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Multidimensional Stein's method for Gamma approximation

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Abstract. Let $F(\nu)$ be the centered Gamma law with parameter $\nu > 0$ and let us denote by $\mathbb{P}_{\mathbb{Y}}$ the probability distribution of a random vector $\mathbb{Y}$. We develop a multidimensional variant of the Stein's method for Gamma approximation that allows to obtain bounds for the Wasserstein distance between the probability distribution of an arbitrary random vector $(X, \mathbb{Y})$ in $\mathbb{R} \times \mathbb{R}^n$ and the probability distribution $F(\nu) \otimes \mathbb{P}_{\mathbb{Y}}$. In the case of random vectors with components in Wiener chaos, these bounds lead to some interesting criteria for the joint convergence of a sequence $\big((X_n, \mathbb{Y}_n), n \geq 1\big)$ to $F(\nu) \otimes \mathbb{P}_{\mathbb{Y}}$, by assuming that $(X_n, n \geq 1)$ converges in law, as $n \to \infty$, to $F(\nu)$ and $(\mathbb{Y}_n, n \geq 1)$ converges in law, as $n \to \infty$, to an arbitrary random vector $\mathbb{Y}$. We illustrate our criteria by two concrete examples.

1. Introduction

The Stein's method represents a popular probabilistic collection of techniques that allows to evaluate the distances between the probability distributions of random variables. Given two random variables F, G, the Stein's method allows to obtain sharp estimates for the quantities of the form

$$\sup_{h \in \mathcal{H}} \Big| \mathbf{E}[h(F)] - \mathbf{E}[h(G)] \Big| \tag{1.1}$$

where $\mathcal{H}$ constitutes a large enough class of functions. Of particular importance is the case when one of the two random variables follows the Gaussian distribution but the cases of other target distributions are also of interest. We refer to the monographs and surveys Chen et al. (2011), Chen

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and Shao (2005), Reinert (2005), Stein (1986) for a detailed description of the techniques of Stein's method and for its applications.

Our work concerns a variant of the Stein's method that allows to measure the distance between two random vectors with the same marginals but with different correlations between their components. This variant has been initiated in Pimentel (2022), where the author, by combining the Stein's heuristics with the tools of Malliavin calculus, obtained bounds for the Wasserstein distance between the probability distribution of a random vector (X, Y) where $X \sim \mathcal{N}(0,1)$ and Y is an arbitrary random variable and $\mathbb{P}_Z \otimes \mathbb{P}_Y$, *i.e.* the law of the vector (Z, Y) where $Z \sim \mathcal{N}(0,1)$ and Z is independent by Y. We denote by $\mathbb{P}_{\mathbb{X}}$ the probability distribution of a random vector $\mathbb{X}$. In some sense, this approach allows to evaluate how far are the components X and Y from being independent. The method has been extended in Tudor (2024+), by giving asymptotic results and by focusing on the case of random vectors with components in Wiener chaos.

Our purpose is to develop a similar theory for the case of the centered Gamma distribution with parameter $\nu > 0$, denoted $F(\nu)$ in the sequel. The basic observation is that if $X \sim F(\nu)$ and X is independent from an arbitrary n-dimensional random vector $\mathbb{Y}$, then

$$\mathbf{E}\Big[2(X+\nu)\frac{\partial f}{\partial x}(X,\mathbb{Y}) - Xf(X,\mathbb{Y})\Big] = 0 \tag{1.2}$$

for a large class of differentiable functions $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$. Also, if (1.2) holds true for a large class of functions f, then $X \sim F(\nu)$ and X is independent of $\mathbb{Y}$. Then, we introduce a multidimensional counterpart of the standard Stein's equation for the Gamma law, i.e.

$$2(x+\nu)\frac{\partial f}{\partial x}(x,\mathbf{y}) - xf(x,\mathbf{y}) = h(x,\mathbf{y}) - \mathbf{E}[h(Z_\nu,\mathbf{y})], \tag{1.3}$$

where $Z_\nu \sim F(\nu)$ and $h : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ belongs to suitable class of functions. We analyze in details the existence, uniqueness and the regularity of the solution to (1.3) and of its partial derivatives. In particular, we prove that there is a unique bounded solution to (1.3) and the infinity norm of this solution and of its first oder partial derivatives are controlled by the infinity norm of h and of its first and second order partial derivatives. By combining the Stein's equation with the Malliavin's integration by parts, we obtain bounds for

$$d_{\mathrm{W}}\Big(\mathbb{P}_{(X,\mathbb{Y})}, \mathbb{P}_{Z_\nu}\otimes\mathbb{P}_{\mathbb{Y}}\Big),$$

where $Z_\nu \sim F(\nu)$, and $X, \mathbb{Y}$ are arbitrary random vectors in $\mathbb{R}$ and $\mathbb{R}^n$ respectively, sufficiently regular in the sense of Malliavin calculus. We also analyze the case of random sequences in Wiener chaos where those general results translate into easy-to-apply criteria for Gamma approximation. We consider a sequence $(X_k, k \geqslant 1)$ and a sequence of n-dimensional random vectors $(\mathbb{Y}_k, k \geqslant 1)$ such that for each $k \geqslant 1$, X_k and the components of $\mathbb{Y}_k$ belongs to a Wiener chaos and we assume that, in distribution, X_k converges to $Z_\nu \sim F(\nu)$ and $\mathbb{Y}_k$ converges to an arbitrary random vector $\mathbb{Y}$ as $k \to \infty$. Under some rather natural conditions (and easy to check in particular case), we prove that the vector $\Big((X_k, \mathbb{Y}_k), k \geqslant 1\Big)$ converges in law, as $k \to \infty$, to $F(\nu)\otimes\mathbb{P}_{\mathbb{Y}}$. We also evaluate the corresponding rate of convergence under the Wasserstein distance. The method is illustrated by some examples. Our results extend the methods and findings in the literature related to the Gamma approximation, see e.g. Arras et al. (2019), Arras et al. (2020), Arras and Swan (2017), Azmoodeh et al. (2020), Azmoodeh et al. (2015), Döbler and Peccati (2018), Döbler (2012), Kusuoka and Tudor (2012), Kusuoka and Tudor (2018), Nourdin and Peccati (2009b), Nourdin and Peccati (2009a).

We organized our paper as follows. In Section 2 we give a characterization for the probability distribution of a random vector whose first marginal is a centered Gamma law and the rest of the vector is independent by the first component. This naturally leads to a multidimensional Stein's equation corresponding to the law of a such multidimensional random vectors. We then analyze in details the solution to the Stein's equation on the whole real line and we prove suitable bounds for this solution and for its partial derivatives. In Section 3 we combine our findings on the solution to the Stein's equation with the techniques of Malliavin calculus, in the spirit of Nourdin and Peccati (2009b), in order to obtain bounds for the Wasserstein distance between an arbitrary random vector and a random vector with the first component following the $F(\nu)$ law and the rest of the vector independent by this first component. Section 4 is devoted to the study of sequences of random variables in Wiener chaos. If $(X_n, n \geq 1)$ converges in distribution to $F(\nu)$ and $(\mathbb{Y}_n, n \geq 1)$ converges in distribution to an arbitrary law U on $\mathbb{R}^n$, we give criteria to check the joint convergence of the vector $((X_n, \mathbb{Y}_n), n \geq 1)$ to $F(\nu) \otimes U$. We illustrate our result by two concrete examples in Section 5. Finally, the Appendix contains the basics of Malliavin calculus.

2. The multidimensional Stein's equation and its solution

Let Z_ν be a random variable following the centered Gamma distribution $F(\nu)$ with parameter $\nu > 0$, whose probability density function is

$$p_\nu(x) := \frac{(x+\nu)^{\frac{\nu}{2}-1} e^{-\frac{x+\nu}{2}}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \mathbf{1}_{(-\nu,+\infty)}(x). \tag{2.1}$$

When ν is an integer, then Z_ν coincides in law with $\sum_{i=1}^{\nu}(N_i^2 - 1)$, where $N_1, ..., N_\nu$ are standard Gaussian independent random variables. We recall that in the standard Stein's method for Gamma approximation, the development of the method is based by the fact that $Z_\nu \sim F(\nu)$ if and only if (see e.g. Döbler and Peccati (2018) or Nourdin and Peccati (2009b))

$$\mathbf{E}\Big[2(Z_\nu + \nu) f'(Z_\nu)\Big] = \mathbf{E}\Big[Z_\nu f(Z_\nu)\Big] \tag{2.2}$$

for any differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{E}\Big[|(Z_\nu + \nu) f'(Z_\nu)|\Big] < \infty$ and $\mathbf{E}\Big[|Z_\nu f(Z_\nu)|\Big] < \infty$. Let us state and prove an analogous result in the multidimensional context. Recall that if $X \sim F(\nu)$, then its characteristic function ψ is the unique solution to the differential equation

$$\forall \lambda \in \mathbb{R}, \quad (1 - 2i\lambda)\psi'(\lambda) + 2\lambda\nu\psi(\lambda) = 0, \tag{2.3}$$

with $\psi(0) = 1$.

Lemma 2.1. *Let $\mathbb{Y} = (Y_1, ..., Y_n)$ be a n-dimensional real-valued random vector.*

(1) *Assume $Z_\nu \sim F(\nu)$ and Z_ν and $\mathbb{Y}$ are independent. Then*

$$\mathbf{E}\Big[2(Z_\nu + \nu)\frac{\partial f}{\partial x}(Z_\nu, \mathbb{Y})\Big] = \mathbf{E}\Big[Z_\nu f(Z_\nu, \mathbb{Y})\Big],$$

for any function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that for every $y \in \mathbb{R}^n$, $\frac{\partial f}{\partial x}(\cdot, y) : \mathbb{R} \to \mathbb{R}$ exists, f and $\frac{\partial f}{\partial x}$ are measurable functions with $\mathbf{E}\Big[\Big|(Z_\nu + \nu)\frac{\partial f}{\partial x}(Z_\nu, \mathbb{Y})\Big|\Big] < \infty$ and $\mathbf{E}\Big[\Big|Z_\nu f(Z_\nu, \mathbb{Y})\Big|\Big] < \infty$.

(2) *Conversely, assume that Z is an integrable random variable such that*

$$\mathbf{E}\Big[2(Z + \nu)\frac{\partial f}{\partial x}(Z, \mathbb{Y})\Big] = \mathbf{E}\Big[Z f(Z, \mathbb{Y})\Big], \tag{2.4}$$

for any function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{E}\Big[\Big|(Z+\nu)\frac{\partial f}{\partial x}(Z, \mathbb{Y})\Big|\Big] < \infty$ and $\mathbf{E}\Big[\Big|Z f(Z, \mathbb{Y})\Big|\Big] < \infty$. Then $Z \sim F(\nu)$ and Z is independent of $\mathbb{Y}$.

Proof: We start by proving point 1. Assume that $Z_\nu \sim F(\nu)$ and Z_ν and $\mathbb{Y}$ are independent. Then, by the Stein's characterization of the centered Gamma law (2.2) and our assumptions on f, we have for $\mathbb{P}_{\mathbb{Y}}$-almost all $\mathbf{y} \in \mathbb{R}^n$, then expectations $\mathbf{E}\Big[|Z_\nu f(x, \mathbf{y})|\Big]$ and $\mathbf{E}\Big[2(Z_\nu + \nu)\frac{\partial f}{\partial x}(Z_\nu, \mathbf{y})\Big]$ are finite and

$$\mathbf{E}\Big[2(Z_\nu+\nu)\frac{\partial f}{\partial x}(Z_\nu,\mathbf{y})\Big]=\mathbf{E}\Big[Z_\nu f(Z_\nu,\mathbf{y})\Big].$$

So,

$$\int_{-\nu}^{\infty}2(x+\nu)\frac{\partial f}{\partial x}(x,\mathbf{y})\mathrm{d}\mathbb{P}_{Z_\nu}(x)=\int_{-\nu}^{\infty}xf(x,\mathbf{y})\mathrm{d}\mathbb{P}_{Z_\nu}(x),$$

for $\mathbb{P}_{\mathbb{Y}}$-almost all $\mathbf{y}\in\mathbb{R}^n$. We integrate the above relation with respect to $\mathbb{P}_{\mathbb{Y}}$ on $\mathbb{R}^n$ and we find for the left-hand side, by using the independence of Z_ν and $\mathbb{Y}$,

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\int_{-\nu}^{\infty}2(x+\nu)\frac{\partial f}{\partial x}(x,\mathbf{y})\mathrm{d}\mathbb{P}_{Z_\nu}(x)\Big)\mathrm{d}\mathbb{P}_{\mathbb{Y}}(\mathbf{y})&=\int_{(-\nu,\infty)\times\mathbb{R}^n}2(x+\nu)\frac{\partial f}{\partial x}(x,\mathbf{y})\mathrm{d}(\mathbb{P}_{Z_\nu}\otimes\mathbb{P}_{\mathbb{Y}})(x,\mathbf{y})\\&=\mathbf{E}\Big[2(Z_\nu+\nu)\frac{\partial f}{\partial x}(Z_\nu,\mathbb{Y})\Big].\end{aligned}$$

Similarly for the right-hand side

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\int_{-\nu}^{\infty}xf(x,\mathbf{y})\mathrm{d}\mathbb{P}_{Z_\nu}(x)\Big)\mathrm{d}\mathbb{P}_{\mathbb{Y}}(\mathbf{y})&=\int_{(-\nu,\infty)\times\mathbb{R}^n}xf(x,\mathbf{y})\mathrm{d}(\mathbb{P}_{Z_\nu}\otimes\mathbb{P}_{\mathbb{Y}})(x,\mathbf{y})\\&=\mathbf{E}\Big[Z_\nu f(Z_\nu,\mathbb{Y})\Big].\end{aligned}$$

Let us now prove the second point of the lemma. By taking a function f with support contained in $(-\infty,-\nu)$, we found that $\mathbb{P}(Z_\nu\le-\nu)=0$. Let φ be the characteristic function of the random vector $(Z,\mathbb{Y})$, *i.e.*

$$\varphi(\lambda_1,\lambda_2):=\mathbf{E}\Big[e^{i(\lambda_1 Z+\lambda_2\cdot\mathbb{Y})}\Big],$$

for $\lambda_1\in\mathbb{R},\lambda_2\in\mathbb{R}^n$. We take the derivative of φ with respect to λ_1 and we apply (2.4) (for the real and imaginary parts of φ). By denoting $g(x,\mathbf{y})=e^{i(\lambda_1 x+\lambda_2\cdot\mathbb{Y})}$, then since Z is integrable, g satisfies the conditions to have (2.4) and we get

$$\begin{aligned}\frac{\partial\varphi}{\partial\lambda_1}(\lambda_1,\lambda_2)&=i\mathbf{E}\Big[Ze^{i(\lambda_1 Z+\lambda_2\cdot\mathbb{Y})}\Big]=i\mathbf{E}[Zg(X,\mathbb{Y})]\\&=i\mathbf{E}\Big[2(Z+\nu)\frac{\partial g}{\partial x}(Z,\mathbb{Y})\Big]=-2\lambda_1\mathbf{E}\Big[(Z+\nu)e^{i(\lambda_1 Z+\lambda_2\cdot\mathbb{Y})}\Big]\\&=2i\lambda_1\frac{\partial\varphi}{\partial\lambda_1}(\lambda_1,\lambda_2)-2\nu\lambda_1\varphi(\lambda_1,\lambda_2).\end{aligned}$$

Consequently, for every $\lambda_1\in\mathbb{R},\lambda_2\in\mathbb{R}^n$, we have

$$(1-2i\lambda_1)\frac{\partial\varphi}{\partial\lambda_1}(\lambda_1,\lambda_2)+2\lambda_1\nu\varphi(\lambda_1,\lambda_2)=0.$$

By noticing that for every $\lambda_2\in\mathbb{R}^n$

$$\varphi(0,\lambda_2)=\mathbf{E}\Big[e^{i\lambda_2\cdot\mathbb{Y}}\Big]=\varphi_{\mathbb{Y}}(\lambda_2),$$

where $\varphi_{\mathbb{Y}}$ stands for the characteristic function of the random vector $\mathbb{Y}$, we obtain from (2.3), for every $\lambda_1\in\mathbb{R},\lambda_2\in\mathbb{R}^n$,

$$\varphi(\lambda_1,\lambda_2)=\psi(\lambda_1)\varphi_{\mathbb{Y}}(\lambda_2).$$

This means that $Z\sim F(\nu)$ and Z is independent of $\mathbb{Y}$. □

Let us introduce the multidimensional Stein's equation for the centered Gamma law $F(\nu)$

$$2(x+\nu)\frac{\partial f}{\partial x}(x,\mathbf{y})-xf(x,\mathbf{y})=h(x,y)-\mathbf{E}\Big[h(Z_\nu,\mathbf{y})\Big],\tag{2.5}$$

with $Z_\nu \sim F(\nu)$ and $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{E}\Big[|h(Z_\nu, \mathbf{y})|\Big] < \infty$ for any $\mathbf{y} \in \mathbb{R}^n$. In order to solve (2.5), let us recall some facts concerning the one-dimensional Stein's equation (on the whole real line) for the Gamma law

$$2(x+\nu)f'(x) - xf(x) = h(x) - \mathbf{E}[h(Z_\nu)], \tag{2.6}$$

with $h : \mathbb{R} \to \mathbb{R}$ satisfying $\mathbf{E}\Big[\Big|h(Z_\nu)\Big|\Big] < \infty$. Recall that the density of this law is given by (2.1) and let us also introduce the function

$$q_\nu(x) := \frac{(-(x+\nu))^{\frac{\nu}{2}-1} e^{-\frac{x+\nu}{2}}}{2^{\frac{\nu}{2}}\Gamma\Big(\frac{\nu}{2}\Big)} \mathbf{1}_{(-\infty,-\nu)}(x). \tag{2.7}$$

Both functions p_ν and q_ν are positive. Note that contrary to p_ν, q_ν does not define a probability measure.

Lemma 2.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be measurable such that $\mathbf{E}\Big[|h(Z_\nu)|\Big] < \infty$. Then the Stein's equation (2.6) admits an unique bounded solution given by*

$$\begin{aligned} f_h(x) =& \Big[\int_{-\nu}^{x} \frac{p_\nu(u)}{2(x+\nu)p_\nu(x)}\Big(h(u) - \mathbf{E}\Big[h(Z_\nu)\Big]\Big)\,\mathrm{d}u\Big]\mathbf{1}_{(-\nu,+\infty)}(x) \\ &- \Big[\int_{x}^{-\nu} \frac{q_\nu(u)}{2(x+\nu)q_\nu(x)}\Big(h(u) - \mathbf{E}[h(Z_\nu)]\Big)\,\mathrm{d}u\Big]\mathbf{1}_{(-\infty,-\nu)}(x) \\ &+ \frac{h(-\nu) - \mathbf{E}\Big[h(Z_\nu)\Big]}{\nu}\mathbf{1}_{\{-\nu\}}(x). \end{aligned} \tag{2.8}$$

Proof: See Section 2.2 in Döbler and Peccati (2018). □

Remark 2.3. We can show that this unique solution is C^1 on $\mathbb{R}$, by using the l'Hôpital's rule several times, as in Döbler and Peccati (2018) when g is C^1 with h' absolutely continuous.

In a first step, we deduce the existence of a unique bounded solution of the multidimensional Stein's equation (2.5).

Proposition 2.4. *Let $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a test function such that $\mathbf{E}\Big[|h(Z_\nu, \mathbf{y})|\Big] < \infty$ for any $\mathbf{y} \in \mathbb{R}^n$. Then (2.5) admits a unique bounded solution which can be expressed as*

$$\begin{aligned} f_h(x,\mathbf{y}) =& \Big[\int_{-\nu}^{x} \frac{p_\nu(u)}{2(x+\nu)p_\nu(x)}\Big(h(u,\mathbf{y}) - \mathbf{E}\Big[h(Z_\nu,\mathbf{y})\Big]\Big)\,\mathrm{d}u\Big]\mathbf{1}_{(-\nu,+\infty)}(x) \\ &- \Big[\int_{x}^{-\nu} \frac{q_\nu(u)}{2(x+\nu)q_\nu(x)}\Big(h(u,\mathbf{y}) - \mathbf{E}\Big[h(Z_\nu,\mathbf{y})\Big]\Big)\,\mathrm{d}u\Big]\mathbf{1}_{(-\infty,-\nu)}(x), \\ &+ \frac{h(-\nu,\mathbf{y}) - \mathbf{E}\Big[h(Z_\nu,\mathbf{y})\Big]}{\nu}\mathbf{1}_{\{-\nu\}}(x). \end{aligned} \tag{2.9}$$

Proof: Let us prove first that (2.9) satisfies (2.5). Define, for any $\mathbf{y} \in \mathbb{R}^n$,

$$h_{\mathbf{y}} : \mathbb{R} \to \mathbb{R}, \quad h_{\mathbf{y}}(x) = h(x,\mathbf{y}). \tag{2.10}$$

Then, the solution to the (one-dimensional) Stein's equation (2.6) corresponding to $h_{\mathbf{y}}$ is given by (2.8). This obviously implies that f_h given by (2.9) solves (2.5), since $f_h(x,\mathbf{y}) = f_{h_{\mathbf{y}}}(x)$ for every $x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^n$.

Let us prove the unicity. Assume f_h, g_h are two bounded solutions to (2.5). Then for every $(x,\mathbf{y}) \in \mathbb{R} \times \mathbb{R}^n$,

$$2(x+\nu)\frac{\partial(f_h - g_h)}{\partial x}(x,\mathbf{y}) - x(f_h - g_h)(x,\mathbf{y}) = 0.$$

By solving the above equations, we get

$$(f_h - g_h)(x, \mathbf{y}) = \begin{cases} c_1(\mathbf{y}) \frac{e^{\frac{x}{2}}}{(x+\nu)^{\frac{\nu}{2}}} \text{ if } (x, \mathbf{y}) \in (-\nu, \infty) \times \mathbb{R}^n \\ c_2(\mathbf{y}) \frac{e^{\frac{x}{2}}}{(-(x+\nu))^{\frac{\nu}{2}}} \text{ if } (x, \mathbf{y}) \in (-\infty, -\nu) \times \mathbb{R}^n, \end{cases}$$

so

$$f_h(x, \mathbf{y}) = \begin{cases} g_h(x, \mathbf{y}) + c_1(\mathbf{y}) \frac{e^{\frac{x}{2}}}{(x+\nu)^{\frac{\nu}{2}}} \text{ if } (x, \mathbf{y}) \in (-\nu, \infty) \times \mathbb{R}^n \\ g_h(x, \mathbf{y}) + c_2(\mathbf{y}) \frac{e^{\frac{x}{2}}}{(-(x+\nu))^{\frac{\nu}{2}}} \text{ if } (x, \mathbf{y}) \in (-\infty, -\nu) \times \mathbb{R}^n \\ 0, \text{ if } x = -\nu, \mathbf{y} \in \mathbb{R}^n. \end{cases}$$

Consequently, f_h is bounded if and only if $c_1(\mathbf{y}) = c_2(\mathbf{y}) = 0$ for every $\mathbf{y} \in \mathbb{R}^n$. □

The next step is to find suitable estimates for the solution (2.9) and for its partial derivatives. We follow the methodology proposed in Döbler and Peccati (2018) and Döbler (2012) in the one-dimensional case. We denote by x_ν the median of the centered Gamma law $F(\nu)$, i.e. the real number $x_\nu \in (-\nu, \infty)$ such that $\mathbb{P}(Z_\nu \leq x_\nu) = \mathbb{P}(Z_\nu \geq x_\nu)$.

Proposition 2.5. *Let $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a C^1 map with bounded partial derivatives. Let f_h be the unique bounded solution to (2.5) corresponding to h. We have the following estimates:*

$$\|f_h\|_\infty \leqslant \left\| \frac{\partial h}{\partial x} \right\|_\infty, \tag{2.11}$$

and for every $j \in \{1, ..., n\}$,

$$\left\| \frac{\partial f_h}{\partial x} \right\|_\infty \leqslant C_2 \left\| \frac{\partial h}{\partial x} \right\|_\infty \quad \text{and} \quad \left\| \frac{\partial f_h}{\partial y_j} \right\|_\infty \leqslant C_1 \left\| \frac{\partial h}{\partial y_j} \right\|_\infty, \tag{2.12}$$

where

$$C_1 = \max \left\{ \frac{2}{\nu}, \frac{1}{\int_{x_\nu}^\infty u p_\nu(u) du} \right\} \text{ and } C_2 = \max \left\{ 1, \frac{2}{\nu} \right\}. \tag{2.13}$$

Proof: Let us start with the proof of the inequality (2.11). Consider the function $h_{\mathbf{y}}$ given by (2.10) and let f_h be the corresponding solution to the Stein's equation (2.6). Then $h_{\mathbf{y}}$ is C^1 on $\mathbb{R}$ with bounded derivatives. By the proof of Theorems 2.1 and 2.3 in Döbler and Peccati (2018) we have, for every $x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^n$,

$$|f_h(x, \mathbf{y})| = |f_{h_{\mathbf{y}}}(x)| \leq \|h'_{\mathbf{y}}\|_\infty S_\nu(x),$$

where $\|h'_{\mathbf{y}}\|_\infty = \sup_{x \in \mathbb{R}} |h'_{\mathbf{y}}(x)|$ and S_ν is a bounded function on $\mathbb{R}$ (it does not depend on $\mathbf{y}$) such that $\|S_\nu\|_\infty \leq 1$. Then $|f_h(x, \mathbf{y})| \leq \|h\|_\infty$, which gives (2.11).

A similar argument allows to conclude the first inequality in (2.12). Indeed, the results in Döbler (2012) imply, for any $x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^n$,

$$\left| \frac{\partial f_h}{\partial x}(x, \mathbf{y}) \right| = |f'_{h_{\mathbf{y}}}(x)| \leq \|h'_{\mathbf{y}}\|_\infty R_\nu(x),$$

where again R_ν is a bounded function on $\mathbb{R}$ not depending on $\mathbf{y} \in \mathbb{R}^n$, such that $\|R_\nu\|_\infty \leq C_2$, with C_2 from (2.13). This immediately implies the first bound in (2.12).

Next we show show that for any measurable and bounded function $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, we have

$$\|f_g\|_\infty \leq C_1 \|g\|_\infty,$$

where f_g is the solution to the multidimensional Stein's equation (2.5) corresponding to g and C_1 is defined by (2.13). Let, for $x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^n$,

$$\tilde{g}(x, \mathbf{y}) := g(x, \mathbf{y}) - \mathbf{E}[g(Z_\nu, \mathbf{y})]$$

Let $x > -\nu$. By (2.9), we have the following two expressions for f_g

$$g_h(x,\mathbf{y}) = \frac{1}{2(x+\nu)p_\nu(x)} \int_{-\nu}^{x} p_\nu(u)\tilde{g}(u,\mathbf{y})\ \mathrm{d}u = \frac{-1}{2(x+\nu)p_\nu(x)} \int_{x}^{+\infty} p_\nu(u)\tilde{g}(u,\mathbf{y})\ \mathrm{d}u,$$

the second equality being a consequence of the fact that $\mathbf{E}\Big[\tilde{g}(Z_\nu,\mathbf{y})\Big] = 0$ for every $\mathbf{y} \in \mathbb{R}^n$. Hence, we conclude that

$$\sup_{\substack{x\in(-\nu,+\infty)\\ \mathbf{y}\in\mathbb{R}^n}} |f_g(x,\mathbf{y})| \leqslant \|\tilde{g}\|_\infty \sup_{x\in(-\nu,+\infty)} \left[\frac{1}{2(x+\nu)p_\nu(x)} \min\left\{\int_{-\nu}^{x} p_\nu(u)\ \mathrm{d}u, \int_{x}^{+\infty} p_\nu(u)\ \mathrm{d}u\right\}\right].$$

By using the equality $2(x+\nu)p_\nu(x) = -\int_{-\nu}^{x} up_\nu(u)\ \mathrm{d}u$ (see Döbler and Peccati (2018) or Kusuoka and Tudor (2012)) and differentiating, we notice the map $x \longmapsto \frac{1}{2(x+\nu)p_\nu(x)} \int_{-\nu}^{x} p_\nu(u)\ \mathrm{d}u$ is an increasing map, going to 0 in $-\nu$ and to $+\infty$ in $+\infty$. The map $x \longmapsto \frac{1}{2(x+\nu)p_\nu(x)} \int_{x}^{+\infty} p_\nu(u)\ \mathrm{d}u$ is a decreasing map, going to $+\infty$ in $-\nu$ and to 0 in $+\infty$. Hence, the minimum is reached exactly where both maps coincides. This is the case if and only if $\int_{-\nu}^{x} p_\nu(u)\ \mathrm{d}u = \int_{x}^{+\infty} p_\nu(u)\ \mathrm{d}u$, so if and only if $\mathbb{P}(Z_\nu \leqslant x) = \mathbb{P}(Z_\nu \geqslant x)$, so the minimum is reached when x is x_ν, the median of the probability distribution $F(\nu)$. Hence, we have

$$\begin{aligned}\sup_{(x,\mathbf{y})\in(-\nu,\infty)\times\mathbb{R}^n} |f_g(x,\mathbf{y})| &\leqslant \|\tilde{g}\|_\infty \frac{1}{4(x_\nu+\nu)p_\nu(x_\nu)} = \|\tilde{g}\|_\infty \frac{1}{2\int_{x_\nu}^{+\infty} up_\nu(u)\ \mathrm{d}u}. \\ &\leq \|g\|_\infty \frac{1}{\int_{x_\nu}^{+\infty} up_\nu(u)\ \mathrm{d}u}.\end{aligned} \tag{2.14}$$

Let $x < -\nu$. We have from (2.9)

$$f_g(x,\mathbf{y}) = \frac{-1}{2(x+\nu)q_\nu(x)} \int_{x}^{-\nu} q_\nu(u)\tilde{h}(u)\ \mathrm{d}u.$$

Notice that

$$2(x+\nu)q_\nu(x) = \int_{x}^{-\nu} uq_\nu(u)\ \mathrm{d}u \leqslant 0,$$

and

$$-\int_{x}^{-\nu} uq_\nu(u)\ \mathrm{d}u \geqslant \nu \int_{x}^{-\nu} q_\nu(u)\ \mathrm{d}u > 0.$$

This implies the following bound for f_g:

$$\sup_{(x,\mathbf{y})\in(-\infty,-\nu)\times\mathbb{R}^n} |f_g(x,\mathbf{y})| \leqslant \frac{\|\tilde{g}\|_\infty}{\nu} \leq \frac{2}{\nu}\|g\|_\infty. \tag{2.15}$$

From (2.14) and (2.15), we get

$$\|f_g\|_\infty \leqslant C_1 \|g\|_\infty \tag{2.16}$$

with C_1 given by (2.13), for any measurable and bounded function $g : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$.

To deal with the second bound in (2.12), by differentiating the Stein's equation (2.5) with respect to y_j, we observe that we have in fact

$$\partial_{y_j} f_h = f_{\partial_{y_j} h}.$$

On the other hand, for every $j \in \{1,...,n\}$, $\frac{\partial h}{\partial y_j}$ is bounded by assumption. Then the bound (2.16) implies the second inequality in (2.12).

□

3. Multidimensional Stein-Malliavin calculus for the Gamma distribution

We use the multidimensional Stein's equation and the properties of its solution proven in Proposition 2.5 in combination with the techniques of Malliavin calculus in order to obtain bounds for the distance between the probability distributions of random vectors. Actually, we will evaluate the Wasserstein distance between the law of an arbitrary random vector $(X, \mathbb{Y})$ in $\mathbb{R} \times \mathbb{R}^n$ and the random vector $(Z_\nu, \mathbb{Y})$, where $Z_\nu \sim F(\nu)$ and Z_ν is independent by $\mathbb{Y}$. We recall the classical Wasserstein distance. Let

$$\mathcal{A} = \{h : \mathbb{R}^n \to \mathbb{R},\ h \text{ is Lipschitz continuous with } \|h\|_{\text{Lip}} \leqslant 1\},$$

where $\|h\|_{\text{Lip}}$ stands for the Lipschitz norm of h, i.e.

$$\|h\|_{\text{Lip}} = \sup_{x,y\in\mathbb{R}^n; x\neq y} \frac{|h(x) - h(y)|}{|x-y|_n},$$

with $|\cdot|_n$ the Euclidean norm in $\mathbb{R}^n$. Then the Wasserstein distance between two integrable random vectors $\mathbb{X}$ and $\mathbb{Y}$ is defined by

$$d_{\text{W}}(\mathbb{P}_{\mathbb{X}}, \mathbb{P}_{\mathbb{Y}}) := \sup_{h\in\mathcal{A}} \Big|\mathbf{E}[h(\mathbb{X})] - \mathbf{E}[h(\mathbb{Y})]\Big|. \tag{3.1}$$

Let us state and prove the main result of this section.

Theorem 3.1. *Let X a centered random variable and $\mathbb{Y} = (Y_1, \cdots, Y_n)$ a centered random vector of $\mathbb{R}^n$ such that $X \in \mathbb{D}^{1,2}$ and $Y_j \in \mathbb{D}^{1,2}$ for every $j \in \{1, \cdots, n\}$. Then there exists a constant $C = C(\nu) > 0$ such that*

$$\begin{aligned} & d_{\text{W}}\Big(\mathbb{P}_{(X,\mathbb{Y})}, \Gamma(\nu)\otimes\mathbb{P}_{\mathbb{Y}}\Big) \\ & \le C_2\ \mathbf{E}\Big[\Big|2(X+\nu) - \big\langle \text{D}(-L)^{-1}X, \text{D}X\big\rangle\Big|\Big] + C_1 \sum_{j=1}^n \mathbf{E}\Big[\Big|\big\langle \text{D}(-L)^{-1}X, \text{D}Y_j\big\rangle\Big|\Big], \end{aligned} \tag{3.2}$$

with C_1, C_2 defined by (2.13).
If moreover we suppose that $X \in \mathbb{D}^{1,4}$ and $Y_j \in \mathbb{D}^{1,4}$ for every $j \in \{1, \cdots, n\}$, then

$$\begin{aligned} & d_{\text{W}}\Big(\mathbb{P}_{(X,\mathbb{Y})}, \Gamma(\nu)\otimes\mathbb{P}_{\mathbb{Y}}\Big) \\ & \le C_2\ \mathbf{E}\Big[\Big(2(X+\nu) - \big\langle \text{D}(-L)^{-1}X, \text{D}X\big\rangle\Big)^2\Big]^{\frac{1}{2}} + C_1 \sum_{j=1}^n \mathbf{E}\Big[\big\langle \text{D}(-L)^{-1}X, \text{D}Y_j\big\rangle^2\Big]^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

Proof: Let suppose first that $h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ with $\|h\|_{\text{Lip}} \leqslant 1$. Let f_h be the corresponding solution to the Stein's equation (2.5). We have, by integrating this Stein's equation with respect to the measure $\mathbb{P}_{(X,\mathbb{Y})}$,

$$\begin{aligned} & \int_{(-\nu,+\infty)\times\mathbb{R}^n} \Big[2(x+\nu)\frac{\partial f_h}{\partial x}(x,\mathbf{y}) - x f_h(x,\mathbf{y})\Big]\ \text{d}\mathbb{P}_{(X,\mathbb{Y})}(x,\mathbf{y}) \\ &= \int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \text{d}\mathbb{P}_{(X,\mathbb{Y})}(x,\mathbf{y}) - \int_{\mathbb{R}^n} \mathbf{E}[h(Z_\nu,\mathbf{y})]\ \text{d}\mathbb{P}_{\mathbb{Y}}(\mathbf{y}) \\ &= \int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \text{d}\mathbb{P}_{(X,\mathbb{Y})}(x,\mathbf{y}) - \int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \text{d}\mathbb{P}_{Z_\nu}\otimes\text{d}\mathbb{P}_{\mathbb{Y}}(x,\mathbf{y}). \end{aligned}$$

In other words,

$$\int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_{(X,\mathbb{Y})}(x,\mathbf{y}) - \int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_{Z_\nu}\otimes \mathrm{d}\mathbb{P}_{\mathbb{Y}}(x,\mathbf{y})$$
$$=2\mathbf{E}\Big[(X+\nu)\frac{\partial f_h}{\partial x}(X,\mathbb{Y})\Big] - \mathbf{E}\Big[Xf_h(X,\mathbb{Y})\Big]. \tag{3.4}$$

Since X is centered, we can write $X=\delta\Big[\mathrm{D}(-L)^{-1}X\Big]$, and by plugging this into (3.4)

$$\int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_{(X,\mathbb{Y})}(x,\mathbf{y}) - \int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_{Z_\nu}\otimes \mathrm{d}\mathbb{P}_{\mathbb{Y}}(x,\mathbf{y})$$
$$=2\mathbf{E}\Big[(X+\nu)\frac{\partial f_h}{\partial x}(X,\mathbb{Y})\Big] - \mathbf{E}\Big[\delta\Big[\mathrm{D}(-L)^{-1}X\Big]f_h(X,\mathbb{Y})\Big]$$
$$=2\mathbf{E}\Big[(X+\nu)\frac{\partial f_h}{\partial x}(X,\mathbb{Y})\Big] - \mathbf{E}\Big[\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}f_h(X,\mathbb{Y})\right\rangle\Big]$$
$$=2\mathbf{E}\Big[(X+\nu)\frac{\partial f_h}{\partial x}(X,\mathbb{Y})\Big] - \mathbf{E}\Big[\frac{\partial f_h}{\partial x}(X,\mathbb{Y})\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}X\right\rangle\Big]$$
$$-\sum_{j=1}^n \mathbf{E}\Big[\frac{\partial f_h}{\partial y_j}(X,\mathbb{Y})\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}Y_j\right\rangle\Big].$$

Since $\|h\|_{\mathrm{Lip}}\leqslant 1$, we have byProposition 2.5,

$$\left\|\frac{\partial f_h}{\partial x}\right\|_\infty \leqslant C_2 \text{ and } \left\|\frac{\partial f_h}{\partial y_j}\right\|_\infty \leqslant C_1.$$

We then obtain

$$\Big|\int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_{(X,\mathbb{Y})}(x,\mathbf{y}) - \int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_{X}\otimes \mathrm{d}\mathbb{P}_{\mathbb{Y}}(x,\mathbf{y})\Big|$$
$$\leqslant C_2\mathbf{E}\Big[\Big|(X+\nu)-\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}X\right\rangle\Big|\Big] + C_1\sum_{j=1}^n \mathbf{E}\Big[\Big|\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}Y_j\right\rangle\Big|\Big].$$

To conclude the bound (3.2), we need to have the above inequality for every $h\in\mathcal{A}$. If h is one of those functions, we approach it by the sequence

$$h_k(x,\mathbf{y}) := \mathbf{E}\Big[h\Big(x+\frac{N}{\sqrt{k}}, \mathbf{y}+\frac{\mathbf{N}}{\sqrt{k}}\Big)\Big],$$

where $N\sim\mathcal{N}(0,1)$ is independent of $\mathbf{N}\sim\mathcal{N}_n(0,I_n)$. Then $(h_k, k\geq 1)$ uniformly converges to h and we still have $\|h_k\|_{\mathrm{Lip}}\leqslant 1$. Consequently, we have:

$$\Big|\int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_{(X,\mathbb{Y})}(x,\mathbf{y}) - \int_{(-\nu,+\infty)\times\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_{Z_\nu}\otimes \mathrm{d}\mathbb{P}_{\mathbb{Y}}(x,\mathbf{y})\Big|$$
$$\leqslant 2\|h-h_k\|_\infty + C_2\mathbf{E}\Big[\Big|(X+\nu)-\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}X\right\rangle\Big|\Big]$$
$$+C_1\sum_{j=1}^n \mathbf{E}\Big[\Big|\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}Y_j\right\rangle\Big|\Big].$$

Taking the limit as $k\to\infty$, (3.2) is obtained. For (3.3), it suffices to notice that if $X, Y_j\in\mathbb{D}^{1,4}$, then the scalar products $\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}X\right\rangle$ and $\left\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}Y_j\right\rangle$ are in L^2, so that we can apply Cauchy–Schwarz's inequality. □

4. Asymptotic independence on Wiener chaos

We now focus on random variables in Wiener chaos and we give an asymptotic variant of the result proven in Theorem 3.1. We will consider a sequence $(X_k, k \geq 1)$ that converges in distribution when $k \to \infty$ to the centered Gamma law $F(\nu)$ and a sequence of random vectors $(\mathbb{Y}_k, k \geq 1)$ converging in law as $k \to \infty$ to an arbitrary random vector $\mathbb{Y}$. We assume that for each $k \geq 1$, X_k and the components of $\mathbb{Y}_k$ belong to a Wiener chaos of fixed order. Under some pretty natural assumptions, we deduce, by using Theorem 3.1 and the properties of random variables in Wiener chaos, that the random sequence $\Big((X_k, \mathbb{Y}_k), k \geq 1\Big)$ converges in law to $(Z_\nu, \mathbb{Y})$, where Z_ν follows the centered Gamma distribution with parameter $\nu > 0$ and Z_ν and $\mathbb{Y}$ are independent. This means that the sequences $(X_k, k \geq 1)$ and $(\mathbb{Y}_k, k \geq 1)$ are asymptotically independent. We obtain bounds to quantify this asymptotic independence under the d_{W}-distance.

Let us first recall, that if a sequence $(X_k, k \geq 1)$ in the qth Wiener chaos converges to $F(\nu)$, then the order q of the chaos must be an even integer. Indeed, if q is odd then we have $\mathbf{E}\Big[X_k^3\Big] = 0$ for any $k \geqslant 1$ and it contradicts the fact that $\mathbf{E}\Big[Z_\nu^3\Big] = 8\nu > 0$ if $Z_\nu \sim F(\nu)$.

Before stating our result, let us recall the following criterion for the Gamma approximation on Wiener chaos. We refer to Nourdin and Peccati (2009a) for its proof.

Theorem 4.1. *Let $(X_k, k \geq 1)$ be a sequence of random variables such that for every $k \geqslant 1$, $X_k = I_q(f_k)$ where q is an even integer and $f_k \in H^{\odot q}$. Assume that $\mathbf{E}\Big[X_k^2\Big] \xrightarrow[k\to\infty]{} 2\nu$. Then the following are equivalent:*

(1) *The sequence $(X_k, k \geq 1)$ converges in distribution to $F(\nu)$.*
(2) *For every $p \in \{1, ..., q-1\}$ with $p \neq \frac{q}{2}$,*

$$\|f_k \otimes_p f_k\|^2_{H^{\otimes 2q-2p}} \xrightarrow[k\to\infty]{} 0 \quad \textit{and} \quad \Big\|f_k \widetilde{\otimes}_{\frac{q}{2}} f_k - c_q f_k\Big\|^2_{H^{\otimes q}} \xrightarrow[k\to\infty]{} 0,$$

where $c_q > 0$ is an explicit constant.
(3) *As $k \to \infty$,*

$$\|\mathrm{D}X_k\|^2_H - 2qX_k - 2q\nu \xrightarrow[k\to\infty]{} 0 \ \textit{in}\ L^2(\Omega).$$

Other equivalent conditions to 1.-3. (not needed in our work) are stated and proven in Nourdin and Peccati (2009a).

We apply this theorem in our context, about asymptotic independence with a component going to a centered Gamma law. We have the following result.

Theorem 4.2. *Let n, q and $q_1, \cdots, q_n$ strictly positive integers such that q is even and $q \geqslant q_j$, for every $j \in \{1, \cdots, n\}$. We consider a sequence $(X_k, k \geqslant 1) = \Big(I_q(f_k), k \geqslant 1\Big)$ converging as k goes to $+\infty$ in distribution to a random variable X following $F(\nu)$. We consider a sequence of random vectors $(\mathbb{Y}_k, k \geq 1) = ((Y_k^1, \cdots, Y_k^n), k \geq 1)$ such that $Y_k^j = I_{q_j}(g_k^j)$ and there exists a random vector $\mathbb{Y} = (Y^1, \cdots, Y^n)$ such that $(\mathbb{Y}_k, k \geqslant 1)$ converges in distribution to $\mathbb{Y}$. We suppose that, for every $j \in \{1, \cdots, n\}$:*

$$\mathbf{E}\Big[X_k Y_k^j\Big] \xrightarrow[k\to\infty]{} 0 \ \textit{and}\ \Big\|f_k \otimes_{\frac{q}{2}} g_k^j\Big\| \xrightarrow[k\to\infty]{} 0. \tag{4.1}$$

Then we have the following convergence in distribution :

$$(X_k, \mathbb{Y}_k) \xrightarrow[k\to+\infty]{(\mathrm{d})} (X, \mathbb{Y}), \tag{4.2}$$

where $X \sim F(\nu)$ and X is independent of $\mathbb{Y}$. Moreover, if we denote

$$\theta_k := \mathbb{P}_{(X_k,\mathbb{Y}_k)} \text{ and } \eta := \mathbb{P}_X \otimes \mathbb{P}_{\mathbb{Y}},$$

then we have the following estimation of the Wasserstein distance between those two measures:

$$\begin{aligned} d_{\mathrm{W}}(\theta_k,\eta) \leqslant & C_2\ \mathbf{E}\Big[\Big(2(X_k+\nu) - \big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}X_k\big\rangle\Big)^2\Big]^{\frac{1}{2}} \\ & + C_1\ \sum_{j=1}^n \mathbf{E}\Big[\Big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}Y_k^j\Big\rangle^2\Big]^{\frac{1}{2}} + d_{\mathrm{W}}\Big(\mathbb{P}_{\mathbb{Y}_k}, \mathbb{P}_{\mathbb{Y}}\Big), \end{aligned} \tag{4.3}$$

with C_1, C_2 given by (2.13).

Proof: For $k \geq 1$, let us consider the probability measure $\eta_k := \mathbb{P}_X \otimes \mathbb{P}_{\mathbb{Y}_k} = F(\nu) \otimes \mathbb{P}_{\mathbb{Y}_k}$. Then we have for every $h \in \mathcal{A}$ (the set defined at the beginning of Section 3),

$$\begin{aligned} & \Big| \int_{\mathbb{R}} \int_{\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\eta_k(x,\mathbf{y}) - \int_{\mathbb{R}} \int_{\mathbb{R}^n} h(x,\mathbf{y})\ \mathrm{d}\eta(x,\mathbf{y}) \Big| \\ & = \Big| \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_X(x)\Big) \mathrm{d}\mathbb{P}_{\mathbb{Y}_k}(\mathbf{y}) - \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_X(x)\Big) \mathrm{d}\mathbb{P}_{\mathbb{Y}}(\mathbf{y}) \Big| \\ & = \Big| \mathbf{E}\Big[\int_{\mathbb{R}} h(x,\mathbb{Y}_k)\ \mathrm{d}\mathbb{P}_X(x)\Big] - \mathbf{E}\Big[\int_{\mathbb{R}} h(x,\mathbb{Y})\ \mathrm{d}\mathbb{P}_X(x)\Big] \Big| \\ & \leqslant d_{\mathrm{W}}(\mathbb{P}_{\mathbb{Y}_k}, \mathbb{P}_{\mathbb{Y}}), \end{aligned}$$

the last inequality being true because the function $\mathbf{y} \longmapsto \int_{\mathbb{R}} h(x,\mathbf{y})\ \mathrm{d}\mathbb{P}_X(x)$ also belongs to $\mathcal{A}$ (on $\mathbb{R}^n$). Hence,

$$d_{\mathrm{W}}(\eta_k,\eta) \leqslant d_{\mathrm{W}}(\mathbb{P}_{\mathbb{Y}_k}, \mathbb{P}_{\mathbb{Y}}). \tag{4.4}$$

Now by using the triangle inequality, (4.4) and the bound (3.3) in Theorem 3.1,

$$\begin{aligned} d_{\mathrm{W}}(\theta_k,\eta) & \leq d_{\mathrm{W}}(\theta_k,\eta_k) + d_{\mathrm{W}}(\eta_k,\eta) \\ & \leq C_2\ \mathbf{E}\Big[\Big(2(X_k+\nu) - \big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}X_k\big\rangle\Big)^2\Big]^{\frac{1}{2}} \\ & + C_1\ \sum_{j=1}^n \mathbf{E}\Big[\Big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}Y_k^j\Big\rangle^2\Big]^{\frac{1}{2}} + d_{\mathrm{W}}\Big(\mathbb{P}_{\mathbb{Y}_k}, \mathbb{P}_{\mathbb{Y}}\Big) \end{aligned}$$

which is nothing else but (4.3). Let us now prove that the right-hand side of (4.3) converges to zero as $k \to \infty$. For the first term, the convergence to zero comes from Theorem 4.1. Indeed,

$$\mathbf{E}\Big[\Big(2(X_k+\nu) - \big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}X_k\big\rangle\Big)^2\Big] = \frac{1}{q^2}\ \mathbf{E}\Big[\Big(\|\mathrm{D}X_k\|^2 - 2qX_k - 2q\nu\Big)^2\Big] \xrightarrow[k\to\infty]{} 0$$

due to point 3. in Theorem 4.1. About the second term, we will use some classical computations about Wiener chaoses. We have:

$$\mathbf{E}\Big[\Big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}Y_k^j\Big\rangle^2\Big] = \frac{1}{q^2}\mathbf{E}\Big[\Big\langle \mathrm{D}X_k, \mathrm{D}Y_k^j\Big\rangle^2\Big].$$

Then, by using the product formula (6.3), and the assumption $q \geqslant q_j$, we conclude that

$$\begin{aligned}
&\mathbf{E}\Big[\Big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}Y_k^j\Big\rangle^2\Big] \\
&= \frac{1}{q^2}\sum_{r=0}^{q_j-1}(r!)^2\binom{q_j-1}{r}^2\binom{q-1}{r}^2(q+q_j-2-2r)!\Big\|f_k \,\widetilde{\otimes}_{r+1}\, g_k^j\Big\|^2 \\
&= \frac{1}{q^2}\sum_{r=1}^{q_j}((r-1)!)^2\binom{q_j-1}{r-1}^2\binom{q-1}{r-1}^2(q+q_j-2r)!\Big\|f_k \,\widetilde{\otimes}_{r}\, g_k^j\Big\|^2.
\end{aligned} \tag{4.5}$$

Suppose first that $q > q_j$. We estimate the norm of the symmetrical tensor product.

$$\begin{aligned}
\Big\|f_k \,\tilde{\otimes}_r\, g_k^j\Big\|^2 \leqslant& \Big\|f_k \otimes_r g_k^j\Big\|^2 = \Big\langle f_k \otimes_r g_k^j, f_k \otimes_r g_k^j\Big\rangle \\
&= \Big\langle f_k \otimes_{q-r} f_k, g_k^j \otimes_{q_j-r} g_k^j\Big\rangle \le \|f_k \otimes_{q-r} f_k\| \cdot \Big\|g_k^j \otimes_{q_j-r} g_k^j\Big\| \\
&\leqslant \|f_k \otimes_{q-r} f_k\| \cdot \|g_j\|^2 \le \frac{\mathbf{E}\Big[\Big(Y_k^j\Big)^2\Big]}{q_j!}\|f_k \otimes_{q-r} f_k\|.
\end{aligned}$$

We notice that, for every $j = 1, ..., d$, $\sup_{k\ge1}\mathbf{E}\Big[\Big(Y_k^j\Big)^2\Big] < \infty$ (see e.g. Remark 6.16 in Janson (1997)). Hence, we obtain :

$$\begin{aligned}
&\mathbf{E}\Big[\Big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}Y_k^j\Big\rangle^2\Big] \\
&\leqslant \frac{\mathbf{E}\Big[\Big(Y_k^j\Big)^2\Big]}{q^2 q_j!}\sum_{r=1}^{q_j}((r-1)!)^2\binom{q_j-1}{r-1}^2\binom{q-1}{r-1}^2(q+q_j-2r)!\|f_k \otimes_{q-r} f_k\|.
\end{aligned}$$

Then, if for every j, $q > q_j$, then by point 2. in Theorem 4.1 and the second condition of (4.1), the whole expression converges to zero as $k \to \infty$, and so we get (4.2).

Suppose now that $q = q_j$. We isolate the term $r = q_j = q$ in 4.5, giving :

$$\begin{aligned}
&\mathbf{E}\Big[\Big\langle \mathrm{D}(-L)^{-1}X_k, \mathrm{D}Y_k^j\Big\rangle^2\Big] \\
&= \frac{1}{q^2}\sum_{r=1}^{q-1}((r-1)!)^2\binom{q-1}{r-1}^4(2(q-r))!\Big\|f_k \,\widetilde{\otimes}_r\, g_k^j\Big\|^2 + ((q-1)!)^2\Big\|f_k \,\widetilde{\otimes}_q\, g_k^j\Big\|^2 \\
&= \frac{1}{q^2}\mathbf{E}\Big[X_k Y_k^j\Big]^2 + \frac{1}{q^2}\sum_{r=1}^{q-1}((r-1)!)^2\binom{q-1}{r-1}^4(2(q-r))!\Big\|f_k \,\widetilde{\otimes}_r\, g_k^j\Big\|^2.
\end{aligned}$$

By both conditions in (4.1), and by point 2. of Theorem 4.1, we also conclude in this case that this expectation goes to zero, and so we have (4.2). □

Remark 4.3. The assumption (4.1) is necessary to get the the result in Theorem 4.2 and it cannot be avoided. To argue this, consider the following trivial example. Let $(W(h), h \in H)$ be an isonormal process and let $h_1, h_2 \in H$ with $\|h_1\|_H = \|h_2\|_H = 1$ and $\langle h_1, h_2\rangle_H = \rho \in (0,1)$. Define

$$X = W(h_1)^2 - 1 = I_2(h_1^{\otimes 2}) \text{ and } Y = W(h_2).$$

Then $X \sim F(1), Y \sim \mathcal{N}(0,1)$ and all the assumptions in the statement of Theorem 4.2, except (4.1), are satisfied. Notice that $h_1^{\otimes 2} \otimes_1 h_2 = \rho h_1$ and $\|h_1^{\otimes 2} \otimes_1 h_2\|_H = \rho\|h_1\|_H = \rho \neq 0$ so (4.1) does not hold true. On the other hand, the components of the vector (X, Y) are not independent since $h_1^{\otimes 2} \otimes_1 h_2$ does not vanish almost everywhere on H, see Üstünel and Zakai (1989).

5. Examples

We illustrate the results stated in Section 4 by two examples. In these examples we consider a two -dimensional sequence of random variables which has on its first component the sequence U_n defined below by (5.1) and which converges in law to $F(1)$. On the second component, we first consider a fixed random variables in the second Wiener chaos and then another sequence, correlated with U_n, which also converges to the centered Gamma law $F(1)$. We obtain the joint convergence of the two-dimensional sequence to a vector with independent components and we derive the associate rate of convergence under the d_{W}-distance.

5.1. *Example 1.* Let $(h_i, i \geq 1)$ be orthonormal elements of the Hilbert space H. For $n \geq 2$, set

$$U_n = I_2\Big(\frac{2}{n-1} \sum_{1 \leq i < j \leq n} h_i \widetilde{\otimes} hj\Big) =: I_2(f_n), \tag{5.1}$$

where I_2 is the multiple integral with respect to an isonormal process $(W(h), h \in H)$. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in H. Then the sequence $(U_n, n \geq 1)$ converges in distribution to the centered Gamma law $F(1)$ (see e.g. Arras et al. (2019)). We show that for any $\varepsilon > 0$, and for n large enough,

$$\mathbf{E}\Big[\Big(2(U_n+1) - \langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}U_n\rangle\Big)^2\Big] \leqslant \frac{(12+\varepsilon)}{n}. \tag{5.2}$$

We notice that

$$\mathrm{D}U_n = \frac{2}{n-1} \sum_{1 \leq i < j \leq n} \Big(I_1(h_i)h_j + I_1(h_j)h_i\Big),$$

and it is not difficult to see that for $n \geq 2$,

$$\langle \mathrm{D}U_n, \mathrm{D}(-L)^{-1}U_n\rangle = \frac{2(n-2)}{n-1}U_n + \frac{2}{n-1}\sum_{i=1}^n I_1(h_i)^2.$$

This gives

$$2(U_n+1) - \langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}U_n\rangle = \frac{2}{n-1}U_n + 2 - \frac{2}{n-1}\sum_{i=1}^n I_1(h_i)^2.$$

Therefore

$$\begin{aligned}\mathbf{E}\Big[\Big(2(U_n+1) - \langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}U_n\rangle\Big)^2\Big] &\leq \frac{4}{(n-1)^2}\mathbf{E}[U_n^2] + 4\mathbf{E}\Big[\Big(1 - \frac{1}{n-1}\sum_{i=1}^n I_1(h_i)^2\Big)^2\Big] \\ &\leq C\frac{1}{n^2} + 12\frac{1}{n}.\end{aligned}$$

So, for n large enough, we get (5.2). In particular (see Nourdin and Peccati (2009b) or Nourdin and Peccati (2012)),

$$d_W(U_n, F(1)) \leqslant \sqrt{\frac{2}{\pi}}\frac{(2\sqrt{3}+\varepsilon)}{\sqrt{n}}. \tag{5.3}$$

We regard the asymptotic behavior of the two-dimensional sequence (U_n, G) where

$$G := 2H_2(W(h_1)) = I_2\Big(h_1^{\otimes 2}\Big).$$

Let us check (4.1). Concerning the first part of (4.1), we have for every $n \geq 1$,

$$\mathbf{E}\Big[U_n G\Big] = \frac{4}{n-1} \sum_{1\leq i<j\leq n} \langle h_i \widetilde{\otimes} h_j, h_1^{\otimes 2}\rangle_{H^{\otimes 2}}$$
$$= \frac{2}{n-1} \sum_{1\leq i<j\leq n} \langle h_i \otimes h_j + h_j \otimes h_i, h_1^{\otimes 2}\rangle_{H^{\otimes 2}}$$
$$= \frac{4}{n-1} \sum_{1\leq i<j\leq n} \mathbf{1}_{\{i=j=1\}} = 0.$$

Also,

$$f_n \otimes_1 h_1^{\otimes 2} = \frac{2}{n-1} \sum_{1\leq i<j\leq n} \Big((h_i \widetilde{\otimes} h_j) \otimes_1 h_1^{\otimes 2}\Big)$$
$$= \frac{1}{n-1} \sum_{1\leq i<j\leq n} \Big((h_i \otimes h_j + h_j \otimes h_i) \otimes_1 h_1^{\otimes 2}\Big)$$
$$= \frac{1}{n-1} \sum_{1\leq i<j\leq n} \Big((h_j \otimes h_1)\mathbf{1}_{\{i=1\}} + (h_i \otimes h_1)\mathbf{1}_{\{j=1\}}\Big) = \frac{1}{n-1} \sum_{j=2}^{n} h_j \otimes h_1 \tag{5.4}$$

and

$$\big\|f_n \otimes_1 h_1^{\otimes 2}\big\|^2_{H^{\otimes 2}} = \frac{2}{(n-1)^2} \sum_{2\leqslant j,k\leqslant n} \langle h_j \otimes h_1, h_k \otimes h_1\rangle_{H^{\otimes 2}}$$
$$= \frac{2}{(n-1)^2} \sum_{2\leqslant j,k\leqslant n} \langle h_j, h_k\rangle_H = \frac{2(n-1)}{(n-1)^2} \xrightarrow[n\to\infty]{} 0. \tag{5.5}$$

It then follows from Theorem 4.2 that for n large enough,

$$d_{\mathrm{W}}\Big(\mathbb{P}_{(U_n,G)}, F(1) \otimes \mathbb{P}_G\Big)$$
$$\leqslant C_2 \mathbf{E}\Big[\Big(2(U_n+1) - \big\langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}U_n\big\rangle\Big)^2\Big]^{\frac{1}{2}} + C_1 \mathbf{E}\Big[\big\langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}G\big\rangle^2\Big]^{\frac{1}{2}}, \tag{5.6}$$

with C_1 and C_2 defined in (2.13). For any two multiple integrals $X = I_2(f)$, $Y = I_2(g)$ with $f, g \in H^{\odot 2}$,

$$\mathbf{E}\Big[\big\langle \mathrm{D}(-L)^{-1}X, \mathrm{D}Y\big\rangle^2\Big] = \mathbf{E}[XY]^2 + 8\|f \otimes_1 g\|^2_{H^{\otimes 2}}.$$

Thus, from (5.4) and (5.5),

$$\mathbf{E}\Big[\big\langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}G\big\rangle^2\Big] = \frac{16}{n-1}. \tag{5.7}$$

By plugging the estimates (5.2) and (5.7) into (5.6), we obtain for n sufficiently large, and for every $\varepsilon > 0$,

$$d_{\mathrm{W}}\Big(\mathbb{P}_{(U_n,G)}, F(1) \otimes \mathbb{P}_G\Big) \leqslant (\sqrt{12}C_2 + 4C_1 + \varepsilon)\frac{1}{\sqrt{n}}.$$

Notice that $C_2 = \max 1, 2 = 2$ and $C_1 = \max\Big\{2, \frac{1}{\int_{x_1}^{\infty} u p_\nu(u)du}\Big\}$, where x_1 is the median of the law $F(1)$ and p_1 is its density. Since x_1 is approximately -0.7, we get that C_1 is approximately $2,65$.

5.2. *Example 2.* Let $H = L^2(\mathbb{R}_+)$ and for $i \geq 1$ set

$$h_i := \mathbf{1}_{[2i,2i+1]}.$$

Then $(h_i, i \geq 1)$ are orthonormal elements in H. Consider the sequence $(U_n, n \geq 1)$ given by (5.1). Now define for $i \geq 1$,

$$g_i := \mathbf{1}_{\left[2i-1+\frac{1}{i},2i+\frac{1}{i^a}\right]}$$

with $a > 0$. The family $(g_i, i \geq 1)$ is also orthogonal in H. Notice that for every $i, k \geqslant 1$,

$$\langle h_i, g_k\rangle_H = \frac{1}{i^a}\mathbf{1}_{\{i=k\}}.$$

We consider the sequence $(V_n, n \geq 1)$ given by

$$V_n = I_2\Big(\frac{2}{n-1}\sum_{1\leq i<j\leq n} g_i\widetilde{\otimes}g_j\Big) =: I_2(\varphi_n). \tag{5.8}$$

We know that both $(U_n, n \geq 1)$ (defined at previous example) and $(V_n, n \geq 1)$ converge to the centered Gamma law $F(1)$. We consider the two-dimensional sequence $((U_n, V_n), n \geq 1)$ and we prove that it converges in law to the vector (F_1, F_2), where $F_1, F_2 \sim F(1)$ and F_1, F_2 are independent. We will also deduce the rate of convergence under the d_{W}-distance associated to this limit theorem.

We compute the quantities in (4.1).

$$\begin{aligned}
\mathbf{E}\Big[U_nV_n\Big] &= \frac{4}{(n-1)^2}\sum_{1\leq i<j\leq n}\sum_{1\leq k<l\leq n}\langle h_i\widetilde{\otimes}h_j, g_k\widetilde{\otimes}g_l\rangle\\
&= \frac{1}{(n-1)^2}\sum_{1\leq i<j\leq n}\sum_{1\leq k<l\leq n}\langle h_i\otimes h_j + h_j\otimes h_i, g_k\otimes g_l + g_l\otimes g_k\rangle\\
&= \frac{1}{(n-1)^2}\sum_{1\leq i<j\leq n}\sum_{1\leq k<l\leq n}\frac{2}{i^aj^a}\Big(\mathbf{1}_{\{i=k,j=l\}} + \mathbf{1}_{\{i=l,j=k\}}\Big)\\
&= \frac{4}{(n-1)^2}\sum_{1\leq i<j\leq n}\frac{1}{(ij)^a} \leq \frac{4}{(n-1)^2}\Big(\sum_{i=1}^{n}\frac{1}{i^a}\Big)^2 \leq \frac{C}{n^{2a}} \xrightarrow[n\to\infty]{} 0.
\end{aligned} \tag{5.9}$$

Next, let us calculate $\|f_n\otimes_1\varphi_n\|^2_{H^{\otimes 2}}$. We compute first the contraction :

$$\begin{aligned}
&f_n\otimes_1\varphi_n \qquad\qquad (5.10)\\
&= \frac{4}{(n-1)^2}\sum_{1\leq i<j\leq n}\sum_{1\leq k<l\leq n}\Big((h_i\widetilde{\otimes}h_j)\otimes_1(g_k\widetilde{\otimes}g_l)\Big)\\
&= \frac{1}{(n-1)^2}\sum_{1\leq i<j\leq n}\sum_{1\leq k<l\leq n}\int_{\mathbb{R}_+}\Big[h_i(x)h_j(u) + h_j(x)h_i(u)\Big]\Big[g_k(y)g_l(u) + g_l(y)g_k(u)\Big]\mathrm{d}u\\
&= \frac{1}{(n-1)^2}\sum_{1\leq i<j\leq n}\sum_{1\leq k<l\leq n}\Big(h_i\otimes g_k\mathbf{1}_{\{j=l\}} + h_i\otimes g_l\mathbf{1}_{\{j=k\}} + h_j\otimes g_k\mathbf{1}_{\{i=l\}} + h_j\otimes g_l\mathbf{1}_{\{i=k\}}\Big)\\
&= \frac{1}{(n-1)^2}\Big[\sum_{\substack{1\leqslant i<j\leq n\\ 1\leqslant k<j\leq n}} h_i\otimes g_k + \sum_{1\leq i<j<l\leq n} h_i\otimes g_l + \sum_{1\leq k<i<j\leq n} h_j\otimes g_k + \sum_{\substack{1\leq i<j\leq n\\ 1\leq i<l\leq n}} h_j\otimes g_l\Big] \qquad (5.11)
\end{aligned}$$

and for n large enough,

$$\Big|f_n\otimes_1\varphi_n\Big| \leq \frac{C}{n}\Big|\sum_{1\leq i,k\leq n} h_i\otimes g_k\Big|$$

and then

$$\|f_n \otimes_1 \varphi_n\|^2_{H^{\otimes 2}} \leqslant \frac{C}{n^2} \sum_{1\le i,k,i',k'\le n} \langle h_i \otimes g_k, h_{i'} \otimes g_{k'}\rangle_{H^{\otimes 2}}$$
$$= \frac{C}{n^2} \sum_{1\le i,k\le n} \frac{1}{i^a k^a} \le \frac{C}{n^{2a}} \xrightarrow[n\to\infty]{} 0.$$

Consequently, from (5.9) and (5.11) it follows that (4.1) is verified and $\Big((U_n, V_n), n \ge 1\Big)$ converges in law, as $n \to \infty$, to $F(1) \otimes F(1)$ and

$$\mathbf{E}\Big[\big\langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}V_n\big\rangle^2\Big] \leqslant \frac{C}{n^{2a}}. \tag{5.12}$$

By Theorem 4.2,

$$d_{\mathrm{W}}\Big(\mathbb{P}_{(U_n,V_n)}, F(1)\otimes F(1)\Big) \leqslant C\mathbf{E}\Big[\Big(2(U_n+1) - \big\langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}U_n\big\rangle_H\Big)^2\Big]^{\frac{1}{2}}$$
$$+ C\mathbf{E}\Big[\langle \mathrm{D}(-L)^{-1}U_n, \mathrm{D}U_n\rangle^2_H\Big]^{\frac{1}{2}} + d_2(\mathbb{P}_{V_n}, F(1)).$$

The estimates (5.2), (5.3) and (5.12) give, for n large,

$$d_{\mathrm{W}}\Big(\mathbb{P}_{(U_n,V_n)}, F(1)\otimes F(1)\Big) \le C\Big(n^{-\frac{1}{2}} + n^{-a}\Big).$$

6. Appendix: Wiener-Chaos and Malliavin derivatives

Here we describe the elements from stochastic analysis that we will need in the paper. Consider H a real separable Hilbert space and $(W(h), h \in H)$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}\Big[W(\varphi)W(\psi)\Big] = \langle \varphi, \psi\rangle_H$. Denote by I_n the multiple stochastic integral with respect to W (see Nualart (2006)). This mapping I_n is actually an isometry between the Hilbert space $H^{\odot n}$(symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{H^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(W(h))$ where $h \in H, \|h\|_H = 1$ and H_n is the Hermite polynomial of degree $n \in \mathbb{N}$

$$\forall x \in \mathbb{R}, \qquad H_n(x) = \frac{(-1)^n}{n!} \exp\Big(\frac{x^2}{2}\Big) \frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(\exp\Big(-\frac{x^2}{2}\Big)\Big).$$

The isometry of multiple integrals can be written as follows: for m, n positive integers,

$$\mathbf{E}\Big[I_n(f)I_m(g)\Big] = n!\langle \tilde{f}, \tilde{g}\rangle_{H^{\otimes n}} \quad \text{if } m = n,$$
$$\mathbf{E}\Big[I_n(f)I_m(g)\Big] = 0 \quad \text{if } m \neq n. \tag{6.1}$$

It also holds that

$$I_n(f) = I_n\big(\tilde{f}\big)$$

where $\tilde{f}$ denotes the symmetrization of f defined by the formula

$$\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{6.2}$$

where $f_n \in H^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}\Big[F\Big]$.
Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n\geq 0} n I_n(f_n)$$

if F is given by (6.2) and it is such that $\sum_{n=1}^{\infty} n^2 n! \|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty$. The pseudo-inverse of L, denoted $(-L)^{-1}$, is given by,

$$(-L)^{-1}(F) = \sum_{n\geq 1} \frac{1}{n} I_n(f_n)$$

for F as in (6.2) with $\mathbf{E}[F] = 0$.
For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I-L)^{\frac{\alpha}{2}} F\|_{L^p(\Omega)}$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(W(h_1), \ldots, W(h_n))$ (g is a smooth function with compact support and $h_i \in H$)

$$\mathrm{D}F = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(W(h_1), \ldots, W(h_n)) h_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}\Big(H\Big)$.

We will intensively use the product formula for multiple integrals. It is well-known that for $f \in H^{\odot n}$ and $g \in H^{\odot m}$

$$I_n(f) I_m(g) = \sum_{r=0}^{n\wedge m} r! \binom{n}{r} \binom{m}{r} I_{m+n-2r}(f \otimes_r g) \tag{6.3}$$

where $f \otimes_r g$ means the r-contraction of f and g (see e.g. Section 1.1.2 in Nualart (2006)).
We also need to introduce the Skorohod integral integral (or the divergence operator), denoted by δ, which is the adjoint operator of D. Its domain is

$$\mathrm{Dom}(\delta) = \Big\{ u \in L^2\Big(\Omega; H\Big), \mathbf{E}\Big[\Big|\langle \mathrm{D}F, u\rangle_H\Big|\Big] \leq C\|F\|_{L^2(\Omega)} \Big\}$$

and we have the duality relationship

$$\forall F \in \mathbb{D}^{1,2}, \forall u \in \mathrm{Dom}(\delta), \quad \mathbf{E}\Big[F\delta(u)\Big] = \mathbf{E}\Big[\langle \mathrm{D}F, u\rangle_H\Big]. \tag{6.4}$$

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