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Extremal process for irreducible multi-type branching Brownian motion

Haojie Hou, Yan-Xia Ren and Renming Song

School of Mathematical Sciences, Peking University, Beijing, 100871, P.R. China.
E-mail address: houhaojie@pku.edu.cn

LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, P.R. China.
E-mail address: yxren@math.pku.edu.cn
URL: https://www.math.pku.edu.cn/teachers/renyx/indexE.htm

Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, IL 61801, U.S.A.
E-mail address: rsong@illinois.edu
URL: http://rsong.web.illinois.edu

Abstract. We first study the convergence of solutions of a system of F-KPP equations related to irreducible multi-type branching Brownian motions with Heaviside-type initial conditions to traveling wave solutions. Then we apply this convergence result to prove that the extremal process of an irreducible multi-type branching Brownian motion converges weakly to a cluster point process.

1. Introduction and notation

1.1. *Background.* A binary branching Brownian motion (BBM) is a continuous-time Markov process which can be defined as follows. Initially, there is a particle at the origin and the particle moves according to a standard Brownian motion. After an exponential time with parameter 1, this particle dies and splits into 2 particles. The offspring move independently according to standard Brownian motion from the place they are born and obey the same branching mechanism as their parent. We denote the law of this branching Brownian motion by $\mathbb{P}$.

The binary branching Brownian motion is related to the F-KPP equation. Let M_t be the right-most position among all the particles alive at time t. McKean (1975) proved that the function

$$u(t,x) := \mathbb{P}\left(M_t \le x\right), \quad t \ge 0, x \in \mathbb{R},$$

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solves the F-KPP equation

$$u_t = \frac{1}{2}u_{xx} + u^2 - u \tag{1.1}$$

with the Heaviside initial condition $u(0,x) = 1_{[0,\infty)}(x)$. Equation (1.1) was first studied by Fisher (1937) and Kolmogorov et al. (1937). Later, Bramson (1983, Theorems A, B and Example 2) studied the asymptotic behaviors of solutions of (1.1) for a class of more general initial conditions. Let u be a solution of (1.1) and $v = 1 - u$. Bramson proved that, under some conditions on $v(0,x)$,

$$v\left(t, \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + x\right) \to 1 - w(x), \quad \text{uniformly in } x \text{ as } t \to \infty,$$

here w is the unique solution (up to translation) of

$$\frac{1}{2}w'' + \sqrt{2}w' + w^2 - w = 0$$

and w is called a traveling wave solution. In the Heaviside case, a probabilistic representation of the limit $w(x)$ was given by Lalley and Sellke (1987). For different proofs of this result, see Bramson et al. (2016); Roberts (2013).

The extremal point process of branching Brownian motion has also been widely studied. Aïdékon et al. (2013) and Arguin et al. (2013) studied this extremal point process using different methods. Suppose that the set of the positions of all particles alive at time t is given by $\{X_u(t) : u \in Z(t)\}$, where $Z(t)$ is the set of particles alive at time t. It is known that

$$D_t = \sum_{u \in Z_t} \left(\sqrt{2}t - X_u(t)\right) e^{\sqrt{2}X_u(t) - 2t}, \quad t \geq 0,$$

is a martingale and has (non-negative) limit D_∞ as $t \to \infty$. As $t \to \infty$, the extremal point process

$$\sum_{u \in Z(t)} \delta_{X_u(t) - \left(\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t\right)}$$

converges in distribution to a decorated Poisson point process $\mathrm{DPPP}(CD_\infty e^{-\sqrt{2}x}\mathrm{d}x, \mathcal{D}_0)$, in the sense of vague topology, where $\mathcal{D}_0$ is a point process. More precisely, this limit has the following description: Given D_∞, let $\mathcal{P} = \sum_k \delta_{b_k}$ be a Poisson point process with intensity $CD_\infty e^{-\sqrt{2}x}\mathrm{d}x$ and let $\mathcal{D}^{(k)} = \sum_n \delta_{\Delta_n^{(k)}}$ be iid copies of $\mathcal{D}_0$, then

$$\sum_{u \in Z(t)} \delta_{X_u(t) - \left(\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t\right)} \stackrel{\mathrm{d}}{\Longrightarrow} \sum_{k,n} \delta_{b_k + \Delta_n^{(k)}}$$

in the sense of vague topology. For the case of branching random walks, see Aïdékon (2013); Hu and Shi (2009); Madaule (2017). For the case of d-dimensional branching Brownian motions, see Berestycki et al. (2024). For the case of super-Brownian motions, see Ren et al. (2021) and Ren et al. (2024).

In this paper, we consider (irreducible) multit-ype branching Brownian motions. Let $S = \{1, ..., d\}$ be the set of all types and $i \to \{p_{\mathbf{k}}(i) : \mathbf{k} = (k_1, ..., k_d)^T \in \mathbb{N}^d\}$ be the offspring distribution of type i particles, here $\mathbb{N} = \{0, 1, ...\}$. Let $a_i > 0, i \in S$, be the branching rate of type i particles. A multi-type branching Brownian motion can be defined as follows: Initially, there is a particle of type i at site x and it moves according a standard Brownian motion. After an exponential time with parameter a_i, it dies and splits into k_1 offspring of type 1, k_2 offspring of type 2, …, k_d offspring of type d with probability $p_{\mathbf{k}}(i)$, where $\mathbf{k} = (k_1, ..., k_d)^T$. The offspring evolve independently, each moves according to a standard Brownian motion and each type j particle reproduces with law $\{p_{\mathbf{k}}(j) : \mathbf{k} \in \mathbb{N}^d\}$ after an exponential distributed lifetime with parameter a_j. This procedure goes on. We denote the law of this process by $\mathbb{P}_{(x,i)}$. We use $\mathbb{E}_{(x,i)}$ to denote the expectation with respect to $\mathbb{P}_{(x,i)}$. The multi-type branching Brownian motion is related to the following system of F-KPP equations:

$$\mathbf{u}_t = \frac{1}{2}\mathbf{u}_{xx} + \Lambda\left(\psi(\mathbf{u}) - \mathbf{u}\right), \tag{1.2}$$

where

$$\mathbf{u}(t,x) := (u_1(t,x), ..., u_d(t,x))^T, \quad \Lambda := \operatorname{diag}\{a_1, ..., a_d\},$$

$$\psi(\mathbf{u}) := (\psi_1(\mathbf{u}), ..., \psi_d(\mathbf{u}))^T, \quad \psi_i(\mathbf{u}) := \sum_{\mathbf{k}\in\mathbb{N}^d} p_{\mathbf{k}}(i)\prod_{j=1}^d u_j^{k_j}.$$

Let

$$\mu_{i,j} := \sum_{\mathbf{k}\in\mathbb{N}^d} p_{\mathbf{k}}(i)k_j < \infty, \quad i,j\in S,$$

be the mean number of type j offspring given birth by a type i particle. Assume that the mean matrix $\mathfrak{M} = (\mu_{i,j})_{i,j\in S}$ is irreducible, i.e., there exists no permutation matrix S such that $S^{-1}\mathfrak{M}S$ is block triangular. We use $N_i(t)$ to denote the number of type i particles alive at time t. Assume that $\mu_{i,j}(t) = \mathbb{E}_{(0,i)}\left(N_j(t)\right) < \infty$. Then $\mathfrak{M}(t) := (\mu_{i,j}(t))_{i,j\in S}$ satisfies (see the paragraph below Ren and Yang (2014, (2)))

$$\mathfrak{M}(t) = \sum_{n=0}^{\infty}\frac{A^n}{n!}t^n, \quad \text{with } A := (a_{i,j})_{i,j\in S} \text{ and } a_{i,j} = a_i\left(\mu_{i,j} - \delta_{i,j}\right). \tag{1.3}$$

For any $\mathbf{u} = (u_1,\cdots u_d)^T$ and $\mathbf{v} = (v_1,\cdots v_d)^T$, define $\langle\mathbf{u},\mathbf{v}\rangle := \sum_{i=1}^d u_i v_i$. According to the Perron-Frobenius theorem, the matrix A admits a unique simple eigenvalue $\lambda^* > 0$, which is larger than the real part of any other eigenvalue, such that the associated left eigenvector $\mathbf{g} = (g_1, ..., g_d)^T$ and right eigenvector $\mathbf{h} = (h_1, ..., h_d)^T$ can be chosen to have all positive coordinates. We normalize $\mathbf{g}$ and $\mathbf{h}$ so that $\langle\mathbf{g},\mathbf{h}\rangle = \langle\mathbf{g},\mathbf{1}\rangle = 1$, where $\mathbf{1} = (1, ..., 1)^T$. We assume that $p_{\mathbf{0}}(i) = 0$ for all $i\in S$, here $\mathbf{0} := (0, ..., 0)^T$, so the system survives with probability 1. We further assume that the offspring distribution $p_{\mathbf{k}}(i)$ satisfies the following moment condition: there exists $\alpha_0\in(0,1]$ such that

$$\sum_{\mathbf{k}\in\mathbb{N}^d} p_{\mathbf{k}}(i)k_j^{1+\alpha_0} < \infty, \quad \forall\, i,j\in S. \tag{1.4}$$

Define

$$\varphi(\mathbf{v}) := \mathbf{1} - \psi(\mathbf{1} - \mathbf{v}).$$

If $\mathbf{u}$ is a solution of (1.2) and $\mathbf{v} := \mathbf{1} - \mathbf{u}$, then $\mathbf{v}$ satisfies

$$\mathbf{v}_t = \frac{1}{2}\mathbf{v}_{xx} + \Lambda\left(\mathbf{1} - \mathbf{v} - \psi(\mathbf{1} - \mathbf{v})\right) = \frac{1}{2}\mathbf{v}_{xx} + \Lambda\left(\varphi(\mathbf{v}) - \mathbf{v}\right). \tag{1.5}$$

Using the relationship between (1.2) and (1.5), by Ren and Yang (2014, Lemma 5)(or Champneys et al. (1995, (1.36)) for two-type irreducible branching Brownian motion), if $\mathbf{v}$ solves (1.5), then for all $i\in S, t>0, x\in\mathbb{R}$, $v_i(t,x)$ has the following probabilistic representation

$$v_i(t,x) = 1 - \mathbb{E}_{(x,i)}\left(\prod_{u\in Z(t)}\left(1 - v_{I_u(t)}(0, X_u(t))\right)\right). \tag{1.6}$$

Here $Z(t)$ is the set of all the particles alive at time t and for $u\in Z(t)$, $I_u(t)$ is the type of u and $X_u(t)$ is the position of u. For duality relations between general branching Markov process and F-KPP equations, one can refer to Ikeda et al. (1968a,b, 1969).

In addition, we assume that for all $i\in S$,

$$\psi_i(\mathbf{u}) = \rho_i\psi_i^L(\mathbf{u}) + (1-\rho_i)\psi_i^{NL}(\mathbf{u}), \tag{1.7}$$

where $\rho_i \in [0,1)$, $\psi_i^L(\mathbf{u}) = \sum_{k=0}^{\infty} q_k^L(i) u_i^k$ is a probability generating function for local branching, and

$$\psi_i^{NL}(\mathbf{u}) = \sum_{\mathbf{k}\in\mathbb{N}^d:k_i=0} q_{\mathbf{k}}^{NL}(i) \prod_{j\neq i} u_j^{k_j}$$

a probability generating function for non-local branching. This means that for all $i \in S$, on each branching event for a particle with type i, there is at most two cases among all the children— either all the children are type i or all the children are not type i. Also, by (1.7), it holds that

$$p_{\mathbf{k}}(i) = \rho_i q_{k_i}^L(i) 1_{\{k_i>0, k_j=0 \text{ for all } j\neq i\}} + (1-\rho_i) q_{\mathbf{k}}^{NL}(i) 1_{\{k_i=0\}}, \quad \mathbf{k}\in\mathbb{N}^d,\ i\in S.$$

This is a technical assumption that will be used in proving the Feynman-Kac formula (and thus following Bramson's method). We believe that this assumption is not essential since this is the only place where it is used. We mention in passing that the branching Brownian motion studied in Champneys et al. (1995) satisfies condition (1.7).

Define

$$W_{\sqrt{2\lambda^*}}(s) := \sum_{u\in Z(s)} h_{I_u(s)} e^{-\sqrt{2\lambda^*}\left(X_u(s)+\sqrt{2\lambda^*}s\right)}, \quad s\geq 0, \tag{1.8}$$

and

$$M_{\sqrt{2\lambda^*}}(s) := \sum_{u\in Z(s)} h_{I_u(s)} \left(X_u(s)+\sqrt{2\lambda^*}s\right) e^{-\sqrt{2\lambda^*}\left(X_u(s)+\sqrt{2\lambda^*}s\right)}, \quad s\geq 0. \tag{1.9}$$

It is proved in Ren and Yang (2014) that $\{W_{\sqrt{2\lambda^*}}(s), s\geq 0\}$ and $\{M_{\sqrt{2\lambda^*}}(s), s\geq 0\}$ are martingales, called the additive and derivative martingales of multit-ype branching Brownian motion, respectively. Note that the assumption (1.4) implies that $\sum_{\mathbf{k}\in\mathbb{N}^d} p_{\mathbf{k}}(i) k_j (\log_+ k_j)^2 < \infty$. By Ren and Yang (2014, Theorem 3),

$$\lim_{s\to\infty} W_{\sqrt{2\lambda^*}}(s) = 0, \quad \mathbb{P}_{(x,i)}\text{-a.s.} \tag{1.10}$$

According to Ren and Yang (2014, Lemma 10, Theorem 5), there is a nonnegative and non-degenerate random variable $M_{\sqrt{2\lambda^*}}(\infty)$ such that

$$\lim_{s\to\infty} M_{\sqrt{2\lambda^*}}(s) = M_{\sqrt{2\lambda^*}}(\infty), \quad \mathbb{P}_{(x,i)}\text{-a.s.} \tag{1.11}$$

1.2. *Main results.* Our first main result is on the convergence of $\mathbf{v}$ to the traveling wave solution for a class of initial value conditions. Our second main result is about the characterization of the extremal process of multi-type branching Brownian motion.

For the initial value of $\mathbf{v}$, we assume that there exist $N_1 < N_2$ and $i_0 \in S$ such that

$$v_i(0,x) \leq 1_{(-\infty,N_2)}(x), \quad \text{for all } i\in S \quad \text{and} \quad v_{i_0}(0,x) \geq 1_{(-\infty,N_1)}(x). \tag{1.12}$$

Let $m(t) := \sqrt{2\lambda^*}t - \frac{3}{2\sqrt{2\lambda^*}}\log t$ for $t>0$.

Theorem 1.1. *Suppose that* $\mathbf{v}$ *solves* (1.5) *with initial value satisfying* (1.12)*, then it holds that for any* $i\in S$ *and* $x\in\mathbb{R}$*,*

$$\lim_{t\to\infty} (1 - v_i(t, m(t)+x)) = \mathbb{E}_{(0,i)}\left(\exp\left\{-C_v(\infty) M_{\sqrt{2\lambda^*}}(\infty) e^{-\sqrt{2\lambda^*}x}\right\}\right),$$

where $M_{\sqrt{2\lambda^*}}(\infty)$ *is given in* (1.11) *and* $C_v(\infty)$ *is defined by*

$$C_v(\infty) := \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty y e^{\sqrt{2\lambda^*}y} \left(\sum_{j=1}^d g_j v_j(r, y+\sqrt{2\lambda^*}r)\right) \mathrm{d}y \in (0,\infty). \tag{1.13}$$

In the case of single-type branching Brownian motions, a formula similar to (1.13) for the constant $C_v(\infty)$ can be found in Arguin et al. (2013, (3.3)), based on the analysis of the F-KPP equation given in Bramson (1983). Let $v_i(0,x) = 1_{(-\infty,0)}(x)$, then by (1.6) we have

$$v_i(t,x) = 1 - \mathbb{E}_{(x,i)}\left(\prod_{u\in Z(t)}\left(1 - 1_{\{X_t(u)<0\}}\right)\right) = \mathbb{P}_{(x,i)}\left(\min_{u\in Z(t)} X_t(u) < 0\right) = \mathbb{P}_{(0,i)}\left(M_t > x\right),$$

where $M_t := \max_{u\in Z(t)} X_u(t)$ and we used the symmetry of Brownian motion in the last equality. Using this, we get the following corollary of Theorem 1.1:

Corollary 1.2. *For any $i \in S$ and $x \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}_{(0,i)}\left(M_t \le m(t) + x\right) = \mathbb{E}_{(0,i)}\left(\exp\left\{-C_\infty M_{\sqrt{2\lambda^*}}(\infty) e^{-\sqrt{2\lambda^*}x}\right\}\right),$$

where

$$C_\infty = \lim_{r\to\infty}\sqrt{\frac{2}{\pi}}\int_0^\infty y e^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_j \mathbb{P}_{(0,j)}\left(M_r > \sqrt{2\lambda^*}r + y\right)\right)\mathrm{d}y. \tag{1.14}$$

Corollary 1.2 says that $M_t - m(t)$, under $\mathbb{P}_{(0,i)}$, converges to a random shift of the Gumbel distribution (shifted by $-C_\infty M_{\sqrt{2\lambda^*}}(\infty)$).

For $j \in S$, define

$$M_t^j := \max_{u\in Z(t): I_u(t)=j} X_u(t).$$

Fix $i_1 \in S$. Taking $v_{i_1}(0,x) = 1_{(-\infty,0)}(x)$, $v_j(0,x) = 0$ for $j \neq i_1$, then by (1.6) we have

$$\begin{aligned} v_i(t,x) &= 1 - \mathbb{E}_{(x,i)}\left(\prod_{u\in Z(t), I_u(t)=i_1}\left(1 - 1_{\{X_t(u)<0\}}\right)\right) \\ &= \mathbb{P}_{(x,i)}\left(\min_{u\in Z(t), I_u(t)=i_1} X_t(u) < 0\right) = \mathbb{P}_{(0,i)}\left(M_t^{i_1} > x\right). \end{aligned}$$

Then we get the following corollary of Theorem 1.1:

Corollary 1.3. *Fix $i_1 \in S$. For any $i \in S$ and $x \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}_{(0,i)}\left(M_t^{i_1} \le m(t) + x\right) = \mathbb{E}_{(0,i)}\left(\exp\left\{-C_\infty^{(i_1)} M_{\sqrt{2\lambda^*}}(\infty) e^{-\sqrt{2\lambda^*}x}\right\}\right),$$

where

$$C_\infty^{(i_1)} := \lim_{r\to\infty}\sqrt{\frac{2}{\pi}}\int_0^\infty y e^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_j \mathbb{P}_{(0,j)}\left(M_r^{i_1} > \sqrt{2\lambda^*}r + y\right)\right)\mathrm{d}y \in (0,\infty).$$

Corollary 1.3 says that $M_t^{i_1} - m(t)$, under $\mathbb{P}_{(0,i)}$, converges to a random shift of the Gumbel distribution (shifted by $-C_\infty^{(i_1)} M_{\sqrt{2\lambda^*}}(\infty)$).

Theorem 1.4. *Define*

$$\mathcal{D}_t := \sum_{u\in Z(t)} \delta_{(X_u(t)-M_t, I_u(t))}, \quad t \ge 0.$$

Under $\mathbb{P}_{(0,i)}\left(\cdot \middle| M_t > \sqrt{2\lambda^*} + z\right)$, $\left(\mathcal{D}_t, M_t - \sqrt{2\lambda^*} - z\right)$ converges in distribution to some $(\mathcal{D}, Y)$ as $t \to \infty$, where Y is an exponential random variable with parameter $\sqrt{2\lambda^*}$, $\mathcal{D}$ does not depend on $i \in S$ and $z \in \mathbb{R}$, and $\mathcal{D}$ and Y are independent.

Now we consider the point measure of the particles' spatial positions, seen from $m(t)$, and types. Define

$$\mathcal{E}_t := \sum_{u\in Z(t)} \delta_{(X_u(t)-m(t),I_u(t))}.$$

We call $\{\mathcal{E}_t, t\geq 0\}$ the extremal point process of the multit-ype branching Brownian motion.

Let $\mathcal{C}_c^+(\mathbb{R}\times S)$ be the set of all functions $\phi:\mathbb{R}\times S\to\mathbb{R}_+$ such that for any $j\in S$, $\phi(\cdot,j)$ is a non-negative continuous function of compact support. The following result gives a description of the limit $\mathcal{E}_\infty$ of $\mathcal{E}_t$ as $t\to\infty$.

Theorem 1.5. *Given $M_{\sqrt{2\lambda^*}}(\infty)$, let $\mathcal{P}\equiv\sum_{k\in\mathbb{N}}\delta_{b_k}$ be a Poisson point process with intensity $C_\infty M_{\sqrt{2\lambda^*}}(\infty)\sqrt{2\lambda^*}e^{-\sqrt{2\lambda^*}x}\mathrm{d}x$, and let $\{\mathcal{D}^{(k)}:k\in\mathbb{N}\}$ be iid copies of $\mathcal{D}$ defined in Theorem 1.4. If $\mathcal{D}^{(k)}:=\sum_{n\in\mathbb{N}}\delta_{\left(\Delta_n^{(k)},\varsigma_n^{(k)}\right)}$, then for any $i\in S$, under $\mathbb{P}_{(0,i)}$, $\mathcal{E}_t$ converges in distribution to*

$$\mathcal{E}_\infty \overset{\mathrm{d}}{=} \sum_{k,n}\delta_{(b_k+\Delta_n^{(k)},\varsigma_n^{(k)})} \quad \textit{as } t\to\infty$$

For $i_1\in S$, define

$$\mathcal{E}_t(i_1) := \sum_{u\in Z(t):I_u(t)=i_1} \delta_{X_u(t)-m(t)}.$$

As a consequence of Theorem 1.5, we have the following corollary:

Corollary 1.6. *$\mathcal{E}_t(i_1)$ converges in distribution to*

$$\mathcal{E}_\infty(i_1) \overset{\mathrm{d}}{=} \sum_k \sum_{n:\varsigma_n^{(k)}=i_1} \delta_{b_k+\Delta_n^{(k)}} \quad \textit{as } t\to\infty,$$

where $b_k,\Delta_n^{(k)}$ and $\varsigma_n^{(k)}$ are defined in Theorem 1.5.

Remark 1.7. Champneys et al. (1995) studied a two-type irreducible branching Brownian motion where the diffusion coefficients can be different. They considered the F-KPP equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} &= \frac{\sigma_1^2}{2}\frac{\partial^2 u_1}{\partial x^2} + (r_1+\theta q_1)\left(\frac{r_1}{r_1+\theta q_1}u_1^2 + \frac{\theta q_1}{r_1+\theta q_1}u_2 - u_1\right),\\ \dfrac{\partial u_2}{\partial t} &= \frac{\sigma_2^2}{2}\frac{\partial^2 u_2}{\partial x^2} + (r_2+\theta q_2)\left(\frac{r_2}{r_2+\theta q_2}u_2^2 + \frac{\theta q_2}{r_2+\theta q_2}u_1 - u_2\right),\end{cases}$$

In this case, they proved that (see Champneys et al. (1995, (1.41))) when the speed c is larger than the minimal speed $c(\theta)$(which is equal to $\sqrt{2\lambda^*}$ in our model), under suitable initial condition different from (1.12), $1-v_i(t,ct+x)\sim\mathbb{E}_{(x,i)}(e^{-W_\lambda(\infty)})$ as $t\to\infty$, where $\lambda=\lambda(c)$ is a deterministic value. They also studied the convergence of the associated martingales (Champneys et al. (1995, (1.39))), the existence and the uniqueness of the traveling wave solution (Champneys et al. (1995, (1.30))) and the speed of the minimal position of branching Brownian motion(Champneys et al. (1995, (1.44))).

These results are generalized in Ren and Yang (2014) although the diffusion coefficients of all types are assumed to be 1. In this paper and for our model, Theorem 1.1 studies the convergence to traveling wave solution where the speed c is equal to the minimal speed. We also study the extremal process of multi-type branching Brownian motion, see Theorem 1.5.

Remark 1.8. The asymptotic behavior above for irreducible multi-type branching Brownian motions is similar to the one obtained in Aïdékon et al. (2013); Arguin et al. (2013) for single-type branching Brownian motions. Belloum and Mallein (2021), Belloum (2022) and Ma and Ren (2023a,b) considered a 2-type reducible branching Brownian motion and their results are quite different. In their model, particles of type 1 move as a Brownian motion with diffusion coefficient σ^2, reproduce with branching rate $\beta+\alpha$ and offspring distribution $\{p_{\mathbf{k}}(1)\}$ satisfying $p_{(2,0)}(1)=\frac{\beta}{\beta+\alpha}, p_{(1,1)}(1)=\frac{\alpha}{\alpha+\beta}$.

Particles of type 2 evolve as a standard branching Brownian motion with branching rate 1 and binary branching. In the special case when $\sigma^2 = 1$, $\beta = 1$ and $\lambda^* = 1$, their results show that the corresponding front $m(t)$ is $\sqrt{2}t - \frac{1}{2\sqrt{2}} \log t$, which is quite different from the irreducible case where the corresponding front for $\lambda^* = 1$ is $\sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$.

1.3. *A comparison to standard F-KPP equation.* The goal of this paper is to study the asymptotic behavior of extremal processes of irreducible multi-type branching Brownian motions following the general strategy of Bramson (1983). Our results are similar to those for single-type branching Brownian motions. However, to adapt Bramson's idea to the multi-type case, there are quite a few difficulties. For standard F-KPP equation (1.1), let $v = 1 - u$. The idea in Bramson (1983) can be roughly divided to two parts:

- First, by the Feynman-Kac formula, for $0 < r < t$,

$$\begin{aligned} v(t,x) &= \mathbf{E}_x \left(\exp\left\{ \int_0^{t-r} (1 - v(t-s, B_s))\, \mathrm{d}s \right\} v(r, B_{t-r}) \right) \\ &\approx \mathbf{E}_x \left(\exp\left\{ \int_0^{t-r} (1 - v(t-s, B_s))\, \mathrm{d}s \right\} v(r, B_{t-r}) 1_{\{B_{t-r} > -\log r\}} \right) \\ &\approx \mathbf{E}_x \left(\exp\left\{ \int_0^{t-r} (1 - v^*(t-r-s, B_s))\, \mathrm{d}s \right\} v^*(0, B_{t-r}) \right), \end{aligned} \tag{1.15}$$

  where $v^*(0,x) = v(r,x)1_{\{x > -\log r\}}$ and $1 - v^*(s,x)$ solves (1.1) with initial value $1 - v^*(0,x)$. Note that the Feynman-Kac formula still holds in the non-local case, see Section 3. A difficulty appears in the last comparability due to non-local branching. In fact, the last comparability in (1.15) used the fact that $f(v)/v = 1 - v$ is decreasing, which does not always hold when f is a multi-variable concave function. This part is discussed in Lemma 5.2.
- Second, define a suitable event $\mathcal{B}_{mid}$ as in (5.6) below. This event consists all trajectories $B.$ such that $B_s > n_{r,t}(t-s)$ for all $s \in [0, t-r]$, where $n_{t,r}(\cdot)$ is defined as in (5.7). Also, for each $y > -\log r$, under $\mathbf{P}_x(\cdot | B_{t-r} = y)$, on the event $\mathcal{B}_{mid}$, $v^*(t-r-s, B_s)$ is very close to 0, while on $(\mathcal{B}_{mid})^c$, $v^*(t-r-s, B_s)$ is far away to 0. Then by properties of Brownian bridge and Kolmogorov et al's convergence theorem (see Kolmogorov et al. (1937)) for $v(t,x)$ with $v(0,x) = 1_{(-\infty,0)}(x)$, it holds that

$$\begin{aligned} v(t,x) &\approx e^{t-r} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/(2(t-r))}}{\sqrt{2\pi(t-r)}} v^*(0,y) \mathbf{E}_x \left(\mathcal{B}_{mid} \middle| B_{t-r} = y \right) \mathrm{d}y \\ &\approx e^{t-r} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/(2(t-r))}}{\sqrt{2\pi(t-r)}} v(r,y) \mathbf{E}_x \left(\mathcal{B}_{mid} \middle| B_{t-r} = y \right) \mathrm{d}y. \end{aligned} \tag{1.16}$$

  Related to the above, there are two main challenges in the multi-type case. First, for multi-type branching Brownian motions, counterpart to Kolmogorov et al's convergence theorem is not yet available. So we need to give sharp estimates for $v_i(t,x)$ with Heaviside initial condition, and this is done in Section 4 using the moment method. Another difficulty is that for the nonlinear term $f(v) = v - v^2$ in the standard F-KPP equation, the fact that $f'(0+) > 0$ is crucial to obtain the first equality in (1.16), while for multi-variable function $f(\mathbf{v})$, it does not necessarily hold that $\inf_{1 \le j \le d} \partial f(\mathbf{0})/\partial v_j > 0$. We deal with this difficulty in Lemma 7.1 and Lemma 7.2.

We should mention that the assumption that the diffusion coefficients for all types are the same is very important for this paper. This ensures that, in the spine motion (X_t, I_t), X_t is independent of I_t and has the same law as a Brownian motion. For the case where the diffusion coefficients are different, it seems quite difficult to get the sharp order of moment and even harder to adapt the ideas of Bramson (1983).

The organization of the paper is as follows. In Section 2, we gives some basic facts on the spine decomposition and a version of the many-to-one formula. In Section 3, we present the non-local Feynman-Kac formula for solutions of (1.5). In Section 4, we first get some upper and lower bounds for $\mathbb{P}_{(0,i)}\left(M_t^{i_1}\geq m(t)+y\right)$ and $\mathbb{P}_{(0,i)}\left(M_t\geq m(t)+y\right)$, and then use these bounds to obtain the tightness for $\left(M_t-m(t),\ t\geq 1,\mathbb{P}_{(0,i)}\right)$ and $\left(M_t^{i'}-m(t),\ t\geq 1,\mathbb{P}_{(0,i)}\right)$ for any $i,i'\in S$. The proof of Theorem 1.1 is given in Section 5. Since some of the technical results used in the proof are similar to the corresponding results in Bramson (1983), we postpone their proofs to the end of the paper. In Section 6, we prove Theorems 1.4 and 1.5 by adapting the ideas of Arguin et al. (2013). In Section 7, we prove technical results whose proofs are postponed to make the paper more readable.

In the remainder of this paper, for a set E, the notation

$$f(x)\lesssim g(x),\quad x\in E$$

means that there exists some constant C independent of $x\in E$ such that $f(x)\leq Cg(x)$ holds for all $x\in E$. Also, the notation $f\asymp g, x\in E$, means $f\lesssim g, x\in E$ and $g\lesssim f, x\in E$.

2. Many-to-one formula and spine decomposition

Let $\mathbf{N}_t:=(N_1(t),...,N_d(t))^T$ and let $\{\mathcal{F}_t\}$ be the natural filtration of the multi-type branching Brownian motion. By Athreya (1968, Proposition 2), under $\mathbb{P}_{(x,i)}$, $e^{-\lambda^* t}\langle \mathbf{N}_t,\mathbf{h}\rangle$ is a mean h_i positive martingale with respect to $\{\mathcal{F}_t\}$. Define $\widehat{\mathbb{P}}_{(x,i)}$ by

$$\left.\frac{\mathrm{d}\widehat{\mathbb{P}}_{(x,i)}}{\mathrm{d}\mathbb{P}_{(x,i)}}\right|_{\mathcal{F}_t}:=\frac{e^{-\lambda^* t}\langle \mathbf{N}_t,\mathbf{h}\rangle}{h_i}. \tag{2.1}$$

According to Ren and Yang (2014, p. 224), the multi-type branching Brownian motion under $\widehat{\mathbb{P}}_{(x,i)}$ has the following *spine decomposition*:

(i) Initially there is a marked particle ξ, called the spine, of type i at site x.

(ii) After an exponential time ζ_ξ with parameter $a_i+\lambda^*$, this marked particle dies and produces $A_1:=A_1(i)$ offspring of type 1, …, $A_d:=A_d(i)$ offspring of type d with probability $\widehat{p}_{\mathbf{A}}(i):=\frac{p_{\mathbf{A}}(i)\langle\mathbf{A},\mathbf{h}\rangle}{(1+a_i^{-1}\lambda^*)h_i}$, where $\mathbf{A}=(A_1,\dots A_d)^T$. Randomly choose one of these $\langle\mathbf{A},\mathbf{1}\rangle$ particles to continue as the spine, with each type j particle being chosen with probability $h_j/\langle\mathbf{A},\mathbf{h}\rangle$.

(iii) The $\langle\mathbf{A},\mathbf{1}\rangle$ offspring particles evolve independently, with the marked (spine) particle repeating step (ii) with law $\widehat{\mathbb{P}}_{(X_\xi(\zeta_\xi),I_\xi(\zeta_\xi))}$ and each unmarked particle of type j, $j\in S$, evolving as a multi-type branching Brownian motion with law $\mathbb{P}_{(X_\xi(\zeta_\xi),j)}$. The process then goes on.

If we only consider the spine process $(X_\xi(t),I_\xi(t))$, then, under $\widehat{\mathbb{P}}_{(x,i)}$, X_ξ is a standard Brownian motion starting from x, I_ξ is an S-valued Markov chain with generator

$$\widehat{G}=(\widehat{g}_{i,j})_{S\times S}\quad\text{with}\quad \widehat{g}_{i,j}:=(a_i+\lambda^*)\left(\frac{\mu_{i,j}h_j}{(1+\lambda^*/a_i)\,h_i}-\delta_{i,j}\right)=\frac{a_i\mu_{i,j}h_j}{h_i}-(a_i+\lambda^*)\delta_{i,j}$$

and X_ξ is independent of $I_\xi(t)$. According to Ren and Yang (2014, (12)), we have

$$\widehat{\mathbb{P}}_{(x,i)}\left(\xi_t=u\middle|\mathcal{F}_t\right)=\frac{h_{I_u(t)}}{\langle\mathbf{N}_t,\mathbf{h}\rangle}1_{\{u\in Z(t)\}}. \tag{2.2}$$

Using (2.2), we give a stronger version of the many-to-one formula in Ren and Yang (2014, Proposition 1). For the case of branching Brownian motions, one can refer to Maillard and Pain (2019, Proposition 4.1) (In Maillard and Pain (2019), there is also a change-of-measure for the spinal movement).

Proposition 2.1. *For any $t>0$ and $u\in Z(t)$, let $H(u,t)$ be a non-negative $\mathcal{F}_t$-measurable random variable. Then*

$$\mathbb{E}_{(x,i)}\left(\sum_{u\in Z(t)} H(u,t)\right)=e^{\lambda^* t}\widehat{\mathbb{E}}_{(x,i)}\left(H(\xi_t,t)\frac{h_i}{h_{I_\xi(t)}}\right).$$

Proof: By (2.1) and (2.2),

$$\mathbb{E}_{(x,i)}\left(\sum_{u\in Z(t)} H(u,t)\right)=e^{\lambda^* t}\widehat{\mathbb{E}}_{(x,i)}\left(\frac{h_i}{\langle \mathbf{N}_t,\mathbf{h}\rangle}\sum_{u\in Z(t)} H(u,t)\right)$$

$$=e^{\lambda^* t}\widehat{\mathbb{E}}_{(x,i)}\left(\sum_{u\in Z(t)}\frac{h_{I_u(t)}}{\langle \mathbf{N}_t,\mathbf{h}\rangle}H(u,t)\frac{h_i}{h_{I_u(t)}}\right)=e^{\lambda^* t}\widehat{\mathbb{E}}_{(x,i)}\left(\sum_{u\in Z(t)}\widehat{\mathbb{P}}_{(x,i)}\left(\xi_t=u\middle|\mathcal{F}_t\right)H(u,t)\frac{h_i}{h_{I_u(t)}}\right)$$

$$=e^{\lambda^* t}\widehat{\mathbb{E}}_{(x,i)}\left(H(\xi_t,t)\frac{h_i}{h_{I_\xi(t)}}\sum_{u\in Z(t)}1_{\{\xi_t=u\}}\right)=e^{\lambda^* t}\widehat{\mathbb{E}}_{(x,i)}\left(H(\xi_t,t)\frac{h_i}{h_{I_\xi(t)}}\right).$$

□

It is easy to deduce from $A\mathbf{h}=\lambda^*\mathbf{h}$ and $\mathbf{g}^T A=\lambda^*\mathbf{g}^T$ that

$$\sum_{j=1}^d \mu_{i,j}h_j=\frac{a_i+\lambda^*}{a_i}h_i,\quad \sum_{j=1}^d \mu_{j,i}a_j g_j=(a_i+\lambda^*)g_i,\quad i=1,2,...,d.$$

Let $\nu_j:=h_j g_j$ and $\nu=(\nu_1,...,\nu_d)^T$, then we see that ν_j solves the equation

$$\nu_j(a_j+\lambda^*)=\sum_{i=1}^d \nu_i\frac{a_i}{h_i}\mu_{i,j}h_j,\quad j=1,...,d,$$

and $\langle \mathbf{1},\nu\rangle=1$, which implies that ν is an invariant measure for $I_\xi(t)$ under $\widehat{\mathbb{P}}_{(x,i)}$.

3. Non-local Feynman-Kac formula

Throughout this paper, $(X_t,t\ge 0;\mathbf{P}_x)$ is a standard Brownian motion starting from x. The Feynman-Kac formula plays an important role in the probabilistic treatment of the F-KPP equation (1.1). The classical Feynman-Kac formula says that a solution of the linear equation

$$u_t=\frac{1}{2}u_{xx}+k(t,x)u$$

can be given by

$$u(t,x)=\mathbf{E}_x\left(e^{\int_0^t k(t-s,X_s)\mathrm{d}s}u(0,X_t)\right). \tag{3.1}$$

If u is a solution to equation (1.1), the (3.1) holds with $k(s,y)=\frac{(u^2-u)}{u}(s,y)=u(s,y)-1$. For our multi-type branching Brownian motion, we will give a similar representation for a solution $\mathbf{v}$ of (1.5) using a non-local Feynman-Kac formula.

By (1.7), we have

$$\varphi_i(\mathbf{v})=\rho_i\left(1-\psi_i^L(\mathbf{1}-\mathbf{v})\right)+(1-\rho_i)\left(1-\psi_i^{NL}(\mathbf{1}-\mathbf{v})\right)=:\rho_i\varphi_i^L(\mathbf{v})+(1-\rho_i)\varphi_i^{NL}(\mathbf{v}). \tag{3.2}$$

First note that (1.5) is equivalent to

$$v_i(t,x)=\mathbf{E}_x\left(v_i(0,X_t)\right)+\mathbf{E}_x\left(\int_0^t a_i\left(\varphi_i\left(\mathbf{v}(t-s,X_s)\right)-v_i(t-s,X_s)\right)\mathrm{d}s\right),\quad i\in S.$$

Let $n_i := \sum_{j=1}^d \mu_{i,j}$ and $p_{i,j} := \mu_{i,j}/(n_i - \mu_{i,i})1_{\{i\neq j\}}$. Since $p_{\mathbf{0}}(i) = 0$ and $\mathfrak{M}$ is irreducible by assumption, we have $n_i \geq 1$ and $n_i > \mu_{i,i}$ for all $i \in S$. Rewrite A given in (1.3) as

$$A = \mathrm{diag}\left\{a_1(n_1-1), a_2(n_2-1), \dots, a_d(n_d-1)\right\} + A^*, \tag{3.3}$$

where $A^* := (a^*_{i,j})$ and $a^*_{i,j} = a_i(n_i - \mu_{i,i})(p_{i,j} - \delta_{i,j})$. Define

$$\mathcal{A} := \mathrm{diag}\left\{\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}, ..., \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right\} + A^*. \tag{3.4}$$

Let I_t be a continuous time Markov chain, independent of X, with generator A^*. We use $\mathbf{P}_{(x,i)}$ to denote the law of (X_t, I_t) and use $\mathbf{E}_{(x,i)}$ to denote the corresponding expectation. Then (1.5) is equivalent to

$$\mathbf{v}_t = \mathcal{A}\mathbf{v} + \Lambda\left(\varphi(\mathbf{v}) - \mathbf{v}\right) - A^*\mathbf{v},$$

which in turn, by (3.2), is equivalent to

$$\begin{aligned} v_i(t,x) &= \mathbf{E}_{(x,i)}\left(v_{I_t}\left(0, X_t\right)\right) + \mathbf{E}_{(x,i)}\left(\int_0^t a_{I_s}\left(\varphi_{I_s}\left(\mathbf{v}(t-s,X_s)\right) - v_{I_s}(t-s,X_s)\right)\mathrm{d}s\right) \\ &\qquad - \mathbf{E}_{(x,i)}\left(\int_0^t \sum_{j=1}^d a^*_{I_s,j} v_j(t-s,X_s)\mathrm{d}s\right) \\ &= \mathbf{E}_{(x,i)}\left(v_{I_t}\left(0,X_t\right)\right) + \mathbf{E}_{(x,i)}\left(\int_0^t a_{I_s}\left(n_{I_s} - \mu_{I_s,I_s} + \frac{\rho_{I_s}\varphi^L_{I_s}(\mathbf{v}(t-s,X_s))}{v_{I_s}(t-s,X_s)} - 1\right) v_{I_s}(t-s,X_s)\mathrm{d}s\right) \\ &\quad + \mathbf{E}_{(x,i)}\left(\int_0^t a_{I_s}\left((1-\rho_{I_s})\varphi^{NL}_{I_s}\left(\mathbf{v}(t-s,X_s)\right) - (n_{I_s} - \mu_{I_s,I_s})\sum_{j=1}^d p_{I_s,j} v_j(t-s,X_s)\right)\mathrm{d}s\right). \end{aligned} \tag{3.5}$$

We will simplify the formula above using the non-local Feynman-Kac formula introduced below. Define a Feynman-Kac semigroup Q_t by

$$Q_t f(x,i) := \mathbf{E}_{(x,i)}\left(f(X_t, I_t)\exp\left\{\int_0^t a_{I_s}\left(n_{I_s} - 1\right)\mathrm{d}s\right\}\right),$$

then (see, for instance, Harris et al. (2022, Lemma 2.1)), Q_t is the mean semigroup of a non-local branching Markov process with spatial motion (X, I), branching rate function $\beta(x,i) = a_i$ and non-local probability distribution $F\left((x,i),\cdot\right)$, on the space $\mathcal{M}(\mathbb{R}\times S)$ of finite measures on $\mathbb{R}\times S$, defined for all $(x,i) \in \mathbb{R}\times S$ by

$$F\left((x,i), \{\delta_x \times (k_1\delta_1 + ... + k_d\delta_d)\}\right) = p_{\mathbf{k}}(i), \quad \mathbf{k} \in \mathbb{N}^d.$$

Put

$$\mathbf{z}(t,x) = (z_1(t,x), \dots, z_d(t,x))^T \quad \text{with } z_i(t,x) := Q_t f(x,i).$$

Then $\mathbf{z}$ solves the linear equation

$$\mathbf{z}_t = \mathcal{A}\mathbf{z} + \mathrm{diag}\left\{a_1(n_1-1), a_2(n_2-1), \dots, a_d(n_d-1)\right\}\mathbf{z}, \quad x \in \mathbb{R}, i \in S.$$

By (3.4) and (3.3), we see that $\mathbf{z}$ solves the equation

$$\mathbf{z}_t = \mathrm{diag}\left\{\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}, ..., \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right\}\mathbf{z} + A\mathbf{z}. \tag{3.6}$$

Let $H(s) = s$ and

$$J((x,k), \mathrm{d}(y,\ell)) := \delta(x-y)a_k(n_k - \mu_{k,k})\left(p_{k,\ell} - \delta_{k,\ell}\right)1_{\{k\neq\ell\}}\mathrm{d}y\mathrm{d}\ell.$$

Define

$$D_J := \left\{t \middle| \left(X_{t-}, I_{t-}\right) \neq (X_t, I_t)\right\} = \left\{t \middle| I_{t-} \neq I_t\right\}.$$

It is easy to check that for any non-negative Borel function f on $(\mathbb{R}\times S)^2$ vanishing on the diagonal and any $x\in\mathbb{R}, i\in S$,

$$\mathbf{E}_{(x,i)}\left(\sum_{s\in D_J, s\le t} f\left((X_{s-}, I_{s-}),(X_s,I_s)\right)\right) = \mathbf{E}_{(x,i)}\left(\int_0^t\int_{\mathbb{R}\times S} f\left((X_s,I_s),(y,\ell)\right) J\left((X_s,I_s),\mathrm{d}(y,\ell)\right)\mathrm{d}s\right),$$

and thus (J,H) is a Lévy system for (X,I). Note that the integrand in the second term of (3.5) is equal to

$$a_{I_s}\left((1-\rho_{I_s})\varphi^{NL}_{I_s}\left(\mathbf{v}(t-s,X_s)\right) - (n_{I_s}-\mu_{I_s,I_s})\sum_{j=1}^d p_{I_s,j}v_j(t-s,X_s)\right)$$
$$= \left(\frac{(1-\rho_{I_s})\varphi^{NL}_{I_s}\left(\mathbf{v}(t-s,X_s)\right)}{\sum_{j\ne I_s}\mu_{I_s,j}v_j(t-s,X_s)} - 1\right) a_{I_s}(n_{I_s}-\mu_{I_s,I_s})\sum_{j=1}^d p_{I_s,j}v_j(t-s,X_s).$$

Applying Chen et al. (2019, Lemma A.1), similar to Chen et al. (2019, (4.8)), (3.5) can be written as

$$v_i(t,x) = \mathbf{E}_{(x,i)}\left(\exp\left\{\sum_{s\in D_J,s\le t}\log\left(\frac{(1-\rho_{I_s})\varphi^{NL}_{I_s}\left(\mathbf{v}(t-s,X_s)\right)}{\sum_{j\ne I_s}\mu_{I_s,j}v_j(t-s,X_s)}\right) + \int_0^t a_{I_s}\left(n_{I_s}-\mu_{I_s,I_s}+\frac{\rho_{I_s}\varphi^L_{I_s}\left(\mathbf{v}(t-s,X_s)\right)}{v_{I_s}(t-s,X_s)}-1\right)\mathrm{d}s\right\} v_{I_t}(0,X_t)\right). \tag{3.7}$$

For any bounded non-negative function f and $\theta>0$, by (1.6),

$$u_i(t,x;\theta) := \mathbb{E}_{(x,i)}\left(\prod_{u\in Z(t)} e^{-\theta f(X_u(t),I_u(t))}\right)$$

solves equation (1.2). By taking derivative with respect to θ and letting $\theta\downarrow 0$, it is easy to see that

$$z_i(t,x) = \mathbb{E}_{(x,i)}\left(\sum_{u\in Z(t)} f(X_u(t),I_u(t))\right)$$

also solves equation (3.6). Therefore, Q_t is also the mean-semigroup of the multi-type branching Brownian motion, i.e., for every bounded measurable function f,

$$Q_t f(x,i) = \mathbb{E}_{(x,i)}\left(\sum_{u\in Z(t)} f(X_u(t),I_u(t))\right).$$

(For two-type irreducible branching Brownian motion, see Champneys et al. (1995, (4.6)).) It follows from Proposition 2.1 that for $\phi(x,i)=h_i$,

$$Q_t\phi(x,i) = e^{\lambda^* t}\phi(x,i). \tag{3.8}$$

Using the definition of Q_t and (3.8), we can easily see that

$$e^{-\lambda^* t}\exp\left\{\int_0^t a_{I_s}\left(n_{I_s}-1\right)\mathrm{d}s\right\}\frac{h_{I_t}}{h_{I_0}}$$

is a non-negative martingale of mean 1 under $\mathbf{P}_{(x,i)}$. Now we define

$$\left.\frac{\mathrm{d}\mathbf{P}^h_{(x,i)}}{\mathrm{d}\mathbf{P}_{(x,i)}}\right|_{\sigma(X_s,I_s,s\le t)} := e^{-\lambda^* t}\exp\left\{\int_0^t a_{I_s}\left(n_{I_s}-1\right)\mathrm{d}s\right\}\frac{h_{I_t}}{h_{I_0}}.$$

Then by the definition of Q_t, we have

$$Q_t f(x,i) = e^{\lambda^* t}\mathbf{E}^h_{(x,i)}\left(f(X_t,I_t)\frac{h_i}{h_{I_t}}\right).$$

Combining this with Proposition 2.1, we get that

$$\left((X,I),\mathbf{P}^h_{(x,i)}\right) \overset{\mathrm{d}}{=} \left((X_\xi,I_\xi),\widehat{\mathbb{P}}_{(x,i)}\right).$$

Now (3.7) can be rewritten as

$$\begin{aligned} v_i(t,x) =& e^{\lambda^* t}h_i\mathbf{E}^h_{(x,i)}\Bigg(\exp\Bigg\{\sum_{s\in D_J, s\le t}\log\left(\frac{(1-\rho_{I_s})\varphi^{NL}_{I_s}\left(\mathbf{v}(t-s,X_s)\right)}{\sum_{j\neq I_s}\mu_{I_s,j}v_j(t-s,X_s)}\right) \\ &-\int_0^t a_{I_s}\left(\mu_{I_s,I_s}-\frac{\rho_{I_s}\varphi^L_{I_s}(\mathbf{v}(t-s,X_s))}{v_{I_s}(t-s,X_s)}\right)\mathrm{d}s\Bigg\}\frac{v_{I_t}(0,X_t)}{h_{I_t}}\Bigg). \end{aligned} \tag{3.9}$$

Define for $0\le r<t\le t_1$,

$$\begin{aligned} R_{t_1}\left((r,t];v\right) :=& \exp\Bigg\{\sum_{s\in D_J, r<s\le t}\log\left(\frac{(1-\rho_{I_s})\varphi^{NL}_{I_s}\left(\mathbf{v}(t_1-s,X_s)\right)}{\sum_{j\neq I_s}\mu_{I_s,j}v_j(t_1-s,X_s)}\right) \\ &-\int_0^t a_{I_s}\left(\mu_{I_s,I_s}-\frac{\rho_{I_s}\varphi^L_{I_s}(\mathbf{v}(t_1-s,X_s))}{v_{I_s}(t_1-s,X_s)}\right)\mathrm{d}s\Bigg\} \end{aligned} \tag{3.10}$$

and

$$R_{t_1}(t;v) := R_{t_1}((0,t];v),\quad R\left((r,t];v\right) := R_t\left((r,t];v\right),\quad R\left(t;v\right) := R\left((0,t];v\right). \tag{3.11}$$

Now for $0<r<t$, by the Markov property, we get from (3.9) that

$$\begin{aligned} v_i(t,x) &= e^{\lambda^*(t-r)}h_i\mathbf{E}^h_{(x,i)}\left(R_t(t-r;v)\frac{v_{I_{t-r}}(r,X_{t-r})}{h_{I_{t-r}}}\right) \\ &= e^{\lambda^*(t-r)}h_i\int_{\mathbb{R}}\frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}}\mathbf{E}^h_{(x,i)}\left(R_t(t-r;v)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\middle| X_{t-r}=y\right)\mathrm{d}y. \end{aligned} \tag{3.12}$$

The above representation (3.12) of v_i will play an important role in this paper.

For any $i\in S$, by Bernoulli's inequality,

$$\varphi_i(\mathbf{v}) = 1-\psi_i(\mathbf{1}-\mathbf{v}) = 1-\sum_{\mathbf{k}\in\mathbb{N}^d}p_{\mathbf{k}}(i)\prod_{j=1}^d(1-v_j)^{k_j} \le 1-\sum_{\mathbf{k}\in\mathbb{N}^d}p_{\mathbf{k}}(i)(1-\sum_{j=1}^d k_jv_j) = \sum_{j=1}^d\mu_{i,j}v_j.$$

Thus, $\mathbf{P}^h_{(x,i)}$-a.s., for any $0\le r<t$ and $t_1\ge t$, $R_{t_1}\left((r,t];v\right)\le 0$.

The assumption (1.4) implies the following estimate on $\rho_i\varphi^L_i(\mathbf{v})$ and $(1-\rho_i)\varphi^{NL}_i(\mathbf{v})$:

Lemma 3.1. *It holds uniformly for all $i\in S$ and $\mathbf{v}\in[0,1]^d$ that*

$$\frac{\rho_i\varphi^L_i(\mathbf{v})}{v_i} = \mu_{i,i}-O\left(\|\mathbf{v}\|^{\alpha_0}\right)\quad\textit{and}\quad\frac{(1-\rho_i)\varphi^{NL}_i(\mathbf{v})}{\sum_{j\neq i}\mu_{i,j}v_j} = 1-O\left(\|\mathbf{v}\|^{\alpha_0}\right). \tag{3.13}$$

The proof is postponed to Subsection 7.1.

In the remainder of this paper, when we consider the spine process (X_ξ, I_ξ) under $\widehat{\mathbb{P}}_{(x,i)}$, we sometimes use $\left((X,I), \mathbf{P}^h_{(x,i)}\right)$ to denote the law of the spine process for simplicity.

4. Estimates in the case of Heaviside initial conditions

In this section, we consider two kinds of initial conditions. The first kind is

$$v_i(0,x) = 1_{(-\infty,0)}(x), \quad \text{for all } i \in S. \tag{4.1}$$

Fix $i' \in S$. The second kind of initial condition is

$$v_{i'}(0,x) = 1_{(-\infty,0)}(x), \quad v_j(0,x) = 0, \quad \text{for } i \neq i'. \tag{4.2}$$

Note that if $\mathbf{v}$ solves (1.5) with initial condition (4.1), then

$$v_i(t,x) = \mathbb{P}_{(x,i)}\left(\min_{u\in Z(t)} X_t(u) < 0\right) = \mathbb{P}_{(0,i)}\left(M_t > x\right);$$

and that if $\mathbf{v}$ solves (1.5) with initial condition (4.2), then

$$v_i(t,x) = \mathbb{P}_{(x,i)}\left(\min_{u\in Z(t), I_u(t)=i'} X_t(u) < 0\right) = \mathbb{P}_{(0,i)}\left(M_t^{i'} > x\right).$$

Recall that $m(t) = \sqrt{2\lambda^*}t - \frac{3}{2\sqrt{2\lambda^*}}\log t$. The first goal of this section is to get estimates on solutions $\mathbf{v}(t,x)$ of (1.5) with Heaviside initial conditions (4.1) or (4.2), and with $x = m(t)+y$, that is to say, we want to get some upper and lower bounds for $\mathbb{P}_{(0,i)}\left(M_t \geq m(t)+y\right)$ and $\mathbb{P}_{(0,i)}\left(M_t^{i'} \geq m(t)+y\right)$ with $y > 0$. See Proposition 4.2 below for the upper bound and Proposition 4.3 for the lower bound. Next, we use Propositions 4.2 and 4.3 to prove that for any $i \in S$, $\left(M_t - m(t),\ t\geq 1; \mathbb{P}_{(0,i)}\right)$ is tight, and that, for any $i, i' \in S$, $\left(M_t^{i'} - m(t),\ t \geq 1; \mathbb{P}_{(0,i)}\right)$ is tight. Counterparts of some of the estimates used to prove tightness can be found in Bramson (1978); Mallein (2015b).

In Bramson's argument, sharp estimates for $\mathbb{P}_{(0,i)}\left(M_t \geq m(t)+y\right)$ and $\mathbb{P}_{(0,i)}\left(M_t^{i'} \geq m(t)+y\right)$ played essential roles. To get these estimates, Bramson used a convergence result of Kolmogorov et al. (1937). However, counterpart to the convergence result of Kolmogorov et al. (1937) is not yet available in the multi-type case. We need to overcome this difficulty.

We first give an estimate on the path of Brownian motion, whose proof is postponed to Subsection 7.2.

Lemma 4.1. *Let $K > 0$, $\alpha < 1/2$ and $t \geq 1$. For any function f satisfying*

$$\sup_{s\leq t}\left(\frac{|f(s)|}{s^\alpha} + \frac{|f(t)-f(s)|}{(t-s)^\alpha}\right) < K,$$

there exists a constant Γ_1 depending only on K and α such that

$$\mathbf{P}_0\left(B_s \geq -y + f(s), s\leq t,\ B_t + y - f(t) \in [z, z+1]\right) \leq \Gamma_1 \frac{(y\wedge\sqrt{t})(z\wedge\sqrt{t})}{t^{3/2}}, \quad y,z\geq 1,$$

where $(B_t, t\geq 0; \mathbf{P}_x)$ is a standard Brownian motion starting from x.

Fix $y, t > 0$, we define for $s \in [0,t]$,

$$h_s^{t,y} := \frac{3}{2\sqrt{2\lambda^*}}\log\left(\frac{t+1}{t-s+1}\right) - y, \quad f_s^{t,y} := \sqrt{2\lambda^*}s - h_s^{t,y}.$$

Then $f_t^{t,y} = \sqrt{2\lambda^*}t - \frac{3}{2\sqrt{2\lambda^*}}\log(t+1) + y \leq m(t) + y$ and $f_0^{t,y} = y > 0$.

The following result gives the upper bound.

Proposition 4.2. *There exists a positive constant C_0 such that for any $y, t \geq 1$ and $i, i' \in S$,*

$$\mathbb{P}_{(0,i)}\left(M_t^{i'} \geq m(t)+y\right) \leq \mathbb{P}_{(0,i)}\left(M_t \geq m(t)+y\right) \leq C_0(y \wedge \sqrt{t})e^{-\sqrt{2\lambda^*}y}.$$

Proof: The first inequality is trivial since $M_t^{i'} \leq M_t$. Now we prove the second inequality. Let $[x]$ be the largest integer less than or equal to x. Then

$$\begin{aligned}
&\mathbb{P}_{(0,i)}\left(M_t \geq m(t)+y\right) \leq \mathbb{P}_{(0,i)}\left(M_t \geq f_t^{t,y}\right)\\
&\leq \sum_{k=0}^{[t]} \mathbb{E}_{(0,i)}\left(\sum_{u\in Z((k+1)\wedge t)} 1_{\left\{\sup_{s\in[k,(k+1)\wedge t]} X_u(s)\geq f_k^{t,y}\right\}} 1_{\left\{X_u(s)\leq f_s^{t,y}, s\leq k\right\}}\right)\\
&= \sum_{k=1}^{[t]} \mathbb{E}_{(0,i)}\left(\sum_{u\in Z((k+1)\wedge t)} 1_{\left\{\sup_{s\in[k,(k+1)\wedge t]} X_u(s)\geq f_k^{t,y}\right\}} 1_{\left\{X_u(s)\leq f_s^{t,y}, s\leq k\right\}}\right) =: \sum_{k=1}^{[t]} D_k. \qquad (4.3)
\end{aligned}$$

Since all components of $\mathbf{h}$ are positive, we have by Proposition 2.1 that

$$D_k \lesssim e^{\lambda^*(k+1)}\widehat{\mathbb{P}}_{(0,i)}\left(\sup_{s\in[k,(k+1)\wedge t]} X_\xi(s) \geq f_k^{t,y},\ X_\xi(s) \leq f_s^{t,y}, s \leq k\right). \qquad (4.4)$$

Note that, under $\widehat{\mathbb{P}}_{(0,i)}$, $X_\xi(t)$ is a standard Brownian motion. Thus,

$$\begin{aligned}
&\widehat{\mathbb{P}}_{(0,i)}\left(\sup_{s\in[k,(k+1)\wedge t]} X_\xi(s) \geq f_k^{t,y},\ X_\xi(s) \leq f_s^{t,y}, s \leq k\right) \qquad (4.5)\\
&= \mathbf{E}_0\left(\mathbf{P}_0\left(B_k \geq f_k^{t,y} - x, B_s \leq f_s^{t,y}, s\leq k\right)\Big|_{x=\sup_{s\in[k,(k+1)\wedge t]} B_s - B_k}\right).
\end{aligned}$$

For any $\lambda \in \mathbb{R}$, define

$$\left.\frac{\mathrm{d}\mathbf{P}_0^\lambda}{\mathrm{d}\mathbf{P}_0}\right|_{\sigma(B_s, s\leq t)} := e^{\lambda B_t - \frac{1}{2}\lambda^2 t}, \qquad (4.6)$$

then under $\mathbf{P}_0^\lambda$, B_t is a Brownian motion with drift λ. Using this change of measure, we get that

$$\begin{aligned}
&\mathbf{P}_0\left(B_k \geq f_k^{t,y} - x, B_s \leq f_s^{t,y}, s \leq k\right) = \mathbf{E}_0^{\sqrt{2\lambda^*}}\left(e^{-\sqrt{2\lambda^*}B_k + \lambda^* k} 1_{\left\{B_k \geq f_k^{t,y}-x, B_s\leq f_s^{t,y}, s\leq k\right\}}\right)\\
&\leq e^{-\sqrt{2\lambda^*}\left(f_k^{t,y}-x\right)+\lambda^* k}\mathbf{P}_0\left(B_k + \sqrt{2\lambda^*}k \geq f_k^{t,y} - x, B_s + \sqrt{2\lambda^*}s \leq f_s^{t,y}, s\leq k\right)\\
&= e^{-\sqrt{2\lambda^*}\left(f_k^{t,y}-x\right)+\lambda^* k}\mathbf{P}_0\left(B_k \leq x + h_k^{t,y}, B_s \geq h_s^{t,y}, s \leq k\right)\\
&\leq e^{-\sqrt{2\lambda^*}\left(f_k^{t,y}-x\right)+\lambda^* k}\mathbf{P}_0\left(B_s \geq h_s^{t,y+1}, s\leq k,\ B_k - h_k^{t,y+1} \in [1, x+1]\right)\\
&=: e^{-\sqrt{2\lambda^*}\left(f_k^{t,y}-x\right)+\lambda^* k} F_k^{t,y}\left([1, x+1]\right). \qquad (4.7)
\end{aligned}$$

Let

$$f(s) := \frac{3}{2\sqrt{2\lambda^*}}\log\left(\frac{t+1}{t-s+1}\right).$$

Note that, since $\log(1+x) \lesssim x^{1/4}, x > 0$, we have for all $s \leq k$ and all $1 \leq k \leq t$,

$$\begin{aligned}
&\frac{|f(s)|}{s^{1/4}} + \frac{|f(k)-f(s)|}{(k-s)^{1/4}} \lesssim \frac{1}{s^{1/4}}\log\left(1 + \frac{s}{t-s+1}\right) + \frac{1}{(k-s)^{1/4}}\log\left(1+\frac{k-s}{t-k+1}\right)\\
&\leq \frac{\log(1+s)}{s^{1/4}} + \frac{\log(1+k-s)}{(k-s)^{1/4}} \lesssim 1.
\end{aligned}$$

Therefore, applying Lemma 4.1 to the function f above with $\alpha = 1/4$, y replaced by $y+1$ and $z = 1, .., [x]+1$, we get

$$F_k^{t,y}([1, x+1]) \leq \sum_{z=1}^{[x]+1} F_k^{t,y}([z, z+1])$$
$$\lesssim \frac{(y+1)\wedge\sqrt{k}}{k^{3/2}}\left(\sum_{z=1}^{[x]+1} z\wedge\sqrt{k}\right) \lesssim \frac{y\wedge\sqrt{t}}{k^{3/2}}(x+1)^2, \quad y\geq 1, x\geq 0,\ 1\leq k\leq [t]. \tag{4.8}$$

Plugging this upper bound into (4.7), we get that

$$\mathbf{P}_0\left(B_k \geq f_k^{t,y} - x, B_s \leq f_s^{t,y}, s\leq k\right) \lesssim e^{-\sqrt{2\lambda^*}(f_k^{t,y}-x)+\lambda^* k}\frac{y\wedge\sqrt{t}}{k^{3/2}}(x+1)^2. \tag{4.9}$$

Note that $0 \leq \sup_{s\in[k,(k+1)\wedge t]} B_s - B_k \leq \sup_{s\in[k,k+1]} B_s - B_k$ which is equal in law to $W := \sup_{s\in[0,1]} B_s$ under $\mathbf{P}_0$. Combining this with (4.5) and (4.9), we get that for all $y\geq 1, 1\leq k\leq \sqrt{t}$,

$$\widehat{\mathbb{P}}_{(0,i)}\left(\sup_{s\in[k,(k+1)\wedge t]} X_\xi(s) \geq f_k^{t,y},\ X_\xi(s)\leq f_s^{t,y}, s\leq k\right)$$
$$\lesssim e^{-\sqrt{2\lambda^*}f_k^{t,y}+\lambda^* k}\left(\frac{y\wedge\sqrt{t}}{k^{3/2}}\right)\mathbf{E}_0\left(e^{\sqrt{2\lambda^*}W}(W+1)^2\right) \asymp e^{-\sqrt{2\lambda^*}f_k^{t,y}+\lambda^* k}\left(\frac{y\wedge\sqrt{t}}{k^{3/2}}\right). \tag{4.10}$$

Combining (4.3), (4.4) and (4.10), we finally get that

$$\mathbb{P}_{(0,i)}\left(M_t \geq m(t)+y\right) \lesssim \sum_{k=1}^{[t]} e^{\lambda^*(k+1)}e^{-\sqrt{2\lambda^*}f_k^{t,y}+\lambda^* k}\left(\frac{y\wedge\sqrt{t}}{k^{3/2}}\right)$$
$$\asymp (y\wedge\sqrt{t})e^{-\sqrt{2\lambda^*}y}\sum_{k=1}^{[t]}\left(\frac{t+1}{t-k+1}\right)^{3/2}\frac{1}{k^{3/2}}, \quad t, y\geq 1.$$

Note that for all $t\geq 1$,

$$\sum_{k=1}^{[t]}\left(\frac{t+1}{t-k+1}\right)^{3/2}\frac{1}{k^{3/2}} \leq \sum_{k=1}^{[t]}\left(\frac{[t]+2}{[t]-k+1}\right)^{3/2}\frac{1}{k^{3/2}}$$
$$\leq 2\sum_{k=1}^{[[t]/2]+1}\left(\frac{3[t]}{[t]-k+1}\right)^{3/2}\frac{1}{k^{3/2}} \leq 2\left(\frac{3[t]}{[t]-[[t]/2]}\right)^{3/2}\sum_{k=1}^{[[t]/2]+1}\frac{1}{k^{3/2}}$$
$$\leq 2\left(\frac{3[t]}{[t]-[t]/2}\right)^{3/2}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}} = 2\times 6^{3/2}\sum_{k=1}^{\infty}\frac{1}{k^{3/2}} \lesssim 1.$$

Therefore, for all $y\geq 1, t\geq 1$ and $i\in S$,

$$\mathbb{P}_{(0,i)}\left(M_t\geq m(t)+y\right) \lesssim (y\wedge\sqrt{t})e^{-\sqrt{2\lambda^*}y},$$

which is the desired result. □

Next, we are going to get a lower bound for $\mathbb{P}_{(0,i)}\left(M_t \geq m(t)+y\right)$. For $i'\in S$, let

$$\mathcal{A}^{t,y}(i') := \#\left\{u\in Z(t):\ I_u(t) = i' \text{ and } \forall s\leq t,\ X_u(s)\leq f_s^{t,y},\ X_u(t)\geq f_t^{t,y}-4\right\}.$$

The following display is used for getting a lower bound of $\mathbb{P}_{(0,i)}\left(M_t\geq m(t)+y\right)$:

$$\mathbb{E}_{(0,i)}\left(\mathcal{A}^{t,y}(i')\right) \gtrsim (y-3)e^{-\sqrt{2\lambda^*}(y-3)} \gtrsim ye^{-\sqrt{2\lambda^*}y}, \quad i, i'\in S, \quad t\geq 1, \quad 4\leq y\leq 4\sqrt{t}. \tag{4.11}$$

To prove the above, we will adapt some ideas from Bramson et al. (2016). The proof will be given in Subsection 7.3.

Here is our result for the lower bound.

Proposition 4.3. *Let* $\beta_0 := 4 + \frac{3}{2\sqrt{2\lambda^*}}\log 2$. *There exists a positive constant* K_0 *such that for any* $t \geq (\beta_0/3)^2, y \in [1, \sqrt{t}]$ *and* $i, i' \in S$,

$$\mathbb{P}_{(0,i)}\left(M_t \geq m(t)+y\right) \geq \mathbb{P}_{(0,i)}\left(M_t^{i'} \geq m(t)+y\right) \geq K_0 y e^{-\sqrt{2\lambda^*}y}.$$

Proof: The first inequality is trivial, so we only need to prove the second inequality. It is easy to see that for all $t \geq 1, y \geq 1$,

$$\mathbb{P}_{(0,i)}\left(M_t^{i'} \geq m(t)+y-\beta_0\right) \geq \mathbb{P}_{(0,i)}\left(\mathcal{A}^{t,y}(i') \geq 1\right).$$

If we can prove that for all $t \geq 1, y \in [4, 4\sqrt{t}]$ and $i, i' \in S$,

$$\mathbb{P}_{(0,i)}\left(\mathcal{A}^{t,y}(i') \geq 1\right) \gtrsim y e^{-\sqrt{2\lambda^*}y}, \tag{4.12}$$

then for any $t \geq (\beta_0/3)^2$ (which is equivalent to $3\sqrt{t} \geq \beta_0$) and $1 \leq y \leq \sqrt{t}$, we have $4 \leq y + \beta_0 \leq 4\sqrt{t}$, and thus

$$\begin{aligned}
&\mathbb{P}_{(0,i)}\left(M_t^{i'} \geq m(t)+y\right) = \mathbb{P}_{(0,i)}\left(M_t^{i'} \geq m(t)+y+\beta_0-\beta_0\right) \geq \mathbb{P}_{(0,i)}\left(\mathcal{A}^{t,y+\beta_0}(i') \geq 1\right) \\
&\gtrsim (y+\beta_0)\, e^{-\sqrt{2\lambda^*}(y+\beta_0)} \gtrsim y e^{-\sqrt{2\lambda^*}y},
\end{aligned}$$

which completes the proof. To prove (4.12), we use the trivial inequality $\mathbb{E}\left(|Y|^{1+\alpha_0}\right)\mathbb{E}(1_A)^{\alpha_0} \geq \mathbb{E}\left(|Y|1_A\right)^{1+\alpha_0}$ to get that for all $t \geq 1, 4 \leq y \leq 4\sqrt{t}$ and $i, i' \in S$,

$$\mathbb{P}_{(0,i)}\left(\mathcal{A}^{t,y}(i') \geq 1\right) \geq \left\{\frac{\left(\mathbb{E}_{(0,i)}\left(\mathcal{A}^{t,y}(i')\right)\right)^{1+\alpha_0}}{\mathbb{E}_{(0,i)}\left(\left(\mathcal{A}^{t,y}(i')\right)^{1+\alpha_0}\right)}\right\}^{1/\alpha_0}.$$

If we can prove that for all $y, t \geq 1$ and $i, i' \in S$,

$$\mathbb{E}_{(0,i)}\left(\left(\mathcal{A}^{t,y}(i')\right)^{1+\alpha_0}\right) \lesssim (y \wedge \sqrt{t}) e^{-\sqrt{2\lambda^*}y}, \tag{4.13}$$

then using (4.11), we get (4.12). Now we prove (4.13).

Step 1 For $u \in Z(t)$, define

$$\Upsilon(u) := \left\{X_u(s) \leq f_s^{t,y},\ s \leq t, X_u(t) \geq f_t^{t,y} - 4\right\}.$$

We first estimate $\mathbb{E}_{(0,i)}\left(\left(\mathcal{A}^{t,y}(i')\right)^{1+\alpha_0}\right)$ from above. Define $\mathcal{A}^{t,y} := \sum_{j=1}^d \mathcal{A}^{t,y}(j)$. It follows from Proposition 2.1 that

$$\begin{aligned}
&\mathbb{E}_{(0,i)}\left(\left(\mathcal{A}^{t,y}(i')\right)^{1+\alpha_0}\right) \leq \mathbb{E}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{1+\alpha_0}\right) = \mathbb{E}_{(0,i)}\left(\sum_{u\in Z(t)} \left(\mathcal{A}^{t,y}\right)^{\alpha_0} 1_{\Upsilon(u)}\right) \\
&= h_i e^{\lambda^* t}\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0}\frac{1}{h_{I_\xi(t)}} 1_{\Upsilon(\xi)}\right) \lesssim e^{\lambda^* t}\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0} 1_{\Upsilon(\xi)}\right).
\end{aligned} \tag{4.14}$$

Recall that $\{I_\xi(t), t \geq 0\}$ is the type of spine. Let τ_k be the k-th time that $I_\xi(s-) \neq I_\xi(s)$ and let $K_t := \sup\{k : \tau_k \leq t\}$. For all $t \geq (\beta_0/3)^2, y \in [1, \sqrt{t}]$ and $i \in S$, we have the following upper bound for $\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0}1_{\Upsilon(\xi)}\right)$:

$$\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0}1_{\Upsilon(\xi)}\right)\lesssim(y\wedge\sqrt{t})e^{-\lambda^*t}e^{-\sqrt{2\lambda^*}y}$$
$$\times\widehat{\mathbb{E}}_{(0,i)}\left(1+\sum_{\ell=1}^{K_t}\left(\frac{1_{\{t/2\geq\tau_\ell\geq1\}}}{\tau_\ell^{3/2}}+\frac{1_{\{t/2<\tau_\ell\leq t-1\}}}{(t-\tau_\ell)^{3/2}}+1_{\{\tau_\ell\leq1\}}+1_{\{\tau_\ell\geq t-1\}}\right)\right).\tag{4.15}$$

The proof of (4.15) is technical and we postpone it to Subsection 7.4. Combining (4.14) and (4.15), we get

$$\mathbb{E}_{(0,i)}\left((\mathcal{A}^{t,y}(i'))^{1+\alpha_0}\right)$$
$$\lesssim\frac{(y\wedge\sqrt{t})}{e^{\sqrt{2\lambda^*}y}}\widehat{\mathbb{E}}_{(0,i)}\left(1+\sum_{\ell=1}^{K_t}\left(\frac{1_{\{t/2\geq\tau_\ell\geq1\}}}{\tau_\ell^{3/2}}+\frac{1_{\{t/2<\tau_\ell\leq t-1\}}}{(t-\tau_\ell)^{3/2}}+1_{\{\tau_\ell\leq1\}}+1_{\{\tau_\ell\geq t-1\}}\right)\right).\tag{4.16}$$

Step 2 Let $0<\underline{a}\leq\overline{a}$ be such that

$$0<\underline{a}\leq\min_{i\in S}(a_i+\lambda^*)\leq\max_{i\in S}(a_i+\lambda^*)\leq\overline{a}.$$

Recall that $D_J:=\{t:I_\xi(t-)\neq I_\xi(t)\}$. We can define two processes $I_t^{\underline{a}}$ and $I_t^{\overline{a}}$ with the same jumping probability as $I_\xi(t)$ and with constant jump rates $\underline{a}$ and $\overline{a}$ respectively. Similarly, we define $D^{\underline{a}}$ and $D^{\overline{a}}$ to be the jumping times of $I_t^{\underline{a}}$ and $I_t^{\overline{a}}$. We can construct a coupling of $(I_\xi(t),I_t^{\underline{a}},I_t^{\overline{a}})$ such that the embedded chain of the three processes are the same and the jump times

$$D_J=\{t_n:0<t_1<t_2<...\},\ D^{\underline{a}}=\{t_n^{\underline{a}}:0<t_1^{\underline{a}}<t_2^{\underline{a}}<...\},\ D^{\overline{a}}=\{t_n^{\overline{a}}:0<t_1^{\overline{a}}<t_2^{\overline{a}}<...\}\tag{4.17}$$

satisfy $t_n^{\overline{a}}\leq t_n\leq t_n^{\underline{a}}$ for every n. More precisely, let $\{Y_n:n=0,1,...\}$ be the embedded chain of $I_\xi(t)$ with $Y_0=I_\xi(0)$. Let $T_0:=t_1$ and $T_n:=t_{n+1}-t_n$ for $n\geq1$. Then by the strong Markov property, given $Y_j,0\leq j\leq n$, T_n is an exponential distribution with parameter $a_{Y_n}+\lambda^*$. Let $T_n^{\overline{a}}:=(a_{Y_n}+\lambda^*)T_n/\overline{a}\leq T_n\leq(a_{Y_n}+\lambda^*)T_n/\underline{a}=:T_n^{\underline{a}}$. Then we see that given $Y_j,0\leq j\leq n$, $T_n^{\overline{a}}$ and $T_n^{\underline{a}}$ are exponential distribution with parameter $\overline{a}$ and $\underline{a}$ respectively. Now for $n\geq1$, define $t_n^{\overline{a}}:=\sum_{j=0}^{n-1}T_j^{\overline{a}}$ and $t_n^{\underline{a}}:=\sum_{j=0}^{n-1}T_j^{\underline{a}}$. Define $K_t^{\overline{a}}:=\sup\{n:t_n^{\overline{a}}\leq t\}$, $K_t^{\underline{a}}:=\sup\{n:t_n^{\underline{a}}\leq t\}$, $I_t^{\overline{a}}:=Y_{K_t^{\overline{a}}}$ and $I_t^{\underline{a}}:=Y_{K_t^{\underline{a}}}$. Then $(I_\xi(t),I_t^{\underline{a}},I_t^{\overline{a}})$ is the desired coupling. Therefore, for any non-negative and non-increasing function f,

$$\sum_{s\in D_J:s\leq t}f(s)=\sum_{n=1}^{K_t}f(t_n)\leq\sum_{n=1}^{K_t}f(t_n^{\overline{a}})\leq\sum_{i=1}^{K_t^{\overline{a}}}f(t_n^{\overline{a}})=\sum_{s\in D^{\overline{a}}:s\leq t}f(s),\tag{4.18}$$

here $K_t\leq\#\left\{D^{\overline{a}}\cap[0,t]\right\}=K_t^{\overline{a}}$ by the coupling. Applying (4.18) to $f(s)=1_{\{s\leq1\}}+s^{-3/2}1_{\{s\in[1,t/2]\}}$ and $f(s)=1_{\{s\leq1\}}$, by the Markov property at time $t-1$, we get that

$$\widehat{\mathbb{E}}_{(0,i)}\left(1+\sum_{\ell=1}^{K_t}\left(\frac{1_{\{t/2\geq\tau_\ell\geq1\}}}{\tau_\ell^{3/2}}+1_{\{\tau_\ell\leq1\}}+1_{\{\tau_\ell\geq t-1\}}\right)\right)$$
$$\leq\widehat{\mathbb{E}}_{(0,i)}\left(1+\sum_{\ell=1}^{K_t}\left(\frac{1_{\{t/2\geq\tau_\ell\geq1\}}}{\tau_\ell^{3/2}}+1_{\{\tau_\ell\leq1\}}\right)\right)+\sup_{j\in S}\widehat{\mathbb{E}}_{(0,j)}\left(\sum_{\ell=1}^{K_1}1\right)$$
$$\leq1+\mathbb{E}\left(\sum_{\ell=1}^{K_t^{\overline{a}}}\frac{1}{(\tau_\ell^{\overline{a}})^{3/2}}1_{\{t/2\geq\tau_\ell^{\overline{a}}\geq1\}}+1_{\{\tau_\ell^{\overline{a}}\leq1\}}\right)+\mathbb{E}\left(\sum_{\ell=1}^{K_1^{\overline{a}}}1\right)$$
$$=1+\overline{a}\int_0^t\left(1_{\{s\leq1\}}+\frac{1}{s^{3/2}}1_{\{s\in(1,t-1]\}}\right)\mathrm{d}s+\overline{a}\int_0^1 1\mathrm{d}s\lesssim1.\tag{4.19}$$

For $\tau_\ell \in (t/2, t-1]$, note that

$$\sum_{\ell=1}^{K_t} \frac{1_{\{t-1\geq \tau_\ell > t/2\}}}{(t-\tau_\ell)^{3/2}} \leq \sum_{k=[t/2]}^{\lceil t-1\rceil -1} \frac{1_{\{k<\tau_\ell\leq (k+1)\}}}{(t-k-1)^{3/2}},$$

where $\lceil t-1\rceil$ is the smallest integer larger than or equal to $t-1$. For each k, using the Markov property at time k, and using (4.18), we have

$$\begin{aligned}\widehat{\mathbb{E}}_{(0,i)}\left(\sum_{\ell=1}^{K_t} \frac{1_{\{t-1\geq \tau_\ell > t/2\}}}{(t-\tau_\ell)^{3/2}}\right) &\leq \sum_{k=[t/2]}^{\lceil t-1\rceil -1} \frac{1}{(t-k-1)^{3/2}} \sup_{j\in S} \widehat{\mathbb{E}}_{(0,j)}\left(\sum_{\ell=1}^{K_1} 1\right)\\ &\leq \sum_{k=[t/2]}^{\lceil t-1\rceil -1} \frac{1}{(t-k-1)^{3/2}} \mathbb{E}\left(\sum_{\ell=1}^{K_1^{\overline{a}}} 1\right) \lesssim \sum_{k=[t/2]}^{\lceil t-1\rceil -1} \frac{1}{(t-k-1)^{3/2}} \lesssim 1. \end{aligned} \tag{4.20}$$

Combining (4.16), (4.19) and (4.20), we get (4.13). The proof is now complete. $\square$

Remark 4.4. As a consequence of (4.18), we have the following useful inequality: for any $r<t$ and any decreasing non-negative function f on $[r,t]$,

$$\begin{aligned}\widehat{\mathbb{E}}_{(0,i)}\left(\exp\left\{-\sum_{s\in D_J: r<s\leq t} f(s)\right\}\right) &\geq \inf_{\ell\in S} \widehat{\mathbb{E}}_{(0,\ell)}\left(\exp\left\{-\sum_{s\in D_J: s\leq t-r} f(s+r)\right\}\right)\\ \geq \inf_{\ell\in S} \widehat{\mathbb{E}}_{(0,\ell)}\left(\exp\left\{-\sum_{s\in D^{\overline{a}}: s\leq t-r} f(s+r)\right\}\right) &= \exp\left\{-\overline{a}\int_0^{t-r}\left(1-e^{-f(s+r)}\right)\mathrm{d}s\right\}. \end{aligned} \tag{4.21}$$

Using the fact that $\#\{D_J\cap[0,t]\} \geq \#\{D^{\underline{a}}\cap[0,t]\}$, we also have that for any $\theta>0$,

$$\begin{aligned}&\widehat{\mathbb{E}}_{(0,i)}\left(\exp\left\{-\theta \sum_{s\in D_J: r<s\leq t} 1\right\}\right) \leq \sup_{\ell\in S} \widehat{\mathbb{E}}_{(0,\ell)}\left(\exp\left\{-\theta\sum_{s\in D^{\underline{a}}: s\leq t-r} 1\right\}\right)\\ &= \exp\left\{-\underline{a}\int_0^{t-r}\left(1-e^{-\theta}\right)\mathrm{d}s\right\} = e^{-\underline{a}(t-r)(1-e^{-\theta})}.\end{aligned}$$

Using Propositions 4.2 and 4.3, we can get the following result:

Theorem 4.5. *For any $i\in S$, $\left(M_t - m(t),\ t\geq 1, \mathbb{P}_{(0,i)}\right)$ is tight. Also, $\left(M_t^{i'} - m(t),\ t\geq 1, \mathbb{P}_{(0,i)}\right)$ is tight for any $i,i'\in S$.*

Proof: Fix $i\in S$. For any $\varepsilon>0$, choose $y>4$ and δ small so that $K_0 y e^{-\sqrt{2\lambda^*}y} \geq \delta$, where K_0 is the constant in Proposition 4.3. Now choose L large so that $\mathbb{E}_{(0,i)}\left((1-\delta)^{\langle \mathbf{N}_L, \mathbf{1}\rangle}\right) < \varepsilon/2$. Indeed, we can find a large n such that $(1-\delta)^n < \varepsilon/4$, therefore,

$$\mathbb{E}_{(0,i)}\left((1-\delta)^{\langle \mathbf{N}_L, \mathbf{1}\rangle}\right) < \frac{\varepsilon}{4} + \mathbb{P}_{(0,i)}\left(\langle \mathbf{N}_L, \mathbf{1}\rangle < n\right),$$

which is less than $\varepsilon/2$ for large L since $\langle \mathbf{N}_L, \mathbf{1}\rangle \to +\infty$ $\mathbb{P}_{(0,i)}$-a.s. Let $b>0$ be a constant such that

$$\mathbb{P}_{(0,i)}\left(\min_{u\in Z(L)} X_u(L) < -b\right) < \frac{\varepsilon}{2}.$$

By Proposition 4.3, for t large enough so that $t-L > \max\{\beta_0^2, y^2\}$ where β_0 is the constant in Proposition 4.3,

$$\inf_{i,i'\in S} \mathbb{P}_{(0,i)}\left(M_{t-L}^{i'} > m(t-L) + y\right) \geq K_0 y e^{-\sqrt{2\lambda^*}y} \geq \delta.$$

Thus,

$$\mathbb{P}_{(0,i)}\left(M_t < m(t-L)-b+y\right) \le \mathbb{P}_{(0,i)}\left(M_t^{i'} < m(t-L)-b+y\right)$$
$$\le \mathbb{P}_{(0,i)}\left(\min_{u\in Z(L)} X_u(L) < -b\right) + \mathbb{P}_{(0,i)}\left(\min_{u\in Z(L)} X_u(L) \ge -b, \max_{u\in Z(L)} M_{t-L}^{i'}(u) < m(t-L)-b+y\right)$$
$$\le \frac{\varepsilon}{2} + \mathbb{E}_{(0,i)}\left(\prod_{u\in Z(L)} \mathbb{P}_{(0,I_u(L))}\left(M_{t-L}^{i'} < m(t-L)+y\right)\right) \le \frac{\varepsilon}{2} + \mathbb{E}_{(0,i)}\left((1-\delta)^{\langle \mathbf{N}_L, \mathbf{1}\rangle}\right) < \varepsilon.$$

From this one can easily see that there exists $\widetilde{y} > 0$ such that for all t large

$$\mathbb{P}_{(0,i)}\left(M_t < m(t)-\widetilde{y}\right) \le \mathbb{P}_{(0,i)}\left(M_t^{i'} < m(t)-\widetilde{y}\right) < \varepsilon.$$

Also, by Proposition 4.2, there exists y^* large enough such that for all $t \ge 1$

$$\mathbb{P}_{(0,i)}\left(M_t^{i'} > m(t)+y^*\right) \le \mathbb{P}_{(0,i)}\left(M_t > m(t)+y^*\right) \le C_0 y^* e^{-\sqrt{2\lambda^*}y^*} < \varepsilon. \tag{4.22}$$

Thus there exists $T > 1$ such that $\left(M_t - m(t),\ t \ge T, \mathbb{P}_{(0,i)}\right)$ and $\left(M_t^{i'} - m(t),\ t \ge T, \mathbb{P}_{(0,i)}\right)$ are tight. Since $\min_{1\le t\le T} M_t^{i'}$ is finite $\mathbb{P}_{(0,i)}$-a.s., for y^* large enough we also have for $1 \le t \le T$,

$$\mathbb{P}_{(0,i)}\left(M_t > m(t)-y^*\right) \ge \mathbb{P}_{(0,i)}\left(M_t^{i'} > m(t)-y^*\right)$$
$$\ge \mathbb{P}_{(0,i)}\left(\min_{1\le t\le T}\left(M_t^{i'} - m(t)\right) > -y^*\right) > 1-\varepsilon.$$

Combining this with (4.22), we get

$$\mathbb{P}_{(0,i)}\left(-y^* < M_t - m(t) < y^*\right) = \mathbb{P}_{(0,i)}\left(M_t - m(t) > -y^*\right) - \mathbb{P}_{(0,i)}\left(M_t - m(t) > y^*\right) > 1-2\varepsilon$$

and

$$\mathbb{P}_{(0,i)}\left(-y^* < M_t^{i'} - m(t) < y^*\right) = \mathbb{P}_{(0,i)}\left(M_t^{i'} - m(t) > -y^*\right) - \mathbb{P}_{(0,i)}\left(M_t^{i'} - m(t) > y^*\right) > 1-2\varepsilon.$$

This completes the proof. □

5. Proof of Theorem 1.1

The main idea of our proof of Theorem 1.1 is similar to that of the corresponding result in Bramson (1983) for single-type branching Brownian motions. However, some parts are much more complicated in the multi-type case. To keep the flow of the argument, we postpone the proofs of some technical results to the appendix.

5.1. *Upper bound for* $\mathbf{v}$. In this subsection, we first give some estimates involving $R(t;v)$ defined in (3.11) and then use these estimates to get some upper bound for solutions of (1.5) with initial condition satisfying (1.12). We roughly follow the arguments of Bramson (1983, Sections 6-8). However, some of the arguments in Bramson (1983) do not work in the multi-type case. We will explain these later in this section.

Let $\delta \in (0, 1/2)$ and $K > 2+\sqrt{2\lambda^*}$ be fixed constants. If L is a function on $[0,t]$, for $t > 4r > 0$, as in Bramson (1983, (6.11), p. 88), we define

$$\theta_{r,t}\circ L(s) := \begin{cases} L\left(s + s^\delta \wedge (t-s)^\delta\right) + K s^\delta \wedge (t-s)^\delta, & r \le s \le t-2r, \\ L(s), & \text{otherwise.} \end{cases}$$

We define $\theta_{r,t}^{-1}$ to be the inverse of $\theta_{r,t}$. Similar to Bramson (1983, (7.6), p. 99, (6.13), p. 88, and (6.14), p. 89), we define

$$\begin{aligned} L_{r,t}(s) &:= m(s) - \frac{s}{t}m(t) - \frac{t-s}{t}\alpha(r), \\ \underline{L}_{r,t}(s) &:= \theta_{r,t}^{-1} \circ L_{r,t}(s), \\ \overline{L}_{r,t}(s) &:= (\theta_{r,t} \circ L_{r,t}(s)) \vee \underline{L}_{r,t}(s) \vee L_{r,t}(s), \end{aligned}$$

and similar to Bramson (1983, (7.44), p. 111) or Ren et al. (2021, (2.11)), we define

$$\underline{\mathcal{M}}'_{r,t}(s) := \begin{cases} \underline{L}_{r,t}(s) + \frac{s}{t}m(t) + \frac{t-s}{t}\alpha(r), & r + r^\delta \le s \le t - 2r, \\ -\infty, & \text{otherwise}, \end{cases}$$

where the function $\alpha(r)$ is either taken to be $-\log r$ or taken to be identically 0. Let

$$\mathcal{B}_{up} := \left\{ X_s > \underline{\mathcal{M}}'_{r,t}(t-s) \text{ for all } s \in [0, t-r] \right\}.$$

Note that when $s \in [0, 2r) \cup [t - r - r^\delta, t]$, $\underline{\mathcal{M}}'_{r,t}(t-s) = -\infty$, therefore,

$$(\mathcal{B}_{up})^c = \left\{ \exists\ s \in [2r, t - r - r^\delta] \text{ such that } X_s \le \underline{\mathcal{M}}'_{r,t}(t-s) \right\}.$$

Similar to Bramson (1983, (7.19)–(7.20), p.102–103), we define

$$\begin{aligned} S^1(r,t) &:= \sup \left\{ s : 2r \le s \le t/2,\ X_s \le \underline{\mathcal{M}}'_{r,t}(t-s) \right\}, \\ S^2(r,t) &:= \inf \left\{ s : t/2 \le s \le t - r,\ X_s \le \underline{\mathcal{M}}'_{r,t}(t-s) \right\}, \\ S(r,t) &:= S^1(r,t) 1_{\{S^1(r,t)+S^2(r,t)>t\}} + S^2(r,t) 1_{\{S^1(r,t)+S^2(r,t)\le t\}}. \end{aligned}$$

We use the convention that $S^1(r,t) = 0$ if $X_s > \underline{\mathcal{M}}'_{r,t}(t-s)$ for all $s \in [2r, t/2]$ and that $S^2(r,t) = t$ if $X_s > \underline{\mathcal{M}}'_{r,t}(t-s)$ for all $s \in [t/2, t-r]$. Next, similar to Bramson (1983, (7.21), p.103), we define for $r_1 \in [r, t/2]$,

$$(\mathcal{B}_{up}^{r_1})^c := \{ r_1 \le S(r,t) \le t - r_1 \} = \left\{ \exists\ s \in [r_1 \vee (2r), t - r_1] \text{ such that } X_s \le \underline{\mathcal{M}}'_{r,t}(t-s) \right\}.$$

Let j_1 be the integer such that $j_1 < t/2 \le j_1 + 1$. Define $G_j = [j, j+1) \cup (t-j-1, t-j], j = 0, \cdots, j_1 - 1$ and $G_{j_1} = [j_1, t - j_1]$. Similar to Bramson (1983, (7.23)–(7.24), p. 103), we define for $j = 0, \cdots, j_1$,

$$\begin{aligned} A_j(r,t) &:= \{ S(r,t) \in G_j \}, \\ A_j^1(r,t) &:= A_j(r,t) \cap \left\{ X_s > -(s \wedge (t-s)) + \frac{s}{t}y + \frac{t-s}{t}x \text{ for all } s \in G_j \right\} \end{aligned}$$

and $A_j^2(r,t) = A_j(r,t) \setminus A_j^1(r,t)$. The following result is Bramson (1983, Lemma 7.1, p. 104).

Lemma 5.1. *For large r, $t > 4r$ and $r_1 \in [r, t/2]$, for any $y \ge -\log r$ and $x \ge m(t)$,*

$$\mathbf{P}_x \left(\mathcal{B}_{up}^{r_1} \middle| X_t = y \right) \lesssim \frac{r_1}{r} \mathbf{P}_x \left(\mathcal{B}_{up} \middle| X_t = y \right).$$

Recall that by the definitions (3.10)–(3.11),

$$\begin{aligned} R_t(t-r; v) = \exp \Bigg\{ &\sum_{s \in D_J, s \le t-r} \log \left(\frac{(1-\rho_{I_s}) \varphi_{I_s}^{NL}(\mathbf{v}(t-s, X_s))}{\sum_{j \ne I_s} \mu_{I_s,j} v_j(t-s, X_s)} \right) \\ &- \int_0^{t-r} a_{I_s} \left(\mu_{I_s,I_s} - \frac{\rho_{I_s} \varphi_{I_s}^L(\mathbf{v}(t-s, X_s))}{v_{I_s}(t-s, X_s)} \right) \mathrm{d}s \Bigg\}. \end{aligned}$$

If $\mathbf{v}$ solves (1.5) with initial value satisfying (1.12), then by (1.6), for any $i \in S$,

$$\begin{aligned} v_i(t,x) &\geq 1 - \mathbb{E}_{(x,i)}\left(\prod_{u\in Z(t):I_u(t)=i_0}\left(1 - 1_{(-\infty,N_1)}(X_u(t))\right)\right) \\ &= 1 - \mathbb{E}_{(0,i)}\left(\prod_{u\in Z(t):I_u(t)=i_0} 1_{\{X_u(t)+x\geq N_1\}}\right) = 1 - \mathbb{E}_{(0,i)}\left(\prod_{u\in Z(t):I_u(t)=i_0} 1_{\{X_u(t)\leq x-N_1\}}\right) \\ &= \mathbb{P}_{(0,i)}\left(M_t^{i_0} > x - N_1\right). \end{aligned} \tag{5.1}$$

By induction, for any $0 \leq x_k, y_k \leq 1$ with $x_k + y_k \leq 1$ for all $1 \leq k \leq n$, it holds that

$$1 - \prod_{k=1}^{n}(1 - x_k - y_k) \leq \left\{1 - \prod_{k=1}^{n}(1 - x_k)\right\} + \left\{1 - \prod_{k=1}^{n}(1 - y_k)\right\}. \tag{5.2}$$

Indeed, it is easy to see that (5.2) holds for $n = 1$. If (5.2) holds for n and $0 \leq x_k, y_k, x_k + y_k \leq 1$ with $1 \leq k \leq n$, then

$$\begin{aligned} 1 - \prod_{k=1}^{n+1}(1 - x_k - y_k) &= (1 - x_{n+1} - y_{n+1})\left(1 - \prod_{k=1}^{n}(1 - x_k - y_k)\right) + x_{n+1} + y_{n+1} \\ &\leq (1 - x_{n+1} - y_{n+1})\left(\left\{1 - \prod_{k=1}^{n}(1 - x_k)\right\} + \left\{1 - \prod_{k=1}^{n}(1 - y_k)\right\}\right) + x_{n+1} + y_{n+1} \\ &\leq (1 - x_{n+1})\left\{1 - \prod_{k=1}^{n}(1 - x_k)\right\} + (1 - y_{n+1})\left\{1 - \prod_{k=1}^{n}(1 - y_k)\right\} + x_{n+1} + y_{n+1} \\ &= \left\{1 - \prod_{k=1}^{n+1}(1 - x_k)\right\} + \left\{1 - \prod_{k=1}^{n+1}(1 - y_k)\right\}, \end{aligned}$$

which implies (5.2).

Suppose that $\mathbf{v}$ satisfies (1.5) and (1.12). For $r > 1$ and $t \geq r$, define $v_j^*(0,x) := v_j(r,x)1_{\{x>-\log r\}}$ for all $j \in S$ and

$$v_i^*(t-r,x) = 1 - \mathbb{E}_{(x,i)}\left(\prod_{u\in Z(t-r)}\left(1 - v^*_{I_u(t-r)}(0, X_u(t-r))\right)\right). \tag{5.3}$$

The next lemma is slightly different from Bramson (1983, Proposition 8.3 (b), p. 136). In Bramson (1983) (see the argument Bramson (1983, from (8.44) to (8.46), p.137)), Bramson used the Feynman-Kac formula to get

$$v(t, m(t) + x) = \int_{\mathbb{R}} v(r,y)\frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}}\mathbf{E}_{m(t)+x}\left(\exp\left\{\int_0^{t-r} k(v(t-r-s, X_s))\mathrm{d}s\right\}\Big|X_{t-r} = y\right)\mathrm{d}y,$$

where $k(x) = f(x)/x$. Then he separated the integral into $\int_{-\infty}^{-\log r} + \int_{-\log r}^{\infty}$. For the $\int_{-\infty}^{-\log r}$ part (see Bramson (1983, (8.45), p. 137)), he used the fact that if $f : [0,1] \to [0,\infty)$ is a concave function with $f(0) = 0$ (which is obviously the case for $f(x) = 1 - \sum_{k=0}^{\infty} p_k(1-x)^k$ with $\{p_k : k \geq 1\}$ a distribution with finite mean), then $k(x) = f(x)/x$ is decreasing in $(0, 1]$. But when f is a multi-variable function like φ_i, it no longer holds that

$$\frac{f(\mathbf{v})}{\nabla f(0)\cdot\mathbf{v}} \leq \frac{f(\widetilde{\mathbf{v}})}{\nabla f(0)\cdot\widetilde{\mathbf{v}}}, \qquad \text{if } 1 \geq v_j \geq \widetilde{v}_j \geq 0 \text{ for all } j \in S.$$

To avoid this difficulty, we deal with the part $\int_{-\infty}^{-\log r}$ by first using probabilistic representation (1.6), Propositions 4.2 and 4.3, and then using the Feynman-Kac formula. This is accomplished in the following lemma.

Lemma 5.2. *Suppose that* $\mathbf{v}$ *satisfies* (1.5) *and* (1.12). *If* r *is large enough, then there exists a positive function* $C(r)$ *with* $\lim_{r\to\infty} C(r) = 1$ *such that for all* $r < x \le \sqrt{t}$,

$$\begin{aligned} &v_i(t, m(t)+x) \le C(r) v_i^* (t-r, m(t)+x) \\ &= C(r) e^{\lambda^*(t-r)} h_i \int_{-\log r}^{\infty} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbf{E}^h_{(m(t)+x,i)} \left(R(t-r; v^*) \frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}} \Big| X_{t-r} = y \right) \mathrm{d}y. \end{aligned}$$

Proof: The equality in the lemma follows from the Feynman-Kac formula (3.12), so we only prove the inequality. Note that $1 \ge v_j^*(0,x) + 1_{\{x \le -\log r\}} \ge v_j(r,x)$, by (1.6) and (5.2), we have

$$\begin{aligned} v_i(t, m(t)+x) &\le v_i^*(t-r, m(t)+x) + 1 - \mathbb{E}_{(m(t)+x,i)} \left(\prod_{u \in Z(t-r)} (1 - 1_{\{X_u(t-r) \le -\log r\}}) \right) \\ &= v_i^*(t-r, m(t)+x) + \mathbb{P}_{(0,i)} \left(M_{t-r} > m(t)+x+\log r \right). \end{aligned} \tag{5.4}$$

By (5.1), Propositions 4.2 and 4.3, for all $r < x \le \sqrt{t}$, if r is large enough so that $r - N_1 \ge \frac{r}{2}$ and $\sqrt{2\lambda^*} r - 1 \le m(t) - m(t-r) \le \sqrt{2\lambda^*} r$, then

$$\begin{aligned} \frac{\mathbb{P}_{(0,i)} \left(M_{t-r} > m(t)+x+\log r \right)}{v_i(t, m(t)+x)} &\le \frac{\mathbb{P}_{(0,i)} \left(M_{t-r} > m(t)+x+\log r \right)}{\mathbb{P}_{(0,i)} \left(M_t^{i_0} > m(t)+x-N_1 \right)} \\ &\lesssim \frac{(m(t)-m(t-r)+x+\log r) e^{-\sqrt{2\lambda^*}(m(t)-m(t-r)+x+\log r)}}{(x-N_1)\, e^{-\sqrt{2\lambda^*}(x-N_1)}} \\ &\lesssim \frac{(\sqrt{2\lambda^*} r + x + \log r) e^{-\sqrt{2\lambda^*}(\sqrt{2\lambda^*} r + \log r)}}{(x-N_1)} \lesssim e^{-\sqrt{2\lambda^*}(\sqrt{2\lambda^*} r + \log r)}, \end{aligned}$$

where in the last inequality we used the fact that for $x > r$ and $r - N_1 \ge r/2$,

$$\frac{(\sqrt{2\lambda^*} r + x + \log r)}{(x-N_1)} \le \frac{(\sqrt{2\lambda^*} + 1 + 1)x}{x/2} \lesssim 1.$$

Therefore, if Γ is a constant such that for large r and $r < x \le \sqrt{t}$,

$$\frac{\mathbb{P}_{(0,i)} \left(M_{t-r} > m(t)+x+\log r \right)}{v_i(t, m(t)+x)} \le \Gamma e^{-\sqrt{2\lambda^*}(\sqrt{2\lambda^*} r + \log r)} < 1,$$

we can choose $C(r)$ to be

$$C(r) := \left(1 - \Gamma e^{-\sqrt{2\lambda^*}(\sqrt{2\lambda^*} r + \log r)} \right)^{-1}.$$

□

The next proposition is similar to Bramson (1983, Proposition 7.3, p. 108). However, many new difficulties and challenges appear in the multi-type case. We will give the proof and some detailed discussions about it in Subsection 7.5.

Proposition 5.3. *Let* $\mathbf{v}$ *be the solution of* (1.5) *with initial value satisfying* (1.12). *Then for* r *large enough,* $t > 4r$, $y > -\log r$ *and* $x \ge m(t)$, *it holds that*

$$\mathbf{E}^h_{(x,i)} \left(R((r, t-r]; v_*) \frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}; (\mathcal{B}_{up})^c \Big| X_t = y \right) \le \frac{1}{r^2} \mathbf{E}^h_{(x,i)} \left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}} \right) \mathbf{P}^h_{(x,i)} \left(\mathcal{B}_{up} \big| X_t = y \right).$$

We now give an upper bound for $v_i(t, m(t)+x)$:

Proposition 5.4. *Suppose that* $\mathbf{v}$ *satisfies* (1.5) *and* (1.12)*. Let* r *be large enough, then for all* $r < x \le \sqrt{t}$*, it holds that*

$$v_i(t, m(t)+x) \le C_{up}(r) e^{\lambda^*(t-r)} h_i \int_{-\log r}^{\infty} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \times \mathbf{P}^h_{(m(t)+x,i)} \left(\mathcal{B}_{up} \middle| X_{t-r} = y\right) \sum_{j=1}^{d} g_j v_j(r,y) \mathrm{d}y,$$

where $C_{up}(r) \downarrow 1$ *as* $r \to \infty$.

Proof: By Lemma 5.2, Proposition 5.3, and the independence of X and I, we have

$$\begin{aligned}
&(C(r)h_i)^{-1} e^{-\lambda^*(t-r)} v_i(t, m(t)+x) \\
&\le \int_{-\log r}^{\infty} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbf{E}^h_{(m(t)+x,i)} \left(R(t-r; v_*) \frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}} \middle| X_{t-r} = y \right) \mathrm{d}y \\
&\le \int_{-\log r}^{\infty} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbf{E}^h_{(m(t)+x,i)} \left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}; \mathcal{B}_{up} \middle| X_{t-r} = y \right) \mathrm{d}y \\
&\quad + \int_{-\log r}^{\infty} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbf{E}^h_{(m(t)+x,i)} \left(R(t-r; v_*) \frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}; (\mathcal{B}_{up})^c \middle| X_{t-r} = y \right) \mathrm{d}y \\
&\le \int_{-\log r}^{\infty} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \times \mathbf{P}^h_{(m(t)+x,i)} \left(\mathcal{B}_{up} \middle| X_{t-r} = y \right) \mathbf{E}^h_{(m(t)+x,i)} \left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}} \right) \mathrm{d}y \\
&\quad + \int_{-\log r}^{\infty} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \left(\frac{1}{r^2} \mathbf{E}^h_{(m(t)+x,i)} \left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}} \right) \mathbf{P}^h_{(m(t)+x,i)} \left(\mathcal{B}_{up} | X_{t-r} = y \right) \right) \mathrm{d}y \\
&= \left(1 + \frac{1}{r^2}\right) \int_{-\log r}^{\infty} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbf{P}^h_{(m(t)+x,i)} \left(\mathcal{B}_{up} \middle| X_{t-r} = y \right) \mathbf{E}^h_{(m(t)+x,i)} \left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}} \right) \mathrm{d}y.
\end{aligned}$$

Note that $\lim_{r\to\infty} \mathbf{P}^h_{(x,i)}(I_r = j) = \nu_j = g_j h_j = \lim_{r\to\infty} \sup_{t>r} \mathbf{P}^h_{(x,i)}(I_t = j)$. Letting

$$C_{up}(r) := C(r) \left(1 + \frac{1}{r^2}\right) \sup_{i,j \in S} \sup_{t>r} \frac{\mathbf{P}^h_{(x,i)}(I_t = j)}{g_j h_j},$$

we get the assertion of the proposition. □

5.2. *Lower bound for* $\mathbf{v}$. Similar to Bramson (1983, (7.42), p. 111 and (7.9), p. 99) or Ren et al. (2021, (2.10)), we define

$$\overline{\mathcal{M}}^x_{r,t}(s) := \begin{cases} \overline{L}_{r,t}(s) + \frac{s}{t} m(t) - \frac{t-s}{t} \log r, & 0 \le s \le t - 2r, \\ \dfrac{x}{2} + m(t), & t - 2r < s \le t, \end{cases}$$

where $\overline{L}_{r,t}$ is defined in the beginning of Subsection 5.1. Define

$$\mathcal{B}_{low} := \left\{ X_s > \overline{\mathcal{M}}^x_{r,t}(t-s),\ s \in [0, t-r] \right\}.$$

Proposition 5.5. *When* r *is large enough, it holds that for all* $t, x > 8r$ *and all* $i \in S$*,*

$$v_i(t, m(t)+x) \ge C_{low}(r) e^{\lambda^*(t-r)} h_i \int_{\mathbb{R}} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbf{P}^h_{(m(t)+x,i)} \left(\mathcal{B}_{low} \middle| X_{t-r} = y \right) \sum_{j=1}^{d} g_j v_j(r,y) \mathrm{d}y,$$

where $C_{low}(r) \uparrow 1$ *as* $r \to \infty$.

The proof is similar to that of Bramson (1983, Proposition 8.3(a)) and is given in Subsection 7.6 below.

5.3. *Proof of Theorem 1.1.*

Proof of Theorem 1.1: Define

$$\Psi^i_{low}(r;t,x) := e^{\lambda^*(t-r)} h_i \int_{\mathbb{R}} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \times \mathbf{P}^h_{(m(t)+x,i)}\left(\mathcal{B}_{low}\Big| X_{t-r}=y\right) \sum_{j=1}^d g_j v_j(r,y)\mathrm{d}y,$$

$$\Psi^i_{up}(r;t,x) := e^{\lambda^*(t-r)} h_i \int_{\mathbb{R}} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \times \mathbf{P}^h_{(m(t)+x,i)}\left(\mathcal{B}_{up}\Big| X_{t-r}=y\right) \sum_{j=1}^d g_j v_j(r,y)\mathrm{d}y.$$

By Propositions 5.5 and 5.4, for all $8r < x \le \sqrt{t}$,

$$C_{up}(r)\Psi^i_{up}(r;t,x) \ge v_i(t,x) \ge C_{low}(r)\Psi^i_{low}(r;t,x)$$

with $C_{low}(r)\uparrow 1$, $C_{up}(r)\downarrow 1$. Note that the proof of Bramson (1983, Proposition 8.3 (c)) only uses probabilities of Brownian bridge. Using the same argument, we get that for all $i\in S$,

$$1 \le \frac{\Psi^i_{up}(r;t,x)}{\Psi^i_{low}(r;t,x)} \le \gamma(r)$$

with $\gamma(r)\downarrow 1$ as $r\to\infty$. Define

$$\Psi(r;t,x) := e^{\lambda^*(t-r)} h_i \int_{\mathbb{R}} \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} \mathbf{P}^h_{(m(t)+x,i)}\left(\mathcal{B}_{mid}\Big| X_{t-r}=y\right) \sum_{j=1}^d g_j v_j(r,y)\mathrm{d}y \tag{5.5}$$

with

$$\mathcal{B}_{mid} := \{X_s > n_{r,t}(t-s) \text{ for all } s\in[0,t-r]\} \tag{5.6}$$

and

$$n_{r,t}(s) := m(t)\frac{s-r}{t-r} + \sqrt{2\lambda^*}r\frac{t-s}{t-r}, \quad s\in[r,t]. \tag{5.7}$$

Then by Bramson (1983, (8.61), p. 144) or Ren et al. (2021, (2.26)),

$$\underline{\mathcal{M}}'_{r,t}(t-s) \le n_{r,t}(t-s) \le \overline{\mathcal{M}}^x_{r,t}(t-s), \quad \text{for } s\in[0,t-r]$$

when r and x are large enough. This yields that (see Bramson (1983, (8.62), p. 144))

$$\frac{1}{\gamma(r)} \le \frac{v_i(t,m(t)+x)}{\Psi^i_{mid}(r;t,x)} \le \gamma(r) \tag{5.8}$$

for $r\ge r_1, 8r<x\le\sqrt{t}$ with r_1 fixed. Therefore, to find the limit of $v_i(t,m(t)+x)$ as $t\to\infty$, we first get the limit of $\Psi^i_{mid}(r;t,x)$ as $t\to\infty$.

Step 1: In this step we study the limit of $\Psi^i_{mid}(r;t,x)$ as $t\to\infty$. Letting $y_1 = y-\sqrt{2\lambda^*}r$ and $x_1 = x - \frac{3}{2\sqrt{2\lambda^*}}\log t$, using Bramson (1983, Lemma 2.2 (a), p. 15), similar to Bramson (1983, (8.63) and (8.64), pp. 144–145), we have that

$$\begin{aligned}\Psi^i_{mid}(r;t,x) &= e^{\lambda^*(t-r)}h_i\int_{\sqrt{2\lambda^*}r}^\infty \frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}}\times\left(1-e^{-2xy_1/(t-r)}\right)\sum_{j=1}^d g_jv_j(r,y)\mathrm{d}y\\
&= e^{\lambda^*(t-r)}h_i\int_0^\infty \frac{e^{-\frac{(x_1-y_1+\sqrt{2\lambda^*}(t-r))^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}}\times\left(1-e^{-2xy_1/(t-r)}\right)\sum_{j=1}^d g_jv_j(r,y_1+\sqrt{2\lambda^*}r)\mathrm{d}y_1\\
&= h_i\int_0^\infty \frac{e^{-\frac{(x_1-y_1)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}}e^{-\sqrt{2\lambda^*}(x_1-y_1)}\times\left(1-e^{-2xy_1/(t-r)}\right)\sum_{j=1}^d g_jv_j(r,y_1+\sqrt{2\lambda^*}r)\mathrm{d}y_1\\
&= \sqrt{\frac{t^3}{2\pi(t-r)^3}}e^{-\sqrt{2\lambda^*}x}h_i\int_0^\infty e^{\sqrt{2\lambda^*}y_1}\\
&\quad\times\left(\sum_{j=1}^d g_jv_j(r,y_1+\sqrt{2\lambda^*}r)\right)e^{-\frac{(x_1-y_1)^2}{2(t-r)}}\times(t-r)\left(1-e^{-2xy_1/(t-r)}\right)\mathrm{d}y_1 \qquad (5.9)\\
&=: C^i_v(r;t,x)xe^{-\sqrt{2\lambda^*}x}.\end{aligned}$$

Therefore, for $r\geq r_1$ and $8r<x\leq\sqrt{t}$,

$$\frac{1}{\gamma(r)}C^i_v(r;t,x)xe^{-\sqrt{2\lambda^*}x}\leq v_i(t,m(t)+x)\leq\gamma(r)C^i_v(r;t,x)xe^{-\sqrt{2\lambda^*}x}. \qquad (5.10)$$

Now we fix r first and replace x by $x(t)$ and suppose that $x(t)\to x$ as $t\to\infty$. Let $x_1(t)=x(t)-\frac{3}{2\sqrt{2\lambda^*}}\log t$. Then we can easily see that for any fixed y_1,

$$e^{-\frac{(x_1(t)-y_1)^2}{2(t-r)}}(t-r)\left(1-e^{-2x(t)y_1/(t-r)}\right)\to 2xy_1, \quad \text{as } t\to\infty,$$

and that $e^{-\frac{(x_1(t)-y_1)^2}{2(t-r)}}(t-r)\left(1-e^{-2x(t)y_1/(t-r)}\right)\leq 2x(t)y_1\leq(2\sup_t x(t))\,y_1$. Note that, for any $j\in S$, by (3.9) and Markov's inequality,

$$\begin{aligned}&\int_0^\infty y_1e^{\sqrt{2\lambda^*}y_1}v_j(r,y_1+\sqrt{2\lambda^*}r)\mathrm{d}y_1\leq\int_0^\infty y_1e^{\sqrt{2\lambda^*}y_1}\times e^{\lambda^*r}h_j\mathbf{E}^h_{(y_1+\sqrt{2\lambda^*}r,j)}\left(\frac{v_{I_r}(0,X_r)}{h_{I_r}}\right)\mathrm{d}y_1\\
&\lesssim e^{\lambda^*r}\int_0^\infty y_1e^{\sqrt{2\lambda^*}y_1}\mathbf{P}_{y_1+\sqrt{2\lambda^*}r}\left(X_r\leq N_2\right)\mathrm{d}y_1\\
&= e^{\lambda^*r}\int_0^\infty y_1e^{\sqrt{2\lambda^*}y_1}\mathbf{P}_0\left(X_r\leq N_2-y_1-\sqrt{2\lambda^*}r\right)\mathrm{d}y_1\\
&\leq e^{\lambda^*r}\int_0^\infty y_1e^{\sqrt{2\lambda^*}y_1}\times e^{-2\sqrt{2\lambda^*}(y_1+\sqrt{2\lambda^*}r-N_2)}\mathbf{E}_0\left(e^{-2\sqrt{2\lambda^*}B_r}\right)\mathrm{d}y_1<\infty. \qquad (5.11)\end{aligned}$$

Therefore, using the dominated convergence theorem, letting $t\to\infty$ in (5.9), we get that when $x(t)\to x$,

$$\begin{aligned}\lim_{t\to\infty}\Psi^i_{mid}(r;t,x(t)) &= \sqrt{\frac{2}{\pi}}xe^{-\sqrt{2\lambda^*}x}h_i\int_0^\infty y_1e^{\sqrt{2\lambda^*}y_1}\left(\sum_{j=1}^d g_jv_j(r,y_1+\sqrt{2\lambda^*}r)\right)\mathrm{d}y_1\\
&=: h_iC_v(r)xe^{-\sqrt{2\lambda^*}x},\end{aligned}$$

or equivalently, by the definition of $C_v^i(r;t,x)$,

$$\lim_{t\to\infty} C_v^i(r;t,x(t)) \to h_i C_v(r), \tag{5.12}$$

where

$$C_v(r) := \sqrt{\frac{2}{\pi}} \int_0^\infty y_1 e^{\sqrt{2\lambda^*} y_1} \left(\sum_{j=1}^d g_j v_j(r, y_1 + \sqrt{2\lambda^*} r) \right) \mathrm{d}y_1. \tag{5.13}$$

Step 2: In this step we use the limit of $\Psi_{mid}^i(r;t,x)$ as $t\to\infty$ to get the limit of $v_i(t,m(t)+x)$ as $t\to\infty$. It is easy to see that for any $r>0$, $C_v(r)\in(0,\infty)$. Indeed, $C_v(r)<\infty$ follows from (5.11). On the other hand, by (5.1),

$$v_j(r, y_1 + \sqrt{2\lambda^*} r) \geq \mathbb{P}_{(0,j)} \left(M_r^{i_0} \geq y_1 + \sqrt{2\lambda^*} r - N_1 \right) > 0,$$

which implies that $C_v(r)>0$.

Therefore, for any $r\geq r_1$ and $x(t)\geq 8r$ with $x(t)\to x$, by (5.10) and (5.12), we get that as $t\to\infty$,

$$\frac{1}{\gamma(r)} \leq \frac{\liminf_{t\to\infty} v_i\left(t, m(t)+x(t)\right)}{h_i C_v(r) x e^{-\sqrt{2\lambda^*} x}} \leq \frac{\limsup_{t\to\infty} v_i\left(t, m(t)+x(t)\right)}{h_i C_v(r) x e^{-\sqrt{2\lambda^*} x}} \leq \gamma(r). \tag{5.14}$$

Now letting $x\to\infty$ in (5.14), we get

$$\begin{aligned} 0 &< \frac{C_v(r)}{\gamma(r)} \leq \liminf_{x\to\infty} \frac{\liminf_{t\to\infty} v_i\left(t, m(t)+x(t)\right)}{h_i x e^{-\sqrt{2\lambda^*} x}} \leq \limsup_{x\to\infty} \frac{\liminf_{t\to\infty} v_i\left(t, m(t)+x(t)\right)}{h_i x e^{-\sqrt{2\lambda^*} x}} \\ &\leq \liminf_{x\to\infty} \frac{\limsup_{t\to\infty} v_i\left(t, m(t)+x(t)\right)}{h_i x e^{-\sqrt{2\lambda^*} x}} \\ &\leq \limsup_{x\to\infty} \frac{\limsup_{t\to\infty} v_i\left(t, m(t)+x(t)\right)}{h_i x e^{-\sqrt{2\lambda^*} x}} \leq C_v(r)\gamma(r) < \infty. \end{aligned} \tag{5.15}$$

Letting $r\to\infty$, using the facts that $\gamma(r)\to 1$ and that the 4 quantities between $C_v(r)/\gamma(r)$ and $C_v(r)\gamma(r)$ in (5.15) are independent of r, we get

$$\begin{aligned} \lim_{r\to\infty} C_v(r) &= \lim_{x\to\infty} \frac{\liminf_{t\to\infty} v_i\left(t, m(t)+x(t)\right)}{h_i x e^{-\sqrt{2\lambda^*} x}} \\ &= \lim_{x\to\infty} \frac{\limsup_{t\to\infty} v_i\left(t, m(t)+x(t)\right)}{h_i x e^{-\sqrt{2\lambda^*} x}} =: C_v(\infty) \in (0,\infty), \end{aligned}$$

which implies that (1.13) holds. Now let $r=[x]/8$ and

$$\begin{aligned} Y_1(x) &:= \left(\frac{C_v(r)}{C_v(\infty)} \vee \frac{C_v(\infty)}{C_v(r)} \right) \gamma(r), \\ Y_2(t;n) &:= 2 \left(\sup_{x>0} \left(x e^{-\sqrt{2\lambda^*} x} \right) \right) \times \sup_{i\in S} \sup \left\{ \left| C_v^i(r;t,x) - h_i C_v(r) \right| : \ x\in[n,n+1) \right\}. \end{aligned}$$

Let $n_0\geq 8r_1$ be large enough such that $\gamma([x]/8)\leq 2$ for all $x\geq n_0$, then by (5.10), for any $n\geq n_0$ and $x\in[n,n+1)$, we have

$$\begin{aligned} & v_i(t, m(t)+x) \leq \gamma(r) C_v^i(r;t,x) x e^{-\sqrt{2\lambda^*} x} \\ &= \gamma(r) \left(C_v^i(r;t,x) - h_i C_v(r) \right) x e^{-\sqrt{2\lambda^*} x} + h_i \gamma(r) C_v(r) x e^{-\sqrt{2\lambda^*} x} \\ &\leq h_i C_v(\infty) x e^{-\sqrt{2\lambda^*} x} Y_1(x) + Y_2(t;n) \end{aligned}$$

and similarly,

$$v_i(t, m(t)+x) \geq h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} \frac{1}{Y_1(x)} - Y_2(t;n).$$

Note that uniformly for all $i \in S$, it holds that

$$Y_1(x) \to 1, \quad \text{as } x \to \infty, \qquad Y_2(t;n) \to 0, \quad \text{as } t \to \infty. \tag{5.16}$$

In conclusion, for all $n \geq n_0$, $x \in [n, n+1)$, $t \geq (n+1)^2$ and all $i \in S$,

$$\begin{aligned} h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} \frac{1}{Y_1(x)} - Y_2(t;n) &\leq v_i(t, m(t)+x) \\ &\leq h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} Y_1(x) + Y_2(t;n) \end{aligned} \tag{5.17}$$

with $Y_1(x), Y_2(t;n)$ satisfying (5.16).

Fix $s > 0, x \in \mathbb{R}$, let

$$x(t,s) := m(t+s) - m(t) + x - \sqrt{2\lambda^*}s = x - \frac{3}{2\sqrt{2\lambda^*}} \log\left(1 + \frac{s}{t}\right).$$

Then $m(t+s) + x = m(t) + x(t,s) + \sqrt{2\lambda^*}s$ and

$$\begin{aligned} 1 - v_i(t+s, m(t+s)+x) &= \mathbb{E}_{(m(t+s)+x,i)} \left(1 - \prod_{u \in Z(s)} \left(1 - v_{I_u(s)}\left(t, X_u(s)\right)\right)\right) \\ = \mathbb{E}_{(0,i)} &\left(1 - \prod_{u \in Z(s)} \left(1 - v_{I_u(s)}\left(t, X_u(s) + \sqrt{2\lambda^*}s + m(t) + x(t,s)\right)\right)\right). \end{aligned}$$

For any $1 > \delta > 0$, let $\varepsilon > 0$ be sufficient small such that for all $x \in (0, \varepsilon)$,

$$e^{-(1+\delta)x} \leq 1 - x \leq e^{-x}.$$

For this $\varepsilon > 0$, let n_0 be sufficient large such that when $x \geq n_0$, we have

$$h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} < \frac{\varepsilon}{2}, \quad \text{and} \quad \frac{1}{Y(x)},\ Y(x) \in [1-\delta, 1+\delta],$$

which implies that

$$h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} \max\left\{\frac{1}{Y(x)}, Y(x)\right\} < \varepsilon.$$

Thus, for any $\delta > 0$, there exists n_0 such that when $x \geq n_0$,

$$\begin{aligned} 1 - h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} \frac{1}{Y_1(x)} &\leq \exp\left\{-h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} \frac{1}{Y_1(x)}\right\} \\ &\leq \exp\left\{-(1-\delta) h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x}\right\}, \\ 1 - h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} Y_1(x) &\geq \exp\left\{-(1+\delta) h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x} Y_1(x)\right\} \\ &\geq \exp\left\{-(1+\delta)^2 h_i C_v(\infty) x e^{-\sqrt{2\lambda^*}x}\right\}. \end{aligned}$$

Since $M_s^- + \sqrt{2\lambda^*}s := \min_{u \in Z(s)} X_u(s) + \sqrt{2\lambda^*}s \to \infty$ (see Ren and Yang (2014, Theorem 4)), there exists $s(\omega)$ such that for $t \geq s \geq s(\omega)$,

$$\begin{aligned} \Delta_u(s,t) :&= X_u(s) + \sqrt{2\lambda^*}s + x(t,s) \\ &\geq X_u(s) + \sqrt{2\lambda^*}s + x - \frac{3}{2\sqrt{2\lambda^*}} \log 2 \geq n_0, \quad \forall u \in Z(s). \end{aligned}$$

It follows from (5.17) that when $t \ge s \ge s(\omega)$,

$$1 - v_{I_u(s)}\left(t, \Delta_u(s,t) + m(t)\right) \ge 1 - h_{I_u(s)} C_v(\infty) \Delta_u(s,t) e^{-\sqrt{2\lambda^*}\Delta_u(s,t)} Y_1(\Delta_u(s,t)) - Y_2(t; [\Delta_u(s,t)])$$

$$\ge \exp\left\{-(1+\delta)^2 h_{I_u(s)} C_v(\infty) \Delta_u(s,t) e^{-\sqrt{2\lambda^*}\Delta_u(s,t)}\right\} - Y_2(t; [\Delta_u(s,t)]),$$

and

$$1 - v_{I_u(s)}\left(t, \Delta_u(s,t) + m(t)\right) \le 1 - h_{I_u(s)} C_v(\infty) \Delta_u(s,t) e^{-\sqrt{2\lambda^*}\Delta_u(s,t)} Y_1(\Delta_u(s,t)) + Y_2(t; [\Delta_u(s,t)])$$

$$\le \exp\left\{-(1-\delta) h_{I_u(s)} C_v(\infty) \Delta_u(s,t) e^{-\sqrt{2\lambda^*}\Delta_u(s,t)}\right\} + Y_2(t; [\Delta_u(s,t)]).$$

Therefore, on the event that $\left\{M_s^- + \sqrt{2\lambda^*}s + x(t,s) \ge n_0\right\}$, it holds that

$$\begin{aligned}
&\prod_{u \in Z(s)} \left(\exp\left\{-(1-\delta) h_{I_u(s)} C_v(\infty) \Delta_u(s,t) e^{-\sqrt{2\lambda^*}\Delta_u(s,t)}\right\} - Y_2(t; [\Delta_u(s,t)])\right) \\
&\ge \prod_{u \in Z(s)} \left(1 - v_{I_u(s)}\left(t, X_u(s) + \sqrt{2\lambda^*}s + m(t) + x(t,s)\right)\right) \\
&\ge \prod_{u \in Z(s)} \left(\exp\left\{-(1+\delta)^2 h_{I_u(s)} C_v(\infty) \Delta_u(s,t) e^{-\sqrt{2\lambda^*}\Delta_u(s,t)}\right\} - Y_2(t; [\Delta_u(s,t)])\right).
\end{aligned} \tag{5.18}$$

Since $x(s,t) \to x$ as $t \to \infty$, we have $\Delta_u(s,t) \to X_u(s) + \sqrt{2\lambda^*}s + x$ as $t \to \infty$. Letting $t \to \infty$ in (5.18), we get from (5.16) that

$$\begin{aligned}
&\limsup_{t\to\infty} \left(1 - v_i(t, m(t) + x)\right) = \limsup_{t\to\infty} \left(1 - v_i(t+s, m(t+s) + x)\right) \\
&\le \mathbb{P}_{(0,i)}\left(M_s^- + \sqrt{2\lambda^*}s + x < n_0\right) \\
&\quad + \mathbb{E}_{(0,i)}\left(\exp\left\{-(1-\delta) C_v(\infty)\left(x W_{\sqrt{2\lambda^*}}(s) + M_{\sqrt{2\lambda^*}}(s)\right) e^{-\sqrt{2\lambda^*}x}\right\} 1_{\left\{M_s^- + \sqrt{2\lambda^*}s + x \ge n_0\right\}}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty} \left(1 - v_i(t, m(t) + x)\right) \\
&\ge \mathbb{E}_{(0,i)}\left(\exp\left\{-(1+\delta)^2 C_v(\infty)\left(x W_{\sqrt{2\lambda^*}}(s) + M_{\sqrt{2\lambda^*}}(s)\right) e^{-\sqrt{2\lambda^*}x}\right\} 1_{\left\{M_s^- + \sqrt{2\lambda^*}s + x \ge n_0\right\}}\right),
\end{aligned}$$

where $\{W_{\sqrt{2\lambda^*}}(s), s \ge 0\}$ is the additive martingale defined by (1.8), and $\{M_{\sqrt{2\lambda^*}}(s), s \ge 0\}$ is the derivative martingale defined by (1.9). By (1.10) and (1.11), letting $s \to \infty$ and noting that $\mathbb{P}_{(0,i)}\left(M_s^- + \sqrt{2\lambda^*}s + x \ge n_0\right) \to 1$ for every fixed x and n_0, we get that

$$\begin{aligned}
&\mathbb{E}_{(0,i)}\left(\exp\left\{-(1-\delta) C_v(\infty) M_{\sqrt{2\lambda^*}}(\infty)\right\}\right) \ge \limsup_{t\to\infty} \left(1 - v_i(t, m(t) + x)\right) \\
&\ge \liminf_{t\to\infty} \left(1 - v_i(t, m(t) + x)\right) \\
&\ge \mathbb{E}_{(0,i)}\left(\exp\left\{-(1+\delta)^2 C_v(\infty) M_{\sqrt{2\lambda^*}}(\infty)\right\}\right).
\end{aligned}$$

Letting $\delta \to 0$, we get the desired convergence. □

6. Extremal Process for multi-type branching Brownian motion

In this section, we study the asymptotic behavior of the extremal process of multi-type branching Brownian motion and prove Theorems 1.4 and 1.5.

Proposition 6.1. *For any $\phi \in \mathcal{C}_c^+(\mathbb{R}\times S)$ and $x\in\mathbb{R}$,*

$$\lim_{t\to\infty}\mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi(y+x,j)\mathcal{E}_t(\mathrm{d}y\mathrm{d}j)\right\}\right)=\mathbb{E}_{(0,i)}\left(\exp\left\{-C(\phi)M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}x}\right\}\right),$$

where

$$C(\phi):=\lim_{r\to\infty}\sqrt{\frac{2}{\pi}}\int_0^\infty ye^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_jv_j(r,y+\sqrt{2\lambda^*}r)\right)\mathrm{d}y\in(0,\infty)\tag{6.1}$$

with $\mathbf{v}$ *a solution of* (1.5) *with initial value* $v_j(0,y)=1-e^{-\phi(-y,j)}$.

Proof: For $L\in\mathbb{R}$, define

$$v_j(0,y;L):=1-\left(e^{-\phi(-y,j)}1_{\{-y\le L\}}\right),\tag{6.2}$$

then

$$\begin{aligned}v_i(t,x;L)&=1-\mathbb{E}_{(x,i)}\left(\prod_{u\in Z(t)}\left(1-v_{I_u(t)}(0,X_u(t);L)\right)\right)\\&=1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\sum_{u\in Z(t)}\phi\left(X_u(t)-x,I_u(t)\right)\right\}1_{\{M_t\le x+L\}}\right).\end{aligned}\tag{6.3}$$

For any fixed L, $v_j(0,y;L)$ satisfies (1.12). Therefore, by Theorem 1.1,

$$\lim_{t\to\infty}\left(1-v_i\left(t,m(t)+x;L\right)\right)=\mathbb{E}_{(0,i)}\left(\exp\left\{-C(\phi;L)M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}x}\right\}\right),$$

with $C(\phi;L)$ defined by

$$C(\phi;L)=\lim_{r\to\infty}\sqrt{\frac{2}{\pi}}\int_0^\infty ye^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_jv_j(r,y+\sqrt{2\lambda^*}r;L)\right)\mathrm{d}y.\tag{6.4}$$

Since

$$\begin{aligned}&0\le v_j(t,x;L)-v_j(t,x)\\&=\mathbb{E}_{(0,j)}\left(\exp\left\{-\sum_{u\in Z(t)}\phi\left(X_u(t)-x,I_u(t)\right)\right\}1_{\{M_t>x+L\}}\right)\le\mathbb{P}_{(0,j)}\left(M_t>x+L\right),\end{aligned}\tag{6.5}$$

we get that

$$\begin{aligned}H_1(r;L)-H_2(r;L) &:= \sqrt{\frac{2}{\pi}}\int_0^\infty ye^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_j v_j(r,y+\sqrt{2\lambda^*}r;L)\right)\mathrm{d}y\\
&\quad -\sqrt{\frac{2}{\pi}}\int_0^\infty ye^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_j \mathbb{P}_{(0,j)}\left(M_r>y+\sqrt{2\lambda^*}r+L\right)\right)\mathrm{d}y\\
&\le \sqrt{\frac{2}{\pi}}\int_0^\infty ye^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_j v_j(r,y+\sqrt{2\lambda^*}r)\right)\mathrm{d}y =: H(r)\\
&\le \sqrt{\frac{2}{\pi}}\int_0^\infty ye^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_j v_j(r,y+\sqrt{2\lambda^*}r;L)\right)\mathrm{d}y = H_1(r;L).\end{aligned}$$

Therefore,

$$H_1(r;L)-H_2(r;L)\le H(r)\le H_1(r;L). \tag{6.6}$$

Note that

$$H_2(r;L)\le e^{-\sqrt{2\lambda^*}L}\sqrt{\frac{2}{\pi}}\int_0^\infty ye^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_j\mathbb{P}_{(0,j)}\left(M_r>y+\sqrt{2\lambda^*}r\right)\right)\mathrm{d}y \overset{r\to\infty}{\longrightarrow} e^{-\sqrt{2\lambda^*}L}C_\infty$$

with C_∞ given in Corollary 1.2. Thus

$$\lim_{L\to+\infty}\limsup_{r\to\infty} H_2(r;L)=0.$$

Also note that $\lim_{r\to\infty} H_1(r;L)=C(\phi;L)$. Since $C(\phi;L)$ is positive and decreasing in L, letting $r\to\infty$ and then $L\to\infty$ in (6.6), we have

$$C(\phi):=\lim_{L\to\infty}C(\phi;L)\le\liminf_{r\to\infty}H(r)\le\limsup_{r\to\infty}H(r)\le\lim_{L\to\infty}C(\phi;L),$$

which implies that $\lim_{r\to\infty}H(r)=C(\phi)$.

Next, for any $\phi\in\mathcal{C}_c^+(\mathbb{R})$ with $\phi\neq 0$, there exist $\ell_0\in S$, $a_{\ell_0}<b_{\ell_0}$ and $c_0>0$ such that $\phi(y,\ell_0)\ge c_0$ for all $y\in[a_{\ell_0},b_{\ell_0}]$. Thus,

$$\begin{aligned}\mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi(y,j)\mathcal{E}_t(\mathrm{d}y\mathrm{d}j)\right\}\right) &\le e^{-c_0}\mathbb{P}_{(0,i)}\left(M_t^{\ell_0}-m(t)\in[a_{\ell_0},b_{\ell_0})\right)\\
&\quad+\left(1-\mathbb{P}_{(0,i)}\left(M_t^{\ell_0}-m(t)\in[a_{\ell_0},b_{\ell_0})\right)\right).\end{aligned}$$

It follows immediately from Corollary 1.3 that

$$\lim_{t\to\infty}\mathbb{P}_{(0,i)}\left(M_t^{\ell_0}-m(t)\in[a_{\ell_0},b_{\ell_0})\right)>0.$$

Thus $C(\phi)>0$. Hence we have shown that $C(\phi)>0$ when $\phi\in\mathcal{C}_c^+(\mathbb{R}\times S)$ and $\phi\neq 0$.

For any $x\in\mathbb{R}$, when L is large enough so that $x+L\ge 1$, by Proposition 4.2, there exists a constant C_0 such that

$$\begin{aligned}&1-v_i\left(t,m(t)+x;L\right)\le 1-v_i\left(t,m(t)+x\right)\\
&\le 1-v_i\left(t,m(t)+x;L\right)+\mathbb{P}_{(0,i)}\left(M_t>m(t)+x+L\right)\\
&\le 1-v_i\left(t,m(t)+x;L\right)+C_0(x+L)e^{-\sqrt{2\lambda^*}(x+L)}.\end{aligned}$$

Letting $t \to \infty$, we get

$$\mathbb{E}_{(0,i)}\left(\exp\left\{-C(\phi;L)M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}x}\right\}\right) \le \liminf_{t\to\infty}\left(1-v_i\left(t,m(t)+x\right)\right)$$
$$\le \limsup_{t\to\infty}\left(1-v_i\left(t,m(t)+x\right)\right)$$
$$\le \mathbb{E}_{(0,i)}\left(\exp\left\{-C(\phi;L)M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}x}\right\}\right) + C_0(x+L)e^{-\sqrt{2\lambda^*}(x+L)}.$$

Next, letting $L \to \infty$, we get the desired result. □

Corollary 6.2. *The point process $\mathcal{E}_t$ converges in distribution to a random measure $\mathcal{E}_\infty$, where the Laplace transform of $\mathcal{E}_\infty$ is given by*

$$\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi(y+x,j)\mathcal{E}_\infty(\mathrm{d}y\mathrm{d}j)\right\}\right) = \mathbb{E}_{(0,i)}\left(\exp\left\{-C(\phi)M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}x}\right\}\right)$$

with $C(\phi)$ given in (6.1).

Proof: Without loss of generality, we assume $x=0$, otherwise we may consider $\widehat{\phi}(\cdot,j)=\phi(x+\cdot,j)$. It suffices to prove the tightness for $\mathcal{E}_t$, which is equivalent to the tightness for $\int \phi(y,j)\mathcal{E}_t(\mathrm{d}y\mathrm{d}j)$. By Proposition 6.1, it suffices to show that $\lim_{\theta\downarrow 0} C(\theta\phi)=0$. Choose m_ϕ so that $\phi(y,j)=0$ for all $y<m_\phi$ and $j\in S$. Let $\|\|\phi\|_\infty := \sup_{x\in\mathbb{R},j\in S}|\phi(x,j)|$, then

$$\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \theta\phi(y,j)\mathcal{E}_t(\mathrm{d}y\mathrm{d}j)\right\}\right) \ge \mathbb{E}_{(0,i)}\left(\exp\left\{-\theta\|\phi\|_\infty\mathcal{E}_t\left((m_\phi,\infty)\times S\right)\right\}\right)$$
$$\ge e^{-\theta\|\phi\|_\infty N}\mathbb{P}_{(0,i)}\left(\mathcal{E}_t\left((m_\phi,\infty)\times S\right)\le N\right).$$

First letting $t\to\infty$, next $\theta\to 0$ and then $N\to\infty$, we only need to prove that for all $i\in S$,

$$\lim_{N\to\infty}\limsup_{t\to\infty}\mathbb{P}_{(0,i)}\left(\mathcal{E}_t\left((m_\phi,\infty)\times S\right)>N\right)=0. \tag{6.7}$$

Suppose that under $\{\mathcal{E}_t\left((m_\phi,\infty)\times S\right)>N\}$, $u_1,...,u_N\in\{u\in Z(t), X_u(t)-m(t)>m_\phi\}$, then

$$\mathbb{P}_{(0,i)}\left(\mathcal{E}_t\left((m_\phi,\infty)\times S\right)>N, M_{t+1}-m(t+1)\le n\right)$$
$$\le \mathbb{P}_{(0,i)}\left(\mathcal{E}_t\left((m_\phi,\infty)\times S\right)>N, \max_{1\le n\le N} M_1^{u_n}+X_{u_n}(t)-m(t+1)\le n\right)$$
$$\le \mathbb{P}_{(0,i)}\left(\mathcal{E}_t\left((m_\phi,\infty)\times S\right)>N, \max_{1\le n\le N} M_1^{u_n}+m(t)+m_\phi-m(t+1)\le n\right)$$
$$\le \left(\sup_{j\in S}\mathbb{P}_{(0,j)}\left(M_1+m(t)+m_\phi-m(t+1)\le n\right)\right)^N.$$

By Proposition 4.2, we have

$$\mathbb{P}_{(0,i)}\left(\mathcal{E}_t\left((m_\phi,\infty)\times S\right)>N\right)$$
$$\le \mathbb{P}_{(0,i)}\left(M_{t+1}-m(t+1)>n\right)+\left(\sup_{j\in S}\mathbb{P}_{(0,j)}\left(M_1+m(t)+m_\phi-m(t+1)\le n\right)\right)^N$$
$$\lesssim ne^{-\sqrt{2\lambda^*}n}+\left(\mathbf{P}_0\left(B_1+m(t)+m_\phi-m(t+1)\le n\right)\right)^N,\quad n,t,N\ge 1. \tag{6.8}$$

For every $n\ge 1$, letting $t\to\infty$ first and then $N\to\infty$ in (6.8), we get that

$$\lim_{N\to\infty}\limsup_{t\to\infty}\mathbb{P}_{(0,i)}\left(\mathcal{E}_t\left((m_\phi,\infty)\times S\right)>N\right)\lesssim ne^{-\sqrt{2\lambda^*}n}.$$

Letting $n\to\infty$, we get (6.7) and thus $\mathcal{E}_t$ converges in distribution to a random point process $\mathcal{E}_\infty$. □

Recall the definition of $\Psi^i_{mid}(r;t,x)$ defined in (5.5). By (5.9), we have

$$\Psi^i_*(r;t,x) := \Psi^i_{mid}\left(r;t,x+\frac{3}{2\sqrt{2\lambda^*}}\log t\right) = \sqrt{\frac{t^3}{2\pi(t-r)^3}}e^{-\sqrt{2\lambda^*}\left(x+\frac{3}{2\sqrt{2\lambda^*}}\log t\right)}h_i\int_0^\infty e^{\sqrt{2\lambda^*}y}$$
$$\times\left(\sum_{j=1}^d g_j v_j(r,y+\sqrt{2\lambda^*}r)\right)e^{-\frac{(x-y)^2}{2(t-r)}}\times(t-r)\left(1-e^{-2\left(x+\frac{3}{2\sqrt{2\lambda^*}}\log t\right)y/(t-r)}\right)\mathrm{d}y$$
$$=\frac{1}{\sqrt{2\pi(t-r)}}e^{-\sqrt{2\lambda^*}x}h_i\int_0^\infty e^{\sqrt{2\lambda^*}y}$$
$$\times\left(\sum_{j=1}^d g_j v_j(r,y+\sqrt{2\lambda^*}r)\right)e^{-\frac{(x-y)^2}{2(t-r)}}\left(1-e^{-2\left(x+\frac{3}{2\sqrt{2\lambda^*}}\log t\right)y/(t-r)}\right)\mathrm{d}y.$$

It follows from (5.8) that

$$\frac{1}{\gamma(r)}\le\frac{v_i(t,\sqrt{2\lambda^*}t+x)}{\Psi^i_*(r;t,x)}\le\gamma(r) \tag{6.9}$$

holds for $r\ge r_1, 8r-\frac{3}{2\sqrt{2\lambda^*}}\log t<x\le\sqrt{t}-\frac{3}{2\sqrt{2\lambda^*}}\log t$ with r_1 fixed.

Lemma 6.3. *Let* $\mathbf{v}$ *solve* (1.5) *with initial value satisfying* (1.12). *Then for any fixed* $x\in\mathbb{R}$,

$$\lim_{t\to\infty}\frac{t^{3/2}}{\frac{3}{2\sqrt{2\lambda^*}}\log t}\Psi^i_*(r;t,x)=h_iC_v(r)e^{-\sqrt{2\lambda^*}x},$$

where $C_v(r)$ *is given by* (5.13).

Proof: The proof is very similar to that of Arguin et al. (2013, Lemma 4.5) and we omit the details. □

Let $\mathbf{v}$ solve (1.5) with initial value satisfying (1.12). By Lemma 6.3 and (6.9), we have for every $x\in\mathbb{R}$,

$$\lim_{t\to\infty}\frac{t^{3/2}}{\frac{3}{2\sqrt{2\lambda^*}}\log t}v_i(t,\sqrt{2\lambda^*}t+x)=h_iC_v(\infty)e^{-\sqrt{2\lambda^*}x}, \tag{6.10}$$

where $C_v(\infty)=\lim_{r\to\infty}C_v(r)$, given by (1.13). Hence for every $x\in\mathbb{R}$,

$$\lim_{t\to\infty}\frac{t^{3/2}}{\frac{3}{2\sqrt{2\lambda^*}}\log t}\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+x\right)=h_iC_\infty e^{-\sqrt{2\lambda^*}x}, \tag{6.11}$$

with C_∞ defined in (1.14). Now we extend (6.10) to the case $v_i(0,y)=1-e^{-\phi(-y,i)}$:

Lemma 6.4. *For any* $\phi\in\mathcal{C}_c^+(\mathbb{R}\times S)$, *let* $\mathbf{v}$ *solve* (1.5) *with* $v_i(0,y)=1-e^{-\phi(-y,i)}$, *then*

$$\lim_{t\to\infty}\frac{t^{3/2}}{\frac{3}{2\sqrt{2\lambda^*}}\log t}v_i(t,\sqrt{2\lambda^*}t+x)=h_iC(\phi)e^{-\sqrt{2\lambda^*}x}$$

with $C(\phi)$ *defined in* (6.1).

Proof: Let $v_i(t,x;L)$ solves (1.5) with initial value (6.2). By (6.10),

$$\lim_{t\to\infty}\frac{t^{3/2}}{\frac{3}{2\sqrt{2\lambda^*}}\log t}v_i(t,\sqrt{2\lambda^*}t+x;L)=h_iC(\phi;L)e^{-\sqrt{2\lambda^*}x}$$

with $C(\phi;L)$ defined in (6.4). By (6.5) and Proposition 4.2, we have

$$\begin{aligned}
&0 \le v_i(t,\sqrt{2\lambda^*}t+x;L) - v_i(t,\sqrt{2\lambda^*}t+x) \\
&\le \mathbb{P}_{(0,i)}\left(M_t > \sqrt{2\lambda^*}t+x+L\right) \lesssim \left(\frac{3}{2\sqrt{2\lambda^*}}\log t + x + L\right) e^{-\sqrt{2\lambda^*}\left(\frac{3}{2\sqrt{2\lambda^*}}\log t+x+L\right)}.
\end{aligned}$$

Note that $C(\phi) = \lim_{L\to\infty} C(\phi;L)$. Letting $t\to\infty$ first and then $L\to+\infty$, we arrive at the desired conclusion. □

Define

$$\overline{\mathcal{E}}_t := \sum_{u\in Z(t)} \delta_{(X_u(t)-\sqrt{2\lambda^*}t, I_u(t))}, \quad \overline{\mathcal{E}}_t - z := \sum_{u\in Z(t)} \delta_{(X_u(t)-\sqrt{2\lambda^*}t-z, I_u(t))},$$

Proposition 6.5. *For any $z\in\mathbb{R}$ and $i\in S$, under $\mathbb{P}_{(0,i)}\left(\cdot\middle|M_t > \sqrt{2\lambda^*}t+z\right)$,*

$$\left(\overline{\mathcal{E}}_t - z, M_t - \sqrt{2\lambda^*}t - z\right)$$

converges in distribution to a limit $(\overline{\mathcal{E}}_\infty, Y)$ independent of z and i, where $\overline{\mathcal{E}}_\infty$ is a point process, Y is an exponential random variable with parameter $\sqrt{2\lambda^*}$ and

$$\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi(y,j)\overline{\mathcal{E}}_\infty(\mathrm{d}y\mathrm{d}j)\right\}; Y > x\right) = \frac{\widetilde{C}(\phi,x)}{C_\infty} - \frac{C(\phi)}{C_\infty}, \tag{6.12}$$

where $C(\phi)$ is given by (6.1), C_∞ is given by (1.13), and

$$\widetilde{C}(\phi,x) := \lim_{r\to\infty}\sqrt{\frac{2}{\pi}}\int_0^\infty y e^{\sqrt{2\lambda^*}y}\left(\sum_{j=1}^d g_j(v_1)_j(r,y+\sqrt{2\lambda^*}r)\right)\mathrm{d}y \tag{6.13}$$

with $\mathbf{v}_1$ being a solution of (1.5) with initial value

$$(v_1)_i(0,y) = 1 - e^{-\phi(-y,i)}1_{\{-y\le x\}}, \quad i\in S.$$

The proof is similar to Ren et al. (2021, Proposition 3.4) and is postponed to Subsection.

Proof of Theorem 1.4: Define $\mathcal{D} := \overline{\mathcal{E}}_\infty - Y$. By Proposition 6.5 and Arguin et al. (2013, Lemma 4.13), also note that $\mathcal{D}_t = (\overline{\mathcal{E}}_t - z) - \left(M_t - \sqrt{2\lambda^*}t - z\right)$, we get that under $\mathbb{P}_{(0,i)}\left(\cdot\middle|M_t > \sqrt{2\lambda^*}+z\right)$, $\mathcal{D}_t$ converges in distribution to $\mathcal{D}$. Also, for all $x>0$,

$$\begin{aligned}
&\mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi(y,j)\mathcal{D}_t(\mathrm{d}y\mathrm{d}j)\right\}; M_t > \sqrt{2\lambda^*}t+z+x\middle|M_t > \sqrt{2\lambda^*}t+z\right) \\
&= \mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi(y-M_t+\sqrt{2\lambda^*}t+z+x,j)\overline{\mathcal{E}}_t(\mathrm{d}y\mathrm{d}j)\right\}\middle|M_t > \sqrt{2\lambda^*}t+z+x\right) \\
&\quad\times\mathbb{P}_{(0,i)}\left(M_t > \sqrt{2\lambda^*}t+z+x\middle|M_t > \sqrt{2\lambda^*}t+z\right) \\
&\to \mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi(y-Y,j)\overline{\mathcal{E}}_\infty(\mathrm{d}y\mathrm{d}j)\right\}\right)\mathbb{P}_{(0,i)}(Y>x) \\
&= \mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi(y,j)\mathcal{D}(\mathrm{d}y\mathrm{d}j)\right\}\right)\mathbb{P}_{(0,i)}(Y>x).
\end{aligned}$$

The desired result follows. □

Proof of Theorem 1.5: By Proposition 6.1 and Corollary 6.2. We only need to show that for any $\phi \in \mathcal{C}_c^+(\mathbb{R}\times S)$,

$$\mathbb{E}_{(0,i)}\Big(\exp\Big\{-C(\phi)M_{\sqrt{2\lambda^*}}(\infty)\Big\}\Big) = \mathbb{E}_{(0,i)}\Big(\exp\Big\{-\sum_{k,n}\phi\big(b_k+\Delta_n^{(k)},\varsigma_n^{(k)}\big)\Big\}\Big).$$

Note that by Campbell's formula,

$$\begin{aligned}
&\mathbb{E}_{(0,i)}\Big(\exp\Big\{-\sum_{k,n}\phi\big(b_k+\Delta_n^{(k)},\varsigma_n^{(k)}\big)\Big\}\Big)\\
&= \mathbb{E}_{(0,i)}\Big(\exp\Big\{-\sum_{k}\int \phi\left(b_k+y,j\right)\mathcal{D}^{(k)}\left(\mathrm{d}y\mathrm{d}j\right)\Big\}\Big)\\
&= \mathbb{E}_{(0,i)}\Big(\prod_k \mathbb{E}_{(0,i)}\Big(\exp\Big\{-\int \phi\left(z+y,j\right)\mathcal{D}\left(\mathrm{d}y\mathrm{d}j\right)\Big\}\Big)\Big|_{z=b_k}\Big)\\
&= \mathbb{E}_{(0,i)}\Big(\exp\Big\{-\int_{\mathbb{R}}\Big(1-\mathbb{E}_{(0,i)}\Big(\exp\Big\{-\int \phi\left(z+y,j\right)\mathcal{D}\left(\mathrm{d}y\mathrm{d}j\right)\Big\}\Big)\Big)C_\infty M_{\sqrt{2\lambda^*}}(\infty)\sqrt{2\lambda^*}e^{-\sqrt{2\lambda^*}z}\mathrm{d}z\Big\}\Big).
\end{aligned}$$

It suffices to show that for every $\phi \in \mathcal{C}_c^+(\mathbb{R}\times S)$,

$$C(\phi) = C_\infty \int_{\mathbb{R}}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi\left(z+y,j\right)\mathcal{D}\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)\right)\sqrt{2\lambda^*}e^{-\sqrt{2\lambda^*}z}\mathrm{d}z. \tag{6.14}$$

Suppose that $\phi(y,j)=0$ for all $y\le m_\phi$ and $j\in S$. Recalling that Y is an exponential random variable with parameter $\sqrt{2\lambda^*}$ and $\overline{\mathcal{E}}_\infty = \mathcal{D}+Y$, we get that

$$\begin{aligned}
&\int_{\mathbb{R}}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi\left(z+y,j\right)\mathcal{D}\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)\right)\sqrt{2\lambda^*}e^{-\sqrt{2\lambda^*}z}\mathrm{d}z\\
&= \int_{m_\phi}^{\infty}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi\left(z+y,j\right)\mathcal{D}\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)\right)\sqrt{2\lambda^*}e^{-\sqrt{2\lambda^*}z}\mathrm{d}z\\
&= e^{-\sqrt{2\lambda^*}m_\phi}\int_0^{\infty}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi\left(z+y+m_\phi,j\right)\mathcal{D}\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)\right)\sqrt{2\lambda^*}e^{-\sqrt{2\lambda^*}z}\mathrm{d}z\\
&= e^{-\sqrt{2\lambda^*}m_\phi}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi\left(Y+y+m_\phi,j\right)\mathcal{D}\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)\right)\\
&= e^{-\sqrt{2\lambda^*}m_\phi}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi\left(y+m_\phi,j\right)\overline{\mathcal{E}}_\infty\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)\right).
\end{aligned} \tag{6.15}$$

Applying Proposition 6.5 with $z=m_\phi$, we get

$$\begin{aligned}
&e^{-\sqrt{2\lambda^*}m_\phi}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi\left(y+m_\phi,j\right)\overline{\mathcal{E}}_\infty\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)\right)\\
&= e^{-\sqrt{2\lambda^*}m_\phi}\lim_{t\to\infty}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int \phi\left(y+m_\phi,j\right)\left(\overline{\mathcal{E}}_t-m_\phi\right)\left(\mathrm{d}y\mathrm{d}j\right)\right\}\Big| M_t>\sqrt{2\lambda^*}t+m_\phi\right)\right)\\
&= e^{-\sqrt{2\lambda^*}m_\phi}\lim_{t\to\infty}\frac{\mathbb{E}_{(0,i)}\left(1-\exp\left\{-\int \phi\left(y,j\right)\overline{\mathcal{E}}_t\left(\mathrm{d}y\mathrm{d}j\right)\right\}; M_t>\sqrt{2\lambda^*}t+m_\phi\right)}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+m_\phi\right)}\\
&= e^{-\sqrt{2\lambda^*}m_\phi}\lim_{t\to\infty}\frac{\mathbb{E}_{(0,i)}\left(1-\exp\left\{-\int \phi\left(y,j\right)\overline{\mathcal{E}}_t\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+m_\phi\right)}.
\end{aligned} \tag{6.16}$$

By (6.11),

$$e^{-\sqrt{2\lambda^*}m_\phi}\lim_{t\to\infty}\frac{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t\right)}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+m_\phi\right)}=1.$$

Therefore, by the probabilistic representation of $(v_2)_i(t,\sqrt{2\lambda^*}t)$ given by (1.6), we continue (6.16) to obtain

$$\begin{aligned}
&e^{-\sqrt{2\lambda^*}m_\phi}\left(1-\mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi\left(y+m_\phi,j\right)\overline{\mathcal{E}}_\infty\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)\right)\\
&=\lim_{t\to\infty}\frac{\mathbb{E}_{(0,i)}\left(1-\exp\left\{-\int\phi\left(y,j\right)\overline{\mathcal{E}}_t\left(\mathrm{d}y\mathrm{d}j\right)\right\}\right)}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t\right)}\\
&=\lim_{t\to\infty}\frac{1}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t\right)}(v_2)_i(t,\sqrt{2\lambda^*}t)=\frac{C(\phi)}{C_\infty},
\end{aligned}\tag{6.17}$$

where $\mathbf{v}_2$ solves (1.5) whose initial value is defined in (7.37) and in the last equality above we used (7.38). Combining (6.15) and (6.17), we get (6.14). The proof is now complete. □

7. Proofs of auxiliary results

7.1. *Proof of Lemma 3.1.*

Proof of Lemma 3.1: We only prove the case for $(1-\rho_i)\varphi_i^{NL}$. For any $\mathbf{v}\in[0,1]^d$, let $F(r):=(1-\rho_i)\varphi_i^{NL}(r\mathbf{v})$ for $r\in[0,1]$, then there exists $\theta\in[0,1]$ such that

$$(1-\rho_i)\varphi_i^{NL}(\mathbf{v})=F(1)-F(0)=F'(\theta)=(1-\rho_i)\nabla\varphi_i^{NL}(\theta\mathbf{v})\cdot\mathbf{v}.$$

Let $J(i):=\{j\in S:j\neq i,\mu_{i,j}>0\}$, then for any $j\notin J(i)$, $p_{\mathbf{k}}(i)=0$ for any $\mathbf{k}\in\mathbb{N}^d$ with $k_j>0$. Therefore, by the trivial inequalities

$$1-\prod_{j\neq i}x_j\le\sum_{j\neq i}(1-x_j),\quad x_j\in[0,1],j\neq i$$

and

$$1-(1-x)^k\le k^{\alpha_0}x^{\alpha_0},\quad k\ge1,x\in[0,1],$$

we have

$$\begin{aligned}
&(1-\rho_i)\left|\varphi_i^{NL}(\theta\mathbf{v})-\nabla\varphi_i^{NL}(0)\cdot\mathbf{v}\right|\le\sum_{\ell\neq i}v_\ell\left|\sum_{\mathbf{k}\in\mathbb{N}^d:k_i=0}k_\ell p_{\mathbf{k}}(i)\prod_{j\neq i}(1-\theta v_j)^{k_j-\delta_{j,\ell}}-\mu_{i,\ell}\right|\\
&=\sum_{\ell\in J(i)}v_\ell\left(\sum_{\mathbf{k}\in\mathbb{N}^d:k_i=0}k_\ell p_{\mathbf{k}}(i)-\sum_{\mathbf{k}\in\mathbb{N}^d:k_i=0}k_\ell p_{\mathbf{k}}(i)\prod_{j\neq i}(1-\theta v_j)^{k_j-\delta_{j,\ell}}\right)\\
&\le\sum_{\ell\in J(i)}v_\ell\sum_{\mathbf{k}\in\mathbb{N}^d:k_i=0}k_\ell p_{\mathbf{k}}(i)\left(1-\prod_{j\neq i}(1-\theta v_j)^{k_j}\right)\le\sum_{\ell\in J(i)}v_\ell\sum_{\mathbf{k}\in\mathbb{N}^d}k_\ell p_{\mathbf{k}}(i)\sum_{j\neq i}\left(1-(1-\theta v_j)^{k_j}\right)\\
&\le\sum_{\ell\in J(i)}v_\ell\sum_{\mathbf{k}\in\mathbb{N}^d}k_\ell p_{\mathbf{k}}(i)\sum_{j\neq i}k_j^{\alpha_0}v_j^{\alpha_0}\le\frac{1}{\min_{\ell\in J(i)}\mu_{i,\ell}}\sum_{\ell\in J(i)}\mu_{i,\ell}v_\ell\sum_{\mathbf{k}\in\mathbb{N}^d}k_\ell p_{\mathbf{k}}(i)\sum_{j=1}^d k_j^{\alpha_0}\cdot\|\mathbf{v}\|^{\alpha_0}\\
&=:\Gamma(i)\left((1-\rho_i)\nabla\varphi_i(0)\cdot\mathbf{v}\right)\|\mathbf{v}\|^{\alpha_0},
\end{aligned}$$

where we used (1.4) at the end of the display above. Thus (3.13) is valid. □

7.2. *Proof of Lemma 4.1.*

Proof of Lemma 4.1: Using $(\inf_{s\le t} B_s,\ \mathbf{P}_0) \overset{\mathrm{d}}{=} (-|B_t|,\ \mathbf{P}_0)$, we can easily get

$$\mathbf{P}_0\left(B_s \ge -y,\ s\le t\right) = \mathbf{P}_0\left(|B_t|\le y\right) \lesssim \frac{y\wedge\sqrt{t}}{\sqrt{t}}, \quad y,t>0. \tag{7.1}$$

Next, we prove that, for any $K_1\in\mathbb{R}$ and $\alpha\in(0,1/2)$, it holds that

$$\mathbf{P}_0\left(B_s \ge -y + K_1 s^{\alpha},\ s\le t\right) \lesssim \frac{y\wedge\sqrt{t}}{\sqrt{t}}, \quad y,t\ge 1. \tag{7.2}$$

If $K_1>0$, then by (7.1),

$$\mathbf{P}_0\left(B_s \ge -y + K_1 s^{\alpha},\ s\le t\right) \le \mathbf{P}_0\left(B_s \ge -y,\ s\le t\right) \lesssim \frac{y\wedge\sqrt{t}}{\sqrt{t}}, \quad y,t\ge 1.$$

If $K_1<0$, then by Mallein (2015a, Lemma 3.6), when $y\le\sqrt{t}$, it holds that

$$\mathbf{P}_0\left(B_s \ge -y + K_1 s^{\alpha},\ s\le t\right) \le \mathbf{P}_0\left(B_j \ge -y - |K_1| j^{\alpha},\ j=1,2,...[t]\right) \lesssim \frac{y}{\sqrt{[t]}} \lesssim \frac{y}{\sqrt{t}}, \quad t,y\ge 1.$$

When $y\ge\sqrt{t}$, we use the trivial upper bound 1.

Now we prove the desired result. When $t\le 3$, we use the trivial upper-bound 1. When $t>3$, by the Markov property at time $t/3$ and (7.2),

$$\begin{aligned}
&\mathbf{P}_0\big(B_s \ge -y + f(s), s\le t,\ B_t + y - f(t)\in[z,z+1]\big)\\
&\quad\le \mathbf{P}_0\left(B_s \ge -y - K s^{\alpha}, s\le \frac{t}{3}\right)\cdot\\
&\qquad\quad \sup_{x\in\mathbb{R}} \mathbf{P}_x\left(B_s \ge -y + f\left(s+\frac{t}{3}\right), s\le \frac{2t}{3},\ B_{2t/3} + y - f(t)\in[z,z+1]\right)\\
&\quad\lesssim \frac{y\wedge\sqrt{t}}{\sqrt{t}}\cdot \sup_{x\in\mathbb{R}} \mathbf{P}_x\left(B_s \ge -y + f\left(s+\frac{t}{3}\right), s\le \frac{2t}{3},\ B_{2t/3} + y - f(t)\in[z,z+1]\right), \quad y,z\ge 1.
\end{aligned}$$

For any $x\in\mathbb{R}$,

$$\begin{aligned}
&\mathbf{P}_x\left(B_s \ge -y + f\left(s+\frac{t}{3}\right), s\le \frac{2t}{3},\ B_{2t/3} + y - f(t)\in[z,z+1]\right)\\
&\le \mathbf{P}_x\left(B_s - B_{2t/3} \ge -(z+1) + f\left(s+\frac{t}{3}\right) - f(t), \frac{t}{3}\le s\le \frac{2t}{3},\ B_0 - B_{2t/3}\in[h-1,h]\right)\\
&= \mathbf{P}_0\left(\widetilde{B}_s \ge -(z+1) + f\left(t-s\right) - f(t), s\le \frac{t}{3},\ \widetilde{B}_{2t/3}\in[h-1,h]\right)
\end{aligned}$$

with $h = x + y - z - f(t)$ and $\widetilde{B}_s := B_{2t/3-s} - B_{2t/3}$ being still a Brownian motion starting from 0. By the Markov property of $\widetilde{B}$ at time $t/3$ and (7.2), we get that

$$\begin{aligned}
&\mathbf{P}_0\left(\widetilde{B}_s \geq -(z+1) + f(t-s) - f(t), s \leq \frac{t}{3},\ \widetilde{B}_{2t/3} \in [h-1, h]\right) \\
&\leq \mathbf{P}_0\left(\widetilde{B}_s \geq -(z+1) - Ks^{\alpha}, s \leq \frac{t}{3}\right) \cdot \sup_{x' \in \mathbb{R}} \mathbf{P}_{x'}\left(\widetilde{B}_{t/3} \in [h-1, h]\right) \\
&\lesssim \frac{(z+1) \wedge \sqrt{t}}{\sqrt{t}} \cdot \sup_{x' \in \mathbb{R}} \int_{h-1}^{h} \frac{e^{(y'-x')^2/(t/3)}}{\sqrt{2\pi t/3}} \mathrm{d}y' \\
&\leq \frac{(2z) \wedge \sqrt{t}}{\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi t/3}} \lesssim \frac{z \wedge \sqrt{t}}{t}.
\end{aligned}$$

Therefore, the desired result is valid. □

7.3. *Lower bound for* $\mathbb{E}_{(0,i)}\left(\mathcal{A}^{t,y}(i')\right)$. In this subsection, we adapt some ideas from Bramson et al. (2016) to prove (4.11).

Proof of (4.11)*:* Since $\lim_{t\to\infty} \widehat{\mathbb{P}}_{(0,i)}\left(I_\xi(t) = i'\right) = g_{i'} h_{i'} > 0$ for all $i, i' \in S$, we have

$$\inf_{i,i' \in S} \inf_{t>1} \widehat{\mathbb{P}}_{(0,i)}\left(I_\xi(t) = i'\right) > 0.$$

Therefore, for $t > 1$, by Proposition 2.1, and the independence of I_ξ and X_ξ, we get that

$$\begin{aligned}
\mathbb{E}_{(0,i)}\left(\mathcal{A}^{t,y}(i')\right) &= e^{\lambda^* t} \widehat{\mathbb{E}}_{(0,i)}\left(\frac{h_i}{h_{I_\xi(t)}} 1_{\left\{X_\xi(s) \leq f_s^{t,y}, s \leq t,\ X_\xi(t) \geq f_t^{t,y} - 4\right\}} 1_{\{I_\xi(t) = i'\}}\right) \\
&\gtrsim e^{\lambda^* t} \mathbf{P}_0\left(B_s \leq f_s^{t,y}, s \leq t,\ B_t \geq f_t^{t,y} - 4\right). \qquad (7.3)
\end{aligned}$$

We first show that for all $4\sqrt{t} - 3 \geq y \geq 1, t \geq 1$,

$$e^{\lambda^* t} \mathbf{P}_0\left(B_s \leq y + \frac{s}{t} m(t),\ s \leq t, \quad B_t \geq y + m(t) - 1\right) \gtrsim y e^{-\sqrt{2\lambda^*} y}. \qquad (7.4)$$

Let $q_t := m(t)/t$. Taking $\lambda = q_t$ in (4.6), we get that for all $t, y \geq 1$,

$$\begin{aligned}
&e^{\lambda^* t} \mathbf{P}_0\left(B_s \leq y + \frac{s}{t} m(t),\ s \leq t, \quad B_t \geq y + m(t) - 1\right) \\
&= e^{\lambda^* t} \mathbf{E}_0^{q_t}\left(e^{-q_t B_t + q_t^2 t/2}; B_s \leq y + q_t s,\ s \leq t, \quad B_t \geq y + q_t t - 1\right) \\
&\geq e^{\lambda^* t} e^{-q_t(y + q_t t) + q_t^2 t/2} \mathbf{P}_0\left(B_s \geq -y,\ s \leq t, \quad B_t \leq -y + 1\right) \\
&\gtrsim e^{-\sqrt{2\lambda^*} y} t^{3/2} \mathbf{P}_0\left(B_s \geq -y\ s \leq t;\ B_t \leq -y + 1\right) = e^{-\sqrt{2\lambda^*} y} t^{3/2} \mathbb{E}_{\mathbb{Q}_y}\left(\frac{y}{R_t}; R_t \leq 1\right),
\end{aligned}$$

where $(R_t, \mathbb{Q}_y)$ is a Bessel-3 process starting from y, and in the last equation we use the following well-known change-of-measure

$$\left.\frac{\mathrm{d}\mathbb{Q}_y}{\mathrm{d}\mathbf{P}_0}\right|_{\sigma(B_s, s \leq t)} = \frac{B_t + y}{y} 1_{\{B_s + y \geq 0,\ s \leq t\}},$$

here $(B_t + y,\ \mathbb{Q}_y)$ is equal in law to $(R_t, \mathbb{Q}_y)$. The density of R_t under $\mathbb{Q}_y$ is given by

$$\frac{x}{y\sqrt{2\pi t}} e^{-(x-y)^2/(2t)}\left(1 - e^{-2xy/t}\right) 1_{\{x>0\}}.$$

Note that $1-e^{-x}\gtrsim x, 0\le x\le 8$, and that $0\le 2xy/t\le 8\sqrt{t}/t\le 8$ and $(x-y)^2/(2t)\le (4\sqrt{t})^2/(2t)\lesssim 1$ for all $0\le x\le 1, 1\le y\le 4\sqrt{t}-3<4\sqrt{t}$ and $t\ge 1$. Thus,

$$\begin{aligned}
&e^{-\sqrt{2\lambda^*}y}t^{3/2}\mathbb{E}_{\mathbb{Q}_y}\left(\frac{y}{R_t};R_t\le 1\right)=ye^{-\sqrt{2\lambda^*}y}t^{3/2}\int_0^1\frac{1}{y\sqrt{2\pi t}}e^{-(x-y)^2/(2t)}(1-e^{-2xy/t})\mathrm{d}x\\
&\gtrsim ye^{-\sqrt{2\lambda^*}y}t^{3/2}\int_0^1\frac{1}{y\sqrt{2\pi t}}\frac{2xy}{t}\mathrm{d}x\\
&\gtrsim ye^{-\sqrt{2\lambda^*}y}\int_0^1 x\mathrm{d}x\gtrsim ye^{-\sqrt{2\lambda^*}y},\quad 1\le y\le 4\sqrt{t}-3,\ t\ge 1.
\end{aligned}$$

Thus (7.4) is true.

Since, for fixed t,

$$\frac{s}{t}m(t)+y-\left(f_s^{t,y}+3\right)=\frac{3}{2\sqrt{2\lambda^*}}\left(\log\left(\frac{t+1}{t-s+1}\right)-\frac{s}{t}\log t\right)-3=:G(s),$$

we see that

$$G'(s)=\frac{3}{2\sqrt{2\lambda^*}}\left(\frac{1}{t-s+1}-\frac{\log t}{t}\right)=0\iff s=t+1-\frac{t}{\log t}.$$

Since $G(0)=0$, $G'(0)<0$ and $G'(t)>0$, $t\ge e$, we have for $t\ge e$ and $s\le t$,

$$G(s)\le G(t)=-3\frac{3}{2\sqrt{2\lambda^*}}\log\left(1+\frac{1}{t}\right)\le 0.$$

Thus

$$\left\{B_s\le y+\frac{s}{t}m(t),\ s\le t\right\}\subset\left\{B_s\le f_s^{t,y}+3,\quad s\le t\right\}. \tag{7.5}$$

On the other hand, since

$$y+m(t)-1-\left(f_t^{t,y}+3-4\right)=\frac{3}{2\sqrt{2\lambda^*}}\log\left(1+\frac{1}{t}\right)\ge 0,$$

we have that

$$\{B_t\ge y+m(t)-1\}\subset\left\{B_t\ge f_t^{t,y}+3-4\right\}. \tag{7.6}$$

Combining (7.5) and (7.6), and noting that $f_s^{t,y}+3=f_s^{t,y+3}$, we get that

$$\mathbf{P}_0\left(B_s\le y+\frac{s}{t}m(t),\ s\le t,\quad B_t\ge y+m(t)-1\right)\le\mathbf{P}_0\left(B_s\le f_s^{t,y+3},s\le t,\ B_t\ge f_t^{t,y+3}-4\right). \tag{7.7}$$

By (7.3), (7.4), and (7.7),

$$\mathbb{E}_{(0,i)}\left(\mathcal{A}^{t,y}(i')\right)\gtrsim(y-3)e^{-\sqrt{2\lambda^*}(y-3)}\gtrsim ye^{-\sqrt{2\lambda^*}y},\quad i,i'\in S,\quad t\ge 1,\quad 4\le y\le 4\sqrt{t},$$

which implies (4.11). □

7.4. *Upper bound for* $\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0}1_{\Upsilon(\xi)}\right)$. In this subsection, we prove (4.15). The proof is divided into two steps.

Proof of (4.15)*:* **Step (i)** Let $\mathcal{G}_t:=\sigma\left(X_\xi(s),I_\xi(s),s\le t\right)$ and let

$$\mathbf{A}\left(I_\xi(\tau_\ell-)\right)=\left(A_1\left(I_\xi(\tau_\ell-)\right),...,A_d\left(I_\xi(\tau_\ell-)\right)\right)^T.$$

By the trivial inequality $\left(\sum_{j=1}^{k} x_j\right)^{\alpha_0} \le \sum_{j=1}^{k} x_j^{\alpha_0}$, on the event $\Upsilon(\xi)$,

$$\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0} \middle| \mathcal{G}_t \cap \left(\mathbf{A}\left(I_\xi(\tau_\ell-)\right), \ell \le K_t\right)\right) \le 1 + \sum_{\ell=1}^{K_t}\sum_{j=1}^{d}\left(A_j\left(I_\xi(\tau_\ell-)\right) - \delta_{j,I_\xi(\tau_\ell)}\right)^{\alpha_0} \times \left\{\mathbb{E}_{(X_\xi(\tau_\ell),j)}\left(\#\left\{u \in Z(t-\tau_\ell):\ \forall s \le t-\tau_\ell, X_u(s) \le f^{t,y}_{s+\tau_\ell}, X_u(t-\tau_\ell) \ge f^{t,y}_t - 4\right\}\right)\right\}^{\alpha_0}. \tag{7.8}$$

Let $z = X_\xi(\tau_\ell) \le f^{t,y}_{\tau_\ell}$, $z' := f^{t,y}_{\tau_\ell} - z \ge 0$ and $r = t - \tau_\ell$, then

$$\begin{aligned}
&\mathbb{E}_{(z,j)}\left(\#\left\{u \in Z(r):\ \forall s \le r, X_u(s) \le f^{t,y}_{s+\tau_\ell}, X_u(r) \ge f^{t,y}_t - 4\right\}\right)\\
&\lesssim e^{\lambda^* r}\mathbf{P}_0\left(B_s \le f^{t,y}_{s+\tau_\ell} - z, s \le r,\ B_r \ge f^{t,y}_t - 4 - z\right)\\
&= e^{\lambda^* r}\mathbf{E}_0^{\sqrt{2\lambda^*}}\left(e^{-\sqrt{2\lambda^*}B_r + \lambda^* r}1_{\{B_s \le f^{t,y}_{s+\tau_\ell} - z, s \le r,\ B_r \ge f^{t,y}_t - 4 - z\}}\right)\\
&\lesssim e^{-\sqrt{2\lambda^*}\left(f^{t,y}_t - 4 - z\right) + 2\lambda^* r}\mathbf{P}_0\left(B_s \ge h^{r,z'}_s, s \le r,\ B_r \le 4 + h^{r,z'}_r\right).
\end{aligned} \tag{7.9}$$

Recall that $h^{t,y}_s = \frac{3}{2\sqrt{2\lambda^*}}\log\left(\frac{t+1}{t-s+1}\right) - y$. It follows from Lemma 4.1 that for $r \ge 1$,

$$\mathbf{P}_0\left(B_s \ge h^{r,z'}_s, s \le r,\ B_r \le 4 + h^{r,z'}_r\right) \le \mathbf{P}_0\left(B_s \ge h^{r,z'+1}_s, s \le r,\ B_r \le 3 + h^{r,z'+1}_r\right) \lesssim \frac{(z'+1)}{r^{3/2}}.$$

For $r < 1$, we use the trivial bound 1. Plugging these into (7.9), together with (7.8), we conclude that on the event $\Upsilon(\xi)$,

$$\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0} \middle| \mathcal{G}_t \cap \left(\mathbf{A}\left(I_\xi(\tau_\ell-)\right), \ell \le K_t\right)\right) \lesssim 1 + \sum_{\ell=1}^{K_t}\sum_{j=1}^{d}\left(A_j\left(I_\xi(\tau_\ell-)\right)\right)^{\alpha_0} \times e^{-\alpha_0\sqrt{2\lambda^*}\left(f^{t,y}_t - 2 - X_\xi(\tau_\ell)\right) + 2\alpha_0\lambda^*(t-\tau_\ell)}\left(\frac{\left(f^{t,y}_{\tau_\ell} - X_\xi(\tau_\ell) + 1\right)}{(t-\tau_\ell)^{3/2}}1_{\{t-\tau_\ell \ge 1\}} + 1_{\{t-\tau_\ell \le 1\}}\right)^{\alpha_0}. \tag{7.10}$$

Note that the distribution of the number of offspring $\mathbf{A}(j)$ of a spine particle of type j is given by

$$\widehat{\mathbb{P}}_{(0,i)}\left(\mathbf{A}(j) = \mathbf{k}\right) = \frac{p_{\mathbf{k}}(j)\langle \mathbf{k}, \mathbf{h}\rangle}{(1+\lambda^*/a_j)h_j} =: \widehat{p}_{\mathbf{k}}(j).$$

Thus, given $\mathcal{G}_t$, the law of $\mathbf{A}\left(I_\xi(\tau_\ell-)\right)$ is equal to $\widehat{\mathbb{P}}_{(0,i)}\left(\cdot\middle| A_{I_\xi(\tau_\ell)}\left(I_\xi(\tau_\ell-)\right) \ge 1, I_\xi(\tau_\ell-), I_\xi(\tau_\ell)\right)$ since there must be at least one particle of type $I_\xi(\tau_\ell)$ among the $\mathbf{A}\left(I_\xi(\tau_\ell-)\right)$ offspring. So for any $\mathbf{k} \in \mathbb{N}^d$,

$$\begin{aligned}
\widehat{\mathbb{P}}_{(0,i)}\left(\mathbf{A}\left(I_\xi(\tau_\ell-)\right) = \mathbf{k}\middle|\mathcal{G}_t\right) &= \frac{\widehat{\mathbb{P}}_{(0,i)}\left(\mathbf{A}\left(I_\xi(\tau_\ell-)\right) = \mathbf{k}, A_{I_\xi(\tau_\ell)}\left(I_\xi(\tau_\ell-)\right) \ge 1\middle| I_\xi(\tau_\ell-), I_\xi(\tau_\ell)\right)}{\widehat{\mathbb{P}}_{(0,i)}\left(A_{I_\xi(\tau_\ell)}\left(I_\xi(\tau_\ell-)\right) \ge 1\middle| I_\xi(\tau_\ell-), I_\xi(\tau_\ell)\right)}\\
&= \frac{1}{\sum_{\mathbf{k}\in\mathbb{N}^d: k_{I_\xi(\tau_\ell)} \ge 1}\widehat{p}_{\mathbf{k}}\left(I_\xi(\tau_\ell-)\right)}\widehat{p}_{\mathbf{k}}\left(I_\xi(\tau_\ell-)\right)1_{\{k_{I_\xi(\tau_\ell)} \ge 1\}}.
\end{aligned}$$

Define

$$\mathcal{S} := \left\{(j_1, j_2) \in S \times S:\ \sum_{\mathbf{k}\in\mathbb{N}^d: k_{j_2} \ge 1}\widehat{p}_{\mathbf{k}}(j_1) > 0\right\}.$$

There exists a constant $c_1 \in (0,1]$ such that for all $(j_1, j_2) \in \mathcal{S}$, $\sum_{\mathbf{k}\in\mathbb{N}^d: k_{j_2}\geq 1} \widehat{p}_{\mathbf{k}}(j_1) \geq c_1$. Note that $\sum_{\mathbf{k}\in\mathbb{N}^d: k_{I_\xi(\tau_\ell)}\geq 1} \widehat{p}_{\mathbf{k}}(I_\xi(\tau_\ell-)) > 0$. Therefore, $(I_\xi(\tau_\ell-), I_\xi(\tau_\ell)) \in \mathcal{S}$ and

$$\begin{aligned}
&\widehat{\mathbb{E}}_{(0,i)}\left(\sum_{j=1}^d (A_j(I_\xi(\tau_\ell)-))^{\alpha_0} \Big| \mathcal{G}_t\right) = \frac{1}{\sum_{\mathbf{k}\in\mathbb{N}^d: k_{I_\xi(\tau_\ell)}\geq 1} \widehat{p}_{\mathbf{k}}(I_\xi(\tau_\ell-))} \sum_{j=1}^d \sum_{\mathbf{k}\in\mathbb{N}^d: k_{I_\xi(\tau_\ell)}\geq 1} \widehat{p}_{\mathbf{k}}(I_\xi(\tau_\ell)-) k_j^{\alpha_0} \\
&\leq \frac{1}{c_1}\sum_{j=1}^d \sum_{\mathbf{k}\in\mathbb{N}^d} \widehat{p}_{\mathbf{k}}(I_\xi(\tau_\ell)-) k_j^{\alpha_0} \lesssim \sup_{\ell\in S}\sum_{j=1}^d \sum_{\mathbf{k}\in\mathbb{N}^d} \widehat{p}_{\mathbf{k}}(\ell) k_j^{\alpha_0} \\
&= \sup_{\ell\in S}\sum_{j=1}^d \sum_{\mathbf{k}\in\mathbb{N}^d} \frac{p_{\mathbf{k}}(\ell)\langle \mathbf{k}, \mathbf{h}\rangle}{(1+\lambda^*/a_\ell)h_\ell} k_j^{\alpha_0} \lesssim \sup_{\ell,j,q\in S} \sum_{\mathbf{k}\in\mathbb{N}^d} p_{\mathbf{k}}(\ell) k_q k_j^{\alpha_0} \\
&\leq \sup_{\ell,j,q\in S} \sum_{\mathbf{k}\in\mathbb{N}^d} p_{\mathbf{k}}(\ell)\left(\frac{k_q^{1+\alpha_0}}{1+\alpha_0} + \frac{\alpha_0 k_j^{1+\alpha_0}}{1+\alpha_0}\right) \lesssim 1, \qquad (7.11)
\end{aligned}$$

where in the last inequality we used the assumption (1.4). By (7.10) and (7.11), on the event $\Upsilon(\xi)$,

$$\begin{aligned}
\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0} | \mathcal{G}_t\right) &\lesssim 1 + \sum_{\ell=1}^{K_t} e^{-\alpha_0\sqrt{2\lambda^*}\left(f_t^{t,y}-2-X_\xi(\tau_\ell)\right)+2\alpha_0\lambda^*(t-\tau_\ell)} \\
&\times \left(\frac{(f_{\tau_\ell}^{t,y} - X_\xi(\tau_\ell)+1)}{(t-\tau_\ell)^{3/2}} 1_{\{t-\tau_\ell\geq 1\}} + 1_{\{t-\tau_\ell\leq 1\}}\right)^{\alpha_0}.
\end{aligned}$$

Step (ii) Note that τ_ℓ is measurable with respect to $\sigma(I_\xi(s): s\geq 0)$, which means that τ_ℓ is independent of X_ξ. For the Brownian motion B, define

$$\Upsilon := \left\{B_s \leq f_s^{t,y},\ s\leq t, B_t \geq f_t^{t,y} - 4\right\}.$$

Then

$$\begin{aligned}
&\widehat{\mathbb{E}}_{(0,i)}\left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0} 1_{\Upsilon(\xi)} \big| \tau_\ell : \ell\geq 1\right) \lesssim \mathbf{P}_0\left(B_s \leq f_s^{t,y},\ s\leq t, B_t\geq f_t^{t,y}-4\right) \\
&\quad + \sum_{\ell=1}^{K_t} \mathbf{E}_0\left(e^{-\alpha_0\sqrt{2\lambda^*}\left(f_t^{t,y}-B_{\tau_\ell}\right)+2\alpha_0\lambda^*(t-\tau_\ell)} \left(\frac{f_{\tau_\ell}^{t,y} - B_{\tau_\ell}+1}{(t-\tau_\ell)^{3/2}} 1_{\{t-\tau_\ell\geq 1\}} + 1_{\{t-\tau_\ell\leq 1\}}\right)^{\alpha_0} 1_\Upsilon\right) \\
&=: L_1 + \sum_{\ell=1}^{K_t} L_2(\ell). \qquad (7.12)
\end{aligned}$$

For L_1, note that the argument leading to (4.9) also works when k is not an integer. Letting $k=t$ and $x=4$ in (4.9), we get

$$L_1 \lesssim e^{-\sqrt{2\lambda^*}(f_t^{t,y}-4)+\lambda^* t} \frac{(y+1)\wedge\sqrt{t}}{t^{3/2}} \times 25 \lesssim e^{-\lambda^* t}(y\wedge\sqrt{t}) e^{-\sqrt{2\lambda^*}y}. \qquad (7.13)$$

Note that τ_ℓ can be regarded as a constant with respect to $\mathbf{E}_0$. For $L_2(\ell)$, we deal with two cases separately. We first deal with the case $t-\tau_\ell \geq 1$. Set $r := \tau_\ell$. By the Markov property,

$$\begin{aligned} L_2(\ell) &= \mathbf{E}_0 \left(e^{-\alpha_0\sqrt{2\lambda^*}\left(f_t^{t,y}-B_r\right)+2\alpha_0\lambda^*(t-r)} \left(\frac{f_r^{t,y}-B_r+1}{(t-r)^{3/2}} \right)^{\alpha_0} 1_\Upsilon \right) \\ &= \mathbf{E}_0 \Bigg(e^{-\alpha_0\sqrt{2\lambda^*}\left(f_t^{t,y}-B_r\right)+2\alpha_0\lambda^*(t-r)} \left(\frac{f_r^{t,y}-B_r+1}{(t-r)^{3/2}} \right)^{\alpha_0} 1_{\left\{B_s\leq f_s^{t,y},\ s\leq r\right\}} \\ &\qquad \times \mathbf{P}_{B_r}\left(B_s \leq f_{s+r}^{t,y},\ s\leq t-r,\ B_{t-r}\geq f_t^{t,y}-4\right) \Bigg). \end{aligned} \tag{7.14}$$

For $z \leq f_r^{t,y}$, set $z' := f_r^{t,y} - z$. Using (4.6), the fact that $f_{s+r}^{t,y} - z = \sqrt{2\lambda^*}s - h_s^{t-r,z'}$ and (4.8) (which is still valid when $k = t$ is not an integer) with $F_{t-r}^{t-r,z'}([1,5])$ defined in (4.7), we get that

$$\begin{aligned} &\mathbf{P}_z\left(B_s \leq f_{s+r}^{t,y},\ s\leq t-r,\ B_{t-r}\geq f_t^{t,y}-4\right) \\ &= \mathbf{E}_0^{\sqrt{2\lambda^*}}\left(e^{-\sqrt{2\lambda^*}B_{t-r}+\lambda^*(t-r)} 1_{\left\{B_s\leq f_{s+r}^{t,y}-z,\ s\leq t-r,\ B_{t-r}\geq f_t^{t,y}-4-z\right\}}\right) \\ &\leq e^{-\sqrt{2\lambda^*}\left(f_t^{t,y}-z\right)+\lambda^*(t-r)} \mathbf{P}_0\left(B_s \leq -h_s^{t-r,z'},\ s\leq t-r,\ B_{t-r}\geq -4-h_{t-r}^{t-r,z'}\right) \\ &= e^{-\sqrt{2\lambda^*}\left(f_t^{t,y}-z\right)+\lambda^*(t-r)} \mathbf{P}_0\left(B_s \geq h_s^{t-r,z'},\ s\leq t-r,\ B_{t-r}\leq 4+h_{t-r}^{t-r,z'}\right) \\ &= e^{-\sqrt{2\lambda^*}\left(f_t^{t,y}-z\right)+\lambda^*(t-r)} \mathbf{P}_0\left(B_s \geq 1+h_s^{t-r,z'+1},\ s\leq t-r,\ B_{t-r}\leq 5+h_{t-r}^{t-r,z'+1}\right) \\ &\lesssim e^{-\sqrt{2\lambda^*}\left(f_t^{t,y}-z\right)+\lambda^*(t-r)} \left(\frac{z'+1}{(t-r)^{3/2}}\right). \end{aligned} \tag{7.15}$$

Combining (7.14) and (7.15), we get that

$$\begin{aligned} L_2(\ell) &\lesssim \mathbf{E}_0 \left(e^{-(1+\alpha_0)\sqrt{2\lambda^*}\left(f_t^{t,y}-B_{\tau_\ell}\right)+(1+2\alpha_0)\lambda^*(t-\tau_\ell)} \left(\frac{f_{\tau_\ell}^{t,y}-B_{\tau_\ell}+1}{(t-\tau_\ell)^{3/2}} \right)^{1+\alpha_0} 1_{\left\{B_s\leq f_s^{t,y},\ s\leq \tau_\ell\right\}} \right) \\ &=: \frac{1}{(t-\tau_\ell)^{3(1+\alpha_0)/2}} e^{-(1+\alpha_0)\sqrt{2\lambda^*}f_t^{t,y}+(1+2\alpha_0)\lambda^*(t-\tau_\ell)} \mathbf{E}_0\left(G 1_{\left\{B_s\leq f_s^{t,y},\ s\leq\tau_\ell\right\}}\right), \end{aligned} \tag{7.16}$$

where $G := e^{(1+\alpha_0)\sqrt{2\lambda^*}B_{\tau_\ell}}(f_{\tau_\ell}^{t,y} - B_{\tau_\ell} + 1)^{1+\alpha_0}$. Next, using (4.6) with $\lambda = \sqrt{2\lambda^*}$, and noticing that $(B_s, \mathbf{P}_0^{\sqrt{2\lambda^*}}) \overset{\mathrm{d}}{=} (B_s + \sqrt{2\lambda^*}s, \mathbf{P}_0)$ and that $f_s^{t,y} = \sqrt{2\lambda^*}s - h_s^{t,y}$, we get that

$$\begin{aligned} &\mathbf{E}_0\left(G 1_{\left\{B_s\leq f_s^{t,y},\ s\leq\tau_\ell\right\}}\right) = \sum_{k=0}^\infty \mathbf{E}_0\left(G 1_{\left\{B_s\leq f_s^{t,y},\ s\leq\tau_\ell\right\}} 1_{\{B_{\tau_\ell}-f_{\tau_\ell}^{t,y}\in[-k-1,-k]\}}\right) \\ &\leq \sum_{k=0}^\infty e^{\alpha_0\sqrt{2\lambda^*}\left(f_{\tau_\ell}^{t,y}-k\right)}(k+2)^{1+\alpha_0} \mathbf{E}_0\left(e^{\sqrt{2\lambda^*}B_{\tau_\ell}} 1_{\left\{B_s\leq f_s^{t,y},\ s\leq\tau_\ell\right\}} 1_{\{B_{\tau_\ell}-f_{\tau_\ell}^{t,y}\in[-k-1,-k]\}}\right) \\ &= e^{\alpha_0\sqrt{2\lambda^*}f_{\tau_\ell}^{t,y}+\lambda^*\tau_\ell} \sum_{k=0}^\infty e^{-\alpha_0\sqrt{2\lambda^*}k}(k+2)^{1+\alpha_0} \mathbf{P}_0\left(B_s\leq -h_s^{t,y},\ s\leq\tau_\ell, B_{\tau_\ell}+h_{\tau_\ell}^{t,y}\in[-(k+1),-k]\right). \end{aligned}$$

When $\tau_\ell \geq 1$, by Lemma 4.1, we have for all $k \geq 0$,

$$\begin{aligned}
&\mathbf{P}_0\left(B_s \leq -h_s^{t,y},\ s \leq \tau_\ell, B_{\tau_\ell} + h_{\tau_\ell}^{t,y} \in [-(k+1), -k]\right)\\
&= \mathbf{P}_0\left(B_s \geq h_s^{t,y},\ s \leq \tau_\ell, B_{\tau_\ell} - h_{\tau_\ell}^{t,y} \in [k, (k+1)]\right)\\
&\leq \mathbf{P}_0\left(B_s \geq h_s^{t,y+1},\ s \leq \tau_\ell,\ B_r - h_{\tau_\ell}^{t,y+1} \in [k+1, k+2]\right)\\
&\lesssim \frac{((y+1)\wedge\sqrt{\tau_\ell})(k+1)}{\tau_\ell^{3/2}} \lesssim \frac{(y\wedge\sqrt{t})(k+1)}{\tau_\ell^{3/2}}.
\end{aligned} \tag{7.17}$$

When $\tau_\ell \leq 1$, we use the trivial upper bound 1. Therefore, using the fact that $\sum_{k=0}^\infty e^{-\alpha_0\sqrt{2\lambda^*}k}(k+2)^{1+\alpha_0}(k+1) < \infty$, we conclude that

$$\begin{aligned}
&\mathbf{E}_0\left(G1_{\left\{B_s \leq f_s^{t,y},\ s \leq \tau_\ell\right\}}\right)\\
&\lesssim e^{\alpha_0\sqrt{2\lambda^*}f_{\tau_\ell}^{t,y} + \lambda^*\tau_\ell}\sum_{k=0}^\infty e^{-\alpha_0\sqrt{2\lambda^*}k}(k+2)^{1+\alpha_0}\left(\frac{(y\wedge\sqrt{t})(k+1)}{\tau_\ell^{3/2}}1_{\{\tau_\ell\geq 1\}} + 1_{\{\tau_\ell \leq 1\}}\right)\\
&\lesssim \left(y\wedge\sqrt{t}\right)e^{\alpha_0\sqrt{2\lambda^*}f_{\tau_\ell}^{t,y} + \lambda^*\tau_\ell}\left(\frac{1}{\tau_\ell^{3/2}}1_{\{\tau_\ell\geq 1\}} + 1_{\{\tau_\ell\leq 1\}}\right).
\end{aligned} \tag{7.18}$$

Combining (7.16) and (7.18), we get that in the case when $t - \tau_\ell \geq 1$,

$$\begin{aligned}
L_2(\ell) &\lesssim \left(y\wedge\sqrt{t}\right)\frac{e^{-(1+\alpha_0)\sqrt{2\lambda^*}f_t^{t,y} + (1+2\alpha_0)\lambda^*(t-\tau_\ell)}e^{\alpha_0\sqrt{2\lambda^*}f_{\tau_\ell}^{t,y}+\lambda^*\tau_\ell}}{(t-\tau_\ell)^{3(1+\alpha_0)/2}}\left(\frac{1}{\tau_\ell^{3/2}}1_{\{\tau_\ell\geq 1\}} + 1_{\{\tau_\ell\leq 1\}}\right)\\
&= \left(y\wedge\sqrt{t}\right)e^{-\sqrt{2\lambda^*}y}\frac{(t+1)^{3/2}(t-\tau_\ell+1)^{3\alpha_0/2}}{(t-\tau_\ell)^{3(1+\alpha_0)/2}}e^{-\lambda^* t}\left(\frac{1}{\tau_\ell^{3/2}}1_{\{\tau_\ell\geq 1\}} + 1_{\{\tau_\ell\leq 1\}}\right)\\
&\lesssim \left(y\wedge\sqrt{t}\right)e^{-\sqrt{2\lambda^*}y}\frac{(t+1)^{3/2}}{(t-\tau_\ell)^{3/2}}e^{-\lambda^* t}\left(\frac{1}{\tau_\ell^{3/2}}1_{\{\tau_\ell\geq 1\}} + 1_{\{\tau_\ell\leq 1\}}\right)\\
&\lesssim \left(y\wedge\sqrt{t}\right)e^{-\sqrt{2\lambda^*}y}e^{-\lambda^* t}\left(\frac{1}{\tau_\ell^{3/2}}1_{\{t/2\geq\tau_\ell\geq 1\}} + \frac{1}{(t-\tau_\ell)^{3/2}}1_{\{t/2<\tau_\ell\leq t-1\}} + 1_{\{\tau_\ell\leq 1\}}\right).
\end{aligned} \tag{7.19}$$

Now we deal with the case $t - \tau_\ell < 1$. For $z = B_{\tau_\ell}$,

$$\begin{aligned}
&\mathbf{P}_z\left(B_s \leq f_{s+\tau_\ell}^{t,y}, s \leq t-\tau_\ell, B_{t-\tau_\ell} \geq f_t^{t,y} - 4\right) \leq \mathbf{P}_0\left(B_{t-\tau_\ell} \geq f_t^{t,y} - 4 - z\right)\\
&= \mathbf{E}_0^{\sqrt{2\lambda^*}}\left(e^{-\sqrt{2\lambda^*}B_{t-\tau_\ell} + \lambda^*(t-\tau_\ell)}1_{\{B_{t-\tau_\ell}\geq f_t^{t,y}-4-z\}}\right) \leq e^{-\sqrt{2\lambda^*}(f_t^{t,y}-4-z)+\lambda^*(t-\tau_\ell)},
\end{aligned}$$

which implies that for all $y, t \geq 1$ and ℓ with $t - \tau_\ell < 1$,

$$\begin{aligned}
&L_2(\ell) \\
&= \mathbf{E}_0 \left(e^{-\alpha_0\sqrt{2\lambda^*}\left(f_t^{t,y} - B_{\tau_\ell}\right) + 2\alpha_0\lambda^*(t-\tau_\ell)} 1_{\left\{B_s \leq f_s^{t,y},\ s\leq t, B_t \geq f_t^{t,y} - 4\right\}} \right) \\
&\lesssim \mathbf{E}_0 \left(e^{-(1+\alpha_0)\sqrt{2\lambda^*}\left(f_t^{t,y} - B_{\tau_\ell}\right) + (1+2\alpha_0)\lambda^*(t-\tau_\ell)} 1_{\left\{B_s \leq f_s^{t,y},\ s\leq \tau_\ell\right\}} \right) \\
&= e^{-(1+\alpha_0)\sqrt{2\lambda^*} f_t^{t,y} + (1+2\alpha_0)\lambda^*(t-\tau_\ell)} \sum_{k=0}^{\infty} \mathbf{E}_0 \left(e^{(1+\alpha_0)\sqrt{2\lambda^*} B_{\tau_\ell}} 1_{\left\{B_s \leq f_s^{t,y},\ s\leq \tau_\ell\right\}} 1_{\left\{B_{\tau_\ell} - f_{\tau_\ell}^{t,y} \in [-(k+1), -k]\right\}} \right) \\
&\lesssim e^{-(1+\alpha_0)\sqrt{2\lambda^*} f_t^{t,y} + (1+2\alpha_0)\lambda^*(t-\tau_\ell)} e^{\alpha_0\sqrt{2\lambda^*} f_{\tau_\ell}^{t,y}} e^{\lambda^*\tau_\ell} \\
&\quad \times \sum_{k=0}^{\infty} e^{-\alpha_0\sqrt{2\lambda^*}k} \mathbf{P}_0^{\sqrt{2\lambda^*}} \left(B_s \leq f_s^{t,y},\ s \leq \tau_\ell, B_{\tau_\ell} - f_{\tau_\ell}^{t,y} \in [-(k+1), -k]\right) \\
&\lesssim e^{-(1+\alpha_0)\sqrt{2\lambda^*} f_t^{t,y} + (1+2\alpha_0)\lambda^*(t-\tau_\ell)} e^{\alpha_0\sqrt{2\lambda^*} f_{\tau_\ell}^{t,y}} e^{\lambda^*\tau_\ell} \sum_{k=0}^{\infty} e^{-\alpha_0\sqrt{2\lambda^*}k} \frac{(y\wedge\sqrt{t})(k+1)}{\tau_\ell^{3/2}},
\end{aligned}$$

where in the last inequality we used (7.17). Therefore, when $t - \tau_\ell < 1$,

$$\begin{aligned}
L_2(\ell) &\lesssim e^{-(1+\alpha_0)\sqrt{2\lambda^*} f_t^{t,y} + (1+2\alpha_0)\lambda^*(t-\tau_\ell)} e^{\alpha_0\sqrt{2\lambda^*} f_{\tau_\ell}^{t,y}} e^{\lambda^*\tau_\ell} \frac{(y\wedge\sqrt{t})}{\tau_\ell^{3/2}} \\
&\lesssim \left(y \wedge \sqrt{t}\right) e^{-\lambda^* t} e^{-\sqrt{2\lambda^*} y}. \qquad (7.20)
\end{aligned}$$

Using (7.12), (7.13), (7.19) and (7.20), taking expectation with respect to $\widehat{\mathbb{P}}_{(0,i)}$, we get

$$\begin{aligned}
&\widehat{\mathbb{E}}_{(0,i)} \left(\left(\mathcal{A}^{t,y}\right)^{\alpha_0} 1_{\Upsilon(\xi)} \right) \lesssim \widehat{\mathbb{E}}_{(0,i)} \left(L_1 + \sum_{\ell=1}^{K_t} L_2(\ell) \right) \\
&\lesssim (y\wedge\sqrt{t}) e^{-\lambda^* t} e^{-\sqrt{2\lambda^*} y} \widehat{\mathbb{E}}_{(0,i)} \left(1 + \sum_{\ell=1}^{K_t} \left(\frac{1_{\{t/2 \geq \tau_\ell \geq 1\}}}{\tau_\ell^{3/2}} + \frac{1_{\{t/2 < \tau_\ell \leq t-1\}}}{(t-\tau_\ell)^{3/2}} + 1_{\{\tau_\ell \leq 1\}} + 1_{\{\tau_\ell \geq t-1\}} \right) \right).
\end{aligned}$$

Then we get (4.15). □

7.5. *Proof of Proposition 5.3*. In this subsection, we will give the proof of Proposition 5.3. To prove Proposition 5.3, we will need a lemma (Lemma 7.4), which is similar to Bramson (1983, Lemma 7.2, p. 105). A key step in the proof of Lemma 7.4 is the inequality (7.27). In Bramson's argument for the analog of (7.27) (see Bramson (1983, (7.32) on p. 107, Proposition 7.1 on p. 97)), the Kolmogorov-Petrovsky-Piscounov theorem (see Bramson (1983, p. 34)) was used, see Bramson (1983, Proposition 3.4 on p. 47 and (3.71) on p. 49). In the multi-type case, the analog of the Kolmogorov-Petrovsky-Piscounov theorem has not been proved yet. So we have to overcome this difficulty.

Lemma 7.1 below is the key to (7.27), which is different from Bramson (1983, (7.32)). Roughly speaking, since $v_i^*(0, y) = v_i(r, y) 1_{\{y > -\log r\}}$ is very close to 1 when $|y| < \log r$, by representation (1.6) for $v_i^*(t, y)$, it suffices to show that, under $\mathbb{P}_{(y,i)}$, the probability of the event that there is at least one particle locating in $[-\log r, \log r]$ is close to 1 when r is large enough. This is easy to prove since we know the behavior of the maximal position M_t very well by Theorem 4.5. Using this, we can get Lemma 7.1 below.

Recall the definition (5.3) of $\mathbf{v}^*$, where $\mathbf{v}$ is a solution to (1.5) with initial value satisfying (1.12). Define

$$m_+(t) := \max\left\{\sqrt{2\lambda^*}t - \frac{3}{2\sqrt{2\lambda^*}}\log_+ t, 0\right\}.$$

Lemma 7.1. *For any $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that when $r > N$,*

$$v_i^*(t,y) \geq 1-\varepsilon, \quad \textit{for all } t \geq 0,\ i \in S,\ y \in [0, m_+(t)].$$

Proof: We first prove that for any $\varepsilon_1 > 0$, there exists $N^* = N^*(\varepsilon_1)$ such that when $r \geq N^*$,

$$\mathbb{P}_{(0,i)}\left(\#\{u \in Z(t) : |X_u(t)| \leq r\} = 0\right) < \varepsilon_1, \quad i \in S, t \geq 0. \tag{7.21}$$

Using Theorem 4.5 and symmetry, we get that, for any $\varepsilon_1 > 0$, there exists N_1 such that

$$\sup_{t\geq 0}\sup_{i\in S}\mathbb{P}_{(0,i)}\left(|M_t - m_+(t)| > N_1\right) = \sup_{t\geq 0}\sup_{i\in S}\mathbb{P}_{(0,i)}\left(|M_t^- + m_+(t)| > N_1\right) < \frac{\varepsilon_1}{2}. \tag{7.22}$$

Here $M_t^- := \inf_{u\in Z(t)} X_u(t)$ is the leftmost position among all the particles. By the Markov property and branching property at time $t/2$, we have

$$\begin{aligned}
&\mathbb{P}_{(0,i)}\left(\#\{u \in Z(t) : |X_u(t)| \leq 2N_1\} = 0\right)\\
&\leq \frac{\varepsilon_1}{2} + \mathbb{P}_{(0,i)}\left(|M_{t/2} - m_+(t/2)| \leq N_1, \#\{u \in Z(t) : |X_u(t)| \leq 2N_1\} = 0\right)\\
&\leq \frac{\varepsilon_1}{2} + \sup_{z:|z-m_+(t/2)|\leq N_1}\sup_{j\in S}\mathbb{P}_{(z,j)}\left(|M_{t/2}^-| > 2N_1\right)\\
&\leq \frac{\varepsilon_1}{2} + \sup_{z:|z-m_+(t/2)|\leq N_1}\sup_{j\in S}\mathbb{P}_{(z,j)}\left(|M_{t/2}^- + m_+(t/2)| > 2N_1 - |z - m_+(t/2)|\right)\\
&\leq \frac{\varepsilon_1}{2} + \sup_{j\in S}\mathbb{P}_{(0,j)}\left(|M_{t/2}^- + m_+(t/2)| > N_1\right) < \varepsilon_1.
\end{aligned}$$

Therefore, (7.21) holds with $N^* = 2N_1$.

Next, we prove that, for any $\varepsilon_2 > 0$, there exists $N' = N'(\varepsilon_2)$ such that when $r \geq N'$,

$$\mathbb{P}_{(y,i)}\left(\#\{u \in Z(t) : |X_u(t)| \leq r\} = 0\right) < \varepsilon_2, \quad i \in S, t \geq 0, y \in [0, m_+(t)]. \tag{7.23}$$

Let $t_0 > 1$ be a constant such that $m_+(t) = m(t)$ for all $t \geq t_0$. When $t \leq t_0$, we use the trivial upper-bound

$$\mathbb{P}_{(y,i)}\left(\#\{u \in Z(t) : |X_u(t)| \leq r\} = 0\right) \leq \mathbf{P}_y\left(|B_t| > r\right)$$

and the tail probability of normal random variables; when $y \leq m(t_0)$, we use the bound

$$\mathbb{P}_{(y,i)}\left(\#\{u \in Z(t) : |X_u(t)| \leq r\} = 0\right) \leq \mathbb{P}_{(0,i)}\left(\#\{u \in Z(t) : |X_u(t)| \leq r - m(t_0)\} = 0\right)$$

and (7.21). So we only deal with the case when $t \geq t_0$ and $y \in [m(t_0), m_+(t)] = [m(t_0), m(t)]$. Suppose that $y = m(s)$ for some $t_0 \leq s \leq t$. Let $\varepsilon_1 = \varepsilon_2/2$, Using (7.21), (7.22) and the fact that $\mathbb{P}_{(0,i)}\left(|M_s^- + m_+(s)| > N_1\right) = \mathbb{P}_{(y,i)}\left(|M_s^-| > N_1\right)$,

$$\begin{aligned}
&\mathbb{P}_{(y,i)}\left(\#\{u \in Z(t) : |X_u(t)| \leq N_1 + N^*\} = 0\right)\\
&\leq \frac{\varepsilon_1}{2} + \mathbb{P}_{(m(s),i)}\left(|M_s^-| \leq N_1, \#\{u \in Z(t-s) : |X_u(t-s)| \leq N_1 + N^*\} = 0\right)\\
&\leq \frac{\varepsilon_2}{4} + \sup_{z:|z|\leq N_1}\sup_{j\in S}\mathbb{P}_{(z,j)}\left(\#\{u \in Z(t-s) : |X_u(t-s)| \leq N_1 + N^*\} = 0\right)\\
&\leq \frac{\varepsilon_2}{4} + \sup_{j\in S}\mathbb{P}_{(0,j)}\left(\#\{u \in Z(t-s) : |X_u(t-s)| \leq N^*\} = 0\right) < \frac{\varepsilon_2}{4} + \frac{\varepsilon_2}{2} < \varepsilon_2,
\end{aligned}$$

which implies (7.23).

For any $\varepsilon > 0$, by (5.1) and Theorem 4.5, when r is large enough, we have for all $x \in [-\log r, \log r]$ and any $i \in S$,

$$v_i(r,x) \geq \mathbb{P}_{(0,i)}\left(M_r^{i_0} > x - N_1\right) \geq \mathbb{P}_{(0,i)}\left(M_r^{i_0} - m(r) > \log r - m(r) - N_1\right) > 1 - \frac{\varepsilon}{2}.$$

Taking $\varepsilon_2 = \frac{\varepsilon}{2}$ in (7.23), we get that for any $i \in S, t \geq 0$ and $y \in [0, m_+(t)]$, as long as $\log r > N' \Leftrightarrow r > e^{N'} =: N$,

$$\begin{aligned}
v_i^*(t,y) &= 1 - \mathbb{E}_{(y,i)}\left(\prod_{u \in Z(t)}\left(1 - v^*_{I_u(t)}(0, X_u(t))\right)\right) \\
&\geq 1 - \mathbb{E}_{(y,i)}\left(\prod_{u \in Z(t)}\left(1 - v_{I_u(t)}(r, X_u(t))1_{\{|X_u(t)| \leq \log r\}}\right)\right) \\
&\geq 1 - \mathbb{E}_{(y,i)}\left(\prod_{u \in Z(t)}\left(1 - \left(1 - \frac{\varepsilon}{2}\right)1_{\{|X_u(t)| \leq \log r\}}\right)\right) \\
&\geq 1 - \frac{\varepsilon}{2} - \mathbb{P}_{(y,i)}\left(\#\left\{u \in Z(t) : |X_u(t)| \leq \log r\right\} = 0\right) > 1 - \varepsilon.
\end{aligned}$$

This completes the proof. □

For single-type BBM, the assumption $p_0 = 0, p_1 \neq 1$ implies that the offspring mean is strictly larger than 1. This fact is used in the inequality $k(v(t-s, \mathfrak{z}_{x,y}(s))) \leq 1/2$ above (7.34) on page 107 of Bramson (1983) to prove the exponential decay in Bramson (1983, Lemma 7.2), where $\mathfrak{z}_{x,y}$ is the Brownian bridge starting at x and ending at y. But for multi-type BBM, there are two cases that will contribute exponential decay: (i) $\mu_{i,i} = 0$ for all $i \in S$. In this case, the assumptions $\mathbf{p_0}(i) = 0$ for all $i \in S$ and $\lambda^* > 0$ imply that there exists $j_0 \in S$ such that $n_{j_0} - \mu_{j_0,j_0} > 1$, which will play a role in getting the exponential decay in Lemma 7.4 below. In Lemma 7.2 below, we give an estimate which will replace the role of the inequality $k(v(t-s, \mathfrak{z}_{x,y}(s))) \leq 1/2$ of Bramson (1983) in the multi-type case. (ii) Another case is that there exists j_1 such that $\mu_{j_1,j_1} > 0$. In this case, we can also get the exponential decay in Lemma 7.4 by using Lemma 7.3 below.

Lemma 7.2. *Suppose that $n_{j_0} - \mu_{j_0,j_0} > 1$. Then for any $\theta > 0$, there exist $C_\theta > 0$ and $\varepsilon = \varepsilon_\theta > 0$ such that for all $t > 0$,*

$$\sup_{x \in \mathbb{R}, i \in S} \mathbf{E}^h_{(x,i)}\left(\exp\left\{-\theta \sum_{s \in D_J : s \leq t} 1_{\{I_s = j_0\}}\right\}\right) \leq C_\theta e^{-\varepsilon t}.$$

Proof: Since I and X are independent, we only consider the case $x = 0$. Let $\{Y_n : n = 0, 1, ...\}$ be the embedded chain of I_t under $\mathbf{P}_{(0,i)}$. We first prove that there exist $C_1, \delta_1 > 0$ such that

$$\sup_{i \in S} \mathbf{P}^h_{(0,i)}\left(Y_0 \neq j_0, \cdots, Y_n \neq j_0\right) \leq C_1 e^{-\delta_1 n}. \tag{7.24}$$

Since $\{Y_n\}$ is irreducible, for each $i \in S$, there exists $L_i \in \mathbb{N}$ such that $\mathbf{P}^h_{(0,i)}(Y_{L_i} \neq j_0) < 1$. Let $L := \max_{i \in S} L_i$. Then

$$\sup_{i \in S} \mathbf{P}^h_{(0,i)}\left(Y_0 \neq j_0, \cdots, Y_L \neq j_0\right) \leq \sup_{i \in S} \mathbf{P}^h_{(0,i)}(Y_{L_i} \neq j_0) =: e^{-\varepsilon_1}.$$

Therefore, for $n \geq L$, we have

$$\sup_{i \in S} \mathbf{P}^h_{(0,i)}\left(Y_0 \neq j_0, \cdots, Y_n \neq j_0\right) \leq e^{-\varepsilon_1} \sup_{i \in S} \mathbf{P}^h_{(0,i)}\left(Y_0 \neq j_0, ..., Y_{n-L} \neq j_0\right),$$

which implies (7.24) with $C_1 := e^{\varepsilon_1}, \delta_1 := \varepsilon_1/L$.

Next, define $U_{j_0} := \inf\{t \in D_J, I_t = j_0\}$. We prove that there exists $\delta_2 > 0$ such that

$$\sup_{i\in S} \mathbf{E}^h_{(0,i)} \left(e^{\delta_2 U_{j_0}}\right) =: C_2 < \infty. \tag{7.25}$$

To this end, it suffices to show that there exist constants $C_3, \varepsilon_2 > 0$ such that for t large enough,

$$\sup_{i\in S} \mathbf{P}^h_{(0,i)} \left(U_{j_0} > t\right) \le C_3 e^{-\varepsilon_2 t}.$$

Recall that in the paragraph containing (4.17), we defined a coupling $(I_t, I^{\underline{a}}_t)$ so that the embedded chain of I and $I^{\underline{a}}$ are the same, and the jump times $D_J = \{t_n : 0 < t_1 < t_2 < \dots\}$ of I and the jumps times $D^{\underline{a}} = \{0 < t^{\underline{a}}_1 < t^{\underline{a}}_2 < \dots\}$ of $I^{\underline{a}}$ satisfy $t_n \le t^{\underline{a}}_n$ for all $n \ge 1$. Let $U^{\underline{a}}_{j_0} := \inf\{t \in D^{\underline{a}}, I^{\underline{a}}_t = j_0\}$, then $U_{j_0} \le U^{\underline{a}}_{j_0}$. For $n \in \mathbb{N}$, on the event that the first hitting time of j_0 by the embedded chain is larger than n, by (7.24) we can bound $\mathbf{P}^h_{(0,i)} \left(U_{j_0} > t\right)$ from above by $C_1 e^{-\delta_1 n}$. On the event that the first hitting time of j_0 by the embedded chain is less than or equal to n, we bound $\mathbf{P}^h_{(0,i)} \left(U_{j_0} > t\right)$ from above by

$$\sup_{i\in S} \sum_{m=1}^n \mathbf{P}^h_{(0,i)} \left(Y^{\underline{a}}_1 + \dots + Y^{\underline{a}}_m > t\right) \le n e^{-\underline{a}t/2} \sup_{i\in S} \left(\mathbf{E}^h_{(0,i)} \left(e^{\underline{a} Y^{\underline{a}}_1/2}\right)\right)^n = n e^{-\underline{a}t/2} 2^n.$$

Taking $n = [\varepsilon_2 t]$ for $0 < \varepsilon_2 \log 2 < \underline{a}/2$, we get

$$\sup_{i\in S} \mathbf{P}^h_{(0,i)} \left(U_{j_0} > t\right) \le C_1 e^{-\delta_1 [\varepsilon_2 t]} + [\varepsilon_2 t] e^{-\underline{a}t/2} e^{\log 2 [\varepsilon_2 t]},$$

which implies (7.25).

Define $V^1_{j_0} := \inf\{t \in D_J, I_t = j_0\}$ and $V^n_{j_0} := \inf\left\{t \in D_J : t > V^{n-1}_{j_0} : I_t = j_0\right\}$ for $n \ge 2$. Set $U^1_{j_0} = V^1_{j_0}$ and $U^n_{j_0} := V^n_{j_0} - V^{n-1}_{j_0}$. By the strong Markov property, $\{U^n_{j_0} : n \ge 1\}$ are independent. Define $S_t := \sum_{s\in D_J : s\le t} 1_{\{I_s = j_0\}} = \sup\left\{n : \sum_{m=1}^n U^m_{j_0} \le t\right\}$, then for any n, $\{S_t = n\} \subset \left\{\sum_{m=1}^{n+1} U^m_{j_0} > t\right\}$. Thus, by (7.25) and the strong Markov property,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}, i\in S} \mathbf{E}^h_{(x,i)} \left(e^{-\theta S_t}\right) \le e^{-\theta n} + \sup_{x\in\mathbb{R}, i\in S} \sum_{\ell=1}^n \mathbf{P}^h_{(x,i)} \left(S_t = \ell\right)\\
&\le e^{-\theta n} + \sup_{x\in\mathbb{R}, i\in S} \sum_{\ell=1}^n \mathbf{P}^h_{(x,i)} \left(\sum_{m=1}^{\ell+1} U^m_{j_0} > t\right) \le e^{-\theta n} + n \sup_{x\in\mathbb{R}, i\in S} \mathbf{P}^h_{(x,i)} \left(\sum_{m=1}^{n+1} U^m_{j_0} > t\right)\\
&\le e^{-\theta n} + n e^{-\delta_2 t} \sup_{i\in S} \mathbf{E}^h_{(0,i)} \left(e^{\delta_2 \sum_{m=1}^{n+1} U^m_{j_0}}\right) \le e^{-\theta n} + n e^{-\delta_2 t} \prod_{m=1}^{n+1} \sup_{i\in S} \mathbf{E}^h_{(0,i)} \left(e^{\delta_2 U_{j_0}}\right)\\
&= e^{-\theta n} + n e^{(n+1)\delta_2 - \delta_2 t}.
\end{aligned}$$

Taking $n = [t/2]$, we get the conclusion of the lemma. □

Lemma 7.3. *Suppose that $\mu_{j_1,j_1} > 0$. Then for any $\theta > 0$, there exist $C^*_\theta > 0$ and $\varepsilon^* = \varepsilon^*_\theta > 0$ such that for all $t > 0$,*

$$\sup_{x\in\mathbb{R}, i\in S} \mathbf{E}^h_{(x,i)} \left(\exp\left\{-\theta \int_0^t 1_{\{I_s = j_1\}} \mathrm{d}s\right\}\right) \le C^*_\theta e^{-\varepsilon^* t}.$$

Proof: We continue the notation in the proof of Lemma 7.2. By Lemma 7.2 with j_0 replaced by j_1 and the fact that $\int_0^t 1_{\{I_s=j_1\}}\mathrm{d}s \geq \sum_{m=1}^{S_t-1} U_{j_1}^m$, we get that for any fixed small $\eta > 0$,

$$\begin{aligned}
&\mathbf{E}_{(x,i)}^h\left(\exp\left\{-\theta\int_0^t 1_{\{I_s=j_1\}}\mathrm{d}s\right\}\right) \leq \mathbf{E}_{(x,i)}^h\left(\exp\left\{-\theta\sum_{m=1}^{S_t-1} U_{j_1}^m\right\}\right)\\
&\leq \mathbf{P}_{(x,i)}^h(S_t < [\eta t]) + \mathbf{E}_{(x,i)}^h\left(\exp\left\{-\theta\sum_{m=1}^{[\eta t]-1} U_{j_1}^m\right\}\right)\\
&\leq e^{\eta t}\sup_{x\in\mathbb{R},i\in S}\mathbf{E}_{(x,i)}^h\left(e^{-S_t}\right) + \left(\mathbf{E}_{(0,j_0)}^h\left(\exp\left\{-\theta U_{j_1}^2\right\}\right)\right)^{[\eta t]-2}\\
&\leq C_1 e^{-(\varepsilon_1-\eta)t} + \left(\mathbf{E}_{(0,j_0)}^h\left(\exp\left\{-\theta U_{j_1}^2\right\}\right)\right)^{\eta t-3}.
\end{aligned}$$

Taking $\eta < \varepsilon_1$, we get the desired result. □

Our goal is to get the upper bound for $v_i(t, m(t)+x)$ for large r and $r < x \leq \sqrt{t}$ in Proposition 5.4. Lemma 5.2 implies that the upper bound is related to $\mathbf{E}_{(x,i)}^h\left(R(t-r;v^*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\middle| X_{t-r}=y\right)$ for $m(t)+r < x \leq m(t)+\sqrt{t}$ and $y > -\log r$. In Lemma 7.4 and Proposition 5.3 below, we will estimate $\mathbf{E}_{(x,i)}^h\left(R(t-r;v^*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};(\mathcal{B}_{up})^c\middle| X_{t-r}=y\right)$ under the condition $x \geq m(t)$ and $y > -\log r$.

Recall that j_1 is the integer such that $j_1 < t/2 \leq j_1+1$.

Lemma 7.4. *Let* $\mathbf{v}$ *be a solution to* (1.5) *with initial value satisfying* (1.12) *and let* $\mathbf{v}_*$ *be given by* (5.3). *Then for* r *large enough,* $t > 4r$ *and* $j_1 \geq j \geq [r+r^\delta]$, *it holds that*

$$\begin{aligned}
&\mathbf{E}_{(x,i)}^h\left(R((r,t-r];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};A_j^1(r,t)\big|X_t=y\right)\\
&\lesssim e^{-j^\delta/C}\mathbf{E}_{(x,i)}^h\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathbf{E}_{(x,i)}^h\left(A_j^1(r,t)\big|X_t=y\right)
\end{aligned}$$

for all $y > -\log r, x \geq m(t)$ *and some constant* C.

Proof: First note that $j_1 \geq j \geq [r+r^\delta]$ implies that $r \leq j - j^\delta/2 < j \leq t-r$ and $r \leq t-j < t-j+j^\delta/2 \leq t-r$. When r is large enough, we have

$$\begin{aligned}
&\mathbf{E}_{(x,i)}^h\left(R((r,t-r];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};A_j^1(r,t)\big|X_t=y\right)\\
&\leq \mathbf{E}_{(x,i)}^h\left(R_{t-r}((j-j^\delta/2,j];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};A_j^1(r,t),S(r,t)=S^1(r,t)\big|X_t=y\right)\\
&\quad+\mathbf{E}_{(x,i)}^h\left(R_{t-r}((t-j,t-j+j^\delta/2];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};A_j^1(r,t),S(r,t)=S^2(r,t)\big|X_t=y\right).
\end{aligned}$$

For the first term, when $s \in [2r, t/2]$, by Bramson (1983, (7.30), p. 106),

$$\underline{\mathcal{M}}'_{r,t}(t-s) = m(s_1) - \left(K + \frac{\alpha(r)-m(t)}{t}\right)s^\delta + o_1(1), \tag{7.26}$$

where $s_1 = t-s-s^\delta+o_2(1)$ and $o_1(1), o_2(1) \to 0$ as $r \to \infty$. Since $K > 2+\sqrt{2\lambda^*}$, for r large enough, we have for any $s \in [2r,t/2], t > 4r$ and $s' \in [2r,t/2]\cap[s,s+1)$,

$$\underline{\mathcal{M}}'_{r,t}(t-s') \leq m(s_1) - 2s^\delta.$$

Now we first prove that, for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $r > N(\varepsilon)$, $i \in S, t > 4r$, $s \in [2r, t/2]$ and $0 \le y' \le \underline{\mathcal{M}}'_{r,t}(t-s') + 2s^\delta \le m(s_1)$ with some $s' \in [2r, t/2] \cap [s, s+1)$,

$$v_i^*(t-r-s, y') \ge 1 - \varepsilon. \tag{7.27}$$

When $y' \in [0, m_+(t-r-s)]$, by Lemma 7.1, we can find $N_1(\varepsilon)$ such that for $r > N_1(\varepsilon)$,

$$v_i^*(t-r-s, y') \ge 1 - \varepsilon. \tag{7.28}$$

Combining the above with (7.26) we get (7.27) when $m_+(t-r-s) > m(s_1)$. If $m_+(t-r-s) \le m(s_1)$, then for $y' \in [m_+(t-r-s), m(s_1)]$, by (5.1) and (5.4),

$$\begin{aligned} v_i^*(t-r-s, y') &\ge v_i(t-s, y') - \mathbb{P}_{(0,i)}\left(M_{t-r-s} > y' + \log r\right) \\ &\ge \mathbb{P}_{(0,i)}\left(M_{t-s}^{i_0} > y' - N_1\right) - \mathbb{P}_{(0,i)}\left(M_{t-r-s} > y' + \log r\right) \\ &\ge \mathbb{P}_{(0,i)}\left(M_{t-s}^{i_0} > m(s_1) - N_1\right) - \mathbb{P}_{(0,i)}\left(M_{t-r-s} > m_+(t-r-s) + \log r\right), \end{aligned} \tag{7.29}$$

where $i_0 \in S$ is the type fixed in (1.12). Note that $t - s - s_1 = s^\delta - o_2(1)$ and $m'(s) \ge \sqrt{\lambda^*}$ for large s, when r is large enough,

$$m(t-s) - m(s_1) + N_1 \ge \sqrt{\lambda^*}(t-s-s_1) + N_1 \ge \frac{\sqrt{\lambda^*}}{2} s^\delta + N_1 \to +\infty.$$

Therefore, by Theorem 4.5, there exists $N_2(\varepsilon)$ such that for $r \ge N_2(\varepsilon)$,

$$\begin{aligned} &\mathbb{P}_{(0,i)}\left(M_{t-s}^{i_0} - m(t-s) > m(s_1) - m(t-s) - N_1\right) \ge 1 - \frac{\varepsilon}{2}, \\ &\mathbb{P}_{(0,i)}\left(M_{t-r-s} - m_+(t-r-s) > \log r\right) \le \frac{\varepsilon}{2}. \end{aligned}$$

Putting these inequalities back to (7.29), we have that $v_i^*(t-r-s, y') \ge 1-\varepsilon$ when $r > N_2(\varepsilon)$ and $y' \in [m_+(t-r-s), m(s_1)]$, which, together with (7.28), implies (7.27).

On the event $A_j^1(r,t) \cap \{S(r,t) = S^1(r,t) \in [j, j+1)\}$, set

$$E_j := \left\{X_s - X_{S^1(r,t)} \le 2j^\delta, \quad \forall\, s \in [j - j^\delta/2, S^1(r,t)]\right\}.$$

Then on $A_j^1(r,t) \cap \{S(r,t) = S^1(r,t) \in [j, j+1)\} \cap E_j$, it holds that

$$X_s \le 2j^\delta + X_{S^1(r,t)} = 2j^\delta + \underline{\mathcal{M}}'_{r,t}\left(t - S^1(r,t)\right).$$

By (7.27), uniformly for $i \in S$, on $A_j^1(r,t) \cap \{S(r,t) = S^1(r,t)\} \cap E_j$,

$$v_i^*(t-s, X_s) \ge 1 - \varepsilon, \quad \text{for } s \in [j - j^\delta/2, j].$$

This implies that on $A_j^1(r,t) \cap \{S(r,t) = S^1(r,t)\} \cap E_j$, for $s \in [j - j^\delta/2, j]$,

$$\frac{(1-\rho_{I_s})\varphi_{I_s}^{NL}\left(\mathbf{v}^*(t-s, X_s)\right)}{\sum_{j \ne I_s} \mu_{I_s,j} v_j^*(t-s, X_s)} \le \frac{1}{(1-\varepsilon)(n_{I_j} - \mu_{I_j,I_j})} 1_{\{I_j = j_0\}} + 1_{\{I_j \ne j_0\}}.$$

For the case $I_j = j_0$, we have $n_{j_0} - \mu_{j_0,j_0} > 1$ by assumption. Choose an $\varepsilon > 0$ sufficient small and an appropriate $\eta < 1$ so that

$$\frac{1}{(1-\varepsilon)(n_{j_0} - \mu_{j_0,j_0})} \le \eta < 1.$$

By Bramson (1983, (7.36)), under the assumption $y > -\log r$ and $x \ge m(t)$, for r large enough, we have

$$\mathbf{P}_x\left(E_j^c \middle| A_j^1(r,t), S(r,t) = S^1(r,t), X_t = y\right) \le e^{-j^\delta/4}$$

Using the independence of X and I, we get

$$\begin{aligned}
&\mathbf{E}^h_{(x,i)}\left(R_{t-r}((j-j^\delta/2,j];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};A^1_j(r,t),S(r,t)=S^1(r,t)\big|X_t=y\right)\\
&\le \mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};E^c_j,A^1_j(r,t),S(r,t)=S^1(r,t)\big|X_t=y\right)\\
&\quad+\mathbf{E}^h_{(x,i)}\Big(\exp\Big\{\sum_{s\in D_J,j-j^\delta/2<s\le j}1_{\{I_s=j_0\}}\log\eta\Big\}\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};E_j,A^1_j(r,t),S(r,t)=S^1(r,t)\big|X_t=y\Big)\\
&\le e^{-j^\delta/4}\mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathbf{P}^h_{(x,i)}\left(A^1_j(r,t),S(r,t)=S^1(r,t)\big|X_t=y\right)\\
&\quad+\mathbf{E}^h_{(x,i)}\Big(\exp\Big\{\sum_{s\in D_J,j-j^\delta/2<s\le j}1_{\{I_s=j_0\}}\log\eta\Big\}\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\Big)\mathbf{P}^h_{(x,i)}\left(A^1_j(r,t),S(r,t)=S^1(r,t)\big|X_t=y\right).
\end{aligned}$$

By Lemma 7.2, and the fact that $\inf_{j\in S}\inf_{r\ge1,t>4r}\mathbf{P}^h_{(x,i)}(I_{t-r}=j)>c>0$, we have

$$\begin{aligned}
&\mathbf{E}^h_{(x,i)}\left(\exp\left\{\sum_{s\in D_J,j-j^\delta/2<s\le j}1_{\{I_s=j_0\}}\log\eta\right\}\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\\
&\le\sum_{j=1}^d\frac{v_j(r,y)}{h_j}\sup_{x\in\mathbb{R},\ell\in S}\mathbf{E}^h_{(x,\ell)}\left(\exp\left\{\sum_{s\in D_J,s\le j^\delta/2}1_{\{I_s=j_0\}}\log\eta\right\}\right)\le C_\eta\sum_{j=1}^d\frac{v_j(r,y)}{h_j}e^{-\varepsilon j^\delta/2}\\
&\le\frac{C_\eta}{c}\mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)e^{-\varepsilon j^\delta/2}.
\end{aligned}$$

Combining the two displays above, we get

$$\begin{aligned}
&\mathbf{E}^h_{(x,i)}\left(R_{t-r}((j-j^\delta/2,j];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};A^1_j(r,t),S(r,t)=S^1(r,t)\big|X_t=y\right)\\
&\le\mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathbf{P}^h_{(x,i)}\left(A^1_j(r,t),S(r,t)=S^1(r,t)\big|X_t=y\right)\left(e^{-j^\delta/4}+\frac{C_\eta}{c}e^{-\varepsilon j^\delta/2}\right)\\
&\lesssim e^{-j^\delta/C}\mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathbf{P}^h_{(x,i)}\left(A^1_j(r,t),S(r,t)=S^1(r,t)\big|X_t=y\right)
\end{aligned}$$

with

$$C=\max\left\{4,\frac{2}{\varepsilon}\right\}.$$

The second term can be treated similarly. Note that the case that $\mu_{j_1,j_1}>0$ is similar by Lemma 7.3. Thus the assertion of the lemma is valid. □

Now we are ready to prove Proposition 5.3.

Proof: Proof of Proposition 5.3: Note that

$$(\mathcal{B}_{up})^c\subset\bigcup_{j=[r+r^\delta]}^{j_1}A_j(r,t)=\bigcup_{j=[r+r^\delta]}^{j_1}A^1_j(r,t)\bigcup\bigcup_{j=[r+r^\delta]}^{j_1}A^2_j(r,t).$$

By Lemma 7.4, and the independence of X and I, for large r, when $y > -\log r$ and $x \geq m(t)$,

$$\begin{aligned}
&\mathbf{E}^h_{(x,i)}\left(R((r,t-r];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};(\mathcal{B}_{up})^c\big|X_t=y\right)\\
&\leq \sum_{j=[r+r^\delta]}^{j_1}\mathbf{E}^h_{(x,i)}\left(R((r,t-r];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};A^1_j(r,t)\big|X_t=y\right)\\
&\quad+\sum_{j=[r+r^\delta]}^{j_1}\mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};A^2_j(r,t)\big|X_t=y\right)\\
&\lesssim \sum_{j=[r+r^\delta]}^{j_1}e^{-j^\delta/C}\mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathbf{P}^h_{(x,i)}\left(A^1_j(r,t)\big|X_t=y\right)\\
&\quad+\sum_{j=[r+r^\delta]}^{j_1}\mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathbf{P}^h_{(x,i)}\left(A^2_j(r,t)\big|X_t=y\right).
\end{aligned}$$

Note that the estimates of the probabilities of $A^1_j(r,t)$ and $A^2_j(r,t)$ only relies on the path of Brownian bridge, using Lemma 5.1 and the argument on Bramson (1983, p. 109), we get

$$\begin{aligned}
&\sum_{j=[r+r^\delta]}^{j_1}e^{-j^\delta/C}\mathbf{P}^h_{(x,i)}\left(A^1_j(r,t)\big|X_t=y\right)\lesssim\sum_{j=[r+r^\delta]}^{j_1}e^{-j^\delta/C}\mathbf{P}^h_{(x,i)}\left(\mathcal{B}^{j+1}_{up}\big|X_t=y\right)\\
&\lesssim\sum_{j=[r+r^\delta]}^{j_1}e^{-j^\delta/C}\frac{j+1}{r}\mathbf{P}^h_{(x,i)}\left(\mathcal{B}_{up}\big|X_t=y\right)\leq\frac{1}{r}\mathbf{P}^h_{(x,i)}\left(\mathcal{B}_{up}\big|X_t=y\right)\times\sum_{j=[r+r^\delta]}^{\infty}(j+1)e^{-j^\delta/C},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=[r+r^\delta]}^{j_1}\mathbf{P}^h_{(x,i)}\left(A^2_j(r,t)\big|X_t=y\right)\leq\sum_{j=[r+r^\delta]}^{j_1}e^{-j/2}\mathbf{P}^h_{(x,i)}\left(\mathcal{B}^{j+1}_{up}\big|X_t=y\right)\\
&\lesssim\frac{1}{r}\mathbf{P}^h_{(x,i)}\left(\mathcal{B}_{up}\big|X_t=y\right)\times\sum_{j=[r+r^\delta]}^{\infty}(j+1)e^{-j/2}.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
&\mathbf{E}^h_{(x,i)}\left(R((r,t-r];v_*)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};(\mathcal{B}_{up})^c\big|X_t=y\right) \qquad (7.30)\\
&\lesssim\mathbf{E}^h_{(x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\times\frac{1}{r}\mathbf{P}^h_{(x,i)}\left(\mathcal{B}_{up}\big|X_t=y\right)\times\left(\sum_{j=[r+r^\delta]}^{\infty}(j+1)e^{-j^\delta/C}+\sum_{j=[r+r^\delta]}^{\infty}(j+1)e^{-j/2}\right).
\end{aligned}$$

As $r\to\infty$, the last term of (7.30) decays faster than r^{-1}, thus the assertion of the proposition is valid. □

7.6. *Proof of Proposition 5.5.*

Proof of Proposition 5.5: When $s\in[0,2r]$, on the set $\mathcal{B}_{low}$, for $t,x>8r$ and r large enough, we have

$$X_s>\overline{\mathcal{M}}^x_{r,t}(t-s)\geq m(t)+4r\geq m(t-s)+4r+\sqrt{2\lambda^*}s+O(1)\geq m(t-s)+r+N_2,$$

where N_2 is the constant in (1.12). Note that for any $y > m(t-s)+r+N_2$, by (1.12), (1.6) and Proposition 4.2,

$$\begin{aligned}
v_i(t-s,y) &= \mathbb{E}_{(y,i)}\left(1-\prod_{u\in Z(t-s)}\left(1-v_{I_u(t-s)}(0,X_u(t-s))\right)\right)\\
&\le \mathbb{E}_{(y,i)}\left(1-\prod_{u\in Z(t-s)}\left(1-1_{(-\infty,N_2)}(X_u(t-s))\right)\right)\\
&= \mathbb{P}_{(0,i)}(M_{t-s} > y-N_2) \le \mathbb{P}_{(0,i)}(M_{t-s}\ge m(t-s)+r) \lesssim re^{-\sqrt{2\lambda^*}r}.
\end{aligned}$$

Using (3.13) and noting that $\log x \sim x-1$ as $x\to 1$, we get that when r is large enough, on $\mathcal{B}_{low}$,

$$\sum_{s\in D_J, s\le 2r}\log\left(\frac{(1-\rho_{I_s})\varphi^{NL}_{I_s}(\mathbf{v}(t-s,X_s))}{\sum_{j\ne I_s}\mu_{I_s,j}v_j(t-s,X_s)}\right) \gtrsim -\sum_{s\in D_J,s\le 2r} r^{\alpha_0}e^{-\alpha_0\sqrt{2\lambda^*}r} =: -\Gamma_1^{NL}(0,2r) \tag{7.31}$$

and

$$-\int_0^{2r} a_{I_s}\left(\mu_{I_s,I_s}-\frac{\rho_{I_s}\varphi^L_{I_s}(\mathbf{v}(t-s,X_s))}{v_{I_s}(t-s,X_s)}\right)\mathrm{d}s \gtrsim -\int_0^{2r} r^{\alpha_0}e^{-\alpha_0\sqrt{2\lambda^*}r}\mathrm{d}s =: -\Gamma_1^L(0,2r). \tag{7.32}$$

Now we deal with the case $s\in[2r,t-r]$. Similar as above, when r is large enough, for all $s\in[2r,t-r]$ and $y > m(s+s^\delta\wedge(t-s)^\delta)$,

$$\begin{aligned}
v_i(s,y) &\le \mathbb{P}_{(0,i)}\left(M_s > y-N_2\right) \le \mathbb{P}_{(0,i)}\left(M_s - m(s) > m(s+s^\delta\wedge(t-s)^\delta)-m(s)-N_2\right)\\
&= \mathbb{P}_{(0,i)}\left(M_s-m(s) > \sqrt{2\lambda^*}\left(s^\delta\wedge(t-s)^\delta\right)+O(1)\right) \lesssim \left(s^\delta\wedge(t-s)^\delta\right)e^{-2\lambda^*\left(s^\delta\wedge(t-s)^\delta\right)}\\
&\lesssim e^{-\lambda^*\left(s^\delta\wedge(t-s)^\delta\right)}.
\end{aligned}$$

In this case, when r is large enough (see the display below Ren et al. (2021, (2.14))),

$$\overline{\mathcal{M}}^x_{r,t}(s) \ge m(s+s^\delta\wedge(t-s)^\delta)+s^\delta\wedge(t-s)^\delta\left(K-\frac{m(t)}{t}-\frac{\log r}{t}\right) > m(s+s^\delta\wedge(t-s)^\delta)$$

since $K>\sqrt{2\lambda^*}$. Therefore, on $\mathcal{B}_{low}$, when r is large enough,

$$\begin{aligned}
\sum_{s\in D_J,2r<s\le t-r}\log\left(\frac{(1-\rho_{I_s})\varphi^{NL}_{I_s}(\mathbf{v}(t-s,X_s))}{\sum_{j\ne I_s}\mu_{I_s,j}v_j(t-s,X_s)}\right) &\gtrsim -\sum_{s\in D_J,2r<s\le t-r} e^{-\lambda^*\left(s^\delta\wedge(t-s)^\delta\right)}\\
&=: -\Gamma_2^{NL}(2r,t-r).
\end{aligned} \tag{7.33}$$

Similarly,

$$\begin{aligned}
-\int_{2r}^{t-r} a_{I_s}\left(\mu_{I_s,I_s}-\frac{\rho_{I_s}\varphi^L_{I_s}(\mathbf{v}(t-s,X_s))}{v_{I_s}(t-s,X_s)}\right)\mathrm{d}s &\gtrsim -\int_{2r}^{\infty} e^{-\lambda^*\left(s^\delta\wedge(t-s)^\delta\right)}\mathrm{d}s\\
&=: -\Gamma_2^L(2r,\infty).
\end{aligned} \tag{7.34}$$

Note that $\mathcal{B}_{low}$ is independent of D_J. By (7.31), (7.32), (7.33) and (7.34), there exists a constant c such that

$$\begin{aligned}
v_i(t,m(t)+x) &\geq e^{\lambda^*(t-r)}h_i\int_{\mathbb{R}}\frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}}\\
&\quad\times \mathbf{E}^h_{(m(t)+x,i)}\left(\exp\left\{-c\Gamma_1(0,2r)-c\Gamma_2(2r,t-r)\right\}\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}};\mathcal{B}_{low}\Big|X_{t-r}=y\right)\mathrm{d}y\\
&= e^{\lambda^*(t-r)}h_i\int_{\mathbb{R}}\frac{e^{-\frac{(m(t)+x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}}\times \mathbf{P}^h_{(m(t)+x,i)}\left(\mathcal{B}_{low}\Big|X_{t-r}=y\right)e^{-c\Gamma_1^L(0,2r)-c\Gamma_2^L(2r,\infty)}\\
&\quad\times \mathbf{E}^h_{(m(t)+x,i)}\left(\exp\left\{-c\Gamma_1^{NL}(0,2r)-c\Gamma_2^{NL}(2r,t-r)\right\}\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathrm{d}y.
\end{aligned}$$

Since when r is large enough, $\mathbf{P}^h_{(x,i)}\left(I_{t-r}=\ell\right)\geq c'>0$ for all $t>8r$ and $i,\ell\in S$, we get that

$$\begin{aligned}
&\mathbf{E}^h_{(m(t)+x,i)}\left(\exp\left\{-c\Gamma_1^{NL}(0,2r)-c\Gamma_2^{NL}(2r,t-r)\right\}\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathrm{d}y\\
&=\mathbf{E}^h_{(m(t)+x,i)}\Bigg(\Bigg(1-\exp\Bigg\{-c\sum_{s\in D_J,s\leq 2r}r^{\alpha_0}e^{-\alpha_0\sqrt{2\lambda^*}r}-c\sum_{s\in D_J,2r<s\leq t-r}e^{-\lambda^*(s^\delta\wedge(t-s)^\delta)}\Bigg\}\Bigg)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\Bigg)\\
&=:\mathbf{E}^h_{(m(t)+x,i)}\left(M(r,t)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\leq\sum_{j=1}^d\frac{v_j(r,y)}{h_j}\mathbf{E}^h_{(m(t)+x,i)}\left(M(r,t)\right)\\
&\leq\frac{1}{c'}\mathbf{E}^h_{(m(t)+x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\mathbf{E}^h_{(m(t)+x,i)}\left(M(r,t)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}^h_{(m(t)+x,i)}\Bigg(\exp\Bigg\{-c\sum_{s\in D_J,s\leq 2r}r^{\alpha_0}e^{-\alpha_0\sqrt{2\lambda^*}r}-c\sum_{s\in D_J,2r<s\leq t-r}e^{-\lambda^*(s^\delta\wedge(t-s)^\delta)}\Bigg\}\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\Bigg)\\
&=\mathbf{E}^h_{(m(t)+x,i)}\left(\left(1-M(r,t)\right)\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\\
&\geq\mathbf{E}^h_{(m(t)+x,i)}\left(\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\times\left(1-\frac{1}{c'}\mathbf{E}^h_{(m(t)+x,i)}\left(M(r,t)\right)\right).
\end{aligned}$$

By the Markov property, we see that

$$\begin{aligned}
&\mathbf{E}^h_{(m(t)+x,i)}\Bigg(\exp\Bigg\{-c\sum_{s\in D_J,s\leq 2r}r^{\alpha_0}e^{-\alpha_0\sqrt{2\lambda^*}r}-c\sum_{s\in D_J,2r<s\leq t-r}e^{-\lambda^*(s^\delta\wedge(t-s)^\delta)}\Bigg\}\Bigg)\\
&\geq\mathbf{E}^h_{(m(t)+x,i)}\Bigg(\exp\Bigg\{-c\sum_{s\in D_J,s\leq 2r}r^{\alpha_0}e^{-\alpha_0\sqrt{2\lambda^*}r}\Bigg\}\Bigg)\\
&\quad\times\inf_{j\in S}\mathbf{E}^h_{(m(t)+x,j)}\Bigg(\exp\Bigg\{-c\sum_{s\in D_J,s\leq -2r+t/2}e^{-\lambda^*(s+2r)^\delta}\Bigg\}\Bigg)\\
&\quad\times\inf_{\ell\in S}\mathbf{E}^h_{(m(t)+x,\ell)}\Bigg(\exp\Bigg\{-c\sum_{s\in D_J,s\leq t/2-r}e^{-\lambda^*(t/2-s)^\delta}\Bigg\}\Bigg).
\end{aligned}\tag{7.35}$$

By (4.21), the product of the first two terms on the right-hand side of (7.35) is bounded from below by

$$\exp\left\{-\overline{a}\int_0^{2r}\left(1-e^{-cr^{\alpha_0}e^{-\alpha_0\sqrt{2\lambda^*}r}}\right)\mathrm{d}s-\overline{a}\int_{2r}^{\infty}\left(1-e^{-ce^{-\lambda^* s^\delta}}\right)\mathrm{d}s\right\}=:F_1(r).$$

For the last term of the right-hand side of (7.35), let $\lceil x\rceil$ be the smaller integer larger than x, then by the Markov property and (4.21),

$$\begin{aligned}
&\mathbf{E}^h_{(m(t)+x,\ell)}\left(\exp\left\{-c\sum_{s\in D_J,s\le t/2-r}e^{-\lambda^*(t/2-s)^\delta}\right\}\right)\\
&\ge \mathbf{E}^h_{(m(t)+x,\ell)}\left(\exp\left\{-\sum_{k=1}^{\lceil t/2-r\rceil}c\sum_{s\in D_J,k-1<s\le k\wedge(t/2-r)}e^{-\lambda^*(t/2-k)^\delta}\right\}\right)\\
&\ge \prod_{k=1}^{\lceil t/2-r\rceil}\inf_{\ell_k\in S}\mathbf{E}^h_{(m(t)+x,\ell_k)}\left(\exp\left\{-c\sum_{s\in D_J,s\le 1\wedge(t/2-r-k+1)}e^{-\lambda^*(t/2-k)^\delta}\right\}\right)\\
&\ge \prod_{k=1}^{\lceil t/2-r\rceil}\exp\left\{-\underline{a}\int_0^{1\wedge(t/2-r-k+1)}\left(1-e^{-ce^{-\lambda^*(t/2-k)^\delta}}\right)\mathrm{d}s\right\}\\
&= \prod_{k=1}^{\lceil t/2-r\rceil}\exp\left\{-\underline{a}\int_{k-1}^{k\wedge(t/2-r)}\left(1-e^{-ce^{-\lambda^*(t/2-k)^\delta}}\right)\mathrm{d}s\right\}\\
&\ge \exp\left\{-\underline{a}\int_0^{t/2-r}\left(1-e^{-ce^{-\lambda^*(t/2-s+1)^\delta}}\right)\mathrm{d}s\right\}\ge \exp\left\{-\underline{a}\int_{r+1}^{\infty}\left(1-e^{-ce^{-\lambda^* s^\delta}}\right)\mathrm{d}s\right\}=:F_2(r).
\end{aligned}$$

By the definition of $M(r,t)$, we conclude that for large r and $t>8r$,

$$\mathbf{E}^h_{(m(t)+x,i)}\left(M(r,t)\right)\le 1-F_1(r)F_2(r).$$

Since $\lim_{t\to\infty}\mathbf{P}^h_{(x,i)}(I_t=j)=\lim_{r\to\infty}\inf_{t>r}\mathbf{P}^h_{(x,i)}(I_t=j)=g_jh_j$, we have

$$\begin{aligned}
&e^{-c\Gamma_1^L(0,2r)-c\Gamma_2^L(2r,\infty)}\\
&\times\mathbf{E}^h_{(m(t)+x,i)}\left(\exp\left\{-c\sum_{s\in D_J,s\le 2r}r^{\alpha_0}e^{-\alpha_0\sqrt{2\lambda^*}r}-c\sum_{s\in D_J,2r<s\le t-r}e^{-\lambda^* s^\delta\wedge(t-s)^\delta}\right\}\frac{v_{I_{t-r}}(r,y)}{h_{I_{t-r}}}\right)\\
&\ge\inf_{j\in S}\inf_{t>r}\frac{\mathbf{P}^h_{(x,i)}(I_t=j)}{g_jh_j}e^{-c\Gamma_1^L(0,2r)-c\Gamma_2^L(2r,\infty)}\left(1-\frac{1}{c'}\left(1-F_1(r)F_2(r)\right)\right)\sum_{j=1}^d g_jv_j(r,y)\\
&=:C_{low}(r)\sum_{j=1}^d g_jv_j(r,y).
\end{aligned}$$

It is easy to see that $C_{low}(r)\uparrow 1$ as $r\to\infty$. The proof is now complete. □

7.7. *Proof of Proposition 6.5.*

Proof of Proposition 6.5: By (6.11), for any $x > 0$,

$$\begin{aligned}&\lim_{t\to\infty}\mathbb{P}_{(0,i)}\left(M_t-\sqrt{2\lambda^*}t-z>x\middle|M_t>\sqrt{2\lambda^*}t+z\right)\\&=\lim_{t\to\infty}\frac{\frac{t^{3/2}}{\frac{3}{2\sqrt{2\lambda^*}}\log t}\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+x+z\right)}{\frac{t^{3/2}}{\frac{3}{2\sqrt{2\lambda^*}}\log t}\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+z\right)}=e^{-\sqrt{2\lambda^*}x}.\end{aligned}\tag{7.36}$$

Then we have under $\mathbb{P}_{(0,i)}\left(\cdot\middle|M_t>\sqrt{2\lambda^*}t+z\right)$, $M_t-\sqrt{2\lambda^*}t-z$ converges in distribution to Y, an exponential random variable with parameter $\sqrt{2\lambda^*}$. For any $\phi\in\mathcal{C}_c^+(\mathbb{R}\times S)$ and $x>0$,

$$\begin{aligned}&\mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi(y,j)\left(\overline{\mathcal{E}}_t-z\right)(\mathrm{d}y\mathrm{d}j)\right\};M_t>\sqrt{2\lambda^*}t+z+x\middle|M_t>\sqrt{2\lambda^*}t+z\right)\\&=\frac{1}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+z\right)}\mathbb{E}_{(0,i)}\left(\prod_{u\in Z(t)}e^{-\phi(X_u(t)-\sqrt{2\lambda^*}t-z,I_u(t))};M_t>\sqrt{2\lambda^*}t+z+x\right)\\&=\frac{1}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+z\right)}\left(1-\mathbb{E}_{(0,i)}\left(\prod_{u\in Z(t)}e^{-\phi(X_u(t)-\sqrt{2\lambda^*}t-z,I_u(t))};M_t\le\sqrt{2\lambda^*}t+z+x\right)\right)\\&\quad-\frac{1}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+z\right)}\mathbb{E}_{(0,i)}\left(1-\prod_{u\in Z(t)}e^{-\phi(X_u(t)-\sqrt{2\lambda^*}t-z,I_u(t))}\right)\\&=:\frac{1}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+z\right)}(v_1)_i(t,\sqrt{2\lambda^*}t+z)-\frac{1}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+z\right)}(v_2)_i(t,\sqrt{2\lambda^*}t+z)\end{aligned}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ solve (1.5) with

$$(v_1)_i(0,y)=1-e^{-\phi(-y,i)}1_{\{-y\le x\}},\quad (v_2)_i(0,y)=1-e^{-\phi(-y,i)},\quad i\in S\tag{7.37}$$

according to (6.3). Using Lemma 6.4 and (6.11), it is easy to see that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+z\right)}(v_1)_i(t,\sqrt{2\lambda^*}t+z)-\lim_{t\to\infty}\frac{1}{\mathbb{P}_{(0,i)}\left(M_t>\sqrt{2\lambda^*}t+z\right)}(v_2)_i(t,\sqrt{2\lambda^*}t+z)\\&=\frac{\widetilde{C}(\phi,x)}{C_\infty}-\frac{C(\phi)}{C_\infty}\end{aligned}\tag{7.38}$$

with $\widetilde{C}(\phi,x)$ being defined by (6.13), and the right-hand side of (7.38) is independent of $z\in\mathbb{R}$ and $i\in S$. Let $x=0$ in (7.38), then

$$\lim_{t\to\infty}\mathbb{E}_{(0,i)}\left(\exp\left\{-\int\phi(y,j)\left(\overline{\mathcal{E}}_t-z\right)(\mathrm{d}y\mathrm{d}j)\right\}\middle|M_t>\sqrt{2\lambda^*}t+z\right)=\frac{\widetilde{C}(\phi,0)}{C_\infty}-\frac{C(\phi)}{C_\infty}.\tag{7.39}$$

Note that $\left(\overline{\mathcal{E}}_t-z\right)$ under $\mathbb{P}_{(0,i)}\left(\cdot|M_t>\sqrt{2\lambda^*}t+z\right)$ is still a point process. We now prove the convergence of $\left(\overline{\mathcal{E}}_t-z\right)$ in distribution under $\mathbb{P}_{(0,i)}\left(\cdot|M_t>\sqrt{2\lambda^*}t+z\right)$. By (7.39), it suffices to prove that

$$\lim_{\theta\downarrow0}\left(\frac{\widetilde{C}(\theta\phi,0)}{C_\infty}-\frac{C(\theta\phi)}{C_\infty}\right)=1.\tag{7.40}$$

By Corollary 6.2, we have $\lim_{\theta\downarrow 0} C(\theta\phi) = 0$. Note that the initial value of $\mathbf{v}_1$ in (7.37) with $x = 0$ satisfies condition (1.12), it follows from Theorem 1.1 that

$$\lim_{t\to\infty} \mathbb{E}_{(0,i)}\left(\prod_{u\in Z(t)} e^{-\theta\phi(X_u(t)-m(t)-z,I_u(t))}; M_t \le m(t)+z\right)$$
$$= \mathbb{E}_{(0,i)}\left(\exp\left\{-\widetilde{C}(\theta\phi,0)M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}z}\right\}\right)$$

and by Corollary 1.2,

$$\lim_{t\to\infty} \mathbb{P}_{(0,i)}\left(M_t \le m(t)+z\right) = \mathbb{E}_{(0,i)}\left(\exp\left\{-C_\infty M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}z}\right\}\right).$$

Also note that

$$\left|\mathbb{P}_{(0,i)}\left(M_t \le m(t)+z\right) - \mathbb{E}_{(0,i)}\left(\prod_{u\in Z(t)} e^{-\theta\phi(X_u(t)-m(t)-z,I_u(t))}; M_t \le m(t)+z\right)\right|$$
$$\le 1 - \mathbb{E}_{(0,i)}\left(\exp\left\{-\theta\int \phi(y-z,j)\mathcal{E}_t(\mathrm{d}y\mathrm{d}j)\right\}\right).$$

Letting $t\to\infty$, we get that

$$\left|\mathbb{E}_{(0,i)}\left(\exp\left\{-C_\infty M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}z}\right\}\right) - \mathbb{E}_{(0,i)}\left(\exp\left\{-\widetilde{C}(\theta\phi,0)M_{\sqrt{2\lambda^*}}(\infty)e^{-\sqrt{2\lambda^*}z}\right\}\right)\right|$$
$$\le 1 - \mathbb{E}_{(0,i)}\left(\exp\left\{-C(\theta\phi)M_{\sqrt{2\lambda^*}}(\infty)e^{\sqrt{2\lambda^*}z}\right\}\right).$$

Let $\theta\downarrow 0$, we get that $\lim_{\theta\downarrow 0}\widetilde{C}(\theta\phi,0) = C_\infty$, which implies (7.40). Combining (7.36), (7.38), (7.40) and the fact that the process (X_t, Y_t) is tight if X_t and Y_t are both tight, which follows from the inequality

$$\inf_{t>0}\mathbb{P}\left(|X_t|\le K, |Y_t|\le K\right) \ge \inf_{t>0}\mathbb{P}\left(|X_t|\le K\right) + \inf_{t>0}\mathbb{P}\left(|Y_t|\le K\right) - 1,$$

we get that under $\mathbb{P}_{(0,i)}$,

$$\left(\overline{\mathcal{E}}_t - z, M_t - \sqrt{2\lambda^*}t - z\right)|_{M_t > \sqrt{2\lambda^*}t+z}$$

converges joint in distribution to $(\overline{\mathcal{E}}_\infty, Y)$, where the joint law is given in (6.12) and is independent of $z\in\mathbb{R}$ and $i\in S$. □

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