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On the Quenched Functional Central Limit Theorem for Stationary Random Fields under Projective Criteria

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Abstract. In this work, we study and establish some quenched functional Central Limit Theorems (CLTs) for stationary random fields under a projective criteria. These results are functional generalizations of the theorems obtained by Zhang et al. (2020) and of the quenched functional CLTs for ortho-martingales established by Peligrad and Volný (2020) to random fields satisfying a Hannan type projective condition. In the work of Zhang et al. (2020), the authors have already proven a quenched functional CLT, however the assumptions were not optimal as they required the existence of a $2+\delta$-moment. In this article, we establish the results under weaker assumptions, namely we only require an Orlicz space condition to hold. The methods used to obtain these generalizations are somewhat similar to the ones used by Zhang et al. (2020) but we improve on them in order to obtain results within the functional framework. Moreover, a Rosenthal type inequality for said Orlicz space is also derived and used to obtain a sufficient condition analogous to that of Theorem 4.4 in the work of Zhang et al. (2020). Finally, we apply our new results to derive some quenched functional CLTs under weak assumptions for a variety of stochastic processes.

1. Introduction

Developments within the Markovian theory led to the question of the conditions under which a central limit theorem could be derived for Markov chains; in particular what restrictions were sufficient on the initial distribution and the transition operator to have this kind of result. Seminal results were obtained by Gordin and Lifšic (1978) (see also Borodin and Ibragimov, 1995; Derriennic and Lin, 2001) for Markov chains endowed with the stationary measure as their initial distribution as well as Kipnis and Varadhan (1986) (see also Derriennic and Lin, 2001) for additive functionals of reversible Markov chains. Additionally, Derriennic and Lin (2001) also obtained a CLT for Markov chains starting from a fixed point (in other words, endowed with δ_x, the Dirac measure at the state x, as their initial distribution). Such theorems are called quenched CLTs. Another way of expressing

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these results is to consider a fixed past and to study the convergence in distribution with respect to that past. The difficulties during the proof arise from the fact that this fixed past causes the process to not be stationary anymore. An extensive literature exists on the subject, one can cite the following works by Barrera et al. (2016); Cuny and Peligrad (2012); Cuny and Merlevède (2014); Cuny and Volný (2013); Dedecker et al. (2014); Peligrad (2015); Volný and Woodroofe (2014). Note that some counterexamples to quenched central limit theorems under specific conditions were found by Ouchti and Volný (2008) and Volný and Woodroofe (2010). Functional versions of these quenched central limit theorems, also called quenched weak invariance principles, have also been the subject of numerous research articles such as the ones by Barrera et al. (2016); Cuny and Merlevède (2014); Cuny and Volný (2013); Peligrad (2015).

Random fields naturally appear as a generalization of sequences of random variables, however extending the one-dimensional results to greater dimensions is much harder than one would think. The first problem we are faced with is to correctly define the notion of past trajectory. The approach we have implemented in this paper is to use the notion of commuting filtrations. In particular, this property is satisfied by filtrations generated by fields of independent random variables or even by fields with independent columns (or, equivalently, independent rows). As a lot of processes can be expressed as a functional of i.i.d. random variables, these types of filtrations are quite common and merit interest. A lot of work has been done under commuting filtrations (see Volný, 2015; Cuny et al., 2016).

As usual, we will require some kind of dependency condition on the studied field. Namely, in this paper, we will use Hannan's projective condition as defined by Hannan (1973). The problem we are interested in has been studied by Cuny and Volný (2013) for time series but it has yet to be investigated for higher dimensions, which is the purpose of this article. Though the problem we focus on hasn't been studied yet, one can note that fields satisfying Hannan's condition have been quite extensively studied and numerous CLTs and functional CLTs, both in the annealed and quenched sense, have been obtained. One could refer to the following works: Volný and Wang (2014); Klicnarová et al. (2016); Zhang et al. (2020).

The proofs for the main theorems in this paper are based upon the use of a martingale-coboundary decomposition that can be found in Volný and Wang (2014) (some more recent and general results can be found in El Machkouri and Giraudo, 2016; Volný, 2018; Giraudo, 2018, see also Gordin, 2009) as well as the central limit theorem and the weak invariance principle established by Peligrad and Volný (2020) for ortho-martingales. Once the main theorems are established, we derive corollaries in the spirit of the results obtained by Zhang et al. (2020). As shown by the previous results in the literature, it will be required to address two situations separately: first when the summations are done over cubic regions of $\mathbb{Z}^d$ and, after that, when the regions are only required to be rectangular.

In the previous work of Zhang et al. (2020), the Rosenthal inequality for Lebesgue spaces (see Hall and Heyde, 1980, Theorem 2.11, p.23) was used to derive a sufficient condition for the quenched CLT and its functional form to hold. In order to obtain an analogous result within our framework, we will make use of a Rosenthal type inequality for this Orlicz space. Given that no such result seems to exist in the literature, we will follow the outline of the proof given by Burkholder (1973) and adapt it to our framework in order to establish the required inequality.

This paper will be structured as follows: in Section 2, we introduce the notations used throughout our article and we present the main results obtained in this work. In particular, we will split the results into two categories: the first one will aggregate theorems dealing with summations over cubic regions only while the other category will deal with results concerning more general rectangular regions. The proofs of these theorems will appear in Section 3 and we will improve on the two applications studied by Zhang et al. (2020) as well as provide some additional examples in Section 4. These examples include linear and Volterra random fields as well as Hölder continuous functions of linear fields, which are a common occurrence in the field of financial mathematics and economics, and also weakly dependent random fields in the sense of Wu (2005) which hold a significant role in mathematical physics and, in particular, within the study of particle systems. Finally, in Section 5, we give the proof of the Rosenthal type inequality for the Orlicz space mentioned throughout this paper.

2. Framework and results

In all that follows, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and all the random variables considered thereafter will be real-valued and defined on that probability space. We start by introducing multiple items of notation that will be used throughout this article: d will be an integer greater than 1, $[x]$ will denote the integer part of a real number x, bold characters will designate multi-indexes and in particular we shall write $\mathbf{0} := (0, \ldots, 0) \in \mathbb{Z}^d$ as well as $\mathbf{1} := (1, \ldots, 1) \in \mathbb{Z}^d$. For any $\boldsymbol{n} \in \mathbb{Z}^d$, we denote $\boldsymbol{n} := (n_1, \ldots, n_d)$ and $|\boldsymbol{n}| := \prod_{i=1}^d n_i$. The set of all positive integers will be denoted by $\mathbb{N}^*$ and the set of integers $\{1, \ldots, d\}$ will be denoted by $[\![1, d]\!]$. In order to define the concept of past trajectory, it is necessary to define an order on $\mathbb{Z}^d$: if $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^d$ are multi-indexes such that for all $k \in [\![1, d]\!], u_k \leq v_k$, then we will write $\boldsymbol{u} \leq \boldsymbol{v}$.

Convergence of fields indexed by $\mathbb{Z}^d$ will be interpreted in the following sense. If $\boldsymbol{n} = (n_1, \ldots, n_d)$ is a multi-index, then the notation $\boldsymbol{n} \to \infty$ is to be interpreted as the convergence of $\min\{n_1, \ldots, n_d\}$ to ∞. Convergence in distribution (resp. almost surely) will be denoted by $\xrightarrow{\mathcal{D}}$ (resp. $\xrightarrow{\text{a.s.}}$).

Before introducing the field we are interested in, we define some transformations on Ω. We let $T_i : \Omega \to \Omega$, $i \in \{1, \ldots, d\}$ be invertible measure-preserving commuting transforms on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we make use of the operator notation (i.e. if U and V are two transformations on Ω, we denote $UV := U \circ V$).

We consider a sigma-field $\mathcal{F}_{\mathbf{0}} \subset \mathcal{F}$ such that $\mathcal{F}_{\mathbf{0}} \subset T^{-\boldsymbol{i}}\mathcal{F}_{\mathbf{0}}$ for all $\boldsymbol{i} \in \mathbb{Z}^d$, and a random variable $X_{\mathbf{0}} \in L_2^0$ where $L_2^0 = L_2^0(\Omega, \mathcal{F}_{\mathbf{0}}, \mathbb{P})$ is the set of all $\mathcal{F}_{\mathbf{0}}$-measurable and square integrable random variables with zero mean.

For every $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, set

$$X_{\boldsymbol{n}} = X_{\mathbf{0}} \circ T^{\boldsymbol{n}}, \tag{2.1}$$

and

$$\mathcal{F}_{\boldsymbol{n}} = T^{-\boldsymbol{n}}\mathcal{F}_{\mathbf{0}}, \tag{2.2}$$

where $T^{\boldsymbol{n}} = T_1^{n_1} \cdots T_d^{n_d}$. As a result, $X_{\boldsymbol{n}}$ is $\mathcal{F}_{\boldsymbol{n}}$-measurable.

Suppose that the family $(\mathcal{F}_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathbb{Z}^d}$ is a commuting filtration, that is, for every integrable random variable X, we have

$$\mathbb{E}_{\boldsymbol{i}}\big[\mathbb{E}_{\boldsymbol{j}}[X]\big] = \mathbb{E}_{\boldsymbol{i} \wedge \boldsymbol{j}}[X],$$

where $\mathbb{E}_{\boldsymbol{i}}[X] = \mathbb{E}[X|\mathcal{F}_{\boldsymbol{i}}]$ and $\boldsymbol{i} \wedge \boldsymbol{j}$ is the coordinate-wise minimum between $\boldsymbol{i}$ and $\boldsymbol{j}$.

We recall the notion of ortho-martingale which was introduced by Cairoli (1969) (see also Khoshnevisan, 2002). We say that a random field $(D_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ is an ortho-martingale difference field if each $D_{\boldsymbol{n}}$ is in $L^1(\mathcal{F}_{\boldsymbol{n}})$ and satisfies the equation $\mathbb{E}_{\boldsymbol{a}}[D_{\boldsymbol{n}}] = 0$ as long as there exists $k \in [\![1, d]\!]$ such that $a_k < n_k$. Then, if $M_{\boldsymbol{n}} := \sum_{\mathbf{0} \leq \boldsymbol{u} \leq \boldsymbol{n}} D_{\boldsymbol{u}}$, the random field $(M_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{Z}_+^d}$ will be called an ortho-martingale.

Suppose also that the random variable $X_{\mathbf{0}}$ is regular with respect to the filtration $\mathcal{F}$, that is $\mathbb{E}[X_{\mathbf{0}}|\mathcal{F}_{-\infty \boldsymbol{e}_i}] = 0$ for every $i \in \{1, \cdots, d\}$, where $\boldsymbol{e}_i$ is the multi-index whose i-th coordinate is equal to 1 and the others are equal to 0 with the convention that $\infty \times 0 = 0$.

We consider the projection operators defined, for any $\boldsymbol{n} \in \mathbb{Z}^d$, by $\mathcal{P}_{\boldsymbol{n}} = \prod_{i=1}^d (\mathbb{E}_{\boldsymbol{n}} - \mathbb{E}_{\boldsymbol{n}-\boldsymbol{e}_i})$, and for every $\omega \in \Omega$, we denote by $\mathbb{P}^\omega$ a regular version of the conditional probability given $\mathcal{F}_{\mathbf{0}}$, that is, $\mathbb{P}^\omega = \mathbb{P}(\cdot|\mathcal{F}_{\mathbf{0}})(\omega)$.

Finally, we introduce the sum that we will be studying, for every $\boldsymbol{n} \in (\mathbb{N}^*)^d$,

$$S_{\boldsymbol{n}} = \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} X_{\boldsymbol{i}} := \sum_{\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{n}} X_{\boldsymbol{i}},$$

and we also set

$$\bar{S}_{\boldsymbol{n}} = S_{\boldsymbol{n}} - R_{\boldsymbol{n}} \quad \text{with} \quad R_{\boldsymbol{n}} = \sum_{i=1}^{d} (-1)^{i-1} \sum_{1 \leq j_1 < \cdots < j_i \leq d} \mathbb{E}_{\boldsymbol{n}^{(j_1, \cdots, j_i)}}[S_{\boldsymbol{n}}],$$

where $\boldsymbol{n}^{(j_1, \cdots, j_d)}$ is the multi-index obtained by replacing with 0 all the $j_1, \cdots, j_i$-th coordinates of the multi-index $\boldsymbol{n}$ and leaving the rest unchanged.

In dimension $d = 1$, this reduces to the following expression:

$$\bar{S}_n = S_n - \mathbb{E}[S_n|\mathcal{F}_0], \quad \text{for } n \in \mathbb{N}^*.$$

This case was investigated by Cuny and Volný (2013) and therefore, we will always consider $d > 1$ in the rest of the paper. In dimension $d = 2$, the definition of $\bar{S}_{\boldsymbol{n}}$ reduces down to

$$\bar{S}_{n,m} = S_{n,m} - \mathbb{E}[S_{n,m}|\mathcal{F}_{n,0}] - \mathbb{E}[S_{n,m}|\mathcal{F}_{0,m}] + \mathbb{E}[S_{n,m}|\mathcal{F}_{0,0}], \quad \text{for } (n, m) \in (\mathbb{N}^*)^2.$$

2.1. *Functional CLT over cubic regions.* Here we present the quenched functional CLT over cubic regions of $\mathbb{Z}^d$. These results expand Theorem 4.1, the second part of Corollary 4.3, and Theorem 4.4 (a) obtained by Zhang et al. (2020) to the functional framework. It is also possible to view these results as an extension to higher dimensions of Theorem 1 established by Cuny and Volný (2013). As noted by Zhang et al. (2020), the proofs of these theorems essentially reduce down to particular cases of the proofs of the functional central limit theorems over rectangular regions of $\mathbb{Z}^d$. The differences in the proofs between the two frameworks will be specified in greater detail in Section 3.

Theorem 2.1. *Assume that $(X_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{Z}^d}$ is defined by (2.1) and that the filtration $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{Z}^d}$ given by (2.2) is commuting. Also, assume that one of the transformations $T_i, 1 \leq i \leq d$, is ergodic and that*

$$\sum_{\boldsymbol{u} \geq \mathbf{0}} \|\mathcal{P}_{\mathbf{0}}(X_{\boldsymbol{u}})\|_2 < \infty. \tag{2.3}$$

Then, for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\left(\frac{1}{n^{d/2}} \bar{S}_{[n\boldsymbol{t}]} \right)_{\boldsymbol{t} \in [0,1]^d} \xrightarrow[n \to \infty]{\mathcal{D}} (\sigma W_{\boldsymbol{t}})_{\boldsymbol{t} \in [0,1]^d} \quad \text{under} \quad \mathbb{P}^\omega,$$

where $\sigma^2 := \mathbb{E}[D_{\mathbf{0}}^2]$ with $D_{\mathbf{0}} = \sum_{\boldsymbol{i} \geq \mathbf{0}} \mathcal{P}_{\mathbf{0}}(X_{\boldsymbol{i}})$, $(W_{\boldsymbol{t}})_{\boldsymbol{t} \in [0,1]^d}$ is a standard Brownian sheet, $[k\boldsymbol{t}] := ([kt_1], \cdots, [kt_d])$ for $k \in \mathbb{Z}$ and the convergence happens in the Skorokhod space $D([0,1]^d)$ endowed with the uniform topology. Moreover, $\sigma^2 = \lim\limits_{n \to \infty} \frac{\mathbb{E}[\bar{S}^2_{n,\ldots,n}]}{n^d}$.

In Theorem 2.1, the random centering $R_{[n\boldsymbol{t}]}$ cannot be avoided without additional hypotheses. As a matter of fact, for $d = 1$, Volný and Woodroofe (2010) constructed an example showing that the CLT for partial sums needs not be quenched. It should also be noticed that, for a stationary ortho-martingale, the existence of a finite second moment is not enough for the validity of a quenched CLT when the summation is taken over rectangles (see Peligrad and Volný, 2020). That being said, the following corollary gives a sufficient condition to get rid of the stochastic centering $R_{\boldsymbol{n}}$ in the previous theorem.

Corollary 2.2. *Assume that the hypotheses of Theorem 2.1 are satisfied and assume in addition that for every $i \in \{1, \ldots, d\}$, it holds*

$$\frac{1}{n^d}\mathbb{E}_{\boldsymbol{0}}\Big[\max_{\boldsymbol{1}\le\boldsymbol{m}\le n\boldsymbol{1}}\big(\mathbb{E}_{\boldsymbol{m}^{(i)}}[S_{\boldsymbol{m}}]\big)^2\Big] \xrightarrow[n\to\infty]{\text{a.s.}} 0$$

where we recall that $\boldsymbol{m}^{(i)}$ is the multi-index obtained by replacing with 0 the i-th coordinate of the multi-index $\boldsymbol{m}$ and leaving the rest unchanged. Then, for almost all $\omega \in \Omega$,

$$\left(\frac{1}{n^{d/2}}S_{[n\boldsymbol{t}]}\right)_{\boldsymbol{t}\in[0,1]^d} \xrightarrow[n\to\infty]{\mathcal{D}} (\sigma W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d} \quad \text{under} \quad \mathbb{P}^\omega, \tag{2.4}$$

where $(W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d}$ is a standard Brownian sheet and the convergence happens in the Skorokhod space $D([0,1]^d)$ endowed with the uniform topology.

To end this section, we give a condition that is easier to verify but still guarantees that the convergence (2.4) holds.

Corollary 2.3. *Assume that $(X_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ is defined by (2.1), that $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ is given by (2.2) and is a commuting filtration, and that one of the transformations $T_i, 1 \le i \le d$, is ergodic. If the following condition is satisfied:*

$$\sum_{\boldsymbol{u}\ge\boldsymbol{1}} \frac{\|\mathbb{E}_{\boldsymbol{1}}(X_{\boldsymbol{u}})\|_2}{|\boldsymbol{u}|^{\frac{1}{2}}} < \infty. \tag{2.5}$$

Then, for almost all $\omega \in \Omega$, the conclusion of Corollary 2.2 holds.

Once again we note that this result is an extension of Corollary 2 in Cuny and Volný (2013) to random fields and an extension of Theorem 2.6 (a) found in Zhang et al. (2020) to the functional framework.

2.2. *Functional CLT over rectangular regions.* In order to obtain a functional CLT when we sum over rectangular regions, a stronger projective condition than (2.3) is necessary. Indeed, Peligrad and Volný (2020) gave a counterexample to a quenched CLT over rectangles for some stationary ortho-martingale under condition (2.3). This leads us to consider a projective condition in an Orlicz space associated with a specific Young function.

Following the work of Krasnosel'skiĭ and Rutitskiĭ (1960), we define the Luxemburg norm associated with the Young function $\Phi : [0, \infty) \to [0, \infty)$ as

$$\|f\|_\Phi = \inf\Big\{t > 0 : \mathbb{E}\big[\Phi(|f|/t)\big] \le 1\Big\}.$$

In everything that follows, we will consider the Young function $\Phi_d : [0, \infty) \to [0, \infty)$ defined for every $x \in [0, \infty)$ by

$$\Phi_d(x) = x^2(\log(1 + x))^{d-1}. \tag{2.6}$$

Theorem 2.4. *Assume that $(X_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ is defined by (2.1) and that the filtration $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ given by (2.2) is commuting. Also, assume that one of the transformations $T_i, 1 \leq i \leq d$, is ergodic and that*

$$\sum_{\boldsymbol{u}\geq\boldsymbol{0}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}})\|_{\Phi_d} < \infty. \tag{2.7}$$

Then, for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\left(\frac{1}{\sqrt{|\boldsymbol{n}|}}\bar{S}_{[\boldsymbol{t}\boldsymbol{n}]}\right)_{\boldsymbol{t}\in[0,1]^d} \xrightarrow[\boldsymbol{n}\to\infty]{\mathcal{D}} (\sigma W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d} \quad \text{under} \quad \mathbb{P}^\omega,$$

where $[\boldsymbol{t}\boldsymbol{n}] := ([t_1n_1], \cdots, [t_dn_d])$, σ^2 is defined in Theorem 2.1, $(W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d}$ is a Brownian sheet, and the convergence happens in the Skorokhod space $D([0,1]^d)$. In addition, $\sigma^2 = \lim_{\boldsymbol{n}\to\infty} \frac{\mathbb{E}[\bar{S}^2_{\boldsymbol{n}}]}{|\boldsymbol{n}|}$.

We remark that this result and the following ones extend Theorem 4.2, the first part of Corollary 4.3, and Theorem 2.6 (b) in Zhang et al. (2020) by obtaining functional versions of these theorems.

Corollary 2.5. *Suppose that the hypotheses of Theorem 2.4 hold and assume that in addition, for every $i \in [\![1, d]\!]$,*

$$\frac{1}{|\boldsymbol{n}|}\mathbb{E}_{\boldsymbol{0}}\left[\max_{\boldsymbol{1}\leq\boldsymbol{m}\leq\boldsymbol{n}} \left(\mathbb{E}_{\boldsymbol{m}^{(i)}}[S_{\boldsymbol{m}}]\right)^2\right] \xrightarrow[\boldsymbol{n}\to\infty]{\text{a.s.}} 0.$$

Then, for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\left(\frac{1}{\sqrt{|\boldsymbol{n}|}}S_{[\boldsymbol{t}\boldsymbol{n}]}\right)_{\boldsymbol{t}\in[0,1]^d} \xrightarrow[\boldsymbol{n}\to\infty]{\mathcal{D}} (\sigma W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d} \quad \text{under} \quad \mathbb{P}^\omega, \tag{2.8}$$

where $(W_{\boldsymbol{t}})_{\boldsymbol{t}\in[0,1]^d}$ is a Brownian sheet and the convergence happens in the Skorokhod space $D([0,1]^d)$.

Corollary 2.6. *Assume that the hypotheses of Theorem 2.4 and (2.5) hold. Then for almost all $\omega \in \Omega$, (2.8) holds.*

This last Corollary not only extends Theorem 4.4 (b) in Zhang et al. (2020) to the functional case but also reduces the required condition even in the classical CLT case.

Corollary 2.7. *Assume that $(X_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ is defined by (2.1), that $(\mathcal{F}_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ is given by (2.2) and is a commuting filtration, and that one of the transformations $T_i, 1 \leq i \leq d$, is ergodic. If the following condition is satisfied:*

$$\sum_{\boldsymbol{u}\geq\boldsymbol{1}} \frac{\|\mathbb{E}_{\boldsymbol{1}}[X_{\boldsymbol{u}}]\|_{\Phi_d}}{\Phi_d^{-1}(|\boldsymbol{u}|)} < \infty. \tag{2.9}$$

Then, for almost all $\omega \in \Omega$, the conclusion of Corollary 2.5 holds.

3. Proofs of the results

Before we prove the previous results, we start by defining some additional notations:

- if $h : \Omega \to \mathbb{R}$ is a measurable function, we will denote by $h_{\boldsymbol{u}}, \boldsymbol{u} \in \mathbb{Z}^d$, the function $h \circ T^{\boldsymbol{u}}$;
- for any $\boldsymbol{n} \in (\mathbb{N}^*)^d$ and for any measurable function $h : \Omega \to \mathbb{R}$, we denote

  $$S_{\boldsymbol{n}}(h) = \sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{n}} h_{\boldsymbol{i}} \quad \text{and} \quad \bar{S}_{\boldsymbol{n}}(h) = S_{\boldsymbol{n}}(h) - R_{\boldsymbol{n}}(h)$$

  where

  $$R_{\boldsymbol{n}}(h) = \sum_{i=1}^{d}(-1)^{i-1} \sum_{1\leq j_1<\cdots<j_i\leq d} \mathbb{E}_{\boldsymbol{n}^{(j_1,\cdots,j_i)}}[S_{\boldsymbol{n}}(h)],$$

  and $\boldsymbol{n}^{(j_1,\cdots,j_i)}$ is the multi-index whose $j_1, \cdots, j_i$−th coordinates are 0 and the others are equal to the corresponding coordinates of $\boldsymbol{n}$;

- for any $i \in [\![1, d]\!]$ and for any $\ell \in \mathbb{N}$, we denote
$$\mathcal{F}_\ell^{(i)} = \bigvee_{\substack{\boldsymbol{k}\in\mathbb{Z}^d \\ k_i \le l}} \mathcal{F}_{\boldsymbol{k}};$$
- we set $L^2 \log^{d-1} L(\mathcal{G})$ to be the set of $\mathcal{G}$-measurable functions $h : \Omega \to \mathbb{R}$ such that $\mathbb{E}\Big[h^2 \max\big(0, \log|h|\big)^{d-1}\Big] < \infty$; if $\mathcal{G} = \mathcal{F}$, we simply write $L^2 \log^{d-1} L(\mathcal{F}) = L^2 \log^{d-1} L$;
- if $h \in L^2 \log^{d-1} L$, then we define the maximal operator $h^* = \sup_{\boldsymbol{m}>\boldsymbol{0}} \frac{1}{|\boldsymbol{m}|} \sum_{\boldsymbol{1}\le \boldsymbol{i} \le \boldsymbol{m}} |h| \circ T^{\boldsymbol{i}}$.

Let us start with the proof of Theorem 2.4 as it is the most general result. Moreover, the computations used in the proof of Theorem 2.1 are a particular case of the computations used in the proof of Theorem 2.4 and will be largely skipped.

The proof of Theorem 2.4 relies on the following important lemma which we will refer to as the Main Lemma in the rest of the paper

Lemma 3.1 (Main Lemma)**.** *For any $\mathcal{F}_{\boldsymbol{0}}$-measurable function $h \in L^2 \log^{d-1} L$ satisfying the following condition:*
$$\sum_{\boldsymbol{u}\ge\boldsymbol{0}} \|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\|_{\Phi_d} < \infty, \tag{3.1}$$
there exists an integrable function g such that for all $\boldsymbol{N} \in (\mathbb{N}^*)^d$,
$$\sqrt{\mathbb{E}_{\boldsymbol{0}}\left[\max_{\boldsymbol{1}\le\boldsymbol{n}\le\boldsymbol{N}} \frac{1}{|\boldsymbol{n}|}\big|\overline{S}_{\boldsymbol{n}}(h)\big|^2\right]} \le g \quad \mathbb{P}-\text{a.s.}$$

To establish this lemma, we shall first obtain the following intermediary lemma.

Lemma 3.2. *For any function $h \in L^2 \log^{d-1} L$, there exists a constant $C > 0$ such that for all $\boldsymbol{u} \in \mathbb{Z}^d$, we have*
$$\left\|\sqrt{\Big(\big|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\big|^2\Big)^*}\right\|_1 \le C\|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\|_{\Phi_d}.$$

Proof of Lemma 3.2: Let $h \in L^2 \log^{d-1} L$, $\boldsymbol{u} \in \mathbb{Z}^d$ and $t > \|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\|_{\Phi_d}$. We let
$$\Omega_t = \Big\{\omega \in \Omega : 4\big(\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\big)^2(\omega) > t^2\Big\}.$$
According to Corollary 1.7 of Chapter 6 in Krengel (1985), there exists a constant $C_d > 0$ such that
$$\begin{aligned}
\mathbb{P}\left(\sup_{\boldsymbol{n}\in(\mathbb{N}^*)^d} \frac{1}{|\boldsymbol{n}|} \sum_{\boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}} \Big(\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}}) \circ T^{\boldsymbol{i}}\Big)^2 > t^2\right) &\le C_d \int_{\Omega_t} \frac{4\big(\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\big)^2}{t^2} \left(\log\left(\frac{4\big(\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\big)^2}{t^2}\right)\right)^{d-1} \mathrm{d}\mathbb{P} \\
&\le 2^{d-1} C_d \int_{\Omega_t} \frac{\big(2\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\big)^2}{t^2} \left(\log\left(1 + \frac{2|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})|}{t}\right)\right)^{d-1} \mathrm{d}\mathbb{P} \\
&\le 2^{d+1} C_d t^{-2} \|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\|_{\Phi_d}^2.
\end{aligned}$$
The last inequality results from the fact that
$$\|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\|_{\Phi_d} = \inf\left\{t > 0 : \mathbb{E}\left[\Phi_d\left(\frac{|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})|}{t}\right)\right] \le 1\right\}.$$
Indeed, by letting $t_0 = \|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\|_{\Phi_d}$, we have
$$\mathbb{E}\left[\big(\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\big)^2 \left(\log\left(1 + \frac{|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})|}{t_0}\right)\right)^{d-1}\right] \le t_0^2.$$

Hence since $t > t_0$,

$$\int_{\Omega} \frac{\left(\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\right)^2}{t^2}\left(\log\left(1+\frac{|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})|}{t}\right)\right)^{d-1} \mathrm{d}\mathbb{P} \le t_0^2 t^{-2}.$$

Therefore, applying this inequality to $h' = 2h \in L^2 \log^{d-1} L$, we get

$$\int_{\Omega_t} \frac{\left(2\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\right)^2}{t^2}\left(\log\left(1+\frac{2|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})|}{t}\right)\right)^{d-1} \mathrm{d}\mathbb{P} \le 4t^{-2}\|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\|_{\Phi_d}^2.$$

Thus

$$\begin{aligned}\left\|\sqrt{\left(\left|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\right|^2\right)^*}\right\|_1 &= \int_0^\infty \mathbb{P}\left(\sup_{\boldsymbol{n}\in(\mathbb{N}^*)^d} \frac{1}{|\boldsymbol{n}|}\sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{n}}\left(\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\circ T^{\boldsymbol{i}}\right)^2 > t^2\right)\mathrm{d}t \\ &\le \int_0^{t_0} 1\mathrm{d}t + \int_{t_0}^\infty \mathbb{P}\left(\sup_{\boldsymbol{n}\in(\mathbb{N}^*)^d} \frac{1}{|\boldsymbol{n}|}\sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{n}}\left(\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\circ T^{\boldsymbol{i}}\right)^2 > t^2\right)\mathrm{d}t \\ &\le (2^{d+1}C_d+1)\|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\|_{\Phi_d}.\end{aligned}$$

□

Proof of the Main Lemma: We consider a measurable function h satisfying the hypotheses of the lemma and we let $\boldsymbol{n}, \boldsymbol{N} \in (\mathbb{N}^*)^d$ such that $\boldsymbol{n} \le \boldsymbol{N}$. Then, we start by studying the quantity $\bar{S}_{\boldsymbol{n}}(h)$ using the following projective decomposition (see Peligrad and Zhang, 2018):

$$S_{\boldsymbol{n}}(h) - R_{\boldsymbol{n}}(h) = \sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{n}} \mathcal{P}_{\boldsymbol{i}}\left(\sum_{\boldsymbol{i}\le \boldsymbol{u}\le \boldsymbol{n}} h_{\boldsymbol{u}}\right) = \sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{n}} \mathcal{P}_{\mathbf{0}}\left(\sum_{\mathbf{0}\le \boldsymbol{u}\le \boldsymbol{n}-\boldsymbol{i}} h_{\boldsymbol{u}}\right)\circ T^{\boldsymbol{i}}.$$

By exchanging the sums, we get

$$\bar{S}_{\boldsymbol{n}}(h) = \sum_{\mathbf{0}\le \boldsymbol{u}\le \boldsymbol{n}-\mathbf{1}}\ \sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{n}-\boldsymbol{u}} \mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\circ T^{\boldsymbol{i}}.$$

Then, recalling that $\boldsymbol{n} \le \boldsymbol{N}$, we obtain

$$\left|\bar{S}_{\boldsymbol{n}}(h)\right| \le \sum_{\mathbf{0}\le \boldsymbol{u}\le \boldsymbol{N}-\mathbf{1}} \max_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{N}}\left|\sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{k}} \mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\circ T^{\boldsymbol{i}}\right|.$$

Note that for all $\boldsymbol{u} \ge \mathbf{0}$, the partial sum $\sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{k}} \mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\circ T^{\boldsymbol{i}}$ is an ortho-martingale. Using Cairoli's inequality for ortho-martingales (see Khoshnevisan, 2002), we find that

$$\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le \boldsymbol{k}\le \boldsymbol{N}}\left|\sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{k}} \mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\circ T^{\boldsymbol{i}}\right|^2\right] \le 2^{2d}\mathbb{E}_{\mathbf{0}}\left[\left(\sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{N}} \mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\circ T^{\boldsymbol{i}}\right)^2\right].$$

By orthogonality, we obtain for all $\boldsymbol{N} \in (\mathbb{N}^*)^d$

$$\sqrt{\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le \boldsymbol{n}\le \boldsymbol{N}}\left|\bar{S}_{\boldsymbol{n}}(h)\right|^2\right]} \le 2^d\sum_{\boldsymbol{u}\ge\mathbf{0}}\sqrt{\sum_{\mathbf{1}\le \boldsymbol{i}\le \boldsymbol{N}} \mathbb{E}_{\mathbf{0}}\left[\left(\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\right)^2\circ T^{\boldsymbol{i}}\right]} \le 2^d\sqrt{|\boldsymbol{N}|}\sum_{\boldsymbol{u}\ge\mathbf{0}}\sqrt{\left(\left|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\right|^2\right)^*}.$$

Since the previous inequality is satisfied for all $\boldsymbol{N} \in (\mathbb{N}^*)^d$, then it also holds

$$\sqrt{\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le \boldsymbol{n}\le \boldsymbol{N}}\frac{1}{|\boldsymbol{n}|}\left|\bar{S}_{\boldsymbol{n}}(h)\right|^2\right]} \le 2^d\sum_{\boldsymbol{u}\ge\mathbf{0}}\sqrt{\left(\left|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\right|^2\right)^*}. \tag{3.2}$$

However, according to Lemma 3.2 and hypothesis (3.1), there exists $C > 0$ such that

$$\left\| \sum_{\boldsymbol{u} \geq \boldsymbol{0}} \sqrt{\left(\left|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\right|^2\right)^*} \right\|_1 \leq C \sum_{\boldsymbol{u} \geq \boldsymbol{0}} \|\mathcal{P}_{\boldsymbol{0}}(h_{\boldsymbol{u}})\|_{\Phi_d} < \infty.$$

This concludes the proof of the main lemma. □

Proof of Theorem 2.4: For any $n \in \mathbb{N}^*$, we let

$$X_{\boldsymbol{0}}^{(n)} = \sum_{\boldsymbol{j} \in [\![-n,0]\!]^d} \mathcal{P}_{\boldsymbol{j}}(X_{\boldsymbol{0}}).$$

Given the regularity of $X_{\boldsymbol{0}}$, the sequence of random variable $(X_{\boldsymbol{0}} - X_{\boldsymbol{0}}^{(n)})_{n \in \mathbb{N}}$ converges almost surely to 0 and using (3.2), we get the inequality

$$\limsup_{\boldsymbol{N} \to \infty} \sqrt{\mathbb{E}_{\boldsymbol{0}}\left[\max_{\boldsymbol{1} \leq \boldsymbol{m} \leq \boldsymbol{N}} \frac{1}{|\boldsymbol{m}|} \left|\overline{S}_{\boldsymbol{m}}(X_{\boldsymbol{0}} - X_{\boldsymbol{0}}^{(n)})\right|^2\right]} \leq 2^d \sum_{\boldsymbol{u} \geq \boldsymbol{0}} \sqrt{\left(\left|\mathcal{P}_{\boldsymbol{0}}\Big((X_{\boldsymbol{0}} - X_{\boldsymbol{0}}^{(n)}) \circ T^{\boldsymbol{u}}\Big)\right|^2\right)^*}$$

for all $n \in \mathbb{N}^*$. Then, using lemma 3.2, there exists a constant C such that

$$\left\| \sqrt{\limsup_{\boldsymbol{N} \to \infty} \mathbb{E}_{\boldsymbol{0}}\left[\max_{\boldsymbol{1} \leq \boldsymbol{m} \leq \boldsymbol{N}} \frac{1}{|\boldsymbol{m}|} \left|\overline{S}_{\boldsymbol{m}}(X_{\boldsymbol{0}} - X_{\boldsymbol{0}}^{(n)})\right|^2\right]} \right\|_1 \leq C \sum_{\boldsymbol{u} \geq \boldsymbol{0}} \left\|\mathcal{P}_{\boldsymbol{0}}\Big((X_{\boldsymbol{0}} - X_{\boldsymbol{0}}^{(n)}) \circ T^{\boldsymbol{u}}\Big)\right\|_{\Phi_d} \xrightarrow[n \to \infty]{} 0.$$

Therefore, there exists an increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \to \infty} \limsup_{\boldsymbol{N} \to \infty} \mathbb{E}_{\boldsymbol{0}}\left[\max_{\boldsymbol{1} \leq \boldsymbol{m} \leq \boldsymbol{N}} \frac{1}{|\boldsymbol{m}|} \left|\overline{S}_{\boldsymbol{m}}(X_{\boldsymbol{0}} - X_{\boldsymbol{0}}^{(n_k)})\right|^2\right] = 0 \quad \text{a.s.} \tag{3.3}$$

Moreover, we also have, for all $n \in \mathbb{N}^*$

$$\frac{1}{|\boldsymbol{N}|} \mathbb{E}_{\boldsymbol{0}}\left[\max_{\boldsymbol{1} \leq \boldsymbol{i} \leq \boldsymbol{N}} \left|R_{\boldsymbol{i}}(X_{\boldsymbol{0}}^{(n)})\right|^2\right] \xrightarrow[\boldsymbol{N} \to \infty]{\text{a.s.}} 0. \tag{3.4}$$

Indeed, using the triangle inequality, it is enough to show that for all $\boldsymbol{i} \in [\![-n, 0]\!]^d$

$$\frac{1}{|\boldsymbol{N}|} \mathbb{E}_{\boldsymbol{0}}\left[\max_{\boldsymbol{1} \leq \boldsymbol{j} \leq \boldsymbol{N}} \left|R_{\boldsymbol{j}}(\mathcal{P}_{\boldsymbol{i}}(X_{\boldsymbol{0}}))\right|^2\right] \xrightarrow[\boldsymbol{N} \to \infty]{\text{a.s.}} 0.$$

This holds true by applying the following lemma.

Lemma 3.3. *For any square integrable $\mathcal{F}_{\boldsymbol{0}}$-measurable function h, the condition*

$$\sum_{\boldsymbol{u} \geq \boldsymbol{1}} \frac{\|\mathbb{E}_{\boldsymbol{1}}[h_{\boldsymbol{u}}]\|_2}{|\boldsymbol{u}|^{1/2}} < \infty \tag{3.5}$$

implies

$$\frac{1}{|\boldsymbol{N}|} \mathbb{E}_{\boldsymbol{0}}\left[\max_{\boldsymbol{1} \leq \boldsymbol{n} \leq \boldsymbol{N}} \left|R_{\boldsymbol{n}}(h)\right|^2\right] \xrightarrow[\boldsymbol{N} \to \infty]{a.s.} 0.$$

We delay the proof of this lemma to later in this section.

Remark that the proof of Proposition 4.1 in Volný and Wang (2014) can be easily adapted to the case of Orlicz spaces; so that for some fixed $n \in \mathbb{N}^*$, we get the following martingale-coboundary decomposition

$$X_{\boldsymbol{0}}^{(n)} = \sum_{S \subset [\![1,d]\!]} h_S^{(n)} \circ \prod_{j \in S^c} (I - T_j),$$

where $h_S^{(n)} \in \bigcap_{i\in S}\left(L^2\log^{d-1}L\big(\mathcal{F}_0^{(i)}\big)\ominus L^2\log^{d-1}L\big(\mathcal{F}_{-1}^{(i)}\big)\right)$ for all $S\subset [\![1,d]\!]$ and using the convention $\prod_{j\in\emptyset}(I-T_j)=I$. Moreover

$$h_{[\![1,d]\!]}^{(n)}=\sum_{\boldsymbol{i}\in\mathbb{Z}^d}\mathcal{P}_{\boldsymbol{0}}\big(X_{\boldsymbol{0}}^{(n)}\circ T^{\boldsymbol{i}}\big).$$

According to the proof of Remark 11 in Peligrad and Volný (2020) (see also the proof of Theorem 7 in the same article), the following almost-sure convergence

$$\mathbb{P}^\omega\left(\max_{\boldsymbol{1}\leq\boldsymbol{m}\leq\boldsymbol{N}}\frac{1}{|\boldsymbol{N}|}\left|S_{\boldsymbol{m}}\big(X_{\boldsymbol{0}}^{(n)}-d_n\big)\right|^2\geq\epsilon\right)\xrightarrow[\boldsymbol{N}\to\infty]{\text{a.s.}}0 \tag{3.6}$$

holds for all $\epsilon>0$, where $d_n=h_{[\![1,d]\!]}^{(n)}$. Moreover, letting $\boldsymbol{N}\in(\mathbb{N}^*)^d$ and $D_0=\sum_{\boldsymbol{i}\in\mathbb{Z}^d}\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{i}})$, we get

$$S_{\boldsymbol{N}}(D_0-d_n)=\sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{N}}\big(D_{\boldsymbol{i}}-D_{\boldsymbol{i}}^{(n)}\big),$$

where

$$D_{\boldsymbol{i}}=\sum_{\boldsymbol{j}\in\mathbb{Z}^d}\mathcal{P}_{\boldsymbol{i}}\big(X_{\boldsymbol{i}-\boldsymbol{j}}\big)\quad\text{and}\quad D_{\boldsymbol{i}}^{(n)}=\sum_{\boldsymbol{j}\in[\![-n,0]\!]^d}\mathcal{P}_{\boldsymbol{i}}\big(X_{\boldsymbol{i}-\boldsymbol{j}}\big).$$

Hence, given that $\big(D_{\boldsymbol{i}}-D_{\boldsymbol{i}}^{(n)}\big)_{\boldsymbol{i}\in\mathbb{Z}^d}$ is an ortho-martingale difference field and according to Cairoli's inequality, we have

$$\mathbb{P}^\omega\left(\frac{1}{\sqrt{|\boldsymbol{N}|}}\max_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{N}}\big|S_{\boldsymbol{i}}(D_0-d_n)\big|>\epsilon\right)\leq\frac{2^{2d}}{\epsilon^2|\boldsymbol{N}|}\sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{N}}\mathbb{E}_{\boldsymbol{0}}\left[\Big(D_{\boldsymbol{i}}-D_{\boldsymbol{i}}^{(n)}\Big)^2\right].$$

Let us note that

$$\sqrt{\frac{1}{|\boldsymbol{N}|}\sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{N}}\mathbb{E}_{\boldsymbol{0}}\left[\Big(D_{\boldsymbol{i}}-D_{\boldsymbol{i}}^{(n)}\Big)^2\right]}\leq\sum_{\boldsymbol{j}\notin[\![-n,0]\!]^d}\sqrt{\frac{1}{|\boldsymbol{N}|}\sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{N}}\mathbb{E}_{\boldsymbol{0}}\left[\big(\mathcal{P}_{\boldsymbol{0}}(X_{-\boldsymbol{j}})\big)^2\circ T^{\boldsymbol{i}}\right]}.$$

According to the ergodic Theorem 1.1 of Chapter 6 in Krengel (1985) for Dunford Schwartz operators and Lemma 7.1 in Dedecker et al. (2014), we have the convergence

$$\lim_{\boldsymbol{N}\to\infty}\frac{1}{|\boldsymbol{N}|}\sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{N}}\mathbb{E}_{\boldsymbol{0}}\left[\big(\mathcal{P}_{\boldsymbol{0}}(X_{-\boldsymbol{j}})\big)^2\circ T^{\boldsymbol{i}}\right]=\mathbb{E}\left[\big(\mathcal{P}_{\boldsymbol{j}}(X_{\boldsymbol{0}})\big)^2\right]\quad\text{a.s.}$$

for all $\boldsymbol{j}\notin[\![-n,0]\!]^d$. Since $\sum_{\boldsymbol{j}\geq\boldsymbol{0}}\|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{j}})\|_2<\infty$, we get

$$\lim_{n\to\infty}\lim_{\boldsymbol{N}\to\infty}\frac{2^{2d}}{\epsilon^2|\boldsymbol{N}|}\sum_{\boldsymbol{1}\leq\boldsymbol{i}\leq\boldsymbol{N}}\mathbb{E}_{\boldsymbol{0}}\left[\Big(D_{\boldsymbol{i}}-D_{\boldsymbol{i}}^{(n)}\Big)^2\right]=0\quad\text{a.s.} \tag{3.7}$$

Combining (3.3), (3.4),(3.6) and (3.7), we obtain that for all $\epsilon>0$,

$$\limsup_{\boldsymbol{N}\to\infty}\mathbb{P}^\omega\left(\frac{1}{\sqrt{|\boldsymbol{N}|}}\max_{\boldsymbol{1}\leq\boldsymbol{m}\leq\boldsymbol{N}}\big|\bar{S}_{\boldsymbol{m}}-S_{\boldsymbol{m}}(D_0)\big|\geq\epsilon\right)=0\quad\text{a.s.}$$

We conclude by noticing that the field $(D_0\circ T^{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ satisfies a functional central limit theorem (according to Theorem 10 in Peligrad and Volný, 2020) and therefore the expected result is obtained by applying Theorem 3.1 in Neuhaus (1971). □

Proof of the Theorem 2.1: The proof of this theorem is very similar to the previous one, with the exception of using Theorem 2.8 instead of Theorem 1.1 of Chapter 6 in Krengel (1985) and Lemma 1.4 in the same Chapter (applied to the abstract maximal operator $Mf := \sup_{n\in\mathbb{N}} \frac{1}{n^d}\sum_{\mathbf{1}\le \boldsymbol{i}\le n\mathbf{1}} |f|\circ T^{\boldsymbol{i}}$, see Definition 1.3 of Chapter 6 and Corollary 2.2 of Chapter 1 in Krengel, 1985) instead of Corollary 1.7 in order to obtain the L^2 versions of lemma 3.2 and the Main Lemma mentioned below. $\square$

Lemma 3.4 (L^2 version of the Main Lemma 3.1)**.** *For any function $h \in L^2(\mathcal{F}_{\mathbf{0}})$ satisfying the following condition:*

$$\sum_{\boldsymbol{u}\ge \mathbf{0}} \|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\|_2 < \infty, \tag{3.8}$$

there exist an integrable function g such that for all $N \in \mathbb{N}^*$,

$$\sqrt{\mathbb{E}_{\mathbf{0}}\left[\max_{1\le n\le N} \frac{1}{n^d}\left|\overline{S}_{n\mathbf{1}}(h)\right|^2\right]} \le g \quad \mathbb{P}-\text{a.s.}$$

Lemma 3.5 (L^2 version of Lemma 3.2)**.** *For all functions $h \in L^2$, there exists a constant $C > 0$ such that for all $\boldsymbol{u} \in \mathbb{Z}^d$, we have*

$$\left\|\sqrt{\left(\left|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\right|^2\right)^\star}\right\|_1 \le C\|\mathcal{P}_{\mathbf{0}}(h_{\boldsymbol{u}})\|_2,$$

where $h^\star = \sup_{n\in\mathbb{N}^*} \frac{1}{n^d}\sum_{\mathbf{1}\le \boldsymbol{i}\le n\mathbf{1}} |h|\circ T^{\boldsymbol{i}}$.

The following proof of Corollary 2.5 can be easily adapted to obtain Corollary 2.2 by using the Theorem 2.1 instead of Theorem 2.4.

Proof of Corollary 2.5: According to Theorem 2.4 and Theorem 3.1 in Neuhaus (1971), it is enough to show

$$\frac{1}{|\boldsymbol{n}|}\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le \boldsymbol{m}\le \boldsymbol{n}} R_{\boldsymbol{m}}^2\right] \xrightarrow[\boldsymbol{n}\to\infty]{\text{a.s.}} 0.$$

Let $\boldsymbol{m} \in \mathbb{Z}^d$, and recall that

$$R_{\boldsymbol{m}} = \sum_{i=1}^{d} (-1)^{i-1} \sum_{1\le j_1<\cdots<j_i\le d} \mathbb{E}_{\boldsymbol{m}^{(j_1,\cdots,j_i)}}[S_{\boldsymbol{m}}],$$

where $\boldsymbol{m}^{(j_1,\ldots,j_i)}$ is the multi-index such that the j_k-th, $1 \le k \le i$ coordinates are zero and the others are equal to the corresponding coordinates of $\boldsymbol{m}$.

Using the triangle inequality, it is enough to prove that the property

$$\frac{1}{|\boldsymbol{n}|}\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le \boldsymbol{m}\le \boldsymbol{n}} \left(\mathbb{E}_{\boldsymbol{m}^{(j_1,\cdots,j_i)}}[S_{\boldsymbol{m}}]\right)^2\right] \xrightarrow[\boldsymbol{n}\to\infty]{\text{a.s.}} 0 \tag{$P(j_1,\ldots,j_i)$}$$

holds for any $j_1 < \cdots < j_i, 1 \le i \le d$. We establish that via induction on i.

The terms satisfying $i = 1$ have this property according to the hypothesis of the corollary. By induction, the corollary will be proven if we can show that if property $P(j_1, \ldots, j_i)$ is verified for all $j_1 < \cdots < j_i$ for some $i < d$, then $P(j_1, \ldots, j_{i+1})$ also holds for all $j_1 < \cdots < j_{i+1}$. For the sake of simplicity and without loss of generality, we will only establish that

$$\forall j \in [\![1, d]\!], P(j) \quad \Longrightarrow \quad P(1, 2).$$

In other words, we use the hypothesis of the corollary to show

$$\frac{1}{|\boldsymbol{n}|}\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le \boldsymbol{m}\le \boldsymbol{n}} \left(\mathbb{E}_{\boldsymbol{m}^{(1,2)}}[S_{\boldsymbol{m}}]\right)^2\right] \xrightarrow[\boldsymbol{n}\to\infty]{\text{a.s.}} 0.$$

According to Jensen's inequality, we have

$$\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{m}\le\boldsymbol{n}}\left(\mathbb{E}_{\boldsymbol{m}^{(1,2)}}[S_{\boldsymbol{m}}]\right)^2\right]=\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{m}\le\boldsymbol{n}}\left(\mathbb{E}_{\boldsymbol{m}^{(1,2)}}\left[\mathbb{E}_{\boldsymbol{m}^{(1)}}[S_{\boldsymbol{m}}]\right]\right)^2\right]\le\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{m}\le\boldsymbol{n}}\left(\mathbb{E}_{\boldsymbol{m}^{(1)}}[S_{\boldsymbol{m}}]\right)^2\right].$$

So

$$\frac{1}{|\boldsymbol{n}|}\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{m}\le\boldsymbol{n}}\left(\mathbb{E}_{\boldsymbol{m}^{(1,2)}}[S_{\boldsymbol{m}}]\right)^2\right]\le\frac{1}{|\boldsymbol{n}|}\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{m}\le\boldsymbol{n}}\left(\mathbb{E}_{\boldsymbol{m}^{(1)}}[S_{\boldsymbol{m}}]\right)^2\right]\xrightarrow[\boldsymbol{n}\to\infty]{\text{a.s.}}0.$$

□

Before continuing with the proof, we establish Lemma 3.3.

Proof of Lemma 3.3: We show that for any $i\in[\![1,d]\!]$ and any $1\le j_1<\cdots<j_i\le d$, we have the convergence

$$\frac{1}{|\boldsymbol{N}|}\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{n}\le\boldsymbol{N}}\left(\mathbb{E}_{\boldsymbol{n}^{(j_1,\dots,j_i)}}[S_{\boldsymbol{n}}(h)]\right)^2\right]\xrightarrow[\boldsymbol{N}\to\infty]{\text{a.s.}}0.$$

In order to do so, we use an induction on $k=d-i$ with d fixed. For $k=0$ and in the same way as in the proof of Lemma 3.2 in Zhang et al. (2020) (see also the proof of Theorem 4.4 in the same article), we establish that

$$\frac{\left(\mathbb{E}_{\mathbf{0}}[S_{\boldsymbol{n}}(h)]\right)^2}{|\boldsymbol{n}|}\xrightarrow[\boldsymbol{n}\to\infty]{\text{a.s.}}0.$$

Hence

$$\frac{1}{|\boldsymbol{N}|}\max_{\mathbf{1}\le\boldsymbol{n}\le\boldsymbol{N}}\left(\mathbb{E}_{\mathbf{0}}[S_{\boldsymbol{n}}(h)]\right)^2\xrightarrow[\boldsymbol{N}\to\infty]{\text{a.s.}}0.$$

Now, we suppose that the desired property holds for some $k-1<d-1$. Without loss of generality, we establish the property only for $(j_1,\dots,j_k)=(1,\dots,k)$; it is enough to show that

$$\frac{1}{|\boldsymbol{N}|}\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{n}\le\boldsymbol{N}}\left(\mathbb{E}_{\boldsymbol{n}^{(j_1,\dots,j_k)}}[S_{\boldsymbol{n}}(h)]-\mathbb{E}_{\boldsymbol{n}^{(j_2,\dots,j_k)}}[S_{\boldsymbol{n}}(h)]\right)^2\right]\xrightarrow[\boldsymbol{N}\to\infty]{\text{a.s.}}0.$$

Let $\boldsymbol{n},\boldsymbol{N}\in(\mathbb{N}^*)^d$ such that $\boldsymbol{n}\le\boldsymbol{N}$, then the following decomposition holds

$$\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{n}\le\boldsymbol{N}}\left(\mathbb{E}_{\boldsymbol{n}^{(j_1,\dots,j_k)}}[S_{\boldsymbol{n}}(h)]-\mathbb{E}_{\boldsymbol{n}^{(j_2,\dots,j_k)}}[S_{\boldsymbol{n}}(h)]\right)^2\right]=\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{n}\le\boldsymbol{N}}\left(\sum_{i=1}^{n_1}P^{(1)}_{\boldsymbol{n}^{(j_1,\dots,j_k)},i}(S_{\boldsymbol{n}}(h))\right)^2\right],$$

where $P^{(1)}_{\boldsymbol{n}^{(j_1,\dots,j_k)},i}(S_{\boldsymbol{n}}(h))=\mathbb{E}_{i\boldsymbol{e}_1+\boldsymbol{n}^{(j_1,\dots,j_k)}}[S_{\boldsymbol{n}}(h)]-\mathbb{E}_{(i-1)\boldsymbol{e}_1+\boldsymbol{n}^{(j_1,\dots,j_k)}}[S_{\boldsymbol{n}}(h)]$ and $\boldsymbol{e}_1$ is the multi-index whose coordinates are all zero except for the first one which is 1.

Since h is $\mathcal{F}_{\mathbf{0}}$-measurable, we have

$$\left|\sum_{i=1}^{n_1}P^{(1)}_{\boldsymbol{n}^{(j_1,\dots,j_k)},i}(S_{\boldsymbol{n}}(h))\right|\le\sum_{\mathbf{1}\le\boldsymbol{u}\le\boldsymbol{N}}\max_{1\le k\le N_1}\left|\sum_{i=1}^{k}P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},i}(h_{\boldsymbol{u}})\right|.$$

Therefore, according to Doob's inequality for martingales, it follows

$$\sqrt{\mathbb{E}_{\mathbf{0}}\left[\max_{\mathbf{1}\le\boldsymbol{n}\le\boldsymbol{N}}\left(\sum_{i=1}^{\boldsymbol{n}_1}P^{(1)}_{\boldsymbol{n}^{(j_1,\dots,j_k)},i}(S_{\boldsymbol{n}}(h))\right)^2\right]}\le2\sum_{\mathbf{1}\le\boldsymbol{u}\le\boldsymbol{N}}\sqrt{\mathbb{E}_{\mathbf{0}}\left[\left(\sum_{i=1}^{N_1}P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},i}(h_{\boldsymbol{u}})\right)^2\right]}.$$

Let $c>0$, we use the following decomposition

$$\frac{1}{\sqrt{|\boldsymbol{N}|}}\sum_{\mathbf{1}\le\boldsymbol{u}\le\boldsymbol{N}}\sqrt{\mathbb{E}_{\mathbf{0}}\left[\left(\sum_{i=1}^{N_1}P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},i}(h_{\boldsymbol{u}})\right)^2\right]}=:\mathrm{I}_{\boldsymbol{N},c}+\mathrm{II}_{\boldsymbol{N},c}$$

where

$$\mathrm{I}_{\boldsymbol{N},c}=\frac{1}{\sqrt{|\boldsymbol{N}|}}\sum_{1\le u_1\le N_1}\sum_{1\le u_2,\dots u_d\le c}\sqrt{\mathbb{E}_{\mathbf{0}}\left[\left(\sum_{i=1}^{N_1}P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},i}(h_{\boldsymbol{u}})\right)^2\right]}$$

and $\mathrm{II}_{\boldsymbol{N},c}$ is the remainder of the initial sum.

Let us show that

$$\limsup_{\boldsymbol{N}\to\infty}\mathrm{I}_{\boldsymbol{N},c}=0\quad\text{ a.s.}$$

Indeed, by an orthogonality argument

$$\mathrm{I}_{\boldsymbol{N},c}\le\frac{c^{d-1}}{\sqrt{|\boldsymbol{N}|}}\sup_{1\le u_2,\dots,u_d\le c}\sum_{u_1\ge 0}\sqrt{\sum_{i=1}^{N_1}\mathbb{E}_{\mathbf{0}}\left[\left(P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},0}(h_{\boldsymbol{u}})\right)^2\circ T_1^i\right]}.$$

However, according to Lemma 7.1 in Dedecker et al. (2014), we get the convergence

$$\lim_{N_1\to\infty}\frac{1}{N_1}\sum_{1\le i\le N_1}\mathbb{E}_{\mathbf{0}}\left[\left(P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},0}(h_{\boldsymbol{u}})\right)^2\circ T_1^i\right]=\mathbb{E}\left[\left(P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},0}(h_{\boldsymbol{u}})\right)^2\,\middle|\,\mathcal{I}_1\right]\quad\text{ a.s.}$$

where $\mathcal{I}_1$ is the invariant σ-algebra of the transformation T_1. Hence

$$\limsup_{\boldsymbol{N}\to\infty}\mathrm{I}_{\boldsymbol{N},c}=0\quad\text{ a.s.}$$

Each sum appearing in $\mathrm{II}_{\boldsymbol{N},c}$, admits at least one direction (different from the first one) for which the index is at least equal to $c+1$. Without loss of generality, we will only treat the case where the second direction has an index at least equal to $c+1$ and all other directions have the full range of indexes. Then

$$\begin{aligned}&\frac{1}{\sqrt{|\boldsymbol{N}|}}\sum_{1\le u_1\le N_1}\sum_{c+1\le u_2\le N_2}\sum_{1\le u_3\le N_3}\cdots\sum_{1\le u_d\le N_d}\sqrt{\mathbb{E}_{\mathbf{0}}\left[\left(\sum_{i=1}^{N_1}P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},i}(h_{\boldsymbol{u}})\right)^2\right]}\\&\qquad\le\sum_{u_2\ge c+1}\sum_{u_1\ge 0}\sum_{u_3\ge 1}\cdots\sum_{u_d\ge 1}(u_2\cdots u_d)^{-\frac{1}{2}}\sqrt{\frac{1}{N_1}\sum_{i=1}^{N_1}\mathbb{E}_{\mathbf{0}}\left[\left(P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},0}(h_{\boldsymbol{u}})\right)^2\circ T_1^i\right]}.\end{aligned}$$

Once again, applying Lemma 7.1 in Dedecker et al. (2014), we obtain

$$\limsup_{\boldsymbol{N}\to\infty}\mathrm{II}_{\boldsymbol{N},c}\le\sum_{u_2\ge c+1}\sum_{u_1\ge 0}\sum_{u_2\ge 1}\cdots\sum_{u_d\ge 1}\frac{\sqrt{\mathbb{E}\left[\left(P^{(1)}_{\boldsymbol{u}^{(j_1,\dots,j_k)},0}(h_{\boldsymbol{u}})\right)^2\,\middle|\,\mathcal{I}_1\right]}}{\sqrt{u_2\cdots u_d}}.$$

Since (3.5) implies (45) in Zhang et al. (2020) (see the proof of Theorem 4.4), we get

$$\lim_{c\to\infty}\limsup_{\boldsymbol{N}\to\infty}\mathrm{II}_{\boldsymbol{N},c}=0\quad\text{ a.s.}$$

This concludes the proof. □

Corollaries 2.6 and 2.3 are direct consequences of this lemma and the previous results.

Proof of Corollary 2.6: This theorem is a consequence of Lemma 3.3 and Theorem 2.4. □

Proof of Corollary 2.3: The theorem is a consequence of Lemma 3.3, Theorem 2.4 and Lemma 3.3 in Zhang et al. (2020) (for $d>2$, see Theorem 4.4 (a) and its proof in Zhang et al., 2020) and Theorem 2.1. □

The rest of this section will be dedicated to proving Corollary 2.7. We start by making a few remarks concerning the Luxemburg norms. Let us note that for $x \geq 0$ and $0 < \lambda \leq \mathrm{e} - 1$,

$$\log\Big(1 + \frac{x}{\lambda}\Big) \log(1+\lambda) \leq \log(1+x).$$

Recall that the function $\Phi_d : [0, \infty) \to [0, \infty)$ is defined by

$$\Phi_d(x) = x^2 (\log(1+x))^{d-1}$$

for all $x \in [0, \infty)$. Then, we deduce the following remarkable property of the function Φ_d. For $x > 0$ and $0 < \lambda \leq \mathrm{e} - 1$,

$$\Phi_d\Big(\frac{x}{\lambda}\Big) = \Big(\frac{x}{\lambda}\Big)^2 \Big(\log\Big(1 + \frac{x}{\lambda}\Big)\Big)^{d-1} \leq \frac{x^2 (\log(1+x))^{d-1}}{\lambda^2 (\log(1+\lambda))^{d-1}} = \frac{\Phi_d(x)}{\Phi_d(\lambda)}. \tag{3.9}$$

Besides, since Φ_d is a convex function, we also have

$$\Phi_d\Big(\frac{x}{\lambda}\Big) = \Phi_d\Big(\frac{x}{\lambda} + \Big(1 - \frac{1}{\lambda}\Big) \cdot 0\Big) \leq \frac{\Phi_d(x)}{\lambda} + \Big(1 - \frac{1}{\lambda}\Big) \Phi_d(0) = \frac{\Phi_d(x)}{\lambda}, \tag{3.10}$$

for $x \geq 0$ and $\lambda \geq 1$.

Obviously, the function Φ_d defined by (2.6) is bijective and we denote by Φ_d^{-1} its inverse function. The following lemma might be well-known but we could not find it in the literature.

Lemma 3.6. *Let $X \in L^2 \log^{d-1} L$. If $\Phi_d^{-1}\big(\mathbb{E}\big[\Phi_d(|X|)\big]\big) \leq \mathrm{e} - 1$, then*

$$\|X\|_{\Phi_d} \leq \Phi_d^{-1}\big(\mathbb{E}\big[\Phi_d(|X|)\big]\big),$$

and if $\mathbb{E}\big[\Phi_d(|X|)\big] \geq 1$, then

$$\|X\|_{\Phi_d} \leq \mathbb{E}\big[\Phi_d(|X|)\big].$$

Proof of Lemma 3.6: If $X = 0$ almost surely, then the property is evident. Else, suppose that $\mathbb{P}(X = 0) \neq 1$, and recall the definition of Luxemburg norm

$$\|X\|_{\Phi_d} = \inf\Big\{\lambda > 0 : \mathbb{E}\Big[\Phi_d\Big(\frac{|X|}{\lambda}\Big)\Big] \leq 1\Big\}.$$

Note that by the properties of Φ_d for any $0 < \lambda \leq \mathrm{e} - 1$

$$\mathbb{E}\Big[\Phi_d\Big(\frac{|X|}{\lambda}\Big)\Big] \leq \frac{\mathbb{E}\big[\Phi_d(|X|)\big]}{\Phi_d(\lambda)}.$$

From this inequality it follows that if λ is the solution to the equation $\mathbb{E}\big[\Phi_d(|X|)\big] = \Phi_d(\lambda)$, we have necessarily that $\mathbb{E}\Big[\Phi_d\Big(\frac{|X|}{\lambda}\Big)\Big] \leq 1$, and then $\|X\|_{\Phi_d} \leq \lambda = \Phi_d^{-1}\big(\mathbb{E}\big[\Phi_d(|X|)\big]\big)$.

For the case $\mathbb{E}\big[\Phi_d(|X|)\big] > 1$, the proof is similar using property (3.10) of Φ_d. $\square$

Lemma 3.7. *Condition (2.9) implies $\sum_{\boldsymbol{u} \geq \boldsymbol{0}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}})\|_{\Phi_d} < \infty$.*

Proof of Lemma 3.7: Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^d$ such that $\boldsymbol{a} \leq \boldsymbol{b}$. Denote by Ψ_d the conjugate function associated with Φ_d defined in the following way

$$\Psi_d(x) = \sup_{y \geq 0} (xy - \Phi_d(y))$$

for $x \geq 0$. By the generalized Holder inequality for Orlicz spaces (see Rao and Ren, 1991, p.58), we have

$$\sum_{\boldsymbol{u}\geq \boldsymbol{1}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}})\|_{\Phi_d} = \sum_{\boldsymbol{n}\geq \boldsymbol{0}} \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n+1}}-\boldsymbol{1}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{v}})\|_{\Phi_d}$$
$$\leq 2\sum_{\boldsymbol{n}\geq \boldsymbol{0}} \inf\left\{\eta>0 : \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n+1}}-\boldsymbol{1}} \Psi_d\left(\frac{1}{\eta}\right) \leq 1\right\} \cdot \inf\left\{\eta>0 : \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n+1}}-\boldsymbol{1}} \Phi_d\left(\frac{\|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{v}})\|_{\Phi_d}}{\eta}\right) \leq 1\right\}, \tag{3.11}$$

where $2^{\boldsymbol{n}} = (2^{n_1}, \dots, 2^{n_d})$. Computing the second term in the sum, we get

$$\inf\left\{\eta>0 : \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n+1}}-\boldsymbol{1}} \Psi_d\left(\frac{1}{\eta}\right) \leq 1\right\} = \frac{1}{\Psi_d^{-1}(|2^{\boldsymbol{n}}|^{-1})}.$$

In order to control the second term in (3.11), we make the following remark: if f is a nondecreasing convex function then for any $\eta > 0$, the following holds

$$f\left(\left\|\sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n+1}}-\boldsymbol{1}} \frac{\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})}{\eta}\right\|_{\Phi_d}\right) = f\left(\left\|\frac{1}{\eta}\sum_{i=0}^{d}(-1)^{d-i} \sum_{1\leq j_1<\cdots<j_i\leq d} \mathbb{E}_{-2^{\boldsymbol{n+1}}(j_1,\dots,j_i)}[X_{\boldsymbol{0}}]\right\|_{\Phi_d}\right)$$
$$\leq \frac{1}{2^d}\sum_{i=0}^{d} \sum_{1\leq j_1<\cdots<j_i\leq d} f\left(\left\|\frac{2^d}{\eta}(-1)^{d-i}\mathbb{E}_{-2^{\boldsymbol{n+1}}(j_1,\dots,j_i)}[X_{\boldsymbol{0}}]\right\|_{\Phi_d}\right)$$
$$\leq f\left(\frac{2^d}{\eta}\|\mathbb{E}_{-2^{\boldsymbol{n}}}[X_{\boldsymbol{0}}]\|_{\Phi_d}\right).$$

Note that in the previous inequalities, the term corresponding to $i = 0$ is by convention $(-1)^d\mathbb{E}_{-2^{\boldsymbol{n+1}}}[X_{\boldsymbol{0}}]$. It is enough to control the second term in (3.11) only when $\mathbb{E}\left[\Phi_d\left(\frac{|\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})|}{\eta}\right)\right] \leq \Phi_d(\mathrm{e}-1)$ for any $\boldsymbol{v} \geq \boldsymbol{0}$, as the other cases can be proved using similar arguments and are left to the reader. By Lemma 3.6 above, if $\mathbb{E}\left[\Phi_d\left(\frac{|\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})|}{\eta}\right)\right] \leq \Phi_d(\mathrm{e}-1)$ for any $\boldsymbol{v} \geq \boldsymbol{0}$, we also have

$$\Phi_d\left(\frac{\|\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})\|_{\Phi_d}}{\eta}\right) \leq \mathbb{E}\left[\Phi_d\left(\frac{|\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})|}{\eta}\right)\right].$$

So

$$\sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n+1}}-\boldsymbol{1}} \Phi_d\left(\frac{\|\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})\|_{\Phi_d}}{\eta}\right) \leq \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n+1}}-\boldsymbol{1}} \mathbb{E}\left[\Phi_d\left(\frac{|\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})|}{\eta}\right)\right]. \tag{3.12}$$

Before proceeding, we state the following lemma which is a version of the Rosenthal inequality in the Orlicz space associated with Φ_d. Its proof will be given in the appendix.

Lemma 3.8. *If $(d_{\boldsymbol{u}})_{\boldsymbol{u}\in(\mathbb{N})^*}$ is an ortho-martingale difference field and $0 < \epsilon < 1$, then there exists two constants $C_1, C_2 > 1$, which only depend on d and ϵ, such that*

$$\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n-1}} \mathbb{E}\left[\Phi_d(|d_{\boldsymbol{u}}|)\right] \leq C_1 \max\left\{\varphi_d^{-1}\left(C_2\left\|\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n-1}} d_{\boldsymbol{u}}\right\|_{\Phi_d}\right), \phi_d\circ f_\epsilon\left(C_2\left\|\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n-1}} d_{\boldsymbol{u}}\right\|_{\Phi_d}\right)\right\},$$

with $\phi_d(x) = x^{d+1}\big(\log(1+x)\big)^{d-1}$, $\varphi_d(x) = x\big(\log(1+x)\big)^{d-1}$ and $f_\epsilon(x) = x^{1/(1-\epsilon)}$ for all $x \geq 0$.

We let $\epsilon = \frac{1}{d+2}$ and we notice that while the function $\phi_d \circ f_\epsilon$ is convex on $[0,\infty)$, that is not the case for φ_d^{-1}. To solve this issue, remark that $\left(\varphi_d'\left(\frac{C_1^{-1}}{2}\right)\right)^{-1} x + \frac{C_1^{-1}}{2} \geq \varphi_d^{-1}(x)$ for all $x \geq 0$ and thus the function $\rho : [0,\infty) \to [0,\infty)$ defined by $\rho(x) = \max\left\{ \left(\varphi_d'\left(\frac{C_1^{-1}}{2}\right)\right)^{-1} x + \frac{C_1^{-1}}{2}, \phi_d \circ f_\epsilon(x) \right\}$ is convex and greater than φ_d^{-1}. Applying Lemma 3.8, we can show that there exists $C_1, C_2 > 1$ such that for any $\eta > 0$,

$$\begin{aligned}
&\sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n}+\boldsymbol{1}}-\boldsymbol{1}} \Phi_d\left(\frac{\|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{v}})\|_{\Phi_d}}{\eta}\right) \\
&\qquad \leq \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n}+\boldsymbol{1}}-\boldsymbol{1}} \mathbb{E}\left[\Phi_d\left(\frac{|\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})|}{\eta}\right)\right] \\
&\qquad \leq C_1 \max\left\{ \varphi_d^{-1}\left(C_2 \left\| \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n}+\boldsymbol{1}}-\boldsymbol{1}} \frac{\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})}{\eta} \right\|_{\Phi_d}\right), \phi_d \circ f_\epsilon\left(C_2 \left\| \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n}+\boldsymbol{1}}-\boldsymbol{1}} \frac{\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})}{\eta} \right\|_{\Phi_d}\right) \right\} \\
&\qquad \leq C_1 \rho\left(C_2 \left\| \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n}+\boldsymbol{1}}-\boldsymbol{1}} \frac{\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})}{\eta} \right\|_{\Phi_d}\right).
\end{aligned}$$

According to the remark above, we get that

$$\rho\left(C_2 \left\| \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n}+\boldsymbol{1}}-\boldsymbol{1}} \frac{\mathcal{P}_{-\boldsymbol{v}}(X_{\boldsymbol{0}})}{\eta} \right\|_{\Phi_d}\right) \leq \rho\left(\frac{2^d C_2}{\eta} \|\mathbb{E}_{-2^{\boldsymbol{n}}}[X_{\boldsymbol{0}}]\|_{\Phi_d}\right).$$

Since ρ is continous and $\rho(0) < C_1^{-1}$, the set $\left\{\mu > 0 : \rho(\mu) \leq C_1^{-1}\right\}$ is non-empty. Moreover

$$C_1 \rho\left(C_2 \frac{2^d}{\eta} \|\mathbb{E}_{-2^{\boldsymbol{n}}}[X_{\boldsymbol{0}}]\|_{\Phi_d}\right) \leq 1 \quad \iff \quad \eta \geq \frac{2^d C_2}{\rho^{-1}(C_1^{-1})} \|\mathbb{E}_{-2^{\boldsymbol{n}}}[X_{\boldsymbol{0}}]\|_{\Phi_d}.$$

Therefore, setting $C = \frac{2^d C_2}{\rho^{-1}(C_1^{-1})}$, we conclude that

$$\begin{aligned}
\inf\left\{\eta > 0 : \sum_{\boldsymbol{v}=2^{\boldsymbol{n}}}^{2^{\boldsymbol{n}+\boldsymbol{1}}-\boldsymbol{1}} \Phi_d\left(\frac{\|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{v}})\|_{\Phi_d}}{\eta}\right) \leq 1\right\} &\leq \inf\left\{\eta > 0 : C_1 \rho\left(C_2 \frac{2^d}{\eta} \|\mathbb{E}_{-2^{\boldsymbol{n}}}[X_{\boldsymbol{0}}]\|_{\Phi_d}\right) \leq 1\right\} \\
&= C \|\mathbb{E}_{-2^{\boldsymbol{n}}}[X_{\boldsymbol{0}}]\|_{\Phi_d}.
\end{aligned}$$

Since $\|\mathbb{E}_{-\boldsymbol{n}}[X_{\boldsymbol{0}}]\|_{\Phi_d}$ is nonincreasing in all directions of $\boldsymbol{n}$, we obtain that for any $\boldsymbol{n}$ such that $n_k > 0$ for all $k \in [\![1, d]\!]$, we have

$$\frac{|2^{\boldsymbol{n}}|}{{\Phi_d}^{-1}(|2^{\boldsymbol{n}}|)} \|\mathbb{E}_{-2^{\boldsymbol{n}}}[X_{\boldsymbol{0}}]\|_{\Phi_d} \leq 2^d \sum_{\boldsymbol{u}=2^{\boldsymbol{n}-\boldsymbol{1}}}^{2^{\boldsymbol{n}}-\boldsymbol{1}} \frac{\|\mathbb{E}_{\boldsymbol{0}}[X_{\boldsymbol{u}}]\|_{\Phi_d}}{{\Phi_d}^{-1}(|\boldsymbol{u}|)}.$$

So, for some positive constant K,

$$\sum_{\boldsymbol{v} \geq \boldsymbol{1}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}})\|_{\Phi_d} \leq K \sum_{\boldsymbol{n} \geq \boldsymbol{1}} \frac{\Phi_d^{-1}(|2^{\boldsymbol{n}}|)}{|2^{\boldsymbol{n}}| {\Psi_d}^{-1}(|2^{\boldsymbol{n}}|^{-1})} \sum_{\boldsymbol{u}=2^{\boldsymbol{n}-\boldsymbol{1}}}^{2^{\boldsymbol{n}}-\boldsymbol{1}} \frac{\|\mathbb{E}_{-\boldsymbol{u}}(X_{\boldsymbol{0}})\|_{\Phi_d}}{{\Phi_d}^{-1}(|\boldsymbol{u}|)}.$$

However, there exists a constant $K' > 0$ such that ${\Phi_d}^{-1}(|2^{\boldsymbol{n}}|) \underset{\boldsymbol{n} \to \infty}{\sim} |2^{\boldsymbol{n}}| {\Psi_d}^{-1}(|2^{\boldsymbol{n}}|^{-1}) K'$. Hence, according to the previous inequalities, we have shown that (2.9) implies

$$\sum_{\boldsymbol{u} \geq \boldsymbol{1}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}})\|_{\Phi_d} < \infty.$$

In the same way, we have for every $i \in [\![1,d]\!]$ and for every $(j_1,\ldots,j_i) \in [\![1,d]\!]^i$ such that $j_1 < \cdots < j_i$,

$$\sum_{\boldsymbol{u}^{(j_1,\ldots,j_i)} \geq \boldsymbol{1}^{(j_1,\ldots,j_i)}} \|\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}^{(j_1,\ldots,j_i)}})\|_{\Phi_d} < \infty.$$

Hence (2.7) is fulfilled. $\square$

Proof of Corollary 2.7: This corollary is a consequence of Theorem 2.4 and Lemmas 3.3 and 3.7. $\square$

4. Examples

In this section, we present various examples of applications of the different results we obtained. First, we will focus on linear processes as well as a particular case of nonlinearity known as the Volterra field. In doing so, we improve on the results by Zhang et al. (2020) by requiring weaker assumptions on both the moment of the innovations and the coefficients which appear in each example. More precisely, we obtain a functional CLT despite only requiring that the i.i.d. innovations belong to the Orlicz space $L^2 \log^{d-1} L$ instead of the Lebesgue space L^q with $q > 2$ as is required by Zhang et al. (2020). Afterward, we shall discuss the case of Hölder continuous functions of linear fields. To the best of the authors' knowledge, it does not appear that quenched central limit theorems have been derived in this context. Finally, we study the case of weakly dependent processes in the sense of Wu (2005) which play an important role in many physical models such as particle systems (see Liggett, 1985; Stroock and Zegarliński, 1992). As far as we know, this class of random fields has been seldom, if ever, investigated for quenched central limit theorems. In that, our theorems provide some innovative convergence results for these processes.

Throughout this section, we will write $a \triangleleft b$ whenever $a \leq Cb$ with $C > 0$ being a constant which can only depend on some fixed parameters. Recall that the function $\Phi_d : [0,\infty) \to [0,\infty)$ is bijective and defined by (2.6).

4.1. *Linear field with independent innovations.* The first application of our results will deal with linear fields as it presents an opportunity to show how our results improve on that of Zhang et al. (2020). It is also a very common type of fields which present a lot of interest in and of themselves. The main argument of the proof relies on Corollary 2.6.

Example 4.1. (Linear field) Let $(\xi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ be a random field of independent, identically distributed random variables, which are centered and satisfy $\mathbb{E}\Big[|\xi_{\boldsymbol{0}}|^2\big(\log(1+|\xi_{\boldsymbol{0}}|)\big)^{d-1}\Big] < \infty$. For $\boldsymbol{k} \geq \boldsymbol{0}$ define

$$X_{\boldsymbol{k}} = \sum_{\boldsymbol{j}\geq\boldsymbol{0}} a_{\boldsymbol{j}}\xi_{\boldsymbol{k}-\boldsymbol{j}},$$

where $a_{\boldsymbol{u}}$ are real coefficients such that $\sum_{\boldsymbol{u}\geq\boldsymbol{0}} a_{\boldsymbol{u}}^2 < \infty$. In addition, assume that

$$\sum_{\boldsymbol{k}\geq\boldsymbol{1}} \frac{1}{\sqrt{|\boldsymbol{k}|}} \Big(\sum_{\boldsymbol{j}\geq\boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}^2\Big)^{\frac{1}{2}} < \infty. \tag{4.1}$$

Then the quenched convergence (2.8) holds.

The results obtained by Zhang et al. (2020) (Remark 6.2 (c)) required the existence of $q-$th moment, with $q > 2$, of the innovation $\xi_{\boldsymbol{0}}$ to obtain the quenched functional CLT; meanwhile, we only require that $\xi_{\boldsymbol{0}}$ satisfy a weaker Orlicz condition to obtain that result. Additionally, we require weaker assumptions on the coefficients $a_{\boldsymbol{u}}, \boldsymbol{u} \in \mathbb{Z}^d$.

Proof of Example 4.1: Let $\boldsymbol{u} \geq \boldsymbol{0}$. According to the independence of the $\xi_{\boldsymbol{n}}$, we have

$$\mathcal{P}_{\boldsymbol{0}}(X_{\boldsymbol{u}}) = a_{\boldsymbol{u}}\xi_{\boldsymbol{0}} \quad \text{and for } \boldsymbol{u} \geq \boldsymbol{1}, \quad \mathbb{E}_{\boldsymbol{1}}[X_{\boldsymbol{u}}] = \sum_{\boldsymbol{j}\geq\boldsymbol{u}-\boldsymbol{1}} a_{\boldsymbol{j}}\xi_{\boldsymbol{u}-\boldsymbol{j}}.$$

Applying Burkholder inequality, we obtain

$$\|\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{u}}]\|_2 = \left\| \sum_{\boldsymbol{j}\geq \boldsymbol{u}-\mathbf{1}} a_{\boldsymbol{j}}\xi_{\boldsymbol{u}-\boldsymbol{j}} \right\|_2 \lhd \sqrt{\sum_{\boldsymbol{j}\geq \boldsymbol{u}-\mathbf{1}} a_{\boldsymbol{j}}^2 \|\xi_{\boldsymbol{u}-\boldsymbol{j}}\|_2^2}.$$

By stationarity of the random field $(\xi_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$, we get that

$$\|\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{u}}]\|_2 \lhd \|\xi_{\mathbf{0}}\|_2 \left(\sum_{\boldsymbol{j}\geq \boldsymbol{u}-\mathbf{1}} a_{\boldsymbol{j}}^2 \right)^{\frac{1}{2}}.$$

Thus, using assumption (4.1) and since $\|\xi_{\mathbf{0}}\|_{\Phi_d} < \infty$, we have shown that condition (2.5) is satisfied. Now, noticing that

$$\sum_{\boldsymbol{u}\geq \mathbf{0}} \|\mathcal{P}_{\mathbf{0}}(X_{\boldsymbol{u}})\|_{\Phi_d} = \frac{\|\xi_{\mathbf{0}}\|_{\Phi_d}}{\|\xi_{\mathbf{0}}\|_2} \sum_{\boldsymbol{u}\geq \mathbf{0}} \|\mathcal{P}_{\mathbf{0}}(X_{\boldsymbol{u}})\|_2$$

whenever $\mathbb{P}(\xi_{\mathbf{0}} = 0) < 1$ and using Lemma 3.3 in Zhang et al. (2020), we find that condition (2.7) is satisfied. To get the result, we simply apply Corollary 2.6. $\square$

4.2. *Nonlinearity: the case of Volterra fields.* As for the linear case, a lot is known about Volterra fields including some quenched limit theorem under a variety of conditions as in Dedecker et al. (2007); Zhang et al. (2020). Applying our results will require a bit more work than in the previous case since the lack of linearity means that we cannot guarantee that Hannan's criterion is satisfied if we only assume that the coefficients of the Volterra field satisfy a condition similar to (4.1). In particular, a new method of proof relying on Corollary 2.5 will be required leading to slightly stronger assumptions than in Example 4.1.

Example 4.2. (Volterra field) Let $(\xi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ be a random field of independent, identically distributed, and centered random variables satisfying $\mathbb{E}\left[|\xi_{\mathbf{0}}|^2\big(\log(1+|\xi_{\mathbf{0}}|)\big)^{d-1}\right] < \infty$. For $\boldsymbol{k} \geq \mathbf{0}$, define

$$X_{\boldsymbol{k}} = \sum_{\boldsymbol{u},\boldsymbol{v}\geq \mathbf{0}} a_{\boldsymbol{u},\boldsymbol{v}}\xi_{\boldsymbol{k}-\boldsymbol{u}}\xi_{\boldsymbol{k}-\boldsymbol{v}}$$

where $a_{\boldsymbol{u},\boldsymbol{v}}$ are real coefficients with $a_{\boldsymbol{u},\boldsymbol{u}} = 0$ and $\sum_{\boldsymbol{u},\boldsymbol{v}\geq \mathbf{0}} a_{\boldsymbol{u},\boldsymbol{v}}^2 < \infty$. In addition, assume that

$$\sum_{\boldsymbol{k}\geq \mathbf{1}} \frac{1}{{\Phi_d}^{-1}(|\boldsymbol{k}|)} \left(\sum_{\boldsymbol{u},\boldsymbol{v}\geq \boldsymbol{k}-\mathbf{1}} a_{\boldsymbol{u},\boldsymbol{v}}^2 \right)^{1/2} < \infty, \tag{4.2}$$

Then the quenched functional CLT in Corollary 2.5 holds.

This result is a generalization of the quenched functional CLT obtained in Zhang et al. (2020) (Example 6.3). Here, we only require an Orlicz space type condition on the innovation $\xi_{\mathbf{0}}$ and we weaken the condition (54) in Zhang et al. (2020) to condition (4.2). Additionally, we can see that (4.2) is not a very tractable condition to work with; therefore we provide the following stronger, but easier to verify, sufficient condition for (4.2) to hold:

$$\sum_{\boldsymbol{k}\geq \mathbf{1}} \frac{\big(\log(|\boldsymbol{k}|)\big)^{\frac{d-1}{2}}}{|\boldsymbol{k}|^{\frac{1}{2}}} \left(\sum_{\boldsymbol{u},\boldsymbol{v}\geq \boldsymbol{k}-\mathbf{1}} a_{\boldsymbol{u},\boldsymbol{v}}^2 \right)^{1/2} < \infty.$$

Proof of Example 4.2: Let $\boldsymbol{k} \geq \mathbf{1}$ and note that

$$\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}] = \sum_{\boldsymbol{u},\boldsymbol{v} \geq \boldsymbol{k}-\mathbf{1}} a_{\boldsymbol{u},\boldsymbol{v}} \xi_{\boldsymbol{k}-\boldsymbol{u}} \xi_{\boldsymbol{k}-\boldsymbol{v}}.$$

Let $(\xi'_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ and $(\xi''_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ be two independent copies of $(\xi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$. By applying a decoupling inequality by de la Peña and Giné (Theorem 3.1.1 in de la Peña and Giné, 1999, p.99), we get for any $t > 0$,

$$\begin{aligned}
\mathbb{E}\big[\Phi_d\big(|\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}]|/t\big)\big] &= \mathbb{E}\left[\Phi_d\left(\frac{1}{t}\left|\sum_{\boldsymbol{u},\boldsymbol{v} \geq \boldsymbol{k}-\mathbf{1}} a_{\boldsymbol{u},\boldsymbol{v}} \xi_{\boldsymbol{k}-\boldsymbol{u}} \xi_{\boldsymbol{k}-\boldsymbol{v}}\right|\right)\right] \\
&\leq \mathbb{E}\left[\Phi_d\left(\frac{C}{t}\left|\sum_{\boldsymbol{u},\boldsymbol{v} \geq \boldsymbol{k}-\mathbf{1}} a_{\boldsymbol{u},\boldsymbol{v}} \xi'_{\boldsymbol{k}-\boldsymbol{u}} \xi''_{\boldsymbol{k}-\boldsymbol{v}}\right|\right)\right],
\end{aligned}$$

with $C > 0$. Hence

$$\|\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}]\|_{\Phi_d} \lhd \left\|\sum_{\boldsymbol{u},\boldsymbol{v} \geq \boldsymbol{k}-\mathbf{1}} a_{\boldsymbol{u},\boldsymbol{v}} \xi'_{\boldsymbol{k}-\boldsymbol{u}} \xi''_{\boldsymbol{k}-\boldsymbol{v}}\right\|_{\Phi_d}.$$

Therefore, using the inequality $\|XY\|_{\Phi_d} \lhd \|X\|_{\Phi_d}\|Y\|_{\Phi_d}$ for any two independent random variables X, Y such that both $\|X\|_{\Phi_d}$ and $\|Y\|_{\Phi_d}$ are finite, and applying the Burkholder inequality for Orlicz spaces (see Dellacherie and Meyer, 1982, p.304, VII - 92), we get

$$\|\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}]\|_{\Phi_d} \lhd \left(\sum_{\substack{\boldsymbol{u},\boldsymbol{v} \geq \boldsymbol{k}-\mathbf{1} \\ \boldsymbol{u}\neq\boldsymbol{v}}} a^2_{\boldsymbol{u},\boldsymbol{v}} \|\xi_{\boldsymbol{k}-\boldsymbol{u}}\|^2_{\Phi_d} \|\xi_{\boldsymbol{k}-\boldsymbol{v}}\|^2_{\Phi_d}\right)^{\frac{1}{2}}.$$

By stationarity and since $\|\xi_{\mathbf{0}}\|_{\Phi_d} < \infty$, we obtain by using assumption (4.2), that condition (2.9) holds. Thus the CLT in Corollary 2.5 holds. $\square$

Here, we cannot relax assumption (4.2) on the coefficients $a_{\boldsymbol{u},\boldsymbol{v}}$ to a condition similar to (4.1) since it would not guarantee that the projective criterion (2.7) is satisfied. Indeed, in the case of Volterra fields we have

$$\mathcal{P}_{\mathbf{0}}(X_{\boldsymbol{k}}) = \sum_{\substack{\boldsymbol{u},\boldsymbol{v} \geq \boldsymbol{k} \\ \langle \boldsymbol{u}-\boldsymbol{k},\boldsymbol{v}-\boldsymbol{k}\rangle = 0}} a_{\boldsymbol{u},\boldsymbol{v}} \xi_{\boldsymbol{k}-\boldsymbol{u}} \xi_{\boldsymbol{k}-\boldsymbol{v}},$$

where $\langle \boldsymbol{i}, \boldsymbol{j}\rangle = \sum_{\ell=1}^d i_\ell j_\ell$ for $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^d$. Therefore, using the independence of the $\xi_{\boldsymbol{n}}$, it holds that

$$\|\mathcal{P}_{\mathbf{0}}(X_{\boldsymbol{k}})\|_{\Phi_d} \rhd \|\xi_{\mathbf{0}}\|_2^2 \sqrt{\sum_{\substack{\boldsymbol{u},\boldsymbol{v} \geq \boldsymbol{k} \\ \langle \boldsymbol{u}-\boldsymbol{k},\boldsymbol{v}-\boldsymbol{k}\rangle = 0}} a^2_{\boldsymbol{u},\boldsymbol{v}}},$$

Now, if we let $g : x \mapsto x^{-\frac{1}{2}} h(x)$ where $h(x) = \int_1^x \frac{1}{(\log(1+y))^2} dy$ and $a_{\boldsymbol{u},\boldsymbol{v}} = \left(\frac{g'(u_2)g'(v_1)}{u_1 v_2}\right)^{1/2}$, then

$$\sum_{\boldsymbol{k} \geq \mathbf{1}} \frac{1}{\sqrt{|\boldsymbol{k}|}} \left(\sum_{\boldsymbol{u},\boldsymbol{v} \geq \boldsymbol{k}-\mathbf{1}} a^2_{\boldsymbol{u},\boldsymbol{v}}\right)^{1/2} < \infty,$$

but Hannan's condition (2.7) fails.

4.3. *Hölderian function of a linear field.* Linear random fields such as the ones presented in Example 4.1 are quite useful for stochastic modeling and are a very common occurrence in the literature of that subject. Despite that, these types of fields might not capture the nonlinear properties of many dynamical systems and thus practicians are often required to introduce some more complex models that lack linearity. As we have seen, Volterra fields are a way to do so; however, in a lot of cases, a better model for a dynamical system can appear through random fields which are regular functions of linear fields. In the following example, we are interested in a type of regularity known as Hölder continuity. Such processes have been studied by Dedecker et al. (2007) for example and an annealed, i.e. nonquenched, central limit theorem under Hannan's condition has been derived. Here, we will provide sufficient conditions for that central limit theorem to be quenched.

Example 4.3. (Hölder function of a linear field) Consider an Hölder continuous function $f : \mathbb{R} \to \mathbb{R}$ of order $\alpha \in (0, 1]$ and let $(\xi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ be a random field of independent, identically distributed, and centered random variables which satisfy the following condition:

$$\begin{cases} \mathbb{E}[\xi_{\mathbf{0}}^2] < \infty \text{ if } \alpha \in (0,1), \\ \mathbb{E}\Big[|\xi_{\mathbf{0}}|^2\big(\log(1+|\xi_{\mathbf{0}}|)\big)^{d-1}\Big] < \infty \text{ if } \alpha = 1. \end{cases}$$

For $\boldsymbol{k} \geq \mathbf{0}$, define

$$X_{\boldsymbol{k}} = f\left(\sum_{\boldsymbol{j}\geq\mathbf{0}} a_{\boldsymbol{j}}\xi_{\boldsymbol{k}-\boldsymbol{j}}\right) - \mathbb{E}\left[f\left(\sum_{\boldsymbol{j}\geq\mathbf{0}} a_{\boldsymbol{j}}\xi_{\boldsymbol{k}-\boldsymbol{j}}\right)\right] \tag{4.3}$$

where $a_{\boldsymbol{u}}$ are real coefficients such that $\sum_{\boldsymbol{u}\geq\mathbf{0}} a_{\boldsymbol{u}}^2 < \infty$. If the coefficients $a_{\boldsymbol{u}}$ also satisfy

$$\sum_{\boldsymbol{k}\geq\mathbf{1}} \frac{1}{{\Phi_d}^{-1}(|\boldsymbol{k}|)}\left(\sum_{\boldsymbol{j}\geq\boldsymbol{k}-\mathbf{1}} a_{\boldsymbol{j}}^2\right)^{\frac{\alpha}{2}} < \infty \tag{4.4}$$

then the quenched functional CLT in Corollary 2.5 holds.

Once again, note that (4.4) is satisfied whenever

$$\sum_{\boldsymbol{k}\geq\mathbf{1}} \frac{\big(\log(|\boldsymbol{k}|)\big)^{\frac{d-1}{2}}}{|\boldsymbol{k}|^{\frac{1}{2}}}\left(\sum_{\boldsymbol{j}\geq\boldsymbol{k}-\mathbf{1}} a_{\boldsymbol{j}}^2\right)^{\frac{\alpha}{2}} < \infty.$$

Before moving on with the rest of the proof we will need the following lemma whose proof will be provided later on.

Lemma 4.4. *For all $\alpha \in (0, 1)$ and for any nonnegative random variable X, we have the following bound*

$$\|X^\alpha\|_{\Phi_d} \leq K_{\alpha,d}\|X\|_2^\alpha$$

with $K_{\alpha,d} > 0$ only depending on α and d.

Proof of Example 4.3: Throughout this proof, we will denote by C_α the Hölder constant associated with f.

Let $\boldsymbol{k} \geq \mathbf{1}$ and consider $(\xi'_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ an independent copy of $(\xi_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$. Using the fact that $\mathbb{E}_{\mathbf{1}}[X'_{\boldsymbol{k}}] = 0$ where $X'_{\boldsymbol{k}}$ is given by (4.3) with the innovations $\xi_{\boldsymbol{n}}$ replaced by $\xi'_{\boldsymbol{n}}$, we deduce that

$$\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}] = \mathbb{E}_{\mathbf{1}}\left[f\left(\sum_{\boldsymbol{j}\geq\mathbf{0}} a_{\boldsymbol{j}}\xi^*_{\boldsymbol{k}-\boldsymbol{j}}\right)\right] - \mathbb{E}_{\mathbf{1}}\left[f\left(\sum_{\boldsymbol{j}\geq\mathbf{0}} a_{\boldsymbol{j}}\xi'_{\boldsymbol{k}-\boldsymbol{j}}\right)\right],$$

where $\xi^*_{\boldsymbol{n}} = \xi_{\boldsymbol{n}}$ for $\boldsymbol{n} \leq \boldsymbol{1}$ and $\xi^*_{\boldsymbol{n}} = \xi'_{\boldsymbol{n}}$ otherwise. Since f is Hölder continuous of order α, we find that

$$\|\mathbb{E}_{\boldsymbol{1}}[X_{\boldsymbol{k}}]\|_{\Phi_d} \leq C_\alpha \left\| \left| \sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}(\xi_{\boldsymbol{k}-\boldsymbol{j}} - \xi'_{\boldsymbol{k}-\boldsymbol{j}}) \right|^\alpha \right\|_{\Phi_d}.$$

First, suppose that $\underline{0 < \alpha < 1}$. Then by Lemma 4.4, there exists $K_{\alpha,d} > 0$ such that

$$\left\| \left| \sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}(\xi_{\boldsymbol{k}-\boldsymbol{j}} - \xi'_{\boldsymbol{k}-\boldsymbol{j}}) \right|^\alpha \right\|_{\Phi_d} \leq K_{\alpha,d} \left\| \sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}(\xi_{\boldsymbol{k}-\boldsymbol{j}} - \xi'_{\boldsymbol{k}-\boldsymbol{j}}) \right\|_2^\alpha.$$

However, according to the Burkholder inequality for the L^2-norm, it holds

$$\left\| \sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}(\xi_{\boldsymbol{k}-\boldsymbol{j}} - \xi'_{\boldsymbol{k}-\boldsymbol{j}}) \right\|_2 \lhd \left(\sum_{\mathbf{j} \geq \mathbf{k}-\mathbf{1}} a_{\mathbf{j}}^2 \|\xi_{\mathbf{k}-\mathbf{j}}\|_2^2 \right)^{\frac{1}{2}} = \|\xi_{\mathbf{0}}\|_2 \left(\sum_{\mathbf{j} \geq \mathbf{k}-\mathbf{1}} a_{\mathbf{j}}^2 \right)^{\frac{1}{2}}$$

Thus

$$\left\| \left| \sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}(\xi_{\boldsymbol{k}-\boldsymbol{j}} - \xi'_{\boldsymbol{k}-\boldsymbol{j}}) \right|^\alpha \right\|_{\Phi_d} \lhd \|\xi_{\mathbf{0}}\|_2^\alpha \left(\sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}^2 \right)^{\frac{\alpha}{2}}. \tag{4.5}$$

Now, suppose that $\underline{\alpha = 1}$, then applying the Burkholder inequality for Orlicz space (see Dellacherie and Meyer, 1982, p.304, VII - 92), we get

$$\left\| \sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}(\xi_{\boldsymbol{k}-\boldsymbol{j}} - \xi'_{\boldsymbol{k}-\boldsymbol{j}}) \right\|_{\Phi_d} \lhd \|\xi_{\mathbf{0}}\|_{\Phi_d} \left(\sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}^2 \right)^{\frac{1}{2}}. \tag{4.6}$$

Combining both inequalities (4.5) and (4.6), we obtain that for any $\alpha \in (0,1]$, the inequality

$$\left\| \left| \sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}(\xi_{\boldsymbol{k}-\boldsymbol{j}} - \xi'_{\boldsymbol{k}-\boldsymbol{j}}) \right|^\alpha \right\|_{\Phi_d} \lhd \begin{cases} \|\xi_{\mathbf{0}}\|_2^\alpha \left(\sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}^2 \right)^{\frac{\alpha}{2}} & \text{if } \alpha \in (0,1), \\ \|\xi_{\mathbf{0}}\|_{\Phi_d}^\alpha \left(\sum_{\boldsymbol{j} \geq \boldsymbol{k}-\boldsymbol{1}} a_{\boldsymbol{j}}^2 \right)^{\frac{\alpha}{2}} & \text{if } \alpha = 1 \end{cases}$$

is satisfied. Therefore, using (4.4) and the moment condition on the random variable $\xi_{\mathbf{0}}$, we obtain that (2.9) is verified and thus the quenched functional CLT in Corollary 2.5 holds. $\square$

In order to prove Lemma 4.4, we give another very useful property of the natural logarithm: if $0 < \alpha < 1$ and $\epsilon \in (0, \alpha]$, then there exists a constant $c_{d,\epsilon} > 0$ such that for any $x > 0$,

$$\big(\log(1 + x^\alpha)\big)^{d-1} \leq c_{d,\epsilon} x^{(d-1)\epsilon}. \tag{4.7}$$

In particular, this implies that the function Φ_d satisfies

$$\Phi_d(x^\alpha) \leq c_{d,\epsilon} x^{2\alpha+(d-1)\epsilon},$$

for any $x > 0$.

Proof of Lemma 4.4: Let $\alpha \in (0,1)$ and X be a positive random variable, and consider $\epsilon \in (0, \alpha]$. Consider $t = \kappa^\alpha > 0$ and remark that

$$\mathbb{E}\left[\Phi_d\left(\frac{X^\alpha}{t}\right)\right] = \mathbb{E}\left[\Phi_d\left(\left(\frac{X}{\kappa}\right)^\alpha\right)\right] \leq c_{d,\epsilon} \mathbb{E}\left[\left(\frac{X}{\kappa}\right)^{2\alpha+(d-1)\epsilon}\right].$$

First, suppose that $\alpha \in [1/2, 1)$. We choose ϵ small enough such that $\delta := 2\alpha + (d-1)\epsilon < 2$ (which is always possible since $\alpha < 1$). Letting $\kappa_0 = c_{d,\epsilon}^{1/\delta}\|X\|_\delta$ (correspondingly $t_0 = c_{d,\epsilon}^{\alpha/\delta}\|X\|_\delta^\alpha$), we have

$$\mathbb{E}\left[\Phi_d\left(\frac{X^\alpha}{t_0}\right)\right] \le c_{d,\epsilon}\mathbb{E}\left[\left(\frac{X}{\kappa_0}\right)^\delta\right] = \frac{\mathbb{E}[X^\delta]}{\|X\|_\delta^\delta} = 1.$$

We deduce from the definition of $\|\cdot\|_{\Phi_d}$ that

$$\|X^\alpha\|_{\Phi_d} \le c_{d,\epsilon}^{\alpha/\delta}\|X\|_\delta^\alpha \le c_{d,\epsilon}^{\alpha/\delta}\|X\|_2^\alpha.$$

Now suppose $\alpha \in (0, 1/2)$. We choose ϵ small enough such that $\delta := 2\alpha + (d-1)\epsilon < 1$ (which is always possible since $\alpha < 1/2$). In this case, we set $\kappa_0 = c_{d,\epsilon}^{1/\delta}\|X\|_1$ (correspondingly $t_0 = c_{d,\epsilon}^{\alpha/\delta}\|X\|_1^\alpha$). Then, using the concavity of the function $x \mapsto x^\delta$, we get

$$\mathbb{E}\left[\Phi_d\left(\frac{X^\alpha}{t_0}\right)\right] \le c_{d,\epsilon}\mathbb{E}\left[\left(\frac{X}{\kappa_0}\right)^\delta\right] \le c_{d,\epsilon}\left(\mathbb{E}\left[\frac{X}{\kappa_0}\right]\right)^\delta = \left(\frac{\mathbb{E}[X]}{\|X\|_1}\right)^\delta = 1.$$

Therefore,

$$\|X^\alpha\|_{\Phi_d} \le c_{d,\epsilon}^{\alpha/\delta}\|X\|_1^\alpha \le c_{d,\epsilon}^{\alpha/\delta}\|X\|_2^\alpha.$$

Combining the discussions above, we conclude that for any $\alpha \in (0, 1)$ there exists $K_{\alpha,d} > 0$ such that

$$\|X^\alpha\|_{\Phi_d} \le K_{\alpha,d}\|X\|_2^\alpha.$$

□

As in the previous example, we cannot relax condition (4.4) to condition (4.1) in this case. In fact, neither (4.1) nor the condition $\sum_{\boldsymbol{i}\ge\boldsymbol{0}} |a_{\boldsymbol{i}}|^\alpha < \infty$ are enough to guarantee that (2.7) holds. Indeed, consider the case $d = 2$ and let

$$a_{u,v} = \begin{cases} g'(u)g'(v) & \text{if } u > 0 \text{ and } v > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $g : x \mapsto x^{-1}\big(\log(1+x)\big)^{-3}$.

Assume that the innovation field $(\xi_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^2}$ is a random field of independent and identically distributed random variables such that $\xi_{0,0}$ follows the standard normal distribution $\mathcal{N}(0,1)$. Consider the Lipschitz (Hölderian of order 1) function $f : x \in \mathbb{R} \mapsto |x| \in \mathbb{R}^+$. Letting $i, j \ge 0$, we have

$$\mathcal{P}_{0,0}(X_{i,j}) = \mathbb{E}_{0,0}\big[f(Y+Z+\zeta_{0,0}) - f(Y+Z+\zeta_{-1,0}) - f(Y+Z+\zeta_{0,-1}) + f(Y+Z+\zeta_{-1,-1})\big]$$

with

$$Y = \sum_{k\ge i+1}\sum_{l\ge j+1} a_{k,l}\xi_{i-k,j-l}, \quad Z = \sum_{k\ge 0}\sum_{l\ge 0} a_{k,l}\xi'_{i-k,j-l} - \sum_{k\ge i}\sum_{l\ge j} a_{k,l}\xi'_{i-k,j-l},$$

$$\zeta_{0,0} = \sum_{k\ge i} a_{k,j}\xi_{i-k,0} + \sum_{l\ge j+1} a_{i,l}\xi_{0,j-l},$$

$$\zeta_{-1,0} = \sum_{k\ge i+1} a_{k,j}\xi_{i-k,0} + \sum_{l\ge j} a_{i,l}\xi'_{0,j-l},$$

$$\zeta_{0,-1} = \sum_{k\ge i} a_{k,j}\xi'_{i-k,0} + \sum_{l\ge j+1} a_{i,l}\xi_{0,j-l},$$

and

$$\zeta_{-1,-1} = \sum_{k\ge i} a_{k,j}\xi'_{i-k,0} + \sum_{l\ge j+1} a_{i,l}\xi'_{0,j-l}.$$

where $(\xi'_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is an independent copy of $(\xi_{i,j})_{(i,j)\in\mathbb{Z}^2}$. Let $x \in \mathbb{R}^{\mathbb{Z}^2}$, then

$$\begin{aligned}\mathbb{E}\big[f(Y+Z+\zeta_{0,0}) - f(Y+Z+\zeta_{-1,0}) - f(Y+Z+\zeta_{0,-1}) + f(Y+Z+\zeta_{-1,-1}) \,\big|\, \xi = x\big] \\ = \mathbb{E}\big[f(y+Z+\zeta^x_{0,0}) - f(y+Z+\zeta^x_{-1,0}) - f(y+Z+\zeta^x_{0,-1}) + f(y+Z+\zeta^x_{-1,-1})\big]\end{aligned}$$

where $y = \sum_{k\geq i+1}\sum_{l\geq j+1} a_{k,l}x_{i-k,j-l}$,

$$\zeta^x_{0,0} = \sum_{k\geq i} a_{k,j}x_{i-k,0} + \sum_{l\geq j+1} a_{i,l}x_{0,j-l},$$

$$\zeta^x_{-1,0} = \sum_{k\geq i+1} a_{k,j}x_{i-k,0} + \sum_{l\geq j} a_{i,l}\xi'_{0,j-l}$$

$$\zeta^x_{0,-1} = \sum_{k\geq i} a_{k,j}\xi'_{i-k,0} + \sum_{l\geq j+1} a_{i,l}x_{0,j-l}$$

and

$$\zeta^x_{-1,-1} = \sum_{k\geq i} a_{k,j}\xi'_{i-k,0} + \sum_{l\geq j+1} a_{i,l}\xi'_{0,j-l}.$$

Remark that $y+Z+\zeta^x_{0,0}$, $y+Z+\zeta^x_{-1,0}$, $y+Z+\zeta^x_{0,-1}$ and $y+Z+\zeta^x_{-1,-1}$ follows respectively the normal distributions $\mathcal{N}\big(m^x_{0,0},\sigma^2_{0,0}\big)$, $\mathcal{N}\big(m^x_{-1,0},\sigma^2_{-1,0}\big)$, $\mathcal{N}\big(m^x_{0,-1},\sigma^2_{0,-1}\big)$ and $\mathcal{N}\big(m^x_{-1,-1},\sigma^2_{-1,-1}\big)$ with

$$\sigma^2_{0,0} = \mathrm{Var}[Z] = \sum_{k\geq 0}\sum_{l\geq 0} a^2_{k,l} - \sum_{k\geq i}\sum_{l\geq j} a^2_{k,l} \quad \text{and} \quad m^x_{0,0} = y + \zeta^x_{0,0},$$

$$\sigma^2_{-1,0} = \mathrm{Var}[Z+\zeta^x_{-1,0}] = \sum_{k\geq 0}\sum_{l\geq 0} a^2_{k,l} - \sum_{k\geq i+1}\sum_{l\geq j} a^2_{k,l} \quad \text{and} \quad m^x_{-1,0} = y + \sum_{k\geq i+1} a_{k,j}x_{i-k,0},$$

$$\sigma^2_{0,-1} = \mathrm{Var}[Z+\zeta^x_{0,-1}] = \sum_{k\geq 0}\sum_{l\geq 0} a^2_{k,l} - \sum_{k\geq i}\sum_{l\geq j+1} a^2_{k,l} \quad \text{and} \quad m^x_{0,-1} = y + \sum_{l\geq j+1} a_{i,l}x_{0,j-l},$$

and

$$\sigma^2_{-1,-1} = \mathrm{Var}[Z+\zeta^x_{-1,-1}] = \sum_{k\geq 0}\sum_{l\geq 0} a^2_{k,l} - \sum_{k\geq i+1}\sum_{l\geq j+1} a^2_{k,l} \quad \text{and} \quad m^x_{-1,-1} = y.$$

Thus, for $(a,b)\in\{-1,0\}^2$,

$$\mathbb{E}[f(y+Z+\zeta^x_{a,b})] = \mathbb{E}[|y+Z+\zeta^x_{a,b}|] = \frac{2\sigma_{a,b}}{\sqrt{2\pi}}e^{-\frac{(m^x_{a,b})^2}{2\sigma^2_{a,b}}} + \frac{|m^x_{a,b}|}{\sqrt{2\pi\sigma^2_{a,b}}}\int_{-|m^x_{a,b}|}^{|m^x_{a,b}|} e^{-\frac{z^2}{2\sigma^2_{a,b}}}dz.$$

Rewriting the right-hand side, we notice that

$$\begin{aligned}&\frac{2\sigma_{a,b}}{\sqrt{2\pi}}e^{-\frac{(m^x_{a,b})^2}{2\sigma^2_{a,b}}} + \frac{|m^x_{a,b}|}{\sqrt{2\pi\sigma^2_{a,b}}}\int_{-|m^x_{a,b}|}^{|m^x_{a,b}|} e^{-\frac{z^2}{2\sigma^2_{a,b}}}dz \\ &\qquad = \sigma_{a,b}\left(\frac{2}{\sqrt{2\pi}}e^{-\frac{(m^x_{a,b})^2}{2\sigma^2_{a,b}}} + \frac{|m^x_{a,b}|}{\sigma_{a,b}}\int_{-\frac{|m^x_{a,b}|}{\sigma_{a,b}}}^{\frac{|m^x_{a,b}|}{\sigma_{a,b}}}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz\right) \\ &\qquad =: \sigma_{a,b}\,g\left(\frac{m^x_{a,b}}{\sigma_{a,b}}\right).\end{aligned}$$

For any $(a,b)\in\{-1,0\}^2$, we let $\Xi_{a,b}=m^{\xi}_{a,b}-Y$ and by applying Taylor's formula, we get

$$\begin{aligned}g\Big(\frac{m^{\xi}_{a,b}}{\sigma_{a,b}}\Big)&=g\Big(\frac{Y}{\sigma_{0,0}}+\Big(\frac{1}{\sigma_{a,b}}-\frac{1}{\sigma_{0,0}}\Big)Y+\frac{\Xi_{a,b}}{\sigma_{a,b}}\Big)\\&=g\Big(\frac{Y}{\sigma_{0,0}}\Big)+\frac{\Xi_{a,b}}{\sigma_{a,b}}g'\Big(\frac{Y}{\sigma_{0,0}}\Big)+\Big(\frac{1}{\sigma_{a,b}}-\frac{1}{\sigma_{0,0}}\Big)Yg'\Big(\frac{Y}{\sigma_{0,0}}\Big)+R_{a,b}.\end{aligned}$$

where $|R|\le\frac{2\Xi^2_{a,b}}{\sigma^2_{a,b}}+2\Big(\frac{1}{\sigma_{a,b}}-\frac{1}{\sigma_{0,0}}\Big)^2Y^2$. Thus

$$\begin{aligned}\|\mathcal{P}_{0,0}(X_{i,j})\|_{\Phi_d}&=\Big\|\sum_{(a,b)\in\{0,1\}^2}(-1)^{a+b}\sigma_{a,b}g\Big(\frac{m^{\xi}_{a,b}}{\sigma_{a,b}}\Big)\Big\|_{\Phi_d}\\&\ge\underbrace{\Big\|g\Big(\frac{Y}{\sigma_{0,0}}\Big)\sum_{(a,b)\in\{0,1\}^2}(-1)^{a+b}\sigma_{a,b}\Big\|_{\Phi_d}}_{=:N_1}-\underbrace{\Big\|g'\Big(\frac{Y}{\sigma_{0,0}}\Big)\sum_{(a,b)\in\{0,1\}^2}(-1)^{a+b}\Xi_{a,b}\Big\|_{\Phi_d}}_{=:N_2}\\&\quad-\underbrace{\Big\|Yg'\Big(\frac{Y}{\sigma_{0,0}}\Big)\sum_{(a,b)\in\{0,1\}^2}(-1)^{a+b}\Big(1-\frac{\sigma_{a,b}}{\sigma_{0,0}}\Big)\Big\|_{\Phi_d}}_{=:N_3}-\underbrace{\Big\|\sum_{(a,b)\in\{0,1\}^2}(-1)^{a+b}\sigma_{a,b}R_{a,b}\Big\|_{\Phi_d}}_{=:N_4}.\end{aligned}$$

We have the following relations

$$\begin{aligned}N_1&=\Big\|g\Big(\frac{Y}{\sigma_{0,0}}\Big)\Big\|_{\Phi_d}|\sigma_{0,0}-\sigma_{-1,0}-\sigma_{0,-1}+\sigma_{-1,-1}|,\\N_2&=\Big\|g'\Big(\frac{Y}{\sigma_{0,0}}\Big)\xi'_{0,0}\Big\|_{\Phi_d}|a_{i,j}|,\\N_3&=\Big\|\frac{Y}{\sigma_{0,0}}g'\Big(\frac{Y}{\sigma_{0,0}}\Big)\Big\|_{\Phi_d}|\sigma_{0,0}-\sigma_{-1,0}-\sigma_{0,-1}+\sigma_{-1,-1}|.\end{aligned}$$

and

$$\begin{aligned}N_4\le{}&\Big\|\Big(\frac{Y}{\sigma_{0,0}}\Big)^2\Big\|_{\Phi_d}\Big|\frac{(\sigma_{0,0}-\sigma_{-1,0})^2}{\sigma_{-1,0}}+\frac{(\sigma_{0,0}-\sigma_{0,-1})^2}{\sigma_{0,-1}}-\frac{(\sigma_{0,0}-\sigma_{-1,-1})^2}{\sigma_{-1,-1}}\Big|\\&+2\Big(\frac{\|\Xi^2_{0,0}\|_{\Phi_d}}{\sigma_{0,0}}+\frac{\|\Xi^2_{-1,0}\|_{\Phi_d}}{\sigma_{-1,0}}+\frac{\|\Xi^2_{0,-1}\|_{\Phi_d}}{\sigma_{0,-1}}\Big).\end{aligned}$$

Before continuing the proof, note that the random variable $Y/\sigma_{0,0}$ follows a centered normal distribution with variance inferior to 1 whenever i and j are large enough. Using the previous relations, it holds that

$$\|\mathcal{P}_{0,0}(X_{i,j})\|_{\Phi_d}\ge|\sigma_{0,0}-\sigma_{-1,0}-\sigma_{0,-1}+\sigma_{-1,-1}|\Bigg(\Big\|g\Big(\frac{Y}{\sigma_{0,0}}\Big)\Big\|_{\Phi_d}-\Big\|\frac{Y}{\sigma_{0,0}}g'\Big(\frac{Y}{\sigma_{0,0}}\Big)\Big\|_{\Phi_d}\Bigg)-N_2-N_4.$$

Now, making use of the equality $u_1-u_2=(u_1^2-u_2^2)/(u_1+u_2)$ for any $u_1,u_2>0$, we obtain

$$\sigma_{0,0}-\sigma_{-1,0}-\sigma_{0,-1}+\sigma_{-1,-1}=\frac{\sigma^2_{0,0}-\sigma^2_{-1,0}}{\sigma_{0,0}+\sigma_{-1,0}}-\frac{\sigma^2_{0,-1}-\sigma^2_{-1,-1}}{\sigma_{0,-1}+\sigma_{-1,-1}}.$$

However,

$$\delta_{i,j}:=|\sigma^2_{0,0}-\sigma^2_{-1,0}|=\sum_{l\ge j}a^2_{i,l}\quad\text{ and }\quad|\sigma^2_{0,-1}-\sigma^2_{-1,-1}|=\delta_{i,j}-a^2_{i,j}.$$

According to the definition of the coefficients $a_{i,j}$, it is possible to show that

$$|\sigma_{0,0}-\sigma_{-1,0}-\sigma_{0,-1}+\sigma_{-1,-1}| \geq \left(\frac{1}{\sigma_{0,0}+\sigma_{-1,0}}-\frac{1}{\sigma_{0,-1}+\sigma_{-1,-1}}\right)\delta_{i,j} \triangleright \big(\log(1+i)\log(1+j)\big)^{-3/2}\sqrt{j}.$$

In particular, it holds that

$$\sum_{(i,j)\geq(0,0)} \|\mathcal{P}_{0,0}(X_{i,j})\|_{\Phi_d} = \infty,$$

while also having both

$$\sum_{(i,j)\geq(0,0)} |a_{i,j}| < \infty \quad \text{and} \quad \sum_{(i,j)\geq(1,1)} \frac{1}{\sqrt{ij}}\left(\sum_{(u,v)\geq(i-1,j-1)} a_{i,j}^2\right)^{\frac{1}{2}} < \infty.$$

4.4. *Weakly Dependent Processes.* In our last example, we study the quenched central limit theorem for weakly dependent random fields in the sense of Wu. Fields of this kind were introduced by Wu (2005) and have many applications in mathematical physics, especially within the study of particle systems (see Liggett, 1985; Stroock and Zegarliński, 1992). Indeed, weakly dependent random fields are particularly well-suited to model physical systems as they can capture, at least partially, the influence of the inputs over the outputs of these systems. In particular, they are well adapted to study the case of nonlinear physical models.

Consider a centered Bernoulli random field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ defined for every $\boldsymbol{i} \in \mathbb{Z}^d$ by$X_{\boldsymbol{i}} := G(\xi_{\boldsymbol{i}-\boldsymbol{s}}; \boldsymbol{s} \geq \boldsymbol{0})$ where $(\xi_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is a field of independent and identically distributed random variables. Now, denote by $(\xi'_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ an independent copy of $(\xi_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ and set, for any $\boldsymbol{i} \in \mathbb{Z}^d$,

$$\begin{cases} \xi^*_{\boldsymbol{i}} = \xi'_{\boldsymbol{i}} & \text{if } \boldsymbol{i} = \boldsymbol{0} \\ \xi^*_{\boldsymbol{i}} = \xi_{\boldsymbol{i}} & \text{otherwise} \end{cases} \quad \text{as well as} \quad \begin{cases} \widetilde{\xi}_{\boldsymbol{i}} = \xi'_{\boldsymbol{i}} & \text{if } i_1 = 0 \text{ and } \boldsymbol{i} \leq \boldsymbol{1} \\ \widetilde{\xi}_{\boldsymbol{i}} = \xi_{\boldsymbol{i}} & \text{otherwise.} \end{cases}$$

Then, the perturbed systems $(X^*_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ and $(\widetilde{X}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ are given by

$$X^*_{\boldsymbol{i}} = G(\xi^*_{\boldsymbol{i}-\boldsymbol{s}}; \boldsymbol{s} \geq \boldsymbol{0}) \quad \text{and} \quad \widetilde{X}_{\boldsymbol{i}} = G(\widetilde{\xi}_{\boldsymbol{i}-\boldsymbol{s}}; \boldsymbol{s} \geq \boldsymbol{0}), \qquad \boldsymbol{i} \in \mathbb{Z}^d.$$

In this subsection, we are interested in two different stability conditions. First, we take a look at the usual notion of weak dependence in the sense of Wu by saying that the random field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is stable whenever

$$\sum_{\boldsymbol{i}\geq\boldsymbol{0}} \delta_{\boldsymbol{i}} < \infty$$

where the terms $\delta_{\boldsymbol{i}}$ are known as the physical dependence coefficients and are defined by

$$\delta_{\boldsymbol{i}} = \|X_{\boldsymbol{i}} - X^*_{\boldsymbol{i}}\|_{\Phi_d}.$$

Under this notion of weak dependence, we have the following quenched functional central limit theorem.

Example 4.5. Suppose that the field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ satisfies

$$\mathbb{E}[X_{\boldsymbol{0}}|\mathcal{G}_n] \xrightarrow[n\to\infty]{\text{a.s.}} 0, \tag{4.8}$$

where $\mathcal{G}_n = \sigma\big(\xi_{\boldsymbol{i}} : \boldsymbol{i} \leq \boldsymbol{1} \text{ and } \exists k \in [\![1,d]\!], i_k \leq n\big)$. Additionally, suppose that

$$\sum_{\boldsymbol{k}\geq\boldsymbol{1}} \frac{1}{\Phi_d^{-1}(|\boldsymbol{k}|)} \sum_{\boldsymbol{j}\geq\boldsymbol{k}-\boldsymbol{1}} \delta_{\boldsymbol{j}} < \infty. \tag{4.9}$$

Then the conclusion of Corollary 2.5 holds.

Note that the condition (4.8) is a stronger condition than the regularity of $X_{\mathbf{0}}$. Moreover, as we have seen in Example 4.2 and Example 4.3, it is possible to give a stronger yet more tractable condition than (4.9) which is stated below. Indeed, if

$$\sum_{\boldsymbol{k}\geq\mathbf{1}}\frac{\big(\log(|\boldsymbol{k}|)\big)^{\frac{d-1}{2}}}{|\boldsymbol{k}|^{\frac{1}{2}}}\sum_{\boldsymbol{j}\geq\boldsymbol{k}-\mathbf{1}}\delta_{\boldsymbol{j}}<\infty$$

then the conclusion of Example 4.5 is verified.

Proof: Let $\boldsymbol{k}\geq\mathbf{1}$ and consider a bijection $\tau:\mathbb{Z}\to\mathbb{Z}^d$ such that for all $n\in\mathbb{Z}$, we have

$$\big(n\geq 1 \iff \tau(n)\in\mathbb{Z}^-_{\mathbf{1}}\big)\quad\text{and}\quad\big(\forall k\in\mathbb{N},\{\boldsymbol{i}\in\mathbb{Z}^d : 1\leq\tau^{-1}(\boldsymbol{i})\leq k^d\}=[\![2-k,1]\!]^d\big)$$

where $\mathbb{Z}^-_{\mathbf{1}}=\{\boldsymbol{i}\in\mathbb{Z}^d:\boldsymbol{i}\leq\mathbf{1}\}$. Since $X_{\mathbf{0}}$ is centered and satisfies (4.8), we find that

$$\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}]=\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}]-\mathbb{E}_{\mathbf{1}}\big[G\big(\xi'_{\boldsymbol{k}-\boldsymbol{s}};\boldsymbol{s}\geq\mathbf{0}\big)\big]=\sum_{n\geq0}\mathbb{E}_{\mathbf{1}}[Y_{\tau(n)}-Y_{\tau(n+1)}],\tag{4.10}$$

where

$$Y_{\tau(n)}=G\big(\zeta^n_{\boldsymbol{k}-\boldsymbol{s}};\boldsymbol{s}\geq\mathbf{0}\big),\quad\text{with}\quad\zeta^n_{\boldsymbol{i}}=\begin{cases}\xi_{\boldsymbol{i}} & \text{if } \tau^{-1}(\boldsymbol{i})>n\\ \xi'_{\boldsymbol{i}} & \text{otherwise.}\end{cases}$$

Remark that, according to stationarity, for any $n\geq 0$

$$\begin{aligned}
&\|\mathbb{E}_{\mathbf{1}}[Y_{\tau(n)}-Y_{\tau(n+1)}]\|_{\Phi_d}\\
&\quad\leq\big\|G\big(\zeta^n_{\boldsymbol{k}-\boldsymbol{s}};\boldsymbol{s}\geq\mathbf{0}\big)-G\big(\zeta^{n+1}_{\boldsymbol{k}-\boldsymbol{s}};\boldsymbol{s}\geq\mathbf{0}\big)\big\|_{\Phi_d}\\
&\quad=\big\|G\big(\zeta^n_{\boldsymbol{k}-\boldsymbol{s}};\boldsymbol{s}\geq\mathbf{0}\big)-G\big(\zeta^n_{\boldsymbol{k}-\boldsymbol{s}},\xi'_{\tau(n+1)};\boldsymbol{s}\geq\mathbf{0},\boldsymbol{s}\neq\boldsymbol{k}-\tau(n+1)\big)\big\|_{\Phi_d}\\
&\quad=\big\|G\big(\xi_{\boldsymbol{k}-\tau(n+1)-\boldsymbol{s}};\boldsymbol{s}\geq\mathbf{0}\big)-G\big(\xi_{\boldsymbol{k}-\tau(n+1)-\boldsymbol{s}},\xi'_{\mathbf{0}};\boldsymbol{s}\geq\mathbf{0},\boldsymbol{s}\neq\boldsymbol{k}-\tau(n+1)\big)\big\|_{\Phi_d}\\
&\quad=\delta_{\boldsymbol{k}-\tau(n+1)}.
\end{aligned}$$

Therefore, using the triangular inequality, we get

$$\|\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}]\|_{\Phi_d}\leq\sum_{n\geq0}\delta_{\boldsymbol{k}-\tau(n+1)}=\sum_{\boldsymbol{j}\geq\boldsymbol{k}-\mathbf{1}}\delta_{\boldsymbol{j}}.$$

Using condition (4.9) we find that the conclusion of Corollary 2.5 is satisfied. □

By considering a stronger notion of stability, we can relax the hypothesis (4.9) to (4.11) as well as condition (4.8) to the simple regularity of $X_{\mathbf{0}}$. In fact, we will say that a random field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is strongly stable whenever

$$\sum_{\boldsymbol{i}\geq\mathbf{0}}\widetilde{\delta}_{\boldsymbol{i}}<\infty$$

with

$$\widetilde{\delta}_{\boldsymbol{i}}=\|X_{\boldsymbol{i}}-\widetilde{X}_{\boldsymbol{i}}\|_{\Phi_d}.$$

Under this stronger assumption, we can show that Hannan's condition (2.7) holds. Then there only remains to satisfy (2.5) for Corollary 2.6 to apply.

Example 4.6. Suppose that the field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is strongly stable and that $X_{\mathbf{0}}$ is regular, then condition (2.7) is also satisfied. If, in addition, we also assume that

$$\sum_{\boldsymbol{k}\geq\mathbf{1}}\frac{1}{\sqrt{|\boldsymbol{k}|}}\sum_{j\geq0}\widetilde{\delta}_{\boldsymbol{k}+(j-1)\boldsymbol{e}_1}<\infty,\tag{4.11}$$

then the conclusion of Corollary 2.5 holds.

Proof: Let $\boldsymbol{k} \geq \mathbf{0}$, then we have the following bound

$$\begin{aligned}\|\mathcal{P}_{\mathbf{0}}(X_{\boldsymbol{k}})\|_{\Phi_d} &= \left\|\prod_{i=1}^{d}(\mathbb{E}_{\mathbf{0}} - \mathbb{E}_{-\boldsymbol{e}_i})[X_{\boldsymbol{k}}]\right\|_{\Phi_d} \\ &\leq 2^{d-1}\|\mathbb{E}_{\mathbf{0}}[X_{\boldsymbol{k}}] - \mathbb{E}_{-\boldsymbol{e}_1}[X_{\boldsymbol{k}}]\|_{\Phi_d} \\ &\leq 2^{d-1}\widetilde{\delta}_{\boldsymbol{k}}.\end{aligned}$$

Hence (2.7) is satisfied. Now suppose that (4.11) holds and $\boldsymbol{k} \geq \mathbf{1}$. Since $G(\xi'_{\boldsymbol{k}-\boldsymbol{s}}; \boldsymbol{s} \geq \mathbf{0})$ is a centered random variable, we have

$$\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}] = \mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}] - \mathbb{E}_{\mathbf{1}}\left[G\left(\xi'_{\boldsymbol{k}-\boldsymbol{s}}; \boldsymbol{s} \geq \mathbf{0}\right)\right] = \sum_{j\geq 0}\mathbb{E}_{\mathbf{1}}[Y_j - Y_{j+1}], \tag{4.12}$$

where

$$Y_j = G\left(\zeta^j_{\boldsymbol{k}-\boldsymbol{s}}; \boldsymbol{s} \geq \mathbf{0}\right), \quad \text{with} \quad \zeta^j_{\boldsymbol{i}} = \begin{cases} \xi'_{\boldsymbol{i}} & \text{if } 1-j < i_1 \leq 1 \text{ and } \boldsymbol{i} \leq \mathbf{1} \\ \xi_{\boldsymbol{i}} & \text{otherwise.}\end{cases}$$

However, using a similar argument as before, we have that

$$\|\mathbb{E}_{\mathbf{1}}[Y_j - Y_{j+1}]\|_2 \leq \widetilde{\delta}_{\boldsymbol{k}+(j-1)\boldsymbol{e}_1}.$$

Therefore, using the triangle inequality, we get

$$\|\mathbb{E}_{\mathbf{1}}[X_{\boldsymbol{k}}]\|_2 \leq \sum_{j\geq 0}\widetilde{\delta}_{\boldsymbol{k}+(j-1)\boldsymbol{e}_1}.$$

And so, using condition (4.11), the conclusion of Corollary 2.6 holds for the stochastic process $(X_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$. □

5. Appendix

In this section, we give the proof of Lemma 3.8. We will follow the outline of the proof given by Burkholder (1973) for the Rosenthal inequality in Lebesgue spaces but first, we need to establish a preliminary lemma concerning the Orlicz norm studied in this document. We start by recalling the definition of the different tools we will require.

Recall that the Luxemburg norm associated with the Young function $\Phi_d : x \in [0,\infty) \mapsto \Phi_d(x) = x^2\left(\log(1+x)\right)^{d-1} \in [0,\infty)$ is defined as

$$\|f\|_{\Phi} = \inf\left\{t > 0 : \mathbb{E}\left[\Phi(|f|/t)\right] \leq 1\right\},$$

and by Ψ_d we denote the conjugate function associated with Φ_d defined in the following way

$$\Psi_d(x) = \sup_{y\geq 0}\left(xy - \Phi_d(y)\right).$$

Besides properties (3.9) and (3.10), the natural logarithm also satisfies

$$\log\left(1+\frac{x}{\lambda}\right)\log(1+\lambda) \geq \log(2)\lambda\log(1+x), \tag{5.1}$$

for all $0 < \lambda \leq 1$ and $x \geq 0$ as well as

$$\log\left(1+\frac{x}{\lambda}\right)\log(1+\lambda) \geq \log(2)\frac{\log(1+x)}{\lambda}, \tag{5.2}$$

for all $\lambda \geq 1$ and $x \geq 0$. The following lemma will help us compute the Orlicz norm associated with Ψ_d of a specific random variable which will appear in the proof of Lemma 3.8.

Lemma 5.1. *Suppose that $h \in L^2 \log L$ takes nonnegative values. If $\|h\|_{\Phi_d} \leq 1$, then*

$$\left\| h\big(\log(1+h)\big)^{d-1} \right\|_{\Psi_d} \leq \|h\|_{\Phi_d}.$$

If $\|h\|_{\Phi_d} > 1$ then for all $\epsilon \in (0,1)$, there exist a positive constant $C_{d,\epsilon}$ depending only on d and ϵ such that

$$\left\| h\big(\log(1+h)\big)^{d-1} \right\|_{\Psi_d} \leq C_{d,\epsilon} \|h\|_{\Phi_d}^{1+\epsilon}. \tag{5.3}$$

Before moving on with the proof of Lemma 5.1, we explicit another useful property of the natural logarithm. For all $x \geq \lambda \geq 1$,

$$\log\Big(1 + \frac{x}{\lambda}\Big) \log(1+\lambda) \geq \log(2) \log(1+x). \tag{5.4}$$

Proof of Lemma 5.1: Let $h \in L^2 \log L$ be a nonnegative function such that $\|h\|_{\Phi_d} \leq 1$ and let $t \in (0,1]$. Using the inequality $\Psi\big(x\big(\log(1+x)\big)^{d-1}/t\big) \leq \Phi(x/t)$ for all $x \geq 0$, we get

$$\mathbb{E}\bigg[\Psi_d\bigg(\frac{h \log^{d-1}(1+h)}{t}\bigg)\bigg] \leq \mathbb{E}\bigg[\Phi_d\Big(\frac{h}{t}\Big)\bigg].$$

Taking $t = \|h\|_{\Phi_d} \leq 1$, we obtain

$$\left\| h\big(\log(1+h)\big)^{d-1} \right\|_{\Psi_d} \leq \|h\|_{\Phi_d}.$$

We now turn to the proof of the second part of Lemma 5.1 and we begin by noticing that if $\|h\|_{\Phi_d} = \infty$, then (5.3) is trivially satisfied. From now on, we will therefore assume that $1 < \|h\|_{\Phi_d} < \infty$. We start by recalling that if we let $\epsilon \in (0,1)$, then there exists a constant $c_{d,\epsilon} > 0$ such that

$$\Big(\log(1 + \|h\|_{\Phi_d})\Big)^{d-1} \leq c_{d,\epsilon} \|h\|_{\Phi_d}^{\epsilon}. \tag{5.5}$$

Now, according to the triangle inequality, we have

$$\left\| h\big(\log(1+h)\big)^{d-1} \right\|_{\Psi_d} \leq \left\| h\big(\log(1+h)\big)^{d-1} \mathbf{1}_{h \leq \|h\|_{\Phi_d}} \right\|_{\Psi_d} + \left\| h\big(\log(1+h)\big)^{d-1} \mathbf{1}_{h > \|h\|_{\Phi_d}} \right\|_{\Psi_d}.$$

To bound the first term in the right-hand side of this inequality, we make use of (5.5) and we obtain

$$\begin{aligned} \left\| h\big(\log(1+h)\big)^{d-1} \mathbf{1}_{h \leq \|h\|_{\Phi_d}} \right\|_{\Psi_d} &\leq \Big(\log(1 + \|h\|_{\Phi_d})\Big)^{d-1} \|h\|_{\Phi_d} \\ &\leq c_{d,\epsilon} \|h\|_{\Phi_d}^{1+\epsilon}. \end{aligned} \tag{5.6}$$

Dealing with the second term, we combine (5.4), (5.5) and the inequality $\Psi_d(\varphi_d(x)) \leq \Phi_d(x)$ where $\varphi_d(x) = x\big(\log(1+x)\big)^{d-1}$ for all $x \geq 0$, in order to get

$$\begin{aligned} \mathbb{E}\bigg[\Psi_d\bigg(\frac{h\big(\log(1+h)\big)^{d-1}}{\log(2)^{1-d} c_{d,\epsilon} \|h\|_{\Phi_d}^{1+\epsilon}}\bigg) \mathbf{1}_{h > \|h\|_{\Phi_d}}\bigg] &\leq \mathbb{E}\Bigg[\Psi_d\Bigg(\frac{h\Big(\log(2)^{-1} \log\Big(1 + \frac{h}{\|h\|_{\Phi_d}}\Big) \log(1 + \|h\|_{\Phi_d})\Big)^{d-1}}{\log(2)^{1-d} c_{d,\epsilon} \|h\|_{\Phi_d}^{1+\epsilon}}\Bigg)\Bigg] \\ &\leq \mathbb{E}\bigg[\Phi_d\Big(\frac{h}{\|h\|_{\Phi_d}}\Big)\bigg] \\ &= 1. \end{aligned}$$

Thus

$$\left\| h\big(\log(1+h)\big)^{d-1} \mathbf{1}_{h > \|h\|_{\Phi_d}} \right\|_{\Psi_d} \leq \log(2)^{1-d} c_{d,\epsilon} \|h\|_{\Phi_d}^{1+\epsilon}. \tag{5.7}$$

Therefore, combining (5.6) and (5.7) we get the desired result. $\square$

We can now prove Lemma 3.8. In order to do so, we will make use of Lemma 3.1 in Burkholder (1973).

Proof of Lemma 3.8: We start by introducing a few items of notation. For all $\boldsymbol{n} \in \mathbb{N}^d$, we denote $M_{\boldsymbol{n}} = \sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} d_{\boldsymbol{u}}$ and $\sigma_{\boldsymbol{n}} = \sqrt{\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} d_{\boldsymbol{u}}^2}$. Our proof will be split into two parts. In the first part, we will make the additional assumption that the ortho-martingale $(M_{\boldsymbol{n}})_{\boldsymbol{n}}$ is nonnegative. Then, in the second part, we will establish the result for real-valued ortho-martingales.

First step: We suppose that $(M_{\boldsymbol{n}})_{\boldsymbol{n}\in(\mathbb{N}^*)^d}$ is a nonnegative ortho-martingale. Let $\boldsymbol{n} \in \mathbb{N}^d$ be fixed and remark that, since $\Phi_d(\sqrt{a+b}) \geq \Phi_d(\sqrt{a}) + \Phi_d(\sqrt{b})$ for all $a, b \geq 0$, it holds that

$$\mathbb{E}\left[\Phi_d\left(\frac{\sigma_{\boldsymbol{n}}}{\eta}\right)\right] \geq \sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}\left[\Phi_d\left(\frac{d_{\boldsymbol{u}}}{\eta}\right)\right]. \tag{5.8}$$

for any $\eta > 0$. Let $X = \max(\sigma_{\boldsymbol{n}}, \max_{\boldsymbol{0}\leq\boldsymbol{u}\leq\boldsymbol{n}} M_{\boldsymbol{u}})$ and suppose that $\|X\|_{\Phi_d} \leq 1$. Applying (5.8) with $\eta = 1$, we get

$$\|\sigma_{\boldsymbol{n}}\|_{\Phi_d} \leq \|X\|_{\Phi_d} \leq 1 \quad \text{and} \quad \eta_0 := \sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}})] \leq 1. \tag{5.9}$$

Setting $\eta_0' = \eta_0 \log(2)^{d-1}$, we find that

$$\begin{aligned}\mathbb{E}\left[\Phi_d\left(\frac{\sigma_{\boldsymbol{n}}}{\eta_0'}\right)\right] &= \frac{1}{{\eta_0'}^2}\mathbb{E}\left[\sigma_{\boldsymbol{n}}^2\left(\log\left(1+\frac{\sigma_{\boldsymbol{n}}}{\eta_0'}\right)\right)^{d-1}\right] \\ &\geq \frac{1}{\eta_0}\mathbb{E}[\Phi_d(\sigma_{\boldsymbol{n}})] \\ &\geq 1.\end{aligned}$$

The second to last inequality holds since according to (5.1), we have

$$\begin{aligned}\left(\log\left(1+\frac{\sigma_{\boldsymbol{n}}}{\eta_0'}\right)\right)^{d-1} &\geq {\eta_0'}^{d-1}\log(2)^{d-1}\frac{\big(\log(1+\sigma_{\boldsymbol{n}})\big)^{d-1}}{\big(\log(1+\eta_0')\big)^{d-1}} \\ &\geq {\eta_0'}^2\log(2)^{d-1}\frac{\big(\log(1+\sigma_{\boldsymbol{n}})\big)^{d-1}}{\eta_0'} \\ &= \frac{{\eta_0'}^2}{\eta_0}\big(\log(1+\sigma_{\boldsymbol{n}})\big)^{d-1}.\end{aligned}$$

From the previous inequality, we deduce that

$$\log(2)^{1-d}\|\sigma_{\boldsymbol{n}}\|_{\Phi_d} \geq \eta_0 = \sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}})]. \tag{5.10}$$

Using Lemma 3.1 in Burkholder (1973), for any $\lambda > 0$

$$\lambda\mathbb{P}\big(X > \sqrt{3}\lambda\big) \leq 3\int_{\{X>\lambda\}} M_{\boldsymbol{n}} \mathrm{d}\mathbb{P}.$$

Therefore

$$\mathbb{E}[\Phi_d(X)] = \int_0^{+\infty} \Phi_d'(u)\mathbb{P}(X > u)\mathrm{d}u \leq 3\sqrt{3}\int_\Omega M_{\boldsymbol{n}} \int_0^{\sqrt{3}X} \frac{\Phi_d'(u)}{u}\mathrm{d}u\mathrm{d}\mathbb{P}.$$

Computing $\int_0^{\sqrt{3}X} \frac{\Phi_d'(u)}{u} \mathrm{d}u$, we find that $\int_0^{\sqrt{3}X} \frac{\Phi_d'(u)}{u} \mathrm{d}u \le 3\sqrt{3}X\big(\log(1+\sqrt{3}X)\big)^{d-1}$. Thus

$$\mathbb{E}[\Phi_d(X)] \le 3^{\frac{d+5}{2}} \int_\Omega M_{\boldsymbol{n}} X\big(\log(1+X)\big)^{d-1} \mathrm{d}\mathbb{P}. \tag{5.11}$$

Applying Holder's inequality for Orlicz spaces, we get

$$\int_\Omega M_{\boldsymbol{n}} X\big(\log(1+X)\big)^{d-1} \mathrm{d}\mathbb{P} \le 2\|M_{\boldsymbol{n}}\|_{\Phi_d} \Big\| X\big(\log(1+X)\big)^{d-1} \Big\|_{\Psi_d}.$$

Using Lemma 5.1, we find that

$$\Big\| X\big(\log(1+X)\big)^{d-1} \Big\|_{\Psi_d} \le \|X\|_{\Phi_d}.$$

Then

$$\int_\Omega M_{\boldsymbol{n}} X\big(\log(1+X)\big)^{d-1} \mathrm{d}\mathbb{P} \le 2\|M_{\boldsymbol{n}}\|_{\Phi_d} \|X\|_{\Phi_d}.$$

Recalling (5.11), we deduce that

$$\mathbb{E}[\Phi_d(X)] \le 2 \cdot 3^{\frac{d+5}{2}} \|M_{\boldsymbol{n}}\|_{\Phi_d} \|X\|_{\Phi_d}.$$

Thus, recalling that $\|X\|_{\Phi_d} \le 1$ and applying Lemma 3.6, we obtain

$$\varphi_d\big(\|X\|_{\Phi_d}\big) \le 2 \cdot 3^{\frac{d+5}{2}} \|M_{\boldsymbol{n}}\|_{\Phi_d}, \tag{5.12}$$

with $\varphi_d(x) = x\big(\log(1+x)\big)^{d-1}$ for all $x \ge 0$. Keeping in mind the inequalities (5.9), (5.10) and (5.12), we obtain

$$\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}})] \le \log(2)^{1-d} \varphi_d^{-1}\Big(2 \cdot 3^{\frac{d+5}{2}} \|M_{\boldsymbol{n}}\|_{\Phi_d}\Big). \tag{5.13}$$

Now, suppose that $\|X\|_{\Phi_d} > 1$. According to (5.8), we have

$$\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}\left[\Phi_d\left(\frac{d_{\boldsymbol{u}}}{\|X\|_{\Phi_d}}\right)\right] \le 1.$$

For any $\boldsymbol{0} \le \boldsymbol{u} \le \boldsymbol{n}-\boldsymbol{1}$ and by making use of inequality (5.2), it holds that

$$\begin{aligned}\mathbb{E}\left[\Phi_d\left(\frac{d_{\boldsymbol{u}}}{\|X\|_{\Phi_d}}\right)\right] &= \mathbb{E}\left[\frac{d_{\boldsymbol{u}}^2}{\|X\|_{\Phi_d}^2}\left(\log\left(1+\frac{d_{\boldsymbol{u}}}{\|X\|_{\Phi_d}}\right)\right)^{d-1}\right] \\ &\ge \mathbb{E}\left[\frac{\log(2)^{d-1}\Phi_d(d_{\boldsymbol{u}})}{\|X\|_{\Phi_d}^{d+1}\big(\log\big(1+\|X\|_{\Phi_d}\big)\big)^{d-1}}\right].\end{aligned}$$

We conclude that

$$\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}})] \le \log(2)^{1-d} \|X\|_{\Phi_d}^{d+1}\Big(\log\big(1+\|X\|_{\Phi_d}\big)\Big)^{d-1} =: \log(2)^{1-d}\phi_d\big(\|X\|_{\Phi_d}\big). \tag{5.14}$$

where $\phi_d(x) = x^{d+1}\big(\log(1+x)\big)^{d-1}$, for all $x \ge 0$. Once again, by the same argument as in the first case, we get

$$\mathbb{E}[\Phi_d(X)] \le 2 \cdot 3^{\frac{d+5}{2}} \|M_{\boldsymbol{n}}\|_{\Phi_d} \Big\| X\big(\log(1+X)\big)^{d-1} \Big\|_{\Psi_d}.$$

However using Lemma 5.1, there exists $C_{d,\epsilon} > 0$ such that

$$\Big\| X\big(\log(1+X)\big)^{d-1} \Big\|_{\Psi_d} \le C_{d,\epsilon} \|X\|_{\Phi_d}^{1+\epsilon},$$

Since log is an increasing function and $\|X\|_{\Phi_d} > 1$, we deduce that

$$1 = \mathbb{E}\left[\Phi_d\left(\frac{X}{\|X\|_{\Phi_d}}\right)\right] \le \frac{\mathbb{E}[\Phi_d(X)]}{\|X\|_{\Phi_d}^2}$$

and so

$$\|X\|_{\Phi_d}^2 \le \mathbb{E}[\Phi_d(X)] \le 2 \cdot 3^{\frac{d+5}{2}} C_{d,\epsilon} \|M_{\boldsymbol{n}}\|_{\Phi_d} \|X\|_{\Phi_d}^{1+\epsilon}.$$

Thus

$$\|X\|_{\Phi_d}^{1-\epsilon} \le 2 \cdot 3^{\frac{d+5}{2}} C_{d,\epsilon} \|M_{\boldsymbol{n}}\|_{\Phi_d}. \tag{5.15}$$

Combining (5.14) and (5.15), we get the following inequality

$$\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}})] \le \log(2)^{1-d} \phi_d \circ f_\epsilon\left(2 \cdot 3^{\frac{d+5}{2}} C_{d,\epsilon} \|M_{\boldsymbol{n}}\|_{\Phi_d}\right). \tag{5.16}$$

where $f_\epsilon(x) = x^{\frac{1}{1-\epsilon}}$ for all $x \ge 0$. Finally, recalling (5.13) and (5.16), there exists $C_1, C_2 > 1$ only depending on d such that

$$\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}})] \le C_1 \max\left\{\varphi_d^{-1}\left(C_2\|M_{\boldsymbol{n}}\|_{\Phi_d}\right), \phi_d \circ f_\epsilon\left(C_2\|M_{\boldsymbol{n}}\|_{\Phi_d}\right)\right\}.$$

Second step: Now suppose that M can take negative values. We let

$$M_{\boldsymbol{u}}^+ = \mathbb{E}\big[\max(M_{\boldsymbol{n}}, 0)|\mathcal{G}_{\boldsymbol{u}}\big] \quad \text{and} \quad M_{\boldsymbol{u}}^- = \mathbb{E}\big[\max(-M_{\boldsymbol{n}}, 0)|\mathcal{G}_{\boldsymbol{u}}\big],$$

with $\boldsymbol{0} \le \boldsymbol{u} \le \boldsymbol{n}$ and $\mathcal{G}_{\boldsymbol{u}} = \sigma\big(M_{\boldsymbol{v}}, \boldsymbol{v} \le \boldsymbol{u}\big)$. Both $M_{\boldsymbol{u}}^+$ and $M_{\boldsymbol{u}}^-$ are ortho-martingales and satisfy the conditions of the first part. Let $\boldsymbol{n} \in (\mathbb{N}^*)^d$, we define

$$M_{\boldsymbol{n}}^+ := \sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} d_{\boldsymbol{u}}^+, \quad M_{\boldsymbol{n}}^- := \sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} d_{\boldsymbol{u}}^-, \quad \sigma_{\boldsymbol{n}}^+ = \sqrt{\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} (d_{\boldsymbol{u}}^+)^2} \quad \text{and} \quad \sigma_{\boldsymbol{n}}^- = \sqrt{\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} (d_{\boldsymbol{u}}^-)^2}.$$

Therefore there exists $C_1, C_2 > 1$ only depending on d such that

$$\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}}^+)] \le C_1 \max\left\{\varphi_d^{-1}\left(C_2\|M_{\boldsymbol{n}}^+\|_{\Phi_d}\right), \phi_d \circ f_\epsilon\left(C_2\|M_{\boldsymbol{n}}^+\|_{\Phi_d}\right)\right\}$$

and

$$\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}}^-)] \le C_1 \max\left\{\varphi_d^{-1}\left(C_2\|M_{\boldsymbol{n}}^-\|_{\Phi_d}\right), \phi_d \circ f_\epsilon\left(C_2\|M_{\boldsymbol{n}}^-\|_{\Phi_d}\right)\right\}.$$

Using the inequalities $\Phi_d(a+b) \le 2^{d+1}(\Phi_d(a) + \Phi_d(b))$ for all $a, b \ge 0$, we obtain

$$\begin{aligned}
\sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(|d_{\boldsymbol{u}}|)] &\le 2^{d+1} \sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}}^+)] + 2^{d+1} \sum_{\boldsymbol{u}=\boldsymbol{0}}^{\boldsymbol{n}-\boldsymbol{1}} \mathbb{E}[\Phi_d(d_{\boldsymbol{u}}^-)] \\
&\le 2^{d+1}\Big(C_1 \max\left\{\varphi_d^{-1}\left(C_2\|M_{\boldsymbol{n}}^+\|_{\Phi_d}\right), \phi_d \circ f_\epsilon\left(C_2\|M_{\boldsymbol{n}}^+\|_{\Phi_d}\right)\right\} \\
&\qquad + C_1 \max\left\{\varphi_d^{-1}\left(C_2\|M_{\boldsymbol{n}}^-\|_{\Phi_d}\right), \phi_d \circ f_\epsilon\left(C_2\|M_{\boldsymbol{n}}^-\|_{\Phi_d}\right)\right\}\Big) \\
&\le 2^{d+2} C_1 \max\left\{\varphi_d^{-1}\left(C_2\|M_{\boldsymbol{n}}\|_{\Phi_d}\right), \phi_d \circ f_\epsilon\left(C_2\|M_{\boldsymbol{n}}\|_{\Phi_d}\right)\right\}.
\end{aligned}$$

The proof of the theorem is then complete. □

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