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Limiting Distributions of Largest Entries of Sample Covariance Matrices from 1-Dependent Normal Populations

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Abstract. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be a random sample coming from a p-dimension 1-dependent Gaussian population. Assume the adjacent entries of the population distribution have a common correlation coefficient ρ_n with $|\rho_n| < 1/2$. We derive that the limiting distribution of the largest off-diagonal entry of the sample covariance matrices is a Gumbel distribution in the ultra-high-dimensional setting where both n and p tend to infinity with $\log p = o\left(n^{1/3}\right)$. And the law of large numbers can be obviously obtained from the limiting distribution. The proofs are completed by using the Chen–Stein Poisson approximation method and the moderation deviation principle.

1. Introduction and Main Results

The rapid progress in computing science and technology has propelled random matrix theory into a pivotal role, offering statistical foundations for high-dimensional data processing. In such applications, the dimension p often far exceeds the sample size n. Consequently, classical multivariate statistical methods, assuming a fixed dimension p become impractical. This evolving landscape necessitates the development of new technical tools and statistical procedures to meet the demands of contemporary data analysis.

This paper is inspired by the result of Fan and Jiang (2019), focusing on the limiting distribution of the maximum off-diagonal entry of sample covariance matrices from the equi-correlated normal population. Consider a random sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ from the p-dimensional population $\boldsymbol{X}$ with

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mean $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$ and correlation coefficient matrix $\boldsymbol{R}$. Let $\mathcal{M}_{n,p} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)' = (X_{k,i})_{1\le k\le n, 1\le i\le p}$ be an $n \times p$ random matrix. Our main objects of interest in the present paper are the following four statistics:

$$\begin{aligned} J_n &= \max_{1\le i<j\le p} \frac{1}{n}\sum_{k=1}^{n} X_{k,i}X_{k,j}, \\ J_n' &= \max_{1\le i<j\le p} \left|\frac{1}{n}\sum_{k=1}^{n} X_{k,i}X_{k,j}\right|, \\ J_{n1} &= \max_{1\le i<j\le p, i<j-1} \frac{1}{n}\sum_{k=1}^{n} X_{k,i}X_{k,j}, \\ J_{n2} &= \max_{1\le i<j\le p, i=j-1} \frac{1}{n}\sum_{k=1}^{n} X_{k,i}X_{k,j}. \end{aligned} \tag{1.1}$$

The first statistic is the largest magnitude of off-diagonal entries of normalized sample covariance matrices when $\boldsymbol{\mu} = \mathbf{0}$. Previous researches have shown that J_n plays a vital role in the multivariate statistical analysis. Jiang (2004) is the first to get the following asymptotic distribution of J_n' under the assumption that p elements in $\boldsymbol{X}$ are independent and identically distributed (i.i.d.).

Lemma 1.1 (Lemma 3.2 of Jiang, 2004)**.** *Suppose that $E|X_{1,1}|^{30+\varepsilon} < \infty$ for some $\varepsilon > 0$. If $n/p \to \gamma$, then*

$$P\left(nJ_n'^2 - 4\log n + \log\log n \le y\right) \to e^{-Le^{-y/2}} \tag{1.2}$$

as $n \to \infty$ for any $y \in \mathbb{R}$, where $L = \left(4\gamma^2\sqrt{2\pi}\right)^{-1}$.

The limiting distribution appearing in (1.2) is called Gumbel distribution. Besides, some strong limit theorems for J_n' were shown by Li and Rosalsky (2006) when n/p is bounded away from 0 and ∞. Li et al. (2010) discovered the similar result to Jiang (2004) under the more relaxed assumption. Lytova (2018) and Tieplova (2017) studied the limiting behavior of the sample covariance matrices constructed by the random tensor data. Jiang and Xie (2020) further considered the limiting distribution of the largest off-diagonal entry of the hypercubic random tensor in the high-dimension case and the ultra-high-dimension case. Xiao and Wu (2013) obtained the asymptotic distribution of maximum deviations of sample covariance matrices.

All of these works assumed that the p components in $\boldsymbol{X}$ are independent and identically distributed (i.i.d.). However, Cai and Jiang (2011) generalized the problem from the independent case to the dependent case under the assumption that $\log p = o\left(n^{1/3}\right)$. Because of the complexity of dependent case, they considered a new statistic

$$V_{n,\tau} = \max_{1\le i<j\le p, j-i\ge\tau} \frac{1}{n}\left|\sum_{k=1}^{n} X_{k,i}X_{k,j}\right|,$$

where $\tau \ge 1$ is a constant. Then Cai and Jiang (2011) proved that $V_{n,\tau}$ asymptotically obeys a Gumbel distribution under the (τ −1)-dependent normal assumption. Furthermore, Fan and Jiang (2019) assumed that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ is a random sample from the p-dimensional equi-correlation normal population, that is, the entries of the population distribution have a common correlation coefficient $\rho_n > 0$. By using the Chen–Stein Poisson approximation method, they showed the limiting distribution of J_n as follows.

Lemma 1.2 (Theorem 2.1 of Fan and Jiang, 2019)**.** *Let $\rho_n \geq 0$ for each $n \geq 1$ and $\sup_{n\geq 1} \rho_n < 1/2$. If $\boldsymbol{\mu} = \mathbf{0}$, $p = p_n \to \infty$ and $\log p = o\left(n^{1/3}\right)$ as $n \to \infty$, then*

$$\begin{cases} 4\sqrt{\log p}\left(\sqrt{n}J_n - \mu_0\right) \xrightarrow{d} \xi, & \rho_n\sqrt{\log p} \to 0, \\ \frac{\sqrt{n}J_n - \mu_0}{\sqrt{2}\rho_n} \xrightarrow{d} \phi + \lambda_0\xi, & \rho_n\sqrt{\log p} \to \lambda_1 \in (0,\infty), \\ \frac{\sqrt{n}J_n - \mu_0}{\sqrt{2}\rho_n} \xrightarrow{d} N(0,1), & \rho_n\sqrt{\log p} \to \infty, \end{cases}$$

where $\phi \sim N(0,1)$, $\lambda_0 = \frac{1}{4\sqrt{2}\lambda_1}$, $\mu_0 = \sqrt{n}\rho_n + \left(2\sqrt{\log p} - \frac{\log\log p}{4\sqrt{\log p}}\right)\sqrt{1-\rho_n^2}$ and the distribution function of ξ is $F(x) = e^{-Ke^{-x/2}}$, $x \in \mathbb{R}$ with $K = \frac{1}{4\sqrt{2\pi}}$.

Compared with Cai and Jiang (2011), the present paper does not delve into the limiting properties of $V_{n,\tau}$. Instead, we focus on a more general statistic J_n. On the application side, it has been demonstrated that the largest entry of sample covariance matrices J_n performs effectively in testing the covariance structure of a high-dimensional random variable. In particular, the equi-correlation structure, as presented in Fan and Jiang (2019), is deemed too restrictive for several applications. In practical applications, it is often observed that the correlation between random variables diminishes as the distance between them increases. Therefore, this paper considers a 1-dependent structure, which is a weaker and more concise dependence model with broader applicability. The significance of this work lies in its ability to test not only the independence of a high-dimensional random variable but also whether the corresponding covariance matrix $\boldsymbol{\Sigma}$ exhibits a tridiagonal structure.

In this paper, we will consider the ultra-high-dimensional case where $\log p = o\left(n^{1/3}\right)$ and assume that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are a random sample from the 1-dependent normal population, that is, two adjacent elements in $\boldsymbol{X}$ are dependent. Let $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ stands for a p-variate normal population with the banded correlation matrix $\boldsymbol{R} = (r_{ij})_{p\times p}$, that is,

$$r_{ij} = \begin{cases} 1, & i = j, \\ \rho_n, & |i-j| = 1, \\ 0, & |i-j| > 1. \end{cases} \tag{1.3}$$

Note that the corresponding correlation matrix $\boldsymbol{R}$ has the tridiagonal structure. By Horn and Johnson (2013), Theorem 6.1.10, we find that $\boldsymbol{R}$ is positive definite if $|\rho_n| < 1/2$.

In addition, Fan et al. (2018) derived the limiting distribution of the maximum spurious correlation using Gaussian approximation techniques of Chernozhukov et al. (2013). Bai et al. (2007) and Johnstone (2001) investigated the asymptotic behaviors of the largest eigenvectors and eigenvalues of sample covariance matrices, respectively. And the limiting behavior of the largest magnitude of off-diagonal entries of the sample correlation matrices has been studied by several authors in various cases, including Cai and Jiang (2012), Li et al. (2012), Liu et al. (2008), Shao and Zhou (2014), Zhou (2007), and Zhao and Zhang (2022).

To investigate the distribution of J_n, we need the following assumptions.

Assumption 1.3. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be a random sample from the population $N_p(\mathbf{0}, \boldsymbol{R})$. The data matrix is given by $\mathcal{M}_{n,p} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)' = (X_{k,i})_{n\times p}$.

Assumption 1.4. $p = p_n \to \infty$ with $\log p = o\left(n^{1/3}\right)$ as $n \to \infty$.

We first show the limiting distributions of J_{n1} and J_{n2}.

Theorem 1.5. *Under Assumptions 1.3 and 1.4, suppose* $\sup_{n\geq 1}|\rho_n| < 1/2$. *Then*

$$4\sqrt{\log p}\left(\sqrt{n}J_{n1} - \mu_1\right) \xrightarrow{d} \xi \quad \text{and} \quad 2\sqrt{2\log p}\left(\frac{\sqrt{n}}{\sqrt{1+\rho_n^2}}J_{n2} - \mu_2\right) \xrightarrow{d} \xi$$

as $n \to \infty$, *where* $\mu_1 = 2\sqrt{\log p} - \frac{\log\,\log p}{4\sqrt{\log p}}$, $\mu_2 = \sqrt{2\log p} - \frac{\log\,\log p}{2\sqrt{2\log p}} + \frac{\log 8}{2\sqrt{2\log p}} + \frac{\rho_n\sqrt{n}}{\sqrt{1+\rho_n^2}}$, *and the distribution function of* ξ *is* $F_\xi(x) = e^{-Ke^{-x/2}}$, $x \in \mathbb{R}$ *with* $K = \frac{1}{4\sqrt{2\pi}}$.

The above theorem implies immediately the following results.

Corollary 1.6. *Under Assumptions 1.3 and 1.4, suppose* $\sup_{n\geq 1}|\rho_n| < 1/2$. *Then*

$$\sqrt{\frac{n}{\log p}}J_{n1} \xrightarrow{P} 2 \quad \text{and} \quad \sqrt{\frac{n}{(1+\rho_n^2)\log p}}\,(J_{n2}-\rho_n) \xrightarrow{P} \sqrt{2} \qquad \text{as } n\to\infty.$$

Corollary 1.7. *Under Assumptions 1.3 and 1.4, suppose* $\rho_n = \rho$ *is fixed with* $|\rho| < 1/2$. *Then*

$$4\sqrt{\log p}\left(\sqrt{n}J_{n1} - 2\sqrt{\log p} + \frac{\log\,\log p}{4\sqrt{\log p}}\right) \xrightarrow{d} \xi,$$

$$2\sqrt{2\log p}\left(\frac{\sqrt{n}}{\sqrt{1+\rho^2}}J_{n2} - \frac{\rho\sqrt{n}}{\sqrt{1+\rho^2}} - \sqrt{2\log p} + \frac{\log\,\log p}{2\sqrt{2\log p}} - \frac{\log 8}{2\sqrt{2\log p}}\right) \xrightarrow{d} \xi$$

as $n \to \infty$, *where* ξ *is given by Theorem 1.5.*

Then, we obtain the limiting distribution of J_n as follows.

Theorem 1.8. *Under Assumptions 1.3 and 1.4, suppose* $\sup_{n\geq 1}|\rho_n| < 1/2$. *If* $\lim_{n\to\infty}\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} = \lambda \in [-\infty,\infty]$, *then the following holds as* $n\to\infty$:

$$\begin{cases} 4\sqrt{\log p}\,(\sqrt{n}J_n - \mu_1) \xrightarrow{d} \xi, & \lambda \in \left[-\infty, 2-\sqrt{2}\right], \\ 2\sqrt{2\log p}\left(\dfrac{\sqrt{n}}{\sqrt{1+\rho_n^2}}J_n - \mu_2\right) \xrightarrow{d} \xi, & \lambda \in \left(2-\sqrt{2}, \infty\right], \end{cases}$$

where μ_1, μ_2 *and* ξ *are given by Theorem 1.5.*

There are two obvious consequences of Theorem 1.8 as follows.

Corollary 1.9. *Under Assumptions 1.3 and 1.4, suppose* $\sup_{n\geq 1}|\rho_n| < 1/2$. *If* $\lim_{n\to\infty}\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} = \lambda \in [-\infty,\infty]$, *then the following holds as* $n\to\infty$:

$$\begin{cases} \sqrt{\dfrac{n}{\log p}}J_n \xrightarrow{P} 2, & \lambda \in \left[-\infty, 2-\sqrt{2}\right], \\ \sqrt{\dfrac{n}{(1+\rho_n^2)\log p}}\,(J_n-\rho_n) \xrightarrow{P} \sqrt{2}, & \lambda \in \left(2-\sqrt{2}, \infty\right]. \end{cases}$$

Corollary 1.10. *Under Assumptions 1.3 and 1.4, suppose* $\rho_n = \rho$ *is fixed with* $|\rho| < 1/2$. *Then the following holds as* $n\to\infty$:

(i) If $\rho \in (-1/2, 0]$, *then*

$$4\sqrt{\log p}\left(\sqrt{n}J_n - 2\sqrt{\log p} + \frac{\log\,\log p}{4\sqrt{\log p}}\right) \xrightarrow{d} \xi,$$

where ξ *is given by Theorem 1.5.*

(ii) If $\rho \in (0, 1/2)$, *then*

$$2\sqrt{2\log p}\left(\frac{\sqrt{n}}{\sqrt{1+\rho^2}}J_n - \frac{\rho\sqrt{n}}{\sqrt{1+\rho^2}} - \sqrt{2\log p} + \frac{\log\,\log p}{2\sqrt{2\log p}} - \frac{\log 8}{2\sqrt{2\log p}}\right) \xrightarrow{d} \xi,$$

where ξ *is given by Theorem 1.5.*

Theorems 1.5 and 1.8 indicate that the limiting distributions of J_{n1}, J_{n2} and J_n are Gumbel distribution. Corollaries 1.6 and 1.9 show the laws of large numbers for J_{n1}, J_{n2} and J_n.

Next, we make a few remarks as follows.

Remark 1.11. For the maximum statistic J_n' as in (1.1), Theorem 1.8 still hold with the limiting distribution $F_\xi(x)$ is replaced by

$$G(x) = \exp\left(-\frac{1}{2\sqrt{2\pi}}e^{-x/2}\right), \quad x \in \mathbb{R}.$$

In fact, we only change $N(0,1)$ to $|N(0,1)|$ in the proof of Theorem 1.8.

Remark 1.12. Cai and Jiang (2011) supposed $\boldsymbol{X}$ comes from $(\tau-1)$-dependent normal populations and showed the asymptotic distribution of $V_{n,\tau}$, where $\tau \geq 1$ is a constant. Obviously, under their assumption, $V_{n,\tau}$ is the maximum of $(p-\tau+1)(p-\tau)/2$ random variables, each of which is the sum of the product of two i.i.d. random variables. In this paper, we relax the condition of $V_{n,\tau}$ and obtain the limiting distribution of J_n under 1-dependent normal populations assumption. Notice that under our assumption, J_n is the maximum of $p(p-1)/2$ random variables, where $(p-1)(p-2)/2$ random variables are the sum of the product of two i.i.d. random variables, and $(p-1)$ random variables are the sum of the product of two dependent random variables. Therefore, our study is more complex and challenging than that of Cai and Jiang (2011). In addition, when $\tau = 2$, Cai and Jiang (2011) obtained the asymptotic behavior of $V_{n,2}$. According to Remark 1.11, it can be inferred that the limiting distribution of J_{n1} is similar to the limiting distribution of $V_{n,2}$. Therefore, we will omit the proof of the limiting distribution of J_{n1} in this paper.

Remark 1.13. Assume random variables $\{\xi_{k,i}; k=1,\dots,n, i=0,1,\dots,p\}$ are i.i.d. as $N(0,1)$. Set

$$\boldsymbol{X}_k = (X_{k,1},\dots,X_{k,p})' = \sqrt{\theta}\,(\xi_0,\dots,\xi_{p-1})' + \sqrt{1-\theta}\,(\xi_1,\dots,\xi_p)' \tag{1.4}$$

for each k. By some calculations, we find that $EX_{k,i}=0$, $\operatorname{Var}(X_{k,i})=1$ and

$$\operatorname{Cov}(X_{k,i},X_{k,j}) = \begin{cases} 0, & |i-j|>1, \\ \sqrt{\theta(1-\theta)}, & |i-j|=1, \end{cases}$$

for $1 \leq i,j \leq p$. And the decomposition structure (1.4) plays a vital role in the proofs of theorems.

Remark 1.14. For the $(\tau-1)$-dependent case $(\tau \geq 3)$, we find that it is difficult to obtain the limiting distribution of J_n using the method presented in this paper. In fact, the proofs of our results mainly rely on the decomposition structure as in Remark 1.13. For the $(\tau-1)$-dependent case, we consider the simplest covariance structure and assume $\boldsymbol{X} \sim N_p(\boldsymbol{0}, \boldsymbol{R}')$, where $\boldsymbol{R}' = (r_{ij}')_{p\times p}$ is a banded matrix defined as follows:

$$r_{ij}' = \begin{cases} 1, & \text{if } i=j, \\ \rho_n, & \text{if } |i-j| \leq \tau-1, \\ 0, & \text{if } |i-j| > \tau-1. \end{cases}$$

However, it is challenging to determine a decomposition structure for the $(\tau-1)$-dependent normal population. Furthermore, it is possible to observe multiple phase transition phenomena when studying the asymptotic behavior of J_n, but we cannot currently determine how these phase transition phenomena occur by the method presented in this paper. We will leave this as future work. If we do not make specific assumptions about the covariance structure of the population and consider a more general dependent structure, we find that the method presented in this paper is no longer applicable. We need to seek new theories and methods.

Remark 1.15. In the proofs, Fan and Jiang (2019) used the following decomposition structure, for each $k=1,\dots,n$,

$$\boldsymbol{X}_k = (X_{k,1},\dots,X_{k,p})' = \sqrt{\rho_n}(\xi,\dots,\xi)' + \sqrt{1-\rho_n}\,(\xi_{k,1},\dots,\xi_{k,p})',$$

where $\{\xi, \xi_{k,i}; k, i = 1, 2, \dots\}$ are i.i.d. as $N(0,1)$. Compared to our decomposition structure as in (1.4), where both terms are related to the subscript i of $X_{k,i}$, the first term of their decomposition structure is fixed as ξ, and only the second term is related to the subscript i of $X_{k,i}$. Therefore, for $\max_{1\le i<j\le p}\sum_{k=1}^{n} X_{k,i}X_{k,j}$, our decomposition structure will lead to larger and more complex calculations. Another technical challenge lies in studying how the phase transition phenomenon occurs. Fan and Jiang (2019) found that if $\rho_n\sqrt{\log p}$ tends to 0, ∞ or a positive constant, the corresponding limiting distribution of J_n is the Gumbel distribution, the normal distribution and a convolution of the two distributions, respectively. In this paper, we present a different and interesting phase transition phenomenon. If $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} \to \lambda \le 2-\sqrt{2}$, then J_{n1} contributes to J_n. On the contrary, if $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} \to \lambda > 2-\sqrt{2}$, then J_{n2} contributes to J_n. It is obvious that there is a phase transition phenomenon at $\lambda = 2-\sqrt{2}$. Consequently, the threshold depends not only on the dimension p but also on the sample size n, and this phase transition phenomenon is different from that of Fan and Jiang (2019).

2. Proof of Main Results

The proofs of Theorems 1.5 and 1.8 are quite complicated. We break them into Sections 2.1–2.4. If $0 \le \theta_n < 1/2$ and $0 \le |\rho_n| < 1/2$, there is a one-to-one correspondence between $\sqrt{\theta_n(1-\theta_n)}$ and $|\rho_n|$. So we will substitute $\sqrt{\theta_n(1-\theta_n)}$ for $|\rho_n|$ in the proofs of theorems.

2.1. *Some Notation.* The random variables

$$\{\xi_k, \xi_{k,i}; k = 1, 2, \dots, i = 0, 1, 2, \dots\} \text{ are i.i.d. as } N(0,1). \tag{2.1}$$

Given $\theta_n \in [0, 1/2)$ for each $n \ge 1$, set

$$a_n = \theta_n,\ b_n = 1-\theta_n,\ c_n = \sqrt{\theta_n(1-\theta_n)},$$

and

$$a_n' = \frac{\theta_n}{\sqrt{1+\theta_n-\theta_n^2}},\ b_n' = \frac{1-\theta_n}{\sqrt{1+\theta_n-\theta_n^2}},\ c_n' = \frac{\sqrt{\theta_n(1-\theta_n)}}{\sqrt{1+\theta_n-\theta_n^2}}.$$

For $x \in \mathbb{R}$ and integer $p \ge 1$, set

$$s_p = \sqrt{4\log p - \log\log p + x}$$

and

$$s_p' = \sqrt{2\log p - \log\log p + \log 8 + x}.$$

Define

$$\eta_{kij} = a_n\xi_{k,i-1}\xi_{k,j-1} + b_n\xi_{k,i}\xi_{k,j} + \operatorname{sgn}(\rho_n)\cdot c_n(\xi_{k,i-1}\xi_{kj} + \xi_{k,i}\xi_{k,j-1}), \tag{2.2}$$

$$\eta_{kij}' = a_n'\xi_{k,i-1}\xi_{k,j-1} + b_n'\xi_{k,i}\xi_{k,j} + \operatorname{sgn}(\rho_n)\cdot c_n'(\xi_{k,i-1}\xi_{kj} + \xi_{k,i}\xi_{k,j-1} - 1), \tag{2.3}$$

$$M_{nij} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{kij} \quad \text{and} \quad M_{nij}' = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{kij}'$$

for all $1 \le i < j \le p$.

2.2. *Auxiliary Results.* In this section, we will show some general results which will be used to prove Theorems 1.5 and 1.8. According to Fan and Jiang (2019), Lemma 3.8, we can get the following lemma, and its proof will not be described in detail in this paper.

Lemma 2.1. *Let M_n be a random variable for each $n \geq 1$ and $g > 0$ be a constant satisfying*

$$\lim_{n\to\infty} P\left(M_n \leq \sqrt{g^2 \log p - \log\log p - h \cdot \log 8 + x}\right) = F(x)$$

for any $x \in \mathbb{R}$, where $h = \{0, 1\}$ and $F(x)$ is a continuous distribution function on $\mathbb{R}$. Then

$$M_n = g\sqrt{\log p} - \frac{\log\log p}{2g\sqrt{\log p}} - \frac{h \cdot \log 8}{2g\sqrt{\log p}} + \frac{1}{2g\sqrt{\log p}} U_n,$$

where U_n converges weakly to a probability measure with distribution function $F(x)$.

Proposition 2.2. *Set $M_n' = \max_{1\leq i<j\leq p, i<j-1} M_{nij}$. Under Assumption 1.4, suppose $\sup_{n\geq1}|\rho_n| < 1/2$. Then*

$$\lim_{n\to\infty} P\left(M_n' \leq s_p\right) = \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right)$$

for any $x \in \mathbb{R}$.

Proposition 2.3. *Set $M_n'' = \max_{1\leq i<j\leq p, i=j-1} M_{nij}'$. Under Assumption 1.4, suppose $\sup_{n\geq1}|\rho_n| < 1/2$. Then*

$$\lim_{n\to\infty} P\left(M_n'' \leq s_p'\right) = \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right)$$

for any $x \in \mathbb{R}$.

Proposition 2.4. *Set $M_n = \max_{1\leq i<j\leq p} M_{nij}$. Under Assumption 1.4, suppose $\sup_{n\geq1}|\rho_n| < 1/2$. If $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} \to \lambda \in \left[-\infty, 2-\sqrt{2}\right]$ as $n \to \infty$, then*

$$\lim_{n\to\infty} P\left(M_n \leq s_p\right) = \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right)$$

for any $x \in \mathbb{R}$.

Proposition 2.5. *Set $M_n = \max_{1\leq i<j\leq p} M_{nij}$. Under Assumption 1.4, suppose $\sup_{n\geq1}|\rho_n| < 1/2$. If $\frac{\rho_n\sqrt{n}}{\sqrt{\log p}} \to \lambda \in \left(2-\sqrt{2}, \infty\right]$ as $n \to \infty$, then*

$$\lim_{n\to\infty} P\left(M_n \leq c_n\sqrt{n} + \sqrt{1+\theta_n-\theta_n^2}\, s_p'\right) = \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right)$$

for any $x \in \mathbb{R}$.

2.3. *Proof of Theorem 1.5.* In this section, we will prove Theorem 1.5.

Proof: Review (2.1). Write

$$x_{k,i} = \sqrt{\theta_n}\xi_{k,i-1} + \operatorname{sgn}(\rho_n) \cdot \sqrt{1-\theta_n}\xi_{k,i}, \quad 1 \leq k \leq n,\ 1 \leq i \leq p. \tag{2.4}$$

It is obvious that the n rows of the matrix $(x_{k,i})_{n\times p}$ are i.i.d. random vectors and $x_{1,i} \sim N(0,1)$ for each $1 \leq i \leq p$. In addition, for $1 \leq i \neq j \leq p$, we have

$$\operatorname{Cov}\left(x_{1,i}, x_{1,j}\right) = \begin{cases} 0, & |i-j| > 1, \\ \operatorname{sgn}(\rho_n)\sqrt{\theta_n\left(1-\theta_n\right)}, & |i-j| = 1, \end{cases}$$

where $\mathrm{sgn}(\rho_n)\sqrt{\theta_n(1-\theta_n)} = \rho_n$. That is, each row of the matrix $(x_{k,i})_{n\times p}$ obeys $N_p(\mathbf{0}, \boldsymbol{R})$, where $\boldsymbol{R}$ is given by (1.3). As a result, $\mathcal{M}_{n,p}$ and $(x_{k,i})_{n\times p}$ have the same distribution. So we assume $\mathcal{M}_{n,p} = (x_{k,i})_{n\times p}$ in the proofs. Then it follows from (2.2) and (2.4) that

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} x_{ki}x_{kj} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{kij} = M_{nij}. \tag{2.5}$$

Then, by Lemma 2.1, Propositions 2.2 and 2.3, we obtain that

$$\begin{aligned} M_n' &= 2\sqrt{\log p} - \frac{\log\log p}{4\sqrt{\log p}} + \frac{1}{4\sqrt{\log p}}U_{n1}, \\ M_n'' &= \sqrt{2\log p} - \frac{\log\log p}{2\sqrt{2\log p}} + \frac{\log 8}{2\sqrt{2\log p}} + \frac{1}{2\sqrt{2\log p}}U_{n2}, \end{aligned}$$

where $U_{n1} \xrightarrow{d} \xi$ and $U_{n2} \xrightarrow{d} \xi$ with distribution function $F_\xi(x) = e^{-\frac{1}{4\sqrt{2\pi}}e^{-x/2}}$ for all $x \in \mathbb{R}$. Notice

$$\sqrt{n}J_{n1} = \max_{1\le i<j\le p, i<j-1}\frac{1}{\sqrt{n}}\sum_{k=1}^{n} x_{ki}x_{kj} = \max_{1\le i<j\le p, i<j-1}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{kij} = M_n',$$

$$\begin{aligned} \frac{\sqrt{n}}{\sqrt{1+\theta_n-\theta_n^2}}J_{n2} &= \max_{1\le i<j\le p, i=j-1}\frac{1}{\sqrt{n+n\theta_n-n\theta_n^2}}\sum_{k=1}^{n} x_{ki}x_{kj} \\ &= M_n'' + \frac{\mathrm{sgn}(\rho_n)\sqrt{n\theta_n(1-\theta_n)}}{\sqrt{1+\theta_n-\theta_n^2}}. \end{aligned}$$

Set

$$\begin{aligned} \mu_1 &= 2\sqrt{\log p} - \frac{\log\ \log p}{4\sqrt{\log p}}, \\ \mu_2 &= \frac{\mathrm{sgn}(\rho_n)\sqrt{n\theta_n(1-\theta_n)}}{\sqrt{1+\theta_n-\theta_n^2}} + \sqrt{2\log p} - \frac{\log\ \log p}{2\sqrt{2\log p}} + \frac{\log 8}{2\sqrt{2\log p}} \\ &= \frac{\rho_n\sqrt{n}}{\sqrt{1+\rho_n^2}} + \sqrt{2\log p} - \frac{\log\ \log p}{2\sqrt{2\log p}} + \frac{\log 8}{2\sqrt{2\log p}}. \end{aligned}$$

Then, we get the following conclusions

$$\begin{aligned} 4\sqrt{\log p}\left(\sqrt{n}J_{n1} - \mu_1\right) &= U_{n1} \xrightarrow{d} \xi, \\ 2\sqrt{2\log p}\left(\frac{\sqrt{n}}{\sqrt{1+\rho_n^2}}J_{n2} - \mu_2\right) &= U_{n2} \xrightarrow{d} \xi, \end{aligned}$$

where ξ is defined as above. These prove the theorem. □

2.4. *Proof of Theorem 1.8.* In this section, we will prove Theorem 1.8.

Proof: We continue to use the notation in the proof of Theorem 1.5. By Lemma 2.1, Propositions 2.4 and 2.5, we have the results as follows.

Case (i): if $\lambda \in \left[-\infty, 2-\sqrt{2}\right]$, then

$$M_n = 2\sqrt{\log p} - \frac{\log\log p}{4\sqrt{\log p}} + \frac{1}{4\sqrt{\log p}}U_{n3},$$

where $U_{n3} \xrightarrow{d} \xi$.

Case (ii): if $\lambda \in \left(2-\sqrt{2}, \infty\right]$, then

$$\frac{M_n}{\sqrt{1+\rho_n^2}} = \frac{\rho_n \sqrt{n}}{\sqrt{1+\rho_n^2}} + \sqrt{2\log p} - \frac{\log\log p}{2\sqrt{2\log p}} + \frac{\log 8}{2\sqrt{2\log p}} + \frac{1}{2\sqrt{2\log p}} U_{n4},$$

where $U_{n4} \xrightarrow{d} \xi$.

Recalling the definitions of μ_1, μ_2, M_n and J_n, we obtain

$$\sqrt{n} J_n = \max_{1\le i<j\le p} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} x_{ki} x_{kj} = \max_{1\le i<j\le p} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \eta_{kij} = M_n.$$

By the above relation, we then arrive at

Case (i): if $\lambda \in \left[-\infty, 2-\sqrt{2}\right]$, then

$$4\sqrt{\log p}\left(\sqrt{n} J_n - \mu_1\right) \xrightarrow{d} \xi,$$

where ξ is defined as above.

Case (ii): if $\lambda \in \left(2-\sqrt{2}, \infty\right]$, then

$$4\sqrt{\log p}\left(\frac{\sqrt{n} J_n}{\sqrt{1+\rho_n^2}} - \mu_2\right) \xrightarrow{d} \xi,$$

where ξ is defined as above. □

3. Proof of Auxiliary Results

3.1. *Some Technical Tools.* To prove Propositions 2.2–2.5, we need some preliminary lemmas. The first is the Chen–Stein Poisson approximation method.

Lemma 3.1 (Theorem 1 of Arratia et al., 1989)**.** *Let $\{\eta_\alpha; \alpha \in I\}$ be random variables on an index set I and $\{B_\alpha; \alpha \in I\}$ be a set of subsets of I, that is, for each $\alpha \in I, B_\alpha \subset I$. For any $t \in \mathbb{R}$, set $\lambda = \sum_{\alpha\in I} P\left(\eta_\alpha > t\right)$. Then we have*

$$\left| P\left(\max_{\alpha\in I} \eta_\alpha \le t\right) - e^{-\lambda} \right| \le \left(1 \wedge \lambda^{-1}\right)\left(b_1 + b_2 + b_3\right),$$

where

$$b_1 = \sum_{\alpha\in I} \sum_{\beta\in B_\alpha} P\left(\eta_\alpha > t\right) P\left(\eta_\beta > t\right),$$

$$b_2 = \sum_{\alpha\in I} \sum_{\alpha\neq\beta\in B_\alpha} P\left(\eta_\alpha > t, \eta_\beta > t\right),$$

$$b_3 = \sum_{\alpha\in I} E|P\left\{\eta_\alpha > t \mid \sigma\left(\eta_\beta; \beta \notin B_\alpha\right)\right\} - P\left(\eta_\alpha > t\right)|,$$

and $\sigma\left(\eta_\beta; \beta \notin B_\alpha\right)$ is the σ-algebra generated by $\{\eta_\beta; \beta \notin B_\alpha\}$. In particular, if η_α is independent of $\{\eta_\beta; \beta \notin B_\alpha\}$ for each α, then b_3 vanishes.

The following lemma is about the moderation deviation of the partial sum of the independent but not necessarily identically distributed random variables.

Lemma 3.2 (Proposition 4.5 of Chen et al., 2013)**.** *Let $\{\eta_i; 1 \le i \le n\}$ be independent random variables with $E\eta_i = 0$ and $Ee^{h_n|\eta_i|} < \infty$ for some $h_n > 0$ and $1 \le i \le n$. Assume that $\sum_{i=1}^n E\eta_i^2 = 1$. Then*

$$P\left(\sum_{i=1}^{n}\eta_i \geq x\right) = [1-\Phi(x)]\cdot\left[1+C_n\left(1+x^3\right)\gamma e^{4x^3\gamma}\right]$$

for all $0 \leq x \leq h_n$ *and* $\gamma = \sum_{i=1}^{n} E\left(|\eta_i|^3 e^{x|\eta_i|}\right)$, *where* $\sup_{n\geq 1}|C_n| \leq C$ *and* C *is an absolute constant.*

To complete the proofs by using the above lemma, we have to control $E\left(|\eta_i|^3 e^{x|\eta_i|}\right)$. By the similar argument to Lemma 3.6 of Fan and Jiang (2019), we obtain the following lemma.

Lemma 3.3. *Let* U, V, W *and* Y *be i.i.d.* $N(0,1)$ *random variables. Let* $\{a_i; i = 1,\dots,5\}$ *be real numbers. Set* $\eta = a_1UW + a_2VY + a_3UY + a_3VW + a_4WY + a_3W^2 - a_5$. *Then*

$$E\left(|\eta|^3 e^{x|\eta|}\right) \leq C\cdot e^{x|a_5|}\cdot\sum_{i=1}^{4}|a_i|^3$$

as $0 < x \leq \frac{1}{12(|a_1|+|a_2|+2|a_3|+|a_4|)}$, *where* C *is a constant not depending on* a_i.

3.2. *Proofs of Propositions 2.2 and 2.3*. The proof of Proposition 2.2 can be obtained from Cai and Jiang (2011), Proposition 6.4. So we only prove Proposition 2.3 in this section.

Lemma 3.4. *Under the conditions of Proposition 2.3, we have*

$$\lim_{n\to\infty}(p-1)P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k12} > s'_p\right) = \frac{1}{4\sqrt{2\pi}}e^{-x/2}$$

for all $x \in \mathbb{R}$.

Proof: Write

$$\sum_{k=1}^{n}\eta'_{k12} = \sum_{k=1}^{n}\left[a'_n\xi_{k0}\xi_{k1} + b'_n\xi_{k1}\xi_{k2} + \operatorname{sgn}(\rho_n)\cdot c'_n\left(\xi_{k0}\xi_{k2}+\xi_{k1}^2-1\right)\right].$$

Obviously,

$$E\left(\sum_{k=1}^{n}\eta'_{k12}\right) = 0 \quad \text{and} \quad \operatorname{Var}\left(\sum_{k=1}^{n}\eta'_{k12}\right) = n. \tag{3.1}$$

Then, set $a' = a'_n/\sqrt{n}$, $b' = b'_n/\sqrt{n}$ and $c' = c'_n/\sqrt{n}$. Define $\eta'_k = a'\xi_{k0}\xi_{k1} + b'\xi_{k1}\xi_{k2} + \operatorname{sgn}(\rho_n)\cdot c'\left(\xi_{k0}\xi_{k2}+\xi_{k1}^2-1\right)$. Then it follows from (3.1) that

$$E\left(\eta'_k\right) = 0 \quad \text{and} \quad \sum_{k=1}^{n}\operatorname{Var}\left(\eta'_k\right) = 1 \tag{3.2}$$

for each k. Furthermore, one has

$$|a'| \leq \frac{1}{2\sqrt{n}}\ ,\ |b'| \leq \frac{1}{\sqrt{n}} \text{ and } |c'| \leq \frac{1}{2\sqrt{n}}.$$

By the Hölder inequality and the fact that $2|\xi_{k1}\xi_{k2}| \leq \xi_{k1}^2 + \xi_{k2}^2$, we then arrive at

$$\begin{aligned}
&Ee^{h|\eta'_k|} \leq E\exp\left[h\left(|a'\xi_{k0}\xi_{k1}| + |b'\xi_{k1}\xi_{k2}| + |c'\xi_{k0}\xi_{k2}| + |c'\xi_{k1}^2| + |c'|\right)\right]\\
\leq& E\exp\left\{h\left[\left(\frac{|a'|}{2}+\frac{|c'|}{2}\right)\xi_{k0}^2 + \left(\frac{|a'|}{2}+\frac{|b'|}{2}+|c'|\right)\xi_{k1}^2 + \left(\frac{|b'|}{2}+\frac{|c'|}{2}\right)\xi_{k2}^2 + |c'|\right]\right\}\\
\leq& C\cdot E\exp\left[h\left(\frac{|a'|}{2}+\frac{|c'|}{2}\right)\xi_{k0}^2\right]E\exp\left[h\left(\frac{|a'|}{2}+\frac{|b'|}{2}+|c'|\right)\xi_{k1}^2\right]E\exp\left[h\left(\frac{|a'|}{2}+\frac{|c'|}{2}\right)\xi_{k2}^2\right]\\
<&\infty
\end{aligned} \tag{3.3}$$

for all h, k, n satisfying $0 < h < h'_n := \frac{2}{5}\sqrt{n}$ and $1 \le k \le n$. By Lemma 3.3, we have

$$\gamma := \sum_{k=1}^{n} E\left(|\eta'_k|^3 e^{s'_p|\eta'_k|}\right) \le \sum_{k=1}^{n} C \cdot \left(|a'|^3 + |b'|^3 + |c'|^3\right) \cdot e^{s'_p|c'|} \le \frac{C}{\sqrt{n}} \cdot e^{s'_p/\sqrt{n}}.$$

Note that $s'_p < h'_n = \frac{2}{5}\sqrt{n}$. Then, by (3.2), (3.3) and Lemma 3.2, we find

$$\begin{aligned} p \cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k12} > s'_p\right) &= P\left(\sum_{k=1}^{n}\eta'_k > s'_p\right) \\ &= p \cdot \left[1 - \Phi\left(s'_p\right)\right] \cdot \left[1 + O\left(1\right)\left(1 + s'^3_p\right)\gamma e^{4s'^3_p\gamma}\right] \\ &= p \cdot \left[1 - \Phi\left(s'_p\right)\right] \cdot \left[1 + O\left(\frac{e^{s'^3_p/\sqrt{n}}}{\sqrt{n}}\right)\right] \\ &= \frac{p}{\sqrt{2\pi}s'_p} e^{-s'^2_p/2}\left(1 + o\left(1\right)\right) \\ &\to \frac{1}{4\sqrt{2\pi}} e^{-x/2} \end{aligned} \tag{3.4}$$

as $n \to \infty$. In the above equality, we use the formula $P\left(N\left(0,1\right) \ge x\right) = \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}$ as $x \to \infty$, and the fact $s'^3_p\gamma = O\left(s'^3_p n^{-1/2} e^{s'_p/\sqrt{n}}\right) \to 0$ as $n \to \infty$ under Assumption 1.4. Then the proof is completed. □

The following lemma is an obvious consequence of Lemma 3.4.

Lemma 3.5. *Under the conditions of Proposition 2.3, we have*

$$\lim_{n\to\infty} p \cdot \left[P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k12} > s'_p\right)\right]^2 = 0.$$

Lemma 3.6. *Under the conditions of Proposition 2.3, we have*

$$\lim_{n\to\infty} p \cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k12} > s'_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k34} > s'_p\right) = 0.$$

Proof: Let P_1 stand for the conditional probability given $\{\xi_{k2}; 1 \le k \le n\}$. By independence,

$$P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k12} > s'_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k34} > s'_p\right) = E\left[P_1\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k12} > s'_p\right)^2\right]. \tag{3.5}$$

Then, given $\{\xi_{k2}; 1 \le k \le n\}$, we have from independence that

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta'_{k12} \sim N\left(0, \sigma^2_{1n}\right), \tag{3.6}$$

where

$$\sigma^2_{1n} = \frac{1}{n}\sum_{k=1}^{n}\left(a'^2_n + b'^2_n\xi^2_{k2} + c'^2_n\xi^2_{k2} + 2c'^2_n\right) = a'^2_n + 2c'^2_n + \frac{1}{n}\sum_{k=1}^{n}\left(b'^2_n + c'^2_n\right)\xi^2_{k2}.$$

Given $\delta \in (0,1)$, set

$$B_\delta = \left\{1 - \delta + \frac{a_n'^2 + 2c_n'^2}{b_n'^2 + c_n'^2} \le \frac{\sigma_{1n}^2}{b_n'^2 + c_n'^2} \le 1 + \delta + \frac{a_n'^2 + 2c_n'^2}{b_n'^2 + c_n'^2}\right\}$$
$$= \left\{1 - \delta \le \frac{1}{n}\sum_{k=1}^{n} \xi_{k2}^2 \le 1 + \delta\right\}.$$

Then, by the large deviation for the sum of i.i.d. random variables, one can get that

$$P\left(B_\delta^c\right) = P\left(\frac{1}{n}\sum_{k=1}^{n} \xi_{k2}^2 \in [1-\delta, 1+\delta]^c\right) \le e^{-nC_\delta}$$

for all $\delta \in (0,1)$, where $C_\delta > 0$ for each $\delta \in (0,1)$.

Observe that $\sigma_{1n}^2 \le (1+\delta)\left(b_n'^2 + c_n'^2\right) + a_n'^2 + 2c_n'^2 = a_n'^2 + b_n'^2 + 3c_n'^2 + \left(b_n'^2 + c_n'^2\right)\delta \le 1 + \delta$ on B_δ. Therefore, we see from (3.6) that, on B_δ,

$$\begin{aligned}
P_1\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} \eta_{k12}' > s_p'\right) &= P_1\left(N\left(0, \sigma_{1n}^2\right) > s_p'\right) \\
&= P_1\left(N(0,1) > \frac{s_p'}{\sigma_{1n}}\right) \\
&\le \exp\left(-\frac{s_p'^2}{2\sigma_{1n}^2}\right) \le \exp\left(-\frac{s_p'^2}{2(1+\delta)}\right).
\end{aligned}$$

We also use the fact that $P(N(0,1) \ge y) \le \frac{1}{\sqrt{2\pi}y} e^{-y^2/2} \le \frac{1}{2} e^{-y^2/2}$ for all $y \ge 1$ in the above inequality. Reviewing (3.5), we then conclude

$$\begin{aligned}
&P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} \eta_{k12}' > s_p', \frac{1}{\sqrt{n}}\sum_{k=1}^{n} \eta_{k34}' > s_p'\right) \\
\le &E\left[P_1\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} \eta_{k12}' > s_p'\right)^2 I_{B_\delta^c}\right] + \exp\left(-\frac{s_p'^2}{1+\delta}\right) \\
\le &P\left(B_\delta^c\right) + \exp\left(-\frac{s_p'^2}{1+\delta}\right) \\
\le &e^{-nC_\delta} + \exp\left(-\frac{s_p'^2}{1+\delta}\right).
\end{aligned}$$

By choosing $\delta > 0$ small enough, we get the desired conclusion. □

The proof of Proposition 2.3.

Proof: Set

$$I' = \{(i,j); 1 \le i < j \le p, i = j - 1\}.$$

For $\alpha = (i,j) \in I'$, define

$$\begin{aligned}
Z_\alpha &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} \eta_{kij}' \\
&= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} \left(a_n'\xi_{k,i-1}\xi_{k,j-1} + b_n'\xi_{k,i}\xi_{k,j} + \operatorname{sgn}(\rho_n) \cdot c_n'\left(\xi_{k,i-1}\xi_{k,j} + \xi_{k,i}\xi_{k,j-1} - 1\right)\right)
\end{aligned}$$

and

$$B'_\alpha = \left\{(k,l) \in I';\ k \in \{i-2, i-1, i, i+1, i+2\}, (k,l) \neq \alpha\right\}.$$

Obviously, random variable Z_α is independent of $\{Z_\beta; \beta \notin B'_\alpha\}$. By Lemma 3.1, we have

$$\left| P\left(\max_{\alpha \in I'} Z_\alpha \leq s'_p\right) - e^{-\lambda_{p1}} \right| \leq v_1 + v_2,$$

where

$$\lambda_{p1} = \sum_{\alpha \in I'} P\left(Z_\alpha > s'_p\right) = (p-1) P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^n \eta'_{k12} > s'_p\right),$$

$$v_1 = \sum_{\alpha \in I'} \sum_{\beta \in B'_\alpha} P\left(Z_\alpha > s'_p\right) P\left(Z_\beta > s'_p\right) \leq (p-1) \cdot 5 \cdot P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^n \eta'_{k12} > s'_p\right)^2$$

and

$$v_2 = \sum_{\alpha \in I'} \sum_{\alpha \neq \beta \in B'_\alpha} P\left(Z_\alpha > s'_p, Z_\beta > s'_p\right) \leq (p-1) \cdot 5 \cdot P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^n \eta'_{k12} > s'_p, \frac{1}{\sqrt{n}} \sum_{k=1}^n \eta'_{k34} > s'_p\right).$$

By Lemmas 3.4–3.6, we obtain that $e^{-\lambda_{p1}} \to \exp\left(-\frac{1}{4\sqrt{2\pi}} e^{-x/2}\right)$, $v_1 \to 0$ and $v_2 \to 0$ as $n \to \infty$. These imply Proposition 2.3. □

3.3. *Proof of Proposition 2.4*. In this section, we will use the Chen–Stein Poisson approximation method (Lemma 3.1) to get the asymptotical distribution of M_n when $\lambda \in \left[-\infty, 2-\sqrt{2}\right]$.

Lemma 3.7. *Under the conditions of Proposition 2.4, suppose $\rho_n \geq 0$. Then*

$$\lim_{n\to\infty} (p-1) \cdot P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^n \eta_{k12} > s_p\right) = 0.$$

Proof: If $\rho_n \geq 0$, then

$$\sum_{k=1}^n (\eta_{k12} - c_n) = \sum_{k=1}^n \left[a_n \xi_{k0}\xi_{k1} + b_n \xi_{k1}\xi_{k2} + c_n\left(\xi_{k0}\xi_{k2} + \xi_{k1}^2 - 1\right)\right].$$

Thus,

$$E\left(\sum_{k=1}^n (\eta_{k12} - c_n)\right) = 0 \quad \text{and} \quad \mathrm{Var}\left(\sum_{k=1}^n (\eta_{k12} - c_n)\right) = n\left(1 + \theta_n - \theta_n^2\right) := n\sigma_{n0}^2. \tag{3.7}$$

It is obvious that $1 \leq \sigma_{n0}^2 < 5/4$ due to $\theta_n \in [0, 1/2)$. Then, we take $a' = a_n/(\sqrt{n}\sigma_{n0})$, $b' = b_n/(\sqrt{n}\sigma_{n0})$ and $c' = c_n/(\sqrt{n}\sigma_{n0})$. Set $\eta'_k = a'\xi_{k0}\xi_{k1} + b'\xi_{k1}\xi_{k2} + c'\left(\xi_{k0}\xi_{k2} + \xi_{k1}^2 - 1\right)$. Then it follows from (3.7) that

$$E\left(\eta'_k\right) = 0 \quad \text{and} \quad \sum_{k=1}^n \mathrm{Var}\left(\eta'_k\right) = 1 \tag{3.8}$$

for each k. Furthermore, we have

$$|a'| \leq \frac{1}{2\sqrt{n}}\ ,\ |b'| \leq \frac{1}{\sqrt{n}} \text{ and } |c'| \leq \frac{1}{2\sqrt{n}}.$$

Then, by using the Hölder inequality, and the fact that $2|\xi_{k1}\xi_{k2}| \le \xi_{k1}^2 + \xi_{k2}^2$, we see

$$\begin{aligned} Ee^{h|\eta_k'|} \le & E \exp\left[h\left(|a'\xi_{k0}\xi_{k1}| + |b'\xi_{k1}\xi_{k2}| + |c'\xi_{k0}\xi_{k2}| + |c'\xi_{k1}^2| + |c'|\right)\right] \\ \le & E \exp\left[h\left(\frac{|a'|}{2} + \frac{|c'|}{2}\right)\xi_{k0}^2\right] \cdot E \exp\left[h\left(\frac{|a'|}{2} + \frac{|b'|}{2} + |c'|\right)\xi_{k1}^2\right] \\ & \cdot E \exp\left[h\left(\frac{|a'|}{2} + \frac{|c'|}{2}\right)\xi_{k2}^2\right] \cdot e < \infty \end{aligned} \tag{3.9}$$

for all h, k, n satisfying $0 < h < h_n' := \frac{2}{5}\sqrt{n}$ and $1 \le k \le n$. Set $x_0 = (s_p - c_n\sqrt{n})/\sigma_{n0}$. By Lemma 3.3, we have

$$\gamma := \sum_{k=1}^{n} E\left(|\eta_k'|^3 e^{x_0|\eta_k'|}\right) \le \sum_{k=1}^{n} C\left(|a'|^3 + |b'|^3 + |c'|^3\right) \cdot e^{x_0|c'|} \le \frac{C}{\sqrt{n}} \cdot e^{x_0|c'|}.$$

Since $x_0 < h_n' = \frac{2}{5}\sqrt{n}$, we see from (3.8), (3.9) and Lemma 3.2 that

$$\begin{aligned} & p \cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12} > s_p\right) \\ = & p \cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\eta_{k12} - c_n) > s_p - c_n\sqrt{n}\right) \\ = & p \cdot P\left(\sum_{k=1}^{n}\eta_k' > x_0\right) \\ = & p \cdot [1 - \Phi(x_0)] \cdot \left[1 + O(1)\left(1 + x_0^3\right)\gamma e^{4x_0^3\gamma}\right] \\ = & p \cdot [1 - \Phi(x_0)] \cdot \left[1 + O\left(\frac{e^{s_p^3/\sqrt{n}}}{\sqrt{n}}\right)\right] \\ = & \frac{p \cdot \sigma_{n0}}{\sqrt{2\pi}(s_p - c_n\sqrt{n})} e^{-\frac{(s_p - c_n\sqrt{n})^2}{2\sigma_{n0}^2}} (1 + o(1)) \\ = & \frac{\sigma_{n0}}{\sqrt{2\pi}(s_p - c_n\sqrt{n})} \exp\left(\log p - \frac{s_p^2}{2\sigma_{n0}^2} - \frac{c_n^2 n}{2\sigma_{n0}^2} + \frac{c_n\sqrt{n}s_p}{\sigma_{n0}^2}\right)(1 + o(1)) \end{aligned} \tag{3.10}$$

as $n \to \infty$. In the above equality, we use the formula $P(N(0,1) \ge x) = \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}$ as $x \to \infty$, and the fact $x_0^3\gamma = O\left(s_p^3 n^{-1/2} e^{s_p/\sqrt{n}}\right) \to 0$ as $n \to \infty$ under Assumption 1.4. If $\lambda \in \left[-\infty, 2 - \sqrt{2}\right]$, then $\sigma_{n0}^2 = 1 + \theta_n - \theta_n^2 = 1 + o(1/\sqrt{n})$. Hence $\sigma_{n0}^{-2} = 1 + o(1/\sqrt{n})$. It follows that

$$\begin{aligned} & \exp\left(\log p - \frac{s_p^2}{2\sigma_{n0}^2} - \frac{c_n^2 n}{2\sigma_{n0}^2} + \frac{c_n\sqrt{n}s_p}{\sigma_{n0}^2}\right) \\ = & \exp\left(\log p - \frac{s_p^2}{2} - \frac{c_n^2 n}{2} + c_n\sqrt{n}s_p + o(1)\right) \to 0 \end{aligned}$$

as n is sufficiently large. Now combining this with (3.10), we prove the lemma. □

The above lemma has the following implication.

Lemma 3.8. *Under the conditions of Proposition 2.4, suppose* $\rho_n \geq 0$. *Then*

$$\lim_{n\to\infty} p \cdot \left[P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \eta_{k12} > s_p \right) \right]^2 = 0.$$

Lemma 3.9. *Under the conditions of Proposition 2.4, suppose* $\rho_n \geq 0$. *Then*

$$\lim_{n\to\infty} p \cdot P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \eta_{k12} > s_p, \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \eta_{k34} > s_p \right) = 0.$$

Proof: Let P_1 stand for the conditional probability given $\{\xi_{k2}; 1 \leq k \leq n\}$. By indenpendence,

$$P\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \eta_{k12} > s_p, \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \eta_{k34} > s_p \right) = E\left[P_1\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\eta_{k12} - c_n) > s_p - c_n\sqrt{n} \right)^2 \right]. \quad (3.11)$$

Then, given $\{\xi_{k2}; 1 \leq k \leq n\}$, we have from independence that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\eta_{k12} - c_n) \sim N\left(0, \sigma_{2n}^2\right), \quad (3.12)$$

where

$$\sigma_{2n}^2 = \frac{1}{n} \sum_{k=1}^{n} \left(a_n^2 + b_n^2 \xi_{k2}^2 + c_n^2 \xi_{k2}^2 + 2c_n^2 \right) = a_n^2 + 2c_n^2 + \frac{1}{n} \sum_{k=1}^{n} \left(b_n^2 + c_n^2 \right) \xi_{k2}^2.$$

For a given $\delta \in (0,1)$, set

$$\begin{aligned} D_\delta &= \left\{ 1 - \delta + \frac{a_n^2 + 2c_n^2}{b_n^2 + c_n^2} \leq \frac{\sigma_{2n}^2}{b_n^2 + c_n^2} \leq 1 + \delta + \frac{a_n^2 + 2c_n^2}{b_n^2 + c_n^2} \right\} \\ &= \left\{ 1 - \delta \leq \frac{1}{n} \sum_{k=1}^{n} \xi_{k2}^2 \leq 1 + \delta \right\}. \end{aligned}$$

Then, by the large deviation for the sum of i.i.d. random variables, we can deduce that

$$P\left(D_\delta^c\right) = P\left(\frac{1}{n} \sum_{k=1}^{n} \xi_{k2}^2 \in [1-\delta, 1+\delta]^c \right) \leq e^{-nC_\delta}$$

for all $\delta \in (0,1)$, where $C_\delta > 0$ for each $\delta \in (0,1)$. Furthermore, notice that $\sigma_{2n}^2 \leq (1+\delta)\left(b_n^2 + c_n^2\right) + a_n^2 + 2c_n^2 = a_n^2 + b_n^2 + 3c_n^2 + \left(b_n^2 + c_n^2\right)\delta < \frac{5}{4} + \delta$ on D_δ. Then, we have from (3.12) that, on D_δ,

$$\begin{aligned} P_1\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\eta_{k12} - c_n) > s_p - c_n\sqrt{n} \right) &= P_1\left(N\left(0, \sigma_{2n}^2\right) > s_p - c_n\sqrt{n} \right) \\ &= P_1\left(N(0,1) > \frac{s_p - c_n\sqrt{n}}{\sigma_{2n}} \right) \\ &\leq \exp\left(-\frac{\left(s_p - c_n\sqrt{n}\right)^2}{2\sigma_{2n}^2} \right) \\ &\leq \exp\left(-\frac{\left(s_p - c_n\sqrt{n}\right)^2}{2\left(5/4 + \delta\right)} \right). \end{aligned}$$

Recalling (3.11), we conclude

$$
\begin{aligned}
&P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12}>s_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k34}>s_p\right)\\
\leq& E\left[P_1\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\eta_{k12}-c_n)>s_p-c_n\sqrt{n}\right)^2 I_{D_\delta^c}\right]+\exp\left(-\frac{\left(s_p-c_n\sqrt{n}\right)^2}{5/4+\delta}\right)\\
\leq& P\left(D_\delta^c\right)+\exp\left(-\frac{\left(s_p-c_n\sqrt{n}\right)^2}{5/4+\delta}\right)\\
\leq& e^{-nC_\delta}+\exp\left(-\frac{\left(s_p-c_n\sqrt{n}\right)^2}{5/4+\delta}\right).
\end{aligned}
$$

By choosing $\delta>0$ small enough, we know the last expectation is identical to $o\left(1/p\right)$. Then the proof is completed. □

Lemma 3.10. *Under the conditions of Proposition 2.4, suppose $\rho_n<0$. Then*

$$
\lim_{n\to\infty}(p-1)\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12}>s_p\right)=0.
$$

Proof: If $\rho_n<0$, then

$$
\sum_{k=1}^{n}(\eta_{k12}+c_n)=\sum_{k=1}^{n}\left[a_n\xi_{k0}\xi_{k1}+b_n\xi_{k1}\xi_{k2}-c_n\left(\xi_{k0}\xi_{k2}+\xi_{k1}^2-1\right)\right].
$$

Obviously,

$$
E\left(\sum_{k=1}^{n}(\eta_{k12}+c_n)\right)=0 \quad \text{and} \quad \mathrm{Var}\left(\sum_{k=1}^{n}(\eta_{k12}+c_n)\right)=n\left(1+\theta_n-\theta_n^2\right).
$$

Review the proof of Lemma 3.7, (3.10), and the definition of σ_{n0}^2. Then, we obtain that,

$$
\begin{aligned}
p\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12}>s_p\right)=&p\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\eta_{k12}+c_n)>s_p+c_n\sqrt{n}\right)\\
\leq& p\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\eta_{k12}+c_n)>s_p\right)\\
=&p\cdot\left[1-\Phi\left(\frac{s_p}{\sigma_{n0}}\right)\right]\cdot(1+o(1))\\
=&\frac{p\cdot\sigma_{n0}}{\sqrt{2\pi}s_p}e^{-\frac{s_p^2}{2\sigma_{n0}^2}}\\
\leq&\frac{\sqrt{5}\left(\log p\right)^{3/5}}{2\sqrt{2\pi}s_p\cdot p^{3/5}}\to 0
\end{aligned}
$$

as $n\to\infty$. This proves the lemma. □

Lemma 3.10 implies the following lemma.

Lemma 3.11. *Under the conditions of Proposition 2.4, suppose $\rho_n < 0$. Then*

$$\lim_{n\to\infty} p \cdot \left[P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12} > s_p\right)\right]^2 = 0.$$

Lemma 3.12. *Under the conditions of Proposition 2.4, suppose $\rho_n < 0$. Then*

$$\lim_{n\to\infty} p \cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12} > s_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k34} > s_p\right) = 0.$$

Proof: Let P_1 stand for the conditional probability given $\{\xi_{k2}; 1 \le k \le n\}$. By independence,

$$P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12} > s_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k34} > s_p\right) = E\left[P_1\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\eta_{k12}+c_n) > s_p + c_n\sqrt{n}\right)^2\right]. \quad (3.13)$$

Reviewing the proof of Lemma 3.9 and the definitions of σ_{2n}^2 and D_δ, we find that $\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\eta_{k12} + c_n) \sim N\left(0, \sigma_{2n}^2\right)$. Since $\sigma_{2n}^2 < 5/4 + \delta$ on D_δ, we have that, on D_δ,

$$\begin{aligned} P_1\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\eta_{k12}+c_n) > s_p + c_n\sqrt{n}\right) &= P_1\left(N\left(0,\sigma_{2n}^2\right) > s_p + c_n\sqrt{n}\right) \\ &\le \exp\left(-\frac{\left(s_p + c_n\sqrt{n}\right)^2}{2\sigma_{2n}^2}\right) \\ &\le \exp\left(-\frac{\left(s_p + c_n\sqrt{n}\right)^2}{2\left(5/4+\delta\right)}\right). \end{aligned}$$

Therefore, we see from (3.13) that

$$\begin{aligned} & P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12} > s_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k34} > s_p\right) \\ =& E\left[P_1\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(\eta_{k12}+c_n) > s_p + c_n\sqrt{n}\right)^2 I_{D_\delta^c}\right] + \exp\left(-\frac{\left(s_p + c_n\sqrt{n}\right)^2}{5/4+\delta}\right) \\ \le& P\left(D_\delta^c\right) + \exp\left(-\frac{\left(s_p + c_n\sqrt{n}\right)^2}{5/4+\delta}\right) \\ \le& e^{-nC_\delta} + \exp\left(-\frac{\left(s_p + c_n\sqrt{n}\right)^2}{5/4+\delta}\right). \end{aligned}$$

Choosing $\delta > 0$ small enough, we know the last expectation is identical to $o(1/p)$. Then we obtain the desired conclusion. □

The proof of Propositon 2.4. Set

$$\begin{aligned} I &= \{(i,j); 1 \le i < j \le p\}, \\ I'' &= \{(i,j); 1 \le i < j \le p, i < j-1\}, \\ B_\alpha &= \{(k,l) \in I;\ \{k,l\} \cap \{i-1, i, i+1, j-1, j, j+1\} \ne \emptyset,\ (k,l) \ne \alpha\}, \\ B''_\alpha &= \left\{(k,l) \in I'';\ \{k,l\} \cap \{i-1, i, i+1, j-1, j, j+1\} \ne \emptyset,\ (k,l) \ne \alpha\right\}. \end{aligned}$$

For $\alpha = (i,j) \in I$, define $X_\alpha = M_{nij}$. Note that X_α is independent of $\{X_\beta; \beta \notin B_\alpha\}$. Review the definitions of I' and B'_α. By Lemma 3.1, we have

$$\left| P\left(\max_{\alpha\in I} X_\alpha \le s_p\right) - e^{-\lambda_{p2}} \right| \le u_1 + u_2,$$

where

$$\begin{aligned}\lambda_{p2} &= \sum_{\alpha\in I} P\left(X_\alpha > s_p\right) = \sum_{\alpha\in I'} P\left(X_\alpha > s_p\right) + \sum_{\alpha\in I''} P\left(X_\alpha > s_p\right) \\ &= (p-1)P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12} > s_p\right) + \frac{(p-1)(p-2)}{2}P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > s_p\right),\end{aligned}$$

$$\begin{aligned}u_1 =& \sum_{\alpha\in I}\sum_{\beta\in B_\alpha} P\left(X_\alpha > s_p\right)P\left(X_\beta > s_p\right) \\ =& \sum_{\alpha\in I'}\sum_{\beta\in B'_\alpha} P\left(X_\alpha > s_p\right)P\left(X_\beta > s_p\right) + \sum_{\alpha\in I''}\sum_{\beta\in B''_\alpha} P\left(X_\alpha > s_p\right)P\left(X_\beta > s_p\right) \\ \le& (p-1)\cdot 5\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12} > s_p\right)^2 + \frac{(p-1)(p-2)}{2}\cdot(6p)\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > s_p\right)^2\end{aligned}$$

and

$$\begin{aligned}u_2 =& \sum_{\alpha\in I}\sum_{\alpha\ne\beta\in B_\alpha} P\left(X_\alpha > s_p, X_\beta > s_p\right) \\ =& \sum_{\alpha\in I'}\sum_{\beta\in B'_\alpha} P\left(X_\alpha > s_p, X_\beta > s_p\right) + \sum_{\alpha\in I''}\sum_{\beta\in B''_\alpha} P\left(X_\alpha > s_p, X_\beta > s_p\right) \\ \le& (p-1)\cdot 5\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12} > s_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k34} > s_p\right) \\ &+ \frac{(p-1)(p-2)}{2}\cdot(6p)\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > s_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k15} > s_p\right).\end{aligned}$$

By Cai and Jiang (2011), Proposition 6.4, we can obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{(p-1)(p-2)}{2}\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > s_p\right) = \frac{1}{4\sqrt{2\pi}}e^{-x/2}, \\ &\lim_{n\to\infty} p^3\cdot\left[P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > s_p\right)\right]^2 = 0, \\ &\lim_{n\to\infty} p^3\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > s_p, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k15} > s_p\right) = 0.\end{aligned}$$

Then, combining the above equalities with Lemmas 3.7–3.12, we have $e^{-\lambda_{p2}} \to \exp\left(-\frac{1}{4\sqrt{2\pi}}e^{-x/2}\right)$, $u_1 \to 0$ and $u_2 \to 0$ as $n\to\infty$. These prove Proposition 2.4. □

3.4. *Proof of Proposition 2.5.* In this section, we will use the Chen–Stein Poisson approximation method (Lemma 3.1) to get the asymptotical distribution of M_n when $\lambda \in \left(2-\sqrt{2}, \infty\right]$. Reviewing the definition of σ_{n0}, set $\tau = c_n\sqrt{n} + \sigma_{n0}s'_p$.

Lemma 3.13. *Under the conditions of Proposition 2.5, we have*

$$\lim_{n\to\infty}(p-1)(p-2)\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13}>\tau\right)=0.$$

Proof: If $\lambda\in\left(2-\sqrt{2},\infty\right]$, then

$$\sum_{k=1}^{n}\eta_{k13}=\sum_{k=1}^{n}\left[a_n\xi_{k0}\xi_{k2}+b_n\xi_{k1}\xi_{k3}+c_n\left(\xi_{k0}\xi_{k3}+\xi_{k1}\xi_{k2}\right)\right].$$

It is obvious that

$$E\left(\sum_{k=1}^{n}\eta_{k13}\right)=0\quad\text{and}\quad\operatorname{Var}\left(\sum_{k=1}^{n}\eta_{k13}\right)=na_n^2+nb_n^2+2nc_n^2=n.\tag{3.14}$$

Then, we take $a=a_n/\sqrt{n}$, $b=b_n/\sqrt{n}$ and $c=c_n/\sqrt{n}$.
Set $\eta_k=a\xi_{k0}\xi_{k2}+b\xi_{k1}\xi_{k3}+c\left(\xi_{k0}\xi_{k3}+\xi_{k1}\xi_{k2}\right)$. It follows from (3.14) that

$$E\left(\eta_k\right)=0\quad\text{and}\quad\sum_{k=1}^{n}\operatorname{Var}\left(\eta_k\right)=1\tag{3.15}$$

for each k. Furthermore, we have

$$|a|\le\frac{1}{2\sqrt{n}}\ ,\ |b|\le\frac{1}{\sqrt{n}}\text{ and }|c|\le\frac{1}{2\sqrt{n}}.$$

Next, by using the Hölder inequality, and the fact that $2|\xi_{k1}\xi_{k2}|\le\xi_{k1}^2+\xi_{k2}^2$, we arrive at

$$\begin{aligned}Ee^{h|\eta_k|}\le&E\,\exp\left[h\left(|a\xi_{k0}\xi_{k2}|+|b\xi_{k1}\xi_{k3}|+|c\xi_{k0}\xi_{k3}|+|c\xi_{k1}\xi_{k2}|\right)\right]\\<&E\,\exp\left[\frac{h}{4\sqrt{n}}\left(2\xi_{k0}^2+3\xi_{k1}^2+2\xi_{k2}^2+3\xi_{k3}^2\right)\right]\\=&E\left[\exp\left(\frac{h}{2\sqrt{n}}\xi_{k0}^2\right)\cdot\exp\left(\frac{3h}{4\sqrt{n}}\xi_{k1}^2\right)\cdot\exp\left(\frac{h}{2\sqrt{n}}\xi_{k3}^2\right)\cdot\exp\left(\frac{3h}{4\sqrt{n}}\xi_{k3}^2\right)\right]<\infty,\end{aligned}\tag{3.16}$$

for all h,k,n satisfying $0<h<h_n:=\frac{2}{3}\sqrt{n}$ and $1\le k\le n$. We see from Lemma 3.3 that

$$\gamma:=\sum_{k=1}^{n}E\left(|\eta_k|^3e^{s_p|\eta_k|}\right)\le\sum_{k=1}^{n}C\left(|a|^3+|b|^3+|c|^3\right)\le\sum_{k=1}^{n}\frac{C}{n^{3/2}}=\frac{C}{\sqrt{n}}.$$

Since $\lambda\in\left(2-\sqrt{2},\infty\right]$, we assume $\rho_n\sqrt{n}=c_n\sqrt{n}=\left(2-\sqrt{2}+\varepsilon\right)\sqrt{\log p}$ for some $\varepsilon>0$. Notice that $\tau=c_n\sqrt{n}+\sigma_{n0}s_p'\ge\left(2-\sqrt{2}+\varepsilon\right)\sqrt{\log p}+s_p'>\frac{2-\sqrt{2}+\varepsilon}{\sqrt{2}}s_p'+s_p'=\left(\sqrt{2}+\varepsilon\right)s_p'$. From (3.15), (3.16) and Lemma 3.2, we then have

$$\begin{aligned}\frac{p^2}{2}\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13}>\tau\right)&<\frac{p^2}{2}\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13}>\left(\sqrt{2}+\varepsilon\right)s_p'\right)\\&=\frac{p^2}{2}\cdot\left[1-\Phi\left(\left(\sqrt{2}+\varepsilon\right)s_p'\right)\right]\cdot\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right]\\&=\frac{p^2}{2\sqrt{2\pi}\left(\sqrt{2}+\varepsilon\right)s_p'}e^{-\frac{\left(\sqrt{2}+\varepsilon\right)^2s_p'^2}{2}}\left[1+o\left(1\right)\right]\\&=\frac{C\cdot\left(\sqrt{\log p}\right)^{1+\varepsilon}}{p^{\varepsilon}}e^{-(1+\varepsilon)x}\left[1+o\left(1\right)\right]\to0\end{aligned}\tag{3.17}$$

as $\lambda\in\left(2-\sqrt{2},\infty\right]$ and $n\to\infty$. This proves the lemma. □

From the above lemma, we immediately have the following result.

Lemma 3.14. *Under the conditions of Proposition 2.5, we have*

$$\lim_{n\to\infty} p^3 \cdot \left[P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > \tau\right)\right]^2 = 0.$$

Lemma 3.15. *Under the conditions of Proposition 2.5, we have*

$$\lim_{n\to\infty} p^3 \cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > \tau, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k15} > \tau\right) = 0.$$

Proof: Let P_2 stand for the conditional probability given $\{\xi_{k0}, \xi_{k1}; 1 \le k \le n\}$. By independence,

$$P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > \tau, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k15} > \tau\right) = E\left[P_2\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > \tau\right)^2\right]. \tag{3.18}$$

If $\lambda \in \left(2-\sqrt{2}, \infty\right]$, then

$$\sum_{k=1}^{n}\eta_{k13} = \sum_{k=1}^{n}\left(a_n\xi_{k0} + c_n\xi_{k1}\right)\xi_{k2} + \sum_{k=1}^{n}\left(b_n\xi_{k1} + c_n\xi_{k0}\right)\xi_{k3}.$$

Given $\{\xi_{k0}, \xi_{k1}; 1 \le k \le n\}$, we have

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} \sim N\left(0, \sigma_{0n}^2\right), \tag{3.19}$$

where

$$\sigma_{0n}^2 = \frac{1}{n}\left[\sum_{k=1}^{n}\left(a_n\xi_{k0} + c_n\xi_{k1}\right)^2 + \sum_{k=1}^{n}\left(b_n\xi_{k1} + c_n\xi_{k0}\right)^2\right] = \frac{1}{n}\sum_{k=1}^{n}\left(\sqrt{a_n^2 + c_n^2}\xi_{k0} + \sqrt{b_n^2 + c_n^2}\xi_{k1}\right)^2.$$

For a given $\delta \in (0,1)$, set

$$A_\delta = \left\{1 - \delta \le \frac{\sigma_{0n}^2}{a_n^2 + b_n^2 + 2c_n^2} \le 1 + \delta\right\}.$$

Notice that $\sqrt{a_n^2 + c_n^2}\xi_{k0} + \sqrt{b_n^2 + c_n^2}\xi_{k1} \overset{d}{=} \sqrt{a_n^2 + b_n^2 + 2c_n^2}\xi_k$ due to (2.1). Thus, $\frac{\sigma_{0n}^2}{a_n^2+b_n^2+2c_n^2} \overset{d}{=} \frac{1}{n}\sum_{k=1}^{n}\xi_k^2$. Then, by the large deviation for the sum of i.i.d. random variables, we obtain

$$P\left(A_\delta^c\right) = P\left(\frac{1}{n}\sum_{k=1}^{n}\xi_k^2 \in [1-\delta, 1+\delta]^c\right) \le e^{-nC_\delta}$$

for all $\delta \in (0,1)$, where $C_\delta > 0$ for each $\delta \in (0,1)$. Furthermore, $\sigma_{0n}^2 \le (1+\delta)\left(a_n^2 + b_n^2 + 2c_n^2\right) = 1 + \delta$ on A_δ. Then, we have from (3.19) that, on A_δ,

$$P_2\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13} > \tau\right) = P_2\left(N\left(0, \sigma_{0n}^2\right) > \tau\right) \le \exp\left(-\frac{\tau^2}{2\sigma_{0n}^2}\right) \le \exp\left(-\frac{\tau^2}{2(1+\delta)}\right).$$

By the same argument as in (3.17), we see from (3.18) that

$$\begin{aligned}
&P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13}>\tau, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k15}>\tau\right)\\
\leq &E\left[P_2\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13}>\tau\right)^2 I_{A_\delta^c}\right]+\exp\left(-\frac{\tau^2}{1+\delta}\right)\\
\leq &P\left(A_\delta^c\right)+\exp\left(-\frac{\left(\sqrt{2}+\varepsilon\right)^2 s_p'^2}{1+\delta}\right)\\
\leq &e^{-nC_\delta}+\exp\left(-\frac{\left(\sqrt{2}+\varepsilon\right)^2 s_p'^2}{1+\delta}\right).
\end{aligned}$$

Choosing $\delta>0$ small enough, we know the last expectation is identical to $o\left(1/p^3\right)$. These prove the lemma. □

The proof of Propositon 2.5.

Proof: Recalling the proof of Proposition 2.4, by Lemma 3.1, we have

$$\left|P\left(\max_{\alpha\in I}X_\alpha\leq\tau\right)-e^{-\lambda_{p3}}\right|\leq m_1+m_2,$$

where

$$\lambda_{p3}=(p-1)P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12}>\tau\right)+\frac{(p-1)(p-2)}{2}P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13}>\tau\right)$$

and

$$m_1\leq(p-1)\cdot 5\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12}>\tau\right)^2+\frac{(p-1)(p-2)}{2}\cdot(6p)\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13}>\tau\right)^2$$

and

$$\begin{aligned}
m_2\leq&(p-1)\cdot 5\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k12}>\tau, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k45}>\tau\right)\\
&+\frac{(p-1)(p-2)}{2}\cdot(6p)\cdot P\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k13}>\tau, \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\eta_{k15}>\tau\right).
\end{aligned}$$

Lemmas 3.4–3.6 and Lemmas 3.13–3.15 say that $e^{-\lambda_{p3}}\to\exp\left(-\frac{1}{4\sqrt{2\pi}}e^{-x/2}\right)$, $m_1\to 0$ and $m_2\to 0$ as $n\to\infty$. Then the proof of Proposition 2.5 is completed. □

4. Applications

The limiting distribution presented in Section 1 holds significance for various statistical applications, one of which involves testing structures of covariance matrices. Let $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ be a random sample from the p-variate normal population $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ with known mean $\boldsymbol{\mu}$ and unknown covariance matrix $\boldsymbol{\Sigma}$.

Example 4.1. An important problem is testing for independence in the Gaussian case. Consider the test

$$H_0:\boldsymbol{\Sigma}=\boldsymbol{I}_p \quad \text{v.s.} \quad H_1:\boldsymbol{\Sigma}\neq\boldsymbol{I}_p,$$

where $\boldsymbol{I}_p$ is the $p\times p$ identity matrix. Based on the limiting law derived in Theorem 1.8, we choose J_n as the test statistic. Under H_0 and Assumption 1.4, we have

$$4\sqrt{\log p}\left(\sqrt{n}J_n-\mu_1\right)\xrightarrow{d}\xi,$$

where μ_1 and ξ are given by Theorem 1.5. For a given $\alpha\in(0,1)$, set

$$q_\alpha=-\log(32\pi)-2\log\log(1-\alpha)^{-1}.$$

We find that q_α is the $(1-\alpha)$-quantile of the distribution $F_\xi(x)$. Then, we will reject H_0 when

$$\sqrt{n}J_n-2\sqrt{\log p}+\frac{\log\log p}{4\sqrt{\log p}}\geq\frac{q_\alpha}{4\sqrt{\log p}}.$$

Example 4.2. As the application of Theorem 1.8, we wish to test whether $\boldsymbol{\Sigma}$ has the tridiagonal structure like $\boldsymbol{R}$ as in (1.3). Consider the testing problem

$$H_0:\boldsymbol{\Sigma}=\boldsymbol{R}\quad \text{v.s.}\quad H_1:\boldsymbol{\Sigma}\neq\boldsymbol{R}.$$

By the same argument as in Example 4.1, we will use J_n as our test statistic. Let q_α denote the $(1-\alpha)$-quantile of the distribution $F_\xi(x)$. By Theorem 1.8, we will show two rejection regions Ω_1 and Ω_2 when $\lambda\in\left[-\infty,2-\sqrt{2}\right]$ and $\lambda\in\left(2-\sqrt{2},\infty\right]$, respectively,

$$\Omega_1=\left\{\sqrt{n}J_n-2\sqrt{\log p}+\frac{\log\log p}{4\sqrt{\log p}}\geq\frac{q_\alpha}{4\sqrt{\log p}}\right\},$$

$$\Omega_2=\left\{\frac{\sqrt{n}J_n}{\sqrt{1+\rho_n^2}}-\frac{\rho_n\sqrt{n}}{\sqrt{1+\rho_n^2}}-\sqrt{2\log p}+\frac{\log\log p}{2\sqrt{2\log p}}-\frac{\log 8}{2\sqrt{2\log p}}\geq\frac{q_\alpha}{2\sqrt{2\log p}}\right\}.$$

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