ALEA, Lat. Am. J. Probab. Math. Stat. **21**, 1495–1525 (2024)
DOI: 10.30757/ALEA.v21-56

# Limit theorems for iid products of positive matrices

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**Abstract.** We study stochastic properties of the norm cocycle associated with iid products of positive matrices. We obtain the almost sure invariance principle (ASIP) with rate $o(n^{1/p})$ under the optimal condition of a moment of order $p > 2$ and the Berry-Esseen theorem with rate $O(1/\sqrt{n})$ under the optimal condition of a moment of order 3. The results are also valid for the matrix norm. For the matrix coefficients, we also have the ASIP but we obtain only partial results for the Berry-Esseen theorem. The proofs make use of coupling coefficients that surprisingly decay exponentially fast to 0 while there is only a polynomial decay in the case of invertible matrices. All the results are actually valid in the context of iid products of matrices leaving invariant a suitable cone.

## 1. Introduction

Let $d \geq 2$ be an integer. Let $G$ be the semi-group of $d$-dimensional positive allowable matrices: by positive, we mean that all entries are greater than or equal to 0, by allowable, we mean that any row and any column admits a strictly positive element. We endow $\mathbb{R}^d$ with the $\ell^1$ norm and $G$ with the corresponding operator norm. We denote both norms by $\|\cdot\|$. Recall that $\|g\| = \sup_{\|x\|=1} \|gx\|$. Define also

$$S^+ := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \,:\, \|x\| = 1 \text{ and } x_i \geq 0, \, \forall i \in \{1, \ldots, d\}\,\}\,. \tag{1.1}$$

Let $\mu$ be a probability on the Borel sets of $G$. Let $(Y_n)_{n\in\mathbb{N}}$ be independent and identically distributed (iid) random variables with law $\mu$ living on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $n \in \mathbb{N}$, set $A_n := Y_n \cdots Y_1$. We wish to study the asymptotic behaviour of the sequences $(\log \|A_n\|)_{n\in\mathbb{N}}$ and $(\log \|A_n x\|)_{n\in\mathbb{N}}$, $x \in S^+$. Other sequences of interest are $(\log\langle A_n x, y\rangle)_{n\in\mathbb{N}}$ for $x, y \in S^+$, where

*Received by the editors January 23rd, 2023; accepted June 11th, 2024.*
2010 *Mathematics Subject Classification.* 60F05, 60B15, 60G50.
*Key words and phrases.* Random walk; Positive matrices; Cocycle; Almost sure invariance principle; Berry-Esseen theorem.
The first author was supported by the ANR project "Rawabranch" number ANR-23-CE40-0008, and the France 2030 framework program, Centre Henri Lebesgue ANR-11-LABX-0020-01.

$\langle \cdot, \cdot \rangle$ stands for the standard inner product on $\mathbb{R}^d$; $(\log \kappa(A_n))_{n\in\mathbb{N}}$, with $\kappa$ the spectral radius, $(\log(\inf_{x\in S^+} \|A_n x\|))_{n\in\mathbb{N}}$ or $(\log \inf_{x,\, y\in S^+} \log\langle A_n x, y\rangle)_{n\in\mathbb{N}}$.

In a series of papers Cuny et al. (2017a,b, 2018, 2023, 2022) we studied the stochastic properties of the norm cocycle (i.e. $(\log \|A_n x\|)_{n\in\mathbb{N}}$) associated with the left random walk on $GL_d(\mathbb{R})$ under optimal or close to optimal moment conditions. The moment conditions required in these works are in particular optimal in case of the almost sure invariance principle (ASIP) with rate, and close to optimal in the case of the Berry-Esseen theorem. We also obtained results for the matrix norm, the matrix coefficients and the spectral radius. A key ingredient to get these results is to obtain a suitable control of some coupling coefficients introduced in Cuny et al. (2017a) (in the context of invertible matrices), under appropriate moment conditions on $\mu$.

In the context of positive matrices, we are even able to control the stronger coupling coefficients $\tilde{\delta}_{p,\infty}(n)$ defined in Section 4 (see (4) for the exact definition). As we shall see, in the context of positive matrices, these couplings coefficients decrease exponentially fast even if $\mu$ has only polynomial moments, in contrast with the case of invertible matrices where the decay is only arithmetical. More precisely we shall prove in Proposition 4.1 below that, when $\mu$ is strictly contracting and almost admits a moment of order $p \geq 1$ (see Section 2 for a definition of these notions), there exists $a \in ]0, 1[$ such that $\tilde{\delta}_{p,\infty}(n) = O(a^n)$ .

As we already mentioned, a suitable control of this kind of coefficients (together with the Markovian structure of the random walk) is one of the main arguments used in Cuny et al. (2018) and Cuny et al. (2023) for obtaining rates in the ASIP, as well as Berry-Esseen type bounds in the case of the left random walk on $GL_d(\mathbb{R})$. We follow this strategy in the context of positive matrices in Section 7, where we obtain rates of order $o(n^{1/p})$ in the ASIP when $\mu$ has a moment of order $p > 2$, and in Section 8 where we obtain rates of order $O(n^{1-p/2})$ for Berry-Esseen type bounds (for the quantities $\log \|A_n\|$ and $\log \|A_n x\|$) when $\mu$ has a moment of order $p \in ]2, 3]$.

Let us mention that the study of iid products of positive matrices benefited from a lot of works. Let us cite, among others, Cohn et al. (1993), Hennion (1997), Hennion and Hervé (2004), Buraczewski et al. (2014), Buraczewski and Mentemeier (2016) or Xiao et al. (2020, 2022a, 2024, 2022b).

Hennion (1997) obtained the strong law of large numbers and the central limit theorem (CLT) under optimal moment conditions in the more general situation of products of dependent positive random matrices satisfying some mixing conditions. All the other above mentioned papers, except Hennion and Hervé (2004) and Xiao et al. (2022b), assume that $\mu$ has an exponential moment, which allows to use in a natural way the Guivarc'h-Nagaev method based on perturbation of operators.

In fact, in the context of products of positive random matrices, Hennion and Hervé (2004) and Xiao et al. (2022b) recently observed that the Guivarc'h-Nagaev method applies under polynomial moment conditions. In particular, Hennion and Hervé obtained the Berry-Esseen theorem with rate $O(1/\sqrt{n})$, under a moment of order $p > 4$ while Xiao, Grama and Liu obtained the same rate (as well as a first order Edgeworth expansion) under a moment of order 3, but assuming an additional technical condition (see their condition **A2**, that we shall discuss at the end of Section 8.2).

The paper is organised as follows. In Section 2, we introduce some notations and definitions and we also recall several key properties in the study of positive matrices. In Section 3 we recall some already known results on the existence of an invariant measure for $\mu$, the law of large numbers, the central limit theorem and the Berry-Esseen theorem for $\log \|A_n x\|$, $\log \|A_n\|$, and some related quantities. In Section 4, we define the coupling coefficients adapted to the case of positive matrices, and show that they decrease geometrically as soon as $\mu$ has a moment of order 1 (see Proposition 4.1). In Section 5, we provide some complementary results on the strong law of large numbers that can be obtained via Proposition 4.1. In Section 6, we provide several identifications of the asymptotic variance $s^2$ in the CLT. Moreover, we show that the known aperiodicity condition (see Definition 6.2) is sufficient for $s^2 > 0$, under a moment of order 2. In Section 7, we obtain the ASIP for the norm cocycle, the matrix norm, the spectral radius and the matrix coefficients under optimal polynomial moment conditions. In Section 8, we obtain the Berry-Esseen theorem for all the above mentioned quantities. The obtained rates are optimal (in terms of moment conditions) in the case of the norm cocyle and the matrix norm, but we have a loss in the case of the spectral radius and the matrix coefficients. In Section 9 we study the regularity of the invariant measure and in Section 10, we provide some deviation inequalities for the norm cocycle and the matrix coefficients. In Section 11, we explain how to generalise our results to matrices leaving invariant a suitable cone (notice that the positive matrices of size $d$ may be seen as the matrices leaving invariant the cone $(\mathbb{R}^+)^d$). Finally, in Section 12, we provide technical results related to the previous section.

In all the paper we denote $\mathbb{N} := \{1, 2, \ldots\}$.

## 2. Norm cocycle and matrix norm

We put on $G$ the topology inherited from (the distance associated with) the norm. Then, $G$ becomes a locally compact space. Let $G^+$ be the sub-semi-group of $G$ whose entries are all strictly positive. Actually, $G^+$ is the interior of $G$.

Notice that for $g \in G$, we actually have $\|g\| = \sup_{x\in S^+} \|gx\|$ and that, if $g = (g_{ij})_{1\le i,\, j\le d}$,

$$\|g\| = \max_{1\le j\le d} \sum_{i=1}^{d} g_{ij} \,. \tag{2.1}$$

For every $g \in G$, set $v(g) = \inf_{x\in S^+} \|gx\|$. If $g = (g_{ij})_{1\le i,\, j\le d}$, we have

$$v(g) = \min_{1\le j\le d} \sum_{i=1}^{d} g_{ij} \,. \tag{2.2}$$

By definition of $G$, $v(g) > 0$ for every $g \in G$.

We then define $N(g) := \max(\|g\|, 1/v(g))$ and $L(g) = \frac{\|g\|}{v(g)}$. Notice that $N(g)^2 \ge L(g) \ge 1$ for every $g \in G$.

We endow $S^+$ with the following metric (see Proposition 11.1 for a proof that it is indeed a metric). For every $x$, $y \in S^+$,

$$d(x, y) = \varphi(m(x, y)m(y, x)) \,,$$

where

$$\varphi(s) = \frac{1-s}{1+s} \qquad \forall s \in [0, 1] \,, \tag{2.3}$$

and

$$m(u, v) = \inf \left\{ \frac{u_i}{v_i} \,:\, i \in \{1, \ldots, d\},\, v_i > 0 \right\} .$$

Notice that the diameter of $S^+$ is 1 and that $d(x, y) = 1$ if and only if there exists $i_0 \in \{1, \ldots, d\}$ such that $x_{i_0} = 0$ and $y_{i_0} > 0$ or $x_{i_0} > 0$ and $y_{i_0} = 0$.

Using that for $u, v \in S^+$, $\max_{1\le i\le d} u_i \le 1$ and $\max_{1\le i\le d} v_i \ge 1/d$, we see that $m(u, v) \le d$.

The semi-group $G$ is acting on $S^+$ as follows.

$$g \cdot x = \frac{gx}{\|gx\|} \qquad \forall (g, x) \in G \times S^+ \,.$$

We then define a cocycle by setting $\sigma(g, x) = \log(\|gx\|)$ for every $(g, x) \in G \times S^+$. The cocycle property reads

$$\sigma(gg', x) = \sigma(g, g' \cdot x) + \sigma(g', x) \,. \tag{2.4}$$

Following Hennion (1997, Lemma 10.6), for every $g \in G$ we define $c(g) := \sup\limits_{x,y\in S^+} d(gx, gy)$.

Let us recall some properties that one may find in Hennion (1997), see his Lemmas 5.2, 5.3 and 10.6 and his Proposition 3.1.

**Proposition 2.1.** *For every* $(g, g', x, y) \in G^2 \times (S^+)^2$ *we have*

(*i*) $|\sigma(g,x)| \le \log N(g)$;
(*ii*) $\|x-y\| \le 2d(x,y)$;
(*iii*) $|\sigma(g,x)-\sigma(g,y)| \le 2L(g)d(x,y)$;
(*iv*) $|\sigma(g,x)-\sigma(g,y)| \le 2\ln\big(1/(1-d(x,y))\big)$ ;
(*v*) $c(gg') \le c(g)c(g')$;
(*vi*) $c(g) \le 1$ *and* $c(g) < 1$ *iff* $g \in G^+$;
(*vii*) $d(g\cdot x, g\cdot y) \le c(g)d(x,y)$.

Let us also mention a closed-form expression for $c(g)$ obtained in Lemma 10.7 of Hennion (1997) (see also Proposition 11.3 below). For every $g = (g_{ij})_{1\le i,j\le d}$ we have

$$c(g) = \max_{1\le i,j,k,\ell\le d} \frac{|g_{ij}g_{k\ell} - g_{i\ell}g_{kj}|}{g_{ij}g_{k\ell} + g_{i\ell}g_{kj}}\,. \tag{2.5}$$

Notice that $(g,x)\to gx$ is continuous on $G\times S^+$ (for the distance on $G$ induced by the operator norm and the distance on $S^+$ induced by $\|\cdot\|$) and does not vanish. Hence, it follows from item (*ii*) that $(g,x)\to g\cdot x$ is continuous on $G\times S^+$ (for the distance on $G$ induced by the operator norm and the distance $d$ on $S^+$).

Let us give some more properties that will be useful in the sequel. Set $e = \{1/d,\dots,1/d\} \in S^+$. For $g\in G$, we denote by $g^t$ the adjoint matrix of $g$.

**Lemma 2.2.** *For every* $(g,x,y) \in G\times (S^+)^2$,

(*i*) $|\sigma(g,x)-\sigma(g,y)| \le \log L(g)$;
(*ii*) $\|ge\| \le \|g\| \le d\|ge\|$;
(*iii*) $\|g\| \le d\|g^t\|$;
(*iv*) $|\sigma(g,x)-\sigma(g,y)| \le 2(2+\log L(g))d(x,y)$.

**Remark.** The inequality in item (*iv*) of Lemma 2.2 is much better that the one in item (*iii*) of Proposition 2.1. This inequality, together with items (*vi*) and (*vii*) of Proposition 2.1 are the main ingredients to prove the exponential decay of the coupling coefficients $\tilde\delta_{p,\infty}(n)$ (see Proposition 4.1).

**Proof.** Items (*i*) and (*ii*) are obvious. Item (*iii*) is an easy consequence of (2.1). Let us prove item (*iv*). Let $x,\,y\in S^+$. Assume that $d(x,y)\le 1/2$. Notice that for every $t\in[0,1/2]$, $\ln(1/(1-t)) \le 2t$. Hence, using item (*iv*) of Proposition 2.1, we see that $|\sigma(g,x)-\sigma(g,y)| \le 4d(x,y)$. If $2d(x,y)\ge 1$, then the desired conclusion follows from item (*i*) of Lemma 2.2. □

**Proposition 2.3.** $(S^+,d)$ *is complete and* $S^{++}$ *is closed where*

$$S^{++} := \Big\{x=(x_1,\dots,x_d)\in\mathbb{R}^d \,:\, \|x\|=1 \text{ and } x_i>0,\ \forall i\in\{1,\dots,d\}\Big\}\,. \tag{2.6}$$

**Remark.** This proposition is probably well known. We did not find a reference for it. However, a hint of proof of completeness is given after Theorem 4.1 of Bushell (1973), for Hilbert's metric given by $d_H(x,y) = -\ln(m(x,y)m(y,x))$. See Proposition 11.1 for a proof in a more general situation.

Let us state some of the assumptions used throughout the paper.

**Definition 2.4.** Let $\mu$ be a Borel probability on $G$ and $p>0$. We say that $\mu$ admits a moment of order $p$ if

$$\int_G (\log(N(g)))^p d\mu(g) < \infty\,.$$

We say that $\mu$ almost admits a moment of order $p$ if

$$\int_G (\log(L(g)))^p d\mu(g) < \infty\,.$$

**Remark.** Clearly, since $L(g) \le N(g)^2$, if $\mu$ admits a moment of order $p \ge 1$, it almost admits a moment of order $p \ge 1$, but the converse is not true in general, see the example in Section 6. Assume now that $\mu$ almost admits a moment of order $p \ge 1$. Then, $\mu$ admits a moment of order $p$ iff $\int_G |\log \|g\||^p d\mu(g) < \infty$ iff $\int_G |\log v(g)|^p d\mu(g) < \infty$.

Similarly, we say that $\mu$ admits or almost admits an exponential moment of order $\gamma > 0$, if there exists $\delta > 0$ such that, respectively,

$$\int_G \mathrm{e}^{\delta\left(\log N(g)\right)^\gamma} d\mu(g) < \infty\,,$$

or

$$\int_G \mathrm{e}^{\delta\left(\log L(g)\right)^\gamma} d\mu(g) < \infty\,.$$

**Definition 2.5.** We say that $\mu$ is *strictly contracting* if there exists $r \in \mathbb{N}$, such that $\mu^{*r}(G^+) > 0$.

Equivalently, the closed semi-group $\Gamma_\mu$ generated by the support of $\mu$ has non empty intersection with $G^+$.

## 3. Previous known results

In this section, we recall some already known results on the existence of an invariant measure for $\mu$, the law of large numbers, the central limit theorem and the Berry-Esseen theorem for $\log \|A_n x\|$, $\log \|A_n\|$, and some related quantities. We will briefly indicate our contribution to these questions, which will be the subject of Sections 5, 6 and 8. Note that our Section 7 about rates of convergence in the almost sure invariance principle seems to be completely new, in the sense that we did not find any previous results related to this question.

We do not pretend to be exhaustive, and we will recall the main results obtained under the least restrictive moment conditions.

3.1. *Invariant measure.* Recall that a Borel (with respect to the distance $d(\cdot,\cdot)$) probability $\nu$ on $S^+$ is said to be $\mu$-invariant if for every Borel non negative function $\varphi$ on $S^+$, $\int_{G\times S^+} \varphi(g\cdot x)d\mu(g)d\nu(x) = \int_{S^+} \varphi(x)d\nu(x)$. It is well known and easy to prove (recall that $(g,x) \to g\cdot x$ is continuous on $G\times S^+$) that the support of a $\mu$-invariant measure $\nu$ is $\Gamma_\mu$-invariant, i.e. satisfies $\Gamma_\mu \cdot \operatorname{supp}\nu \subset \operatorname{supp}\nu$ (recall that $\Gamma_\mu$ is the closed semi-group generated by the support of $\mu$).

As recalled below, when $\mu$ is strictly contracting, it admits a unique $\mu$-invariant probability on $S^+$. We need some further notations to describe its support.

Let $g \in G^+$. By the Perron-Frobenius theorem (see Theorem 1.1.1 of Lemmens and Nussbaum (2012)), there exists a unique $x \in S^{++}$ such that $gx = \kappa(g)x$, where $\kappa(g)$ is the spectral radius of $g$. We denote that vector by $u_g$. Notice the following bound that will useful in the sequel,

$$\kappa(g) \ge v(g) \qquad \forall g \in G\,. \tag{3.1}$$

Following Buraczewski et al. (2014) (see (2.4) there) we define

$$\Lambda_\mu = \overline{\{u_g \,:\, g \in \Gamma_\mu \cap G^+\}}\,,$$

where the closure is taken with respect to $d$. By Proposition 2.3, $\Lambda_\mu \subset S^{++}$.

It follows from Lemma 4.2 of Buraczewski et al. (2014) that $\Lambda_\mu$ is $\Gamma_\mu$-invariant (i.e. $\Gamma_\mu\cdot\Lambda_\mu \subset \Lambda_\mu$).

We recall the following result of Hennion and Hervé (2008).

**Proposition 3.1** (Hennion-Hervé). *Assume that $\mu$ is strictly contracting. Then, there exists a unique $\mu$-invariant probability $\nu$ on $S^+$. Moreover* $\operatorname{supp}\nu = \Lambda_\mu$.

The existence and uniqueness of an invariant probability for strictly contracting $\mu$ are proved in Theorem 2.1 of Hennion and Hervé (2008) and the characterization of the support of the invariant measure follows from Lemma 4.3 of Buraczewski et al. (2014). Since there is no explicit proof of the latter fact in Buraczewski et al. (2014), let us give an argument.

For every $n \in \mathbb{N}$, set $B_n := Y_1 \cdots Y_n$ (with $(Y_n)_{n\in\mathbb{N}}$ iid with law $\mu$). It follows from the proof of Theorem 2.1 of Hennion and Hervé (2008) that, $\mathbb{P}$-almost surely, for every $x \in S^+$, $(B_n \cdot x)_{n\in\mathbb{N}}$ converges to some random variable $Z$ whose law $\nu$ is $\mu$-invariant. Then $\operatorname{supp}\nu$ is $\Gamma_\mu$-invariant and $\Lambda_\mu \subset \operatorname{supp}\nu$ by Lemma 4.2 of Buraczewski et al. (2014). Now, since $\Gamma_\mu \cdot \Lambda_\mu \subset \Lambda_\mu$, for every $x \in \Lambda_\mu$, $B_n \cdot x \in \Lambda_\mu$ $\mathbb{P}$-almost surely, for every $n \in \mathbb{N}$. Hence $Z \in \Lambda_\mu$ $\mathbb{P}$-almost surely (recall that $\Lambda_\mu$ is closed for $d$), which implies that $\nu(\Lambda_\mu) = 1$, hence that $\operatorname{supp}\nu \subset \Lambda_\mu$.

3.2. *The strong law of large numbers.* Recall that $(Y_n)_{n\in\mathbb{N}}$ is a sequence of iid random variables taking values in $G$, with law $\mu$ and living on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that $A_n := Y_n \cdots Y_1$.

We denote by $\tilde{\mu}$ the pushforward measure of $\mu$ by the application $g \mapsto g^t$.

We first recall the version of Kingman's subadditive ergodic theorem relevant to our setting (see Kingman (1973, Theorems 1 and 2)). The fact that $\lambda_\mu$ in the next proposition is constant follows from Kolmogorov's $0-1$ law.

**Proposition 3.2** (Kingman). *Assume that $\int_G \big|\log\|g\|\big| d\mu(g) < \infty$. Then, $(\frac{1}{n}\log\|A_n\|)_{n\geq 1}$ converges $\mathbb{P}$-a.s. and in $L^1$ to some constant $\lambda_\mu \in \mathbb{R}$.*

**Remark.** Using that $\|g\| \geq v(g)$ for every $g \in G^+$, we see that $\log^-\|g\| \leq \log^- v(g)$, where $\log^-(x) = \max(-\log x, 0)$ for every $x > 0$. In particular, if $\mu$ or $\tilde{\mu}$ admits a moment of order 1, then, $\int_G \big|\log\|g\|\big| d\mu(g) < \infty$.

The proposition implies in particular that $\lambda_\mu = \lim_{n\to+\infty} \mathbb{E}(\log\|A_n\|)/n$.

Notice that $\int_G \big|\log\|g\|\big| d\mu(g) < \infty$ if and only if $\int_G \big|\log\|g\|\big| d\tilde{\mu}(g) < \infty$. Hence, applying the proposition to $\tilde{\mu}$, using item $(iii)$ of Lemma 2.2 and the fact that $Y_1^t \cdots Y_n^t$ has same law as $Y_n^t \cdots Y_1^t$, we infer that $\lambda_\mu = \lambda_{\tilde{\mu}}$.

We now recall the strong law of large numbers for the coefficients and the spectral radius given in Hennion (1997), see his Theorem 2 (note that Hennion considers the case of products of stationary and $\alpha$-mixing positive matrices, as in Cohn et al. (1993)).

**Theorem 3.3** (Hennion). *Assume that $\mu$ is strictly contracting and that $\tilde{\mu}$ admits a moment of order 1. Then*

$$\left( \sup_{x,\, y\in S^+} \left| \frac{\log\langle y, A_n x\rangle}{n} - \lambda_\mu \right| \right)_{n\in\mathbb{N}} \underset{n\to+\infty}{\longrightarrow} 0 \ \ \mathbb{P}\text{-a.s.} \tag{3.2}$$

*and*

$$\lim_{n\to+\infty} \frac{\log\kappa(A_n)}{n} = \lambda_\mu \ \ \mathbb{P}\text{-a.s.}$$

Note that the strong law of large numbers for $\inf_{x,\, y\in S^+}\log\langle y, A_n x\rangle$ and $\log v(A_n)$ can be easily deduced from (3.2) (see the proof of Proposition 5.3).

Note also that a previous law of large number for $\log\langle y, A_n x\rangle$ was established by Cohn et al. (1993) under a stronger moment assumption than that given in Theorem 3.3.

In Proposition 5.3, we prove that the convergence for $\log\kappa(A_n)$ also holds in $L^1$.

In Proposition 5.1, we consider the case where $\mu$ is strictly contracting and admits a moment of order 1, and prove that, under this assumption, the strong law of large numbers (almost sure and in $L^1$) holds for $\log\|A_n x\|$, $\log v(A_n)$ and $\log\kappa(A_n)$.

3.3. *The central limit theorem.* Let us recall the CLT for the coefficients and the spectral radius given in Hennion (1997), see his Theorem 3. We use the notation $\Rightarrow$ to mean "convergence in distribution" (as $n \to \infty$).

**Theorem 3.4** (Hennion)**.** *Assume that $\mu$ is strictly contracting and that $\tilde{\mu}$ admits a moment of order 2. Then, for any sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$, of elements of $S^+$,*

$$\frac{1}{\sqrt{n}}\left(\log\langle y_n, A_n x_n\rangle - n\lambda_\mu\right) \Rightarrow \mathcal{N}(0, s^2) \tag{3.3}$$

*for some $s^2 \geq 0$. Moreover $\frac{1}{\sqrt{n}}(\log \kappa(A_n) - n\lambda_\mu) \Rightarrow \mathcal{N}(0, s^2)$.*

Note that a previous CLT for $\log\langle y, A_n x\rangle$ was established by Cohn et al. (1993) under a stronger moment assumption than that given in Theorem 3.4.

Note also that, as a particular case of Theorem 3.4, one gets the CLT for $\log \|A_n x\|$ (taking $x_n = x$ and $y_n = \mathbf{1}$, the vector whose coordinates are all equal to 1) and for $\log \|A_n\|$ (taking $x_n = y_n = \mathbf{1}$ , see Lemma 2.2 (ii)). In Proposition 6.5, we give a more precise statement of (3.3). In addition, we also prove the CLT for $\log v(A_n)$.

In Proposition 6.4, we consider the case where $\mu$ is strictly contracting and admits a moment of order 2, and prove that, under this assumption, the CLT holds for $\log \|A_n x\|$, $\log \|A_n\|$, $\log v(A_n)$ and $\log \kappa(A_n)$. In addition, our proof allows us to identify the asymptotic variance $s^2$ in several ways and to characterize the fact that $s^2 > 0$. The obtained characterization is the same as in Buraczewski et al. (2014) and Buraczewski and Mentemeier (2016) but its proof does not require exponential moments as in those works.

3.4. *The Berry-Esseen theorem.* Let us recall the Berry-Esseen theorem for $\log \|A_n x\|$ proved by Hennion and Hervé (2004) (see their Theorem 3.3 (i) with $Z = x$).

**Theorem 3.5** (Hennion-Hervé)**.** *Assume that $\mu$ is strictly contracting and that $\mu$ and $\tilde{\mu}$ admit a moment of order $p > 4$. Then, for any $x \in S^+$, $\frac{1}{\sqrt{n}}(\log \|A_n x\| - n\lambda_\mu) \Rightarrow \mathcal{N}(0, s^2)$ (where $s^2$ does not depend on $x$). Moreover, if $s^2 > 0$,*

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\big(\log \|A_n x\| - n\lambda_\mu \leq t\sqrt{n}\big) - \phi_s(t)\right| = O\left(\frac{1}{\sqrt{n}}\right),$$

*where $\phi_s(t) = \mathbb{P}(sZ \leq t)$ with $Z$ a standard normal variable.*

In Theorem 8.1, we will prove that the conclusion of Theorem 3.5 still holds when $\mu$ has a moment of order $p = 3$ (not assuming any moment condition on $\tilde{\mu}$). In addition, we will prove that the result still holds with $\log \|A_n\|$ instead of $\log \|A_n x\|$, and we will also give rates of convergence when $\mu$ has a moment of order $p \in (2, 3)$.

In Proposition 8.2 we gives rates in the Berry-Esseen inequality for the quantities $\log v(A_n)$ and $\log \kappa(A_n)$, when $\mu$ admits a moment of order $p > 2$. These rates are optimal when $p \leq 1 + \sqrt{3}$ and (possibly) suboptimal when $p > 1 + \sqrt{3}$. As a consequence of Proposition 8.5, we get the rate $O(n^{-1/2} \log n)$ for these quantities when $\mu$ admits an exponential moment of order 1. Similar rates are obtained for $\log\langle y, A_n x\rangle$ in Proposition 8.6, by assuming some moment conditions on $\mu$ and $\tilde{\mu}$.

Note that the rate $O(n^{-1/2} \log n)$ for $\log \kappa(A_n)$ and $\log\langle y, A_n x\rangle$ was previously obtained by Xiao et al. (2024, Theorem 2.3) when $\mu$ admits an exponential moment of order 1, by assuming the extra condition **A4** of that paper. In a recent preprint, Xiao et al. (2022b) obtained the rates $O(n^{-1/2})$ when $\mu$ admits a moment of order 3, by assuming the additional condition **A2** of that paper. In this paper, their condition **A2** is denoted condition $(C)$ (see (8.19)). We obtain the same rate under a weaker extra condition than condition $(C)$, see our Theorem 8.9. In Theorem 8.9, we also consider the case of moments of order $p \in (2, 3)$.

## 4. Coupling coefficients

For every $p \geq 1$ and every $n \in \mathbb{N}$ define

$$\delta_{p,\infty}(n) := \sup_{x,y\in S^+} \mathbb{E}\left(|\sigma(Y_n, A_{n-1}\cdot x) - \sigma(Y_n, A_{n-1}\cdot y)|^p\right) .$$

Those coefficients have been introduced in Cuny et al. (2017a), in the setting of products of iid matrices in $GL_d(\mathbb{R})$ (except that the supremum is taken over $\{x \in \mathbb{R}^d : \|x\| = 1\}$ rather than $S^+$), and proved to be very useful in Cuny et al. (2018) and Cuny et al. (2023), see also Cuny et al. (2017b).

We shall see that those coefficients decrease exponentially fast to 0, as soon as $\mu$ (almost) admits a moment of order 1, while we obtained only a polynomial speed of convergence in the case of $GL_d(\mathbb{R})$.

Actually, we will prove the result for the stronger coefficients

$$\tilde{\delta}_{p,\infty}(n) := \mathbb{E}\Big(\sup_{x,y\in S^+} |\sigma(Y_n, A_{n-1}\cdot x) - \sigma(Y_n, A_{n-1}\cdot y)|^p\Big) .$$

**Proposition 4.1.** *Assume that $\mu$ is strictly contracting and almost admits a moment of order $p \geq 1$. Then, there exists $0 < a < 1$ such that*

$$\delta_{p,\infty}(n) \leq \tilde{\delta}_{p,\infty}(n) = O(a^n) , \tag{4.1}$$

*and*

$$\sup_{x,y\in S^+} \sup_{n\in\mathbb{N}} |\sigma(A_n, x) - \sigma(A_n, y)| \in L^p . \tag{4.2}$$

*In particular,*

$$\sup_{n\in\mathbb{N}} |\log \|A_n\| - \log v(A_n)| \in L^p . \tag{4.3}$$

**Proof.** Let $n \in \mathbb{N}$. By item $(iv)$ of Lemma 2.2 and item $(vii)$ of Proposition 2.1, for every $x$, $y \in S^+$, we have

$$|\sigma(Y_n, A_{n-1}\cdot x) - \sigma(Y_n, A_{n-1}\cdot y)| \leq (4+2\log L(Y_n))d(A_{n-1}\cdot x, A_{n-1}\cdot y) \leq (4+2\log L(Y_n))c(A_{n-1}) .$$

Let $r \in \mathbb{N}$ be as in Definition 2.5. Then, by item $(vi)$ of Proposition 2.1, there exists $\varepsilon > 0$ such that

$$\mu^{*r}(c(g) \leq 1-\varepsilon) =: \gamma > 0 . \tag{4.4}$$

Hence, if $m = [(n-1)/r]$,

$$\mathbb{E}\big[\big(c(A_{n-1})\big)^p\big] \leq \prod_{k=1}^{m} \mathbb{E}\big[\big(c(Y_{kr}\cdots Y_{(k-1)r+1})\big)^p\big] \leq \big(\gamma(1-\varepsilon)^p + 1 - \gamma\big)^m .$$

This proves the desired exponential convergence of $(\tilde{\delta}_{p,\infty}(n))_{n\in\mathbb{N}}$.

Using the cocycle property, we see that for every $n \in \mathbb{N}$ and every $x, y \in S^+$,

$$|\sigma(A_n, x) - \sigma(A_n, y)| \leq \sum_{k=1}^{n} |\sigma(Y_k, A_{k-1}\cdot x) - \sigma(Y_k, A_{k-1}\cdot y)|$$
$$\leq \sum_{k=1}^{\infty} |\sigma(Y_k, A_{k-1}\cdot x) - \sigma(Y_k, A_{k-1}\cdot y)| .$$

Using the triangle inequality in $L^p$, we infer that

$$\mathbb{E}\Big[\sup_{x,y\in S^+}\sup_{n\in\mathbb{N}}|\sigma(A_n,x)-\sigma(A_n,y)|^p\Big]\le\Big(\sum_{k=1}^{\infty}\big(\tilde{\delta}_{p,\infty}(k)\big)^{1/p}\Big)^p$$
$$\le r^p\mathbb{E}\big[\big(2(2+\log L(Y_1))\big)^p\big]\Big(\sum_{m\ge 0}\big(\gamma(1-\varepsilon)^p+1-\gamma\big)^{m/p}\Big)^p$$
$$\le\frac{2^p r^p\mathbb{E}\big[\big(2+\log L(Y_1)\big)^p\big]}{\Big(1-\big(\gamma(1-\varepsilon)^p+1-\gamma\big)^{1/p}\Big)^p},$$

proving (4.2). To prove (4.3), note first that for every $g\in G$,

$$|\log\|g\|-\log v(g)|=\sup_{x\in S^+}\log\|gx\|-\inf_{y\in S^+}\log\|gy\|=\sup_{x\in S^+}\log\|gx\|+\sup_{y\in S^+}(-\log\|gy\|)$$
$$=\sup_{x,y\in S^+}(\log\|gx\|-\log\|gy\|)\le\sup_{x,y\in S^+}|\sigma(g,x)-\sigma(g,y)|.$$

Therefore (4.3) follows from (4.2). □

## 5. Additional results on the law of large numbers

In this section, we give complementary results to those stated in Theorem 3.3 of Section 3 (Hennion (1997)).

We first consider the case where $\mu$ admits a moment of order 1.

**Proposition 5.1.** *Assume that $\mu$ is strictly contracting and that $\mu$ admits a moment of order 1. Then, for every $x\in S^+$,*

$$\lim_{n\to+\infty}\frac{\sigma(A_n,x)}{n}=\lim_{n\to+\infty}\frac{\log v(A_n)}{n}=\lim_{n\to+\infty}\frac{\log\kappa(A_n)}{n}=\lambda_\mu\qquad\mathbb{P}\text{-a.s.},\tag{5.1}$$

*where $\lambda_\mu=\int_{G\times S^+}\sigma(g,x)d\mu(g)d\nu(x)$. Moreover, the convergences also hold in $L^1$ and, we even have*

$$\Big\|\sup_{x\in S^+}\Big|\frac{\sigma(A_n,x)}{n}-\lambda_\mu\Big|\Big\|_1\underset{n\to+\infty}{\longrightarrow}0\ \text{and}\ \sup_{x\in S^+}\Big|\frac{\sigma(A_n,x)}{n}-\lambda_\mu\Big|\underset{n\to+\infty}{\longrightarrow}0\ \mathbb{P}\text{-a.s.}$$

**Remark.** The $\mathbb{P}$-a.s. and $L^1$ convergence of $(\frac{1}{n}\log v(A_n))_{n\in\mathbb{N}}$ when $\int_G|\log v(g)|d\mu(g)<\infty$ (which holds if $\mu$ admits a moment of order 1) follow from Kingman's subadditive ergodic Theorem applied to $(-\log v(A_n))_{n\in\mathbb{N}}$. The formula for $\lambda_\mu$ may be derived from the formula in the middle of page 1568 of Hennion (1997).

**Proof.** By Proposition 3.2 and the remark after it, we have the $\mathbb{P}$-a.s. and $L^1$ convergence of $((\log\|A_n\|)/n)_{n\in\mathbb{N}}$ to $\lambda_\mu$.

By (4.3), we infer the $L^1$ convergence for $((\log v(A_n))/n)_{n\in\mathbb{N}}$. To prove the almost sure convergence, define first $Z:=\sup_{n\in\mathbb{N}}|\log\|A_n\|-\log v(A_n)|$. By (4.3), $Z\in L^1$ and, for every $\varepsilon>0$, by Fubini's theorem,

$$\sum_{n\in\mathbb{N}}\mathbb{P}(|\log\|A_n\|-\log v(A_n)|\ge\varepsilon n)\le\frac{\mathbb{E}(Z)}{\varepsilon}<\infty.$$

The $\mathbb{P}$-a.s. convergence for $((\log v(A_n))/n)_{n\in\mathbb{N}}$ then follows from the one for $((\log\|A_n\|))/n)_{n\in\mathbb{N}}$ and the Borel-Cantelli lemma.

The convergences for $((\log\kappa(A_n))/n)_{n\in\mathbb{N}}$ follow from the bounds $v(A_n)\le\kappa(A_n)\le\|A_n\|$ (see (3.1) for the first bound).

Finally, notice that for every $n \in \mathbb{N}$,

$$\sup_{x \in S^+} |\sigma(A_n, x) - n\lambda_\mu| \le \max(|\log \|A_n\| - n\lambda_\mu|, |\log v(A_n) - n\lambda_\mu|)\,,$$

which proves the remaining convergences.

Hence, it remains to identify $\lambda_\mu$. From the above, using the $\mu$-invariance of $\nu$, we infer that

$$\begin{aligned}\int_{G\times S^+} \sigma(g,x) d\mu(g) d\nu(x) &= \frac{1}{n}\int_{S^+} \mathbb{E}\Big(\sum_{k=1}^{n} \sigma(Y_k, A_{k-1}\cdot x)\Big) d\nu(x) \\ &= \frac{1}{n}\int_{S^+} \mathbb{E}(\sigma(A_n, x)) d\nu(x) \underset{n\to+\infty}{\longrightarrow} \lambda_\mu\,.\end{aligned}$$

□

We shall now consider the case of matrix coefficients when $\tilde\mu$ has a moment of order 1, as in Theorem 3.3 (recall that $\tilde\mu$ stands for the pushforward measure of $\mu$ by the map $g \to g^t$). The proof will rely on Lemma 5.2 below, which is essentially contained in Lemma 2.1 of Hennion and Hervé (2008) (see also Lemma 6.3 of Buraczewski et al. (2014) for (5.4)). We need also some further notations. As in Hennion and Hervé (2008), set

$$T := \inf\{n \in \mathbb{N} \,:\, Y_n \cdots Y_1 \in G^+\}\,. \tag{5.2}$$

From item $(i)$ of Hennion and Hervé (2008, Lemma 2.1), note that $\mu$ is strictly contracting if and only if $\mathbb{P}(T < \infty) = 1$.

**Lemma 5.2.** *Assume that $\mu$ is strictly contracting. With the above notations,*

$$\inf_{n\ge T} \inf_{x,\, y\in S^+} \frac{\langle y, A_n x\rangle}{\|Y_1^t \cdots Y_n^t y\|} > 0 \qquad \mathbb{P}\text{-}a.s. \tag{5.3}$$

*and*

$$\inf_{n\in\mathbb{N}} \inf_{x\in S^+} \frac{\|A_n x\|}{\|A_n\|} = \inf_{n\in\mathbb{N}} \frac{v(A_n)}{\|A_n\|} > 0 \qquad \mathbb{P}\text{-}a.s. \tag{5.4}$$

Inequality (5.3) is just a reformulation of item $(iii)$ of Lemma 2.1 from Hennion and Hervé (2008), and (5.4) follows from (5.3) and the fact that for every $n \in \mathbb{N}$ and every $x \in S^+$, using items $(ii)$ and $(iii)$ of Lemma 2.2, it holds

$$\frac{\|A_n x\|}{\|A_n\|} \ge \frac{\langle e, A_n x\rangle}{d^2 \|A_n^t e\|}\,.$$

We are now in position to state the results when $\tilde\mu$ admits a moment of order 1. Compared to Hennion's result (Theorem 3.3 of Section 3), we add the $L^1$ convergence for $\log \kappa(A_n)$. For the sake of completeness, we give a self contained proof of the next proposition (some of the arguments will be used in the next sections).

**Proposition 5.3.** *Assume that $\mu$ is strictly contracting and that $\tilde\mu$ admits a moment of order 1. Then (3.2) holds with $\lambda_\mu = \int_{G\times S^+} \sigma(g,x) d\tilde\mu(g) d\tilde\nu(x)$ ($\tilde\nu$ being the only $\tilde\mu$-invariant probability on $S^+$). In particular,*

$$\left(\left|\frac{\inf_{x,\, y\in S^+} \log\langle y, A_n x\rangle}{n} - \lambda_\mu\right|\right)_{n\in\mathbb{N}} \underset{n\to+\infty}{\longrightarrow} 0 \ \ \mathbb{P}\text{-}a.s.$$

*Moreover, $((\log \|A_n\| - n\lambda_\mu)/n)_{n\in\mathbb{N}}$ and $((\log \kappa(A_n) - n\lambda_\mu)/n)_{n\in\mathbb{N}}$ converge $\mathbb{P}$-a.s. and in $L^1$ to 0; and $((\log v(A_n) - n\lambda_\mu)/n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.s. to 0.*

**Proof.** First notice that Proposition 3.2 applies, which yields the $\mathbb{P}$-a.s. and $L^1$ convergence for $((\log \|A_n\|)/n)_{n\in\mathbb{N}}$ and for $((\log \|A_n^t\|)/n)_{n\in\mathbb{N}}$ by item $(iii)$ of Lemma 2.2.

By Lemma 5.2, there exists a random variable $W > 0$ such that, for every $x, y \in S^+$ and every $n \in \mathbb{N}$, on the set $\{T \le n\}$ (recall that $T$ is defined in (5.2)),

$$0 \le \log \|A_n\| - \log\langle y, A_n x\rangle \le \log W + \log \|A_n\| - \log \|Y_1^t \cdots Y_n^t y\| \,. \tag{5.5}$$

Let $\varepsilon > 0$. Using that $(Y_1, \dots, Y_n)$ and $(Y_n, \dots, Y_1)$ have the same law, we get

$$\sum_{n\ge 1} \mathbb{P}\big( \sup_{y\in S^+} \big| \log \|Y_1^t \cdots Y_n^t y\| - \log \|Y_1^t \cdots Y_n^t e\| \big| \ge \varepsilon n\big)$$
$$\le \sum_{n\ge 1} \mathbb{P}\big( \sup_{y\in S^+} \sup_{m\in\mathbb{N}} \big| \log \|Y_m^t \cdots Y_1^t y\| - \log \|Y_m^t \cdots Y_1^t e\| \big| \ge \varepsilon n\big) < \infty \,,$$

where we used Proposition 4.1 for $\tilde{\mu}$.

By the Borel-Cantelli lemma, using item $(ii)$ of Lemma 2.2, we infer that

$$\frac{\sup_{y\in S^+} \big| \log \|Y_1^t \cdots Y_n^t y\| - \log \|A_n^t\| \big|}{n} \underset{n\to+\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{5.6}$$

Combining this with (5.5) (recall that $\mathbb{P}(T < \infty) = 1$ and that $\|g\| \le d\|g^t\|$ for every $g \in G$) we obtain that

$$\sup_{x,\, y\in S^+} \frac{\big| \log \|A_n\| - \log\langle y, A_n x\rangle \big|}{n} \underset{n\to+\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.}$$

This gives the desired convergence for the coefficients. The $\mathbb{P}$-a.s. convergences for $(\log \kappa(A_n)/n)_{n\in\mathbb{N}}$ and $(\log v(A_n)/n)_{n\in\mathbb{N}}$ follow from the inequalities

$$\frac{\inf_{x,\, y\in S^+} \log\langle y, A_n x\rangle}{n} \le \frac{\log v(A_n)}{n} \le \frac{\log \kappa(A_n)}{n} \le \frac{\log \|A_n\|}{n} \,.$$

The $L^1$ convergence for $(\log \kappa(A_n)/n)_{n\in\mathbb{N}}$, follows from Proposition 5.1 applied to $\tilde{\mu}$, using item $(iii)$ of Lemma 2.2 and noticing that $(Y_1, \dots, Y_n)$ has the same law as $(Y_n, \dots, Y_1)$. ☐

Under our assumptions, one cannot expect the $L^1$ convergence in Proposition 5.3 for $v(A_n)$.

For instance take $\mu$ such that for every $k \in \mathbb{N}$, $\mu(\{g_k\}) = \frac{1}{3k(k+1)}$ and $\mu(\{h\}) = \mu(\{\mathrm{Id}\}) = 1/3$, with $g_k = \begin{pmatrix} 2^{-k} & 1/2 \\ 0 & 1/2 \end{pmatrix}$ and $h = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. Then, for every $g \in \operatorname{supp}\mu$, $\|g\| \le 1$, which implies that for every $g \in \Gamma_\mu$ (the closed semi-group generated by the support of $\mu$), $v(g) \le \|g\| \le 1$. Moreover, using (2.2), $v(g_k) = 2^{-k}$ and $v(g_k^t) = 1/2$. In particular, $\tilde{\mu}$ admits a moment of order 1 while $\mu$ does not, since $\mathbb{E}(\log v(Y_1)) \le \sum_{k\in\mathbb{N}} \frac{-k\log 2}{3k(k+1)} = -\infty$.

For every integer $n \ge 2$, set $\Lambda_n := \{Y_2 = \cdots = Y_n = \mathrm{Id}\}$. Then,

$$\mathbb{E}(\log v(A_n)) \le \mathbb{E}(\log v(Y_1)\mathbf{1}_{\Lambda_n}) = 3^{-(n-1)}\mathbb{E}(\log v(Y_1)) = -\infty.$$

Similarly, even if $\mu$ and $\tilde{\mu}$ are strictly contracting and admit a moment of order 1, we may not have $L^1$ convergence for the coefficients. For instance, let $\mu$ be such that $\mu(\{\mathrm{Id}\}) = \mu(\{h\}) = 1/2$. Then, $\mu^{*n}(\{\mathrm{Id}\}) \ge 2^{-n}$ and, with $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$, $\mu^{*n}(\{g \in G : \langle e_1, g e_2\rangle = 0\}) > 0$, so that $\mathbb{E}(\log\langle e_1, A_n e_2\rangle) = -\infty$.

## 6. Additional results on the CLT and the asymptotic variance

In this section, we give complementary results to those stated in Theorem 3.4 of Section 3 (Hennion, 1997). In particular, our proof allows us to identify the asymptotic variance $s^2$ in several ways and to characterize the fact that $s^2 > 0$.

We start by proving a martingale-coboundary decomposition. In the case of invertible matrices, such a decomposition was only available for $p \geq 2$, while here it holds as soon as $p \geq 1$.

**Proposition 6.1.** *Assume that $\mu$ is strictly contracting and admits a moment of order $p \geq 1$. There exists a continuous and bounded function $\psi$ on $X$ such that $\big(\sigma(Y_n, A_{n-1} \cdot x) - \lambda_\mu + \psi(A_n \cdot x) - \psi(A_{n-1} \cdot x)\big)_{n\in\mathbb{N}}$ is a sequence of martingale differences in $L^p$. If moreover $W_0$ is a random variable with law $\nu$, independent of $(Y_n)_{n\in\mathbb{N}}$, then $\big(\sigma(Y_n, A_{n-1} \cdot W_0) - \lambda_\mu + \psi(A_n \cdot W_0) - \psi(A_{n-1} \cdot W_0)\big)_{n\in\mathbb{N}}$ is a* stationary *and ergodic sequence of martingale differences in $L^p$.*

**Remark.** The function $\psi$ in the theorem is given by

$$\psi(x) := \sum_{n\geq 1} \Big( \int_{G\times G} \sigma(g, g' \cdot x) d\mu(g) d\mu^{*(n-1)}(g') - \lambda_\mu \Big) . \tag{6.1}$$

**Proof.** Let $\psi$ be given by (6.1). The fact that $\psi$ is well-defined and continuous follows from Proposition 4.1.

Then, notice that

$$\sigma(g, x) - \lambda_\mu = \sigma(g, x) - \int_G \sigma(g', x) d\mu(g') + \int_G \sigma(g', x) d\mu(g') - \lambda_\mu$$

and, using the definition of $\psi$,

$$\int_G \sigma(g, x) d\mu(g) - \lambda_\mu + \int_G \psi(g \cdot x) d\mu(g) = \psi(x) .$$

Now, $\Big(\sigma(Y_n, A_{n-1} \cdot x) - \int_G \sigma(g, A_{n-1} \cdot x) d\mu(g)\Big)_{n\in\mathbb{N}}$ is a sequence of martingale differences in $L^p$ (notice that $x \mapsto \int_G \sigma(g, x) d\mu(g)$ is bounded). Moreover,

$$\int_G \sigma(g, A_{n-1} \cdot x) d\mu(g) - \lambda_\mu + \psi(A_n \cdot x) - \psi(A_{n-1} \cdot x) = \psi(A_n \cdot x) - \int_G \psi(g A_{n-1} \cdot x)\, d\mu(g),$$

and the right-hand side defines a sequence of bounded martingale differences.

The final statement follows from the fact that $((Y_n, A_{n-1} \cdot W_0))_{n\in\mathbb{N}}$ is a stationary and (uniquely) ergodic Markov chain. □

**Definition 6.2.** We say that a probability $\mu$ on $G$ is aperiodic if the group generated by $\{\log \kappa(g) : g \in \Gamma_\mu\}$ is dense in $\mathbb{R}$.

**Proposition 6.3.** *Assume that $\mu$ is strictly contracting and that $\mu$ admits a moment of order 2. Then, there exists $s^2 \geq 0$ such that, with $W_0$ as in Proposition 6.1,*

$$\frac{1}{n} \mathbb{E}[(\sigma(A_n, W_0) - n\lambda_\mu)^2] \underset{n\to+\infty}{\longrightarrow} s^2 \tag{6.2}$$

*and $\frac{1}{\sqrt{n}}(\sigma(A_n, W_0) - n\lambda_\mu) \Rightarrow \mathcal{N}(0, s^2)$. In addition, if there do not exist $m \in \mathbb{N}$ and $\psi_m$ continuous on $S^+$ such that*

$$\sigma(g, x) - m\lambda_\mu = \psi_m(x) - \psi_m(g \cdot x) \qquad \text{for } \mu^{\otimes m} \otimes \nu\text{-almost every } (g, x) \in G \times S^+ , \tag{6.3}$$

*then $s^2 > 0$. In particular, if $\mu$ is aperiodic, then $s^2 > 0$.*

**Remark.** Under the assumptions of the proposition we actually have the functional central limit theorem. Moreover, it is well known that the variance is given by

$$\begin{aligned} s^2 &= \mathbb{E}(\sigma(A_1, W_0)^2) + 2\sum_{n\geq 2} \mathbb{E}(\sigma(A_1, W_0)\sigma(A_n, W_0)) \\ &= \int_{G\times S^+} \sigma^2(g,x)d\mu(g)d\nu(x) + 2\sum_{n\geq 2}\int_{G^2\times S^+} \sigma(g,x)\sigma(g'g,x)d\mu^{*(n-1)}(g')d\mu(g)d\nu(x)\,. \end{aligned}$$

**Proof.** For every $n \in \mathbb{N}$, set $D_n := \sigma(Y_n, A_{n-1}\cdot W_0) - \lambda_\mu + \psi(A_n\cdot W_0) - \psi(A_{n-1}\cdot W_0)$. By Proposition 6.1, $(D_n)_{n\in\mathbb{N}}$ is a stationary and ergodic sequence of martingale differences in $L^2$. In particular, $(D_1+\cdots+D_n)/\sqrt{n} \Rightarrow \mathcal{N}(0,s^2)$, with $s^2 = \mathbb{E}(D_1^2) = \mathbb{E}((D_1+\cdots+D_n)^2)/n$. Hence, the CLT with the description of the variance follows from the following reformulation of Proposition 6.1:

$$\sigma(A_n, W_0) - n\lambda_\mu = (D_1+\cdots+D_n) + \psi(W_0) - \psi(A_n\cdot W_0)\,. \tag{6.4}$$

Assume now that $s^2 = 0$. Then

$$\int_G (\sigma(g,x) - \lambda_\mu - \psi(x) + \psi(g\cdot x))^2\, d\mu(g)d\nu(x) = 0\,.$$

Hence, (6.3) holds with $m=1$ and $\psi_1 = \psi$. Let $m>1$. Notice that $\mu^{*m}$ is strictly contracting and admits a moment of order $p$ and that the unique $\mu^{*m}$-invariant measure is the unique $\mu$-invariant measure. Notice also that $\lambda_{\mu^{*m}} = m\lambda_\mu$. Applying the above argument to $\mu^{*m}$, we infer that there exists a continuous $\psi_m$ satisfying (6.3).

Using that $\psi_m$ is continuous, we see that (6.3) holds for every $g\in \operatorname{supp}\mu^{*m}$ and every $x\in\operatorname{supp}\nu$. Let $g\in\operatorname{supp}\mu^{*m}\subset\Gamma_\mu$. Then, $u_g\in\Lambda_\mu\subset\operatorname{supp}\nu$ (recall that $u_g$ has been defined before (3.1)). Since $g\cdot u_g = u_g$ and $\sigma(g,u_g) = \log\kappa(g)$, we infer that $\psi_m(g\cdot u_g) = \psi_m(u_g)$ and that $\log\kappa(g) = m\lambda_\mu$. Hence, $\log\kappa(\Gamma_\mu)\subset\lambda_\mu\mathbb{N}$ and $\mu$ cannot be aperiodic. □

Let us now give the CLT for $\sigma(A_n,x)$, $\log\|A_n\|$, $\log v(A_n)$ and $\log\kappa(A_n)$. Below and in the rest of the section, we shall use the notation: $\phi_s(t) = \mathbb{P}(sZ\leq t)$ with $Z$ a standard normal variable.

**Proposition 6.4.** *Assume that $\mu$ is strictly contracting and admits a moment of order 2. Then, the following limit exists*

$$s^2 := \lim_{n\to+\infty}\frac{1}{n}\mathbb{E}((\log\|A_n\| - n\lambda_\mu)^2)\,, \tag{6.5}$$

*and we even have*

$$\begin{aligned} s^2 &= \lim_{n\to+\infty}\mathbb{E}[(\sigma(A_n,W_0) - n\lambda_\mu)^2] \\ &= \lim_{n\to+\infty}\frac{1}{n}\sup_{x\in S^+}\mathbb{E}((\sigma(A_n,x) - n\lambda_\mu)^2) \\ &= \lim_{n\to+\infty}\frac{1}{n}\mathbb{E}((\log v(A_n) - n\lambda_\mu)^2)\,, \end{aligned}$$

*and*

$$s^2 = \lim_{n\to+\infty}\frac{1}{n}\mathbb{E}((\log\kappa(A_n) - n\lambda_\mu)^2)\,. \tag{6.6}$$

*Moreover the CLT in Proposition 6.3 also holds if we replace $\sigma(A_n,W_0)$ with $\sigma(A_n,x)$, $\log\|A_n\|$, $\log v(A_n)$ or $\log\kappa(A_n)$ and we also have*

$$\sup_{x\in S^+}\sup_{t\in\mathbb{R}}\left|\mathbb{P}(\sigma(A_n,x) - n\lambda_\mu \leq t\sqrt{n}) - \phi_s(t))\right| \underset{n\to+\infty}{\longrightarrow} 0\,.$$

*If we assume that $\tilde\mu$ is strictly contracting and admits a moment of order 2 then the CLTs for $(\log\|A_n\|)_{n\in\mathbb{N}}$ and $(\log\kappa(A_n))_{n\in\mathbb{N}}$ still hold with $s^2$ given by (6.5) (or equivalently by (6.6)).*

**Remark.** When it is assumed that $\tilde{\mu}$ admits a moment of order 2, we do not know whether $s^2$ is also equal to any the above limits other than (6.5) or (6.6) (we even do not know whether the limits themselves exist) but we will see that the CLTs still hold.

**Proof.** We start with the case where $\mu$ is strictly contracting and admits a moment of order 2. The different expressions of $s^2$ follow from (6.2), Proposition 4.1 and the fact that for every real random variables $U$, $V$, $|\mathbb{E}(U^2) - \mathbb{E}(V^2)| \le \|U - V\|_2(\|U\|_2 + \|V\|_2)$. Next, note that we can deduce the CLT for $(\sigma(A_n, x))_{n\in\mathbb{N}}$ by using (4.2). To get the CLT for $(\log \|A_n\|)_{n\in\mathbb{N}}$, it suffices to notice that, for any $x \in S^+$,

$$\log \|A_n\| - \log v(A_n) \ge \log \|A_n\| - \sigma(A_n, x) \ge 0\,, \tag{6.7}$$

and to use the fact that by (4.3), $\sup_{n\in\mathbb{N}}(\log \|A_n\| - \log v(A_n))$ is in $L^2$. The CLT for $(\log v(A_n))_{n\in\mathbb{N}}$ follows from the CLT for $(\log \|A_n\|)_{n\in\mathbb{N}}$ and (4.3). Finally, the CLT for $(\log \kappa(A_n))_{n\in\mathbb{N}}$ follows from the fact that $v(A_n) \le \kappa(A_n) \le \|A_n\|$ and (4.3). To get the last convergence, we use previous arguments and Inequality (6.8) below which is stated in Kuchibhotla et al. (2020, equation (1)): Let $U, V$ and $R$ be random variables with $|U - V| \le R$ $\mathbb{P}$-a.s. For any $\psi : \mathbb{R} \to \mathbb{R}$ and any $\varepsilon > 0$, we have

$$\sup_{t\in\mathbb{R}} |\mathbb{P}(U \le t) - \psi(t)| \le \sup_{t\in\mathbb{R}} |\mathbb{P}(V \le t) - \psi(t)| + \mathbb{P}(R > \varepsilon) + \sup_{t\in\mathbb{R}} |\psi(t - \varepsilon) - \psi(t + \varepsilon)|\,. \tag{6.8}$$

Assume now that $\tilde{\mu}$ is strictly contracting and admits a moment of order 2. Applying the first part of Proposition 6.4 to $\tilde{\mu}$, we obtain a CLT for $(\|Y_n^t \cdots Y_1^t\|)_{n\in\mathbb{N}}$ which, by item $(iii)$ of Lemma 2.2, implies a CLT for $(\|Y_1 \cdots Y_n\|)_{n\in\mathbb{N}}$. Similarly, since for any matrix $\kappa(g^t) = \kappa(g)$, we infer the convergence in law for $(\log \kappa(A_n))_{n\in\mathbb{N}}$.

The fact that $s^2$ is still given by (6.5) or (6.6) also follows from the above arguments. □

We also have a (functional) CLT for the coefficients. As noticed in the previous section, one cannot expect in general to identify $s^2$ thanks to the matrix coefficients as in Proposition 6.4.

**Proposition 6.5.** *Assume that $\mu$ or $\tilde{\mu}$ is strictly contracting and admits a moment of order 2. Then, with $s^2$ given either by (6.5) or (6.6),*

$$\sup_{x,\, y\in S^+} \sup_{t\in\mathbb{R}} \left|\mathbb{P}\left(\log\langle x, A_n y\rangle - n\lambda_\mu \le t\sqrt{n}\right) - \phi_s(t)\right| \underset{n\to+\infty}{\longrightarrow} 0\,,$$

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\Big(\inf_{x,\, y\in S^+} \log\langle x, A_n y\rangle - n\lambda_\mu \le t\sqrt{n}\Big) - \phi_s(t)\right| \underset{n\to+\infty}{\longrightarrow} 0\,. \tag{6.9}$$

*In particular, we also have a CLT for $(\sigma(A_n, W_0))_{n\in\mathbb{N}}$, $(\log v(A_n))_{n\in\mathbb{N}}$ or $(\sigma(A_n, x))_{n\in\mathbb{N}}$.*

**Proof.** We prove (6.9), the other convergences follow from the fact that for every $u$, $v \in S^+$ and any $n \in \mathbb{N}$,

$$\inf_{x,\, y\in S^+} \log\langle x, A_n y\rangle \le \log\langle u, A_n v\rangle \le \sigma(A_n, v) \le \log \|A_n\|\,, \tag{6.10}$$

$$\inf_{x,\, y\in S^+} \log\langle x, A_n y\rangle \le \log v(A_n) \le \log \|A_n\|\,. \tag{6.11}$$

We start with the case where $\tilde{\mu}$ is strictly contracting and admits a moment of order 2. We proceed as for the proof of Proposition 5.3. By Proposition 4.1 applied to $\tilde{\mu}$,

$$\sum_{n\in\mathbb{N}} \mathbb{P}\big(\sup_{y\in S^+} \big|\log \|Y_1^t \cdots Y_n^t\| - \log \|Y_1^t \cdots Y_n^t y\|\,\big| \ge \varepsilon\sqrt{n}\big)$$
$$\le \sum_{n\in\mathbb{N}} \mathbb{P}\big(\sup_{y\in S^+} \sup_{m\in\mathbb{N}} \big|\log \|Y_m^t \cdots Y_1^t\| - \log \|Y_m^t \cdots Y_1^t y\|\,\big| \ge \varepsilon\sqrt{n}\big) < \infty\,.$$

In particular, since for any $g \in G$, $\|g\| \le d\|g^t\|$,

$$\mathbb{P}\big(\sup_{y\in S^+} \big|\log\|A_n\| - \log\|Y_1^t\cdots Y_n^t y\|\big| \ge \varepsilon\sqrt{n}\big) \underset{n\to+\infty}{\longrightarrow} 0\,. \tag{6.12}$$

To conclude it remains to use Inequality (6.8) with $U := (\inf_{x,\,y\in S^+}\log\langle x, A_n y\rangle) - n\lambda_\mu)/\sqrt{n}$, $V := (\log\|A_n\| - n\lambda_\mu)/\sqrt{n}$ and

$$R := (|U| + |V|)\mathbf{1}_{\{T>n\}} + |\log W| + \log d + \sup_{y\in S^+}\big|\log\|A_n\| - \log\|Y_1^t\cdots Y_n^t y\|\big|\,,$$

where $T$ is defined by (5.2) and $W$ is the positive random variable defined in (5.5). By (5.5) again, $|U-V| \le R$ and (6.9) follows from Inequality (6.8), using Proposition 6.3 and the fact that $\mathbb{P}(T<\infty) = 1$.

Assume now that $\mu$ is strictly contracting and admits a moment of order 2. Notice that, for every $n\in\mathbb{N}$, $\inf_{x,y\in S^+}\log\langle x, Y_n\cdots Y_1 y\rangle = \inf_{x,y\in S^+}\log\langle Y_1^t\cdots Y_n^t x, y\rangle$ and that the latter has same law as $\inf_{x,y\in S^+}\log\langle x, Y_n^t\cdots Y_1^t y\rangle$. Hence, it suffices to apply the already proven part of the proposition to $\tilde\mu$, using (6.10) and (6.11). □

## 7. The almost sure invariance principle

**Theorem 7.1.** *Let $p \ge 2$. Assume that $\mu$ is strictly contracting and admits a moment of order $p$. Let $s^2$ be as in Proposition 6.3. Then, one can redefine the process $(\sigma(A_n, W_0))_{n\in\mathbb{N}}$ on another probability space on which there exist iid variables $(N_n)_{n\in\mathbb{N}}$ with law $\mathcal{N}(0, s^2)$, such that*

$$|\sigma(A_n, W_0) - n\lambda_\mu - (N_1+\cdots+N_n)| = o(\sqrt{n\log\log n}) \quad \mathbb{P}\text{-a.s. if } p=2$$

$$\text{and} \quad |\sigma(A_n, W_0) - n\lambda_\mu - (N_1+\cdots+N_n)| = o(n^{1/p}) \quad \mathbb{P}\text{-a.s. if } p>2$$

**Remark.** It is not necessary here that $s^2 > 0$.

**Proof.** When $p > 2$, the result follows from Theorem 1 of Cuny et al. (2018) by taking into account (4.1). The case $p = 2$ follows from (6.4) and the ASIP for martingales with stationary and ergodic increments in $L^2$, see Strassen (1964). □

Proceeding as in the proof of Proposition 5.1 (using in particular the argument yielding (7.1) below) and using Lemma 4.1 of Berkes et al. (2014), Proposition 4.1 and (6.7), one can prove that the above theorem holds if we replace $(\sigma(A_n, W_0))_{n\in\mathbb{N}}$ with any of the following sequences: $(\sigma(A_n, x))_{n\in\mathbb{N}}$ (for a given $x \in S^+$), $(\log\|A_n\|)_{n\in\mathbb{N}}$, $(\log\kappa(A_n))_{n\in\mathbb{N}}$ or $(\log v(A_n))_{n\in\mathbb{N}}$.

Let us give the ASIP for the matrix coefficients.

**Theorem 7.2.** *Let $p \ge 2$. Assume that $\mu$ is strictly contracting and that $\mu$ and $\tilde\mu$ admit a moment of order $p$. Then, for every $x, y \in S^+$, one can redefine the process $(\log\langle y, A_n x\rangle)_{n\in\mathbb{N}}$ on another probability space on which there exist iid variables $(N_n)_{n\in\mathbb{N}}$ with law $\mathcal{N}(0, s^2)$, such that*

$$|\log\langle y, A_n x\rangle - n\lambda_\mu - (N_1+\cdots+N_n)| = o(\sqrt{n\log\log n}) \quad \mathbb{P}\text{-a.s. if } p=2$$

$$\text{and} \quad |\log\langle y, A_n x\rangle - n\lambda_\mu - (N_1+\cdots+N_n)| = o(n^{1/p}) \quad \mathbb{P}\text{-a.s. if } p>2.$$

**Proof.** We proceed as for the proof of Proposition 5.3. Since $\tilde\mu$ almost admits a moment of order $p \ge 1$, using (4.2), for every $\varepsilon > 0$, we have

$$\sum_{n\ge1}\mathbb{P}\Big(\sup_{y\in S^+}\Big|\log\|Y_1^t\cdots Y_n^t\| - \log\|Y_1^t\cdots Y_n^t y\|\Big| \ge \varepsilon n^{1/p}\Big) < \infty\,.$$

By the Borel-Cantelli lemma, we then infer that

$$\frac{\sup_{y\in S^+}\Big|\log\|Y_1^t\cdots Y_n^t\| - \log\|Y_1^t\cdots Y_n^t y\|\Big|}{n^{1/p}} \underset{n\to+\infty}{\longrightarrow} 0 \qquad \mathbb{P}\text{-a.s.} \tag{7.1}$$

We finish the proof by using similar arguments as those developed in the proof of Proposition 5.3 replacing (5.6) by (7.1). □

**Remark.** In the proof we used that $\tilde{\mu}$ almost admits a moment of order $p$, hence it may seem that one can weaken the conditions of Theorem 7.2. It turns out that if $\mu$ admits a moment of order $p$ and if $\tilde{\mu}$ almost admit a moment of order $p$, then $\tilde{\mu}$ admits a moment of order $p$. This follows from the fact that for every $g \in G$, $v(g^t)| \le \|g^t\| \le d\|g\|$ and $\frac{1}{v(g^t)} \le \frac{\|g\|}{v(g^t)} \frac{1}{v(g)}$.

In the case of exponential moments, combining ideas from Cuny et al. (2018) and Cuny et al. (2020), it is possible to obtain logarithmic rates in the ASIP. This is done in the article Cuny et al. (2024) where it is proved that if $\mu$ is strictly contracting and has a subexponential moment of order $\gamma \in (0,1]$ then the conclusion of Theorem 7.1 holds with rate $O((\log n)^{2+1/\gamma})$.

## 8. The Berry-Esseen theorem

In this section, we obtain the Berry-Esseen theorem for the norm cocycle and the matrix norm, when $\mu$ admits a moment of order $p \in ]2,3]$. We get the rate of convergence $n^{1-p/2}$ which corresponds to the rate in the setting of sums of iid random variables.

As far as we know the only rate of that type under polynomial moment condition has been obtained by Hennion and Hervé (2004). More precisely, they required a moment of order $p > 4$ for $\mu$ and $\tilde{\mu}$ to get the rate $n^{-1/2}$ (see Theorem 3.5 of Section 3).

We also obtain Berry-Esseen type results (with possibly suboptimal rates) for the spectral radius and the quantity $\log v(A_n)$ under stronger moment assumptions. In addition, we get Berry-Esseen type results for the matrix coefficients under polynomial or exponential moment conditions. Finally, assuming that $\mu$ has a moment moment of order $p \in (2,3]$ and satisfies some extra moment condition, to be defined later, we prove that the spectral radius and the matrix coefficients satisfy Berry-Esseen type estimates with rate of order $n^{1-p/2}$.

In this section, we use the notation $\phi_s(t) = \mathbb{P}(sZ \le t)$ with $Z$ a standard normal variable.

### 8.1. *Berry-Esseen for the norm cocycle and the matrix norm.*

**Theorem 8.1.** *Let $p \in (2,3]$. Assume that $\mu$ is strictly contracting and admits a moment of order $p$. Assume that $s^2 > 0$ with $s^2$ as in Proposition 6.3. Then, setting $v_n = \left(\frac{1}{n}\right)^{p/2-1}$, we have*

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\big(\sigma(A_n, W_0) - n\lambda_\mu \le t\sqrt{n}\big) - \phi_s(t)\right| = O(v_n)\,, \tag{8.1}$$

$$\sup_{x\in S^+} \sup_{t\in\mathbb{R}} \left|\mathbb{P}\big(\sigma(A_n, x) - n\lambda_\mu \le t\sqrt{n}\big) - \phi_s(t)\right| = O(v_n)\,, \tag{8.2}$$

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\big(\log\|A_n\| - n\lambda_\mu \le t\sqrt{n}\big) - \phi_s(t)\right| = O(v_n)\,. \tag{8.3}$$

Hennion and Hervé (2004) obtained the rate (8.1) with $v_n = 1/\sqrt{n}$ when $\mu$ and $\tilde{\mu}$ admit a moment of order $p > 4$ (see Theorem 3.5 of Section 3).

**Proof.** The proof of (8.1) and of (8.2) follow the one of Theorem 2.1 of Cuny et al. (2023) with $T = n^{p/2-1}$, using the estimate (4.1) instead of their estimate Cuny et al. (2023, (3.12)). Indeed, using (4.1), one can prove that for $R_1$ and $U_2 - U_2^*$ defined in Cuny et al. (2023, (3.4), (3.5) and (4.15)) we have, for any $p \ge 2$, $\|R_1\|_p = O(1)$ and $\|U_2 - U_2^*\|_p = O(1)$ provided that $\mu$ has a moment of order $p$, whereas in the case of $GL_d(\mathbb{R})$, under the same moment condition on $\mu$, the above quantities were of order $m^{1/p}$ in Cuny et al. (2023) (see their Lemmas 4.3 and 4.6). Consequently for positive matrices, analyzing the proofs of Lemmas 4.10 and 4.11 of Cuny et al. (2023), we infer that when $\mu$ has a moment of order $q = r$, the inequalities stated in Cuny et al.

(2023, Lemmas 4.10 and 4.11) hold by replacing their right hand sides by $|t|^r/m^{(p-2)/2} + |t|/m^{1/2+\eta}$ (with $\eta > 0$). Following the proof of Cuny et al. (2023, Theorem 2.1) by taking into account the previous upper bounds and selecting $T = n^{p/2-1}$, the result follows.

The proof of (8.3) requires some extra arguments. Notice that for every $x \in S^+$ and every $n \in \mathbb{N}$, $\|A_n x\| = \langle e, A_n x\rangle = \langle A_n^t e, x\rangle$ and that, by items $(ii)$ and $(iii)$ of Lemma 2.2, $\|A_n^t e\|/d \le \|A_n\| \le d^2\|A_n^t e\|$. Hence,

$$\int_{S^+} \big| \log\|A_n\| - \log\|A_n x\| \big| d\nu(x) \le 2\log d + \sup_{y\in S^+} |\log\langle y, x\rangle| d\nu(x) < \infty\,. \tag{8.4}$$

Hence, we are in position to redo the proof of the bound Cuny et al. (2023, (2.4)) (see their Section 3.1.2) since (8.4) is the precise analogue of Cuny et al. (2023, (3.30)). □

**Remarks.** By some arguments already mentionned, (8.3) also holds if $\tilde\mu$ is strictly contracting and admits a moment of order $p \in (2,3]$. Let us notice that (8.1) follows also from Theorem 2.3 of Jirak (2023), since the Assumptions 2.1 there are satisfied due to the exponential convergence of the coefficients $\delta_{\infty,p}$ in Proposition 4.1.

Finally, let us mention that Xiao et al. (2022b) obtained (8.2) and (8.3) for $p = 3$ under their condition **A2**, see their Theorem 1.2 (see also Theorem 2.1 of Xiao et al. (2024) by the same authors, when $\mu$ has a subexponential moment). We will comment on this condition **A2** at the end of Section 8.2.

### 8.2. *Berry-Esseen for the spectral radius and the matrix coefficients.*

**Proposition 8.2.** *Let $p \in (2,3]$. Assume that $\mu$ is strictly contracting, admits a moment of order $p$ and almost admits a moment of order $q \in [p, \max(p,(p-2)/(3-p))]$. Assume that $s^2 > 0$ with $s^2$ as in Proposition 6.3. Set $v_n = \left(\frac{1}{n}\right)^{p/2-1}$ if $p \in (2, 1+\sqrt{3}]$ and $v_n = \left(\frac{1}{n}\right)^{q/2(q+1)}$ if $p \in (1+\sqrt{3}, 3]$. Then,*

$$\sup_{t\in\mathbb{R}} \Big| \mathbb{P}\big(\log v(A_n) - n\lambda_\mu \le t\sqrt{n}\big) - \phi_s(t)\Big| = O(v_n) \tag{8.5}$$

*and*

$$\sup_{t\in\mathbb{R}} \Big| \mathbb{P}\big(\log \kappa(A_n) - n\lambda_\mu \le t\sqrt{n}\big) - \phi_s(t)\Big| = O(v_n)\,. \tag{8.6}$$

**Remark.** When $p \le 1+\sqrt{3}$ the condition on $q$ reads $q = p$ hence is satisfied. When $p = 3$ the condition on $q$ reads $q \ge p$. (8.6) also hold if $\tilde\mu$ satisfies the assumptions of the proposition, by the arguments developed in the proof of Proposition 6.4.

**Proof.** Since $\mu$ admits a moment of order $p$, by Proposition 4.1 and Markov's inequality, there exists $C > 0$ such that for every $x > 0$ and every $n \in \mathbb{N}$, $\mathbb{P}(|\log\|A_n\| - \log v(A_n)| \ge x) \le C/x^q$. Hence, (8.5) follows from Theorem 8.1 and Lemma 8.3 below (which is Cuny et al. (2022, Lemma 2)) with $U_n = \log v(A_n) - n\lambda_\mu$, $V_n = \log\|A_n\| - n\lambda_\mu$, $R_n = \log v(A_n) - \log\|A_n\|$, and (up to some multiplicative constants) $a_n = n^{(p-2)/2}$, $b_n = n^{q/2(q+1)}$ and $c_n = (\sqrt{n}/b_n)^q)$. Finally, (8.6) follows from the fact that $v(A_n) \le \kappa(A_n) \le \|A_n\|$ and the same arguments as above.

**Lemma 8.3.** *Let $(U_n)_{n\in\mathbb{N}}$, $(V_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ be three sequences of random variables. Assume that $|U_n - V_n| \le |R_n|$ $\mathbb{P}$-a.s. and that there exist three sequences of positive numbers $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ going to infinity as $n\to\infty$, and a positive constant $s$ such that, for any integer $n$,*

$$\sup_{t\in\mathbb{R}} \big|\mathbb{P}(V_n \le t\sqrt{n}) - \phi(t/s)\big| \le \frac{1}{a_n}\,, \quad \textit{and} \quad \mathbb{P}(|R_n| \ge \sqrt{2\pi n}s/b_n) \le \frac{1}{c_n}\,.$$

*Then, for any integer $n$,*

$$\sup_{t\in\mathbb{R}}\big|\mathbb{P}(U_n \le t\sqrt{n}) - \phi(t/s)\big| \le \frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n}\,.$$

□

We shall now improve the rates under a strengthening of our integrability condition. The proof will rely on the following large deviation result.

**Lemma 8.4.** *Assume that $\mu$ is strictly contracting and almost admits some exponential moment of order $\gamma \in (0, 1]$. Then, there exist $\eta$, $\delta > 0$ such that*

$$\mathbb{P}(\max_{1\le k\le n}\big|\log v(A_k) - \log\|A_k\|\,\big| \ge \eta n) \le \mathrm{e}^{-\delta n^{\gamma}}\,.$$

**Proof.** For every $n \in \mathbb{N}$, using that $\|\cdot\|$ is submultiplicative and that $v$ is supermultiplicative, we see that, setting $\tau := \mathbb{E}(\log\|Y_1\|/v(Y_1))$,

$$\max_{1\le k\le n}\big|(\log(\|A_k\|) - \log(v(A_k))\,\big| \le \max_{1\le k\le n}\Big|\sum_{i=1}^{k}\big[\log\big(\|Y_i\|/v(Y_i)\big) - \tau\big]\Big| + n\tau\,.$$

Then the desired result follows from Theorem 2.1 of Fan et al. (2017), see their estimate (2.7) applied in the independent case (in particular the quantities in (2.3) and (2.4) of Fan et al. (2017) are identical). □

**Proposition 8.5.** *Assume that $\mu$ is strictly contracting, admits a moment of order $p \in (2, 3]$ and almost admits an exponential moment of order $\gamma \in (0, 1]$. Assume that $s^2 > 0$ with $s^2$ as in Proposition 6.3. Set $v_n = \dfrac{(\log n)^{1/\gamma}}{n^{(p-2)/2}}$. Then,*

$$\sup_{t\in\mathbb{R}}\Big|\mathbb{P}\big(\log v(A_n) - n\lambda_{\mu} \le t\sqrt{n}\big) - \phi_s(t)\Big| = O(v_n)$$

*and*

$$\sup_{t\in\mathbb{R}}\Big|\mathbb{P}\big(\log \kappa(A_n) - n\lambda_{\mu} \le t\sqrt{n}\big) - \phi_s(t)\Big| = O(v_n)\,. \tag{8.7}$$

**Remark.** (8.7) also holds if $\tilde{\mu}$ satisfies the assumptions of the proposition.

**Proof.** Let $\varepsilon \in (0, 1)$ be such that (4.4) holds. Let $x, y \in S^{+}$. Let $n \in \mathbb{N}$. Let $\omega \in \Omega$. Let $1 \le m < [n/r]$ be such that $c(Y_{mr}\cdots Y_{(m-1)r+1})(\omega) \le 1-\varepsilon$. Using the cocycle property and several items of Proposition 2.1 (in particular item $(iv)$), we see that

$$\begin{aligned}
&|\sigma(A_n, x) - \sigma(A_n, y)| \\
&\quad\le |\sigma(Y_n\cdots Y_{mr+1}, A_{mr}\cdot x) - \sigma(Y_n\cdots Y_{mr+1}, A_{mr}\cdot y)| + |\sigma(A_{mr}, x) - \sigma(A_{mr}, y)| \\
&\qquad\le 2\ln\big(1/(1 - d(A_{mr}\cdot x, A_{mr}\cdot y))\big) + \log\|A_{mr}\| - \log v(A_{mr}) \\
&\qquad\qquad\le 2\ln(1/\varepsilon) + \log\|A_{mr}\| - \log v(A_{mr})\,.
\end{aligned}$$

Define

$$\Gamma_m := \{\exists k \in 1, \dots, m \,:\, c(Y_{kr}\cdots Y_{(k-1)r+1}) \le 1-\varepsilon\}\,. \tag{8.8}$$

Taking the supremum over $x$ and the infimum over $y$, we infer that on $\Gamma_m$,

$$\log\|A_n\| - \log v(A_n) \le 2\ln(1/\varepsilon) + \max_{1\le k\le m}\big(\log\|A_{kr}\| - \log v(A_{kr})\big)\,.$$

Hence, for $\eta m \ge 4\ln(1/\varepsilon)$, using Lemma 8.4, we have

$$\begin{aligned}
\mathbb{P}(\log\|A_n\| - \log v(A_n) \ge \eta m) &\le \mathbb{P}(\Gamma_m^c) + \mathbb{P}\big(\max_{1\le k\le m}(\log\|A_{kr}\| - \log v(A_{kr})) \ge \eta m/2\big) \\
&\le \alpha^m + C_{\eta}\mathrm{e}^{-\delta_{\eta} m^{\gamma}}\,,
\end{aligned}$$

where $\alpha := \mathbb{P}(\Gamma_1^c)$.

Taking $m = [C(\log n)^{1/\gamma}] + 1$, with $C$ large enough, we infer that the right-hand side is bounded by $D/\sqrt{n}$, and we conclude using Theorem 8.1 and Lemma 8.3 applied with $U_n = \log \|A_n\|$, $V_n = \log v(A_n)$, $R_n = \log \|A_n\| - \log v(A_n)$ and (up to some multiplicative constants) $a_n = n^{(p-2)/2}$, $b_n = \sqrt{n}/(\log n)^{1/\gamma}$ and $c_n = \sqrt{n}$. □

**Proposition 8.6.** *Let $p \in (2, 3]$. Assume that $\mu$ is strictly contracting and admits a moment of order $p$. Assume that $s^2 > 0$ with $s^2$ as in Proposition 6.3. Assume moreover that $\tilde{\mu}$ almost admits a moment of order $q \in [p, \max(p, (p-2)/(3-p))]$ (resp. an exponential moment of order $\gamma \in (0, 1]$). Then, for every $x \in S^+$, the conclusion of Proposition 8.2 (resp. Proposition 8.5) holds with $\inf_{y\in S^+}\langle y, A_n x\rangle$ in place of $\kappa(A_n)$.*

**Proof.** For every $0 < \delta \le 1$, define

$$G_\delta := \{g \in G : \langle y, g \cdot x\rangle \ge \delta \quad \forall x, y \in S^+\}\,. \tag{8.9}$$

Notice that $\cup_{\delta\in(0,1]} G_\delta = G^+$, so that when $\mu$ is strictly contracting, there exist $r \ge 1$ and $\delta \in (0, 1]$ for which $\mu^{*r}(G_\delta) > 0$.

Let $p_0 = \mathbb{P}\big(\langle y, A_r \cdot x\rangle < 1/n_0 : x, y \in S^+\big)$. Note that $p_0 \in [0, 1)$ for $n_0$ large enough.

For $n > r$, let $1 \le m \le [n/r]$ be a positive integer.

Next note that, for any $g \in G_\delta$ and any $g' \in G$ and any $x, y \in S^+$, setting $x' = g'x/\|g'x\|$,

$$\langle y, gg' \cdot x\rangle = \Big\langle y, \frac{gg'x}{\|gg'x\|}\Big\rangle = \langle y, g \cdot x'\rangle \ge \delta\,. \tag{8.10}$$

This implies that if, for some integer $k \in [m, [n/r]]$ $Y_{kr} \dots Y_{(k-1)r+1} \in G_{1/n_0}$, for $x$, $y \in S^+$, we have

$$\langle y, A_n x\rangle \ge \langle Y_{kr+1}^t \cdots Y_n^t y, \frac{A_{kr}x}{\|A_{kr}x\|}\rangle \|A_{kr}x\| \ge (1/n_0)\|Y_{kr+1}^t \cdots Y_n^t y\| \,\frac{\|A_n x\|}{\|Y_n \cdots Y_{kr+1}\|}\,. \tag{8.11}$$

Therefore, if we define

$$\Delta_{n,m} := \{\omega \in \Omega \,|\, \exists k \in [m, [n/r] - 1] : (Y_{kr} \cdots Y_{(k-1)r+1})(\omega) \in G_{1/n_0}\}\,, \tag{8.12}$$

we get that, on the set $\Delta_{n,m}$ and using $\|g\| \le d\|g^t\|$,

$$\inf_{x,\,y\in S^+} \big(\log\langle y, A_n x\rangle - \log \|A_n x\|\big) \ge -\log(n_0) - \log d + \min_{mr\le \ell\le n-1} \big(\log v(Y_{\ell+1}^t \cdots Y_n^t) - \log \|Y_{\ell+1}^t \cdots Y_n^t\|\big)\,. \tag{8.13}$$

Notice that all the above quantities are non positive and that $\min_{mr\le\ell\le n}\big(\log v(Y_{\ell+1}^t \cdots Y_n^t) - \log \|Y_{\ell+1}^t \cdots Y_n^t\|\big)$ has the same law as $\min_{1\le\ell\le n-mr}\big(\log v(Y_\ell^t \cdots Y_1^t) - \log \|Y_\ell^t \cdots Y_1^t\|\big)$.

Note that

$$\mathbb{P}(\Delta_{n,m}^c) = p_0^{[n/r-m]}\,. \tag{8.14}$$

Next, assume that $\tilde{\mu}$ almost admits a moment of order $q$, with $q$ as in the proposition and take $m = 1$. Combining the above computations, for every $a > \log n_0 + \log d$ and every $x \in S^+$, we have

$$\mathbb{P}\big(\big|\inf_{y\in S^+} \log\langle y, A_n x\rangle - \log \|A_n x\|\,\big| \ge 2a\big) \le \mathbb{P}(\Delta_{n,m}^c) + \frac{\mathbb{E}\Big(\sup_{n\in\mathbb{N}} \big|\log(v(Y_n^t \cdots Y_1^t) - \log \|Y_n^t \cdots Y_1^t\|\,\big|^q\Big)}{a^q}\,.$$

Hence, using Proposition 4.1, one may finish the proof as the proof of Proposition 8.2.

Assume now that $\tilde{\mu}$ almost admits some exponential moment of order $\gamma \in (0,1]$ and let $x \in S^+$ be fixed. We wish to apply Theorem 8.1 combined with Lemma 8.3 applied to $U_n = \log \|A_n x\|$, $V_n = \inf_{y \in S^+} \log\langle y, A_n x\rangle$, $R_n = U_n - V_n$ and, up to some multiplicative constants, the sequences $a_n$, $b_n$ and $c_n$ given at the end of the proof of Proposition 8.5.

To do so, it is enough to find $K > 0$ large enough (independent from $n$) and $m$, such that

$$\mathbb{P}(\Delta_{n,m}^c) + (\mathbb{P}\big(\max_{1\le \ell \le n-mr} |\log \|Y_\ell^t \cdots Y_1^t\| - \log v(Y_\ell^t \cdots Y_1^t)| \ge \eta[K(\log n)^{1/\gamma}]\big) = O(1/\sqrt{n}), \quad (8.15)$$

where $\eta$ is given in Lemma 8.4.

Taking $m = [(n - K(\log n)^{1/\gamma})/r] - 1$ and using Lemma 8.4 we have

$$\mathbb{P}\big(\max_{1\le \ell \le n-mr} |\log \|Y_\ell^t \cdots Y_1^t\| - \log v(Y_\ell^t \cdots Y_1^t)| \ge \eta[K(\log n)^{1/\gamma}]\big) \le \mathrm{e}^{-\delta[(K(\log n)^{1/\gamma})]^\gamma} .$$

To conclude one may take $K = \max((2\delta)^{-1/\gamma}, 2^{-1} r(\log(1/p_0))^{-1})$ that implies also that $\mathbb{P}(\Delta_{n,m}^c) = O(1/\sqrt{n})$. □

To get the results of Proposition 8.6 for the quantity $\inf_{x,\, y \in S^+}\langle y, A_n x\rangle$ instead of $\inf_{y\in S^+}\langle y, A_n x\rangle$, we make an additional assumption on $\mu$.

**Proposition 8.7.** *Let $p \in (2,3]$. Assume that $\mu$ is strictly contracting and admits a moment of order $p$. Assume that $s^2 > 0$ with $s^2$ as in Proposition 6.3. Assume moreover that $\mu$ and $\tilde{\mu}$ almost admit a moment of order $q \in [p, \max(p,(p-2)/(3-p))]$ (resp. an exponential moment of order $\gamma \in (0,1]$). Then the conclusion of Proposition 8.2 (resp. Proposition 8.5) holds with $\inf_{x,\, y\in S^+}\langle y, A_n x\rangle$ in place of $\kappa(A_n)$.*

**Proof.** The proof is very close to the proof of Proposition 8.6, hence we only give the main step. We keep the same notations. Starting from (8.11), we get that, on the set where $Y_{mr}\cdots Y_{(m-1)r+1} \in G_{1/n_0}$ (recall that $G_{1/n_0}$ is defined in (8.9)),

$$\inf_{x,\, y\in S^+} \big(\log\langle y, A_n x\rangle\big) - \log \|A_n\| \ge$$
$$- \log n_0 - \log d + \big(\log v(A_n) - \log \|A_n\|\big) + \big(\log v(Y_{mr+1}^t \cdots Y_n^t) - \log \|Y_{mr+1}^t \cdots Y_n^t\|\big) .$$

Hence, the only difference with the proof of Proposition 8.6 is that we need to handle the term $\log v(A_n) - \log \|A_n\|$ but this may be done, as in the proof of Proposition 8.6, using Lemma 4.1 when $\mu$ almost admits a moment of order $q$ and Lemma 8.4 when $\mu$ almost admits some exponential moment of order $\gamma \in (0,1]$. □

Our next result requires a new type of moment condition.

**Definition 8.8.** Let $p > 0$. We say that $\mu$ admits a full moment of order $p$ if

$$\int_G \sup_{x,y\in S^+} |\log\langle x, gy\rangle|^p d\mu(g) < \infty .$$

Before stating our result, let us compare that new condition with the moment conditions previously defined.

Clearly, if $\mu$ admits a full moment of order $p > 0$, it admits a moment of order $p$ and the converse cannot be true in general.

Assume now (until the statement of the next theorem) that $\mu$ admits a moment of order $p > 0$. Then, one easily sees that it admits a full moment of order $p$ if and only if

$$\int_G \sup_{x,y\in S^+} (\log^-\langle x, gy\rangle)^p d\mu(g) < \infty , \quad \text{where } \log^-(z) = -\log(z)\mathbf{1}_{]0,1]}(z) .$$

For $g \in G$, denote $a_{ij}(g) := \langle e_i, g e_j\rangle$, where $(e_i)_{1\le i\le d}$ stands for the canonical basis of $\mathbb{R}^d$. Then, $(a_{ij}(g))_{1\le i,j\le d}$ are just the entries of $g$.

Using that for every $x, y \in S^+$, $d^2 \max_{1\le i,j\le d} a_{ij}(g) \ge \langle x, gy\rangle \ge \min_{1\le i,j\le d} a_{ij}(g)/d^2$, we see that $\mu$ admits a full moment of order $p$ if and only if

$$\int_G \big(\log^-(a_{ij}(g))\big)^p d\mu(g) < \infty \quad \forall 1 \le i,j \le d\,. \tag{8.16}$$

**Theorem 8.9.** *Let $p \in (2,3]$. Assume that $\mu$ is strictly contracting, admits a moment of order $p$ and admits a full moment of order $p-2$. Assume that $s^2 > 0$ with $s^2$ as in Proposition 6.3. Then the conclusion of Theorem 8.1 holds with $\log\big(\inf_{x,\,y\in S^+}\langle y, A_n x\rangle\big)$, or $\log v(A_n)$ instead of $\log\|A_n\|$.*

**Remark.** It will follow from the proof that the conclusion of the theorem remains true if only $\mu^{*r}$ admits a full moment of order $p-2$ for some $r \in \mathbb{N}$. A key ingredient of the proof is the basic estimate (8.18) which does not extend in a straightforward way to the situation of general cones considered in Section 11. The needed generalization appears in Lemma 12.5.

**Proof.** Using that

$$\log\|A_n\| \ge \log v(A_n) \ge \log\big(\inf_{x,\,y\in S^+}\langle y, A_n x\rangle\big) \tag{8.17}$$

and Theorem 8.1, we see that it is enough to prove the desired result for $\log\big(\inf_{x,\,y\in S^+}\langle y, A_n x\rangle\big)$.

Notice that for any matrices $f, g, h \in G$, we have

$$a_{ij}(fgh) = \sum_{k,\ell=1}^{d} a_{ik}(f)a_{k,\ell}(g)a_{\ell j}(h) \ge \min_{1\le r,s\le d} a_{rs}(f)\,\|g\| \min_{1\le r,s\le d} a_{rs}(h)\,. \tag{8.18}$$

Set $\gamma(f) := \min_{1\le r,s\le d} a_{rs}(f)$. For every $n \in \mathbb{N}$, $n \ge 2$, and every $x, y \in S^+$, using (8.18), we see that

$$\begin{aligned}\log\big(\inf_{x,\,y\in S^+}\langle y, A_n x\rangle\big) &\ge \log\big(\min_{1\le i,j\le d} a_{ij}(A_n)\big) - 2\log d\\ &\ge \log\gamma(Y_n) + \log\|Y_{n-1}\cdots Y_2\| - 2\log d + \log\gamma(Y_1)\,,\end{aligned}$$

Hence, using (8.17) and Theorem 8.1 again, we see that is is enough to prove the desired Berry-Esseen type bound for $\Delta_n := \log\gamma(Y_n) + \log\|Y_{n-1}\cdots Y_2\| + \log\gamma(Y_1)$.

We first handle the term $\log\|Y_{n-1}\cdots Y_2\|$. Using the fact that $Y_{n-1}\cdots Y_2$ has same law as $A_{n-2}$, we see that for every $t \in \mathbb{R}$,

$$\begin{aligned}&\Big|\mathbb{P}(\log\|Y_{n-1}\cdots Y_2\| - (n-2)\lambda_\mu \le \sqrt{n}t) - \phi_s(t)\Big|\\ &\le \sup_{u\in\mathbb{R}}\Big|\mathbb{P}(\log\|A_{n-2}\| - (n-2)\lambda_\mu \le \sqrt{n-2}u) - \phi_s(u)\Big| + |\phi_s(\sqrt{(n-2)/n}t) - \phi_s(t)|\,.\end{aligned}$$

Since $\phi_s$ is a Lipshitz function, there exists $C_s > 0$, such that $|\phi_s(\sqrt{(n-2)/n}t) - \phi_s(t)| \le C_s|t|\big(1-(1-2/n)^{1/2}\big) \le \frac{CC_s|t|}{n}$.

In particular, using Theorem 8.1, we see that there exists $C > 0$ such that, if $|t| \le \sqrt{n}$,

$$\Big|\mathbb{P}(\log\|Y_{n-1}\cdots Y_2\| - (n-2)\lambda_\mu \le \sqrt{n}t) - \phi_s(t)\Big| \le C(n^{1-p/2} + C_s n^{-1/2})\,.$$

When $|t| \ge \sqrt{n}$, we have, denoting by $W_s$ a centered normal variable with variance $s^2$ and using Tchebychev's inequality,

$$\begin{aligned}&\Big|\mathbb{P}(\log\|Y_{n-1}\cdots Y_2\| - (n-2)\lambda_\mu \le \sqrt{n}t) - \phi_s(t)\Big|\\ &\le \mathbb{P}(|\log\|Y_{n-1}\cdots Y_2\| - (n-2)\lambda_\mu| \ge \sqrt{n}|t|) + \mathbb{P}(|W_s| \ge |t|)\\ &\le \frac{\mathbb{E}\big((\log\|Y_{n-1}\cdots Y_2\| - (n-2)\lambda_\mu)^2\big)}{nt^2} + \mathbb{P}(|W_s| \ge \sqrt{n}) \le \frac{C_s}{n}\,,\end{aligned}$$

where we used the fact that $\mathbb{E}\big((\log\|Y_{n-1}\cdots Y_2\| - (n-2)\lambda_\mu)^2\big) = O_s(n)$, by (6.5).

Denote by $\tau_n$ the law of $\log\gamma(Y_n)+\log\gamma(Y_1)-2\lambda_\mu$. Since $\mu$ admits a full moment of order $p-2$, we have

$$L:=\sup_{n\in\mathbb{N}}\int_{\mathbb{R}}|x|^{p-2}d\tau_n(x)<\infty\,.$$

Using that $(Y_1,Y_n)$ is independent from $Y_{n-1}\cdots Y_2$, we then obtain that for every $t\in\mathbb{R}$,

$$\begin{aligned}\Big|\mathbb{P}(\Delta_n-n\lambda_\mu\le\sqrt{n}t)-\phi_s(t)\Big|&=\Big|\int_{\mathbb{R}}\mathbb{P}(\log\|Y_{n-1}\cdots Y_2\|-(n-2)\lambda_\mu\le\sqrt{n}t-x)d\tau_n(x)-\phi_s(t)\Big|\\&\le C_s n^{1-p/2}+\int_{\mathbb{R}}\big|\phi_s(t-x/\sqrt{n})-\phi_s(t)\big|d\tau_n(x)\ .\end{aligned}$$

Using again that $\phi_s$ is Lipschitz and the fact that $u\le u^\alpha$ for every $u\in[0,1]$ and $\alpha\in(0,1]$, we see that

$$\begin{aligned}\int_{\mathbb{R}}\big|\phi_s(t-x/\sqrt{n})-\phi_s(t)\big|d\tau_n(x)&\le C_s\int_R\Big(\frac{x}{\sqrt{n}}\Big)^{p-2}d\tau_n(x)+2\tau_n(\mathbb{R}\backslash[-\sqrt{n},\sqrt{n}])\\&\le C_sLn^{1-p/2}+2Ln^{1-p/2}\,,\end{aligned}$$

where we used Markov's inequality for the last bound. □

As already mentionned, Xiao et al. (2022b), proved the Berry-Essen theorem with rate $n^{-1/2}$ when $\mu$ is strictly contracting with a moment of order 3 under the following condition:

*There exists $C>0$ such that for every $g\in\operatorname{supp}\mu$ and every $1\le j\le d$,*

$$\min_{1\le i\le d}a_{ij}(g)\ge C\max_{1\le i\le d}a_{ij}(g)\,. \tag{8.19}$$

We shall refer to that condition as condition $(C)$. Recall that condition $(C)$ is called condition **A2** in Xiao et al. (2022b).

By the next lemma, the above conditions are stronger that the ones required in Theorem 8.9.

**Lemma 8.10.** *Assume that $\mu$ admits a moment of order 1 and satisfies condition $(C)$. Then, $\mu$ admits a full moment of order 1.*

**Proof.** By condition $(C)$, for every $1\le i,j\le d$,

$$a_{ij}(g)\ge(C/d)\sum_{\ell=1}^d a_{\ell j}(g)\ge(C/d)v(g)$$

and we infer that (8.16) holds. □

## 9. Regularity of the invariant measure

We prove here regularity properties of the invariant measure under various moment conditions.

**Theorem 9.1.** *Assume that $\tilde\mu$ is strictly contracting and almost admits a moment of order $p\ge 1$. Then*

$$\int_{S^+}\sup_{y\in S^+}|\log\langle y,x\rangle|^p\,d\nu(x)<\infty\,. \tag{9.1}$$

**Remark.** In the case of invertible matrices, Benoist and Quint (2016) proved that if $\mu$ has a moment of order $p>1$, then $\sup_{y\in X}\int_X|\log\langle y,x\rangle|^{p-1}\,d\nu(x)<\infty$. In view of Theorem 9.1, one may wonder whether one could have $\sup_{y\in X}\int_X|\log\langle y,x\rangle|^p\,d\nu(x)<\infty$ in the case of invertible matrices (there is no hope to have the supremum inside the integral in this setting).

**Proof.** By Fubini's theorem, it is enough to prove that

$$\sum_{n\geq 1} n^{p-1}\nu\big(\{x\in S^+ : \sup_{y\in S^+} |\log\langle y,x\rangle| \geq cn\}\big) < \infty\,,$$

for some $c>0$. Using that $\nu$ is $\mu$-invariant, it suffices to prove that

$$\sum_{n\geq 1} n^{p-1}\mathbb{P}\big(\sup_{x,\,y\in S^+} \big|\log\langle y, A_n\cdot x\rangle\big| \geq cn\big) < \infty\,. \tag{9.2}$$

Now, on $\Delta_{n,1}$ (recall its definition (8.12)), by (8.13), we have

$$\big|\log\langle y, A_n\cdot x\rangle\big| \leq \log n_0 + \log d + \max_{1\leq k\leq n}\big|\log v(Y_k^t\cdots Y_n^t) - \log\|Y_k^t\cdots Y_n^t\|\big|\,. \tag{9.3}$$

Since $\mathbb{P}(\Delta_{n,1}^c) = \eta^{[n/r-1]}$ with $\eta\in[0,1)$, it is clear that (9.2) will hold if we can find some $c>0$ such that

$$\sum_{n\geq 1} n^{p-1}\mathbb{P}\Big(\max_{1\leq k\leq n}\big|\log v(Y_n^t\cdots Y_1^t) - \log\|Y_n^t\cdots Y_1^t\|\big| \geq cn\Big) < \infty\,. \tag{9.4}$$

By Proposition 4.1, since $\tilde\mu$ almost admits a moment of order $p$,

$$\sup_{n\geq 1}\big|\log v(Y_n^t\cdots Y_1^t) - \log\|Y_n^t\cdots Y_1^t\|\big| \in L^p,$$

which yields (9.4). □

**Theorem 9.2.** *Assume that $\tilde\mu$ is strictly contracting and almost admits an exponential moment of order $\gamma\in(0,1]$. Then, there exists $\delta>0$ such that*

$$\int_{S^+}\sup_{y\in S^+} \mathrm{e}^{\delta|\log\langle y,x\rangle|^\gamma}\, d\nu(x) < \infty\,. \tag{9.5}$$

**Remark.** Inequality (9.5) has been proved in Proposition 3.3 of Xiao et al. (2024) with the supremum outside the integral, under stronger conditions. On another hand, they also obtained (9.5) with respect to their measures $\nu_s$, see Xiao et al. (2024) for the definition.

**Proof.** Proceeding as above, the theorem will be proved if we can show that there exist $\delta,\eta>0$ such that

$$\sum_{n\geq 1}\mathrm{e}^{\delta n^\gamma}\mathbb{P}\Big(\sup_{x,y\in S^+}\big|\log\langle y, A_n\cdot x\rangle\big| \geq \eta n\Big) < \infty\,. \tag{9.6}$$

We conclude thanks to (9.3) and Lemma 8.4. □

## 10. Deviation inequalities

We now provide deviation estimates, in the style of Baum-Katz.

**Proposition 10.1.** *Assume that $\mu$ is strictly contracting and admits a moment of order $p\geq 1$. Let $\alpha\in(1/2,1]$ such that $\alpha\geq 1/p$. For any $\varepsilon>0$, we have*

$$\sum_{n\geq 1} n^{\alpha p-2}\sup_{x\in S^+}\mathbb{P}\big(\max_{1\leq k\leq n}|\sigma(A_k,x) - k\lambda_\mu| \geq n^\alpha\varepsilon\big) < \infty\,. \tag{10.1}$$

**Remark.** Using Proposition 4.1, Inequality (6.7) and the fact that for $Z\in L^p$, $p\geq 1$,

$$\sum_{n\geq 1} n^{p\alpha-1}\mathbb{P}(Z\geq n^\alpha\varepsilon) < \infty, \quad \text{for any } \varepsilon>0 \text{ and any } \alpha>0,$$

one can prove similar results for

$$\log\|A_n\| - n\lambda_\mu,\quad \log\kappa(A_n) - n\lambda_\mu,\quad \log v(A_n) - n\lambda_\mu \quad\text{or}\quad \sup_{x\in S^+}|\log\|A_n x\| - n\lambda_\mu|.$$

In addition to its own interest, let us recall that Proposition 10.1 applied with $\alpha = 1/p$ (hence $1 \leq p < 2$) implies the Marcinkiewicz-Zygmund strong law of large numbers: for every $x \in S^+$,

$$\frac{\sigma(A_n, x) - n\lambda_\mu}{n^{1/p}} \underset{n\to+\infty}{\longrightarrow} 0 \qquad \mathbb{P}\text{-a.s.} \tag{10.2}$$

Indeed, by (10.1) with $\alpha = 1/p$, $\sum_{n\geq 0} \mathbb{P}(\max_{1\leq k\leq 2^n} |\sigma(A_k, x) - k\lambda_\mu| \geq 2^{n/p}) < \infty$ and (10.2) follows by the Borel-Cantelli lemma.

Proposition 10.1 is the version for positive matrices of Theorem 4.1 of Cuny et al. (2017b), stated for invertible matrices. The proof is exactly the same. Let us mention the key ingredients: The result concerns a cocycle for which, when $p \geq 2$, the function $\psi$ in (6.1) is well defined and bounded and $\sup_{k\geq 1} \sup_{x\in S^+} \|\mathbb{E}((\sigma(Y_k, A_{k-1}\cdot x))^2|\mathcal{F}_{k-1})\|_\infty < \infty$; and, when $1 \leq p < 2$, one can control the coefficients $\delta_{1,\infty}(n)$.

Concerning the matrix coefficients, the following result holds.

**Proposition 10.2.** *Assume that $\mu$ is strictly contracting and that $\mu$ and $\tilde\mu$ admit a moment of order $p \geq 1$. Let $\alpha \in (1/2, 1]$ such that $\alpha \geq 1/p$. For any $\varepsilon > 0$, we have*

$$\sum_{n\geq 1} n^{\alpha p-2} \mathbb{P}\big(\sup_{x,\, y\in S^+} |\log\langle y, A_n x\rangle - n\lambda_\mu| \geq n^\alpha \varepsilon\big) < \infty\,.$$

**Remark.** One cannot expect to have a maximum over $1 \leq k \leq n$ inside the probability, since one may have $\mathbb{P}(\log\langle y, A_1 x\rangle = -\infty) > 0$, for some $x,\, y \in S^+$.

**Proof.** On the set $\Delta_{n,1}$ defined by (8.12), we get by using (8.13) with $m = 1$,

$$\sup_{x,\, y\in S^+} |\log\langle y, A_n x\rangle - n\lambda_\mu| \\ \leq \sup_{x\in S^+} |\log\|A_n x\| - n\lambda_\mu| + \max_{1\leq k\leq n} \big|\log v(Y_k^t \cdots Y_n^t) - \log\|Y_k^t \cdots Y_n^t\|\big|.$$

To conclude, we apply the remark after Proposition 10.1 and the fact that the random variables $\max_{1\leq k\leq n} \big|\log v(Y_k^t \cdots Y_n^t) - \log\|Y_k^t \cdots Y_n^t\|\big|$ and $\max_{1\leq k\leq n} \big|\log v(Y_k^t \cdots Y_1^t) - \log\|Y_k^t \cdots Y_1^t\|\big|$ have the same law, combined with Proposition 4.1 applied to $\tilde\mu$. □

## 11. Generalization to cones

In this section we show how to extend the previous results to general cones. In the previous sections we studied products of positive matrices, that is products of matrices leaving invariant the cone $(\mathbb{R}^+)^d$. In this section we consider more general cones. This type of generalization was also investigated in Buraczewski et al. (2014).

There are many examples of closed solid cones as the ones considered below. For instance, the Lorentz (or ice-cream) cone: $\{(x_1, \ldots, x_n, z) \in \mathbb{R}^{n+1} : z \geq 0,\, x_1^2 + \ldots + x_n^2 \leq z^2\}$. The linear operators (of matrices) leaving invariant the Lorentz cone have been studied in details by Loewy and Schneider (1975).

Another example is the cone $K_S$ of positive semi-definite matrices of order $n$ viewed as a cone of the vector space of symmetric matrices of order $n$. Examples of operators leaving invariant $K_S$ are given by $M \mapsto A^t M A$ where $A$ is a matrix of size $n$ or $M \mapsto \mathrm{tr}(MR_0)S_0$, with $R_0,\, S_0 \in K_S$ and convex combinations of those.

Let $d \geq 2$. We endow $V = \mathbb{R}^d$ with its usual inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|_2$.

Let $K$ be a closed proper convex cone with non empty interior of $\mathbb{R}^d$. We recall that a cone of $\mathbb{R}^d$ is a set of $\mathbb{R}^d$ stable by multiplication by non-negative real numbers and that it is proper if $K \cap (-K) = \{0\}$.

We shall call such cones *closed solid cones*, as in Lemmens and Nussbaum (2012), page 3.

We associate with $K$ its dual cone $K^* := \{x^* \in V^* : \langle x^*, x\rangle \geq 0 \quad \forall x \in K\}$.

By Lemma 1.2.4 of Lemmens and Nussbaum (2012), $K^*$ is also a closed solid cone. Moreover, for every $x^* \in \mathrm{int}(K^*)$, (the interior of $K^*$) $\langle x^*, x\rangle > 0$ for every $x \in K\backslash\{0\}$ and $\Sigma_{x^*} := \{x \in K : \langle x^*, x\rangle = 1\}$ is a compact convex set.

We define a partial order on $V$ by setting for every $x$, $y \in V$, $x \preceq_K y$ if $y - x \in K$.

In the sequel we will need to work with a monotone norm for $K$, that is a norm compatible with $\preceq_K$ in the sense of (11.2) below.

Let us fix once and for all $x_0^* \in \mathrm{int}(K^*)$. Then, for every $x \in V$, set

$$\|x\|_{x_0^*} = \sup_{x^* \in K^* : x^* \preceq_{K^*} x_0^*} |\langle x^*, x\rangle| . \tag{11.1}$$

By Lemma 12.4, $\|\cdot\|_{x_0^*}$ is a norm on $V$ and, using the definition of $K^*$,

$$\|x\|_{x_0*} \leq \|y\|_{x_0^*} \qquad \text{for } x, y \text{ such that } 0 \preceq_K x \preceq_K y . \tag{11.2}$$

Notice also that

$$\|x\|_{x_0^*} = \langle x_0^*, x\rangle \qquad \forall x \in K . \tag{11.3}$$

Recall that $(K^*)^* = K$. Hence fixing once and for all some $x_0 \in \mathrm{int}(K)$, with $\langle x_0^*, x_0\rangle = 1$, one defines also a monotone norm on $V^*$ by setting

$$\|x^*\|_{x_0} := \sup_{x \preceq_K x_0} |\langle x^*, x\rangle| \qquad \forall x^* \in V^* .$$

Then, for every $x^* \in K^*$, $\|x^*\|_{x_0} = \langle x^*, x_0\rangle$.

Set

$$S^+ := K \cap \{x \in V : \|x\|_{x_0^*} = 1\} = \{x \in K : \langle x_0^*, x\rangle = 1\} = \Sigma_{x_0^*}$$

and

$$S^{++} := \mathrm{int}(K) \cap \{x \in V : \|x\|_{x_0^*} = 1\} = \{x \in \mathrm{int}(K) : \langle x_0^*, x\rangle = 1\} .$$

Notice that those definitions are consistent with (1.1) and (2.6), taking $x_0^* = (1, \dots, 1)$.

We shall now define an application $d$ on $(K\backslash\{0\})^2$ that will make $(S^+, d)$ a metric space.

We first define an equivalence relation $\sim_K$ on $K$, by setting for every $x$, $y$, $x \sim_K y$ if there exists $0 < \alpha \leq \beta$ such that $\alpha x \preceq_K y \preceq \beta x$. The equivalence classes for $\sim_K$ are called *parts* of $K$. By Lemma 12.2, $\mathrm{int}(K)$ is a part of $K$.

Given $x$, $y \in K\backslash\{0\}$, set

$$m(x, y) = \sup\{\lambda \geq 0 : \lambda y \preceq_K x\} .$$

This definition is consistent with the definition of the function $m$ defined in Section 1 when $K = (\mathbb{R}^+)^d$.

Notice that if some $\lambda > 0$ is such that $\lambda y \preceq_K x$ then $x - \lambda y \in K$, hence $x/\lambda - y \in K$. So $m(x, y) < +\infty$ since $K$ is closed and $K \cap (-K) = \{0\}$.

In particular, using again that $K$ is closed,

$$m(y, x)m(x, y)y \preceq_K m(y, x)x \preceq_K y \quad \text{so that} \quad m(y, x)m(x, y) \leq 1.$$

Then, we define for every $x$, $y \in K\backslash\{0\}$,

$$d(x, y) = \varphi(m(x, y)m(y, x)) ,$$

where $\varphi$ is given by (2.3).

It follows from the definition of $\sim_K$ that $x \sim_K y$ if and only if $m(x, y)m(y, x) > 0$ if and only if $d(x, y) < 1$.

Note that $d(x,y) = \tanh\big((1/2)d_H(x,y)\big)$ where $d_H$ is introduced page 26 of Lemmens and Nussbaum (2012). Actually, $d_H$ is defined when $x \sim_K y$ and when one does not have $x \sim_K y$ then one sets $d_H(x,y) = +\infty$.

**Proposition 11.1.** *$(S^+, d)$ is a complete metric space and $S^{++}$ is closed. Moreover, there exists $C_{x_0} > 0$ such that*

$$\|x - y\|_{x_0^*} \le C_{x_0^*} \frac{d(x,y)}{1 - d(x,y)} \qquad \forall (x,y) \in S^+. \tag{11.4}$$

**Remark.** When $x \sim_K y$ the right-hand side of (11.4) is finite. Otherwise, $d(x,y) = 1$ and the right-hand side of (11.4) has to be interpreted as $+\infty$.

**Proof.** We first prove that $(S^+, d)$ is a metric space. Let $x, y, z \in S^+$ be such that $x \sim_K y$ and $y \sim_K z$. By Proposition 2.1.1 of Lemmens and Nussbaum (2012), $d_H(x,z) \le d_H(x,y) + d_H(y,z)$. Using that $u \mapsto \tanh(u/2)$ is subadditive, the inequality remains true with $d$ in place of $d_H$. If we do not have $x \sim_K y$ and $y \sim_K z$, then $m(x,y)m(y,x) = 0$ or $m(y,z)m(z,y) = 0$, hence $d(x,y) = 1$ or $d(y,z) = 1$ so that the triangle inequality is still satisfied.

The fact that $d$ is a distance on $S^+$ then follows from (other statements of) Proposition 2.1.1 of Lemmens and Nussbaum (2012). The fact that $(S^+, d)$ is complete follows from Lemma 2.5.4 of Lemmens and Nussbaum (2012). Indeed, if $(x_n)_{n\in\mathbb{N}} \subset S^+$ is a Cauchy sequence for $d$, then $d(x_p, x_q) < 1$, say for $q$, $p \ge N$, so that $(x_n)_{n\ge N}$ is included in a part $P$ of $K$. But, by Lemma 2.5.4 of Lemmens and Nussbaum (2012), $S^+ \cap P$ is complete for $d$.

Let us explain why $S^{++}$ is closed. Using similar arguments as above we see that it is enough to prove that $\mathrm{int}(K)$ is a part of $K$, but this follows from Lemma 12.2.

Inequality (11.4) follows from (2.21) page 47 of Lemmens and Nussbaum (2012), using the relation between $d_H$ and $d$. □

We shall now define the analogue of the positive matrices.

Let

$$G := \{g \in M_d(\mathbb{R}) \,:\, g(K\backslash\{0\}) \subset K\backslash\{0\},\, g(\mathrm{int}(K)) \subset \mathrm{int}(K)\}\,.$$

It follows from Lemma 12.3 below that

$$G := \{g \in M_d(\mathbb{R}) \,:\, g^t(K^*\backslash\{0\}) \subset K^*\backslash\{0\},\, g^t(\mathrm{int}(K^*)) \subset \mathrm{int}(K^*)\}\,.$$

In particular, $g \in G$ is allowable in the sense of Buraczewski et al. (2014) (see $a$) page 1527). Hence, the allowability condition in Buraczewski et al. (2014) is redundant.

We endow $M_d(\mathbb{R})$ with the norm: $\|g\|_{x_0^*} := \sup_{x\in K,\, \|x\|_{x_0^*}=1} \|gx\|_{x_0^*}$. The fact that this is indeed a norm follows from the fact that $K$ has non empty interior (i.e. $K - K = V$). Notice that for $g \in G$,

$$\|g\|_{x_0^*} = \sup_{x\in K,\, \langle x_0^*, x\rangle = 1} \langle x_0^*, gx\rangle\,.$$

Define also

$$G^+ := \{g \in G \,:\, g(K\backslash\{0\}) \subset \mathrm{int}(K)\}\,.$$

By Lemma 10.1,

$$G^+ := \{g \in G \,:\, g^t(K^*\backslash\{0\}) \subset \mathrm{int}(K^*)\}\,.$$

Define for every $g \in G$

$$v_{x_0^*}(g) = \inf_{x\in K,\, \|x\|_{x_0^*}=1} \|gx\|_{x_0^*}\,,$$

Notice that for $g \in G$, $v(g) = \inf_{x\in K,\, \langle x_0^*, x\rangle=1} \langle x_0^*, gx\rangle$.

We then define $N_{x_0^*}(g) := \max(\|g\|_{x_0^*}, 1/v_{x_0^*}(g))$ and $L_{x_0^*}(g) := \frac{\|g\|_{x_0^*}}{v_{x_0^*}(g)}$.

The semi-group $G$ is acting on $S^+$ as follows.

$$g \cdot x = \frac{gx}{\|gx\|_{x_0^*}} = \frac{gx}{\langle x_0^*, gx\rangle} \qquad \forall (g,x) \in G \times S^+ .$$

We then define a cocyle by setting $\sigma(g,x) = \log(\|gx\|_{x_0^*})$ for every $(g,x) \in G \times S^+$.

For every $g \in G$ set

$$c(g) := \sup_{x,\, y \in K\setminus\{0\}} d(gx, gy) .$$

**Proposition 11.2.** *For every $(g, g', x, y) \in G^2 \times (S^+)^2$ we have*

- *(i)* $|\sigma(g,x)| \le \log N(g)$;
- *(ii)* $|\sigma(g,x) - \sigma(g,y)| \le 2C_{x_0^*} L(g) d(x,y)$ *if* $d(x,y) \le 1/2$;
- *(iii)* $|\sigma(g,x) - \sigma(g,y)| \le 2\ln\big(1/(1-d(x,y))\big)$ ;
- *(iv)* $c(gg') \le c(g)c(g')$;
- *(v)* $c(g) \le 1$ *and* $c(g) < 1$ *iff* $g \in G^+$;
- *(vi)* $d(g\cdot x, g\cdot y) \le c(g) d(x,y)$.

**Remark.** The constant $C > 0$ appearing in item $(ii)$ is the same as in (11.4).

**Proof.** Item $(i)$ is obvious. Item $(ii)$ may be proved exactly as item $(i)$ of Lemma 5.3 of Hennion (1997), using (11.4).

Let us prove Item $(iii)$. Let $x, y \in S^+$. Assume that $x \sim_K y$, since otherwise the right-hand side in item $(iii)$ equals $+\infty$ and the inequality is clear. We have $m(x,y)y \preceq_K x$ and $m(y,x)x \preceq_K y$. Since $g \in G$, $m(x,y)gy \preceq_K gx$ and $m(y,x)gx \preceq_K gy$. Using that $\|\cdot\|_{x_0^*}$ is monotone we infer that $m(x,y)\|gy\|_{x_0^*} \le \|gx\|_{x_0^*}$ and $m(y,x)\|gx\|_{x_0^*} \le \|y\|_{gx_0^*}$. Hence

$$m(x,y) \le \frac{\|gx\|_{x_0^*}}{\|y\|_{x_0^*}} \le 1/m(y,x) .$$

Then, the proof may be finished as the proof of item $(ii)$ of Lemma 5.3 of Hennion (1997).

The proof of Item $(iv)$ may be done exactly as in Hennion (1997).

Item $(v)$ follows from Proposition 11.3 below and Item $(vi)$ may be proved as in Hennion (1997).
□

We may define as above a distance $d^*$ on $K^*$, based on a function $m^* : K^* \times K^* \to \mathbb{R}^+$, to which we associate a function $c^*$ on the set

$$G^* := \{g \in M_d(\mathbb{R}) \,:\, g(K^*\backslash\{0\} \subset K^*\backslash\{0\},\, g(\mathrm{int}(K) \subset \mathrm{int}(K)\} .$$

Notice that by Lemma 12.3, $G^* = \{g^t \,:\, g \in G\}$.

Set $S^{*+} := \{x^* \in K^* \,:\, \langle x^*, x_0\rangle = 1\}$ and denote by $\mathcal{E}(S^{*+})$ the extreme points of $S^{*+}$. Denote also $\mathcal{E}(S^+)$ the extreme points of $S^+$.

**Proposition 11.3.** *For every $g \in G$, we have*

$$c(g) = \sup_{x,y\in S^+, x^*,y^* \in S^{*+}} \frac{\langle x^*, gx\rangle\langle y^*, gy\rangle - \langle x^*, gy\rangle\langle y^*, gx\rangle}{\langle x^*, gx\rangle\langle y^*, gy\rangle + \langle x^*, gy\rangle\langle y^*, gx\rangle} \tag{11.5}$$

$$= \sup_{x,y\in \mathcal{E}(S^+), x^*,y^* \in \mathcal{E}(S^{*+})} \frac{\langle x^*, gx\rangle\langle y^*, gy\rangle - \langle x^*, gy\rangle\langle y^*, gx\rangle}{\langle x^*, gx\rangle\langle y^*, gy\rangle + \langle x^*, gy\rangle\langle y^*, gx\rangle} . \tag{11.6}$$

*The suprema in (11.5) and (11.6) are taken over the $(x, y, x^*, y^*)$ such that $\langle x^*, gx\rangle\langle y^*, gy\rangle > 0$. In particular $c(g) \le 1$ and $c(g) < 1$ if and only if $g \in G^+$.*

**Remarks.** When $K = (\mathbb{R}^+)^d$ (11.6) is just (2.5). For $g \in G$, (11.9) implies that $c^*(g^t) = c(g)$.

**Proof.** As in (2.7) page 35 of Lemmens and Nussbaum (2012), noticing that they denote by $m(x/y)$ the quantity $m(x, y)$, we have

$$m(x,y) = \inf_{x^* \in S^{*+}} \frac{\langle x^*, x\rangle}{\langle x^*, y\rangle} = \inf_{x^* \in \mathcal{E}(S^{*+})} \frac{\langle x^*, x\rangle}{\langle x^*, y\rangle}\,. \tag{11.7}$$

Here and in the sequel, it is implicit that we take the infimum over the $x^*$ such that $\langle x^*, y\rangle > 0$.

Hence, we have

$$m(x,y)m(y,x) = \inf_{x^*,y^* \in S^{*+}} \frac{\langle x^*, x\rangle\langle y^*, y\rangle}{\langle x^*, y\rangle\langle y^*, x\rangle} = \inf_{x^*,y^* \in \mathcal{E}(S^{*+})} \frac{\langle x^*, x\rangle\langle y^*, y\rangle}{\langle x^*, y\rangle\langle y^*, x\rangle}\,.$$

Extending naturally $\varphi$ to a non decreasing function on $[0, +\infty[$, we infer that

$$d(x,y) = \sup_{x^*,y^* \in S^{*+}} \varphi\Big(\frac{\langle x^*, x\rangle\langle y^*, y\rangle}{\langle x^*, y\rangle\langle y^*, x\rangle}\Big) = \sup_{x^*,y^* \in \mathcal{E}(S^{*+})} \varphi\Big(\frac{\langle x^*, x\rangle\langle y^*, y\rangle}{\langle x^*, y\rangle\langle y^*, x\rangle}\Big)\,. \tag{11.8}$$

For every $g \in G$, we have

$$\sup_{x,y \in S^+} d(gx, gy) = \sup_{x,y\in S^+, x^*,y^* \in S^{*+}} \varphi\Big(\frac{\langle x^*, gx\rangle\langle y^*, gy\rangle}{\langle x^*, gy\rangle\langle y^*, gx\rangle}\Big) \tag{11.9}$$

$$= \sup_{x,y\in S^+, x^*,y^* \in \mathcal{E}(S^{*+})} \varphi\Big(\frac{\langle x^*, gx\rangle\langle y^*, gy\rangle}{\langle x^*, gy\rangle\langle y^*, gx\rangle}\Big) \tag{11.10}$$

$$= \sup_{x^*,y^* \in \mathcal{E}(S^{*+})} d^*(g^t x^*, g^t y^*)$$

$$= \sup_{x^*,y^* \in \mathcal{E}(S^{*+}), x,y \in \mathcal{E}(S^+)} \varphi\Big(\frac{\langle x^*, gx\rangle\langle y^*, gy\rangle}{\langle x^*, gy\rangle\langle y^*, gx\rangle}\Big)\,,$$

where we used (11.8) for $d^*$ to obtain the last equality.

Then, (11.5) and (11.6) follow by noticing that for every $s, t, u, v \geq 0$, with $uv > 0$

$$\varphi(st/uv) = \frac{uv - st}{st + uv}\,.$$

The fact that $c(g) \leq 1$ is obvious.

Let $g \in G^+$. Then, $\langle x^*, gx\rangle > 0$ for every $x \in K\backslash\{0\}$ and $x^* \in K^*\backslash\{0\}$. Hence, the continuous function (for either $d$ or $\|\cdot\|$) $(x, y, x^*, y^*) \mapsto \frac{\langle x^*,gx\rangle\langle y^*,gy\rangle - \langle x^*,gy\rangle\langle y^*,gx\rangle}{\langle x^*,gx\rangle\langle y^*,gy\rangle + \langle x^*,gy\rangle\langle y^*,gx\rangle}$ defined on the compact $(S^+)^2 \times (S^{*+})^2$ takes values in $[-1, 1[$. So, $c(g) < 1$.

Assume now that $g \in G\backslash G^+$. By assumption, there exists $x \in S^+$ such that $gx \in K\backslash \mathrm{int}(K)$. By Lemma 12.1, there exists $y^* \in S^{*+}$ such that $\langle y^*, gx\rangle = 0$. Since $gx \neq 0$ and $g^t y^* \neq 0$, there exist $x^* \in S^{*+}$ and $y \in S^+$ such that $\langle y^*, gy\rangle > 0$ and $\langle x^*, gx\rangle > 0$. Hence, $c(g) = 1$. ☐

We shall now consider the analogous statements as those given in Lemma 2.2. Only item (*ii*) requires a proof.

**Lemma 11.4.** *There exists $C > 0$ such that for every $g \in G$,*

$$\|gx_0\|_{x_0^*} \leq \|g\|_{x_0^*} \leq C\|gx_0\|_{x_0^*}\,.$$

**Proof.** Since $\langle x_0^*, x_0\rangle = 1$, $\|gx_0\|_{x_0^*} \le \|g\|_{x_0^*}$. Let $x \in K$ be such that $\langle x_0^*, x\rangle = 1$. Let $g \in G$. Using Lemma 12.2 with the cone $K^*$ there exists $\varepsilon > 0$ such that $g^t x_0^* \preceq_{K^*} \frac{\|g^t x_0^*\|_{x_0}}{\varepsilon} x_0^*$. Hence, using that $gx \in K$ and Lemma 12.1,

$$\begin{aligned}\|gx\|_{x_0^*} = \langle x_0^*, gx\rangle = \langle g^t x_0^*, x\rangle &\le \frac{\|g^t x_0^*\|_{x_0}}{\varepsilon}\langle x_0^*, x\rangle \\ = \frac{\langle g^t x_0^*, x_0\rangle}{\varepsilon} &= \frac{\langle x_0^*, gx_0\rangle}{\varepsilon} = \frac{\|gx_0\|_{x_0^*}}{\varepsilon}\,.\end{aligned}$$

□

All the results of the previous sections hold true for a cocycle satisfying all the properties listed in Proposition 2.1 and Lemma 2.2, replacing the quantities $N(g)$ and $L(g)$ in the moment conditions by the quantities $N_{x_0^*}(g)$ and $L_{x_0^*}(g)$.

## 12. Technical results

The next lemma is just Lemma 1.2.4 of Lemmens and Nussbaum (2012).

**Lemma 12.1.** *Let $K$ be a closed solid cone. Then*

$$\mathrm{int}(K^*) = \{x^* \in K^* \,:\, \langle x^*, x\rangle > 0\,,\ \forall x \in K\backslash\{0\}\}\,.$$

The next lemma follows from the proof Lemma 1.2.4 of Lemmens and Nussbaum (2012). We recall the arguments.

**Lemma 12.2.** *Let $\|\cdot\|$ be a norm on $V = \mathbb{R}^d$. Let $K$ be a closed solid cone. Then, for every $x \in \mathrm{int}(K)$, there exists $\varepsilon > 0$, such that for every $y \in K \cap \bar{B}_{\|\cdot\|}(0,1)$, where $\bar{B}_{\|\cdot\|}(0,1)$ is the closure of the unit ball $B_{\|\cdot\|}(0,1)$, we have $y \preceq \frac{1}{\varepsilon}x$. Then $\|y\| \le \frac{1}{\varepsilon}$. In particular, $\mathrm{int}(K)$ is a part of $K$.*

**Proof.** Let $x \in \mathrm{int}(K)$. There exists $\varepsilon > 0$ such that $\bar{B}_{\|\cdot\|}(x,\varepsilon) \subset \mathrm{int}(K)$. Let $y \in \bar{B}_{\|\cdot\|}(0,1)$. Then, $x - \varepsilon y \in K$, which means precisely that $y \preceq \frac{1}{\varepsilon}x$. In particular, if $x$, $y \in \mathrm{int}(K)$, $x \sim_K y$.

It remains to prove that for every $(x,y) \in \mathrm{int}(K) \times K$, $x \sim_K y \Rightarrow y \in \mathrm{int}(K)$.

Hence, let $x \in \mathrm{int}(K)$. There exists $\varepsilon > 0$ such that $B_{\|\cdot\|}(x,\varepsilon) \subset K$.

Let $y \in K$ be such that $y \sim_K x$. There exists $\alpha > 0$ such that $x \preceq_K \alpha y$. So $\alpha y - x \in K$ and

$$\alpha y = x + \alpha y - x \in \cup_{z\in K}(z + B_{\|\cdot\|}(x,\varepsilon))\,,$$

which is an open subset of $K$. □

**Lemma 12.3.** *Let $g \in M_d(\mathbb{R})$ and let $K$ be a closed solid cone of $E$.*

*(i) $g(K\backslash\{0\}) \subset K\backslash\{0\}$ if and only if $g^t(\mathrm{int}(K^*)) \subset \mathrm{int}(K^*)$;*
*(ii) $g(\mathrm{int}(K)) \subset \mathrm{int}(K)$ if and only if $g^t(K^*\backslash\{0\}) \subset K^*\backslash\{0\}$.*

**Proof.** Assume that $g(K\backslash\{0\}) \subset K\backslash\{0\}$. Let $x^* \in \mathrm{int}(K^*)$ and $x \in K\backslash\{0\}$. We have

$$\langle g^t x^*, x\rangle = \langle x^*, gx\rangle > 0\,,$$

by Lemma 12.1. Using Lemma 12.1 again, we see that $g^t x^* \in \mathrm{int}(K^*)$.

Assume that $g^t(\mathrm{int}(K^*)) \subset \mathrm{int}(K^*)$. Let $x \in K\backslash\{0\}$ and $x^* \in \mathrm{int}(K^*)$. We have

$$\langle x^*, gx\rangle = \langle g^t x^*, x\rangle > 0\,.$$

Hence $gx \in K^{**} = K$ (see Exercise 2.31 of Boyd and Vandenberghe (2004)) and $gx \neq 0$, which proves item $(i)$.

Item $(ii)$ is just item $(i)$ for $K^*$ using that $K^{**} = K$. □

**Lemma 12.4.** *$\|\cdot\|_{x_0^*}$ defined by (11.1) is a norm for every $x_0^* \in \mathrm{int}(K^*)$.*

**Proof.** By Lemma 1.2.5 of Lemmens and Nussbaum (2012), the set $\{x^* \in K : x^* \preceq_{K^*} x_0^*\}$ is bounded, hence $\|\cdot\|_{x_0^*}$ is finite on $V$. The fact that $\|\cdot\|_{x_0^*}$ satisfies the triangular inequality and is positively homogeneous are obvious.

Assume that $x \in E$, is such that $\|x\|_{x_0^*} = 0$. By Lemma 12.2 applied to $K^*$ (with $x = x_0^*$), for every $x^* \in K^*$, $\langle x^*, x\rangle = 0$. Since $K^*$ has non empty interior, $K^* - K^* = V^*$ and $x = 0$. □

**Lemma 12.5.** *There exists $C > 0$ such that for every $(f, g, h) \in G^3$ and every $(x, x^*) \in S \times S^{*+}$, we have*

$$\langle x^*, fghx\rangle \geq C \inf_{(u,u^*)\in S^+\times S^{*+}} \langle u^*, fu\rangle \|g\|_{x_0^*} \inf_{(v,v^*)\in S^+\times S^{*+}} \langle v^*, hv\rangle\,.$$

**Proof.** Let $f, g, h, x, x^*$ be as in the lemma. By (11.7) and the definition of $S^{*+}$, we have

$$m(hx, x_0) = \inf_{x^*\in S^{*+}} \langle x^*, hx\rangle \geq \inf_{(v,v^*)\in S^+\times S^{*+}} \langle v^*, hv\rangle\,.$$

Similarly,

$$m^*(f^*x^*, x_0^*) = \inf_{x\in S^+} \langle f^*x^*, x\rangle \geq \inf_{(u,u^*)\in S^+\times S^{*+}} \langle v^*, fv\rangle\,. \tag{12.1}$$

Using the definition of the function $m$,

$$\Big(\inf_{(v,v^*)\in S^+\times S^{*+}} \langle v^*, hv\rangle\Big) x_0 \preceq_K hx\,.$$

Using that $fg \in G$, we infer that

$$\langle x^*, fghx\rangle \geq \inf_{(u,u^*)\in S^+\times S^{*+}} \langle u^*, fu\rangle\, \langle x^*, fgx_0\rangle\,.$$

Similarly, using the definition of the function $m^*$, (12.1), and the fact that $g \in G$, we infer that

$$\langle x^*, fghx\rangle = \langle f^*x^*, ghx\rangle \geq \inf_{(u,u^*)\in S^+\times S^{*+}} \langle u^*, fu\rangle\, \langle x_0^*, gx_0\rangle \inf_{(v,v^*)\in S^+\times S^{*+}} \langle v^*, hv\rangle\,,$$

and we conclude thanks to Lemma 11.4. □

## Acknowledgements

The authors are very grateful to the referees and an associate editor for their helpful suggestions, which have considerably improved the presentation of the article.

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