V}_p}b(x,z)\phi_v(x,z){\nu_v}^{(K)}(s)(dx,dz)\,ds\\
&+\mu\int_0^t\int_{\mathcal{V}_p}b(x,z)\left[\phi_v(x,z)-\int_{\mathcal{X}}m(z,e)\phi_v(x,e)\,de\right]{\nu_v}^{(K)}(s)(dx,dz)\,ds\\
&+\int_0^t\int_{\mathcal{V}_p}\left(d(z)+c\,\langle(\nu_v^{(K)})^x(s),1\rangle\right)\phi_v(x,z){\nu_v}^{(K)}(s)(dx,dz)ds\\
&\quad+\int_0^t\int_{\mathcal{V}_p\times\mathcal{D}}\beta(s,y,x,\langle(\nu_v^{(K)})^x(s),1\rangle,z)\,\phi_v(x,z){\nu_v}^{(K)}(s)(dx,dz)\,{\nu_u}^{(K)}(s)(dy)ds\\
&\quad-\int_0^t\int_{\mathcal{V}_c\times E}\eta(s,y,x,z)\phi_v(x,z)\,{\nu_c}^{(K)}(s)(dy,dz)\,dx\,ds
\end{aligned}\tag{4.5}$$

is a càdlàg martingale with predictable quadratic variation

$$\langle M_K^{v,\phi_v}\rangle_t = \frac{(1-\mu)}{K}\int_0^t\int_{\mathcal{V}_p} b(x,z)\,\phi_v(x,z)^2\,{\nu_v}^{(K)}(s)(dx,dz)\,ds$$
$$+\frac{\mu}{K}\int_0^t\int_{\mathcal{V}_p\times\mathcal{X}} b(x,z)m(z,e)\,\phi_v(x,e)^2\,de\,\nu_v^{(K)}(s)(dx,dz)\,ds$$
$$+\frac{1}{K}\int_0^t\int_{\mathcal{V}_p}\Big(d(z)+c\,\langle({\nu_v}^{(K)})^x(s),1\rangle\Big)\,\phi_v(x,z)^2\,{\nu_v}^{(K)}(s)(dx,dz)$$
$$+\frac{1}{K}\int_0^t\int_{\mathcal{V}_p\times\mathcal{D}}\beta(s,y,x,\langle(\nu_v^{(K)})^x(s),1\rangle,z)\,\phi_v(x,z)^2\,{\nu_v}^{(K)}(s)(dx,dz)\,{\nu_u}^{(K)}(s)(dy)ds$$
$$+\frac{1}{K}\int_0^t\int_{\mathcal{V}_c\times E}\eta(s,y,x,z)\,\phi_v(x,z)^2\,{\nu_c}^{(K)}(s)(dy,dz)\,dx\,ds. \tag{4.6}$$

(2) *For the uncharged vector population:*

$$M_K^{u,\phi_u}(t) = \langle{\nu_u}^{(K)}(t),\phi_u\rangle - \langle{\nu_u}^{(K)}(0),\phi_u\rangle - \int_0^t\int_{\mathcal{D}}\mathcal{L}^u\phi_u(y)\,{\nu_u}^{(K)}(s)(dy)\,ds$$
$$+\int_0^t\int_{\mathcal{V}_p\times\mathcal{D}}\beta(s,y,x,\langle(\nu_v^{(K)})^x(s),1\rangle,z)\,\phi_u(y){\nu_v}^{(K)}(s)(dx,dz)\,{\nu_u}^{(K)}(s)(dy)\,ds$$
$$-\int_0^t\int_{\mathcal{V}_c\times E}\eta(s,y,x,z)\phi_u(y){\nu_c}^{(K)}(s)(dy,dz)\,dx\,ds - \int_0^t\int_{\mathcal{V}_c}\gamma(z)\phi_u(y)\,{\nu_c}^{(K)}(s)(dy,dz)ds \tag{4.7}$$

is a càdlàg martingale with predictable quadratic variation

$$\langle M_K^{u,\phi_u}\rangle_t = \frac{1}{K}\int_0^t\int_{\mathcal{D}}\sigma^u(y)^2\,|\nabla\phi_u(y)|^2\,{\nu_u}^{(K)}(s)(dy)\,ds$$
$$+\frac{1}{K}\int_0^t\int_{\mathcal{V}_p\times\mathcal{D}}\beta(s,y,x,\langle(\nu_v^{(K)})^x(s),1\rangle,z)\,\phi_u(y)^2\,{\nu_v}^{(K)}(s)(dx,dz)\,{\nu_u}^{(K)}(s)(dy)ds$$
$$+\frac{1}{K}\int_0^t\int_{\mathcal{V}_c\times E}\eta(s,y,x,z)\phi_u(y)^2{\nu_c}^{(K)}(s)(dy,dz)\,dx\,ds + \frac{1}{K}\int_0^t\int_{\mathcal{V}_c}\gamma(z)\phi_u(y)^2\,{\nu_c}^{(K)}(s)(dy,dz)\,ds. \tag{4.8}$$

(3) *For the charged-vector population:*

$$M_K^{c,\phi_c}(t) = \langle{\nu_c}^{(K)}(t),\phi_c\rangle - \langle{\nu_c}^{(K)}(0),\phi_c\rangle - \int_0^t\int_{\mathcal{V}_c}\mathcal{L}^c\phi_c(y,z)\,{\nu_c}^{(K)}(s)(dy,dz)\,ds$$
$$-\int_0^t\int_{\mathcal{V}_p\times\mathcal{D}}\beta(s,y,x,\langle(\nu_v^{(K)})^x(s),1\rangle,z)\,\phi_c(y,z){\nu_v}^{(K)}(s)(dx,dz)\,{\nu_u}^{(K)}(s)(dy)\,ds$$
$$+\int_0^t\int_{\mathcal{V}_c\times E}\eta(s,y,x,z)\phi_c(y,z){\nu_c}^{(K)}(s)(dy,dz)\,dx\,ds + \int_0^t\int_{\mathcal{V}_c}\gamma(z)\phi_c(y,z)\,{\nu_c}^{(K)}(s)(dy,dz)ds \tag{4.9}$$

is a càdlàg martingale with quadratic variation

$$\begin{aligned}\langle M_K^{c,\phi_c}\rangle_t &= \frac{1}{K}\int_0^t\int_{\mathcal{V}_c}\sigma^c(y)^2|\nabla_y\phi_c(y,z)|^2\,{\nu_c}^{(K)}(s)(dy,dz)\,ds\\ &+\frac{1}{K}\int_0^t\int_{\mathcal{V}_p\times\mathcal{D}}\beta(s,y,x,\langle(\nu_v^{(K)})^x(s),1\rangle,z)\,\phi_c(y,z)^2{\nu_v}^{(K)}(s)(dx,dz)\,{\nu_u}^{(K)}(s)(dy)\,ds\\ &+\frac{1}{K}\int_0^t\int_{\mathcal{V}_c\times E}\eta(s,y,x,z)\phi_c(y,z)^2{\nu_c}^{(K)}(s)(dw,du)\,dx\,ds\\ &+\frac{1}{K}\int_0^t\int_{\mathcal{V}_c}\gamma(y)\phi_c(y,z)^2\,{\nu_c}^{(K)}(s)(dy,dz)\,ds. \end{aligned}\tag{4.10}$$

Proof: The proof of this lemma is a replica of Proposition A.8 taking into consideration the extra powers of K appearing in the generators. □

In order to show tightness, and given the control given by Proposition A.5, it is enough (see for example Aldous (1978) or Joffe and Métivier (1986)) to show that for each $\alpha\in\{v,u,c\}$, and $\varepsilon>0$ there exist $\delta>0$ and $\rho>0$ such that:

$$\sup_{K\in\mathbb{N}}\mathbb{P}\left(\omega(M_K^{\alpha,1},\delta,T)>\rho\right)\le\varepsilon,\ \text{and}\ \sup_{K\in\mathbb{N}}\mathbb{P}\left(\omega(\langle M_K^{\alpha,1}\rangle,\delta,T)>\rho\right)\le\varepsilon,\tag{4.11}$$

where $M_K^{\alpha,1}$ are given as in Lemma 4.2 with $\phi_\alpha=1$, and for $\delta>0$, $\omega(X,\delta,T)$ denotes the standard modulus of continuity:

$$\omega(X,\delta,T)=\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}|X(t)-X(s)|.$$

Let us show how to control de modulus of continuity for the predictable quadratic variation process $\langle M_K^{\alpha,1}\rangle$.

Fix $\delta>0$, and consider stopping times $\tau_K,{\tau'}_K$ such that:

$$0\le\tau_K\le{\tau'}_K\le\tau_K+\delta\le T.$$

By Doob's inequality, there exists $C>0$ such that:

$$\mathbb{E}\left[\langle M_K^{\alpha,1}\rangle_{{\tau'}_K}-\langle M_K^{\alpha,1}\rangle_{\tau_K}\right]\le\sum_{\alpha'}\mathbb{E}\left[C\int_{\tau_K}^{\tau_K+\delta}\left(\langle\nu_{\alpha'}^{(K)}(s),1\rangle+\langle\nu_{\alpha'}^{(K)}(s),1\rangle^2\right)\,ds\right],\tag{4.12}$$

where we used the fact that the quadratic variation processes are given explicitly in (4.6), (4.8) and (4.10), together with the technique used in the proof of Proposition A.5. By (4.3), and a redefinition of the constant C, we obtain:

$$\mathbb{E}\left[\langle M_K^{\alpha,1}\rangle_{{\tau'}_K}-\langle M_K^{\alpha,1}\rangle_{\tau_K}\right]\le C\delta.$$

This concludes the proof of uniform tightness of the sequence Q_α^K.

4.1. *Proof of Theorem 3.3.* We start by noticing that since both populations are comparable in order, i.e. $\lambda=1$, the time rescaling of the diffusive generators for the population of vectors reduces to normal speed. In this section we will mainly follow the approach given in the proof of Theorem 4.2 from Champagnat and Méléard (2007). This approach requires to control the quadratic variation of some relevant martingales and show that, as a consequence of the vanishing quadratic variation, the martingales themselves also vanish as $K\to\infty$. Lemma 4.2 introduced the above mentioned martingales and their quadratic variations.

4.1.1. *Characterization of limit points for Theorem 3.3.* By tightness we know that for $\alpha \in \{v, u, c\}$ the sequence of measures Q^K_α contains at least a convergent sub-sequence. Let us denote by Q the limiting law in $\mathbb{P}(\mathcal{D}([0,T], \mathcal{M}_F))$ of any such sub-sequence, which by an abuse of notation we still denote by Q^K. We first want to argue that every process ξ_α with law Q is almost surely strongly continuous. We can see that this is indeed the case from the following estimate:

$$\sup_{t\in[0,T]} |\langle \nu^{(K)}_\alpha(t), \phi_\alpha\rangle - \langle \nu^{(K)}_\alpha(t-), \phi_\alpha\rangle| \leq \frac{\|\phi_\alpha\|_\infty}{K} \tag{4.13}$$

which is true because at every jump event we either kill, create, or re-distribute one virus. In order to show that Q only charges the continuous processes we follow the lines of Step 5, page 54 of Bansaye and Méléard (2015). Since, for each ϕ_α, the mapping $t \mapsto \sup_{t\in[0,T]} |\langle \xi_\alpha(t), \phi_\alpha\rangle - \langle \xi_\alpha(t-), \phi_\alpha\rangle|$ is continuous on $\mathcal{D}([0,T], \mathcal{M}_F)$ we can deduce that Q only charges continuous processes from $[0,T]$ to M_F endowed with the vague topology. To extend this reasoning to the case of M_F being endowed with the weak topology we can follow the lines of Step 6, page 24, in Champagnat et al. (2008).

We now show that the limits ξ_v, ξ_u and ξ_c indeed satisfy the system of IDE's. To do this we introduce the following quantities:

$$\begin{aligned} M^{v,\phi_v}_t(\xi_v,\xi_u,\xi_c) := {}& \langle \xi_v(t), \phi_v\rangle - \langle \xi_v(0), \phi_v\rangle - \int_0^t \int_{\mathcal{V}_p} b(x,z)\phi_v(x,z)\,\xi_v(s)(dx,dz)\,ds \\ & + \mu \int_0^t \int_{\mathcal{V}_p} b(x,z)\left[\phi_v(x,z) - \int_{\mathcal{X}} m(z,e)h(x,e)\,de\right] \xi_v(s)(dx,dz)\,ds \\ & + \int_0^t \int_{\mathcal{V}_p} \left[d(z) + c\langle \xi_v(s)^x, 1\rangle\right] \phi_v(x,z)\xi_v(s)(dx,dz)ds \\ & + \int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v(s)^x,1\rangle, z)\phi_v(x,z)\,\xi_v(s)(dx,dz)\,\xi_u(s)(dy)\,ds \\ & - \int_0^t \int_{\mathcal{I}\times E} \eta(s,y,x,z)\phi_v(x,z)\,\xi_c(s)(dy,dz)\,dx\,ds, \end{aligned}$$

$$\begin{aligned} M^{u,\phi_u}_t(\xi_v,\xi_u,\xi_c) := {}& \langle \xi_u(t), \phi_u\rangle - \langle \xi_u(0), \phi_u\rangle - \int_0^t \int_{\mathcal{D}} \mathcal{L}^u\phi_u(y)\,\xi_u(s)(dy)\,ds \\ & + \int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle \xi_v(s)^x,1\rangle, z)\phi_u(y)\xi_v(s)(dx,dz)\,\xi_u(s)(dy)\,ds \\ & - \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,w,x,u)f(w)\xi_c(s)(dw,du)\,dx\,ds - \int_0^t \int_{\mathcal{V}_c} \gamma(u)f(w)\,\xi_c(s)(dw,du)ds, \end{aligned}$$

and

$$\begin{aligned} M^{c,\phi_c}_t(\xi_v,\xi_u,\xi_c) := {}& \langle \xi_c(t), \phi_c\rangle - \langle \xi_c(0), \phi_c\rangle - \int_0^t \int_{\mathcal{V}_c} \mathcal{L}^c\phi_c(w,u)\,\xi_c(s)(dw,du)\,ds \\ & - \int_0^t \int_{\mathcal{V}_c\times\mathcal{D}} \beta(s,y,x,\langle \xi_v(s)^x,1\rangle, z)g(y,z)\xi_v(s)(dx,dz)\,\xi_u(s)(dy)\,ds \\ & + \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,w,x,u)\phi_c(w,u)\xi_c(s)(dw,du)\,dx\,ds + \int_0^t \int_{\mathcal{V}_c} \gamma(u)\phi_c(w,u)\,\xi_c(s)(dw,du)\,ds. \end{aligned}$$

In order to verify that the processes indeed satisfy the system of IDE's, our goal is to show that we have:

$$\mathbb{E}\left[|M^{\alpha,\phi_\alpha}_t(\xi_v,\xi_u,\xi_c)|\right] = 0, \tag{4.14}$$

for all $\alpha \in \{v,u,c\}$, and any $t \geq 0$.

Notice that the following relations hold:

$$M_K^{\alpha,\phi_\alpha}(t) = M_t^{\alpha,\phi_\alpha}({\nu_v}^{(K)}(t), {\nu_u}^{(K)}(t), {\nu_c}^{(K)}(t)), \qquad \mathbb{E}\left(|M_K^{\phi_\alpha}(t)|^2\right) = \mathbb{E}\left[\langle M_K^{\phi_\alpha}\rangle_t\right],$$

for all $\alpha \in \{v,u,c\}$. Notice that by the non-negativity of the LHS of (4.14), to prove (4.14) it is enough to show that the expectation of its square vanishes. By Assumption 1, Proposition 4.2, and estimates similar to the ones used in (4.12), we have:

$$\mathbb{E}\left[|M_K^{\phi_\alpha}(t)|^2\right] \leq \frac{C^\alpha_{\phi_\alpha,t}}{K},$$

where the constants $C^\alpha_{\phi_\alpha,t}$ may depend on time and the uniform norm of the functions ϕ_α, but do not depend on the scaling parameter K.

To conclude we need to show:

$$\lim_{K\to\infty} \mathbb{E}\left[|M_t^{\alpha,\phi_\alpha}({\nu_v}^{(K)}(t), {\nu_u}^{(K)}(t), {\nu_c}^{(K)}(t)|\right] = \mathbb{E}\left[|M_t^{\alpha,\phi_\alpha}(\xi_v,\xi_u,\xi_c)|\right],$$

along any convergent sub-sequence of $\{({\nu_v}^{(K)}(t), {\nu_u}^{(K)}(t), {\nu_c}^{(K)}(t))\}_{K\geq 1}$. This is true if we are able to show uniform integrability for each collection $\{M_t^{\alpha,\phi_\alpha}({\nu_v}^{(K)}(t), {\nu_u}^{(K)}(t), {\nu_c}^{(K)}(t))\}_{K\geq 1}$, for all $\alpha \in \{v,u,c\}$.

In order to do so, fix $\alpha \in \{v,u,c\}$, and let us show that $\{M_t^\alpha(\nu_v,\nu_u,\nu_c)\}_{K\geq 1}$ is indeed uniformly integrable. First notice, using Assumption 1 and uniform estimates over the parameters and the test functions, that for any (ν_1,ν_2,ν_3) we have:

$$|M_t^{\alpha,\phi_\alpha}(\nu_1,\nu_2,\nu_3)| \leq C(t,\phi_v,\phi_u,\phi_c) \sup_{t\in[0,T]} \left(1 + \langle \nu_1, 1\rangle^2 + \langle \nu_2, 1\rangle^2 + \langle \nu_3, 1\rangle^2\right),$$

and by (4.4) we have uniform integrability. This finishes the proof.

4.2. *Proof of Theorem 3.6.* Given the assumptions of Theorem 3.6, it is not straightforward to adapt the proof of the characterization of limit points of Theorem 3.3 to this case. The main difficulty is that we cannot control the quadratic variations (4.8) and (4.10) as directly as in the case $\lambda = 1$. However, we can make use of the averaging principle for slow-fast systems introduced in Kurtz (1992), and avoid the need to directly control those quadratic variations. We postpone the proof of Theorem 3.6 to Section 4.2.2. First we introduce the context of Kurtz's averaging principle for martingale problems.

4.2.1. *Averaging for martingale problems.* The proof of Theorem 3.6 is based on Theorem 2.1 from Kurtz (1992). For the sake of readability we will now state this theorem without proof. In the following $l_m(\mathcal{S}_2)$ denotes the space of measures $\mu \in \mathbb{R}_+ \times \mathcal{S}_2$ such that $\mu([0,t]\times\mathcal{S}_2) = t$ for all $t \geq 0$. We refer the reader to Kurtz (1992) for relevant definitions and a proof of the theorem.

Theorem 4.3 (T. Kurtz, 1992)**.** *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be complete separable metric spaces, and set $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$. For each K, let $\{(X_K, Y_K)\}$ be a stochastic process with sample paths in $D([0,\infty),\mathcal{S})$ adapted to a filtration $\{\mathcal{F}_t^K\}$. Assume that $\{X_K\}$ satisfies the compact containment condition, that is, for each $\varepsilon > 0$ and $T > 0$, there exists a compact set $\mathcal{C} \in \mathcal{S}_1$ such that*

$$\inf_K \mathbb{P}\left[X_K(t) \in \mathcal{C}, \forall\, t \leq T\right] \geq 1 - \varepsilon, \tag{4.15}$$

and assume that $\{Y_K(t) : t \geq 0, K \in \mathbb{N}\}$ *is relatively compact (as a collection of* $\mathcal{S}_2$*-valued random variables). Suppose the existence of an operator* $A : \mathcal{D}(A) \subset \mathcal{C}_b(\mathcal{S}_1) \to \mathcal{C}_b(\mathcal{S})$ *such that, for every* $f \in \mathcal{D}(A)$*, there exists a process* ϵ_K^f *for which*

$$f(X_K(t)) - \int_0^t Af(X_K(s), Y_K(s))ds + \epsilon_K^f(t) \tag{4.16}$$

is an $\mathcal{F}_t^K$*-martingale. Let* $\mathcal{D}(A)$ *be dense in* $\mathcal{C}_b(\mathcal{S}_1)$ *in the topology of uniform convergence on compact sets. Suppose that for each* $f \in \mathcal{D}(A)$ *and each* $T > 0$*, there exists* $p > 1$ *such that*

$$\sup_K \mathbb{E}\left[\int_0^T \mid Af(X_K(t), Y_K(t)) \mid^p dt\right] < \infty,$$

and

$$\lim_{K\to\infty} \mathbb{E}\left[\sup_{t\leq T} \mid \epsilon_K^f(t) \mid\right] = 0. \tag{4.17}$$

Let Γ_K *be the* $l_m(\mathcal{S}_2)$*-valued random variable given by*

$$\Gamma_K([0,t] \times B) = \int_0^t \mathbb{1}_{\{B\}}(Y_K(s))ds.$$

Then $\{X_K, \Gamma_K\}$ *is relatively compact in* $D([0,\infty), \mathcal{S}_1) \times l_m(\mathcal{S}_2)$ *and for any limit point* (X, Γ) *there exists a filtration* $\{\mathcal{G}_t\}$ *such that*

$$f(X(t)) - \int_0^t \int_{\mathcal{S}_2} Af(X(s), y)\Gamma(ds \times dy)$$

is a $\mathcal{G}_t$*-martingale for each* $f \in \mathcal{D}(A)$*.*

The way we will apply this theorem in the proof Theorem 3.6 is mainly based on the following proposition based on Example 2.3 from Kurtz (1992).

Proposition 4.4. *Let* $B : \mathcal{D}(B) \subset \mathcal{C}_b(\mathcal{S}_2) \to \mathcal{C}_b(\mathcal{S})$ *be such that there exists a countable subset* $\hat{D} \subset \mathcal{D}(B)$ *such that:*

$$\overline{\{(g, Bg) : g \in \hat{D}\}} = \overline{\{(g, Bg) : g \in \mathcal{D}(B)\}}, \tag{4.18}$$

where both closures are taken in $\mathcal{C}_b(\mathcal{S}_2) \times \mathcal{C}_b(\mathcal{S}_1 \times \mathcal{S}_2)$ *with respect to the topology of uniform convergence. Moreover, assume that there exists* $\mu_K \in \mathbb{R}$*, such that for all* $g \in \mathcal{D}(B)$*, the quantity*

$$g(Y_K(t)) - \int_0^t \mu_K Bg(X_K(s), Y_K(s))ds + \delta_K^g(t)$$

is an $\mathcal{F}_t^K$*-martingale,* $\mu_K \to \infty$*, and for each* $T > 0$

$$\lim_{K\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{\mu_K} \mid \delta_K^g(t) \mid\right] = 0.$$

Additionally, suppose that for every $s \leq t$ *there exists a unique measure* $\pi_{X(s)} \in \mathbb{P}(\mathcal{S}_2)$ *such that:*

$$\int_{\mathcal{S}_2} Bg(X(s), y)\pi_{X(s)}(dy) = 0. \tag{4.19}$$

Then, under the assumptions of Theorem 4.3 the limiting process X *is a solution of the martingale problem for the generator*

$$G_A f(x) = \int_{\mathcal{S}_2} Af(x, y)\pi_x(dy). \tag{4.20}$$

Proof: By Theorem 4.3 we have:

$$\int_{[0,t]\times\mathcal{S}_2} Bg(X(s),y)\Gamma(ds\times dy) \tag{4.21}$$

is a martingale. But (4.21) is continuous and of bounded variation and therefore constant. As a consequence, for each $g\in\mathcal{D}(B)$, with probability one

$$\int_{[0,t]\times\mathcal{S}_2} Bg(X(s),y)\Gamma(ds\times dy)=0 \tag{4.22}$$

for all $t>0$. Then by Lemma 1.4 of Kurtz (1992) there exists a $\mathbb{P}(\mathcal{S}_2)$-valued process $\{\pi_t^{(o)}\}$ such that:

$$\int_{[0,t]\times\mathcal{S}_2} h(s,y)\Gamma(ds\times dy)=\int_0^t\left(\int_{\mathcal{S}_2} h(s,y)\pi_s^{(o)}(dy)\right)\Gamma(ds\times\mathcal{S}_2)$$

with probability one, and for all Borel-measurable h in $[0,\infty)\times\mathcal{S}_2$.

By (4.18) we have that $\Gamma\in l_m(\mathcal{S}_2)$, and as a consequence (4.22) can be written as:

$$\int_0^t\left(\int_{\mathcal{S}_2} Bg(X(s),y)\pi_s^{(o)}(dy)\right)ds=0$$

for all t a.s., and hence

$$\int_{\mathcal{S}_2} Bg(X(s),y)\pi_s^{(o)}(dy)=0 \tag{4.23}$$

almost everywhere Lebesgue almost surely.

At this point assumption (4.19) implies that the measure $\pi_s^{(o)}$ is stationary for the process with generator $B_{X(s)}$ given by:

$$B_{X(s)}g(y)=Bg(X(s),y)$$

for all $g\in\mathcal{D}(B_{X(s)})$. This means that we can take $\pi_s^{(o)}=\pi_{X(s)}$. If follows, by Theorem 4.3, that X is the process associated to the martingale problem of the generator G_A given by (4.20). □

4.2.2. *Characterization of limit points for Theorem 3.6.* Let us define $X_K(t)$ and $Y_K(t)$ by:

$$X_K(t):=\nu_v^{(K)}(t)\in\mathcal{M}_F(\mathcal{V}_p)$$
$$Y_K(t):=({\nu_u}^{(K)}(t),{\nu_c}^{(K)}(t))\in\mathcal{M}_F(\mathcal{D})\times\mathcal{M}_F(\mathcal{V}_c)=:\mathcal{M}_F^{\text{vec}}.$$

In order to be able to apply Theorem 4.3 we need to proceed as follows:

(1) Verify that $\{\nu_v^{(K)}(t):t\geq 0\}$ satisfies the compact containment condition.
(2) Verify that $\{({\nu_u}^{(K)}(t),{\nu_c}^{(K)}(t)):t\geq 0,K\in\mathbb{N}\}$ is relatively compact.
(3) Identify the operator A, and related processes of Kurt's result.

Compact containment. Fix $\varepsilon>0$ and $T>0$. Consider the set $\mathcal{C}(T,\varepsilon)\subset\mathcal{M}_F(\mathcal{V}_p)$ given by:

$$\mathcal{C}(T,\varepsilon)=\{\mu\in\mathcal{M}_p(\mathcal{V}_p):\mu(\mathcal{V}_p)<\frac{C_T}{\varepsilon}\}$$

where C_T is equal to the RHS of (A.6) for $p=1$.

Remark 4.5. Notice that, as a consequence of $\mathcal{M}_F(\mathcal{V}_p)$ being Polish, the set $\mathcal{C}(T,\varepsilon)$ is sequentially compact and hence compact.

By Markov's inequality we have:

$$\begin{aligned}\mathbb{P}\left[\nu_v^{(K)}(t) \in \mathcal{C}(T,\varepsilon) : \forall\, t \in [0,T]\right] &= 1 - \mathbb{P}\left[\exists t \in [0,T] : \nu_v^{(K)}(t) \notin \mathcal{C}(T,\varepsilon)\right] \\ &\geq 1 - \mathbb{P}\left[\sup_{t\in[0,T]} \langle \nu_v^{(K)}(t), 1\rangle \geq \frac{C_T}{\varepsilon}\right] \\ &\geq 1 - \frac{\varepsilon}{C_T}\mathbb{E}\left[\sup_{t\in[0,T]} \langle \nu_v^{(K)}(t), 1\rangle\right] \geq 1-\varepsilon\end{aligned}$$

which shows (4.15).

Relative compactness for the fast system. We want to show that for every $t \geq 0$ the sequence $\{(\nu_u{}^{(K)}(t), \nu_c{}^{(K)}(t)) : K \in \mathbb{N}\}$ is relatively compact as a sequence of $\mathcal{M}_F^{\mathrm{vec}}$-valued random variables. By Corollary 1.2 in Kurtz (1992) it is enough to show that

$$\sup_{K\in\mathbb{N}} \mathbb{E}\left[\langle (\nu_u{}^{(K)}(t), \nu_c{}^{(K)}(t)), 1\rangle\right] < \infty \tag{4.24}$$

and that for each $\varepsilon > 0$, there exists a compact $\mathcal{C} \subseteq \bar{\mathcal{D}} \times \bar{\mathcal{D}} \times \mathcal{X}$ such that:

$$\limsup_{K\to\infty} \mathbb{P}\left[\langle (\nu_u{}^{(K)}(t), \nu_c{}^{(K)}(t)), 1_{\mathcal{C}^c}\rangle > \varepsilon\right] \leq \varepsilon$$

where $\mathcal{C}^c$ denotes the complement of $\mathcal{C}$ and

$$\langle (\nu_u{}^{(K)}(t), \nu_c{}^{(K)}(t)), 1\rangle := \langle \nu_u{}^{(K)}(t), 1\rangle + \langle \nu_c{}^{(K)}(t), 1\rangle.$$

Proposition A.5 gives (4.24), and we conclude by choosing $\mathcal{C} = \bar{\mathcal{D}} \times \bar{\mathcal{D}} \times \mathcal{X}$.

Identification of the operators A and B. Before identifying the operators A and B we introduce the following proposition.

Proposition 4.6. *Under Assumption 2, for any cylindrical function $F_v : \mathcal{M}_p(\mathcal{V}_p) \to \mathbb{R}$ of the form:*

$$F_v(\nu) = F(\langle \nu, \phi_v\rangle),$$

with $F \in \mathcal{C}^2(\mathbb{R})$ and ϕ_v measurable and bounded. We have that

$$\begin{aligned}M_K^{F_v}(t) &= F_v(\nu_v^{(K)}(t)) - K(1-\mu)\int_0^t \int_{\mathcal{V}_p} b(x,z)\left(F_v(\nu_v^{(K)}(s) + \frac{1}{K}\delta_{(x,z)}) - F_v(\nu_v^{(K)}(s))\right)\nu_v^{(K)}(s)(dx,dz)\,ds \\ &- K\mu\int_0^t \int_{\mathcal{V}_p\times\mathcal{X}} b(x,z)m(z,e)\left(F_v(\nu_v^{(K)}(s) + \frac{1}{K}\delta_{(x,e)}) - F_v(\nu_v^{(K)}(s))\right) de\,\nu_v^{(K)}(s)(dx,dz)\,ds \\ &+ K\int_0^t \int_{\mathcal{V}_p}\left(d(z) + c\,\langle (\nu_v^{(K)})^x, 1\rangle\right)\left(F_v(\nu_v^{(K)}(s) - \frac{1}{K}\delta_{(x,z)}) - F_v(\nu_v^{(K)}(s))\right)\nu_v^{(K)}(s)(dx,dz)ds \\ &+ K\int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle (\nu_v^{(K)})^x, 1\rangle, z)\left(F_v(\nu_v^{(K)}(s) - \frac{1}{K}\delta_{(x,z)}) - F_v(\nu_v^{(K)}(s))\right)\nu_v^{(K)}(s)(dx,dz)\,\nu_u{}^{(K)}(s)(dy)ds \\ &\quad - K\int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x)\left(F_v(\nu_v^{(K)}(s) + \frac{1}{K}\delta_{(x,z)}) - F_v(\nu_v^{(K)}(s))\right)\nu_c{}^{(K)}(s)(dy,dz)\,dx\,ds\end{aligned}$$

is a càdlàg martingale with predictable quadratic variation

$$\langle M_K^{F_v}\rangle_t = K(1-\mu)\int_0^t\int_{\mathcal{V}_p} b(x,z)\left(F_v(\nu_v^{(K)}(s)+\frac{1}{K}\delta_{(x,z)})-F_v(\nu_v^{(K)}(s))\right)^2\nu_v(s)(dx,dz)\,ds$$
$$+K\mu\int_0^t\int_{\mathcal{V}_p\times\mathcal{X}} b(x,z)m(z,e)\left(F_v(\nu_v^{(K)}(s)+\frac{1}{K}\delta_{(x,e)})-F_v(\nu_v^{(K)}(s))\right)^2 de\,\nu_v^{(K)}(s)(dx,dz)\,ds$$
$$+K\int_0^t\int_{\mathcal{V}_p}\left(d(z)+c\,\langle(\nu_v^{(K)})^x,1\rangle\right)\left(F_v(\nu_v^{(K)}(s)-\frac{1}{K}\delta_{(x,z)})-F_v(\nu_v^{(K)}(s))\right)^2\nu_v^{(K)}(s)(dx,dz)ds$$
$$+\int_0^t K\int_{\mathcal{V}_p\times\mathcal{D}}\beta(s,y,x,\langle(\nu_v^{(K)})^x,1\rangle,z)\left(F_v(\nu_v^{(K)}(s)-\frac{1}{K}\delta_{(x,z)})-F_v(\nu_v^{(K)}(s))\right)^2\nu_v^{(K)}(s)(dx,dz)\,{\nu_u}^{(K)}(s)(dy)ds$$
$$+K\int_0^t\int_{\mathcal{V}_c\times E}\eta(s,y,x)\left(F_v(\nu_v^{(K)}(s)+\frac{1}{K}\delta_{(x,z)})-F_v(\nu_v^{(K)}(s))\right)^2{\nu_c}^{(K)}(s)(dy,dz)\,dx\,ds.$$

Proof: The proof of this proposition is in the same spirit than the proof of Proposition A.6. It is an application of Dynkin's theorem and the specific form of the generator (1.5). □

We can use Taylor's theorem to expand $F(\langle\nu_v^{(K)}(s)\pm\frac{1}{K}\delta_{(x,z)},\phi_v\rangle)$ around $\langle\nu_v,\phi_v\rangle$ to obtain:

$$\left(F_v(\nu_v^{(K)}(s)\pm\frac{1}{K}\delta_{(x,z)})-F_v(\nu_v^{(K)}(s))\right)$$
$$=\pm\frac{1}{K}\phi_v(x,z)F_v'(\langle\nu_v^{(K)}(s),\phi_v\rangle)+\frac{1}{2K^2}\phi_v^2(x,z)F_v''(\langle\nu_v^{(K)}(s),\phi_v\rangle)+o(1/K^2), \tag{4.25}$$

where $o(h^q)$ represents a function $G(h)$ satisfying

$$\lim_{h\to 0}\frac{G(h)}{h^q}=0$$

i.e., a function satisfying Peano's form of the remainder of Taylor's theorem.

From (4.25), we can see that the martingale for the virus population becomes:

$$F_v(\nu_v^{(K)}(t))-\int_0^t AF_v(\nu_v^{(K)}(s),({\nu_u}^{(K)}(s),{\nu_c}^{(K)}(s)))\,ds+\epsilon_K^{F_v}(t)$$

where

$$AF_v(\nu_v,(\nu_u,\nu_c))$$
$$=F'(\langle\nu_v,\phi_v\rangle)\int_{\mathcal{V}_p}b(x,z)\phi_v(x,z)\nu_v(dx,dz)+\mu F'(\langle\nu_v,\phi_v\rangle)\int_{\mathcal{V}_p}\phi_v(x,z)\left[\phi_v(x,z)-\int_{\mathcal{X}}m(z,e)\phi_v(x,e)\,de\right]\nu_v(dx,dz)$$
$$+F'(\langle\nu_v,\phi_v\rangle)\int_{\mathcal{V}_p}(d(z)+c\,\langle\nu_v^x,1\rangle)\,\phi_v(x,z)\nu_v(dx,dz)-F'(\langle\nu_v,\phi_v\rangle)\int_E\left(\int_{\mathcal{V}_c}\eta(y,x)\phi_v(x,z)\,\nu_c(dy,dz)\right)dx$$
$$+F'(\langle\nu_v,\phi_v\rangle)\int_{\mathcal{V}_p}\left(\int_{\mathcal{D}}\beta(y,x,\langle(\nu_v^{(K)})^x,1\rangle,z)\,\nu_u(dy)\right)\phi_v(x,z)\nu_v(dx,dz)$$

with domain:

$$\mathcal{D}(A)=\{F_v: F\in\mathcal{C}^2(\mathbb{R}),\phi_v\text{ is measurable and bounded}\}\subset\mathcal{C}_b(\mathcal{M}_F(\mathcal{V}_p)). \tag{4.26}$$

Notice that under the assumptions of Theorem 3.6, the generator A satisfies (4.16) with $p=3$, and $\epsilon_K^{F_v}(t)$ satisfies (4.17).

Remark 4.7. Notice that $\mathcal{D}(A)$ generates the set $\mathcal{C}_b(\mathcal{M}_F(\mathcal{V}_p))$.

We now introduce the set of cylindrical functions on M_F^{vec}. These are functions that correspond to triplets:

$$\boldsymbol{\phi}_{uc}=\{\phi_v=0,\phi_u=\phi_u,\phi_c=\phi_c\}. \tag{4.27}$$

i.e., functions of the form:

$$F_{\text{vec}}((\nu_u, \nu_c)) := F_{\boldsymbol{\phi}_{uc}}((\nu_v, \nu_u, \nu_c)),$$

where $F \in \mathcal{C}^2(\mathbb{R}^3)$ only depends on the last two coordinates, i.e., $F(x, y, z) = G(y, z)$ for some $G \in \mathcal{C}^2(\mathbb{R}^2)$, and the admissible triplet $\boldsymbol{\phi}_{uc}$ is given as in (4.27) with ϕ_u and ϕ_c given as in Definition 1.1 .

In a similar way we now present the martingale for the vector population. Notice that to apply Theorem 4.3 and Example 2.3 from Kurtz (1992), we do not need to show that the quadratic variation of the fast process vanishes as K goes to infinity. We first introduce the following additional notation:

$$[F_{\text{vec}}(\nu_u, \nu_c)]^{K,\lambda,c}_{y,z} := F_{\text{vec}}(\nu_u - \tfrac{1}{K^\lambda}\delta_y, \nu_c + \tfrac{1}{K^\lambda}\delta_{y,z}) - F_{\text{vec}}(\nu_u, \nu_c) \tag{4.28}$$

and

$$[F_{\text{vec}}(\nu_u, \nu_c)]^{K,\lambda,u}_{y,z} := F_{\text{vec}}(\nu_u + \tfrac{1}{K^\lambda}\delta_y, \nu_c - \tfrac{1}{K^\lambda}\delta_{y,z}) - F_{\text{vec}}(\nu_u, \nu_c). \tag{4.29}$$

We then have the following proposition:

Proposition 4.8. *Under Assumption 2, for the joint process* $({\nu_u}^{(K)}, {\nu_c}^{(K)})$*,*

$$\begin{aligned}
M_K^{F_{vec}}(t) &= F_{vec}({\nu_u}^{(K)}(t), {\nu_c}^{(K)}(t)) - K^{1-\lambda}\int_0^t \partial_u F_{vec}({\nu_u}^{(K)}(s), {\nu_c}^{(K)}(s)) \int_{\mathcal{D}} \mathcal{L}^u\phi_u(y)\, {\nu_u}^{(K)}(s)(dy)\, ds \\
&- K^{1-2\lambda}\int_0^t \partial_u^2 F_{vec}({\nu_u}^{(K)}(s), {\nu_c}^{(K)}(s)) \int_{\mathcal{D}} \sigma^u(y)^2\, |\nabla\phi_u(y)|^2\, {\nu_u}^{(K)}(s)(dy)\, ds \\
&- K^{1-\lambda}\int_0^t \partial_c F_{vec}({\nu_u}^{(K)}(s), {\nu_c}^{(K)}(s)) \int_{\mathcal{V}_c} \mathcal{L}^c\phi_c(y,z)\, {\nu_c}^{(K)}(s)(dy,dz)\, ds \\
&- K^{1-2\lambda}\int_0^t \partial_c^2 F_{vec}({\nu_u}^{(K)}(s), {\nu_c}^{(K)}(s)) \int_{\mathcal{V}_c} \sigma^c(y)^2 |\nabla_y\phi_c(y,z)|^2\, {\nu_c}^{(K)}(s)(dy,dz)\, ds \\
&- K\int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle\nu_v^x,1\rangle,z) \left(\left[F_{vec}({\nu_u}^{(K)}(s), {\nu_c}^{(K)}(s))\right]^{K,\lambda,c}_{y,z}\right) \nu_v^{(K)}(s)(dx,dz)\, {\nu_u}^{(K)}(s)(dy)\, ds \\
&- K\int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x) \left(\left[F_{vec}({\nu_u}^{(K)}(s), {\nu_c}^{(K)}(s))\right]^{K,\lambda,u}_{y,z}\right) {\nu_c}^{(K)}(s)(dy,dz)\, dx\, ds \\
&- K\int_0^t \int_{\mathcal{V}_c} \gamma(z) \left(\left[F_{vec}({\nu_u}^{(K)}(s), {\nu_c}^{(K)}(s))\right]^{K,\lambda,c}_{y,z}\right) {\nu_c}^{(K)}(s)(dy,dz)\, ds
\end{aligned}$$

is a càdlàg martingale.

Using a Taylor expansion analogous to that in (4.25), we reveal the following form for the martingale of the fast system:

$$F_{\text{vec}}({\nu_u}^{(K)}(t), {\nu_c}^{(K)}(t)) - \int_0^t K^{1-\lambda} B F_{\text{vec}}({\nu_v}^{(K)}(s), ({\nu_u}^{(K)}(s), {\nu_c}^{(K)}(s)))\, ds + \delta_K^{F_{\text{vec}}}(t)$$

where the operator $BF_{\text{vec}}(\nu_v, (\nu_u, \nu_c))$ is given as follows:

$$BF_{\text{vec}}(\nu_v, (\nu_u, \nu_c)) = \partial_u F_{\text{vec}}(\nu_u, \nu_c) \int_{\mathcal{D}} \mathcal{L}^u \phi_u(y)\, \nu_u(dy) + \partial_c F_{\text{vec}}(\nu_u, \nu_c) \int_{\mathcal{V}_c} \mathcal{L}^c \phi_c(y,z)\, \nu_c(dy,dz)$$

$$- \partial_u F_{\text{vec}}(\nu_u, \nu_c) \int_{\mathcal{V}_p \times \mathcal{D}} \beta(s, y, x, \langle \nu_v^x, 1\rangle, z) \phi_u(y) \nu_v(dx, dz)\, \nu_u(dy)$$

$$+ \partial_c F_{\text{vec}}(\nu_u, \nu_c) \int_{\mathcal{V}_p \times \mathcal{D}} \beta(s, y, x, \langle \nu_v^x, 1\rangle, z) \phi_c(y,z) \nu_v(dx, dz)\, \nu_u(dy$$

$$+ \partial_u F_{\text{vec}}(\nu_u, \nu_c) \int\limits_{\mathcal{V}_c \times E} \eta(s, y, x) \phi_u(y) \nu_c(dy, dz)\, dx - \partial_c F_{\text{vec}}(\nu_u, \nu_c) \int\limits_{\mathcal{V}_c \times E} \eta(s, y, x) \phi_c(y,z) \nu_c(dy, dz)\, dx$$

$$+ \partial_u F_{\text{vec}}(\nu_u, \nu_c) \int_{\mathcal{V}_c} \gamma(z) \phi_u(y) \nu_c(dy, dz) - \partial_c F_{\text{vec}}(\nu_u, \nu_c) \int_{\mathcal{V}_c} \gamma(z) \phi_c(y,z) \nu_c(dy, dz) \tag{4.30}$$

with domain:

$$\mathcal{D}(B) := \{F_{\text{vec}} : F \in \mathcal{C}^2(\mathbb{R}), \phi_u \in \mathcal{C}_0^2(\mathcal{D}), \phi_c \in \mathcal{C}_0^{2,0}(\mathcal{V}_c)\} \subset \mathcal{C}_b(\mathcal{M}_F^{\text{vec}})$$

and

$$\delta_K^{F_{\text{vec}}}(t) = O(K^{1-2\lambda})$$

Notice that, since $\lambda > 0$, indeed we have:

$$\lim_{K \to \infty} K^{\lambda - 1} \mathbb{E}\left[\sup_{t \in [0,T]} \mid \delta_K^{F_{\text{vec}}}(t) \mid \right] = 0$$

for any $T > 0$.

As a consequence we have the following martingale:

$$\int_0^t \int_{\mathcal{M}_F} BF_{\text{vec}}(\xi_v(s), (\xi_u, \xi_c)) \Gamma(ds \times d(\xi_u, \xi_c)) \tag{4.31}$$

where $\{\xi_v, \xi_u, \xi_c\}$ is a limit of a converging sub-sequence, and Γ is given as in Theorem 4.3.

Following Example 2.3 of Kurtz (1992) we need to find a countable subset of the domain $\mathcal{D}(B)$ such that (4.18) is satisfied, and as a consequence (4.31) can be re-written as:

$$\int_0^t \left(\int_{\mathcal{M}_F} BF_{\text{vec}}(\xi_v(s), (\xi_u, \xi_c)) \Pi_{\xi_v(s)}(d(\xi_u, \xi_c)) \right) ds = 0 \tag{4.32}$$

Countable set. In order to find the desired countable set that verifies (4.18) we proceed as follows:

First, notice that by the compactness of the closure of $\mathcal{D}$, denoted by $\bar{\mathcal{D}}$, and $\mathcal{X}$ we know that the space $\mathcal{M}_F^{\text{vec}}$ is a locally compact separable and metrizable space (this can be found for example in Theorem 1.14 of Li (2011)). We denote by $\overline{\mathcal{M}_F^{\text{vec}}}$ the one point compactification of $\mathcal{M}_F^{\text{vec}}$. That is:

$$\overline{\mathcal{M}_F^{\text{vec}}} = \mathcal{M}_F^{\text{vec}} \cup \{\mu_\infty\}$$

where we have extended the weak topology by imposing $\mu_n \to \mu_\infty$ if and only if $\mu_n(\bar{\mathcal{D}} \times \bar{\mathcal{D}} \times \mathcal{X}) \to \infty$.

Second, we have that $\mathcal{C}(\overline{\mathcal{M}_F^{\text{vec}}})$ is Polish, and hence separable, with the uniform norm (see for example Kechris (1995)). As a consequence $\mathcal{C}_{b,0}(\mathcal{M}_F^{\text{vec}})$ (bounded continuous functions vanishing at μ_∞) is separable as well (because the one point compactification of $\mathcal{M}_F^{\text{vec}}$ is metrizable by for example Theorem 5.3 in Kechris (1995)).

Third, we know that $\mathcal{D}(B)$ generates the whole $\mathcal{C}_b(\mathcal{M}_F^{\text{vec}})$. As a consequence $\hat{\mathcal{D}}(B)$, given by the restriction of $\mathcal{D}(B)$ to the set generating only the functions vanishing at infinity, is separable as well under the topology of uniform convergence.

Finally, by construction, the countable dense set $\hat{D}$ (witnessing the separability of $\hat{\mathcal{D}}(B)$) is the set that verifies (4.18).

Conclusion from the averaging principle. From (4.32) we conclude that

$$\int_{\mathcal{M}_F} BF_{\text{vec}}(\xi_v(t),(\xi_u,\xi_c))\Pi_{\xi_v(t)}(d(\xi_u,\xi_c)) = 0$$

for all t almost surely. By Example 2.3 from Kurtz (1992), we obtain that

$$F_v(\xi_v(t)) - \int_0^t AF_v(\xi_v(s))\,ds \tag{4.33}$$

is a Martingale, where

$$\begin{aligned}
&AF_v(\xi_v)\\
&= F'(\langle \xi_v,\phi_v\rangle)\int_{\mathcal{V}_p} b(x,z)\phi_v(x,z)\nu_v(dx,dz) + \mu F'(\langle \xi_v,\phi_v\rangle)\int_{\mathcal{V}_p} b(x,z)\left[\phi_v(x,z) - \int_{\mathcal{X}} m(z,e)\phi_v(x,e)\,de\right]\nu_v(dx,dz)\\
&+ F'(\langle \xi_v,\phi_v\rangle)\int_{\mathcal{V}_p}(d(z)+c\,\langle \nu_v^x,1\rangle)\,\phi_v(x,z)\nu_v(dx,dz)\\
&+ F'(\langle \xi_v,\phi_v\rangle)\int_{\mathcal{V}_p}\left(\int_{\mathcal{M}_F^{\text{vec}}}\int_{\mathcal{D}} \beta(s,y,x,\langle \nu_v^x,1\rangle,z)\,\Pi^B_{\xi_v(s)}(\nu_u(dy)\times\mathcal{M}_F(\mathcal{V}_c))\right)\phi_v(x,z)\nu_v(dx,dz)\\
&- F'(\langle \xi_v,\phi_v\rangle)\int_E\left(\int_{\mathcal{M}_F^{\text{vec}}}\int_{\mathcal{V}_c}\eta(s,y,x)\phi_v(x,z)\,\Pi^B_{\xi_v(s)}(\mathcal{M}_F(\mathcal{D})\times\nu_c(dy,dz))\right)dx
\end{aligned}$$

where $\Pi^B_{\xi_v(s)}$ is stationary for the generator (4.30).

Remark 4.9. Notice that the presence of the function $F \in \mathcal{C}^2(\mathbb{R})$ only through it first derivative implies that the process $\{\xi_v\}$ is deterministic, so that the martingale in (4.33) is equal to zero. As a consequence of this observation we can use the special case $F(x) = x$, which gives an expression that indeed corresponds to the RHS of (3.14).

To conclude the characterization of the limit points for the case $\lambda < 1$, notice that the same arguments given for the case $\lambda = 1$ in Section 4.1.1 can be easily adapted to this case.

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Appendix A. Construction of the process

Here we provide the technical details that guarantee the well-definedness of our process. We first present the following proposition, which is necessary to control the growth of the process and avoid explosions:

Proposition A.1. *Under Assumption 1 there exists a positive constant C such that for every $\nu \in \mathcal{M}_p := \mathcal{M}_p(E\times\mathcal{X})\times\mathcal{M}_p(\mathcal{D})\times\mathcal{M}_p(\mathcal{D}\times\mathcal{X})$ the global jump rate, i.e. the rate at which a jump event takes place, is bounded by:*

$$C\langle 1,\nu\rangle\,(1+\langle 1,\nu\rangle)\,.$$

Proof: For $\boldsymbol{\nu} = (\nu_v, \nu_u, \nu_c) \in \mathcal{M}_p$, we denote by $R(\boldsymbol{\nu})$ its total jump rate. This rate is bounded from above by:

$$R(\boldsymbol{\nu}) \leq \int_{E\times\mathcal{X}} b(x,z)\nu_v(dx,dz) + \int_{E\times\mathcal{X}} \left[d(z) + c\int_{\mathcal{X}} \nu_v^{(x)}(de)\right]\nu_v(dx,dz) + \int_{\mathcal{V}_p\times\mathcal{D}} \bar{\beta}\,\nu_v(dx,dz)\,\nu_u(dy)$$
$$+ \int_{\mathcal{V}_c\times E} \bar{\eta}\,\nu_c(dy,dz)\,\zeta_E(dx) + \int_{\mathcal{V}_c} \gamma(z)\,\nu_c(dy,dz), \tag{A.1}$$

We conclude using Assumption 1 and setting:

$$C = \max\left\{\bar{b}, \bar{d}, \bar{\gamma}, \bar{\beta}, \bar{\eta}\,|E|\,, c\right\}.$$

□

A.1. *Path-wise construction of the process.* In this section we rigorously define the Markov processes on path space $\mathcal{D}([0,\infty), \mathcal{M}_F)$, with generator $\mathcal{L}$ given by (1.5). Following Bansaye and Méléard (2015), we provide a specific construction in terms of Poisson point measures.

Let us introduce some additional notation. Let $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. Let λ_l be the Lebesgue measure on $\mathbb{R}_+$, λ_c the counting measure on $\mathbb{N}^*$ and ζ_E the counting measure on E. Moreover, for all $t \geq 0$, we introduce the following notation:

$$\mathcal{A}(t) = [0,t]\times\mathbb{N}^*\times\mathbb{R}_+.$$

Definition A.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a sufficiently large probability space. On this probability space we consider the following independent random elements:

- Q_{inf} a Poisson random measure on $\mathbb{R}_+\times\mathbb{N}^*\times\mathbb{N}^*\times\mathbb{R}_+$, with intensity $\lambda_l(ds)\otimes\lambda_c(di)\otimes\lambda_c(dj)\otimes\lambda(d\theta)$,
- Q_{dis} a Poisson random measure on $\mathbb{R}_+\times E\times\mathbb{N}^*\times\mathbb{R}_+$, with intensity $\lambda_l(ds)\otimes\zeta_E(dx)\otimes\lambda_c(dj)\otimes\lambda(d\theta)$,
- Q_{los}, Q_d, Q_{cb} Poisson random measures on $\mathbb{R}_+\times\mathbb{N}^*\times\mathbb{R}_+$, with intensity $\lambda_l(ds)\otimes\lambda_c(di)\otimes\lambda(d\theta)$,
- Q_{bm}, a Poisson random measure on $\mathbb{R}_+\times\mathbb{N}^*\times\mathcal{X}\times\mathbb{R}_+$, with intensity $\lambda_l(ds)\otimes\lambda_c(di)\otimes\overline{m}(dz)\otimes\lambda(d\theta)$.
- For $\alpha\in\{u,c\}$, the set $\{W^{i,\alpha}; i\geq 1\}$ denotes a family of independent Brownian motions in $\mathbb{R}^d$.

Moreover, we enlarge the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the canonical filtration generated by $\{Q_{inf}, Q_{dis}, Q_d, Q_{cd}, Q_{los}, Q_{bm}\}$ and the families $\{W^{i,\alpha}; i\geq 1\}$ for $\alpha\in\{u,c\}$.

Recall the notation introduced in Section 1. We then have the following representations:

- Individuals (in the virus population) born from clonal births up to time t are given by
$$\boldsymbol{\nu}_{cb}(t) = \int_{\mathcal{A}(t)} \left(\delta_{x_i(s^-),z_i(s^-)}, 0, 0\right)\,\mathbb{1}_{\{i\leq N_v(s^-)\}}\mathbb{1}_{\{\theta\leq(1-\mu)b(x_i(s^-),z_i(s^-))\}}Q_{cb}(ds\,di\,d\theta).$$
- Individuals born with mutations are given by
$$\boldsymbol{\nu}_{bm}(t) = \int_{\mathcal{A}(t)\times\mathcal{X}} \left(\delta_{x_i(s^-),z}, 0, 0\right)\,\mathbb{1}_{\{i\leq N_v(s^-)\}}\mathbb{1}_{\{\theta\leq\mu b(x_i(s^-),z_i(s^-))m(z_i(s^-),z)\}}Q_{bm}(ds\,di\,dz\,d\theta).$$
- Individuals who died before time $t\geq 0$ are given by
$$\boldsymbol{\nu}_d(t) = \int_{\mathcal{A}(t)} \left(\delta_{x_i(s^-),z_i(s^-)}, 0, 0\right)\,\mathbb{1}_{\{i\leq N_v(s^-)\}}\mathbb{1}_{\left\{\theta\leq d(z_i(s^-))+cN_{x_i(s^-)}(t)\right\}}Q_d(ds\,di\,d\theta).$$

- Viruses being charged on a vector are represented by:

$$\boldsymbol{\nu}_{inf}(t) = \int_{\mathcal{A}(t)\times\mathbb{N}^*} \left(-\delta_{x_i(s^-),z_i(s^-)}, -\delta_{Y_j(s^-)}, \delta_{Y_j(s^-),z_i(s^-)}\right) \mathbb{1}_{\{i\le N_v(s^-)\}}\mathbb{1}_{\{j\le N_u(s^-)\}}$$
$$\times\, \mathbb{1}_{\left\{\theta\le\beta(s^-,Y_j(s^-),x_i(s^-),N_{x_i(s^-)},z_i(s^-))\right\}} Q_{inf}(ds\,di\,dj\,d\theta).$$

- Viruses who have been unloaded on a host plant up to time t are given by

$$\boldsymbol{\nu}_{dis}(t) = \int\limits_{\mathcal{A}(t)\times E} \left(\delta_{x,z_j(s^-)}, \delta_{Y_j(s^-)}, -\delta_{Y_j(s^-),z_i(s^-)}\right) \mathbb{1}_{\{j\le N_c(s^-)\}}\mathbb{1}_{\{\theta\le\eta(s^-,Y_j(s^-),x,z_j(s^-))\}} Q_{dis}(ds\,dx\,dj\,d\theta).$$

- Finally, viruses dying on vectors up to time t:

$$\boldsymbol{\nu}_{los}(t) = \int_{\mathcal{A}(t)} \left(0, \delta_{Y_i(s-)}, -\delta_{Y_i(s^-),z_i(s-)}\right) \mathbb{1}_{\{i\le N_c(s^-)\}}\mathbb{1}_{\{\theta\le\gamma(z_i(s^-))\}} Q_{los}(ds\,di\,d\theta).$$

Definition A.3. The process $\boldsymbol{\nu}(t) = (\nu_v(t), \nu_u(t), \nu_c(t); t \ge 0)$ is defined as the $\mathcal{F}_t$-adapted solution to the equation:

$$\begin{aligned}\langle \boldsymbol{\phi}, \boldsymbol{\nu}(t)\rangle &= \langle \boldsymbol{\phi}, \boldsymbol{\nu}(0)\rangle + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_{cb}(t)\rangle + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_{bm}(t)\rangle - \langle \boldsymbol{\phi}, \boldsymbol{\nu}_{d}(t)\rangle \\ &\quad + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_{inf}(t)\rangle + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_{dis}(t)\rangle + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_{los}(t)\rangle \\ &\quad + \int_0^t \langle \mathcal{L}^u\boldsymbol{\phi}, \boldsymbol{\nu}(s)\rangle\, ds + \int_0^t \langle \mathcal{L}^c\boldsymbol{\phi}, \boldsymbol{\nu}(s)\rangle\, ds + \int_0^t \sum_{i=1}^{N_u(s^-)} \sigma^u(Y_i^u(s^-))\nabla\phi_u(Y_i^u(s^-))\cdot dW_s^{i,u} \\ &\quad + \int_0^t \sum_{i=1}^{N_c(s^-)} \sigma^c(Y_i^c(s^-))\nabla_y\,\phi_c(Y_i^c(s^-), z_i(s^-))\cdot dW_s^{i,c}, \end{aligned} \tag{A.2}$$

for all $\boldsymbol{\phi} \in \Phi(\mathcal{M}_p)$, and $\mathcal{L}^\alpha\boldsymbol{\phi}$ is given as in Remark 1.2.

The following proposition gives conditions to guarantee that a solution to (A.2) follows the dynamics given by the generator $\mathcal{L}$ given in (1.5).

Proposition A.4. *Let $(\boldsymbol{\nu}_t)_{t\ge0} = (\nu_v(t), \nu_u(t), \nu_c(t); t \ge 0)$ be a solution of (A.2) such that for all $T > 0$ we have*

$$\mathbb{E}\left[\sup_{t\le T} < \mathbf{1}, \boldsymbol{\nu}_t >^2\right] < \infty,$$

where $\mathbf{1} \in \Phi(\mathcal{M}_p)$ denotes the admissible triplet with all functions equal to the constant 1. Then, under Assumption 1, the process $(\boldsymbol{\nu}_t)_{t\ge0}$ is Markov with infinitesimal generator $\mathcal{L}$ given by (1.5).

Proof: The fact that it is Markov is standard. We need to verify that the generator is the one we claimed. We will use Itô's lemma (see for example Theorem 5.1 in Ikeda and Watanabe (2014)) applied to (A.2) to find an expression for $F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(t))$. Let us split this into two, one part coming from jump events, and the other one from diffusion, as follows:

$$F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(t)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(0)) = (F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(t)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(0)))_{\text{Jump}} + (F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(t)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(0)))_{\text{Diff}}.$$

The way to verify that this corresponds to the generator $\mathcal{L}$ is to take expectations and differentiate with respect to time. We refer to Bansaye and Méléard (2015), Proposition 6.3 in particular, for the same procedure in the absence of diffusion. Let us do this for the diffusive part of the generator.

Notice that by Itô's lemma, the diffusive part is given by:

$$
\begin{aligned}
&(F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(t)) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu}(0)))_{\mathrm{Diff}} \\
&\quad = \int_0^t (\partial_u F)_{\boldsymbol{\phi}}(\boldsymbol{\nu}(s))\langle \mathcal{L}^u \boldsymbol{\phi}, \boldsymbol{\nu}(s)\rangle \, ds + \int_0^t (\partial_u F)(\boldsymbol{\nu}(s)) \sum_{i=1}^{N_u(s^-)} \sigma^u(Y_i^u(s^-))\nabla\phi_u(Y_i^u(s^-)) \cdot dW_s^{i,u} \\
&\quad + \frac{1}{2}\int_0^t (\partial_u^2 F)(\boldsymbol{\nu}(s)) \sum_{i=1}^{N_u(s^-)} \sigma^u(Y_i^u(s^-))^2 \mid \nabla\phi_u(Y_i^u(s^-)) \mid^2 \, ds + \int_0^t (\partial_c F)(\boldsymbol{\nu}(s))\langle \mathcal{L}^c \boldsymbol{\phi}, \boldsymbol{\nu}(s)\rangle \, ds \\
&\quad + \int_0^t (\partial_c F)(\boldsymbol{\nu}(s)) \sum_{i=1}^{N_c(s^-)} \sigma^c(Y_i^c(s^-))\nabla_y \, \phi_c(Y_i^c(s^-), z_i(s^-)) \cdot dW_s^{i,c} \\
&\quad + \frac{1}{2}\int_0^t (\partial_c^2 F)(\boldsymbol{\nu}(s)) \sum_{i=1}^{N_c(s^-)} \sigma^c(Y_i^c(s^-))^2 \mid \nabla_y \, \phi_c(Y_i^c(s^-), z_i(s^-)) \mid^2 \, ds.
\end{aligned}
$$

In order to verify that we get the second part of the generator $\mathcal{L}$, we have to proceed as before. Notice however, that by taking expectations the Itô integrals vanish. Differentiating what is left at $t = 0$ leads to the RHS of (1.4). □

Now we show the well-definedness of the process $(\boldsymbol{\nu}_t)_{t\geq 0}$. That is, Theorem 1.6. Moreover we show that a control of the p-th moment at time zero can be extended to later times. More precisely, we show the following proposition:

Proposition A.5. *Let $\boldsymbol{\nu}_0 = (\nu_v(0), \nu_u(0), \nu_c(0))$ be such that for some $p \geq 1$, we have:*

$$\mathbb{E}\left[\langle \boldsymbol{\nu}_0, \mathbf{1}\rangle^p\right] < \infty. \tag{A.3}$$

Then, under Assumption 1, the process $(\boldsymbol{\nu}_t)_{t\geq 0} = (\nu_v(t), \nu_u(t), \nu_c(t) : t \geq 0)$ satisfies:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \langle \boldsymbol{\nu}_t, \mathbf{1}\rangle^p\right] < \infty. \tag{A.4}$$

In particular, if (A.3) holds for $p = 1$, we also have that the process ν_t is well defined.

Proof: To show (A.4), because the number of vectors is invariant under the dynamics, it is enough to show:

$$\mathbb{E}\left[\sup_{t\in[0,T]} \langle \nu_v(t), 1\rangle^p\right] < \infty.$$

In order to do so, we use a stopping time argument. Let us define τ_n as follows:

$$\tau_n = \inf\left\{t \geq 0 : \langle \nu_v(t), 1\rangle \geq n\right\}. \tag{A.5}$$

Hence, by (A.2) we have:

$$
\begin{aligned}
&\sup_{s\in[0,t\wedge\tau_n]} \langle \nu_v(s), 1\rangle^p \leq \langle \nu_v(0), 1\rangle^p \\
&\quad + \int_0^{t\wedge\tau_n} \int_{\mathbb{N}^*\times\mathbb{R}_+} [(\langle \nu_v(s-), 1\rangle + 1)^p - \langle \nu_v(s-), 1\rangle^p] \, \mathbb{1}_{\{i\leq N_v(s^-)\}} \mathbb{1}_{\{\theta\leq(1-\mu)b(x_i(s^-), z_i(s^-))\}} Q_{cb}(ds\, di\, d\theta) \\
&\quad + \int_0^{t\wedge\tau_n} \int_{\mathbb{N}^*\times\mathbb{R}_+\times E} [(\langle \nu_v(s-), 1\rangle + 1)^p - \langle \nu_v(s-), 1\rangle^p] \, \mathbb{1}_{\{j\leq N_c(s^-)\}} \mathbb{1}_{\{\theta\leq\eta(s^-, Y_j(s^-), x, z_j(s^-))\}} Q_{dis}(ds\, dx\, dj\, d\theta) \\
&\quad + \int_0^{t\wedge\tau_n} \int_{\mathbb{N}^*\times\mathcal{X}\times\mathbb{R}_+} [(\langle \nu_v(s-), 1\rangle + 1)^p - \langle \nu_v(s-), 1\rangle^p] \, \mathbb{1}_{\{i\leq N_v(s^-)\}} \mathbb{1}_{\{\theta\leq\mu b(x_i(s^-), z_i(s^-)) m(z_i(s^-), z)\}} Q_{bm}(ds\, di\, dz\, d\theta).
\end{aligned}
$$

where the diffusion terms vanished due to the presences of derivatives of the constant function 1, and we have dropped the integral terms with a negative contribution (death terms, and loading terms).

Taking expectations and using the bounds given by Assumption 1 we obtain:

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]}\langle\nu_v(s),1\rangle^p\right] \leq \mathbb{E}\left[\langle\nu_v(0),1\rangle^p\right] + C\mathbb{E}\left[\int_0^{t\wedge\tau_n} N_v(s)\left[(\langle\nu_v(s),1\rangle+1)^p - \langle\nu_v(s),1\rangle^p\right]\right] ds$$
$$+ C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\left[(\langle\nu_v(s),1\rangle+1)^p - \langle\nu_v(s),1\rangle^p\right]\right] ds.$$
$$\leq C\left(1+\mathbb{E}\left[\int_0^{t\wedge\tau_n} N_v(s)\left[(\langle\nu_v(s),1\rangle+1)^p - \langle\nu_v(s),1\rangle^p\right] ds\right]\right)$$

where the constant C changed its value incorporating the constant given by (A.3).

Using the fact that $N_v(s) = \langle\nu_v(s),1\rangle$, and the simple inequality $(x+1)^p - x^p \leq C_p(1+x^{p-1})$, we obtain:

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]}\langle,\nu_v(s),1\rangle^p\right] \leq C_p\left(1+\mathbb{E}\left[\int_0^t\left[\langle\nu_v(s\wedge\tau_n),1\rangle + \langle\nu_v(s\wedge\tau_n),1\rangle^p\right] ds\right]\right)$$
$$\leq C_p\left(1+\mathbb{E}\left[\int_0^t\langle\nu_v(s\wedge\tau_n),1\rangle^p\, ds\right]\right)$$

where again the constant C_p changed its value incorporating new constants.

By Gronwalls inequality we then have:

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]}\langle\nu_v(s),1\rangle^p\right] \leq C_p e^{C_p t} \tag{A.6}$$

for some constant C_p independent of n.

From (A.6) we can deduce that τ_n goes to infinity as $n\to\infty$ a.s. We then apply Fatou's lemma to conclude:

$$\mathbb{E}\left[\sup_{s\in[0,t]}\langle\nu_v(s),1\rangle^p\right] \leq \liminf_{n\to\infty}\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]}\langle\nu_v(s),1\rangle^p\right] \leq C_p e^{C_p t}.$$

To conclude the well-definedness of the process ν_t, following Champagnat and Méléard (2007), one has to construct the process step by step, where the time steps are given by a sequence of jump instants T_n exponentially distributed with law:

$$R(\nu_{n-1})e^{-R(\nu_{n-1})t}$$

and where the total jump rate $R(\nu)$ is given by (A.1).

It is then enough to check that the sequence T_n goes to infinity almost surely. This follows from

$$\mathbb{E}\left[\sup_{t\in[0,T]}\langle\boldsymbol{\nu}_t,1\rangle\right] < \infty$$

which is a consequence of (A.4) when $p=1$. □

A.2. *Relevant martingales.* Now we will introduce a few martingales that are relevant when computing scaling limits. Let us start from a simple application of Dynkin's theorem:

Proposition A.6. *Let $\nu_0 = (\nu_v(0), \nu_u(0), \nu_c(0))$ be such that for some $p \geq 2$ we have:*

$$\mathbb{E}\left[\langle \nu_v(0), 1\rangle^p\right] < \infty.$$

Let also $F \in \mathcal{C}^2(\mathbb{R}^3)$ and $\boldsymbol{\phi} \in \Phi(\mathcal{M}_p)$, be such that there exists $C > 0$, possibly dependent on F and $\boldsymbol{\phi}$, such that:

$$|F_{\boldsymbol{\phi}}(\nu)| + |\mathcal{L}F_{\boldsymbol{\phi}}(\nu)| \leq C\,\left(1 + \langle \nu, \mathbf{1}\rangle^p\right) \tag{A.7}$$

where $F_{\boldsymbol{\phi}}$ is given as in Definition 1.3.

Then, under Assumption 1, we have that the process

$$M_t(F_{\boldsymbol{\phi}}) = F_{\boldsymbol{\phi}}(\boldsymbol{\nu}_t) - F_{\boldsymbol{\phi}}(\boldsymbol{\nu}_0) - \int_0^t \mathcal{L}F_{\boldsymbol{\phi}}(\boldsymbol{\nu}_s)\,ds \tag{A.8}$$

is a càdlàg martingale.

Proof: From Proposition A.4 and Dynkin's theorem we know that $M_t(F_\phi)$ is a local martingale. Hence, it is enough to show that the R.H.S. of (A.8) is integrable. This is a consequence of Assumption (A.7) and Proposition A.5. □

Remark A.7. Simple but tedious computations show that for, $1 \leq q \leq p-1$, $\phi_v \in \mathcal{C}_b^2(\mathcal{V}_p), \phi_u \in \mathcal{C}_b^2(\mathcal{D}) \cap D(\mathcal{L}^u)$ and $\phi_c \in \mathcal{C}_b^2(\mathcal{V}_c) \cap D(\mathcal{L}^c)$, the functions

$$F_v^q(\nu_v) := \left(\langle \nu_v, \phi_v\rangle\right)^q, \quad F_u^q(\nu_u) := \left(\langle \nu_u, \phi_u\rangle\right)^q, \text{ and } F_c^q(\nu_c) := \left(\langle \nu_c, \phi_c\rangle\right)^q$$

satisfy (A.7).

The following result is a consequence of Proposition A.6 and Remark A.7

Proposition A.8. *Let $\boldsymbol{\nu}_0 = (\nu_v(0), \nu_u(0), \nu_c(0))$ be such that (A.3) holds for $p = 3$. For every $\phi_v \in \mathcal{C}_b^2(\mathcal{V}_p), \phi_u \in D(\mathcal{L}^u)$ and $\phi_c \in D(\mathcal{L}^c)$, under Assumption 1, we have the following càdlàg martingales:*

(1) *For the virus population:*

$$\begin{aligned} M_t^{\phi_v} &= \langle \nu_v(t), \phi_v\rangle - \langle \nu_v(0), \phi_v\rangle - \int_0^t \int_{\mathcal{V}_p} b(x,z)\phi_v(x,z)\nu_v(s)(dx,dz)\,ds \\ &+ \mu \int_0^t \int_{\mathcal{V}_p} b(x,z)\left[\phi_v(x,z) - \int_{\mathcal{X}} m(z,z')\phi_v(x,z')\,dz'\right]\nu_v(s)(dx,dz)\,ds \\ &+ \int_0^t \left[\int_{\mathcal{V}_p} \left(d(z) + c\langle {\nu_v}^x, 1\rangle\right)\phi_v(x,z)\nu_v(s)(dx,dz) + \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle {\nu_v}^x,1\rangle,z)\phi_v(x,z)\nu_v(s)(dx,dz)\,\nu_u(s)(dy)\right] ds \\ &- \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x,z)\phi_v(x,z)\,\nu_c(s)(dy,dz)\,dx\,ds \end{aligned} \tag{A.9}$$

is a càdlàg martingale with predictable quadratic variation

$$\begin{aligned} \langle M^{\phi_v}\rangle_t &= (1-\mu)\int_0^t \int_{\mathcal{V}_p} b(x,z)\,\phi_v(x,z)^2\,\nu_v(s)(dx,dz)\,ds + \mu\int_0^t \int_{\mathcal{V}_p\times\mathcal{X}} b(x,z)m(z,z')\,\phi_v(x,z')^2\,dz'\nu_v(s)(dx,dz)\,ds \\ &+ \int_0^t \int_{\mathcal{V}_p} \left(d(z) + c\langle {\nu_v}^x, 1\rangle\right)\,\phi_v(x,z)^2\,\nu_v(s)(dx,dz) + \int_0^t \int_{\mathcal{V}_c\times E} \eta(s,y,x,z)\,\phi_v(x,z)^2\,\nu_c(s)(dy,dz)\,dx\,ds \\ &+ \int_0^t \int_{\mathcal{V}_p\times\mathcal{D}} \beta(s,y,x,\langle {\nu_v}^x,1\rangle,z)\,\phi_v(x,z)^2\,\nu_v(s)(dx,dz)\,\nu_u(s)(dy)ds. \end{aligned} \tag{A.10}$$

(2) *For the free-vector population:*

$$M_t^{\phi_u} = \langle \nu_u(t), \phi_u \rangle - \langle \nu_u(0), \phi_u \rangle - \int_0^t \int_{\mathcal{D}} \mathcal{L}^u \phi_u(y)\, \nu_u(s)(dy)\, ds$$
$$- \int_0^t \int_{\mathcal{V}_c \times E} \eta(s, y, x, z) \phi_u(y) \nu_c(s)(dy, dz)\, dx\, ds$$
$$+ \int_0^t \left[\int_{\mathcal{V}_p \times \mathcal{D}} \beta(s, y, x, \langle {\nu_v}^x, 1 \rangle, z) \phi_u(y) \nu_v(s)(dx, dz)\, \nu_u(s)(dy) - \int_{\mathcal{V}_c} \gamma(z) \phi_u(y)\, \nu_c(s)(dy, dz) \right] ds \tag{A.11}$$

is a càdlàg martingale with predictable quadratic variation

$$\langle M^{\phi_u} \rangle_t = \int_0^t \int_{\mathcal{D}} \sigma^u(y)^2 \, |\nabla \phi_u(y)|^2 \, \nu_u(s)(dy)\, ds + \int_0^t \int_{\mathcal{V}_c \times E} \eta(s, y, x, z) \phi_u(y)^2 \nu_c(s)(dy, dz)\, dx\, ds$$
$$+ \int_0^t \int_{\mathcal{V}_p \times \mathcal{D}} \beta(s, y, x, \langle {\nu_v}^x, 1 \rangle, z)\, \phi_u(y)^2 \, \nu_v(s)(dx, dz)\, \nu_u(s)(dy) ds$$
$$+ \int_0^t \int_{\mathcal{V}_c} \gamma(z) \phi_u(y)^2 \, \nu_c(s)(dy, dz)\, ds.$$

(3) *For the charged-vector population:*

$$M_t^{\phi_c} = \langle \nu_c(t), \phi_c \rangle - \langle \nu_c(0), \phi_c \rangle - \int_0^t \int_{\mathcal{V}_c} \mathcal{L}^c \phi_c(y, z)\, \nu_c(s)(dy, dz)\, ds$$
$$+ \int_0^t \int_{\mathcal{V}_c \times E} \eta(s, y, x, z) \phi_c(y, z) \nu_c(s)(dw, du)\, dx\, ds$$
$$- \int_0^t \left[\int_{\mathcal{V}_p \times \mathcal{D}} \beta(s, y, x, \langle {\nu_v}^x, 1 \rangle, z) \phi_c(y, z) \nu_v(s)(dx, dz)\, \nu_u(s)(dy) - \int_{\mathcal{V}_c} \gamma(y) \phi_c(y, z)\, \nu_c(s)(dy, dz) \right] ds \tag{A.12}$$

is a càdlàg martingale with quadratic variation

$$\langle M^{\phi_c} \rangle_t = \int_0^t \int_{\mathcal{V}_c} \sigma^c(y)^2 |\nabla_y \phi_c(y, z)|^2 \, \nu_c(s)(dy, dz)\, ds + \int_0^t \int_{\mathcal{V}_c \times E} \eta(s, y, x, z) \phi_c(y, z)^2 \nu_c(s)(dy, dz)\, dx\, ds$$
$$+ \int_0^t \left[\int_{\mathcal{V}_p \times \mathcal{D}} \beta(s, y, x, \langle {\nu_v}^x, 1 \rangle, z) \phi_c(y, z)^2 \nu_v(s)(dx, dz)\, \nu_u(s)(dy) + \int_{\mathcal{V}_c} \gamma(y) \phi_c(y, z)^2 \, \nu_c(s)(dy, dz) \right] ds.$$

Proof: Assume $p \geq 3$. By Theorem A.6 and Remark A.7 with $q = 1$, we have that (A.9), (A.11), and (A.12) are càdlàg martingales. For the predictable quadratic variation we just use Remark A.7 with $q = 2$, and Itô formula together with the Doob-Meyer decomposition. See Bansaye and Méléard (2015) for examples on a similar context. □

Appendix B. Existence and uniqueness of the averaged stationary solutions

This section deals with the issue of existence and uniqueness for the evolution system (3.14) - (3.15). and subsequently establishes the uniqueness of the measure defined by equations (3.17)-(3.18).

B.1. *Construction of the triplet solution* (g_v, g_u, g_c). Recall that in this setting η is assumed to be independent of the virus phenotype, so the strong formulation of (3.14)-(3.15) then reads

$$
\begin{aligned}
&\partial_t g_v(t,x,z) = \mathfrak{G}\,(t,x,z,g_v,g_u)\, g_v(t,x,z) + \mathcal{M}[\,g_v\,](t,x,z) + \int_{\mathcal{D}} \eta(t,y,x)\, g_c(t,y,z)\, dy, && \text{for } t>0, x\in E, z\in\mathcal{X}\\
&\Delta g_u(t,y) - \left[\int_{\mathcal{X}} A(t,y,z,g_v)\, dz\right] g_u(y) + B(t,y)\int_{\mathcal{X}} g_c(y,z)\, dz = 0, && \text{for } t>0, y\in\mathcal{D}\\
&\Delta_y g_c(t,y,z) + A(t,y,z,g_v)\, g_u(y) - B(t,y)\, g_c(y,z) = 0, && \text{for } t>0, y\in\mathcal{D}, z\in\mathcal{X}\\
&\nabla g_u(t,y)\cdot\vec{n}(y) = 0 && \text{for } t>0, y\in\partial D\\
&\nabla_y g_c(t,y,z)\cdot\vec{n}(y) = 0 && \text{for } t>0, (y,z)\in\partial D\times\mathcal{X}\\
&g_v(0,x,z) = g_0(x,z) && \text{for } (x,z)\in E\times\mathcal{X}\\
&\int_{\mathcal{D}}\left(g_u(t,y) + \int_{\mathcal{X}} g_c(t,y,z)\, dz\right) dy = V_0 && \text{for } t>0,
\end{aligned}
$$

where the operator $\mathcal{M}$ is defined by :

$$
\forall\, \varphi \in \mathscr{C}(\mathbb{R}^+, L^2(E\times\mathcal{X})), \quad \mathcal{M}[\,\varphi\,](t,x,z) := \mu\int_{\mathcal{X}} b(x,z')m(z',z)\,\varphi(t,x,z')\, dz',
$$

and the functions $\mathfrak{G}, A$ and B stand for

$$
\begin{aligned}
&\mathfrak{G}(t,x,z,g_v,g_u) := (1-\mu)b(x,z) - \left[d(z) + \kappa\int_{\mathcal{X}} g_v(t,x,z')dz'\right] - \left[\int_{\mathcal{D}} \beta\left(y,x,\int_{\mathcal{X}} g_v(t,x,z')dz', z\right) g_u(t,y)\, dy\right]\\
&A(t,y,z,g_v) := \left[\int_E \beta\left(y,x,\int_{\mathcal{X}} g_v, z\right) g_v(t,x,z)\zeta_E(dx),\right],\\
&B(t,y) = \left(\int_E \eta(t,y,x)\,\zeta_E(dx)\right).
\end{aligned}
$$

Note that the above system is degenerate in multiple ways.

First let us make some straightforward observations that will simplify the proof of existence and uniqueness of a solution of the above system. Let us first observe that the following function

$$
g(t,y) := g_u(t,y) + \int_{\mathcal{X}} g_c(t,y,z)\, dz
$$

is a $L^1(\mathcal{D})$ harmonic function with homogeneous Neumann boundary condition, i.e. g satisfies

$$
\begin{cases}
\Delta g(t,y) = 0, & y\in\mathcal{D},\\
\nabla g(t,y)\cdot\vec{n}(y) = 0, & y\in\partial\mathcal{D}
\end{cases}
$$

By the strong maximum principle, we deduce that for all t g is a constant, i.e. for all t it holds $g(t,\cdot) \equiv C_0 = \frac{V_0}{|\mathcal{D}|}$. Going back to the equation satisfied by g_u, g_c, and using now that $\int_{\mathcal{X}} g_c(t,y,z)\, dz = C_0 - g_u(t,y)$, we get the following set of equations to solve,

$$
\begin{aligned}
&\partial_t g_v(t,x,z) = \mathfrak{G}\,(t,x,z,g_v,g_u)\, g_v(t,x,z) + \mathcal{M}[\,g_v\,](t,x,z) + \left(\int_{\mathcal{D}} \eta(t,y,x)\, g_c(t,y,z)\, dy\right), && \text{for } t>0, x\in E, z\in\mathcal{X}\\
&\Delta g_u(t,y) - \left[\int_{\mathcal{X}} A(t,y,z,g_v)\, dz + B(t,y)\right] g_u(t,y) = -C_0 B(t,y) && \text{for } t>0,\ y\in\mathcal{D}\\
&\Delta_y g_c(t,y,z) - B(t,y) g_c(t,y,z) = -A(t,y,z,g_v)\, g_u(t,y) && \text{for } t>0,\ (y,z)\in\mathcal{D}\times\mathcal{X}\\
&g_v(0,x,z) = g_0(x,z) && \text{for } (x,z)\in E\times\mathcal{X}\\
&\nabla g_u(t,y)\cdot\vec{n}(y) = 0, && \text{for } t>0,\ y\in\partial\mathcal{D}.\\
&\nabla_y g_c(t,y,z)\cdot\vec{n}(y) = 0, && \text{for } t>0,\ y\in\partial\mathcal{D}\times\mathcal{X}
\end{aligned}
$$

From the above system, we can see that the elliptic PDE satisfied by g_u is coupled to the system only through the coefficient $A(t,y,z,g_v)$ and as soon as this coefficient is known, since for all $t\geq 0, B(t,\cdot)\in L^\infty(\mathcal{D})$ is a given non degenerate non negative function, from standard elliptic theory (see for example Brezis (2011); Wu et al. (2006)), there exists a unique smooth positive solution to this elliptic problem. Similarly, the function g_c can be deduced from the knowledge of A and g_u. The main difficulty in the construction of a solution (g_v, g_u, g_c) of the above system is then the construction of a positive function g_v.

To do so, let us first introduce the following auxiliary elliptic equation: For a given $\varphi \in \mathscr{C}(\mathbb{R}^+ \times \bar{\mathcal{D}}, L^2(\bar{\mathcal{X}}))$, $\varphi \geq 0$, let us consider

$$\Delta u(t,y) - \left[\int_{\mathcal{X}} \varphi(t,y,z)\, dz + B(t,y)\right] u(t,y) = -C_0 B(t,y) \qquad \text{for} \quad t>0,\, y \in \mathcal{D} \tag{B.1}$$

$$\nabla u(t,y) \cdot \vec{n}(y) = 0, \qquad \text{for} \quad t>0,\, y \in \partial\mathcal{D}. \tag{B.2}$$

Next, let us introduce the following degenerate elliptic problem. For a given pair (φ, ψ) such that $\varphi \in \mathscr{C}(\mathbb{R}^+ \times \bar{\mathcal{D}}, L^2(\bar{\mathcal{X}}))$, $\psi \in \mathscr{C}(\mathbb{R}^+, \mathscr{C}^{0,\alpha}(\mathcal{D}) \cap L^\infty(\mathcal{D}))$ and $\varphi, \psi \geq 0$, let us consider

$$\Delta_y v(t,y,z) - B(t,y) v(t,y,z) = -\varphi(t,y,z)\, \psi(t,y) \qquad \text{for} \quad t>0,\, (y,z) \in \mathcal{D}\times\mathcal{X} \tag{B.3}$$

$$\nabla_y v(t,y,z) \cdot \vec{n}(y) = 0, \qquad \text{for} \quad t>0,\, y \in \partial\mathcal{D}\times\mathcal{X} \tag{B.4}$$

Note that by making the coupling $\psi = u$, we recover the weakly coupled system satisfied by g_u, g_c. From standard elliptic theory (see for example Brezis (2011); Wu et al. (2006)), since for all $t \geq 0, B(t,\cdot) \in L^\infty(\mathcal{D})$ is a given non degenerate non negative function, we can check that there exists a unique smooth positive solution u to the elliptic problem (B.1)-(B.2). Indeed, since for all $t \geq 0$ fixed, $\varphi(t,\cdot,\cdot) \in \mathscr{C}(\bar{\mathcal{D}}, L^2(\mathcal{X}))$ we have $\int_{\mathcal{X}} \varphi(t,\cdot,z)\, dz \in L^\infty(\mathcal{D}) \geq 0$ and then for all fixed t, say $t = t_0$, there exists a unique positive solution to (B.1) -(B.2), $u(t_0,\cdot)$, such that $u(t_0,\cdot) \in L^\infty(\bar{\mathcal{D}}) \cap \mathscr{C}^{0,\alpha}(\mathcal{D})$ for all $\alpha \in (0,1)$. In addition, by a straightforward use of the maximum principle, we see that for all $t \geq 0, \|u(t,\cdot)\|_\infty \leq C_0$.
The regularity of $u(t_0,\cdot)$ with respect to the variable t_0 can be deduced from the regularity of the functions φ and B in these variable. This can be seen by observing that the function $h(\cdot) = u(t_0,\cdot) - u(t_1,\cdot)$ satisfies

$$\Delta h(y) - \int_{\mathcal{X}} \varphi(t_1,y,z)\, dz h(y) = -C_0(B(t_0,y) - B(t_1,y)) - \int_{\mathcal{X}} [\varphi(t_0,y,z) - \varphi(t_1,y,z)]\, dz u(t_0,y) \quad \text{for} \quad y \in \mathcal{D},$$

which by using elliptic regularity and $u \leq C_0$ then implies that for all $p > 1$

$$\|h\|_{W^{1,p}(\mathcal{D})} \leq C_p(\mathcal{D}) C_0 \left[\|B(t_0,\cdot) - B(t_1,\cdot)\|_p + \left\|\left(\int_{\mathcal{X}} (\varphi(t_0,\cdot,z) - \varphi(t_1,\cdot,z))\, dz\right)\right\|_p\right].$$

For $p > d$, using Sobolev embedding, Brezis (2011), the latter inequality yields

$$\|h\|_{L^\infty(\mathcal{D})} \leq C_0 C(\mathcal{D},p) \left[\|B(t_0,\cdot) - B(t_1,\cdot)\|_p + \left\|\left(\int_{\mathcal{X}} (\varphi(t_0,\cdot,z) - \varphi(t_1,\cdot,z))\, dz\right)\right\|_p\right].$$

So u is continuous in t uniformly with respect to y as soon as the functions B and φ are continuous in time uniformly with respect to y.

Similarly, we can check that a positive solution $v \in \mathscr{C}(\mathbb{R}^+ \times \bar{\mathcal{D}}, L^1(\mathcal{X})) \cap \mathscr{C}(\mathbb{R}^+, L^2(\mathcal{D}\times\mathcal{X}))$ to (B.3)-(B.4) exists. Indeed, for any t fixed, says $t = t_0$, since $B(t_0,y) \geq_{\not\equiv} 0, \varphi(t,\cdot,\cdot) \in \mathscr{C}(\mathcal{D}, L^2(\mathcal{X}))$ and $\psi \in L^\infty$ then for almost every $z_0 \in X$, from standard theory (Lax-Milgram theorem and linear elliptic regularity Brezis (2011); Wu et al. (2006)) there exists a unique positive smooth solution $v(t_0,\cdot,z_0)$ to (B.3)-(B.4), $v(t_0,\cdot,z_0) \in L^\infty(\bar{D}) \cap \mathscr{C}^{0,\alpha}(\mathcal{D})$ for all $\alpha \in (0,1)$. In addition, since B is such that $\lambda_p(\Delta + B(t,\cdot)) > 0$, we get the following $L^2(\mathcal{D})$ estimate

$$\|v(t_0,\cdot,z_0)\|_2 \le \frac{\|\psi(t_0,\cdot)\|_\infty}{\lambda_p(t_0)}\|\varphi(t_0,\cdot,z_0)\|_2. \tag{B.5}$$

Indeed, by multiplying by v the equation satisfied by v and integrating over $\mathcal{D}$, we get after integrating by parts,

$$\int_{\mathcal{D}} |\nabla v(t_0,y,z_0)|^2 + \int_{\mathcal{D}} B(t_0,y)v(t_0,y,z_0)^2\,dy = \int_{\mathcal{D}} \varphi(t_0,y,z_0)\psi(t_0,y)v(t_0,y,z_0)\,dy.$$

By using the H^1 variationnal characterisation of λ_p and the Cauchy-Schwartz inequality we get,

$$\lambda_p(t_0)\int_{\mathcal{D}} v(t_0,y,z_0)^2\,dy \le \|v(t_0,\cdot,z_0)\|_2\|\varphi(t_0,\cdot,z_0)\psi(t_0,y)\|_2,$$

which prove the above estimate.

Now, as for u, the regularity of $v(\cdot,y,\cdot)$ with respect to the variables t_0, z_0 can be obtained through the regularity of the coefficients of the PDE, that is the regularity in t,z of the function φ and ψ. Namely, observe that from (B.5), by setting $C_2(t_0) := \left(\frac{\|\psi(t_0,\cdot)\|_\infty}{\lambda_p(t_0)}\right)^2$ we get

$$\|v(t_0,\cdot,z_0)\|_2^2 \le C_2(t_0)\|\varphi(t_0,\cdot,z_0)\|_2^2$$

and thus by integrating with respect to z and using Fubini's theorem, we get for all $t > 0$

$$\int_{\mathcal{D}}\left(\int_{\mathcal{X}} v^2(t,y,z)\,dz\right)dy \le C_2(t)\int_{\mathcal{D}\times\mathcal{X}} \varphi^2(t,y,z)\,dydz < +\infty.$$

To obtain $v \in \mathscr{C}(\mathbb{R}^+, L^2(\mathcal{D}\times\mathcal{X}))$, we estimate $h(\cdot,\cdot) = v(t_0,\cdot,\cdot) - v(t_1,\cdot,\cdot)$. A straightforward computation yields

$$\Delta h(y) - B(t_0,y)h(y) = -v(t_1,y,z)(B(t_0,y) - B(t_1,y)) - \varphi(t_0,y,z)\psi(t_0,y) + \varphi(t_1,y,z)\psi(t_1,y)$$

for $y \in \mathcal{D}, z \in \mathcal{X}$, and by a standard argument we get

$$\|h\|_2^2 \le \frac{1}{\lambda_p^2(t_0)}\left[\|v(t_1,\cdot,z)(B(t_0,\cdot) - B(t_1,\cdot))\|_2 + \|\varphi(t_0,\cdot,z)\psi(t_0,\cdot) - \varphi(t_1,\cdot,z)\psi(t_1,\cdot)\,\|_2\right]^2$$

which after using Young's inequality and integrating over $\mathcal{X}$ gives

$$\|h\|_2^2 \le \frac{2}{\lambda_p^2(t_0)}\left[\int_{\mathcal{X}\times\mathcal{D}} v^2(t_1,y,z)(B(t_0,y) - B(t_1,y))^2\,dy\,dz + \int_{\mathcal{X}\times\mathcal{D}} (\varphi(t_0,y,z)\psi(t_0,y) - \varphi(t_1,y,z)\psi(t_1,y)\,)^2\,dy\,dz\right]$$

The first integral of the right hand side can be estimated as follows

$$\begin{aligned}\int_{\mathcal{X}\times\mathcal{D}} v^2(t_1,y,z)(B(t_0,y) - B(t_1,y))^2\,dy\,dz &\le \|(B(t_0,\cdot) - B(t_1,\cdot))^2\|_\infty \int_{\mathcal{X}\times\mathcal{D}} v^2(t_1,y,z)\,dy\,dz \\ &\le C_2(t_1)\|(B(t_0,\cdot) - B(t_1,\cdot))^2\|_\infty \int_{\mathcal{X}\times\mathcal{D}} \varphi^2(t_1,y,z)\,dy\,dz \\ &\le \bar{C}_2 C_\varphi\|(B(t_0,\cdot) - B(t_1,\cdot))^2\|_\infty,\end{aligned}$$

where $\bar{C}_2 := \sup_{t\in[t_0,t_1]} C_2(t)$ and $C_\varphi := \sup_{t\in[t_0,t_1]}\int_{\mathcal{X}\times\mathcal{D}} \varphi^2(t,y,z)\,dy\,dz$. As for the second integral, it is estimated as follows. Using Young's inequality we have

$$\begin{aligned}\int_{\mathcal{X}\times\mathcal{D}} (\varphi(t_0,y,z)\psi(t_0,y) - \varphi(t_1,y,z)\psi(t_1,y)\,)^2\,dy\,dz \le 2\Bigg[&\int_{\mathcal{X}\times\mathcal{D}} (\varphi(t_0,y,z) - \varphi(t_1,y,z))^2\psi^2(t_0,y) \\ &+ \int_{\mathcal{X}\times\mathcal{D}} \varphi^2(t_1,y,z)(\psi(t_0,y) - \psi(t_1,y)\,)^2\,dy\,dz\Bigg]\end{aligned}$$

Then by estimating both integrals, it comes

$$\int_{\mathcal{X}\times\mathcal{D}} (\varphi(t_0,y,z)u(t_0,y)-\varphi(t_1,y,z)u(t_1,y)\,)^2\,dy\,dz$$
$$\leq 2\left[\|\psi(t_0,\cdot)\|_\infty^2\|\varphi(t_0,\cdot,\cdot)-\varphi(t_1,\cdot,\cdot)\|_2 + C_\varphi\|(\psi(t_0,\cdot)-\psi(t_1,\cdot)\,)^2\|_\infty\right].$$

Hence, we have

$$\|h\|_2^2 \leq \frac{2}{\lambda_p^2(t_0)}\left[\bar{C}_2 C_\varphi\|(B(t_0,\cdot)-B(t_1,\cdot))^2\|_\infty + 2\|\psi(t_0,\cdot)\|_\infty^2\|\varphi(t_0,\cdot,\cdot)-\varphi(t_1,\cdot,\cdot)\|_2\right]$$
$$+\frac{2}{\lambda_p^2(t_0)}\left[2C_\varphi\|(\psi(t_0,\cdot)-\psi(t_1,\cdot)\,)^2\|_\infty\right],$$

showing that $v\in\mathscr{C}(\mathbb{R}^+,L^2(\mathcal{D}\times\mathcal{X}))$.

Let us denote $\mathcal{K}^+,\mathcal{K}_1^+$ and $\mathcal{K}_2^+$ respectively the positive cone of $\mathscr{C}(\mathbb{R}^+\times\mathcal{D},L^2(\mathcal{X}))$, $\mathscr{C}(\mathbb{R}^+,\mathscr{C}^{0,\alpha}(\mathcal{D})$ and $\mathscr{C}(\mathbb{R}^+,L^2(\mathcal{D}\times\mathcal{X}))$ then from the above existence results, we can see that the following maps $\mathcal{T}_1$ and $\mathcal{T}_2$ are well defined:

$$\begin{array}{cccc}\mathcal{T}_1: & \mathcal{K}^+ & \to & \mathscr{C}(\mathbb{R}^+,\mathscr{C}^{0,\alpha}(\mathcal{D})\cap L^\infty(\bar{\mathcal{D}}))\cap\mathcal{K}_1^+\\ & \varphi & \mapsto & \mathcal{T}_1[\varphi]:=u\end{array}$$

$$\begin{array}{cccc}\mathcal{T}_2: & \mathcal{K}^+\times\mathcal{K}_1^+ & \to & \mathscr{C}(\mathbb{R}^+,L^2(\mathcal{D}\times\mathcal{X}))\cap\mathcal{K}_2^+\\ & (\varphi,\psi) & \mapsto & \mathcal{T}_2[\varphi,\psi]:=v\end{array}$$

where u and v are respectively the unique solution given by (B.2)-(B.2) and the unique solution to (B.3)–(B.4). Observe that by definition, $\mathcal{T}_1$ is bounded, in the sense that $\|\mathcal{T}_1[\varphi]\|_\infty\leq C_0$ for all φ. In addition, arguing as above we can see that for a fixed ψ we get a Lipschitz estimate on the map $\mathcal{T}_2$. More precisely, let φ_1,φ_2 be two functions of $\mathcal{K}^+$, then

$$\int_{\mathcal{D}}\int_{\mathcal{X}}(\mathcal{T}_2(\varphi_1,\psi)-\mathcal{T}_2(\varphi_2,\psi))^2\,dydz \leq 2\|\psi(t_0,\cdot)\|_\infty^2\int_{\mathcal{X}}\int_{\mathcal{D}}(\varphi_1(t,\cdot,\cdot)-\varphi_2(t,\cdot,\cdot))^2\,dydz.$$

Let us also define $\mathcal{T}_\beta$ and $\mathcal{T}_\eta$ the following maps:

$$\begin{array}{cccc}\mathcal{T}_\beta: & \mathscr{C}(\mathbb{R}^+\times\mathbb{E},L^2(\mathcal{X})) & \to & \mathscr{C}(\mathbb{R}^+\times\mathcal{D},L^2(\mathcal{X}))\\ & \Psi & \mapsto & \mathcal{T}_\beta[\psi]:=\displaystyle\int_E\beta\left(y,x,\int_{\mathcal{X}}\Psi,z\right)\Psi(t,x,z)\zeta_E(dx)\end{array}$$

$$\begin{array}{cccc}\mathcal{T}_\eta: & \mathscr{C}(\mathbb{R}^+\times\mathcal{D},L^2(\mathcal{X})) & \to & \mathscr{C}(\mathbb{R}^+\times\mathbb{E},L^2(\mathcal{X}))\\ & \Psi & \mapsto & \mathcal{T}_\eta[\psi]:=\displaystyle\int_{\mathbb{E}}\eta\,(t,y,x)\,\Psi(t,y,z)\,dy\end{array}$$

Equipped with these maps, for $w\in\mathscr{C}(\mathbb{R}^+,\mathscr{C}(E,L^2(\mathcal{X})),w\geq 0$ let us defined the following positive operator

$$\Xi[w]:=\mathcal{T}_\eta\Big[\mathcal{T}_2\Big[\mathcal{T}_\beta[w],\mathcal{T}_1\left[\mathcal{T}_\beta[w]\right]\Big]\Big].$$

Then, the construction of the solution of g_v can be redefined as finding $w\in\mathscr{C}^1(\mathbb{R}^+,\mathscr{C}(E,L^2(\mathcal{X})))$, $w\geq 0$ such that

$$\begin{array}{lll}\partial_t w(t,x,z)=\mathfrak{G}\,(t,x,z,w,\mathcal{T}_1[\mathcal{T}_\beta[w]])\,w(t,x,z)+\mathcal{M}[\,w\,](t,x,z)+\Xi[w](t,x,z)\,, & \text{for} & t>0,x\in E,z\in\mathcal{X}\\ w(0,x,z)=g_0(x,z) & \text{for} & (x,z)\in E\times\mathcal{X},\end{array}$$

The existence of a positive solution of such a problem is rather classical as it reduces to solving a nonlinear ODE problem in an abstract space using Cauchy-Lipschitz like approach coupled with the search of a fixed point of a compact map. We only sketch the argument. We first freeze $\mathcal{T}_1[\mathcal{T}_\beta[w]] =: \psi$ and solve the following problem:

$$\begin{aligned}\partial_t w(t,x,z) &= \mathfrak{G}\,(t,x,z,w,\psi)\,w(t,x,z) + \mathcal{M}[\,w\,](t,x,z)\\ &\quad + \mathcal{T}_\eta\Big[\mathcal{T}_2\Big[\mathcal{T}_\beta[w],\psi\Big]\Big](t,x,z), \quad \text{for } t>0, (x,z)\in E\times\mathcal{X},\\ w(0,x,z) &= g_0(x,z), \quad \text{for } (x,z)\in E\times\mathcal{X}.\end{aligned}$$

Thanks to properties of $\mathcal{T}_\eta, \mathcal{T}_\beta, \mathfrak{G}$ and the Lipschitz continuous property of $\mathcal{T}_2[w,\psi]$, we can construct a unique positive solution to the above system w_ψ. The existence of g_v is then obtained by looking for the fixed point of the bounded compact map $\mathcal{T}_1[\mathcal{T}_\beta[w_\psi]] = \psi$. The uniqueness of the triplet (g_v, g_u, g_c) is obtained by a standard argument using Grownall's inequality.

B.2. *Consequence on the measure* Π. Let us now consider a pair of generic (measure-valued) solutions to the system (3.17)-(3.18). By Lebesgue's decomposition we can write it as

$$\xi_u(t)(dy) = g_u(t,y)\,dy + {\xi_u}^{(sd)}(t)(dy) + {\xi_u}^{(sc)}(t)(dy), \tag{B.6}$$

and

$$\xi_c(t)(dy,dz) = g_c(t,y,z)\,dy\,dz + {\xi_c}^{(sd)}(t)(dy,dz) + {\xi_c}^{(sc)}(t)(dy,dz), \tag{B.7}$$

where g_u and g_c are the unique strong solutions that we just found for (3.17)-(3.18), and the superscripts sd and sc indicate that the measures are the singular discrete and singular continuous parts to their respective Haar measure. However, from the assumptions of Theorem 3.6 we know that at time zero the discrete and continuous singular parts are zero. In particular this implies that for these measures the total population of vectors V_0 equals zero and as a consequence the measures remain zero for later times.

Appendix C. Summary of most commonly used mathematical notation

$\mathcal{X}, \mathcal{D}$: A compact subset of $\mathbb{R}^n$, resp. a bounded C^3 domain in $\mathbb{R}^d$
E: A finite subset of $\mathcal{D}$.
$\mathcal{V}_p, \mathcal{V}_u, \mathcal{V}_c$: The set $E\times\mathcal{X}$, reps. $\mathcal{D}$, resp. $\mathcal{D}\times\mathcal{X}$
$\mathcal{M}_F(X), \mathcal{M}_p(X)$: Set of finite measures on a space X, respectively Set of finite point measures on a space X
$\mathcal{M}_F$: The set of measures given by $\mathcal{M}_F(\mathcal{V}_p)\times\mathcal{M}_F(\mathcal{V}_u)\times\mathcal{M}_F(\mathcal{V}_c)$
$\mathcal{M}_p$: The set of measures given by $\mathcal{M}_p(\mathcal{V}_p)\times\mathcal{M}_p(\mathcal{V}_u)\times\mathcal{M}_p(\mathcal{V}_c)$
$l_m(X)$: The space of measures $\mu\in\mathbb{R}_+\times X$ such that $\mu([0,t]\times X) = t$ for all $t\geq 0$.
L^p, L^∞: functional space of $1\leq p<+\infty$ integrable function, the space bounded function.
$\mathcal{C}^k, \mathscr{C}^k$: Function space of k continuously differentiable functions.
$\mathscr{C}^{k,\alpha}$: Hölder Function space of order $\alpha\in(0,1)$ of k continuously differentiable functions .
$\mathscr{C}^k(A,B)$: functional space of maps from A into B that are $\mathcal{C}^k$, $B = L^p, L^\infty, \mathscr{C}^k$ or $\mathscr{C}^{k,\alpha}$.
ζ_E: Counting measure on the discrete set E.
$\mathcal{L}^\alpha$: Infinitesimal generator of Ito diffusion corresponding to $\alpha\in\{u,c\}$. See expression (1.2).
$\boldsymbol{\phi}$: An admissible triplet $\{\phi_v,\phi_u,\phi_c\}$ in the sense of Definition 1.1 .
$[F(\nu_u,\nu_c)]_{y,z}^{K,\lambda,\alpha}$: Discrete gradient wrt the α variable. See display (4.28) .
$\mathbb{D}([0,T],X)$: Skorokhod space of X-valued trajectories.
$\mathbb{C}([0,T],X)$: Space of X-valued continuous trajectories.

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