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Moderate deviations for functionals over infinitely many Rademacher random variables

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Abstract. In this paper, moderate deviations for normal approximation of functionals over infinitely many Rademacher random variables are derived. They are based on a bound for the Kolmogorov distance between a general Rademacher functional and a Gaussian random variable, continued by an intensive study of the behavior of operators from the Malliavin–Stein method along with the moment generating function of the mentioned functional. As applications, subgraph counting in the Erdős–Rényi random graph and infinite weighted 2-runs are studied.

1. Introduction and applications

The theory of moderate deviations goes back to H. Cramér in 1938: Knowing that for an independently identically distributed (i.i.d.) sequence $(X_k)_{k\in\mathbb{N}}$ of random variables such that $\mathbb{E}[X_k] = 0, \mathbb{E}(X_k^2) = 1, W_n := n^{-1/2}(X_1 + ... + X_n)$ and Φ the standard normal distribution function, the statement

$$\frac{\mathbb{P}(W_n > x)}{1-\Phi(x)} \longrightarrow 1 \quad \text{for} \quad x = O(1) \tag{1.1}$$

is valid, he was asking what happens if x depends on $n \in \mathbb{N}$ such that $x \to \infty$ for $n \to \infty$? Can we find an interval such that (1.1) holds for $0 \le x \le I(n), I(n) \to \infty$? The answer was given by himself: Under the assumption $\mathbb{E}[e^{t_0|X_1|}] \le c < \infty$ for some $t_0 > 0$ and a constant c,

$$\frac{\mathbb{P}(W_n > x)}{1-\Phi(x)} \le 1 + A\, n^{-1/2}(1 + x^3) \quad \text{for} \quad 0 \le x \le a\, n^{1/6}, \tag{1.2}$$

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and A and a are positive constants depending only on t_0 and c. The result is optimal (see e.g. Cramér (2022) and Petrov (1975)). Reminiscent of (1.2) for a sequence $(Y_n)_{n\in\mathbb{N}}$ of random variables, such that $Y_n \overset{d}{\to} Y$, the moderate deviation of Cramér-type is given by

$$\frac{\mathbb{P}(Y_n > x)}{\mathbb{P}(Y > x)} = 1 + \text{error term} \to 1$$

with range $0 \le x \le a_n$, where $a_n \to \infty$ for $n \to \infty$.
Zhang (2023) was able to prove Cramér-type moderate deviations for unbounded exchangeable pairs. He developed them by stopping the proof of the corresponding Berry–Esseen-type inequalities, he had obtained before by Stein's method, at a certain point and continuing differently. Stein's method is a powerful tool by itself to derive upper bounds for differences of probability distributions, originally for the normal distribution and later extended to other distributions. Zhang rearranged the fragments of the so called Stein-equation and the bound of its solution, a technique that was already seen in Chen et al. (2013), Fang et al. (2020), Raič (2007) and Shao et al. (2021).
Our ambition is to prove a similar result for functionals over infinitely many independent Rademacher random variables taking values $+1$ and -1 only. This type of result intersects with Fang and Koike (2023), where the authors obtain Cramér-type moderate deviations via p-Wasserstein bounds, and we will refer to that. For Rademacher functionals a Kolmogorov bound in the context of normal approximation was shown recently by Theorem 3.1 in Eichelsbacher et al. (2023) such that the bounding terms can be expressed in terms of operators of the so called Malliavin–Stein method. Normal approximation of functionals over infinitely many Rademacher random variables was derived already in Nourdin et al. (2010), Krokowski et al. (2017), Krokowski and Thäle (2017) and Döbler and Krokowski (2019). Theorem 3.1 in Eichelsbacher et al. (2023) will be our starting point.

1.1. *Application to infinite weighted 2-runs.* To begin with, we introduce some of the possible applications for our theorem. In what follows, we use the usual big-O notation $O(.)$ with the meaning that the implicit constant does not depend on the parameters in brackets. For a sequence $a = (a_i)_{i\in\mathbb{Z}}$ and $p > 0$ we write $\|a\|_{l^p(\mathbb{Z})} := \left(\sum_{i\in\mathbb{Z}} |a_i|^p\right)^{1/p}$.

Due to their simple dependence structure, runs, and more generally weighted or incomplete U-statistics, lend themselves to normal approximations, see Rinott and Rotar (1997), where an exchangeable pair coupling is employed for a normal approximation. In Rinott and Rotar (1997) the authors studied even degenerate weighted U-statistics, where either weights are considered which ensures a weak dependence or kernel functions are considered which depend on the sample size n in a specific way. See also Nowicki and Wierman (1988), where subgraph counts in random graphs are considered, see Subsection 1.2. Here we consider infinite weighted 2-runs, where random variables are possibly depending on the whole infinite sequence of i.i.d. Rademacher random variables.

Let $X = (X_i)_{i\in\mathbb{Z}}$ be a double-sided sequence of i.i.d. Rademacher random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$ and let for each $n \in \mathbb{N}$, $(a_i^{(n)})_{i\in\mathbb{Z}}$ be a double-sided summable sequence of real numbers. Usually 2-runs are definded with a square-summable sequence but this will be not enough.
The sequence $(F_n)_{n\in\mathbb{N}}$ of standardized infinite weighted 2-runs is then defined as

$$F_n := \frac{G_n - \mathbb{E}[G_n]}{\sqrt{\mathrm{Var}(G_n)}}, \quad G_n := \sum_{i\in\mathbb{Z}} a_i^{(n)} \xi_i \xi_{i+1}, \quad n \in \mathbb{N},$$

where $\xi_i := (X_i + 1)/2$ for $i \in \mathbb{Z}$. More generally one can consider an infinite weighted d-run defined by

$$G_n(d) := \sum_{i\in\mathbb{Z}} a_i^{(n)} \xi_i \cdots \xi_{i+d-1},$$

which is a weighted degenerate U-statistic of degree d. However, since the analysis for any d is of the cost of a quite cumbersome notation, we will focus on the case where $d = 2$ (2-runs).

For recent results on 2-runs combined with Malliavin–Stein method see Eichelsbacher et al. (2023), Krokowski et al. (2016) and Nourdin et al. (2010). Our moderate deviation is given as follows.

Theorem 1.1. *Recall the definition of F_n from above. Then*

$$\frac{\mathbb{P}(F_n > z)}{1-\Phi(z)} = 1 + O(1)(1+z^2)\gamma_n(z), \tag{1.3}$$

for $0 \le z \le \min\{C_n^{-1/3}, C_n^{-2}, \mathrm{Var}(G_n)^{1/2}\}$ such that $(1+z^2)\gamma_n(z) \le 1$, where O(1) is bounded by a constant only depending on the coefficient sequence $(a_i^{(n)})_{i\in\mathbb{Z}}$ and

$$\gamma_n(z) := e^{O(1)z(\mathrm{Var}(G_n))^{-1/2}}\Big((1+z^{1/2}+z)C_n\Big), \quad C_n := \frac{\left\|a^{(n)}\right\|^2_{l^4(\mathbb{Z})}}{\mathrm{Var}(G_n)}.$$

Remark 1.2. The constant C_n has an important meaning. It is the order of the corresponding Berry-Esseen bound of the Kolmogorov distance $\sup_{z\in\mathbb{R}} |\mathbb{P}(F_n \le z) - \Phi(z)|$ between the distribution function of F_n and the standard normal distribution function in Theorem 1.1 in Eichelsbacher et al. (2023). Depending on the coefficient sequence, C_n can behave differently: By (4.1) and (4.20) C_n is in general bounded by a constant, but it can be a constant itself (see e.g. $a_i^{(n)} = \frac{1}{i^2}$). So, to make (1.3) tend to 0 and the range increase in n, the condition $C_n \to 0$ for $n \to \infty$ is sufficient. We give now examples, where this is the case and where the resulting rate is optimal.

Example 1.3. We consider $a_i^{(n)} = \mathbb{1}_{\{|i|\le n\}}\, \forall\, i \in \mathbb{Z}$, which is obviously a summable sequence. Then $\left\|a^{(n)}\right\|^2_{l^4(\mathbb{Z})} = O\big(n^{1/2}\big)$, $\mathrm{Var}(G_n) = O(n)$ and $C_n = O\big(n^{-1/2}\big)$. The moderate deviation we get is

$$\frac{\mathbb{P}(F_n > z)}{1-\Phi(z)} = 1 + O(1)(1+z^2)\gamma_n(z), \tag{1.4}$$

for $0 \le z \le n^{1/6}$ such that $(1+z^2)\gamma_n(z) \le 1$, where

$$\gamma_n(z) := e^{O(1)zn^{-1/2}}\Big((1+z^{1/2}+z)n^{-1/2}\Big).$$

In order to discuss the quality of this result, we use a lower bound of the Kolmogorov distance known from Theorem 1(c) in Englund (1981), which got later refined by Corollary 3.12 in Rednoß (2023). Since G_n is almost surely an integer between $-n$ and n, said results imply that the Kolmogorov distance for normal approximation of F_n is bounded from below by $c_0 \cdot (\mathrm{Var}(G_n))^{-\frac{1}{2}}$ for some constant $c_0 > 0$. As $\mathrm{Var}(G_n)$ is of order n, we conclude that C_n being of order $n^{-\frac{1}{2}}$ is optimal.

Example 1.4. We generalize the previous example to $a_i^{(n)} = n^{-\beta}\mathbb{1}_{\{|i|\le n^\alpha\}}\, \forall\, i \in \mathbb{Z}, \alpha \in \mathbb{R}, \beta > 0$. Then $\left\|a^{(n)}\right\|^2_{l^4(\mathbb{Z})} = O\big(n^{(\alpha-4\beta)/2}\big)$, $\mathrm{Var}(G_n) = O\big(n^{\alpha-2\beta}\big)$ and $C_n = O\big(n^{-\alpha/2}\big)$. If we choose $\beta = 0$ and $\alpha \ge 1$ the moderate deviation we get is of the same form as (1.4) with range $0 \le z \le n^{\alpha/6}$ respectively $0 \le z \le (\mathrm{Var}(G_n))^{\alpha/6}$. Using the same argumentation as in the previous example, we see that the rate of C_n is again optimal.

1.2. *Application to subgraph counts in the Erdős–Rényi random graph.* As a further application we derive a Cramér-type moderate deviation result for the subgraph counting statistic in the Erdős–Rényi random graph $G(n, p)$.

Consider a random graph on $n \in \mathbb{N}$ vertices. Each possible edge between two vertices is included with probability $p \in (0, 1)$ independently of all other edges, where p may depend on n even though we will not make this visible in our notation.

Let G_0 be a fixed graph with at least one edge. A subgraph $H \subset G(n,p)$ is called a *copy* of G_0 in $G(n,p)$ if it is isomorphic to G_0. Note that we are calling two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ *isomorphic* if there is an edge-preserving bijection $f : V_1 \to V_2$ between their sets of vertices, such that two vertices $v, w \in V_1$ are joined by an edge $\{v, w\} \in E_1$ in G_1 if and only if the vertices $f(v), f(w) \in V_2$ are joined by an edge $\{f(v), f(w)\} \in E_2$ in G_2.

We are interested in the standardized number W of copies of G_0 in $G(n,p)$.

It is well known under which necessary and sufficient assumption W is asymptotically normal. To state this condition and further results, we will use the notation $q := 1 - p$ as well as

$$\Psi_{\min} := \Psi_{\min}(G_0) := \min_{\substack{H \subset G_0 \\ e_H \geq 1}} \{n^{v_H} \cdot p^{e_H}\},$$

with $H \subset G_0$ a subgraph of G_0, v_H the number of vertices of H, and e_H the number of edges of H. Further, a_H denotes the number of automorphisms of H. Some very important yet easy to prove bounds for $\Psi_{\min}$ are

$$n^2 p^{e_{G_0}} \leq \Psi_{\min} \leq n^2 p.$$

Theorem 2 in Ruciński (1988) states that W is asymptotically normal if and only if

$$q \cdot \Psi_{\min} \xrightarrow{n \to \infty} \infty.$$

The best known convergence rate regarding the bounded Wasserstein distance d_{bWass} is presented in Theorem 2 in Barbour et al. (1989): There is a constant C_{G_0} only depending on G_0 so that

$$d_{\text{bWass}}(W, N) \leq \frac{C_{G_0}}{\sqrt{q \Psi_{\min}}},$$

where $N \sim \mathcal{N}(0,1)$ is standard normal distributed, and

$$d_{\text{bWass}}(W, N) = \sup_{h \in \mathcal{H}} \left| \mathbb{E}h(W) - \mathbb{E}h(N) \right|$$

with $\mathcal{H}$ being the set of all bounded Lipschitz-functions with $\|h\|_\infty + \|h'\|_\infty \leq 1$. The same bound holds true for the Wasserstein distance, where $\mathcal{H}$ is the collection of Lipschitz-functions with Lipschitz-constant 1. The result can be shown by only a slight modification of the original proof in Barbour et al. (1989). For the Kolmogorov distance, the same order of the bound can be obtained. This was a long standing problem that has been lately solved using different approaches by Privault and Serafin (2020), Zhang (2019), Eichelsbacher and Rednoß (2023) and Eichelsbacher et al. (2023).

A refined Cramér-type moderate deviation result for the subgraph counting statistic in the Erdős–Rényi random graph is shown in Proposition 2.3 in Fang et al. (2020): For any fixed constant $c_0 > 0$ there is a constant C_{c_0, G_0} only depending on c_0 and G_0, so that

$$\max\left\{ \left| \frac{\mathbb{P}(W < -t)}{\Phi(-t)\exp(-\frac{\gamma t^3}{6})} - 1 \right|, \left| \frac{\mathbb{P}(W > t)}{(1 - \Phi(t))\exp(\frac{\gamma t^3}{6})} - 1 \right| \right\} \leq C_{c_0, G_0} \cdot \left(\frac{\Psi_{\min}}{q \cdot p^{2e_{G_0}}} \right)^{\frac{5}{2}} \cdot \frac{1 + t^2}{n^6}$$

for

$$0 \leq t \leq c_0 \cdot \left(\frac{q \cdot p^{2e_{G_0}}}{\Psi_{\min}} \right)^{\frac{5}{4}} \cdot n^3,$$

where Φ is the distribution function of the standard normal distribution, and $\gamma = \mathbb{E}[W^3]$.

Another Cramér-type moderate deviation result is known from Theorem 3.1 in Zhang (2019): There is a constant C_{G_0} only depending on G_0, so that

$$\left| \frac{\mathbb{P}(W > t)}{1 - \Phi(t)} - 1 \right| \leq C_{G_0} \cdot (1 + t^2) \cdot b_n(p, t) \tag{1.5}$$

with

$$b_n(p,t) := \begin{cases} \dfrac{1+t}{\sqrt{\Psi_{\min}}}, & 0 < p < \frac{1}{2}, \\[2ex] \dfrac{1+\frac{t}{\sqrt{q}}}{n\sqrt{p}}, & \frac{1}{2} < p < 1, \end{cases}$$

for all

$$0 \le t \le \frac{q \cdot p^{e_{G_0}}}{\sqrt{\Psi_{\min}}} \cdot n^2$$

that satisfy

$$(1+t^2) b_n(p,t) \le 1.$$

For any subgraph $H \subset G_0$ with at least one edge but more than two vertices, there is $n^{v_H} p^{e_H} \geq n^2 p \cdot \frac{n^{v_H - 2}}{2^{e_H - 1}} \geq n^2 p \cdot \frac{n}{2^{e_{G_0}-1}}$ in case of $\frac{1}{2} < p < 1$. Hence, we know that $\Psi_{\min} = n^2 p$ for $\frac{1}{2} < p < 1$ and $n \geq 2^{e_{G_0}-1}$. We further know that $\frac{1+t}{\sqrt{\Psi_{\min}}} \leq \frac{1+\frac{t}{\sqrt{q}}}{\sqrt{q\Psi_{\min}}} \leq 2 \cdot \frac{1+t}{\sqrt{\Psi_{\min}}}$ in case of $0 < p < \frac{1}{2}$. Therefore, as long as $n \geq 2^{e_{G_0}-1}$, (1.5) can be rewritten without loss of sharpness in the resulting rate:

$$\left| \frac{\mathbb{P}(W > t)}{1 - \Phi(t)} - 1 \right| \le 2 \cdot C_{G_0} \cdot \frac{(1+t^2)\left(1+\frac{t}{\sqrt{q}}\right)}{\sqrt{q \Psi_{\min}}}. \tag{1.6}$$

Further, Theorem 5.2 in Fang and Koike (2023) presents a moderate deviation result for bounded and locally dependent random variables. Application to the context of subgraph counting yields that there exist constants c_{G_0} and C_{G_0} so that

$$\left| \frac{\mathbb{P}(W > t)}{1 - \Phi(t)} - 1 \right| \le C_{G_0} \cdot (1+t) \cdot (1 + |\log(\Delta)| + t^2) \cdot \Delta \tag{1.7}$$

for $0 \leq t \leq \Delta^{-\frac{1}{3}}$ with $\Delta = \frac{\Psi_{\min}}{n^3 p^{2e_{G_0}} q} + \left(\frac{\Psi_{\min}}{n^3 p^{2e_{G_0}} q}\right)^{\frac{3}{2}} \sqrt{n} \log(n) \leq c_{G_0}$. As one can show that $\frac{1}{\sqrt{q\Psi_{\min}}} = o(\Delta)$, result (1.6) from Zhang (2019) is stronger than (1.7) from Fang and Koike (2023).

With our approach we are able to improve on the result of Zhang (2019). We will prove the following result.

Theorem 1.5 (Subgraph counts in the Erdős–Rényi random graph — general result)**.** *Let G_0 be a graph with at least one edge. Let W be the standardized number of copies of G_0 in the Erdős–Rényi random graph $G(n,p)$. And assume that $n \geq 4v_{G_0}^2$. Then for all $t \geq 0$ there is*

$$\left| \frac{\mathbb{P}(W > t)}{1 - \Phi(t)} - 1 \right| \le 50 c_{G_0} \cdot \exp\Big(c_{G_0} \cdot t^2 \cdot s(t)\Big) \cdot (1+t^2) \cdot s(t),$$

with

$$s(t) = \left(1 + \frac{t}{\min\{\sqrt{\Psi_{\min}}, 1\}}\right) \frac{1}{\sqrt{q\Psi_{\min}}} \cdot \exp\left(\frac{5Dt}{\sigma}\right) \tag{1.8}$$

and

$$c_{G_0} = \frac{2^{\frac{9}{2} + \frac{15}{2} e_{G_0}} \cdot (v_{G_0}!)^4 \cdot e_{G_0}}{a_{G_0}^{\frac{3}{2}}}. \tag{1.9}$$

The result holds for all $t \geq 0$. Similar to the approach by Zhang (2019), the result can be simplified by restricting t to be smaller than a suitable bound:

Corollary 1.6 (Subgraph counts in the Erdős–Rényi random graph — bounded domain)**.** *Let G_0 be a graph with at least one edge. Let W be the standardized number of copies of G_0 in the Erdős–Rényi random graph $G(n,p)$. Let $c_1, c_2 > 0$ be positive numbers. Assume that*

$$n \geq 4v_{G_0}^2, \qquad 0 \leq t \leq c_1 \cdot \frac{n^2 \cdot p^{e_{G_0}} \cdot \sqrt{q}}{\sqrt{\Psi_{\min}}}, \qquad t^2 \cdot s(t) \leq c_2,$$

with $s(t)$ as given in (1.8). Then

$$\left| \frac{\mathbb{P}(W > t)}{1 - \Phi(t)} - 1 \right| \leq C_{c_1,c_2,G_0} \cdot \frac{1 + t^3}{\sqrt{q\Psi_{\min}}},$$

where

$$C_{c_1,c_2,G_0} = 100(1 + c_1) \cdot c_{G_0} \cdot e^{5c_1 \hat{c}_{G_0} + c_2 c_{G_0}},$$

with $\hat{c}_{G_0} = \sqrt{2} \cdot \frac{\sqrt{v_{G_0}!} \cdot v_{G_0}^2 \cdot e_{G_0}}{\sqrt{a_{G_0}}}$ and c_{G_0} as given in (1.9).

In case of $\liminf_{n\to\infty} q > 0$, our result is of the same order as the result of Zhang (2019) presented in (1.6). However, in case of $\liminf_{n\to\infty} q = 0$, our result yields the better rate.

We give an overview how the remaining parts of this paper are structured. In Section 2 we list important operators used in Malliavin calculus and give a short introduction to Stein's method. The new moderate deviations for general non-linear functionals of possibly infinite Rademacher random variables are presented in Section 3. The proof of Theorem 1.1 is shown in Section 4 and the proofs of Theorem 1.5 and Corollary 1.6 follow in Section 5.

2. Preliminaries

In this section we list all the definitions and notions we deal with, in particular the operators from Malliavin calculus. Since this is just a summary, we refer to Nourdin and Peccati (2012) for details and to Nourdin (2024) for further results related to the topic. For the setting of Bernoulli processes see also Privault (2008).

Throughout the upcoming definitions we will need the following notations: Let $l^2(\mathbb{N})$ be the space of real square-summable sequences. Moreover, by $l^2(\mathbb{N})^{\otimes p}$ we mean the pth tensor product of $l^2(\mathbb{N})$ for $p \in \mathbb{N}$. Relevant subsets are $l^2(\mathbb{N})^{\circ p}$, the symmetric functions in $l^2(\mathbb{N})^{\otimes p}$, and $l_0^2(\mathbb{N})^{\circ p}$, the symmetric functions in $l^2(\mathbb{N})^{\otimes p}$ which vanish on diagonals.

We start with $X = (X_k)_{k\in\mathbb{N}}$, a sequence of Rademacher random variables, e.g. $\forall k \in \mathbb{N}$:

$$\mathbb{P}(X_k = 1) = p_k \in (0,1),$$
$$\mathbb{P}(X_k = -1) = q_k = 1 - p_k,$$

and, if needed, the standardized random variable

$$Y_k = \frac{X_k - p_k + q_k}{2\sqrt{q_k p_k}}.$$

We are interested in random variables $F \in L^2(\Omega, \sigma(X), \mathbb{P})$ and we will use that in our setting F can be written as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n),$$

where

$$J_n(f) = \sum_{(i_1,\dots,i_n)\in\Delta^n} f(i_1, \dots, i_n) Y_{i_1} Y_{i_2} \dots Y_{i_n}$$

with $f \in l_0^2(\mathbb{N})^{\circ n}$ and $\Delta_n := \{(i_1, ..., i_n) \in \mathbb{N}^n : i_j \neq i_k \text{ for } j \neq k\}$. $J_n(f)$ is called the nth discrete multiple integral. For $F = f(X) = f(X_1, X_2, ...) \in L^1(\Omega, \sigma(X), \mathbb{P})$ we define the discrete gradient $D_k F$ of F at kth coordinate:

$$D_k F := \sqrt{p_k q_k}(F_k^+ - F_k^-),$$
$$DF := (D_1 F, D_2 F, ...),$$

where $F_k^+ := f(X_1, ..., X_{k-1}, +1, X_{k+1}, ...)$ and $F_k^- := f(X_1, ..., X_{k-1}, -1, X_{k+1}, ...)$, $k \in \mathbb{N}$. For the main result we will consider

$$F \in \mathbb{D}^{1,2} := \{F \in L^2(\Omega, \sigma(X), \mathbb{P}) | \, \mathbb{E}[\|DF\|_{l^2(\mathbb{N})}^2] < \infty\},$$

where

$$\mathbb{E}[\|DF\|_{l^2(\mathbb{N})}^2] := \mathbb{E}\left[\sum_{k \in \mathbb{N}} (D_k F)^2\right].$$

Next we define the divergence operator δ, also known as Skorokhod operator, and its domain $Dom(\delta)$. For $u := (u_k)_{k \in \mathbb{N}} \in (L^2(\Omega))^{\mathbb{N}}$ with

$$u_k := \sum_{n=1}^{\infty} J_{n-1}(f_n(\cdot, k)),$$

where $f_n \in l_0^2(\mathbb{N})^{\circ n-1} \otimes l^2(\mathbb{N})$ for $n \in \mathbb{N}$, we say that $u \in Dom(\delta)$, if

$$\sum_{n=1}^{\infty} n! \left\| \widetilde{f_n} \mathbb{1}_{\Delta_n} \right\|_{l^2(\mathbb{N})^{\otimes n}}^2 < \infty.$$

By $\tilde{f}(k_1, ..., k_n) := \frac{1}{n!} \sum_{\sigma \in \mathcal{G}_n} f(k_{\sigma(1)}, ..., k_{\sigma(n)})$ we mean the canonical symmetrization of a function f in n variables such that $\mathcal{G}_n$ is the symmetric group on $\{1, ..., n\}$. Then, for $u \in Dom(\delta)$, the operator δ is given by

$$\delta(u) := \sum_{n=1}^{\infty} J_n\left(\widetilde{f_n} \mathbb{1}_{\Delta_n}\right).$$

Another way to characterize δ is by the duality

$$\mathbb{E}[\langle DF, u \rangle] = \mathbb{E}[F \delta(u)], \quad F \in \mathbb{D}^{1,2}, u \in Dom(\delta), \tag{2.1}$$

such that we can identify δ as the adjoint operator of D. Furthermore we can rewrite its domain to

$$Dom(\delta) = \left\{ u \in L^2(\Omega, l^2(\mathbb{N})) \middle| \exists \, C_u > 0 \, \forall \, F \in \mathbb{D}^{1,2} : |\mathbb{E}[\langle DF, u \rangle]| \leq C_u \sqrt{\mathbb{E}[F^2]} \right\}.$$

For

$$F \in Dom(L) = \left\{ F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n) \in L^2(\Omega, \sigma(X), \mathbb{P}) \middle| \sum_{n=1}^{\infty} n^2 n! \, \|f_n\|_{l^2(\mathbb{N})^{\otimes n}}^2 < \infty \right\}$$

we define by

$$LF := \sum_{n=1}^{\infty} -n J_n(f_n), \quad L^{-1} F := \sum_{n=1}^{\infty} -\frac{1}{n} J_n(f_n)$$

the Ornstein–Uhlenbeck operator L and the pseudo-inverse Ornstein–Uhlenbeck operator L^{-1}. You can show that $F \in Dom(L)$ is equivalent to $F \in \mathbb{D}^{1,2}$ and $DF \in Dom(\delta)$; in this case, it holds that

$$L = -\delta D. \tag{2.2}$$

At last we recall the main ideas of Stein's method for normal approximation, starting with the important characterisation

$$Z \sim \mathcal{N}(0,1) \Leftrightarrow \mathbb{E}[f'(Z) - Zf(Z)] = 0 \tag{2.3}$$

for all continuous differentiable f such that the appearing expectations exist. So if a random variable is in some sense close to $\mathcal{N}(0,1)$, it is likely that the expectation in (2.3) is close to 0. This motivates the Stein-equation, written in the case of Kolmogorov distance, namely

$$f_z'(F) - Ff_z(F) = \mathbb{1}_{\{F \leq z\}} - \Phi(z),$$

respectively

$$\mathbb{E}[f_z'(F) - Ff_z(F)] = \mathbb{P}(F \leq z) - \Phi(z).$$

The solution to this equation is given, see Lemma 2.1 in Barbour and Chen (2005), by

$$f_z(w) = \begin{cases} \frac{\Phi(w)(1-\Phi(z))}{p(w)} & w \leq z, \\ \frac{\Phi(z)(1-\Phi(w))}{p(w)} & w > z, \end{cases} \tag{2.4}$$

where $p(w) = e^{-w^2/2}/\sqrt{2\pi}$ is the density of $\mathcal{N}(0,1)$. Later on we will use the following bounds, see Lemma 2.3 in Chen et al. (2011):

$$\frac{1-\Phi(w)}{p(w)} \leq \min\left\{\frac{1}{w}, \frac{\sqrt{2\pi}}{2}\right\}, \quad w > 0, \tag{2.5}$$

$$|wf_z(w)| \leq 1, \quad w \in \mathbb{R}, \tag{2.6}$$

and

$$|wf_z(w)| \leq 1 - \Phi(z), \quad w < 0 \leq z. \tag{2.7}$$

Note that (2.7) follows with (2.4) and (2.5) by writing

$$\begin{aligned} |wf_z(w)| &= |w| \sqrt{2\pi} e^{w^2/2} \Phi(w)(1-\Phi(z)) \\ &= (1-\Phi(z))(1-\Phi(|w|))\sqrt{2\pi}\,|w|\, e^{|w|^2/2} \\ &\leq (1-\Phi(z)), \end{aligned}$$

where we also used the symmetry of Φ. We will need this more precise bound of $|wf_z(w)|$ for the main result in Section 3, where we distinguish different cases for $w \in \mathbb{R}$.

3. Main Results

Now we present the main theorem of our paper.

Theorem 3.1 (Moderate deviations for Rademacher functionals)**.** *Let* $F \in \mathbb{D}^{1,2}$ *with* $\mathbb{E}[F] = 0, \operatorname{Var}(F) = 1$*, and*

$$Ff_z(F) + \mathbb{1}_{\{F > z\}} \in \mathbb{D}^{1,2} \quad \forall\, z \in \mathbb{R},$$

$$\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right| \in Dom(\delta).$$

Assume that there exists a constant $A > 0$ *and increasing functions* $\gamma_1(t), \gamma_2(t)$ *such that* $e^{tF} \in \mathbb{D}^{1,2}$ *and*

(A1) $$\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right] \leq \gamma_1(t)\mathbb{E}\left[e^{tF}\right],$$

(A2) $$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right|\right)\right| e^{tF}\right] \leq \gamma_2(t)\mathbb{E}\left[e^{tF}\right],$$

for all $0 \le t \le A$*. For* $d_0 \ge 0$*, let*

$$A_0(d_0) := \max\left\{0 \le t \le A : \frac{t^2}{2}(\gamma_1(t) + \gamma_2(t)) \le d_0\right\}.$$

Then, for any $d_0 \ge 0$*,*

$$\left|\frac{\mathbb{P}(F > z)}{1 - \Phi(z)} - 1\right| \le 25e^{d_0}(1 + z^2)(\gamma_1(z) + \gamma_2(z))$$

provided that $0 \le z \le A_0(d_0)$*.*

In consequence, the following result is achieved.

Theorem 3.2. *Under the assumptions from Theorem 3.1, there is*

$$\left|\frac{\mathbb{P}(F > z)}{1 - \Phi(z)} - 1\right| \le 25e^{\frac{z^2}{2}(\gamma_1(z)+\gamma_2(z))}(1 + z^2)(\gamma_1(z) + \gamma_2(z))$$

for all $0 \le z \le A$*.*

As a first application we treat the i.i.d.-case: For our sequence $(X_i)_{i\in\mathbb{N}}$ of Rademacher random variables we consider the standardized nth partial sum

$$F := F_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i := \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{X_i - (2p-1)}{2\sqrt{pq}}.$$

The classical result can be received:

Corollary 3.3. *Recall the definition of* F_n *from above. Then*

$$\frac{\mathbb{P}(F_n > z)}{1 - \Phi(z)} = 1 + O(1)(1 + z^2)\gamma_n(z), \tag{3.1}$$

for $0 \le z \le n^{1/6}$ *such that* $(1 + z^2)\gamma_n(z) \le 1$*, where O(1) is bounded by a constant and*

$$\gamma_n(z) := e^{O(1)zn^{-1/2}}\frac{(1 + z)}{\sqrt{n}}.$$

Remark 3.4. We obtain the optimal range $0 \le z \le n^{1/6}$ from (1.2), but there is no $\log(n)$ in our error term compared to Corollary 2.2 in Fang and Koike (2023). The proof of Corollary 3.3 is basically a shorter version of the proof of Theorem 1.1.

For the proof of Theorem 3.1 we will need two auxiliary lemmas.

Lemma 3.5 (Bound for the moment generating function)**.** *Under the assumptions of Theorem 3.1, for* $0 \le t \le A$*, we have*

$$\mathbb{E}\left[e^{tF}\right] \le \exp\left\{\frac{t^2}{2}(1 + \gamma_1(t) + \gamma_2(t))\right\}. \tag{3.2}$$

Then, for $0 \le t \le A_0(d_0)$*,*

$$\mathbb{E}\left[e^{tF}\right] \le e^{d_0}e^{t^2/2}. \tag{3.3}$$

Proof: Let $h(t) := \mathbb{E}\left[e^{tF}\right]$. We recall that $\mathbb{E}\left[e^{tF}\right] < \infty$ is implied by $e^{tF} \in \mathbb{D}^{1,2}$ for $0 \le t \le A$, and so, by the continuity of the exponential funtion, we have $h'(t) = \mathbb{E}\left[Fe^{tF}\right]$. It follows with (2.2) and (2.1) that

$$\begin{aligned}\mathbb{E}\left[Fe^{tF}\right] &= \mathbb{E}\left[(LL^{-1}F)e^{tF}\right] \\ &= \mathbb{E}\left[(-\delta DL^{-1}F)e^{tF}\right] \\ &= \mathbb{E}\left[\langle De^{tF}, -DL^{-1}F\rangle\right].\end{aligned} \tag{3.4}$$

Now we consider the k-th component of De^{tF}, which gives us

$$\begin{aligned} D_k e^{tF} &= \sqrt{p_k q_k}\left[e^{tF_k^+} - e^{tF_k^-}\right] \\ &= t\sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} e^{tu} du \\ &= t\sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} \left[e^{tu} - e^{tF}\right] du + te^{tF} D_k F \\ &=: tR_k + te^{tF} D_k F. \end{aligned}$$

If we define $R := (R_1, R_2, \ldots)$, we can go on from (3.4) by writing

$$\begin{aligned} \mathbb{E}\left[Fe^{tF}\right] &= \mathbb{E}\left[\langle tR, -DL^{-1}F\rangle\right] + \mathbb{E}\left[\langle te^{tF} DF, -DL^{-1}F\rangle\right] \\ &\leq t\mathbb{E}\left[e^{tF}\right] + t\mathbb{E}\left|\langle R, -DL^{-1}F\rangle\right| + t\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right]. \end{aligned} \tag{3.5}$$

Without loss of generality $F_k^- \leq F \leq F_k^+$ since for the other case we just have to change the sign. Then we can bound R_k as follows.

$$\begin{aligned} R_k &= \sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} \left[e^{tu} - e^{tF}\right] du \\ &\leq \sqrt{p_k q_k}\int_{F_k^-}^{F_k^+} \left[e^{tF_k^+} - e^{tF_k^-}\right] du \\ &= \sqrt{p_k q_k}\left[e^{tF_k^+} - e^{tF_k^-}\right]\int_{F_k^-}^{F_k^+} du \\ &= D_k e^{tF} \cdot \frac{1}{\sqrt{p_k q_k}} D_k F \end{aligned}$$

and by combining both cases

$$|R_k| \leq \frac{1}{\sqrt{p_k q_k}} D_k e^{tF} \cdot D_k F. \tag{3.6}$$

By condition (A2) and (3.6) we get

$$\begin{aligned} t\mathbb{E}\left|\langle R, -DL^{-1}F\rangle\right| &\leq t\mathbb{E}\left[\langle |R|, \left|DL^{-1}F\right|\rangle\right] \\ &\leq t\mathbb{E}\left[\langle De^{tF}, \frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right|\rangle\right] \\ &\leq t\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right|\right)\right| e^{tF}\right] \\ &\leq t\gamma_2(t)\mathbb{E}\left[e^{tF}\right]. \end{aligned} \tag{3.7}$$

By condition (A1), for $0 \leq t \leq A$,

$$t\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right] \leq t\gamma_1(t)\mathbb{E}\left[e^{tF}\right]. \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we have for $0 \leq t \leq A$,

$$\begin{aligned} h'(t) &= \mathbb{E}\left[Fe^{tF}\right] \\ &\leq th(t) + \{t(\gamma_1(t) + \gamma_2(t))\}h(t) \\ &= \{1 + \gamma_1(t) + \gamma_2(t)\}\, t\, h(t). \end{aligned}$$

Having in mind that $h(0) = 1$, and γ_1 and γ_2 are increasing, we complete the proof of (3.2) by solving the foregoing differential inequality:

$$\begin{aligned}\log(h(t)) &= \int_0^t \frac{h'(s)}{h(s)} ds \\ &\leq \int_0^t (1+\gamma_1(s)+\gamma_2(s)) s ds \\ &\leq \int_0^t (1+\gamma_1(t)+\gamma_2(t)) s ds \\ &= \frac{t^2}{2}(1+\gamma_1(t)+\gamma_2(t)),\end{aligned}$$

now we apply exp(.) on both sides. At last, (3.3) follows immediately from (3.2) by definition of $A_0(d_0)$. □

Lemma 3.6. *Under the assumptions of Theorem 3.1, we have for* $0 \leq z \leq A_0(d_0)$,

$$\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| F e^{F^2/2} \mathbb{1}_{\{0 \leq F \leq z\}}\right] \leq 6 e^{d_0}(1+z^2)\gamma_1(z) \tag{3.9}$$

and

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}} DF \left|DL^{-1}F\right|\right)\right| F e^{F^2/2} \mathbb{1}_{\{0 \leq F \leq z\}}\right] \leq 6 e^{d_0}(1+z^2)\gamma_2(z). \tag{3.10}$$

Proof: Same as Zhang (2023) we apply the idea in Lemma 5.2 in Chen et al. (2013) for this proof. For $a \in \mathbb{R}$, denote $[a] = \max\{n \in \mathbb{N} : n \leq a\}$. Next, we define $H := 1 - \langle DF, -DL^{-1}F\rangle$.

$$\mathbb{E}\left[|H| F e^{F^2/2} \mathbb{1}_{\{0 \leq F \leq z\}}\right] = \sum_{j=1}^{[z]} \mathbb{E}\left[|H| F e^{F^2/2} \mathbb{1}_{\{j-1 \leq F \leq j\}}\right] + \mathbb{E}\left[|H| F e^{F^2/2} \mathbb{1}_{\{[z] \leq F \leq z\}}\right].$$

For the first term we get

$$\begin{aligned}\sum_{j=1}^{[z]} \mathbb{E}\left[|H| F e^{F^2/2} \mathbb{1}_{\{j-1 \leq F \leq j\}}\right] &\leq \sum_{j=1}^{[z]} j e^{(j-1)^2/2 - j(j-1)} \mathbb{E}\left[|H| e^{jF} \mathbb{1}_{\{j-1 \leq F \leq j\}}\right] \\ &\leq 3 \sum_{j=1}^{[z]} j e^{-j^2/2} \mathbb{E}\left[|H| e^{jF} \mathbb{1}_{\{j-1 \leq F \leq j\}}\right]\end{aligned}$$

and similarly, for the second

$$\begin{aligned}\mathbb{E}\left[|H| F e^{F^2/2} \mathbb{1}_{\{[z] \leq F \leq z\}}\right] &\leq z e^{[z]^2/2 - [z]z} \mathbb{E}\left[|H| e^{zF} \mathbb{1}_{\{[z] \leq F \leq z\}}\right] \\ &\leq 3 z e^{-z^2/2} \mathbb{E}\left[|H| e^{zF} \mathbb{1}_{\{[z] \leq F \leq z\}}\right].\end{aligned}$$

For both terms, we used similar manipulations, namely for $j - 1 \leq F \leq j$:

- $e^{(j-1)^2/2 - j(j-1)} = e^{j^2/2 - j + 1/2 - j^2 + j} = e^{-j^2/2} e^{1/2} \leq 3 e^{-j^2/2}$.
- $e^{(j-F)^2/2} \leq e^{1/2} \Leftrightarrow e^{F^2/2} \leq e^{-j^2/2 + jF + 1/2} \Leftrightarrow e^{F^2/2} \leq e^{(j-1)^2/2 - j(j-1)} e^{jF}$.

And for $[z] \leq F \leq z$:

- $e^{[z]^2/2 - [z]z} = e^{[z]^2/2 - [z]z + z^2/2 - z^2/2} = e^{(z-[z])^2/2} e^{-z^2/2} \leq 3 e^{-z^2/2}$.
- $e^{(z-F)^2/2} \leq e^{(z-[z])^2/2} \Leftrightarrow e^{F^2/2} \leq e^{(z-[z])^2/2 + zF - z^2/2} \Leftrightarrow e^{F^2/2} \leq e^{[z]^2/2 - [z]z} e^{zF}$.

By condition (A1) and (3.3), and recalling that γ_1 is increasing, for any $0 \leq x \leq z \leq A_0(d_0)$

$$\begin{aligned}e^{-x^2/2} \mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{xF}\right] &\leq \gamma_1(x) \mathbb{E}[e^{xF - x^2/2}] \\ &\leq e^{d_0} \gamma_1(x) \leq e^{d_0} \gamma_1(z).\end{aligned}$$

By the foregoing inequalities,

$$\mathbb{E}\left[|H|\, F e^{F^2/2} \mathbb{1}_{\{0\le F\le z\}}\right] \le 3e^{d_0}\gamma_1(z)\left(\sum_{j=1}^{[z]} j + z\right) \le 6e^{d_0}(1+z^2)\gamma_1(z).$$

The other statement of the lemma can be shown analogously. □

Now we are ready to prove our two theorems.

Proof of Theorem 3.1: We note at first that

$$|\mathbb{P}(F>z)-(1-\Phi(z))| = |1-\mathbb{P}(F\le z)-(1-\Phi(z))| = |\Phi(z)-\mathbb{P}(F\le z)|.$$

By Stein's method and the proof of Theorem 3.1 in Eichelsbacher et al. (2023) we have for $z\in\mathbb{R}$

$$|\mathbb{P}(F>z)-(1-\Phi(z))| = \left|\mathbb{E}[f_z'(F)-Ff_z(F)]\right| \le J_1+J_2$$

with

$$J_1 := \mathbb{E}\left|f_z'(F)\big(1-\langle DF,-DL^{-1}F\rangle\big)\right|,$$
$$J_2 := \mathbb{E}\left[(Ff_z(F)+\mathbb{1}_{\{F>z\}})\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right].$$

For the upcoming estimation we can split J_2 into two terms, namely

$$|J_2| \le \mathbb{E}\left[\left|\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right|\,\left|Ff_z(F)+\mathbb{1}_{\{F>z\}}\right|\right] \le J_{21}+J_{22}$$

with

$$J_{21} := \mathbb{E}\left[\left|\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right|\,|Ff_z(F)|\right],$$
$$J_{22} := \mathbb{E}\left[\left|\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right|\mathbb{1}_{\{F>z\}}\right].$$

Using the same arguments as in the proof of Proposition 4.1 in Zhang (2023), in particular (2.4), (2.6) and (2.7), we have

$$J_{21} \le J_{23}+J_{24}+J_{25}$$

with

$$J_{23} := (1-\Phi(z))\cdot\mathbb{E}\left[\left|\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right|\mathbb{1}_{\{F<0\}}\right],$$
$$J_{24} := \sqrt{2\pi}\cdot(1-\Phi(z))\cdot\mathbb{E}\left[\left|\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right| Fe^{F^2/2}\mathbb{1}_{\{0\le F\le z\}}\right],$$
$$J_{25} := \mathbb{E}\left[\left|\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right|\mathbb{1}_{\{F>z\}}\right] = J_{22}.$$

Thus,

$$|J_2| \le J_{23}+J_{24}+2\cdot J_{25}. \tag{3.11}$$

For J_{23}, by condition (A2) with $t=0$ and noting that γ_2 is increasing,

$$\mathbb{E}\left[\left|\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right|\mathbb{1}_{\{F<0\}}\right] \le \mathbb{E}\left[\left|\delta\Big(\frac{1}{\sqrt{pq}}DF\,|DL^{-1}F|\Big)\right|\right]$$
$$\le \gamma_2(0) \le e^{d_0}\gamma_2(z). \tag{3.12}$$

For J_{24}, by Lemma 3.6, we have

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|Fe^{F^2/2}\mathbb{1}_{\{0\le F\le z\}}\right]\le 6e^{d_0}(1+z^2)\gamma_2(z). \tag{3.13}$$

For J_{25}, by condition (A2) and (3.3), for $0\le z\le A_0(d_0)$,

$$\begin{aligned}
J_{25} &= \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F>z\}}e^{zF}e^{-zF}\right]\\
&\le \mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F>z\}}e^{zF}\right]e^{-z^2}\\
&\le \gamma_2(z)\mathbb{E}\left[e^{zF}\right]e^{-z^2}\\
&\le e^{d_0}\gamma_2(z)e^{-z^2/2}.
\end{aligned}$$

We recall that for $z>0$

$$e^{-z^2/2}\le\sqrt{2\pi}\cdot(1+z)\cdot(1-\Phi(z))\le\frac{3\sqrt{2\pi}}{2}\cdot(1+z^2)\cdot(1-\Phi(z)).$$

Then, for $0\le z\le A_0(d_0)$,

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|\mathbb{1}_{\{F>z\}}\right]\le\frac{3e^{d_0}\sqrt{2\pi}}{2}(1+z^2)\gamma_2(z)(1-\Phi(z)). \tag{3.14}$$

Therefore, combining "(3.11) – (3.14)", for $0\le z\le A_0(d_0)$, we have

$$|J_2|\le\left(1+6\sqrt{2\pi}+3\sqrt{2\pi}\right)e^{d_0}(1+z^2)\gamma_2(z)(1-\Phi(z))\le 25e^{d_0}(1+z^2)\gamma_2(z)(1-\Phi(z)).$$

For the remaining term J_1 we have a similar approach after using again Stein's equation:

$$\begin{aligned}
|J_1| &= \mathbb{E}\left|f_z'(F)\big(1-\langle DF,-DL^{-1}F\rangle\big)\right|\\
&\le \mathbb{E}\left[\left|f_z'(F)\right|\left|1-\langle DF,-DL^{-1}F\rangle\right|\right]\\
&\le J_{11}+J_{12}+J_{13}
\end{aligned}$$

with

$$\begin{aligned}
J_{11} &:= \mathbb{E}\left[|Ff_z(F)|\left|1-\langle DF,-DL^{-1}F\rangle\right|\right],\\
J_{12} &:= \mathbb{E}\left[(1-\Phi(z))\left|1-\langle DF,-DL^{-1}F\rangle\right|\right],\\
J_{13} &:= \mathbb{E}\left[\mathbb{1}_{\{F>z\}}\left|1-\langle DF,-DL^{-1}F\rangle\right|\right].
\end{aligned}$$

From here on we can identify any of these terms with a corresponding term from the first part of the proof, namely $J_{21}-J_{25}$. Therefore, combining these modified estimations, for $0\le z\le A_0(d_0)$, we have

$$|J_1|\le\left(1+1+6\sqrt{2\pi}+3\sqrt{2\pi}\right)e^{d_0}(1+z^2)\gamma_1(z)(1-\Phi(z))\le 25e^{d_0}(1+z^2)\gamma_1(z)(1-\Phi(z)).$$

All in all, we have shown, for $0\le z\le A_0(d_0)$,

$$|\mathbb{P}(F>z)-(1-\Phi(z))|\le 25e^{d_0}(1+z^2)(\gamma_1(z)+\gamma_2(z))(1-\Phi(z))$$

or equivalently

$$\left|\frac{\mathbb{P}(F>z)}{1-\Phi(z)}-1\right|\le 25e^{d_0}(1+z^2)(\gamma_1(z)+\gamma_2(z)).$$

□

Proof of Theorem 3.2: Let $0 \leq z_0 \leq A$ be fixed. Choose $d_0 = \frac{z_0^2}{2}(\gamma_1(z_0) + \gamma_2(z_0))$. Per definition there is $0 \leq z_0 \leq A_0(d_0)$. Hence, we may apply Theorem 3.1, which then implies

$$\begin{aligned}\left|\frac{\mathbb{P}(F > z_0)}{1 - \Phi(z_0)} - 1\right| &\leq 25e^{d_0}(1 + z_0^2)(\gamma_1(z_0) + \gamma_2(z_0)) \\ &= 25e^{\frac{z_0^2}{2}(\gamma_1(z_0)+\gamma_2(z_0))}(1 + z_0^2)(\gamma_1(z_0) + \gamma_2(z_0)) \,. \end{aligned}$$ □

4. Proofs I: Infinite weighted 2-runs

Proof of Theorem 1.1: In what follows, we show that all the assumptions of Theorem 3.1 are verified. Note that although here the Rademacher random variables are indexed by $\mathbb{Z}$ instead of $\mathbb{N}$, Theorem 3.1 can be fully carried to this setting. Since the coefficient sequence $(a_i^{(n)})_{i\in\mathbb{Z}}$ is in $l^1(\mathbb{Z})$ we have $\left\|a^{(n)}\right\|_{l^p(\mathbb{Z})} < \infty \,\forall\, p \in \mathbb{N}$. By definition $\mathbb{E}[F] = 0$ and $\mathrm{Var}(F) = 1$, and the rewritten random variable

$$F := F_n = \frac{1}{\sqrt{\mathrm{Var}(G_n)}} \sum_{i\in\mathbb{Z}} a_i^{(n)} \left[\xi_i\xi_{i+1} - \frac{1}{4}\right] = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}} \sum_{i\in\mathbb{Z}} a_i^{(n)} \left[X_i + X_iX_{i+1} + X_{i+1}\right]$$

is bounded. In particular we will use $\mathbb{E}\,|G_n| \leq \left\|a^{(n)}\right\|_{l^1(\mathbb{Z})}$ and

$$\frac{1}{16}\left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2 \leq \mathrm{Var}(G_n) = \frac{3}{16}\sum_{i\in\mathbb{Z}}(a_i^{(n)})^2 + \frac{1}{8}\sum_{i\in\mathbb{Z}} a_i^{(n)}a_{i+1}^{(n)} \leq \frac{5}{16}\left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2 . \tag{4.1}$$

Mostly with the summability of $(a_i^{(n)})_{i\in\mathbb{Z}}$, $|e^x - 1| \leq |x|\, e^{|x|} \,\forall\, x \in \mathbb{R}$ and equation (12.2) in Privault (2008) we see that $F \in \mathbb{D}^{1,2}$ and $e^{tF} \in \mathbb{D}^{1,2} \,\forall\, t \in \mathbb{R}$. Regarding the assumptions that $\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right| \in Dom(\delta)$ and $Ff_z(F) + \mathbb{1}_{\{F>z\}} \in \mathbb{D}^{1,2} \,\forall\, z \in \mathbb{R}$, we follow the argumentation of Eichelsbacher et al. (2023), see in particular Remark 3.5 in there. F is an element of the sum of the first and second Rademacher chaos, see the beginning of the proof of Theorem 1.1 in Eichelsbacher et al. (2023), and by hypercontractivity we find that $F \in L^4(\Omega)$. Following the calculations in the proof of Lemma 3.7 in Döbler and Krokowski (2019) with $u_k = \frac{1}{\sqrt{p_kq_k}}D_kF\left|D_kL^{-1}F\right|$ for $k \in \mathbb{Z}$, it can be shown that assumption (2.14) in Proposition 2.2 in Krokowski et al. (2017) is satisfied. This implies that $u = \frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right| \in Dom(\delta)$. Further, it implies that $\mathbb{E}[(Ff_z(F) + \mathbb{1}_{\{F>z\}})\delta(u)] = \mathbb{E}[\langle D(Ff_z(F) + \mathbb{1}_{\{F>z\}}), u\rangle]$, which is why we do not need to verify whether $Ff_z(F) + \mathbb{1}_{\{F>z\}}$ is an element of $\mathbb{D}^{1,2} \,\forall\, z \in \mathbb{R}$. Now we start to compute the terms appearing in (A1) and (A2). By definition

$$F_k^+ = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(\sum_{\substack{i\in\mathbb{Z}\\ i\neq k-1,k}} a_i^{(n)}\left[X_i + X_iX_{i+1} + X_{i+1}\right] + a_{k-1}^{(n)}(2X_{k-1} + 1) + a_k^{(n)}(2X_{k+1} + 1)\right),$$

$$F_k^- = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(\sum_{\substack{i\in\mathbb{Z}\\ i\neq k-1,k}} a_i^{(n)}\left[X_i + X_iX_{i+1} + X_{i+1}\right] - a_{k-1}^{(n)} - a_k^{(n)}\right),$$

and we get (for fixed $n \in \mathbb{N}$)

$$\begin{aligned} D_kF &= \frac{1}{2}\left(F_k^+ - F_k^-\right) \\ &= \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(a_{k-1}^{(n)}(X_{k-1} + 1) + a_k^{(n)}(X_{k+1} + 1)\right). \end{aligned}$$

Further we obtain

$$-L^{-1}F = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}} \sum_{i\in\mathbb{Z}} a_i^{(n)} \left[X_i + \frac{1}{2} X_i X_{i+1} + X_{i+1} \right]$$

and so

$$-D_k L^{-1}F = \frac{1}{8\sqrt{\mathrm{Var}(G_n)}} \left(a_{k-1}^{(n)}(X_{k-1}+2) + a_k^{(n)}(X_{k+1}+2) \right).$$

With these expressions we can compute the scalar product

$$\begin{aligned}\langle DF, -DL^{-1}F\rangle = \frac{1}{32\,\mathrm{Var}(G_n)} \Bigg(\sum_{k\in\mathbb{Z}} &(a_{k-1}^{(n)})^2 \left[X_{k-1}^2 + 3X_{k-1} + 2 \right] \\ &+ (a_k^{(n)})^2 \left[X_{k+1}^2 + 3X_{k+1} + 2 \right] \\ &+ a_{k-1}^{(n)} a_k^{(n)} \left[3X_{k-1} + 2X_{k-1}X_{k+1} + 3X_{k+1} + 4 \right] \Bigg).\end{aligned}$$

According to (2.13) in Krokowski et al. (2017) it holds that

$$1 = \mathbb{E}[\langle DF, -DL^{-1}F\rangle] = \frac{1}{32\,\mathrm{Var}(G_n)} \sum_{k\in\mathbb{Z}} 3(a_{k-1}^{(n)})^2 + 4a_{k-1}^{(n)} a_k^{(n)} + 3(a_k^{(n)})^2.$$

We use the Cauchy–Schwarz inequality for

$$\mathbb{E}\left[\left| 1 - \langle DF, -DL^{-1}F\rangle \right| e^{tF} \right] \le \left(\mathbb{E}\left[\left(1 - \langle DF, -DL^{-1}F\rangle \right)^2 e^{tF} \right] \right)^{\frac{1}{2}} \left(\mathbb{E}\left[e^{tF} \right] \right)^{\frac{1}{2}}$$

and deal with the double sum resulting from the square of

$$\begin{aligned}\langle DF, -DL^{-1}F\rangle - 1 = \frac{1}{32\,\mathrm{Var}(G_n)} \Bigg(\sum_{k\in\mathbb{Z}} &(a_{k-1}^{(n)})^2 3X_{k-1} + (a_k^{(n)})^2 3X_{k+1} \\ &+ a_{k-1}^{(n)} a_k^{(n)} \left[3X_{k-1} + 2X_{k-1}X_{k+1} + 3X_{k+1} \right] \Bigg).\end{aligned}$$

Then we can write $\left(1 - \langle DF, -DL^{-1}F\rangle\right)^2 = B_1 + \ldots + B_9$ with

$$B_1 = \frac{9}{1024(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_{k-1}^{(n)})^2 (a_{l-1}^{(n)})^2 X_{k-1} X_{l-1},$$

$$B_2 = \frac{9}{1024(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_k^{(n)})^2 (a_l^{(n)})^2 X_k X_l,$$

$$B_3 = \frac{9}{1024(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_{k-1}^{(n)})^2 (a_l^{(n)})^2 X_{k-1} X_l,$$

$$B_4 = \frac{9}{1024(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_k^{(n)})^2 (a_{l-1}^{(n)})^2 X_k X_{l-1},$$

$$B_5 = \frac{3}{1024(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_{k-1}^{(n)})^2 (a_{l-1}^{(n)})(a_l^{(n)}) X_{k-1} \left[3X_{l-1} + 2X_{l-1}X_{l+1} + 3X_{l+1}, \right]$$

$$B_6 = \frac{3}{1024(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_k^{(n)})^2 (a_{l-1}^{(n)})(a_l^{(n)}) X_k \left[3X_{l-1} + 2X_{l-1}X_{l+1} + 3X_{l+1} \right],$$

$$B_7 = \frac{3}{1024(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_{k-1}^{(n)})(a_k^{(n)})(a_{l-1}^{(n)})^2 X_{l-1} \left[3X_{k-1} + 2X_{k-1}X_{k+1} + 3X_{k+1} \right],$$

$$B_8 = \frac{3}{1024(\mathrm{Var}(G_n))^2} \sum_{k,l\in\mathbb{Z}} (a_{k-1}^{(n)})(a_k^{(n)})(a_l^{(n)})^2 X_l \left[3X_{k-1} + 2X_{k-1}X_{k+1} + 3X_{k+1}\right],$$

where $B_1 = B_2 = B_3 = B_4$ and $B_5 = B_6 = B_7 = B_8$ by symmetry and change of variables. The last missing term is given by

$$B_9 = \frac{1}{1024(\mathrm{Var}(G_n))^2} \Bigg(\sum_{k,l\in\mathbb{Z}} (a_{k-1}^{(n)})(a_k^{(n)})(a_{l-1}^{(n)})(a_l^{(n)}) \cdot [3X_{k-1} + 2X_{k-1}X_{k+1} + 3X_{k+1}] \cdot [3X_{l-1} + 2X_{l-1}X_{l+1} + 3X_{l+1}] \Bigg).$$

So basically we have to deal with three classes of subterms in total. Since they will be multiplied with e^{tF}, we have to study

$$\mathbb{E}[X_i e^{tF}], \quad \mathbb{E}[X_iX_j e^{tF}], \quad \mathbb{E}[X_iX_jX_k e^{tF}], \quad \mathbb{E}[X_iX_jX_kX_l e^{tF}]$$

for $i \neq j \neq k \neq l$ — if two or more indices are equal, it is just one of the terms from before or immediately $\mathbb{E}[e^{tF}]$. This is done in the following lemma and we will refer to it, in particular the inequalities shown in its proof.

Lemma 4.1. *In the setting of Theorem 1.1 we have for* $i \neq j \neq k \neq l \in \mathbb{Z}$

$$\begin{aligned} \left|\mathbb{E}[X_i e^{tF}]\right| \leq\ & \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \cdot \left(\left|a_{i-1}^{(n)}\right| + \left|a_i^{(n)}\right|\right) \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ & + \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i\}\\ n_1\in\{i-2,i+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ & + \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.2}$$

$$\begin{aligned} \left|\mathbb{E}[X_iX_j e^{tF}]\right| \leq\ & \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \cdot \left|a_{\min(i,j)}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \mathbb{1}_{\{|i-j|=1\}} \\ & + \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1\}\\ n_1\in\{i-2,i+1,j-2,j+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ & + \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i,j-1,j\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.3}$$

$$\begin{aligned} \left|\mathbb{E}[X_iX_jX_k e^{tF}]\right| \leq\ & \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1,j,k-1,k\}\\ n_1\in\{i-2,i+1,j-2,j+1,k-2,k+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ & + \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i,j-1,j,k-1,k\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.4}$$

$$\begin{aligned} \left|\mathbb{E}[X_iX_jX_kX_l e^{tF}]\right| \leq\ & \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1,j,k-1,k,l-1,l\}\\ n_1\in\{i-2,i+1,j-2,j+1,k-2,k+1,l-2,l+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ & + \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i,j-1,j,k-1,k,l-1,l\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.5}$$

Proof: The first key element of our strategy is to split F into F_a, the summands that depend on the X's multiplied with e^{tF}, and F_u, the summands that are independent. We should have in mind that F_a and F_u are not necessarily independent from each other. To get this dependency structure under control we will make use of several Taylor expansions of the exponential. Note that there are remainder functions $r_1, r_2 : \mathbb{R} \to \mathbb{R}$ such that $e^x = 1 + x \cdot r_1(x) = 1 + x + \frac{x^2}{2} \cdot r_2(x)$ with $|r_1(x)|, |r_2(x)| \leq e^{\max\{0,x\}} \leq e^{|x|}$ for all $x \in \mathbb{R}$. So, the second key element is an iterated Taylor expansion on e^{tF} according to the following scheme: For a finite index set I, let there be real numbers $(x_j)_{j\in I}$, $(y_j)_{j\in I}$ and z. Then by iterated Taylor expansion there is

$$\begin{aligned} e^{z+\sum_{j\in I} x_j} &= 1 \cdot e^z + \sum_{j\in I} x_j \cdot e^z + \frac{1}{2}\Big(\sum_{j\in I} x_j\Big)^2 r_2\Big(\sum_{j\in I} x_j\Big) \cdot e^z \\ &= 1 \cdot e^z + \sum_{j\in I} x_j \cdot 1 \cdot e^{z-y_j} + \sum_{j\in I} x_j \cdot y_j r_1(y_j) \cdot e^{z-y_j} + \frac{1}{2}\Big(\sum_{j\in I} x_j\Big)^2 r_2\Big(\sum_{j\in I} x_j\Big) \cdot e^z. \end{aligned} \tag{4.6}$$

We will use the short notation $A_k := a_k^{(n)} \left[X_k + X_k X_{k+1} + X_{k+1}\right]$ for the upcoming computations. In the case of $\mathbb{E}[X_i e^{tF}]$:

$$F_a = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}(A_{i-1} + A_i)$$

and by independence

$$\begin{aligned} \mathbb{E}[X_i e^{tF}] = \mathbb{E}[X_i e^{tF_a} e^{tF_u}] &= \mathbb{E}[X_i e^{tF_u}] + t\mathbb{E}[X_i F_a e^{tF_u}] + t^2\mathbb{E}[X_i F_a^2 r_2(tF_a) e^{tF_u}/2] \\ &= t\mathbb{E}[X_i F_a e^{tF_u}] + t^2\mathbb{E}[X_i F_a^2 r_2(tF_a) e^{tF_u}/2], \end{aligned}$$

where we chose $z = F_u$ and $\sum_{j\in I} x_j = F_a$ — note that I will increase with every case since the number of multiplied X's increases. For the first order term we split F_u in the same manner as before, $F_u = F_{u_a} + F_{u_u}$, such that $F_{u_a} = (A_{i-2} + A_{i+1})/4\sqrt{\mathrm{Var}(G_n)}$ and use the iteration from (4.6). Then

$$X_i F_a = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\Big(a_{i-1}^{(n)}(X_{i-1}X_i + X_{i-1} + 1) + a_i^{(n)}(1 + X_{i+1} + X_i X_{i+1})\Big)$$

and

$$\begin{aligned} t\mathbb{E}[X_i F_a e^{tF_u}] &= t\mathbb{E}\left[X_i F_a e^{tF_{u_a}} e^{tF_{u_u}}\right] \\ &= t\mathbb{E}\left[X_i F_a e^{tF_{u_u}}\right] + t^2\mathbb{E}\left[X_i F_a F_{u_a} r_1(tF_{u_a}) e^{tF_{u_u}}\right], \end{aligned}$$

so $y_j = F_{u_a}$. From here on we get e^{tF} back by bounding the difference of the independent part and F, e.g.

$$|F_{u_u} - F| = |F_{u_a} + F_a| \leq c = O\left(\frac{1}{\sqrt{\mathrm{Var}(G_n)}}\right) \to 0 \text{ as } n \to \infty.$$

Since the exact constant is not important, we always write just c if we use that type of estimation, and in the same way C for prefactors. Thus

$$\begin{aligned} t\left|\mathbb{E}[X_i F_a e^{tF_u}]\right| \leq& \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \cdot \left(\left|a_{i-1}^{(n)}\right| + \left|a_i^{(n)}\right|\right) \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ &+ \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i\} \\ n_1\in\{i-2,i+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.7}$$

For the second order term we just bound

$$t^2 \left|\mathbb{E}[X_i F_a^2 r_2(tF_a) e^{tF_u}/2]\right| \leq \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \tag{4.8}$$

In the case of $\mathbb{E}[X_i X_j e^{tF}]$:

$$F_a = \begin{cases} \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}(A_i + A_{j-1} + A_j)\,, & i = j+1, \\ \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}(A_{i-1} + A_i + A_{j-1} + A_j)\,, & |i-j| \geq 2, \\ \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}(A_{i-1} + A_i + A_j)\,, & j = i+1, \end{cases}$$

and

$$\mathbb{E}[X_i X_j e^{tF}] = t\mathbb{E}[X_i X_j F_a e^{tF_u}] + t^2\mathbb{E}[X_i X_j F_a^2 r_2(tF_a) e^{tF_u}/2].$$

For the first order term we compute as a preparation

$$\begin{aligned} X_i X_j F_a = a_{i-1}^{(n)}(X_{i-1}X_iX_j + X_{i-1}X_j + X_j) + a_i^{(n)}(X_j + X_{i+1}X_j + X_iX_{i+1}X_j) \\ + a_{j-1}^{(n)}(X_iX_{j-1}X_j + X_iX_{j-1} + X_i) \\ + a_j^{(n)}(X_i + X_iX_{j+1} + X_iX_jX_{j+1}). \end{aligned}$$

In particular we have to consider the special case $|i-j| = 1$ and assume $i = j+1$. If not, we just have to swap i and j. Under our assumption the last equation reduces to

$$\begin{aligned} X_i X_j F_a = a_{i-1}^{(n)}(X_i + 1 + X_j) + a_i^{(n)}(X_j + X_{i+1}X_j + X_iX_{i+1}X_j) \\ + a_{j-1}^{(n)}(X_iX_{j-1}X_j + X_iX_{j-1} + X_i). \end{aligned}$$

From here on we assume that the indices apperaring in upcoming F_a's and F_{u_a}'s are all different. If not, there is only an effect on the number of coefficients and so the constants, but not on the order of our bound. Having that in mind we split F_u in the same manner as before, $F_u = F_{u_a} + F_{u_u}$, such that $F_{u_a} = (A_{i-2} + A_{i+1} + A_{j-2} + A_{j+1})/4\sqrt{\mathrm{Var}(G_n)}$. Then

$$t\mathbb{E}[X_iX_jF_a e^{tF_u}] = t\mathbb{E}[X_iX_jF_a e^{tF_{u_u}}] + t^2\mathbb{E}[X_iX_jF_aF_{u_a} r_1(tF_{u_a}) e^{tF_{u_u}}]$$

and thus for $i = j+1$

$$\begin{aligned} t\left|\mathbb{E}[X_iX_jF_a e^{tF_u}]\right| \leq & \frac{Ct}{4\sqrt{\mathrm{Var}(G_n)}} \cdot \left|a_{i-1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct} \\ & + \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1\} \\ n_1\in\{i-2,i+1,j-2,j+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \end{aligned} \tag{4.9}$$

In the case $|i-j| \geq 2$ the first term of the last inequality does not appear since all the indices in $X_iX_jF_a$ are different:

$$t\left|\mathbb{E}[X_iX_jF_a e^{tF_u}]\right| \leq \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1,j\} \\ n_1\in\{i-2,i+1,j-2,j+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \tag{4.10}$$

For the second order term we just bound

$$t^2\left|\mathbb{E}[X_iX_jF_a^2 r_2(tF_a) e^{tF_u}/2]\right| \leq \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i,j-1,j\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right] e^{ct}. \tag{4.11}$$

In the case of $\mathbb{E}[X_iX_jX_ke^{tF}]$:

$$F_a = \frac{1}{4\sqrt{\operatorname{Var}(G_n)}}(A_{i-1} + A_i + A_{j-1} + A_j + A_{k-1} + A_k)$$

and

$$\mathbb{E}[X_iX_jX_ke^{tF}] = t\mathbb{E}[X_iX_jX_kF_ae^{tF_u}] + t^2\mathbb{E}[X_iX_jX_kF_a^2r_2(tF_a)e^{tF_u}/2].$$

For the first order term we compute as a preparation

$$\begin{aligned} X_iX_jX_kF_a =& a_{i-1}^{(n)}(X_{i-1}X_iX_jX_k + X_{i-1}X_jX_k + X_jX_k) \\ &+ a_i^{(n)}(X_jX_k + X_{i+1}X_jX_k + X_iX_{i+1}X_jX_k) \\ &+ a_{j-1}^{(n)}(X_iX_{j-1}X_jX_k + X_iX_{j-1}X_k + X_iX_k) \\ &+ a_j^{(n)}(X_iX_k + X_iX_{j+1}X_k + X_iX_jX_{j+1}X_k) \\ &+ a_{k-1}^{(n)}(X_iX_jX_{k-1}X_k + X_iX_jX_{k-1} + X_iX_j) \\ &+ a_k^{(n)}(X_iX_j + X_iX_jX_{k+1} + X_iX_jX_kX_{k+1}). \end{aligned}$$

And by our assumption $i \neq j \neq k$ in every summand at least one X will remain. We split $F_u = F_{u_a} + F_{u_u}$, such that $F_{u_a} = (A_{i-2} + A_{i+1} + A_{j-2} + A_{j+1} + A_{k-2} + A_{k+1})/4\sqrt{\operatorname{Var}(G_n)}$. Then

$$\begin{aligned} t\mathbb{E}[X_iX_jX_kF_ae^{tF_u}] &= t\mathbb{E}[X_iX_jX_kF_ae^{tF_{u_a}}e^{tF_{u_u}}] \\ &= t\mathbb{E}[X_iX_jX_kF_ae^{tF_{u_u}}] + t^2\mathbb{E}[X_iX_jX_kF_aF_{u_a}r_1(tF_{u_a})e^{tF_{u_u}}] \\ &= t^2\mathbb{E}[X_iX_jX_kF_aF_{u_a}r_1(tF_{u_a})e^{tF_{u_u}}] \end{aligned}$$

by independence and thus

$$t\left|\mathbb{E}[X_iX_jX_kF_ae^{tF_u}]\right| \leq \frac{Ct^2}{16\operatorname{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1,j,k-1,k\} \\ n_1\in\{i-2,i+1,j-2,j+1,k-2,k+1\}}} \left|a_{m_1}^{(n)}\right|\left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \tag{4.12}$$

For the second order term we just bound

$$t^2\left|\mathbb{E}[X_iX_jX_kF_a^2r_2(tF_a)e^{tF_u}/2]\right| \leq \frac{Ct^2}{32\operatorname{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i,j-1,j,k-1,k\}} \left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \tag{4.13}$$

In the case of $\mathbb{E}[X_iX_jX_kX_le^{tF}]$:

$$F_a = \frac{1}{4\sqrt{\operatorname{Var}(G_n)}}(A_{i-1} + A_i + A_{j-1} + A_j + A_{k-1} + A_k + A_{l-1} + A_l)$$

and

$$\mathbb{E}[X_iX_jX_kX_le^{tF}] = t\mathbb{E}[X_iX_jX_kX_lF_ae^{tF_u}] + t^2\mathbb{E}[X_iX_jX_kX_lF_a^2r_2(tF_a)e^{tF_u}/2].$$

For the first order term we compute as a preparation

$$\begin{aligned} X_iX_jX_kX_lF_a =& a_{i-1}^{(n)}(X_{i-1}X_iX_jX_kX_l + X_{i-1}X_jX_kX_l + X_jX_kX_l) \\ &+ a_i^{(n)}(X_jX_kX_l + X_{i+1}X_jX_kX_l + X_iX_{i+1}X_jX_kX_l) \\ &+ a_{j-1}^{(n)}(X_iX_{j-1}X_jX_kX_l + X_iX_{j-1}X_kX_l + X_iX_kX_l) \\ &+ a_j^{(n)}(X_iX_kX_l + X_iX_{j+1}X_kX_l + X_iX_jX_{j+1}X_kX_l) \end{aligned}$$

$$
\begin{aligned}
&+ a_{k-1}^{(n)}(X_iX_jX_{k-1}X_kX_l + X_iX_jX_{k-1}X_l + X_iX_jX_l)\\
&+ a_{k}^{(n)}(X_iX_jX_l + X_iX_jX_{k+1}X_l + X_iX_jX_kX_{k+1}X_l)\\
&+ a_{l-1}^{(n)}(X_iX_jX_kX_{l-1}X_l + X_iX_jX_kX_{l-1} + X_iX_jX_k)\\
&+ a_{k}^{(n)}(X_iX_jX_k + X_iX_jX_kX_{l+1} + X_iX_jX_kX_lX_{l+1}).
\end{aligned}
$$

And by our assumption $i \neq j \neq k \neq l$ in every summand at least one X will remain. F_{u_a} is given by $F_{u_a} = (A_{i-2} + A_{i+1} + A_{j-2} + A_{j+1} + A_{k-2} + A_{k+1} + A_{l-2} + A_{l+1}) / 4\sqrt{\mathrm{Var}(G_n)}$ this time. Then

$$
t\mathbb{E}[X_iX_jX_kX_lF_ae^{tF_u}] = t^2\mathbb{E}[X_iX_jX_kX_lF_aF_{u_a}r_1(tF_{u_a})e^{tF_{u_u}}]
$$

by independence and thus

$$
t\left|\mathbb{E}[X_iX_jX_kX_lF_ae^{tF_u}]\right| \leq \frac{Ct^2}{16\,\mathrm{Var}(G_n)} \cdot \sum_{\substack{m_1\in\{i-1,i,j-1,j,k-1,k,l-1,l\}\\ n_1\in\{i-2,i+1,j-2,j+1,k-2,k+1,l-2,l+1\}}} \left|a_{m_1}^{(n)}\right|\left|a_{n_1}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \tag{4.14}
$$

For the second order term we just bound

$$
t^2\left|\mathbb{E}[X_iX_jX_kX_lF_a^2r_2(tF_a)e^{tF_u}/2]\right| \leq \frac{Ct^2}{32\,\mathrm{Var}(G_n)} \cdot \sum_{m_2,n_2\in\{i-1,i,j-1,j,k-1,k,l-1,l\}} \left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right| \cdot \mathbb{E}\left[e^{tF}\right]e^{ct}. \tag{4.15}
$$

□

Now we are ready to deal with all three classes of subterms and choose B_2, B_6 and B_9 as representatives:

First class of subterms, $B_1 - B_4$**:**

$$
\mathbb{E}[B_2e^{tF}] = \frac{9}{1024(\mathrm{Var}(G_n))^2}\Bigg(\sum_{k\in\mathbb{Z}}(a_k^{(n)})^4\mathbb{E}[e^{tF}] + \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}}(a_k^{(n)})^2(a_l^{(n)})^2\mathbb{E}[X_kX_le^{tF}] + \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|\geq 2}}(a_k^{(n)})^2(a_l^{(n)})^2\mathbb{E}[X_kX_le^{tF}]\Bigg). \tag{4.16}
$$

The first one is the easiest by

$$
B_{21} := \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{k\in\mathbb{Z}}(a_k^{(n)})^4\mathbb{E}[e^{tF}] = \frac{9}{1024(\mathrm{Var}(G_n))^2}\left\|a^{(n)}\right\|^4_{l^4(\mathbb{Z})}\mathbb{E}[e^{tF}].
$$

By using (4.9), (4.10) and (4.11) it remains to bound

$$
\begin{aligned}
B_{22} &:= \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}}(a_k^{(n)})^2(a_l^{(n)})^2\mathbb{E}[X_kX_le^{tF}]\\
&\leq \frac{9}{1024(\mathrm{Var}(G_n))^2}(B_{221} + B_{222} + B_{223})\,\mathbb{E}\left[e^{tF}\right]e^{ct},\\
B_{23} &:= \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{\substack{k,l\in\mathbb{Z}\\|k-l|\geq 2}}(a_k^{(n)})^2(a_l^{(n)})^2\mathbb{E}[X_kX_le^{tF}]\\
&\leq \frac{9}{1024(\mathrm{Var}(G_n))^2}(B_{231} + B_{232})\,\mathbb{E}\left[e^{tF}\right]e^{ct},
\end{aligned}
$$

such that

$$B_{221} := \frac{Ct}{4\sqrt{\operatorname{Var}(G_n)}} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}} (a_k^{(n)})^2 (a_l^{(n)})^2 \left|a_{\min(k,l)}^{(n)}\right|,$$

$$B_{222} := \frac{Ct^2}{16\operatorname{Var}(G_n)} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}} (a_k^{(n)})^2 (a_l^{(n)})^2 \cdot \sum_{\substack{m_1\in\{k-1,k,l-1,l\}\\ n_1\in\{k-2,k+1,l-2,l+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right|,$$

$$B_{223} := \frac{Ct^2}{32\operatorname{Var}(G_n)} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|=1}} (a_k^{(n)})^2 (a_l^{(n)})^2 \cdot \sum_{m_2,n_2\in\{k-1,k,l-1,l\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right|,$$

$$B_{231} := \frac{Ct^2}{16\operatorname{Var}(G_n)} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|\geq 2}} (a_k^{(n)})^2 (a_l^{(n)})^2 \cdot \sum_{\substack{m_1\in\{k-1,k,l-1,l\}\\ n_1\in\{k-2,k+1,l-2,l+1\}}} \left|a_{m_1}^{(n)}\right| \left|a_{n_1}^{(n)}\right|,$$

$$B_{232} := \frac{Ct^2}{32\operatorname{Var}(G_n)} \cdot \sum_{\substack{k,l\in\mathbb{Z}\\|k-l|\geq 2}} (a_k^{(n)})^2 (a_l^{(n)})^2 \cdot \sum_{m_2,n_2\in\{k-1,k,l-1,l\}} \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right|.$$

Then by the inequality of arithmetic and geometric means, from here on AM-GM inequality

$$\begin{aligned}
B_{221} &= \frac{Ct}{4\sqrt{\operatorname{Var}(G_n)}} \left(\sum_{k\in\mathbb{Z}} (a_k^{(n)})^2 (a_{k-1}^{(n)})^2 \left|a_{k-1}^{(n)}\right| + \sum_{k\in\mathbb{Z}} (a_k^{(n)})^2 (a_{k+1}^{(n)})^2 \left|a_k^{(n)}\right| \right) \\
&= \frac{Ct}{4\sqrt{\operatorname{Var}(G_n)}} \left(\sum_{k\in\mathbb{Z}} \sqrt[5]{\left|a_k^{(n)}\right|^{10} \left|a_{k-1}^{(n)}\right|^{10} \left|a_{k-1}^{(n)}\right|^{5}} + \sum_{k\in\mathbb{Z}} \sqrt[5]{\left|a_k^{(n)}\right|^{10} \left|a_{k+1}^{(n)}\right|^{10} \left|a_k^{(n)}\right|^{5}} \right) \\
&\leq \frac{Ct}{20\sqrt{\operatorname{Var}(G_n)}} \left(\sum_{k\in\mathbb{Z}} 2\left|a_k^{(n)}\right|^5 + 3\left|a_{k-1}^{(n)}\right|^5 + \sum_{k\in\mathbb{Z}} 3\left|a_k^{(n)}\right|^5 + 2\left|a_{k+1}^{(n)}\right|^5 \right) \\
&\leq \frac{Ct}{\sqrt{\operatorname{Var}(G_n)}} \left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^5 .
\end{aligned}$$

If we look at $B_{222}-B_{232}$, we can change the order of summation since all summands are non-negative. And we become even bigger if we add the missing indices:

$$B_{223} \leq \frac{Ct^2}{32\operatorname{Var}(G_n)} \sum_{k,l\in\mathbb{Z}} \sum_{m_2\in\{k-1,k,l-1,l\}} \sum_{n_2\in\{k-1,k,l-1,l\}} (a_k^{(n)})^2 (a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right|.$$

From here on we treat different cases, but every time we can use the AM-GM inequality:
Case 1: $m_2 \in \{k-1,k\}$ and $n_2 \in \{k-1,k\}$:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} (a_k^{(n)})^2 (a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| &= \sum_{k\in\mathbb{Z}} (a_k^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \sum_{l\in\mathbb{Z}} (a_l^{(n)})^2 \\
&\leq C \left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4 \left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2 .
\end{aligned}$$

Case 2: $m_2 \in \{l-1,l\}$ and $n_2 \in \{l-1,l\}$:

$$\begin{aligned}
\sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} (a_k^{(n)})^2 (a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| &= \sum_{l\in\mathbb{Z}} (a_l^{(n)})^2 \left|a_{m_2}^{(n)}\right| \left|a_{n_2}^{(n)}\right| \sum_{k\in\mathbb{Z}} (a_k^{(n)})^2 \\
&\leq C \left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4 \left\|a^{(n)}\right\|_{l^2(\mathbb{Z})}^2 .
\end{aligned}$$

Case 3: $m_2 \in \{k-1, k\}$ and $n_2 \in \{l-1, l\}$:

$$\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}(a_k^{(n)})^2(a_l^{(n)})^2\left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right| = \sum_{k\in\mathbb{Z}}(a_k^{(n)})^2\left|a_{m_2}^{(n)}\right|\sum_{l\in\mathbb{Z}}(a_l^{(n)})^2\left|a_{n_2}^{(n)}\right| \leq C\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6.$$

Case 4: $m_2 \in \{l-1, l\}$ and $n_2 \in \{k-1, k\}$:

$$\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(a_k^{(n)})^2(a_l^{(n)})^2\left|a_{m_2}^{(n)}\right|\left|a_{n_2}^{(n)}\right| = \sum_{l\in\mathbb{Z}}(a_l^{(n)})^2\left|a_{m_2}^{(n)}\right|\sum_{k\in\mathbb{Z}}(a_k^{(n)})^2\left|a_{n_2}^{(n)}\right| \leq C\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6.$$

According to (4.1) in case 1 and 2 the norm $\|a^{(n)}\|_{l^2(\mathbb{Z})}^2$ vanishes directly with the variance in the prefactor. Summarizing for B_{223}:

$$B_{223} \leq \frac{Ct^2}{32}\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4 + \frac{Ct^2}{32\,\mathrm{Var}(G_n)}\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6.$$

Analogously we get basically bounds of the same order for B_{222}, B_{231} and B_{232}. Combining our bounds for the subterms of (4.16) gives us

$$\mathbb{E}[B_2e^{tF}] \leq Ce^{ct}\left(t^2\frac{\|a^{(n)}\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} + (1+t^2)\frac{\|a^{(n)}\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + t\frac{\|a^{(n)}\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}}\right)\mathbb{E}[e^{tF}].$$

Second class of subterms, $B_5 - B_8$: We write $\mathbb{E}[B_6e^{tF}] = B_{61} + B_{62} + B_{63}$ such that

$$B_{61} := \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_{l-1}^{(n)})(a_l^{(n)})\mathbb{E}[X_kX_{l-1}e^{tF}],$$
$$B_{62} := \frac{6}{1024(\mathrm{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_{l-1}^{(n)})(a_l^{(n)})\mathbb{E}[X_kX_{l-1}X_{l+1}e^{tF}],$$
$$B_{63} := \frac{9}{1024(\mathrm{Var}(G_n))^2}\sum_{k,l\in\mathbb{Z}}(a_k^{(n)})^2(a_{l-1}^{(n)})(a_l^{(n)})\mathbb{E}[X_kX_{l+1}e^{tF}].$$

B_{61} and B_{63} are analogous to B_{22} and B_{23} since they have the same structure: Two coefficients with k-index, two coefficients with l-index, one X with k-index and one X with l-index. And with that the arguments are the same. Looking at the remaining B_{62} two indices of the X's are equal if and only if $k = l+1$ or $k = l-1$. In the latter case B_{62} reduces to

$$\frac{C}{(\mathrm{Var}(G_n))^2}\sum_{k\in\mathbb{Z}}(a_k^{(n)})^3(a_{k+1}^{(n)})\mathbb{E}[X_{k+2}e^{tF}],$$

so we can use (4.7) and (4.8) giving us upper bounds of order

$$O\left(t\frac{\|a^{(n)}\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}}\right) \quad \text{and} \quad O\left(t^2\frac{\|a^{(n)}\|_{l^6(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right).$$

And the same for $k = l+1$. At last, if neither $k = l-1$ nor $k = l+1$ we can use (4.12) and (4.13) giving us upper bounds of order

$$O\left(t^2 \frac{\|a^{(n)}\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right) \quad \text{and} \quad O\left(t^2 \frac{\|a^{(n)}\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2}\right).$$

Combining our bounds for B_{61}, B_{62} and B_{63} we get

$$\mathbb{E}[B_6 e^{tF}] \le Ce^{ct}\left(t^2 \frac{\|a^{(n)}\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} + t^2 \frac{\|a^{(n)}\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + t \frac{\|a^{(n)}\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}} + t^2 \frac{\|a^{(n)}\|_{l^6(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right) \mathbb{E}[e^{tF}].$$

Third class of subterms: It consists only of B_9 and so we have to deal with $\mathbb{E}[B_9 e^{tF}]$. Multiplying all the X's inside we get products of lengths two, three and four. The first two cases are already solved and a product of length four appears only one time, namely

$$\frac{C}{(\mathrm{Var}(G_n))^2} \sum_{k,l \in \mathbb{Z}} (a_{k-1}^{(n)})(a_k^{(n)})(a_{l-1}^{(n)})(a_l^{(n)}) \mathbb{E}[X_{k-1}X_{k+1}X_{l-1}X_{l+1}e^{tF}].$$

We have two pairs of two equal indices of the X's if and only if $k = l$ and then we are in the situation of B_{21}. Note that it is impossible that three or more indices are equal. If two indices are equal and two indices are different, e.g. $k-1 = l+1$ we are in the $2X$-case and can use (4.9), (4.10) and (4.11). At last, if all four indices are different, most of the work is done by (4.14) and (4.15) leading to upper bounds of order

$$O\left(t^2 \frac{\|a^{(n)}\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right) \quad \text{and} \quad O\left(t^2 \frac{\|a^{(n)}\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2}\right).$$

Combining our bounds from all the different cases we get

$$\mathbb{E}[B_9 e^{tF}] \le Ce^{ct}\left(t^2 \frac{\|a^{(n)}\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} + (1+t^2) \frac{\|a^{(n)}\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + t \frac{\|a^{(n)}\|_{l^5(\mathbb{Z})}^5}{(\mathrm{Var}(G_n))^{5/2}} + t^2 \frac{\|a^{(n)}\|_{l^6(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3}\right) \mathbb{E}[e^{tF}].$$

Summarizing everything we have done so far a bound as in condition (A1) is obtained by

$$\begin{aligned}
\mathbb{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle\right| e^{tF}\right] &\le \left(\mathbb{E}\left[\left(1 - \langle DF, -DL^{-1}F\rangle\right)^2 e^{tF}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}} \\
&= \left(\mathbb{E}\left[\sum_{i=1}^{9} B_i e^{tF}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}} \\
&\le \sum_{i=1}^{9} \left(\mathbb{E}\left[B_i e^{tF}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}} \\
&\le \widetilde{\gamma_1}(t) \mathbb{E}\left[e^{tF}\right]
\end{aligned}$$

such that

$$\begin{aligned}
\widetilde{\gamma_1}(t) &= Ce^{ct}\left(t \frac{\|a^{(n)}\|_{l^3(\mathbb{Z})}^3}{(\mathrm{Var}(G_n))^{3/2}} + (1+t) \frac{\|a^{(n)}\|_{l^4(\mathbb{Z})}^2}{\mathrm{Var}(G_n)} + t^{1/2} \frac{\|a^{(n)}\|_{l^5(\mathbb{Z})}^{5/2}}{(\mathrm{Var}(G_n))^{5/4}} + t \frac{\|a^{(n)}\|_{l^6(\mathbb{Z})}^3}{(\mathrm{Var}(G_n))^{3/2}}\right) \\
&=: Ce^{ct}\left(tC_{n,1} + (1+t)C_{n,2} + t^{1/2}C_{n,3} + tC_{n,4}\right).
\end{aligned}$$

We now move on and show that a bound as in condition (A2) exists: Again, by the Cauchy–Schwarz inequality

$$\mathbb{E}\left[\left|\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right|e^{tF}\right] \le \left(\mathbb{E}\left[\left(\delta\left(\frac{1}{\sqrt{pq}}DF\left|DL^{-1}F\right|\right)\right)^2 e^{tF}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}}.$$

By Corollary 9.9 in Privault (2008) it is $\delta(u) = \sum_{k=0}^{\infty} Y_k u_k$, where Y_k is the kth centered and standardized Rademacher random variable. In our case $Y_k = X_k$ and the corollary can be applied since $u_k := D_kF\left|D_kL^{-1}F\right|/\sqrt{p_kq_k}$ does not depend on X_k. Then

$$\begin{aligned}\left(\mathbb{E}\left[(\delta(u))^2 e^{tF}\right]\right)^{\frac{1}{2}} &= \left(\sum_{k,l\in\mathbb{Z}} \mathbb{E}\left[u_k u_l X_k X_l e^{tF}\right]\right)^{\frac{1}{2}} \\ &\le \left(\sum_{k\in\mathbb{Z}} \mathbb{E}\left[u_k^2 e^{tF}\right]\right)^{\frac{1}{2}} + \left(\sum_{\substack{k,l\in\mathbb{Z}\\ k\neq l}} \mathbb{E}\left[u_k u_l X_k X_l e^{tF}\right]\right)^{\frac{1}{2}}. \end{aligned} \tag{4.17}$$

For the upcoming computations we recall

$$\begin{aligned} D_kF &= \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}\left(a_{k-1}^{(n)}(X_{k-1}+1) + a_k^{(n)}(X_{k+1}+1)\right), \\ -D_kL^{-1}F &= \frac{1}{8\sqrt{\mathrm{Var}(G_n)}}\left(a_{k-1}^{(n)}(X_{k-1}+2) + a_k^{(n)}(X_{k+1}+2)\right). \end{aligned}$$

and so

$$\begin{aligned} D_kF(-D_kL^{-1}F) = \frac{1}{32\,\mathrm{Var}(G_n)}\Big(&(a_{k-1}^{(n)})^2\left[3X_{k-1}+3\right] + (a_k^{(n)})^2\left[3X_{k+1}+3\right] \\ &+ a_{k-1}^{(n)}a_k^{(n)}\left[3X_{k-1} + 2X_{k-1}X_{k+1} + 3X_{k+1} + 4\right]\Big). \end{aligned}$$

The square of the righthandside is of a familiar form: Every summand consists of a product of length four of coefficients with index $k \pm ...$ multiplied with something bounded, and so as before we get immediately or by the AM-GM-inequality

$$\sum_{k\in\mathbb{Z}} \mathbb{E}\left[u_k^2 e^{tF}\right] \le C\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2}\mathbb{E}\left[e^{tF}\right].$$

For the remaining term of (4.17) we adapt the strategy that is used in the proof of Lemma 4.1 — see its beginning for a detailed explanation. We remind on

$$A_k = a_k^{(n)}\left[X_k + X_kX_{k+1} + X_{k+1}\right]$$

so that

$$F_a = \frac{1}{4\sqrt{\mathrm{Var}(G_n)}}(A_{k-2} + A_{k-1} + A_k + A_{k+1} + A_{l-2} + A_{l-1} + A_l + A_{l+1})$$

and by Taylor expansion

$$\begin{aligned} \mathbb{E}[u_ku_lX_kX_le^{tF}] &= \mathbb{E}[u_ku_lX_kX_le^{tF_a}e^{tF_u}] \\ &= \mathbb{E}[u_ku_lX_kX_le^{tF_u}] + t\mathbb{E}[u_ku_lX_kX_lF_ae^{tF_u}] + t^2\mathbb{E}[u_ku_lX_kX_lF_a^2r_2(tF_a)e^{tF_u}/2]. \end{aligned}$$

0-order-term: By independence

$$\mathbb{E}[u_ku_lX_kX_le^{tF_u}] = \mathbb{E}[u_ku_lX_kX_l]\mathbb{E}[e^{tF_u}].$$

Since by definition X_k and X_l respectively u_k are independent, we just have to check whether the same goes for X_k and u_l, which is leading to two cases.
Case 1: $l \notin \{k-1, k+1\}$

$$\mathbb{E}[u_k u_l X_k X_l] = \mathbb{E}[X_k]\mathbb{E}[u_k u_l X_l] = 0.$$

Case 2: $l \in \{k-1, k+1\}$

$$\sum_{k\in\mathbb{Z}} \mathbb{E}[u_k u_l X_k X_l]\mathbb{E}[e^{tF_u}] \leq C \frac{\left\|a^{(n)}\right\|^4_{l^4(\mathbb{Z})}}{(\mathrm{Var}(G_n))^2}\mathbb{E}\left[e^{tF}\right] e^{ct},$$

following our usual argumentation.
1st-order-term: We split F_u in the same manner as before, $F_u = F_{u_a} + F_{u_u}$, such that $F_{u_a} = (A_{k-3} + A_{k+2} + A_{l-3} + A_{l+2})/4\sqrt{\mathrm{Var}(G_n)}$ and use another Taylor expansion of degree 1. Then

$$\begin{aligned} t\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_u}] &= t\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_{u_a}} e^{tF_{u_u}}] \\ &= t\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_{u_u}}] + t^2\mathbb{E}[u_k u_l X_k X_l F_a F_{u_a} r_1(tF_{u_a}) e^{tF_{u_u}}]. \end{aligned} \tag{4.18}$$

Note that F_{u_u} is — as part of F_u — independent of X_k, X_l, u_k and u_l, but also independent of F_a since we removed F_{u_a}, the depending part of F_u. As a consequence

$$\mathbb{E}[u_k u_l X_k X_l F_a e^{tF_{u_u}}] = \mathbb{E}[u_k u_l X_k X_l F_a]\mathbb{E}[e^{tF_{u_u}}].$$

Our next observation is $\mathbb{E}[u_k u_l X_k X_l F_a] = 0$ for $|k-l| \geq 5$ — in this case all appearing indices are different and the claim follows ultimately from independence. We treat the remaining case $|k-l| \leq 4$ as four subcases $|k-l| = i$ for $i \in \{1,2,3,4\}$, but here we just write down the first one $|k-l| = 1$ as the others are analogous and so there outcome. In the mentioned case, if $l = k+1$, we receive

$$t\,|u_k u_l X_k X_l F_a| \leq \frac{Ct}{(\mathrm{Var}(G_n))^{5/2}} \sum_{\substack{i_1,i_2\in\{k-1,k\} \\ i_3,i_4\in\{k,k+1\} \\ i_5\in\{k-2,k-1,k,k+1,k+2\}}} \left|a^{(n)}_{i_1}\right|\left|a^{(n)}_{i_2}\right|\left|a^{(n)}_{i_3}\right|\left|a^{(n)}_{i_4}\right|\left|a^{(n)}_{i_5}\right|$$

and very similar for $l = k-1$. For both every summand consists of a product of five coefficients with index $k \pm ...$, and so we get

$$\sum_{\substack{k,l\in\mathbb{Z} \\ k\neq l \\ |k-l|=1}} t\left|\mathbb{E}[u_k u_l X_k X_l F_a]\mathbb{E}[e^{tF_{u_u}}]\right| \leq Ct \frac{\left\|a^{(n)}\right\|^5_{l^5(\mathbb{Z})}}{(\mathrm{Var}(G_n))^{5/2}}\mathbb{E}\left[e^{tF}\right] e^{ct}.$$

Having in mind that $|r_1(tF_{u_a})| \leq e^{t|F_{u_a}|}$ we can bound the second term of (4.18) by using

$$t^2\,|u_k u_l X_k X_l F_a F_{u_a}| \leq \frac{Ct^2}{(\mathrm{Var}(G_n))^3} \sum_{\substack{i_1,i_2\in\{k-1,k\} \\ i_3,i_4\in\{l-1,l\} \\ i_5\in\{k-2,k-1,k,k+1,l-2,l-1,l,l+1\} \\ i_6\in\{k-3,k+2,l-3,l+2\}}} \left|a^{(n)}_{i_1}\right|\left|a^{(n)}_{i_2}\right|\left|a^{(n)}_{i_3}\right|\left|a^{(n)}_{i_4}\right|\left|a^{(n)}_{i_5}\right|\left|a^{(n)}_{i_6}\right|$$

and every summand consists of a product of length six of either three coefficients with index $k \pm ...$ and three coefficients with index $l \pm ...$, or four coefficients with index $k \pm ...$ and two coefficients with index $l \pm ...$or the other way around. Combining the cases we get

$$\sum_{\substack{k,l\in\mathbb{Z} \\ k\neq l}} t^2\left|\mathbb{E}[u_k u_l X_k X_l F_a F_{u_a} r_1(tF_{u_a}) e^{tF_{u_u}}]\right| \leq Ct^2 \left(\frac{\left\|a^{(n)}\right\|^4_{l^4(\mathbb{Z})}}{(\mathrm{Var}(G_n))^2} + \frac{\left\|a^{(n)}\right\|^6_{l^3(\mathbb{Z})}}{(\mathrm{Var}(G_n))^3} \right) \mathbb{E}\left[e^{tF}\right] e^{ct}.$$

2nd-order-term: Finally, having in mind that $|r_2(tF_a)| \leq e^{t|F_a|}$ we can bound the last term of our original Taylor expansion by using

$$t^2 \left|u_k u_l X_k X_l F_a^2\right| \leq \frac{Ct^2}{(\mathrm{Var}(G_n))^3} \sum_{\substack{i_1,i_2\in\{k-1,k\}\\ i_3,i_4\in\{l-1,l\}\\ i_5,i_6\in\{k-2,k-1,k,k+1,l-2,l-1,l,l+1\}}} \left|a_{i_1}^{(n)}\right| \left|a_{i_2}^{(n)}\right| \left|a_{i_3}^{(n)}\right| \left|a_{i_4}^{(n)}\right| \left|a_{i_5}^{(n)}\right| \left|a_{i_6}^{(n)}\right|$$

and get analogously the bound

$$\sum_{\substack{k,l\in\mathbb{Z}\\ k\neq l}} t^2 \left|\mathbb{E}[u_k u_l X_k X_l F_a^2 r_2(tF_a) e^{tF_u}/2]\right| \leq Ct^2 \left(\frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^4}{(\mathrm{Var}(G_n))^2} + \frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^6}{(\mathrm{Var}(G_n))^3} \right) \mathbb{E}\left[e^{tF}\right] e^{ct}.$$

Summarizing everything we have done so far a bound as in condition (A2) is obtained by

$$\begin{aligned}
\mathbb{E}\left[|\delta(u)|\, e^{tF}\right] &\leq \left(\mathbb{E}\left[(\delta(u))^2\, e^{tF}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}} \\
&\leq \left(\left(\sum_{k\in\mathbb{Z}} \mathbb{E}\left[u_k^2 e^{tF}\right] \right)^{\frac{1}{2}} + \left(\sum_{\substack{k,l\in\mathbb{Z}\\ k\neq l}} \mathbb{E}\left[u_k u_l X_k X_l e^{tF}\right] \right)^{\frac{1}{2}} \right) \left(\mathbb{E}\left[e^{tF}\right]\right)^{\frac{1}{2}} \\
&\leq \widetilde{\gamma}_2(t)\mathbb{E}\left[e^{tF}\right]
\end{aligned}$$

such that

$$\begin{aligned}
\widetilde{\gamma}_2(t) &= Ce^{ct} \left(t \frac{\left\|a^{(n)}\right\|_{l^3(\mathbb{Z})}^3}{(\mathrm{Var}(G_n))^{3/2}} + (1+t) \frac{\left\|a^{(n)}\right\|_{l^4(\mathbb{Z})}^2}{\mathrm{Var}(G_n)} + t^{1/2} \frac{\left\|a^{(n)}\right\|_{l^5(\mathbb{Z})}^{5/2}}{(\mathrm{Var}(G_n))^{5/4}} \right) \\
&=: Ce^{ct}\left(tC_{n,1} + (1+t)C_{n,2} + t^{1/2}C_{n,3}\right).
\end{aligned}$$

In a final step we want to simplify our bounds by comparing the constants $C_{n,i}$ with each other. To do so, we will use, for $m \geq 2$:

$$\begin{aligned}
\left\|a^{(n)}\right\|_{l^m(\mathbb{Z})}^m &= \sum_{k\in\mathbb{Z}} \left(|a_k|^{m-1} |a_k|\right) \\
&\leq \sqrt{\sum_{k\in\mathbb{Z}} \left(|a_k|^{m-1}\right)^2 \sum_{k\in\mathbb{Z}} |a_k|^2} \\
&= \sqrt{\sum_{k\in\mathbb{Z}} \left(|a_k|^{m-1}\right)^2} \cdot C \cdot (\mathrm{Var}(G_n))^{1/2} && (4.19) \\
&\leq \sum_{k\in\mathbb{Z}} |a_k|^{m-1} \cdot C \cdot (\mathrm{Var}(G_n))^{1/2} \\
&= \left\|a^{(n)}\right\|_{l^{m-1}(\mathbb{Z})}^{m-1} \cdot C \cdot (\mathrm{Var}(G_n))^{1/2} && (4.20)
\end{aligned}$$

by the Cauchy–Schwarz inequality. Then (4.19) and (4.20) imply

$$C_{n,1} = \sum_{k\in\mathbb{Z}} |a_k|^3 (\mathrm{Var}(G_n))^{-3/2} \leq \left(\sum_{k\in\mathbb{Z}} |a_k|^4 \right)^{1/2} (\mathrm{Var}(G_n))^{-1} = C_{n,2},$$

$$C_{n,3} = \left(\sum_{k\in\mathbb{Z}} |a_k|^5\right)^{1/2} (\mathrm{Var}(G_n))^{-5/4} \le \left(\sum_{k\in\mathbb{Z}} |a_k|^4 (\mathrm{Var}(G_n))^{1/2}\right)^{1/2} (\mathrm{Var}(G_n))^{-5/4} = C_{n,2},$$

$$C_{n,4} = \left(\sum_{k\in\mathbb{Z}} |a_k|^6\right)^{1/2} (\mathrm{Var}(G_n))^{-3/2} \le \left(\sum_{k\in\mathbb{Z}} |a_k|^5 (\mathrm{Var}(G_n))^{1/2}\right)^{1/2} (\mathrm{Var}(G_n))^{-3/2} \le C_{n,2}.$$

So, we choose $\gamma_1(t) = \gamma_2(t) := Ce^{ct}\big((1 + t^{1/2} + t)C_n\big)$ for (A1) and (A2), and $C_n := C_{n,2}$. □

5. Proofs II: Subgraph counts in the Erdős–Rényi random graph

There are $\binom{n}{2}$ possible edges in the Erdős-Rényi random graph. Hence, we can describe it by using as many Rademacher random variables: Let E be the set of all possible edges of $G(n,p)$ and let $(X_k)_{k\in E}$ be a set of independent Rademacher random variables, in which $X_k = 1$ indicates the presence of edge k in $G(n,p)$. Thus, $\mathbb{P}(X_k = 1) = p$ for all $k \in E$. In our calculations, we will make use of the following rescaled versions of $(X_k)_{k\in E}$: For $k \in E$ we define $B_k = \frac{1}{2}(X_k + 1)$, which is Bernoulli($p$)-distributed, and $Y_k = (pq)^{-\frac{1}{2}}(B_k - p)$, which is standardized. Further, for any subset $A \subset E$, we shorten the notation for the product of $(B_k)_{k\in A}$ or $(Y_k)_{k\in A}$ by defining $B_A := \prod_{k\in A} B_k$ and $Y_A := \prod_{k\in A} Y_k$, where $B_\emptyset = Y_\emptyset = 1$ by convention. For any integrable random variable Z, let $Z^c := Z - \mathbb{E}[Z]$ denote the centered version of Z.

Let G_0 be a graph with at least one edge. Neither the standardized number of copies of G_0 in $G(n,p)$ nor $\Psi_{\min}$ do depend on the number of isolated vertices of G_0, see Lemma 4.3 in Eichelsbacher and Rednoß (2023). Hence, without loss of generality, we may assume that G_0 does not have isolated vertices. This way, every copy of G_0 in $G(n,p)$ can be identified by its set of edges. Therefore, it will be useful to simplify our notation:

For any set of edges $\Gamma \subset E$ let graph(Γ) denote the graph consisting of all edges given by Γ and all necessary vertices. Let $v_\Gamma := v_{\mathrm{graph}(\Gamma)}$, $e_\Gamma := e_{\mathrm{graph}(\Gamma)}$, and $a_\Gamma := a_{\mathrm{graph}(\Gamma)}$. We will call Γ to be a *copy* of G_0 if graph(Γ) is a copy of G_0. Let $\mathcal{M} := \{\Gamma \subset E \,|\, \Gamma \text{ is a copy of } G_0\}$ be the set of all possible copies of G_0 in $G(n,p)$, and let $\mathcal{M}_k := \{\Gamma \in \mathcal{M} \,|\, k \in \Gamma\}$ be the set of all possible copies that contain a specific edge $k \in E$. Further, for any non-empty $A \subset E$ let $\mathcal{M}_A := \bigcup_{k\in A} \mathcal{M}_k$ be the set of all possible copies that contain at least one of the edges given by A. This last definition will be mainly used in case of $A = \Gamma \in \mathcal{M}$ being a copy of G_0. In this case, $\mathcal{M}_\Gamma$ contains all possible copies of G_0 that have at least one edge in common with Γ. We will call $\mathcal{M}_\Gamma$ to be the *neighborhood* of Γ, and the elements of $\mathcal{M}_\Gamma$ to be the *neighbors* of Γ. Due to symmetry, the cardinality of the neighborhood $\mathcal{M}_\Gamma$ does not depend on the choice of $\Gamma \in \mathcal{M}$. We will denote this cardinality by D.

The number of copies of G_0 in $G(n,p)$ is given by $\sum_{\Gamma\in\mathcal{M}} B_\Gamma$. Its standardization is

$$W := \frac{1}{\sigma}\sum_{\Gamma\in\mathcal{M}} (B_\Gamma - \mathbb{E}[B_\Gamma]) = \frac{1}{\sigma}\sum_{\Gamma\in\mathcal{M}} B_\Gamma^c,$$

where $\sigma^2 := \mathrm{Var}(\sum_{\Gamma\in\mathcal{M}} B_\Gamma)$. From Lemma 4.2 in Eichelsbacher and Rednoß (2023) we know that

$$\sigma^2 \ge \frac{q}{2\cdot v_{G_0}!\cdot a_{G_0}} \cdot \frac{n^{2v_{G_0}} p^{2e_{G_0}}}{\Psi_{\min}} \tag{5.1}$$

for $n \ge 4v_{G_0}^2$.

The main goal of this section is to prove Theorem 1.5 and Corollary 1.6, which have been presented in our introductory section. These proofs are postponed to the end of this section. Before, we present several lemmas that deal with some basic yet very important properties of our random variables. These results will be useful repeatedly in the proof of Theorem 1.5.

We start by exploring the behavior of our most important operators, the discrete gradient D_k with $k \in E$ and the pseudo-inverse Ornstein-Uhlenbeck operator L^{-1}.

Lemma 5.1. *For any subset $A \subset E$ and any edge $k \in E$ there is*

(i) $$D_k B_A^c = \sqrt{pq} \cdot \mathbb{1}_{\{k\in A\}} \cdot B_{A\setminus\{k\}},$$

(ii) $$-D_k L^{-1} B_A^c = \sqrt{pq} \cdot \mathbb{1}_{\{k\in A\}} \cdot \sum_{\{k\}\subset\alpha\subset A} \frac{p^{|A|-|\alpha|}}{|A|\cdot\binom{|A|-1}{|\alpha|-1}} B_{\alpha\setminus\{k\}}.$$

In particular, all expressions above are non-negative.

Remark 5.2. In the application of Lemma 5.1 (ii) it will be useful to keep in mind that

$$\sum_{\{k\}\subset\alpha\subset A} \frac{1}{|A|\cdot\binom{|A|-1}{|\alpha|-1}} = \sum_{\alpha\subset A\setminus\{k\}} \frac{1}{|A|\cdot\binom{|A|-1}{|\alpha|}} = \sum_{i=0}^{|A|-1} \left(\frac{1}{|A|} \sum_{\substack{\alpha\subset A\setminus\{k\}\\ |\alpha|=i}} \frac{1}{\binom{|A|-1}{i}} \right) = \sum_{i=0}^{|A|-1} \frac{1}{|A|} = 1$$

for any non-empty subset $A \subset E$ and any edge $k \in A$.

Proof of Lemma 5.1: First, we note that $D_k B_A^c = D_k(B_A - \mathbb{E}[B_A]) = D_k B_A$. If $k \notin A$, B_A is independent of X_k so that $D_k B_A = 0$. If $k \in A$, there is $B_A = B_{A\setminus\{k\}} \cdot B_k = B_{A\setminus\{k\}} \cdot \frac{1}{2}(X_k + 1)$ so that $D_k B_A = \sqrt{pq} \cdot (B_{A\setminus\{k\}} \cdot 1 - B_{A\setminus\{k\}} \cdot 0)$. This proves (i).

To prove the second part of our statement, we need to represent B_A with regard to the standardized random variables $Y_\ell = (pq)^{-\frac{1}{2}}(B_\ell - p)$, $\ell \in E$. By expansion of the product, we see that

$$B_A^c = B_A - \mathbb{E}[B_A] = \prod_{\ell\in A} \left((pq)^{\frac{1}{2}} Y_\ell + p \right) - p^{|A|} = \sum_{\emptyset\neq\alpha\subset A} p^{|A|-\frac{1}{2}|\alpha|} q^{\frac{1}{2}|\alpha|} Y_\alpha.$$

Now we are able to describe how the pseudo-inverse Ornstein–Uhlenbeck operator works on this expression. We get

$$-L^{-1} B_A^c = \sum_{\emptyset\neq\alpha\subset A} \frac{p^{|A|-\frac{1}{2}|\alpha|} q^{\frac{1}{2}|\alpha|}}{|\alpha|} Y_\alpha.$$

Having applied the operator, we now want to transform the result back to a representation using the Bernoulli variables $(B_\ell)_{\ell\in E}$. Thus,

$$-L^{-1} B_A^c = \sum_{\emptyset\neq\alpha\subset A} \left(\frac{p^{|A|-\frac{1}{2}|\alpha|} q^{\frac{1}{2}|\alpha|}}{|\alpha|} \cdot \prod_{\ell\in\alpha} \left((pq)^{-\frac{1}{2}} (B_\ell - p) \right) \right) = \sum_{\emptyset\neq\alpha\subset A} \sum_{\hat\alpha\subset\alpha} \frac{p^{|A|-|\hat\alpha|}}{|\alpha|} (-1)^{|\alpha|-|\hat\alpha|} B_{\hat\alpha}.$$

The double sum can be rearranged so that we can first choose $\hat\alpha$ as an arbitrary subset of A, and then fill up the rest $\tilde\alpha := \alpha\setminus\hat\alpha \subset A\setminus\hat\alpha$. However, using this rearrangement, we have to explicitly exclude the case in which $\hat\alpha = \tilde\alpha = \emptyset$. This leads to

$$\begin{aligned}
-L^{-1} B_A^c &= \sum_{\hat\alpha\subset A} \sum_{\substack{\tilde\alpha\subset A\setminus\hat\alpha\\ \hat\alpha\cup\tilde\alpha\neq\emptyset}} \frac{p^{|A|-|\hat\alpha|}}{|\hat\alpha|+|\tilde\alpha|} (-1)^{|\tilde\alpha|} B_{\hat\alpha} \\
&= \sum_{\emptyset\neq\tilde\alpha\subset A} \frac{p^{|A|}}{|\tilde\alpha|} (-1)^{|\tilde\alpha|} B_\emptyset + \sum_{\emptyset\neq\hat\alpha\subset A} \sum_{\tilde\alpha\subset A\setminus\hat\alpha} \frac{p^{|A|-|\hat\alpha|}}{|\hat\alpha|+|\tilde\alpha|} (-1)^{|\tilde\alpha|} B_{\hat\alpha} \\
&= p^{|A|} \sum_{i=1}^{|A|} \binom{|A|}{i} \frac{(-1)^i}{i} + \sum_{\emptyset\neq\hat\alpha\subset A} \left(p^{|A|-|\hat\alpha|} B_{\hat\alpha} \sum_{i=0}^{|A|-|\hat\alpha|} \binom{|A|-|\hat\alpha|}{i} \frac{(-1)^i}{|\hat\alpha|+i} \right).
\end{aligned}$$

The first term above is not random but deterministic and hence not of interest, as it will disappear after the application of the discrete gradient. To handle the second term, we use the Beta-function, which is usually defined via $\mathrm{Beta}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ for all $x, y \in \mathbb{C}$ that have a positive real part. It is well known that in case of $x, y \in \mathbb{N}$ there is $\sum_{i=0}^{x-1} \binom{x-1}{i} \frac{(-1)^i}{y+i} = \mathrm{Beta}(x,y) = (x+y-1)^{-1} \cdot \binom{x+y-2}{y-1}^{-1}$. Application of these results on $x = |A| - |\hat{\alpha}| + 1$ and $y = |\hat{\alpha}|$ yields

$$\sum_{i=0}^{|A|-|\hat{\alpha}|} \binom{|A|-|\hat{\alpha}|}{i} \frac{(-1)^i}{|\hat{\alpha}|+i} = \frac{1}{|A| \cdot \binom{|A|-1}{|\hat{\alpha}|-1}},$$

so that

$$-L^{-1}B_A^c = p^{|A|} \sum_{i=1}^{|A|} \binom{|A|}{i} \frac{(-1)^i}{i} + \sum_{\emptyset \neq \hat{\alpha} \subset A} \frac{p^{|A|-|\hat{\alpha}|}}{|A| \cdot \binom{|A|-1}{|\hat{\alpha}|-1}} B_{\hat{\alpha}}.$$

When applying the discrete gradient, the deterministic term cancels itself out, while the second term can be handled the same way as shown in the first part of this proof. We arrive at

$$\begin{aligned} -D_k L^{-1} B_A^c &= \sqrt{pq} \cdot \sum_{\emptyset \neq \hat{\alpha} \subset A} \frac{p^{|A|-|\hat{\alpha}|}}{|A| \cdot \binom{|A|-1}{|\hat{\alpha}|-1}} B_{\hat{\alpha} \setminus \{k\}} \mathbb{1}_{\{k \in \hat{\alpha}\}} \\ &= \sqrt{pq} \cdot \mathbb{1}_{\{k \in A\}} \cdot \sum_{\{k\} \subset \alpha \subset A} \frac{p^{|A|-|\alpha|}}{|A| \cdot \binom{|A|-1}{|\alpha|-1}} B_{\alpha \setminus \{k\}}. \end{aligned}$$

This proves the second part of our statement. □

Next, we want to improve our understanding of the correlation of our Bernoulli random variables $\{B_A\}_{A \subset E}$.

Lemma 5.3. *For every $i \in \mathbb{N}$ let $A_i \subset E$ be a set of edges. And let $I \subset \mathbb{N}$ be a finite index set. Then the following statements hold:*

(*i*) $\quad 0 \leq \mathbb{E}[B_{A_1}]\mathbb{E}[B_{A_2}] \leq \mathbb{E}[B_{A_1 \cup A_2}],$

(*ii*) $\quad 0 \leq \mathbb{E}[B_{A_1}^c B_{A_2}^c] \leq \mathbb{E}[B_{A_1 \cup A_2}],$

(*iii*) $\quad 0 \leq \mathbb{E}\Big[\Big|\prod_{i \in I} B_{A_i}^c\Big|\Big] \leq 2^{|I|}\mathbb{E}[B_{\bigcup_{i \in I} A_i}].$

Proof: Obviously, $\mathbb{E}[B_{A_i}] = p^{|A_i|}$ is non-negative for all $i \in \mathbb{N}$. Further,

$$\mathbb{E}[B_{A_1}]\mathbb{E}[B_{A_2}] = p^{|A_1|+|A_2|} \leq p^{|A_1 \cup A_2|} = \mathbb{E}[B_{A_1 \cup A_2}].$$

This proves (*i*). To varify (*ii*), see

$$\mathbb{E}[B_{A_1}^c B_{A_2}^c] = \mathbb{E}\big[\big(B_{A_1} - \mathbb{E}[B_{A_1}]\big) \cdot \big(B_{A_2} - \mathbb{E}[B_{A_2}]\big)\big] = \mathbb{E}[B_{A_1 \cup A_2}] - \mathbb{E}[B_{A_1}]\mathbb{E}[B_{A_1}].$$

Hence, (*ii*) follows from (*i*). Finally, by expansion of the product, we see

$$\mathbb{E}\Big[\Big|\prod_{i \in I} B_{A_i}^c\Big|\Big] \leq \sum_{\hat{I} \subset I} \left(\mathbb{E}\Big[\prod_{i \in \hat{I}} B_{A_i}\Big] \cdot \prod_{i \in I \setminus \hat{I}} \mathbb{E}[B_{A_i}]\right) \leq \sum_{\hat{I} \subset I} \mathbb{E}[B_{\bigcup_{i \in I} A_i}] = 2^{|I|}\mathbb{E}[B_{\bigcup_{i \in I} A_i}],$$

where we again used our result from (*i*). This finishes the proof. □

We will later have to deal with sums of powers of p, in which the structure of the sum is determined via the neighborhood structure of the copies of G_0. These sums together with the variance σ^2 are the main reason for the appearance of $\Psi_{\min}$ in our results, and will now be examined in detail.

Definition 5.4. Let $m \in \mathbb{N}$ be a natural number, and let $M = (\Gamma_i)_{i=1}^m \subset \mathcal{M}$ be an ordered set of copies, in which each copy is a neighbor of at least one of its predecessors, i.e. $\Gamma_1 \in \mathcal{M}$ and $\Gamma_i \in \bigcup_{j=1}^{i-1} \mathcal{M}_{\Gamma_j}$ for all $2 \leq i \leq m$. We then call M to be a *set of connected copies* of size m.

Lemma 5.5. *Let m and $\hat{m}$ be natural numbers, and let $\bar{m} = m + \hat{m}$. Then*

(i)
$$\sum_{M:|M|=m \text{ con. cop.}} p^{|\bigcup_{\Gamma\in M}\Gamma|} \leq c_m \cdot \frac{n^{m\cdot v_{G_0}} \cdot p^{m\cdot e_{G_0}}}{\Psi_{\min}^{m-1}},$$

(ii)
$$\sum_{\substack{M:|M|=m \text{ con. cop.} \\ \hat{M}:|\hat{M}|=\hat{m} \text{ con. cop.}}} p^{|\bigcup_{\Gamma\in M\cup\hat{M}}\Gamma|} \leq c_{\bar{m}} \cdot \frac{n^{\bar{m}\cdot v_{G_0}} \cdot p^{\bar{m}\cdot e_{G_0}}}{\Psi_{\min}^{\bar{m}-2} \cdot \min\{\Psi_{\min}, 1\}},$$

where the sums run over all sets of connected copies of size m or $\hat{m}$, respectively, and where the constants are given by $c_k := (v_{G_0}!)^{k-1} \cdot 2^{\frac{1}{2}k(k-1)\cdot e_{G_0}} \cdot a_{G_0}^{-k}$ for $k \in \mathbb{N}$.

This lemma summarizes a counting technique that is commonly used in the field of random subgraph counting, see e.g. (3) in Ruciński (1988), (3.10) in Barbour et al. (1989), and (3.10) in Janson et al. (2000). However, in contrast to these references, we formulate the result of this counting strategy as an independent lemma for an arbitrary number of connected copies of G_0.

Proof of Lemma 5.5: To prove (i), let us first assume that $m = 1$. In this case, every set of connected copies M consists of only one copy Γ_1 that can be arbitrarily chosen among all copies in $\mathcal{M}$. Therefore, we are interested in the size of $\mathcal{M}$: There are $\binom{n}{v_{G_0}}$ possibilities to choose v_{G_0} vertices. On every set of v_{G_0} vertices, we can find $\frac{v_{G_0}!}{a_{G_0}}$ copies of G_0. So there are $\frac{v_{G_0}!}{a_{G_0}} \cdot \binom{n}{v_{G_0}} \leq \frac{n^{v_{G_0}}}{a_{G_0}}$ possible choices for $\Gamma_1 \in \mathcal{M}$. Hence,

$$\sum_{\substack{M:|M|=1 \\ \text{con. cop.}}} p^{|\bigcup_{\Gamma\in M}\Gamma|} = \sum_{\Gamma\in\mathcal{M}} p^{|\Gamma|} \leq \frac{n^{v_{G_0}}}{a_{G_0}} \cdot p^{e_{G_0}} = c_1 \cdot \frac{n^{1\cdot v_{G_0}} \cdot p^{1\cdot e_{G_0}}}{\Psi_{\min}^0}.$$

Next, we assume that (i) holds true for some fixed $m \in \mathbb{N}$ and we want to prove that (i) still holds if m is increased by 1. If $\hat{M}$ is a set of connected copies of size $m+1$, then the first m elements of $\hat{M}$ form a set of connected copies of size m. Hence,

$$\sum_{\substack{\hat{M}:|\hat{M}|=m+1 \\ \text{con. cop.}}} p^{|\bigcup_{\Gamma\in\hat{M}}\Gamma|} = \sum_{\substack{M:|M|=m \\ \text{con. cop.}}} p^{|\bigcup_{\Gamma\in M}\Gamma|} \sum_{\Gamma_{m+1}\in\bigcup_{\Gamma\in M}\mathcal{M}_\Gamma} p^{|\Gamma_{m+1}\setminus\bigcup_{\Gamma\in M}\Gamma|}.$$

Given M a set of connected copies of size m, Γ_{m+1} has to be a neighbor of at least one $\Gamma \in M$. In particular, the intersection $\Gamma_{m+1} \cap \bigcup_{\Gamma\in M}\Gamma$ may not be empty. To understand the resulting structure, we fix a non-empty subset $H \subset \bigcup_{\Gamma\in M}\Gamma$ and we count the possible choices for Γ_{m+1}, whose intersection with $\bigcup_{\Gamma\in M}\Gamma$ equals H. Assuming that H is small enough that it can be completed to a copy of G_0, there are $\binom{n-v_H}{v_{G_0}-v_H}$ possibilities to complete the v_H vertices of H to a set of v_{G_0} vertices. On such a set there are $\frac{v_{G_0}!}{a_{G_0}}$ copies of G_0. Hence, we can find at most $\frac{v_{G_0}!}{a_{G_0}} \cdot \binom{n-v_H}{v_{G_0}-v_H} \leq \frac{v_{G_0}!}{a_{G_0}} \cdot n^{v_{G_0}-v_H}$ possible copies $\Gamma_{m+1} \in \mathcal{M}$ with $H = \Gamma_{m+1} \cap \bigcup_{\Gamma\in M}\Gamma$. Further note that $|\Gamma_{m+1}\setminus\bigcup_{\Gamma\in M}\Gamma| = |\Gamma_{m+1}\setminus H| = e_{G_0} - e_H$. Thus,

$$\sum_{\substack{M:|M|=m \\ \text{con. cop.}}} p^{|\bigcup_{\Gamma\in M}\Gamma|} \sum_{\Gamma_{m+1}\in\bigcup_{\Gamma\in M}\mathcal{M}_\Gamma} p^{|\Gamma_{m+1}\setminus\bigcup_{\Gamma\in M}\Gamma|} \leq \sum_{\substack{M:|M|=m \\ \text{con. cop.}}} p^{|\bigcup_{\Gamma\in M}\Gamma|} \sum_{H \underset{G_0}{\subset} \bigcup_{\Gamma\in M}\mathcal{M}_\Gamma} v_{G_0}! \cdot a_{G_0}^{-1} \cdot n^{v_{G_0}-v_H} \cdot p^{e_{G_0}-e_H},$$

where $H \underset{G_0}{\subset} \bigcup_{\Gamma\in M}\mathcal{M}_\Gamma$ denotes that H is a non-empty subset of $\bigcup_{\Gamma\in M}\mathcal{M}_\Gamma$ while additionally being isomorphic to a subset of G_0. There are less than $2^{m\cdot e_{G_0}}$ sets with these properties. For each of them there is $n^{v_H}p^{e_H} \geq \Psi_{\min}$, so that

$$\sum_{\substack{M:|M|=m \\ \text{con. cop.}}} p^{|\bigcup_{\Gamma\in M}\Gamma|} \sum_{H \underset{G_0}{\subset} \bigcup_{\Gamma\in M}\mathcal{M}_\Gamma} v_{G_0}! \cdot a_{G_0}^{-1} \cdot n^{v_{G_0}-v_H} \cdot p^{e_{G_0}-e_H} \leq \sum_{\substack{M:|M|=m \\ \text{con. cop.}}} p^{|\bigcup_{\Gamma\in M}\Gamma|} \cdot 2^{m\cdot e_{G_0}} \cdot \frac{v_{G_0}!}{a_{G_0}} \cdot \frac{n^{v_{G_0}}p^{e_{G_0}}}{\Psi_{\min}}.$$

As we assumed that (i) holds true for m, we arrive at

$$\sum_{\substack{M:|M|=m\\ \text{con. cop.}}} p^{|\bigcup_{\Gamma\in M}\Gamma|}\cdot 2^{m\cdot e_{G_0}}\cdot\frac{v_{G_0}!}{a_{G_0}}\cdot\frac{n^{v_{G_0}}p^{e_{G_0}}}{\Psi_{\min}} \le c_m\cdot\frac{n^{m\cdot v_{G_0}}\cdot p^{m\cdot e_{G_0}}}{\Psi_{\min}^{m-1}}\cdot 2^{m\cdot e_{G_0}}\cdot\frac{v_{G_0}!}{a_{G_0}}\cdot\frac{n^{v_{G_0}}p^{e_{G_0}}}{\Psi_{\min}}$$

$$= c_{m+1}\cdot\frac{n^{(m+1)\cdot v_{G_0}}\cdot p^{(m+1)\cdot e_{G_0}}}{\Psi_{\min}^{(m+1)-1}}.$$

This finishes the proof of (i). To prove the second part, we want to adapt our strategy from above. Let $M=(\Gamma_i)_{i=1}^m\subset\mathcal{M}$ be a set of connected copies of size m, and $\hat{M}=(\hat{\Gamma}_i)_{i=1}^{\hat{m}}\subset\mathcal{M}$ be a set of connected copies of size $\hat{m}$. Further, let $\bar{M}:=(\Gamma_1,\ldots,\Gamma_m,\hat{\Gamma}_1,\ldots,\hat{\Gamma}_{\hat{m}})$ be the set created by joining M and $\hat{M}$. If we could be sure that $\bar{M}$ is a set of connected copies, we could directly apply (i). However, we can not generally assume that $\bar{M}$ is a set of connected copies. In fact, $\bar{M}$ is a set of connected copies of size $m+\hat{m}$ if and only if $\hat{\Gamma}_1\in\bigcup_{i=1}^m\mathcal{M}_{\Gamma_i}$. Hence, we have to modify our strategy from (i) only with respect to $\hat{H}=\hat{\Gamma}_1\cap\bigcup_{i=1}^m\mathcal{M}_{\Gamma_i}$, which can be empty in this case. Therefore, the lower bound $n^{v_{\hat{H}}}\cdot p^{e_{\hat{H}}}\ge\Psi_{\min}$ does not necessarily hold. We have to take the case $\hat{H}=\emptyset$ into account, which results in $n^{v_{\hat{H}}}\cdot p^{e_{\hat{H}}}\ge\min\{\Psi_{\min},1\}$. This proves the second part of the statement. □

Before we turn to the main proofs, we want to exemplify two bounding strategies in connection with the moment generating function. We will repeatedly use these strategies, later.

Lemma 5.6. *Let $A_1,A_2\subset\mathcal{M}$ be sets of copies of G_0, and let F be a non-negative functional of $\{B_\Gamma\}_{\Gamma\in A_2}$, so that F is a random variable independent from $\{B_\Gamma\}_{\Gamma\in\mathcal{M}\setminus\hat{A}_2}$, where $\hat{A}_2:=\bigcup_{\Gamma\in A_2}\mathcal{M}_\Gamma$ is the set of all copies of G_0 that depend on at least one element from A_2, in particular $A_2\subset\hat{A}_2$. Then for all $t\ge 0$*

$$\mathbb{E}\Big[F\cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus A_1}B_\Gamma^c}\Big]\le\mathbb{E}[F]\cdot e^{\frac{t}{\sigma}|A_1\cup\hat{A}_2|}\cdot\mathbb{E}[e^{tW}].$$

In particular, there is

$$\mathbb{E}\Big[e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus A_1}B_\Gamma^c}\Big]\le e^{\frac{t}{\sigma}|A_1|}\cdot\mathbb{E}[e^{tW}].$$

Proof: There is

$$\begin{aligned}\mathbb{E}\Big[F\cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus A_1}B_\Gamma^c}\Big]&=\mathbb{E}\Big[F\cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\hat{A}_2\setminus A_1}B_\Gamma^c}\cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus(A_1\cup\hat{A}_2)}B_\Gamma^c}\Big]\\&\le\mathbb{E}\Big[F\cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\hat{A}_2\setminus A_1}(1-p^{e_{G_0}})}\cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus(A_1\cup\hat{A}_2)}B_\Gamma^c}\Big]\\&=\mathbb{E}[F]\cdot e^{\frac{t}{\sigma}|\hat{A}_2\setminus A_1|(1-p^{e_{G_0}})}\cdot\mathbb{E}\Big[e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus(A_1\cup\hat{A}_2)}B_\Gamma^c}\Big].\end{aligned}$$

Further,

$$\mathbb{E}\Big[e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus(A_1\cup\hat{A}_2)}B_\Gamma^c}\Big]=\mathbb{E}\Big[e^{-\frac{t}{\sigma}\sum_{\Gamma\in A_1\cup\hat{A}_2}B_\Gamma^c}\cdot e^{tW}\Big]\le e^{\frac{t}{\sigma}|A_1\cup\hat{A}_2|p^{e_{G_0}}}\cdot\mathbb{E}[e^{tW}].$$

This proves the first inequality. The second inequality follows from the special case of $F=1$ with $A_2=\emptyset$. □

We are now prepared to prove Theorem 1.5 and Corollary 1.6.

Proof of Theorem 1.5: This proof is based on our theoretical results presented in Theorem 3.2. Since W depends on only finitely many Rademacher random variables we may apply this theorem as long as we find suitable functions γ_1 and γ_2. In the application of this theorem, the quantity $U = (U_k)_{k\in E}$ with

$$U_k := \frac{1}{\sqrt{pq}} \cdot D_k W \cdot |D_k L^{-1} W| = \frac{1}{\sqrt{pq}} \cdot D_k W \cdot (-D_k L^{-1} W), \tag{5.2}$$

$k \in E$, is of special importance, where $|D_k L^{-1} W| = -D_k L^{-1} W$ holds true due to the results from Lemma 5.1.

Due to the application of the discrete gradient, U_k does not depend on Y_k. For this special case it is known by Corollary 9.9 in Privault (2008) that $\delta(U) = \sum_{k\in E} Y_k U_k$. We further note that $\mathbb{E}[\sqrt{pq}\sum_{k\in E} U_k] = 1$, which is implied by (2.13) in Krokowski et al. (2017) in case of f being the identity.

These considerations allow us to rephrase Theorem 3.1 with the notation introduced in (5.2): To apply this theorem, we need to construct functions γ_1 and γ_2 so that

(A1') $$\mathbb{E}\Big[\Big|\sqrt{pq}\sum_{k\in E} U_k^c\Big| \cdot e^{tW}\Big] \le \gamma_1(t) \cdot \mathbb{E}[e^{tW}],$$

(A2') $$\mathbb{E}\Big[\Big|\sum_{k\in E} Y_k U_k\Big| \cdot e^{tW}\Big] \le \gamma_2(t) \cdot \mathbb{E}[e^{tW}],$$

for $0 \le t \le A$, where $A \ge 0$.

For the construction of suitable γ_1 and γ_2, a further decomposition of $(U_k)_{k\in E}$ will be usefull, which can be achieved by application of Lemma 5.1: We decompose

$$\begin{aligned} U_k &= \frac{1}{\sqrt{pq}} \cdot D_k W \cdot (-D_k L^{-1} W) \\ &= \frac{1}{\sqrt{pq}\cdot\sigma^2} \cdot \sum_{\Gamma_1,\Gamma_2\in\mathcal{M}} D_k B^c_{\Gamma_1} \cdot \big(-D_k L^{-1} B^c_{\Gamma_2}\big) \\ &= \frac{\sqrt{pq}}{\sigma^2} \cdot \sum_{\Gamma_1,\Gamma_2\in\mathcal{M}_k} \sum_{\{k\}\subset\alpha_2\subset\Gamma_2} \frac{p^{|\Gamma_2|-|\alpha_2|}}{|\Gamma_2|\cdot\binom{|\Gamma_2|-1}{|\alpha_2|-1}} B_{(\Gamma_1\cup\alpha_2)\setminus\{k\}} \\ &= \frac{\sqrt{pq}}{\sigma^2} \cdot \sum_{\Gamma_1,\Gamma_2\in\mathcal{M}_k} V_{k,\Gamma_1,\Gamma_2} \end{aligned} \tag{5.3}$$

with

$$V_{k,\Gamma_1,\Gamma_2} := \sum_{\{k\}\subset\alpha_2\subset\Gamma_2} \frac{p^{|\Gamma_2|-|\alpha_2|}}{|\Gamma_2|\cdot\binom{|\Gamma_2|-1}{|\alpha_2|-1}} B_{(\Gamma_1\cup\alpha_2)\setminus\{k\}} \ge 0. \tag{5.4}$$

Construction of γ_1: To verify that assumption (A1') can be fulfilled with some function γ_1—that is yet to be constructed—we start by applying the Cauchy–Schwarz inequality and using the decomposition of U_k from (5.3). This results in

$$\begin{aligned} \mathbb{E}\Big[\Big|\sqrt{pq}\sum_{k\in E} U_k^c\Big| \cdot e^{tW}\Big] &\le \mathbb{E}\Big[\Big(\sqrt{pq}\sum_{k\in E} U_k^c\Big)^2 e^{tW}\Big]^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}} \\ &= \Big(pq\sum_{k,\ell\in E} \mathbb{E}\Big[U_k^c U_\ell^c e^{tW}\Big]\Big)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}} \\ &= \Big(\frac{p^2q^2}{\sigma^4} \cdot \sum_{k,\ell\in E} \sum_{\Gamma_1,\Gamma_2\in\mathcal{M}_k} \sum_{\Gamma_3,\Gamma_4\in\mathcal{M}_\ell} \mathbb{E}\Big[V^c_{k,\Gamma_1,\Gamma_2} V^c_{\ell,\Gamma_3,\Gamma_4} e^{tW}\Big]\Big)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}} \end{aligned}$$

for all $t \geq 0$. Next, we apply the iterated Taylor expansion that was introduced in (4.6) to e^{tW} by choosing $x_\Gamma = \frac{t}{\sigma} B_\Gamma^c$ and $y_\Gamma = \frac{t}{\sigma} \sum_{\Gamma_5 \in \mathcal{M}_\Gamma \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}} B_{\Gamma_5}^c$ for $\Gamma \in I = \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}$, and $z = \frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}} B_\Gamma^c$. Due to the underlying dependency structure this expansion results in

$$\frac{p^2q^2}{\sigma^4} \cdot \sum_{k,\ell \in E} \sum_{\Gamma_1,\Gamma_2 \in \mathcal{M}_k} \sum_{\Gamma_3,\Gamma_4 \in \mathcal{M}_\ell} \mathbb{E}\Big[V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c e^{tW}\Big] = S_{1,1}(t) + S_{1,2}(t) + S_{1,3}(t) + S_{1,4}(t)$$

with

$$S_{1,1}(t) := \frac{p^2q^2}{\sigma^4} \cdot \sum_{\substack{k,\ell \in E \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell}} \mathbb{E}\Big[V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c\Big] \cdot \mathbb{E}\Big[e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}} B_\Gamma^c}\Big],$$

$$S_{1,2}(t) := \frac{p^2q^2}{\sigma^4} \cdot \frac{t}{\sigma} \cdot \sum_{\substack{k,\ell \in E \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5 \in \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}}} \mathbb{E}\Big[V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c \cdot B_{\Gamma_5}^c\Big] \cdot \mathbb{E}\Big[e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5}} B_\Gamma^c}\Big],$$

$$S_{1,3}(t) := \frac{p^2q^2}{\sigma^4} \cdot \frac{t^2}{\sigma^2} \cdot \sum_{\substack{k,\ell \in E \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5 \in \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} \\ \Gamma_6 \in \mathcal{M}_{\Gamma_5} \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}}} \mathbb{E}\Bigg[V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c \cdot B_{\Gamma_5}^c B_{\Gamma_6}^c \cdot r_1\Big(\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M}_{\Gamma_5} \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}} B_\Gamma^c\Big) \cdot e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5}} B_\Gamma^c}\Bigg],$$

$$S_{1,4}(t) := \frac{p^2q^2}{\sigma^4} \cdot \frac{t^2}{2\sigma^2} \cdot \sum_{\substack{k,\ell \in E \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5,\Gamma_6 \in \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}}} \mathbb{E}\Bigg[V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c \cdot B_{\Gamma_5}^c B_{\Gamma_6}^c \cdot r_2\Big(\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}} B_\Gamma^c\Big) \cdot e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}} B_\Gamma^c}\Bigg],$$

so that

$$\mathbb{E}\Big[\Big|\sqrt{pq} \sum_{k \in E} U_k^c\Big| \cdot e^{tW}\Big] \leq \big(S_{1,1}(t) + S_{1,2}(t) + S_{1,3}(t) + S_{1,4}(t)\big)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}}.$$

The advantage of this expansion based on Zhang (2019) is that $S_{1,1}$ and $S_{1,2}$ each possess a comfortable dependency structure that allowed us to split the expectations, while $S_{1,3}$ and $S_{1,4}$ are of second order and therefore allow more rough estimates without a loss in the resulting rate, as we will see later in this proof.

We now have to derive bounds for $S_{1,1}$, $S_{1,2}$, $S_{1,3}$, and $S_{1,4}$.

To derive a bound for $S_{1,1}$, we first fix $k, \ell \in E$, $\Gamma_1, \Gamma_2 \in \mathcal{M}_k$, and $\Gamma_3, \Gamma_4 \in \mathcal{M}_\ell$. If $\Gamma_1 \cup \Gamma_2$ is disjoint from $\Gamma_3 \cup \Gamma_4$, then V_{k,Γ_1,Γ_2} and $V_{\ell,\Gamma_3,\Gamma_4}$ are independent, so that $\mathbb{E}\Big[V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c\Big] = 0$. Otherwise, using our knowledge from Lemma 5.3 about the correlation behavior of the random variables $\{B_A\}_{A \subset E}$ we see that $\mathbb{E}[V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c] \geq 0$ and

$$\begin{aligned}
&\mathbb{E}\Big[V^c_{k,\Gamma_1,\Gamma_2}V^c_{\ell,\Gamma_3,\Gamma_4}\Big] \\
&\quad= \sum_{\substack{\{k\}\subset\alpha_2\subset\Gamma_2\\ \{\ell\}\subset\alpha_4\subset\Gamma_4}} \frac{\mathbb{E}[B_{\Gamma_2\setminus\alpha_2}]}{|\Gamma_2|\cdot\binom{|\Gamma_2|-1}{|\alpha_2|-1}} \frac{\mathbb{E}[B_{\Gamma_4\setminus\alpha_4}]}{|\Gamma_4|\cdot\binom{|\Gamma_4|-1}{|\alpha_4|-1}} \mathbb{E}\Big[B^c_{(\Gamma_1\cup\alpha_2)\setminus\{k\}}B^c_{(\Gamma_3\cup\alpha_4)\setminus\{\ell\}}\Big] \\
&\quad\le \sum_{\substack{\{k\}\subset\alpha_2\subset\Gamma_2\\ \{\ell\}\subset\alpha_4\subset\Gamma_4}} \frac{1}{|\Gamma_2|\cdot\binom{|\Gamma_2|-1}{|\alpha_2|-1}} \frac{1}{|\Gamma_4|\cdot\binom{|\Gamma_4|-1}{|\alpha_4|-1}} \mathbb{E}\Big[B_{\big((\Gamma_1\cup\alpha_2)\setminus\{k\}\big)\cup(\Gamma_2\setminus\alpha_2)\cup\big((\Gamma_3\cup\alpha_4)\setminus\{\ell\}\big)\cup(\Gamma_4\setminus\alpha_4)}\Big] \\
&\quad\le \sum_{\substack{\{k\}\subset\alpha_2\subset\Gamma_2\\ \{\ell\}\subset\alpha_4\subset\Gamma_4}} \frac{1}{|\Gamma_2|\cdot\binom{|\Gamma_2|-1}{|\alpha_2|-1}} \frac{1}{|\Gamma_4|\cdot\binom{|\Gamma_4|-1}{|\alpha_4|-1}} p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|-2} \;=\; p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|-2},
\end{aligned}$$

where in the last step we used what we already noted in Remark 5.2. The second inequality from Lemma 5.6 implies

$$\mathbb{E}\Big[e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4}}B^c_\Gamma}\Big] \le e^{\frac{t}{\sigma}|\mathcal{M}_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4}|}\cdot\mathbb{E}\big[e^{tW}\big] \le e^{\frac{4Dt}{\sigma}}\cdot\mathbb{E}\big[e^{tW}\big].$$

Putting these partial results together yields $S_{1,1}(t)\ge 0$ and

$$\begin{aligned}
S_{1,1}(t) &\le \frac{p^2q^2}{\sigma^4}\cdot\sum_{\substack{k,\ell\in E\\ \Gamma_1,\Gamma_2\in\mathcal{M}_k\\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell}} \mathbb{1}_{\{(\Gamma_1\cup\Gamma_2)\cap(\Gamma_3\cup\Gamma_4)\neq\emptyset\}} p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|-2}\cdot e^{\frac{4Dt}{\sigma}}\cdot\mathbb{E}\big[e^{tW}\big] \\
&\le \frac{q^2}{\sigma^4}\cdot 2\sum_{\substack{\Gamma_1\in\mathcal{M}\\ \Gamma_2\in\mathcal{M}_{\Gamma_1}\\ \Gamma_3\in\mathcal{M}_{\Gamma_1\cup\Gamma_2}\\ \Gamma_4\in\mathcal{M}_{\Gamma_3}}} \sum_{\substack{k\in\Gamma_1\cap\Gamma_2\\ \ell\in\Gamma_3\cap\Gamma_4}} p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|}\cdot e^{\frac{4Dt}{\sigma}}\cdot\mathbb{E}\big[e^{tW}\big],
\end{aligned}$$

where we used that due to symmetry the sum over $\mathbb{1}_{\{(\Gamma_1\cup\Gamma_2)\cap(\Gamma_3\cup\Gamma_4)\neq\emptyset\}}$ is smaller than or equal to 2 times the sum over $\mathbb{1}_{\{(\Gamma_1\cup\Gamma_2)\cap\Gamma_3\neq\emptyset\}} = \mathbb{1}_{\{\Gamma_3\in\mathcal{M}_{\Gamma_1\cup\Gamma_2}\}}$. We further reordered the indices to point out that the first sum runs over a subset of all sets of connected copies of size 4 in the sense of Definition 5.4, while the second sum runs over not more than $e^2_{G_0}$ summands. Lemma 5.5 therefore yields

$$\begin{aligned}
S_{1,1}(t) &\le \frac{q^2}{\sigma^4}\cdot 2\cdot\frac{(v_{G_0}!)^3\cdot 2^{6e_{G_0}}}{a^4_{G_0}}\cdot\frac{n^{4v_{G_0}}\cdot p^{4e_{G_0}}}{\Psi^3_{\min}}\cdot e^2_{G_0}\cdot e^{\frac{4Dt}{\sigma}}\cdot\mathbb{E}\big[e^{tW}\big] \\
&\le c_{1,1}\cdot\frac{1}{\Psi_{\min}}\cdot e^{\frac{4Dt}{\sigma}}\cdot\mathbb{E}\big[e^{tW}\big]
\end{aligned}$$

with $c_{1,1} := \dfrac{2^{3+6e_{G_0}}\cdot(v_{G_0}!)^5\cdot e^2_{G_0}}{a^2_{G_0}}$. For the last step we used the lower bound for σ presented in (5.1), which we may use since we assume that $n\ge 4v^2_{G_0}$.

The calculations for $S_{1,2}$ are largely analogous, except for we have to use part (iii) instead of part (ii) from Lemma 5.3. We find

$$|S_{1,2}(t)| \le \frac{p^2q^2}{\sigma^4}\cdot\frac{t}{\sigma}\cdot 2\sum_{\substack{k,\ell\in E\\ \Gamma_1,\Gamma_2\in\mathcal{M}_k\\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell\\ \Gamma_5\in\mathcal{M}_{\Gamma_3\cup\Gamma_4}}} \mathbb{E}\Big[\Big|V^c_{k,\Gamma_1,\Gamma_2}V^c_{\ell,\Gamma_3,\Gamma_4}\cdot B^c_{\Gamma_5}\Big|\Big]\cdot\mathbb{E}\Big[e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5}}B^c_\Gamma}\Big]$$

$$\leq \frac{p^2q^2}{\sigma^4}\cdot\frac{t}{\sigma}\cdot 2\sum_{\substack{k,\ell\in E\\ \Gamma_1,\Gamma_2\in\mathcal{M}_k\\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell\\ \Gamma_5\in\mathcal{M}_{\Gamma_3\cup\Gamma_4}}} \mathbb{1}_{\{(\Gamma_1\cup\Gamma_2)\cap(\Gamma_3\cup\Gamma_4\cup\Gamma_5)\neq\emptyset\}} 2^3 p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5|-2}\cdot e^{\frac{t}{\sigma}|\mathcal{M}_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5}|}\cdot\mathbb{E}[e^{tW}]$$

$$\leq \frac{q^2}{\sigma^4}\cdot\frac{t}{\sigma}\cdot 2^4\Bigg(2\sum_{\substack{\Gamma_1\in\mathcal{M}\\ \Gamma_2\in\mathcal{M}_{\Gamma_1}\\ \Gamma_3\in\mathcal{M}_{\Gamma_1\cup\Gamma_2}\\ \Gamma_4\in\mathcal{M}_{\Gamma_3}\\ \Gamma_5\in\mathcal{M}_{\Gamma_3\cup\Gamma_4}}}\sum_{\substack{k\in\Gamma_1\cap\Gamma_2\\ \ell\in\Gamma_3\cap\Gamma_4}} p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5|} + 2\sum_{\substack{\Gamma_1\in\mathcal{M}\\ \Gamma_2\in\mathcal{M}_{\Gamma_1}\\ \Gamma_5\in\mathcal{M}_{\Gamma_1\cup\Gamma_2}\\ \Gamma_3\in\mathcal{M}_{\Gamma_5}\\ \Gamma_4\in\mathcal{M}_{\Gamma_3}}}\sum_{\substack{k\in\Gamma_1\cap\Gamma_2\\ \ell\in\Gamma_3\cap\Gamma_4}} p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5|}\Bigg)\cdot e^{\frac{5Dt}{\sigma}}\cdot\mathbb{E}[e^{tW}]$$

$$\leq \frac{q^2}{\sigma^4}\cdot\frac{t}{\sigma}\cdot 2^6\cdot\frac{(v_{G_0}!)^4\cdot 2^{10e_{G_0}}}{a_{G_0}^5}\cdot\frac{n^{5v_{G_0}}\cdot p^{5e_{G_0}}}{\Psi_{\min}^4}\cdot e_{G_0}^2\cdot e^{\frac{5Dt}{\sigma}}\cdot\mathbb{E}[e^{tW}]$$

$$\leq c_{1,2}\cdot\frac{t}{\sqrt{q\Psi_{\min}}}\cdot\frac{1}{\Psi_{\min}}\cdot e^{\frac{5Dt}{\sigma}}\cdot\mathbb{E}[e^{tW}]$$

with $c_{1,2} := \dfrac{2^{\frac{17}{2}+10e_{G_0}}\cdot(v_{G_0}!)^{\frac{13}{2}}\cdot e_{G_0}^2}{a_{G_0}^{\frac{5}{2}}}$.

To construct a bound for $S_{1,3}$ we have to slightly modify our approach. Due to symmetry, we can rename the variables if necessary to make sure that $\Gamma_5\in\mathcal{M}_{\Gamma_3\cup\Gamma_4}$. However, due to the influence of the remainder function r_1 the relevant expectation can be non-zero even if $\Gamma_1\cup\Gamma_2$ and $\Gamma_3\cup\Gamma_4\cup\Gamma_5\cup\Gamma_6$ are disjoint. This disadvantage can later be compensated for by using the additional σ^{-2}. We start by taking the absolute to get rid of the remainder r_1. Recall that $t\geq 0$ and $r_1(x)\leq e^{|x|}$ for all $x\in\mathbb{R}$, so that

$$\Bigg|r_1\Bigg(\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}_{\Gamma_5}\setminus\mathcal{M}_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4}} B_\Gamma^c\Bigg)\Bigg|\leq e^{\frac{Dt}{\sigma}}.$$

Hence, by application of Lemma 5.6 and Lemma 5.3, there is

$$|S_{1,3}(t)|\leq\frac{p^2q^2}{\sigma^4}\cdot\frac{t^2}{\sigma^2}\cdot 2\sum_{\substack{k,\ell\in E\\ \Gamma_1,\Gamma_2\in\mathcal{M}_k\\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell\\ \Gamma_5\in\mathcal{M}_{\Gamma_3\cup\Gamma_4}\\ \Gamma_6\in\mathcal{M}_{\Gamma_5}\setminus\mathcal{M}_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4}}}\mathbb{E}\Big[\big|V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c\cdot B_{\Gamma_5}^c B_{\Gamma_6}^c\big|\cdot e^{\frac{Dt}{\sigma}}\cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5}} B_\Gamma^c}\Big]$$

$$\leq\frac{p^2q^2}{\sigma^4}\cdot\frac{t^2}{\sigma^2}\cdot 2\sum_{\substack{\Gamma_1\in\mathcal{M}\\ \Gamma_2\in\mathcal{M}_{\Gamma_1}\\ \Gamma_3\in\mathcal{M}\\ \Gamma_4\in\mathcal{M}_{\Gamma_3}\\ \Gamma_5\in\mathcal{M}_{\Gamma_3\cup\Gamma_4}\\ \Gamma_6\in\mathcal{M}_{\Gamma_5}\setminus\mathcal{M}_{\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4}}}\sum_{\substack{k\in\Gamma_1\cap\Gamma_2\\ \ell\in\Gamma_3\cap\Gamma_4}}\mathbb{E}\Big[\big|V_{k,\Gamma_1,\Gamma_2}^c V_{\ell,\Gamma_3,\Gamma_4}^c\cdot B_{\Gamma_5}^c B_{\Gamma_6}^c\big|\Big]\cdot e^{\frac{Dt}{\sigma}}\cdot e^{\frac{6Dt}{\sigma}}\cdot\mathbb{E}[e^{tW}]$$

$$\leq\frac{q^2}{\sigma^4}\cdot\frac{t^2}{\sigma^2}\cdot 2\sum_{\substack{\Gamma_1\in\mathcal{M}\\ \Gamma_2\in\mathcal{M}_{\Gamma_1}\\ \Gamma_3\in\mathcal{M}\\ \Gamma_4\in\mathcal{M}_{\Gamma_3}\\ \Gamma_5\in\mathcal{M}_{\Gamma_3\cup\Gamma_4}\\ \Gamma_6\in\mathcal{M}_{\Gamma_5}}} e_{G_0}^2 2^4 p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5\cup\Gamma_6|}\cdot e^{\frac{7Dt}{\sigma}}\cdot\mathbb{E}[e^{tW}].$$

As already mentioned, we can not be sure that $\Gamma_1\cup\Gamma_2$ and $\Gamma_3\cup\Gamma_4\cup\Gamma_5\cup\Gamma_6$ are not disjoint. Therefore, in the sum above Γ_3 runs over the the complete set $\mathcal{M}$, so that (Γ_1,Γ_2) and $(\Gamma_3,\Gamma_4,\Gamma_5,\Gamma_6)$ are sets of connected copies while in general their union $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6)$ is not. This case is covered by the second inequality from Lemma 5.5, so that

$$\begin{aligned}|S_{1,3}(t)| &\le \frac{q^2}{\sigma^4} \cdot \frac{t^2}{\sigma^2} \cdot 2^5 \cdot \frac{(v_{G_0}!)^5 \cdot 2^{15 e_{G_0}}}{a_{G_0}^6} \cdot \frac{n^{6 v_{G_0}} \cdot p^{6 e_{G_0}}}{\Psi_{\min}^4 \cdot \min\{\Psi_{\min}, 1\}} \cdot e_{G_0}^2 \cdot e^{\frac{7Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\ &\le c_{1,3} \cdot \frac{t^2}{q \min\{\Psi_{\min}, 1\}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{7Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]\end{aligned}$$

with $c_{1,3} := \dfrac{2^{8+15e_{G_0}} \cdot (v_{G_0}!)^8 \cdot e_{G_0}^2}{a_{G_0}^3}$.

The calculations for $S_{1,4}$ are largely analogous to $S_{1,3}$ again. We get

$$\begin{aligned}|S_{1,4}(t)| &\le \frac{p^2 q^2}{\sigma^4} \cdot \frac{t^2}{2\sigma^2} \cdot \sum_{\substack{k,\ell \in E \\ \Gamma_1, \Gamma_2 \in \mathcal{M}_k \\ \Gamma_3, \Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5, \Gamma_6 \in \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}}} \mathbb{E}\Big[\big|V^c_{k,\Gamma_1,\Gamma_2} V^c_{\ell,\Gamma_3,\Gamma_4} \cdot B^c_{\Gamma_5} B^c_{\Gamma_6}\big| \cdot e^{\frac{4Dt}{\sigma}} \cdot e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}} B^c_\Gamma}\Big] \\ &\le \frac{p^2 q^2}{\sigma^4} \cdot \frac{t^2}{2\sigma^2} \cdot \sum_{\substack{\Gamma_1 \in \mathcal{M} \\ \Gamma_2 \in \mathcal{M}_{\Gamma_1} \\ \Gamma_3 \in \mathcal{M} \\ \Gamma_4 \in \mathcal{M}_{\Gamma_3} \\ \Gamma_5, \Gamma_6 \in \mathcal{M}_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4}}} \sum_{\substack{k \in \Gamma_1 \cap \Gamma_2 \\ \ell \in \Gamma_3 \cap \Gamma_4}} \mathbb{E}\Big[\big|V^c_{k,\Gamma_1,\Gamma_2} V^c_{\ell,\Gamma_3,\Gamma_4} \cdot B^c_{\Gamma_5} B^c_{\Gamma_6}\big|\Big] \cdot e^{\frac{4Dt}{\sigma}} \cdot e^{\frac{6Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\ &\le \frac{q^2}{\sigma^4} \cdot \frac{t^2}{2\sigma^2} \Bigg(2 \sum_{\substack{\Gamma_1 \in \mathcal{M} \\ \Gamma_2 \in \mathcal{M}_{\Gamma_1} \\ \Gamma_3 \in \mathcal{M} \\ \Gamma_4 \in \mathcal{M}_{\Gamma_3} \\ \Gamma_5, \Gamma_6 \in \mathcal{M}_{\Gamma_3 \cup \Gamma_4}}} e_{G_0}^2 2^4 p^{|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6|} + 2 \sum_{\substack{\Gamma_1 \in \mathcal{M} \\ \Gamma_2 \in \mathcal{M}_{\Gamma_1} \\ \Gamma_3 \in \mathcal{M} \\ \Gamma_4 \in \mathcal{M}_{\Gamma_3} \\ \Gamma_5 \in \mathcal{M}_{\Gamma_1 \cup \Gamma_2} \\ \Gamma_6 \in \mathcal{M}_{\Gamma_3 \cup \Gamma_4}}} e_{G_0}^2 2^4 p^{|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6|}\Bigg) e^{\frac{10Dt}{\sigma}} \mathbb{E}[e^{tW}] \\ &\le \frac{q^2}{\sigma^4} \cdot \frac{t^2}{\sigma^2} \cdot 2^5 \cdot \frac{(v_{G_0}!)^5 \cdot 2^{15 e_{G_0}}}{a_{G_0}^6} \cdot \frac{n^{6 v_{G_0}} \cdot p^{6 e_{G_0}}}{\Psi_{\min}^4 \cdot \min\{\Psi_{\min}, 1\}} \cdot e_{G_0}^2 \cdot e^{\frac{10Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\ &\le c_{1,4} \cdot \frac{t^2}{q \min\{\Psi_{\min}, 1\}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{10Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]\end{aligned}$$

with $c_{1,4} := c_{1,3} = \dfrac{2^{8+15e_{G_0}} \cdot (v_{G_0}!)^8 \cdot e_{G_0}^2}{a_{G_0}^3}$.

Having bounded $S_{1,1}$, $S_{1,2}$, $S_{1,3}$, and $S_{1,4}$, we note that $c_{1,3} = \max\{c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}\}$ because of $a_{G_0} \le v_{G_0}!$. In conclusion, for all $t \ge 0$ it holds that

$$\begin{aligned}\mathbb{E}\Big[\Big|\sqrt{pq} \sum_{k \in E} U^c_k\Big| \cdot e^{tW}\Big] &\le \big(S_{1,1}(t) + S_{1,2}(t) + S_{1,3}(t) + S_{1,4}(t)\big)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}} \\ &\le \Bigg(c_{1,1} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\ &\quad + c_{1,2} \cdot \frac{t}{\sqrt{q \Psi_{\min}}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{5Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\ &\quad + c_{1,3} \cdot \frac{t^2}{q \min\{\Psi_{\min}, 1\}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{7Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\ &\quad + c_{1,4} \cdot \frac{t^2}{q \min\{\Psi_{\min}, 1\}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{10Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]\Bigg)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}}\end{aligned}$$

$$\leq \sqrt{c_{1,3}} \cdot \left(1 + \frac{t}{\sqrt{\Psi_{\min}}} + 2\frac{t^2}{\min\{\Psi_{\min}, 1\}}\right)^{\frac{1}{2}} \frac{1}{\sqrt{q\Psi_{\min}}} \cdot e^{\frac{5Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]$$
$$\leq \gamma_1(t) \cdot \mathbb{E}[e^{tW}]$$

with

$$\gamma_1(t) := c_{A1} \cdot \left(1 + \frac{t}{\min\{\sqrt{\Psi_{\min}}, 1\}}\right) \frac{1}{\sqrt{q\Psi_{\min}}} \cdot e^{\frac{5Dt}{\sigma}}, \tag{5.5}$$

where $c_{A1} := \sqrt{2c_{1,3}} = \dfrac{2^{\frac{9}{2}+\frac{15}{2}e_{G_0}} \cdot (v_{G_0}!)^4 \cdot e_{G_0}}{a_{G_0}^{\frac{3}{2}}}$.

Construction of γ_2: To verify that assumption (A2') can be fulfilled with some function γ_2 we again start by applying the Cauchy–Schwarz inequality and using the decomposition of U_k from (5.3), which results in

$$\begin{aligned}\mathbb{E}\Big[\Big|\sum_{k\in E} Y_k U_k\Big| \cdot e^{tW}\Big] &\leq \mathbb{E}\Big[\Big(\sum_{k\in E} Y_k U_k\Big)^2 e^{tW}\Big]^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}} \\ &= \Big(\sum_{k,\ell\in E} \mathbb{E}\Big[Y_k Y_\ell U_k U_\ell e^{tW}\Big]\Big)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}} \\ &= \left(\frac{1}{\sigma^4} \cdot \sum_{k,\ell\in E} \sum_{\substack{\Gamma_1,\Gamma_2\in\mathcal{M}_k \\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell}} \mathbb{E}\Big[B_k^c B_\ell^c \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot e^{tW}\Big]\right)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}}\end{aligned}$$

for all $t \geq 0$. We split the sum over $k, \ell \in E$ into two parts. The first part

$$S_{2,1}(t) := \frac{1}{\sigma^4} \cdot \sum_{\substack{k\in E \\ \Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\in\mathcal{M}_k}} \mathbb{E}\Big[(B_k^c)^2 \cdot V_{k,\Gamma_1,\Gamma_2} V_{k,\Gamma_3,\Gamma_4} \cdot e^{tW}\Big]$$

contains all terms in which there is $k = \ell$. In the second part, in which there is $k \neq \ell$, we decompose $e^{tW} = e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c} \cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c}$ and we use the Taylor expansion on $e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c}$. This yields the three terms

$$S_{2,2}(t) := \frac{1}{\sigma^4} \cdot \sum_{\substack{k,\ell\in E, k\neq\ell \\ \Gamma_1,\Gamma_2\in\mathcal{M}_k \\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell}} \mathbb{E}\Big[B_k^c B_\ell^c \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c}\Big],$$

$$S_{2,3}(t) := \frac{1}{\sigma^4} \cdot \frac{t}{\sigma} \cdot \sum_{\substack{k,\ell\in E, k\neq\ell \\ \Gamma_1,\Gamma_2\in\mathcal{M}_k \\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell \\ \Gamma_5\in\mathcal{M}_{\{k,\ell\}}}} \mathbb{E}\left[B_k^c B_\ell^c \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot B_{\Gamma_5}^c \cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c}\right],$$

$$S_{2,4}(t) := \frac{1}{\sigma^4} \cdot \frac{t^2}{2\sigma^2} \cdot \sum_{\substack{k,\ell\in E\, k\neq\ell \\ \Gamma_1,\Gamma_2\in\mathcal{M}_k \\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell \\ \Gamma_5,\Gamma_6\in\mathcal{M}_{\{k,\ell\}}}} \mathbb{E}\left[B_k^c B_\ell^c \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot B_{\Gamma_5}^c B_{\Gamma_6}^c \cdot r_2\Big(\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c\Big) \cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c}\right],$$

so that

$$\mathbb{E}\Big[\Big|\sum_{k\in E} Y_k U_k\Big| \cdot e^{tW}\Big] \leq \big(S_{2,1}(t) + S_{2,2}(t) + S_{2,3}(t) + S_{2,4}(t)\big)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}}.$$

To derive a bound for $S_{2,1}$, we first note that V_{k,Γ_1,Γ_2} and V_{k,Γ_3,Γ_4} are non-negative according to (5.4), so that $S_{2,1}(t) \geq 0$. An upper bound for $S_{2,1}(t)$ is derived by application of the first inequality from Lemma 5.6. We then use that V_{k,Γ_1,Γ_2} and V_{k,Γ_3,Γ_4} are independent from B_k according to (5.4), and we finish by applying Lemma 5.3, Remark 5.2, and Lemma 5.5. This yields

$$
\begin{aligned}
S_{2,1}(t) &\leq \frac{1}{\sigma^4} \cdot \sum_{\substack{k\in E\\ \Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\in\mathcal{M}_k}} \mathbb{E}\Big[(B_k^c)^2 \cdot V_{k,\Gamma_1,\Gamma_2} V_{k,\Gamma_3,\Gamma_4}\Big] \cdot e^{\frac{t}{\sigma}|\mathcal{M}_{\{k\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4}|} \cdot \mathbb{E}[e^{tW}] \\
&\leq \frac{1}{\sigma^4} \cdot \sum_{\substack{k\in E\\ \Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\in\mathcal{M}_k}} \mathbb{E}\big[(B_k^c)^2\big] \cdot \mathbb{E}\big[V_{k,\Gamma_1,\Gamma_2} V_{k,\Gamma_3,\Gamma_4}\big] \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq \frac{1}{\sigma^4} \cdot \sum_{\substack{k\in E\\ \Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\in\mathcal{M}_k}} pq \cdot \sum_{\substack{\{k\}\subset\alpha_2\subset\Gamma_2\\ \{k\}\subset\alpha_4\subset\Gamma_4}} \frac{\mathbb{E}[B_{\Gamma_2\setminus\alpha_2}]}{|\Gamma_2|\binom{|\Gamma_2|-1}{|\alpha_2|-1}} \frac{\mathbb{E}[B_{\Gamma_4\setminus\alpha_4}]}{|\Gamma_4|\binom{|\Gamma_4|-1}{|\alpha_4|-1}} \mathbb{E}\Big[B_{(\Gamma_1\cup\alpha_2)\setminus\{k\}} B_{(\Gamma_3\cup\alpha_4)\setminus\{k\}}\Big] e^{\frac{4Dt}{\sigma}} \mathbb{E}[e^{tW}] \\
&\leq \frac{1}{\sigma^4} \cdot \sum_{\substack{\Gamma_1\in\mathcal{M}\\ \Gamma_2,\Gamma_3,\Gamma_4\in\mathcal{M}_{\Gamma_1}}} \sum_{k\in\mathcal{M}_{\Gamma_1}} pq \cdot p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|-1} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq \frac{q}{\sigma^4} \cdot e_{G_0} \cdot \frac{(v_{G_0}!)^3 \cdot 2^{6e_{G_0}}}{a_{G_0}^4} \cdot \frac{n^{4v_{G_0}} \cdot p^{4e_{G_0}}}{\Psi_{\min}^3} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq c_{2,1} \cdot \frac{1}{q\Psi_{\min}} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]
\end{aligned}
$$

with $c_{2,1} := \dfrac{2^{2+6e_{G_0}} \cdot (v_{G_0}!)^5 \cdot e_{G_0}}{a_{G_0}^2}$. For the last step we again used the lower bound for σ presented in (5.1).

Regarding $S_{2,2}$, we first focus on the random variables inside the expectation. Let $k, \ell \in E$ with $k \neq \ell$, $\Gamma_1, \Gamma_2 \in \mathcal{M}_k$ and $\Gamma_3, \Gamma_4 \in \mathcal{M}_\ell$. V_{k,Γ_1,Γ_2} and $e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c}$ are independent of B_k^c. If $k \notin \Gamma_3 \cup \Gamma_4$, then $V_{\ell,\Gamma_3,\Gamma_4}$ is independent of B_k^c, too. In this case, we could split the expectation and obtain $\mathbb{E}[B_k^c] = 0$ as a factor. In the opposite case, in which $k \in \Gamma_3 \cup \Gamma_4$, we only have to regard those summands of $V_{\ell,\Gamma_3,\Gamma_4}$ that contain B_k as a factor. However, then there is $B_k^c \cdot B_k = q \cdot B_k$. Hence, $S_{2,2}(t) \geq 0$. Further, application of our main technical Lemmas 5.6, 5.3, 5.5, as well as inequality (5.1) leads to

$$
\begin{aligned}
S_{2,2}(t) &\leq \frac{1}{\sigma^4} \cdot \sum_{\substack{k,\ell\in E, k\neq\ell\\ \Gamma_1,\Gamma_2\in\mathcal{M}_k\\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell}} \mathbb{E}\Big[q^2 \cdot B_k B_\ell \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot e^{\frac{t}{\sigma}\sum_{\Gamma\in\mathcal{M}\setminus\mathcal{M}_{\{k,\ell\}}} B_\Gamma^c}\Big] \cdot \mathbb{1}_{\{k\in\Gamma_3\cup\Gamma_4\}} \mathbb{1}_{\{\ell\in\Gamma_1\cup\Gamma_2\}} \\
&\leq \frac{q^2}{\sigma^4} \cdot \sum_{\substack{k,\ell\in E, k\neq\ell\\ \Gamma_1,\Gamma_2\in\mathcal{M}_k\\ \Gamma_3,\Gamma_4\in\mathcal{M}_\ell}} \mathbb{E}\Big[B_k B_\ell \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4}\Big] \cdot e^{\frac{t}{\sigma}|\mathcal{M}_{\{k,\ell\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4}|} \cdot \mathbb{E}[e^{tW}] \cdot \mathbb{1}_{\{k\in\Gamma_3\cup\Gamma_4\}} \mathbb{1}_{\{\ell\in\Gamma_1\cup\Gamma_2\}} \\
&\leq \frac{q^2}{\sigma^4} \cdot 2^2 \sum_{\substack{k,\ell\in E, k\neq\ell\\ \Gamma_1,\Gamma_3\in\mathcal{M}_k\cap\mathcal{M}_\ell\\ \Gamma_2\in\mathcal{M}_k\\ \Gamma_4\in\mathcal{M}_\ell}} p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq \frac{q^2}{\sigma^4} \cdot 4 \sum_{\substack{\Gamma_1\in\mathcal{M}\\ \Gamma_2,\Gamma_3,\Gamma_4\in\mathcal{M}_{\Gamma_1}}} \sum_{\substack{k\in\Gamma_1\cap\Gamma_2\cap\Gamma_3\\ \ell\in\Gamma_1\cap\Gamma_3\cap\Gamma_4}} p^{|\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]
\end{aligned}
$$

$$\leq \frac{q^2}{\sigma^4} \cdot 4 \cdot e_{G_0}^2 \cdot \frac{(v_{G_0}!)^3 \cdot 2^{6e_{G_0}}}{a_{G_0}^4} \cdot \frac{n^{4v_{G_0}} \cdot p^{4e_{G_0}}}{\Psi_{\min}^3} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]$$
$$\leq c_{2,2} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]$$

with $c_{2,2} := \dfrac{2^{4+6e_{G_0}} \cdot (v_{G_0}!)^5 \cdot e_{G_0}^2}{a_{G_0}^2}$.

Similar calculations yield

$$|S_{2,3}(t)| \leq \frac{t}{\sigma^5} \cdot 2 \sum_{\substack{k,\ell \in E, k \neq \ell \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5 \in \mathcal{M}_\ell}} \left(\left| \mathbb{E}\left[B_k^c B_\ell^c \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot B_{\Gamma_5} \cdot e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\{k,\ell\}}} B_\Gamma^c} \right] \right| + \left| \mathbb{E}\left[B_k^c B_\ell^c \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot \mathbb{E}[B_{\Gamma_5}] \cdot e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\{k,\ell\}}} B_\Gamma^c} \right] \right| \right)$$
$$\leq \frac{t}{\sigma^5} \cdot 2 \sum_{\substack{k,\ell \in E, k \neq \ell \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4,\Gamma_5 \in \mathcal{M}_\ell}} \left(\left| \mathbb{E}\left[q^2 \cdot B_k B_\ell \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot B_{\Gamma_5} \cdot e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\{k,\ell\}}} B_\Gamma^c} \right] \right| \cdot \mathbb{1}_{\{k \in \Gamma_3 \cup \Gamma_4 \cup \Gamma_5\}} + \left| \mathbb{E}\left[q^2 \cdot B_k B_\ell \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot \mathbb{E}[B_{\Gamma_5}] \cdot e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\{k,\ell\}}} B_\Gamma^c} \right] \right| \cdot \mathbb{1}_{\{k \in \Gamma_3 \cup \Gamma_4\}} \right)$$
$$\leq \frac{q^2 t}{\sigma^5} \cdot 2 \sum_{\substack{k,\ell \in E, k \neq \ell \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4,\Gamma_5 \in \mathcal{M}_\ell}} p^{|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5|} \cdot \left(e^{\frac{5Dt}{\sigma}} \cdot \mathbb{1}_{\{k \in \Gamma_3 \cup \Gamma_4 \cup \Gamma_5\}} + e^{\frac{4Dt}{\sigma}} \cdot \mathbb{1}_{\{k \in \Gamma_3 \cup \Gamma_4\}} \right) \cdot \mathbb{E}[e^{tW}]$$
$$\leq \frac{q^2 t}{\sigma^5} \cdot 2 \cdot 5 \sum_{\substack{\Gamma_1 \in \mathcal{M} \\ \Gamma_2,\Gamma_3 \in \mathcal{M}_{\Gamma_1} \\ \Gamma_4,\Gamma_5 \in \mathcal{M}_{\Gamma_3}}} \sum_{\substack{k \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \\ \ell \in \Gamma_3 \cap \Gamma_4 \cap \Gamma_5}} p^{|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5|} \cdot e^{\frac{5Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]$$
$$\leq \frac{10 q^2 t}{\sigma^5} \cdot e_{G_0}^2 \cdot \frac{(v_{G_0}!)^4 \cdot 2^{10e_{G_0}}}{a_{G_0}^5} \cdot \frac{n^{5v_{G_0}} \cdot p^{5e_{G_0}}}{\Psi_{\min}^4} \cdot e^{\frac{5Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]$$
$$\leq c_{2,3} \cdot \frac{t}{\sqrt{q\Psi_{\min}}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{5Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]$$

with $c_{2,3} := \dfrac{5 \cdot 2^{\frac{7}{2}+10e_{G_0}} \cdot (v_{G_0}!)^{\frac{13}{2}} \cdot e_{G_0}^2}{a_{G_0}^{\frac{5}{2}}}$, and

$$|S_{2,4}(t)| \leq \frac{t^2}{2\sigma^6} \cdot \sum_{\substack{k,\ell \in E\, k \neq \ell \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5,\Gamma_6 \in \mathcal{M}_{\{k,\ell\}}}} \mathbb{E}\left[\left| B_k^c B_\ell^c \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot B_{\Gamma_5}^c B_{\Gamma_6}^c \right| \cdot e^{\frac{2Dt}{\sigma}} \cdot e^{\frac{t}{\sigma} \sum_{\Gamma \in \mathcal{M} \setminus \mathcal{M}_{\{k,\ell\}}} B_\Gamma^c} \right]$$
$$\leq \frac{t^2}{2\sigma^6} \cdot \sum_{\substack{k,\ell \in E\, k \neq \ell \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5,\Gamma_6 \in \mathcal{M}_{\{k,\ell\}}}} \mathbb{E}\left[\left| B_k^c B_\ell^c \right| \cdot V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot 2B_{\Gamma_5} \cdot 2B_{\Gamma_6} \right] \cdot e^{\frac{2Dt}{\sigma}} \cdot e^{\frac{6Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]$$

$$
\begin{aligned}
&\leq \frac{2t^2}{\sigma^6} \cdot \sum_{\substack{k,\ell \in E\, k \neq \ell \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5,\Gamma_6 \in \mathcal{M}_{\{k,\ell\}}}} \mathbb{E}\big[|B_k^c|\big]\mathbb{E}\big[|B_\ell^c|\big]\mathbb{E}\Big[V_{k,\Gamma_1,\Gamma_2} V_{\ell,\Gamma_3,\Gamma_4} \cdot B_{\Gamma_5} B_{\Gamma_6}\Big| B_k = B_\ell = 1\Big] \cdot e^{\frac{8Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq \frac{2t^2}{\sigma^6} \cdot \sum_{\substack{k,\ell \in E\, k \neq \ell \\ \Gamma_1,\Gamma_2 \in \mathcal{M}_k \\ \Gamma_3,\Gamma_4 \in \mathcal{M}_\ell \\ \Gamma_5,\Gamma_6 \in \mathcal{M}_{\{k,\ell\}}}} 2pq \cdot 2pq \cdot p^{|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6| - 2} \cdot e^{\frac{8Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq \frac{8q^2t^2}{\sigma^6} \cdot \Bigg(2 \sum_{\substack{\Gamma_1,\Gamma_3 \in \mathcal{M} \\ \Gamma_2,\Gamma_5 \in \mathcal{M}_{\Gamma_1} \\ \Gamma_4,\Gamma_6 \in \mathcal{M}_{\Gamma_3}}} e_{G_0}^2 p^{|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6|} + 2 \sum_{\substack{\Gamma_1,\Gamma_3 \in \mathcal{M} \\ \Gamma_2,\Gamma_5,\Gamma_6 \in \mathcal{M}_{\Gamma_1} \\ \Gamma_4 \in \mathcal{M}_{\Gamma_3}}} e_{G_0}^2 p^{|\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6|} \Bigg) \cdot e^{\frac{8Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq \frac{32q^2t^2}{\sigma^6} \cdot e_{G_0}^2 \cdot \frac{(v_{G_0}!)^5 \cdot 2^{15e_{G_0}}}{a_{G_0}^6} \cdot \frac{n^{6v_{G_0}} \cdot p^{6e_{G_0}}}{\Psi_{\min}^4 \cdot \min\{\Psi_{\min}, 1\}} \cdot e^{\frac{8Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq c_{2,4} \cdot \frac{t^2}{q \min\{\Psi_{\min}, 1\}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{8Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}]
\end{aligned}
$$

with $c_{2,4} := \dfrac{2^{8+15e_{G_0}} \cdot (v_{G_0}!)^8 \cdot e_{G_0}^2}{a_{G_0}^3}$.

Note that $c_{2,4} = \max\{c_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}\}$. Bringing together the bounds for $S_{2,1}$, $S_{2,2}$, $S_{2,3}$, and $S_{2,4}$ results in

$$
\begin{aligned}
\mathbb{E}\Big[\Big|\sum_{k \in E} Y_k U_k\Big| \cdot e^{tW}\Big] &\leq \big(S_{2,1}(t) + S_{2,2}(t) + S_{2,3}(t) + S_{2,4}(t)\big)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}} \\
&\leq \Bigg(c_{2,1} \cdot \frac{1}{q\Psi_{\min}} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\quad + c_{2,2} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\quad + c_{2,3} \cdot \frac{t}{\sqrt{q\Psi_{\min}}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{5Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\quad + c_{2,4} \cdot \frac{t^2}{q \min\{\Psi_{\min}, 1\}} \cdot \frac{1}{\Psi_{\min}} \cdot e^{\frac{8Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \Bigg)^{\frac{1}{2}} \cdot \mathbb{E}[e^{tW}]^{\frac{1}{2}} \\
&\leq \sqrt{c_{2,4}} \cdot \left(2 + \frac{t}{\sqrt{\Psi_{\min}}} + \frac{t^2}{\min\{\Psi_{\min}, 1\}}\right)^{\frac{1}{2}} \frac{1}{\sqrt{q\Psi_{\min}}} \cdot e^{\frac{4Dt}{\sigma}} \cdot \mathbb{E}[e^{tW}] \\
&\leq \gamma_2(t) \cdot \mathbb{E}[e^{tW}]
\end{aligned}
$$

with

$$
\gamma_2(t) := c_{A2} \cdot \left(1 + \frac{t}{\min\{\sqrt{\Psi_{\min}}, 1\}}\right) \frac{1}{\sqrt{q\Psi_{\min}}} \cdot e^{\frac{4Dt}{\sigma}}, \tag{5.6}
$$

where $c_{A2} := \sqrt{2c_{2,4}} = \dfrac{2^{\frac{9}{2}+\frac{15}{2}e_{G_0}} \cdot (v_{G_0}!)^4 \cdot e_{G_0}}{a_{G_0}^{\frac{3}{2}}} = c_{A1}$ for all $t \geq 0$.

Conclusion: We have constructed $\gamma_1(t)$ and $\gamma_2(t)$ for $t \geq 0$, see (5.5) and (5.6), so that the assumptions (A1') and (A2') that we formulated at the beginning of this proof of Theorem 1.5 on

p. 1364 are fulfilled. Finally, note that

$$\gamma_1(t) + \gamma_2(t) \leq 2c_{G_0} \cdot s(t)$$

with $s(t)$ and c_{G_0} as defined in the statement of Theorem 1.5. Theorem 3.2 yields the desired result. □

Proof of Corollary 1.6: If we restrict t to be smaller than $c_1 \cdot \frac{n^2 p^{e_{G_0}} \sqrt{q}}{\sqrt{\Psi_{\min}}}$, where $c_1 > 0$ is an arbitrary positive number, we can further simplify our results. Under this restriction, due to $n^2 p^{e_{G_0}} \leq \Psi_{\min}$, there is $t \leq c_1 \cdot \frac{n^2 p^{e_{G_0}} \sqrt{q}}{\sqrt{\Psi_{\min}}} \leq c_1 \cdot \sqrt{q\Psi_{\min}}$ and hence

$$\frac{t}{\min\{\sqrt{\Psi_{\min}}, 1\}} \leq \max\left\{c_1 \cdot \sqrt{q}\,, t\right\} \leq c_1 \cdot \sqrt{q} + t.$$

On the other hand, since we are working in the case of $n \geq 4v_{G_0}^2$, we can use explicit bounds for D and σ from Lemma 4.2 in Eichelsbacher and Rednoß (2023) to show that $\frac{D}{\sigma} \leq \hat{c}_{G_0} \cdot \frac{\sqrt{\Psi_{\min}}}{n^2 p^{e_{G_0}} \sqrt{q}}$, where $\hat{c}_{G_0} = \sqrt{2} \cdot \frac{\sqrt{v_{G_0}! \cdot v_{G_0}^2 \cdot e_{G_0}}}{\sqrt{a_{G_0}}}$ is a constant that only depends on G_0. This implies that $e^{\frac{5Dt}{\sigma}} \leq e^{5c_1 \hat{c}_{G_0}}$ for $0 \leq t \leq c_1 \cdot \frac{n^2 p^{e_{G_0}} \sqrt{q}}{\sqrt{\Psi_{\min}}}$.

Additionally, in the case we focus on, there is $e^{c_{G_0} \cdot t^2 \cdot s(t)} \leq e^{c_{G_0} c_2}$.

Application of these bounds to the result of Theorem 1.5 leads to

$$\begin{aligned} \left| \frac{\mathbb{P}(W > t)}{1 - \Phi(t)} - 1 \right| &\leq 50 c_{G_0} \cdot e^{c_{G_0} c_2} \cdot (1 + t^2) \cdot (1 + c_1 \sqrt{q} + t) \cdot \frac{1}{\sqrt{q\Psi_{\min}}} \cdot e^{5c_1 \hat{c}_{G_0}} \\ &\leq 50 c_{G_0} \cdot e^{5c_1 \hat{c}_{G_0} + c_2 c_{G_0}} \cdot (1 + c_1 \sqrt{q}) \cdot (1 + t^2) \cdot (1 + t) \cdot \frac{1}{\sqrt{q\Psi_{\min}}}, \end{aligned}$$

To finish the proof, note that $(1 + t^2)(1 + t) \leq 2(1 + t^3)$. □

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