ALEA, Lat. Am. J. Probab. Math. Stat. 21, 1823–1833 (2024)
DOI: 10.30757/ALEA.v21-68

Survival of one dimensional renewal contact process

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Abstract. The renewal contact process, introduced in 2019 by Fontes, Marchetti, Mountford, and Vares, extends the Harris contact process in $\mathbb{Z}^d$ by allowing the possible cure times to be determined according to independent renewal processes (with some interarrival distribution μ) and keeping the transmission times determined according to independent exponential times with a fixed rate λ. We investigate sufficient conditions on μ to have a process with a finite critical value λ_c for any spatial dimension $d \geq 1$. In particular, we show that λ_c is finite when μ is continuous with bounded support or when μ is absolutely continuous and has a decreasing hazard rate.

1. Introduction

The classical contact process, introduced by Harris (1974), is a model for the spread of infectious diseases and has been intensively studied (see for instance Durrett (1995) and Liggett, 1985). Several variants and extensions appeared in the literature, including the Renewal Contact Process, which is the model studied in this paper and was introduced in Fontes et al. (2019, 2020), motivated by questions regarding long range percolation.

In both models, the sites of $\mathbb{Z}^d$ represent individuals that can be healthy or infected, and the state of the population at time t is represented by a configuration $\xi_t \in \{0,1\}^{\mathbb{Z}^d}$, where $\xi_t(x) = 0$ means that the individual x is healthy at time t and $\xi_t(x) = 1$ means that the individual x is sick at time t. The classical model considers a Markovian evolution: Infected individuals become healthy at rate 1 independently of everything else, and healthy individuals become sick at a rate equal to a given parameter λ times the number of infected neighbors. The Renewal Contact Process extends this model by allowing the possible cure times to be determined by i.i.d. renewal processes with an interarrival distribution μ on $[0, \infty)$ such that $\mu\{0\} < 1$. This model will be denoted by RCP(μ). This additional flexibility comes with the cost of losing the Markov property, which brings new difficulties.

Received by the editors June 11th, 2023; accepted September 24th, 2024.

1991 *Mathematics Subject Classification.* 60K35, 60K05, 82B43.

Key words and phrases. Contact process, percolation, renewal process.

Rafael Santos has been supported by CNPq, Brazil grant 151736/2022-7. Maria Eulália Vares has been supported by CNPq, Brazil grant 310734/2021-5, and FAPERJ, Brazil grant CNE E-26/200.442/2023.

In Harris (1978), the classical contact process was alternatively defined using a percolation structure, known as *Harris graphical construction*, based on a system of infinitely many independent Poisson point processes. Besides being a very useful tool, Harris construction immediately suggests the consideration of more general percolation structures, with Poisson point processes being replaced by more general point processes. Natural variants include the RCP(μ) and the models in Hilário et al. (2022), where, besides the contact process on dynamic edges introduced in Linker and Remenik (2020), the authors consider a version of RCP with renewals on the transmission marks. (See also Fontes et al. (2023), where the results of Fontes et al. (2020) have been strongly refined, and Fontes (2023) for a short survey.)

For sake of completeness, we now recall the basic definition, based on two independent sequences of point processes:

• $(\mathcal{R}_x)_{x\in\mathbb{Z}^d}$ is a family of independent renewal processes with interarrival distribution μ. $\mathcal{R}_x$ describes the cure marks at x. When the specific vertex is not important we will denote simply by $\mathcal{R}$ the renewal process constructed in the same way as $\mathcal{R}_x$.

• $(\mathcal{N}_{x,y})_{(x,y)\in\mathcal{E}}$ is a family of independent Poisson processes with rate λ. Here $\mathcal{E}$ denotes the set of ordered pairs of nearest neighbors in $\mathbb{Z}^d$. $\mathcal{N}_{x,y}$ describes the times when the vertex x tries to infect y.

The RCP(μ) is described in terms of (time oriented) paths, also called *infection paths*. A path from (x, s) to (y, t) for $x, y \in \mathbb{Z}^d$ and $s < t$ is a càdlàg function $\gamma : [s, t] \to \mathbb{Z}^d$ so that: (i) $\gamma(s) = x$; (ii) $\gamma(t) = y$; (iii) $\forall u \in [s, t], u \notin \mathcal{R}_{\gamma(u)}$; and (iv) $\forall u \in [s, t]$, if $\gamma(u-) \neq \gamma(u)$, then $u \in N_{\gamma(u-),\gamma(u)}$. In particular, if $A \subset \mathbb{Z}^d$, our RCP(μ) with infection parameter λ and initially infected set $A \subset \mathbb{Z}^d$ is defined by

$$\xi_t^A(y) \;=\; 1 \iff \exists \text{ a path from } (x, 0) \text{ to } (y, t), \text{for some } x \in A.$$

Among the main questions of interest regarding RCP(μ) we want to better understand the survival and extinction of the infection depending on the parameter λ and the distribution μ. In this direction, given μ, the critical parameter for RCP(μ) is defined as

$$\lambda_c(\mu) := \inf\{\lambda : P(\tau^{\{0\}} = \infty) > 0\},$$

where $\tau^{\{0\}} := \inf\{t : \xi_t^{\{0\}} \equiv 0\}$ (with the usual convention that $\inf \emptyset = \infty$). The articles Fontes et al. (2019) and Fontes et al. (2021) (this last one for finite volumes) obtained sufficient conditions on μ to assure that $\lambda_c(\mu) = 0$ (when μ has a heavy tail), while Fontes et al. (2020) obtained sufficient conditions on μ to assure that $\lambda_c(\mu) > 0$, and which were significantly relaxed in Fontes et al. (2023). In particular, the authors in Fontes et al. (2023) proved that $\lambda_c(\mu) > 0$ whenever $\int x^\alpha \mu(dx) < \infty$ for some $\alpha > 1$. Even combining the results of Fontes et al. (2019) and Fontes et al. (2023), there remains an important gap where we do not know whether the critical parameter is zero or not.

As we can see from the previous paragraph, the study of sufficient conditions to assure that $\lambda_c(\mu)$ is zero or positive is already intense. Nevertheless, it is also natural to ask whether $\lambda_c(\mu)$ is finite or infinite and this question was very little explored in the literature until now. Naturally if μ is degenerate (there exists $c > 0$ such that $\mu\{c\} = 1$), the infection always dies out (at time c) and consequently $\lambda_c(\mu) = \infty$. It is natural to conjecture that except for this degenerate case (simultaneous extinction), the critical parameter should be finite, but we still do not have a proof for that in the literature. Since $\lambda_c(\mu)$ is non-increasing in d, it is enough to consider the one-dimensional case. When $d \geq 2$, this problem becomes much simpler since we can construct an infinite infection path using each vertex only once (i.e. through a coupling with supercritical oriented percolation), avoiding dependencies within each renewal process. This idea was explored in the proof of Theorem 1.3 (ii) in Hilário et al. (2022), which states that the critical parameter is finite for a more general version of the renewal contact process. The proof for RCP(μ) is similar.

Considering $d = 1$, and obviously excluding the cases where we already know that $\lambda_c(\mu) = 0$, the only general result that we are aware of regarding $\lambda_c(\mu) < \infty$ is the one described in Remark 2.2 of

Fontes et al. (2023). It argues that if μ has a density and a bounded decreasing hazard rate, then $\lambda_c(\mu)$ is finite. The proof uses a coupling of the RCP(μ) with the classical contact process using the construction described in Fontes et al. (2020). In particular, if μ and $\tilde{\mu}$ are absolutely continuous distributions on $[0, \infty)$ with hazard rates h_μ and $h_{\tilde{\mu}}$ such that $h_\mu(t) \leq h_{\tilde{\mu}}(t)$ for any $t \geq 0$ and moreover $h_\mu(t)$ is decreasing in t, then $\lambda_c(\mu) \leq \lambda_c(\tilde{\mu})$. Our Theorem 1.2 below improves this by allowing the hazard rate to be unbounded near the origin. This covers for example the case where the times between cure marks have a Weibull distribution with shape parameter smaller than one.

Theorem 1.1 is the other main result of this paper. It states that $\lambda_c(\mu)$ is finite whenever μ is continuous (i.e. $\mu\{x\} = 0$ for all x) and has a bounded support.

Theorem 1.1. *Let μ be a continuous distribution such that $\mu[0, b] = 1$ for some $b < \infty$. Then $\lambda_c(\mu) < \infty$.*

Theorem 1.2. *Let μ be an absolutely continuous distribution on $[0, \infty)$ with density f and distribution function F such that its hazard rate $\frac{f(t)}{1-F(t)}$ is decreasing in t. Then $\lambda_c(\mu) < \infty$.*

In Section 2 we have the proof of Theorem 1.1 and Section 3 is dedicated to the proof of Theorem 1.2. Both proofs involve coupling with supercritical oriented percolation. Differently from Hilário et al. (2022), we need to find a way to do it using the same vertices more than once to construct an infinite path, and therefore it is necessary to control the dependencies that appear. Each proof addresses this problem in a different manner, and we still could not find a way to deal with it for any non-degenerate μ, keeping us from proving the conjecture mentioned before.

2. μ continuous and with bounded support

This section is devoted to proving Theorem 1.1. Before this, we need to state and prove some useful lemmas and a few extra definitions are needed.

To begin, we state a lemma from Hilário et al. (2022) that gives a uniform estimate for renewal processes, and which will be important to many of the proofs presented in this section.

Lemma 2.1. [*Hilário et al. (2022), Theorem* 4.1(*ii*)] *Let μ be a continuous distribution on $[0, \infty)$ and let $\mathcal{R}$ be a renewal process with interarrival distribution μ started from some $\mathfrak{T} \leq 0$. Then, given $p_0 > 0$ there exists $w_0 = w_0(p_0) > 0$ such that uniformly on $\mathfrak{T}$ we have:*

$$\sup_{t \geq 0} P(\mathcal{R} \cap [t, t + w_0] \neq \emptyset) \leq p_0.$$

The next lemma gives us control over the number of renewal marks that appear on the timeline of a vertex during a finite period of time. For any finite set A, we denote its cardinality by $|A|$.

Lemma 2.2. *Assume that $\mu\{0\} < 1$ and let $\mathcal{R}$ be a renewal process with interarrival distribution μ started from some $\mathfrak{T} \leq 0$. Then, for each $p_0 > 0$ and $s > 0$ there exists $K_0 = K_0(p_0, s)$ such that uniformly on $\mathfrak{T}$ we have:*

$$\sup_{t \geq 0} P(\ |\ \mathcal{R} \cap [t, t + s]\ |\ > K_0) \leq p_0.$$

Lemmas 2.1 and 2.2 together allow us to state a lemma that gives control over how close the cure marks of two adjacent vertices inside a finite time interval are from each other.

Lemma 2.3. *Let μ be a continuous distribution on $[0, \infty)$ and let $\mathcal{R}$, $\tilde{\mathcal{R}}$ be two independent renewal processes with interarrival distribution μ started from some $\mathfrak{T} \leq 0$. Then given $p_1 > 0$ and $s > 0$ there exists $w_1 = w_1(p_1, s) > 0$ such that uniformly on $\mathfrak{T}$ we have:*

$$\sup_{t \geq 0} P\Big(\mathrm{d}\big(\mathcal{R} \cap [t, t + s], \tilde{\mathcal{R}} \cap [t, t + s]\big) \leq w_1\Big) \leq p_1,$$

where $\mathrm{d}(A,B) = \inf\{|x-y| \colon x \in A, y \in B\}$ *stands for the usual minimal distance between two subsets* A, B *of the real line, understood as infinity if one of them is empty.*

Now the next two lemmas state that if we choose λ large enough, the infection can survive inside specific space-time boxes with high probability. For both lemmas, consider the RCP(μ) where μ is a continuous distribution such that $\mu[0,b] = 1$.

For convenience we set the following definition.

Definition 2.4. (a) Given bounded subsets of $\mathbb{Z} \times \mathbb{R}$, C, D and H, we say there is a crossing from C to D in H if there exists a path $\gamma : [s,t] \to H$ such that $(\gamma(s), s) \in C$ and $(\gamma(t), t) \in D$.
(b) Given $b > 0, t \geq 0$ and $x \in \mathbb{Z}$, consider the space-time boxes

$$B_h(x,t) = [x, x+3] \times [t, t+b], \quad B_v(x,t) = [x, x+1] \times [t, t+3b],$$

and set

$$C_h(x,t) = \{\text{there is a crossing from } (x,t) \text{ to } \{x+3\} \times [t, t+b] \text{ in } B_h(x,t)\},$$
$$C_v(x,t) = \{\text{there is a crossing from } (y,t) \text{ to } \{x, x+1\} \times \{t+3b\} \text{ in } B_v(x,t), y = x, x+1\}.$$

Lemma 2.5. *Fix* $b > 0$ *and* $\epsilon \in (0,1)$. *There exists* $\lambda_h = \lambda_h(\epsilon) < \infty$ *such that:*

$$\inf_{t \geq 0, x \in \mathbb{Z}} P(C_h(x,t)) \geq 1 - \epsilon, \ \text{if } \lambda \geq \lambda_h.$$

Lemma 2.6. *Fix* $b > 0$ *and* $\epsilon \in (0,1)$. *There exists* $\lambda_v = \lambda_v(\epsilon) < \infty$ *such that:*

$$\inf_{t \geq 0, x \in \mathbb{Z}} P(C_v(x,t)) \geq 1 - \epsilon, \ \text{if } \lambda \geq \lambda_v.$$

We start by proving the above lemmas.

Proof: *(Lemma 2.2)* Let $(X_j)_{j \geq 1}$ be a sequence of i.i.d random variables with distribution μ. Then, we have

$$\sup_{t \geq 0} P(\mid \mathcal{R} \cap [t, t+s] \mid > K_0) \leq P\Big(\sum_{j=1}^{K_0} X_j < s\Big) = P\Big(\frac{1}{K_0}\sum_{j=1}^{K_0} X_j < \frac{s}{K_0}\Big). \tag{2.1}$$

Since the random variables $(X_j)_{j \geq 1}$ are non-negative and not identically null, the strong law of large numbers tells us that $\frac{1}{K}\sum_{j=1}^{K} X_j$ converges almost surely to $E[X_1] \in (0, \infty]$, as $K \to \infty$. It then follows at once from (2.1) that for each $p_0 > 0$ and $s > 0$, we can choose K_0 as in the statement.

□

Proof: *(Lemma 2.3)* The uniformity on $t \geq 0$ (and consequently also for $\mathfrak{T} \leq 0$) follows at once from its validity in Lemmas 2.1 and 2.2. Given $p_1 > 0$ we use Lemma 2.2 for $\tilde{\mathcal{R}}$ with $p_0 = p_1/2$ and let $\tilde{K}_0 = K_0(p_1/2, s)$. We then use Lemma 2.1 with $p_0 = p_1/2\tilde{K}_0$, letting $w_1 = \frac{1}{2}w_0(\frac{p_1}{2\tilde{K}_0})$. It then follows that

$$P\Big(\mathrm{d}\big(\mathcal{R} \cap [t, t+s], \tilde{\mathcal{R}} \cap [t, t+s]\big) \leq w_1\Big) \leq \frac{p_1}{2\tilde{K}_0}\tilde{K}_0 + \frac{p_1}{2} = p_1.$$

□

Proof: *(Lemma 2.5)* Fix $w_0 \in (0, b)$ satisfying Lemma 2.1 for $p_0 = 1 - (1-\epsilon)^{\frac{1}{6}}$ and define the events $F_i = F_i(x,t)$ and $G_i = G_i(x,t)$ for $i = 0, 1, 2$ as the following:

$$F_i = \Big\{\mathcal{R}_{x+i} \cap [t, t+w_0] = \emptyset\Big\},$$
$$G_i = \Big\{\mathcal{N}_{x+i, x+i+1} \cap [t + \tfrac{iw_0}{3}, t + \tfrac{(i+1)w_0}{3}] \neq \emptyset\Big\}.$$

Note that the occurrence of $\bigcap_{i=0}^{2}(F_i \cap G_i)$ implies the occurrence of $C_h(x,t)$, as illustrated in Figure 2.1. So, our next step will focus on finding some useful lower bounds on the probabilities of the events F_i and G_i (the latter one depending on λ). Applying Lemma 2.1 we have that, for $i = 0, 1, 2$,

$$\inf_{t\geq 0, x\in\mathbb{Z}} P(F_i) = \inf_{t\geq 0} P(\mathcal{R}\cap[t, t+w_0] = \emptyset) \geq (1-\epsilon)^{\frac{1}{6}}. \tag{2.2}$$

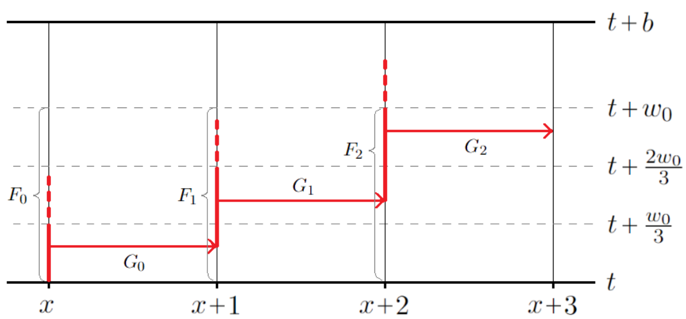

FIGURE 2.1. Illustration of the events F_i and G_i occurring together, for $i = 0, 1, 2$, implying the occurrence of $C_h(x,t)$. The red arrows indicate transmission marks (described by the events G_0, G_1 and G_2) and the thick red lines indicate a path without cure marks (consequence of the events F_0, F_1 and F_2).

Now let Y_λ denote a random variable with exponential distribution with rate λ. Since $w_0 > 0$, we can fix λ_h such that $P(Y_{\lambda_h} \leq \frac{w_0}{3}) = (1-\epsilon)^{\frac{1}{6}}$, i.e., $\lambda_h = \frac{-3}{w_0}\log(1-(1-\epsilon)^{\frac{1}{6}})$. If $\lambda \geq \lambda_h$, by the Markov property of the Poisson process, we have that for $i = 0, 1, 2$,

$$\inf_{t\geq 0, x\in\mathbb{Z}} P(G_i) = P\Big(Y_\lambda \leq \frac{w_0}{3}\Big) \geq (1-\epsilon)^{\frac{1}{6}}. \tag{2.3}$$

But the events $F_0, G_0, F_1, G_1, F_2, G_2$ are all independent, and therefore the conclusion follows from (2.2) and (2.3). □

Proof: *(Lemma 2.6)* Since $B_v(x,t)$ contains only two sites, the path has to keep jumping between these two sites until time $t+3b$, avoiding cure marks, as illustrated in Figure 2.2.

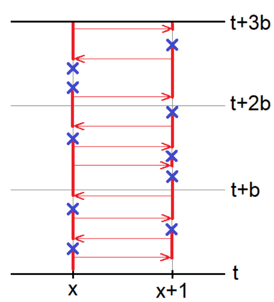

FIGURE 2.2. Illustration of a vertical crossing inside the box $B_v(x,t)$. The red arrows indicate transmission marks, the blue X's indicate cure marks and the thick red lines indicate a path without cure marks.

Fix $p_0 = p_1 = \frac{1-\sqrt{1-\epsilon}}{5}$ and $s = 3b$. Apply Lemmas 2.1, 2.2 and 2.3 for $\mathcal{R}_x$ and $\mathcal{R}_{x+1}$, letting $\tilde{w}_0 = w_0(p_0)$, $\tilde{K}_0 = K_0(p_0, s)$ and $\tilde{w}_1 = w_1(p_1, s)$. Now defining by H_t the intersection of the events

$$\Big\{\mathcal{R}_x \cap [t, t+\tilde{w}_0] = \emptyset,\ \mathcal{R}_{x+1}\cap[t, t+\tilde{w}_0] = \emptyset\Big\},$$

$$\Big\{ |\ \mathcal{R}_x \cap [t, t+3b]\ | \leq \tilde{K}_0,\ |\ \mathcal{R}_{x+1} \cap [t, t+3b]\ | \leq \tilde{K}_0 \Big\},$$

$$\Big\{ \mathrm{d}\Big(\mathcal{R}_x \cap [t, t+3b], \mathcal{R}_{x+1} \cap [t, t+3b] \Big) > \tilde{w}_1 \Big\},$$

it follows from these three lemmas that $\inf_{t \geq 0} P(H_t) \geq \sqrt{1-\epsilon}$. The occurrence of H_t guarantees that a path with the desired crossing property can be constructed by suitably hopping between x and $x+1$ at most $2\tilde{K}_0$ times to reach time $t+3b$ and that it will always encounter an interval of length at least $\tilde{w} = \tilde{w}_0 \wedge \tilde{w}_1$ without cure marks to execute each hop. Fix λ_v such that $P(Y_{\lambda_v} \leq \tilde{w}) = (1-\epsilon)^{\frac{1}{4\tilde{K}_0}}$, i.e., $\lambda_v = \frac{-1}{\tilde{w}} \log(1-(1-\epsilon)^{\frac{1}{4\tilde{K}_0}})$, and hence it follows by the Markov property of the Poisson process that for any $\lambda \geq \lambda_v$,

$$\inf_{t \geq 0, x \in \mathbb{Z}} P(C_v(x,t)) \geq \inf_{t \geq 0} P(H_t) \big[P(Y_{\lambda_v} \leq \tilde{w}) \big]^{2\tilde{K}_0} \geq 1 - \epsilon.$$

□

Now we are finally ready to prove Theorem 1.1.

Proof: *(Theorem 1.1)* Given $x \in \mathbb{Z}$, $t > 0$ and $b > 0$ such that $\mu[0,b] = 1$, consider the space-time boxes $K(x,t) = [x, x+3] \times [t, t+3b]$, which will be called blocks in our proof. The block $K(x,t)$ is said to be *good* if $C_h(x,t) \cap C_h(x, t+2b) \cap C_v(x,t) \cap C_v(x+2,t)$ occurs, and *bad* otherwise. That is, a block will be called good when four specific smaller boxes inside it are crossed in the sense of the corresponding events C_h and C_v. This is illustrated on the left side image of Figure 2.3, in which $K(0,0)$, $K(2,0)$ and $K(0,3b)$ are good blocks and $K(2,3b)$ is a bad block.

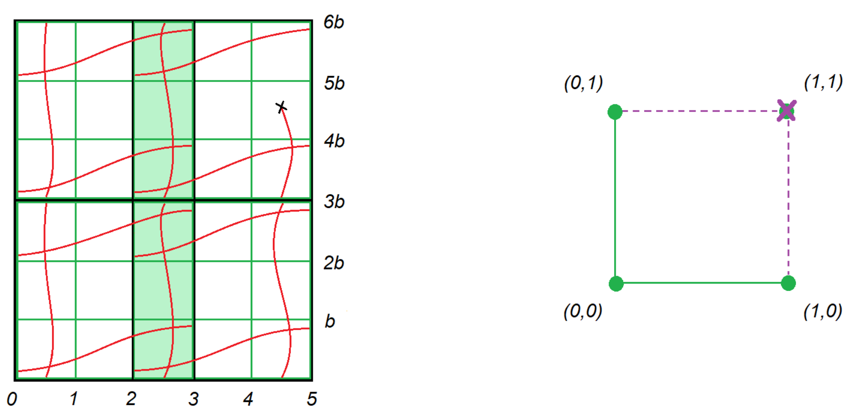

FIGURE 2.3. Example of the relation between the RCP(μ) and the percolation model on G, mapping four blocks into four vertices. On the left side image, each curved red line does not represent a path itself, but symbolizes that the associated horizontal or vertical box are being crossed, with exception of the one that reach an $\times$ symbol, and the painted green boxes in the middle are indicating the regions where two blocks overlaps each other. On the right side image we have the corresponding sites and edges of G where the green dots (lines) indicate the open sites (edges) and the purple $\times$ symbol (dashed line) indicates the closed sites (edges).

Let's consider a graph G in $\mathbb{Z}^2$ with oriented edges $\langle z, z+(1,0) \rangle$ and $\langle z, z+(0,1) \rangle$. Define a percolation system on G related to the RCP(μ) by stating that the vertex (x,y) is open if $K(2x, 3yb)$ is a good block in the RCP(μ), and closed otherwise. Moreover, each edge is open if both endvertices are open. Note that in this construction, the blocks are fitted together (and overlap in the horizontal direction) allowing us to concatenate paths of adjacent good blocks, and implying that when the oriented edge model percolates, the associated RCP(μ) will survive with positive probability. See Figure 2.3 for an example of this percolation model being constructed. Given $\epsilon > 0$, it follows from Lemmas 2.5, 2.6 and union bound that the probability of a site of G being open is at least $1 - \epsilon$ if we take $\lambda \geq \lambda_h(\frac{\epsilon}{4}) \vee \lambda_v(\frac{\epsilon}{4})$.

Now we will argue that in this oriented percolation model, the edges that do not share endvertices are independent. It is a consequence of three facts:

- The independence properties of cure (transmission) marks on (from) different vertices.
- The transmission marks are determined by Poisson processes.
- The time between two consecutive cure marks in a vertex is bounded by b, which means that any time interval of length b will have at least one cure mark in every vertex. It implies that the cure marks after time $t + b$ are independent of the cure marks before time t, for any $t \geq 0$.

It follows that we have a finite range dependent bond model. Applying Theorem 0.0(ii) of Liggett et al. (1997) and choosing $\epsilon > 0$ small enough in the beginning, it follows that the RCP survives with positive probability if $\lambda \geq \lambda_h(\frac{\epsilon}{4}) \vee \lambda_v(\frac{\epsilon}{4})$. □

3. μ with decreasing hazard rate

This section is devoted to proving Theorem 1.2. Again, we start by stating and proving some useful lemmas. Analogously to those in Section 2, they provide suitable lower bounds for the probability of crossing events. The difference is that taking advantage of the assumption on the hazard rate, this can be done uniformly on the history of the renewal processes.

The first two lemmas guarantee that for any fixed $p > 0$, we can find a fixed non-trivial interval so that, uniformly over the history of $\mathcal{R}_x$ up to time t, the conditional probability of no mark of the renewal process in this interval will be at least $1 - p$.

Lemma 3.1. *Consider μ satisfying the assumptions of Theorem 1.2 and let $\mathcal{R}$ be a renewal process with interarrival distribution μ started from some $\mathfrak{T} \leq 0$. Then, given $p_0 > 0$ there exists $w_0 = w_0(p_0) > 0$ such that uniformly on $\mathfrak{T}$ we have:*

$$\sup_{t \geq 0} \sup_{k \geq 0} P\big(\mathcal{R} \cap [t+k, t+k+w_0] \neq \emptyset \mid \mathcal{R} \cap [t, t+k] = \emptyset\big) \leq p_0.$$

Lemma 3.2. *Consider μ satisfying the assumptions of Theorem 1.2 and let $\mathcal{R}$ be a renewal process with interarrival distribution μ started from some $\mathfrak{T} \leq 0$. Given $t \geq 0$, let A_t denote the time elapsed since the last renewal of $\mathcal{R}$ previous to time t (letting $A_t = t - \mathfrak{T}$ if $\mathcal{R} \cap [\mathfrak{T}, t)$ is empty). Then, for any $p_0 > 0$, there exists $w_0 = w_0(p_0) > 0$ such that uniformly on $\mathfrak{T}$ we have*

$$\sup_{t \geq 0} \sup_{0 < a \leq t - \mathfrak{T}} P\big(\mathcal{R} \cap [t, t+w_0] \neq \emptyset \mid A_t = a\big) \leq p_0.$$

Lemma 3.2 allows us to have some control on how close the cure marks of two adjacent vertices inside a finite time interval can be from each other, uniformly on the history of the cure marks in these vertices. From now on, let $A_{x,t}$ be the time elapsed since the last renewal of $\mathcal{R}_x$ previous to time t (letting $A_{x,t} = t - \mathfrak{T}$ if $\mathcal{R}_x \cap [\mathfrak{T}, t)$ is empty).

Lemma 3.3. *Let μ satisfy the assumptions of Theorem 1.2 and let $\mathcal{R}_0$, $\mathcal{R}_1$ be two independent renewal processes with interarrival distribution μ started from some $\mathfrak{T} \leq 0$. Then, given $p_1 > 0$ and $s > 0$ there exists $w_1 = w_1(p_1, s) > 0$ such that uniformly on the triplet $\{t, A_{0,t}, A_{1,t}\}$ for $t \geq 0$, we have*

$$P\Big(\mathrm{d}\big(\mathcal{R}_0 \cap [t, t+s], \mathcal{R}_1 \cap [t, t+s]\big) \leq w_1 \Big| A_{0,t}, A_{1,t}\Big) \leq p_1, \text{ almost surely.}$$

The next two lemmas state that if we choose λ large enough, the infection can survive inside specific space-time boxes with high probability, uniformly on the history of the renewal processes. For both lemmas, consider the RCP(μ) with μ being defined as in Theorem 1.2 and recall Definition 2.4.

Lemma 3.4. *Let $C_h(x,t)$ be as in Definition 2.4 with $b = 1$ and let $\epsilon \in (0,1)$. There exists $\lambda_h = \lambda_h(\epsilon) < \infty$ such that, uniformly on the 5−tuple $\{t, A_{x,t}, A_{x+1,t}, A_{x+2,t}, A_{x+3,t}\}$ for $t \geq 0$, we have*

$$P\Big(C_h(x,t)\Big|(A_{x+i,t})_{i=0}^3\Big) \geq 1-\epsilon, \text{ almost surely, if } \lambda > \lambda_h.$$

Lemma 3.5. *Let $C_v(x,t)$ be as in Definition 2.4 with $b = 1$ and let $\epsilon \in (0,1)$. There exists $\lambda_v = \lambda_v(\epsilon) < \infty$ such that, uniformly on the triplet $\{t, A_{x,t}, A_{x+1,t}\}$ for $t \geq 0$, we have*

$$P\Big(C_v(x,t)\Big|A_{x,t}, A_{x+1,t}\Big) \geq 1-\epsilon, \text{ almost surely, if } \lambda > \lambda_v.$$

We now prove the lemmas.

Proof: *(Lemma 3.1)* Let $(X_j)_{j\geq 0}$ be a sequence of i.i.d. random variables with distribution μ, common density f and common distribution function F, such that X_0 describes the time until the first renewal mark of $\mathcal{R}$ and X_j, for $j \geq 1$, describe the time between the j-th and $(j+1)$-th renewal marks of $\mathcal{R}$. In addition, denote by $(F^{(j)})_{j\geq 0}$ the distribution function of $X_0 + X_1 + \ldots + X_j$, by B_t the residual life process of $\mathcal{R}$ at time t, i.e, the remaining time between t and the next renewal mark of $\mathcal{R}$, and by E_j the event {there are exactly j renewal marks of $\mathcal{R}$ up to time t}. Then, for any $t \geq 0$, $k > 0$ and $w > 0$, we have that

$$P\big(\mathcal{R} \cap [t+k, t+k+w] \neq \emptyset \mid \mathcal{R} \cap [t, t+k] = \emptyset\big) = \sum_{j=0}^{\infty} P(B_t \leq k+w | B_t > k, E_j) P(E_j)$$

$$= P(E_0) P(X_0 \leq t+k+w | X_0 > t+k) + \sum_{j=1}^{\infty} P(E_j) \int_0^t P(B_{t-s} \leq k+w | B_{t-s} > k) dF^{(j-1)}(s)$$

$$= P(E_0) \frac{P(X \in [t+k, t+k+w])}{P(X \geq t+k)} + \sum_{j=1}^{\infty} P(E_j) \int_0^t P(X_j \leq t+k+w-s | X_j > t+k-s) dF^{(j-1)}(s)$$

$$= P(E_0) \int_{t+k}^{t+k+w} \frac{f(u)}{1-F(t+k)} du + \sum_{j=1}^{\infty} P(E_j) \int_0^t \Big[\int_{t+k-s}^{t+k+w-s} \frac{f(u)}{1-F(t+k-s)} du\Big] dF^{(j-1)}(s)$$

$$\leq P(E_0) \int_{t+k}^{t+k+w} \frac{f(u)}{1-F(u)} du + \sum_{j=1}^{\infty} P(E_j) \int_0^t \Big[\int_{t+k-s}^{t+k+w-s} \frac{f(u)}{1-F(u)} du\Big] dF^{(j-1)}(s). \tag{3.1}$$

Since the hazard rate of μ is decreasing in t, we have that both integrals of $\frac{f(u)}{1-F(u)}$ in (3.1) are smaller than or equal to

$$\int_0^w \frac{f(u)}{1-F(u)} du,$$

and then we easily see that (3.1) is bounded above by

$$\frac{1}{1-F(w)} \int_0^w f(u) du \Big[P(E_0) + \sum_{j=1}^{\infty} P(E_j) \int_0^{\infty} dF^{(j-1)}(s)\Big]$$

$$= \frac{F(w)}{1-F(w)} \sum_{j=0}^{\infty} P(E_j) = \frac{F(w)}{1-F(w)}.$$

Since F is a continuous distribution function on $[0,\infty)$, we can take $w_0 = w_0(p_0) > 0$ such that $F(w_0) \leq \frac{p_0}{2}$ and then:

$$\sup_{t\geq 0}\sup_{k\geq 0} P\big(\mathcal{R}\cap[t+k,t+k+w_0]\neq\emptyset \mid \mathcal{R}\cap[t,t+k]=\emptyset\big) \leq \frac{F(w_0)}{1-F(w_0)} \leq \frac{p_0}{2-p_0} \leq p_0.$$

□

Proof: *(Lemma 3.2)* The statement is equivalent to the existence of $w_0 = w_0(p_0)$ such that, uniformly on $\mathfrak{T}$, we have

$$\sup_{a\geq 0} \frac{\mu([a,a+w_0])}{\mu([a,\infty))} \leq p_0.$$

But for any fixed $w > 0$ and $a > 0$,

$$\frac{\mu([a,a+w])}{\mu([a,\infty))} = \int_a^{a+w} \frac{f(u)}{1-F(a)}du \leq \int_a^{a+w} \frac{f(u)}{1-F(u)}du.$$

Since the hazard rate of μ is decreasing, the last integral above is bounded by

$$\int_0^w \frac{f(u)}{1-F(u)}du \leq \frac{1}{1-F(w)}\int_0^w f(u)du = \frac{F(w)}{1-F(w)},$$

and the conclusion follows as in Lemma 3.1. □

Remark 3.6. The bound on the number of renewal marks of $\mathcal{R}_x$ in a finite time interval, that was obtained in Lemma 2.2, also holds almost surely when we condition to $A_{x,t}$, uniformly on $\{t, A_{x,t}\}$ (enlarging K_0 by one if necessary). It occurs because after the first renewal mark of $\mathcal{R}_x$ inside $[t,t+s]$ appears, we are able to forget all information of $\mathcal{R}_x$ previous to time t.

Proof: *(Lemma 3.3)* Given $p_1 > 0$ we use Lemma 2.2 for $\mathcal{R}_1$ with $p_0 = p_1/2$ and let $\tilde{K}_0 = K_0(p_1/2, s)$. We then use Lemma 3.2 with $p_0 = p_1/2\tilde{K}_0$, letting $w_1 = \frac{1}{2}w_0(\frac{p_1}{2\tilde{K}_0})$. It then follows that

$$P\Big(\mathrm{d}\big(\mathcal{R}_0\cap[t,t+s],\mathcal{R}_1\cap[t,t+s]\big)\leq w_1\Big|A_{0,t},A_{1,t}\Big) \leq \frac{p_1}{2\tilde{K}_0}\tilde{K}_0 + \frac{p_1}{2} = p_1, \text{ almost surely.}$$

The uniformity on $t \geq 0$ follows at once from its validity in Lemmas 2.2 and 3.2; and the uniformity on $A_{0,t}, A_{1,t}$ follows from Lemma 3.2. □

Proof: *(Lemma 3.4)* Fix $w_0 \in (0,1)$ satisfying Lemma 3.2 for $p_0 = 1-(1-\epsilon)^{\frac{1}{6}}$ and define the events $F_i = F_i(x,t)$ and $G_i = G_i(x,t)$ for $i = 0,1,2$ as the following:

$$F_i = \Big\{\mathcal{R}_{x+i}\cap[t,t+w_0]=\emptyset\Big\},$$
$$G_i = \Big\{\mathcal{N}_{x+i,x+i+1}\cap[t+\tfrac{iw_0}{3}, t+\tfrac{(i+1)w_0}{3}]\neq\emptyset\Big\}.$$

Note that the occurrence of $\bigcap_{i=0}^{2}(F_i\cap G_i)$ implies the occurrence of $C_h(x,t)$, as in the proof of Lemma 2.5 (see Figure 2.1). Applying Lemma 3.2 we have that uniformly on $\{t, A_{x+i,t}\}$, for $i = 0,1,2$ and $t \geq 0$,

$$P(F_i|A_{x+i,t}) = P(\mathcal{R}_x\cap[t,t+w_0]=\emptyset|A_{x,t}) \geq (1-\epsilon)^{\frac{1}{6}}, \text{ almost surely.} \tag{3.2}$$

Now let Y_λ denote a random variable with exponential distribution with rate λ. Since $w_0 > 0$, we can fix λ_h such that $P(Y_{\lambda_h} \leq \frac{w_0}{3}) = (1-\epsilon)^{\frac{1}{6}}$. If $\lambda \geq \lambda_h$, by the Markov property of the Poisson process, we have that for $i = 0, 1, 2$ and any $t \geq 0$,

$$P(G_i) = P\Big(Y_\lambda \leq \frac{w_0}{3}\Big) \geq (1-\epsilon)^{\frac{1}{6}}. \tag{3.3}$$

But the events $F_0, G_0, F_1, G_1, F_2, G_2$ are all independent, and therefore the conclusion follows from (3.2) and (3.3). □

Proof: *(Lemma 3.5)* Since $B_v(x,t)$ contains only two sites, the path might have to jump between them until time $t+3$, avoiding cure marks, as in the proof of Lemma 2.6 (see Figure 2.2).

Fix $p_0 = p_1 = \frac{1-\sqrt{1-\epsilon}}{5}$ and $s = 3$. Apply Lemmas 2.2, 3.2 and 3.3 for $\mathcal{R}_x$ and $\mathcal{R}_{x+1}$, letting $\tilde{w}_0 = w_0(p_0)$, $\tilde{K}_0 = K_0(p_0, s)$ and $\tilde{w}_1 = w_1(p_1, s)$. Defining H_t as the intersection of the events

$$\Big\{\mathcal{R}_x \cap [t, t+\tilde{w}_0] = \emptyset,\ \mathcal{R}_{x+1} \cap [t, t+\tilde{w}_0] = \emptyset\Big\},$$

$$\Big\{\mid \mathcal{R}_x \cap [t,t+3] \mid \leq \tilde{K}_0,\ \mid \mathcal{R}_{x+1} \cap [t,t+3] \mid \leq \tilde{K}_0\Big\},$$

$$\Big\{\mathrm{d}\Big(\mathcal{R}_x \cap [t,t+3], \mathcal{R}_{x+1} \cap [t,t+3]\Big) > \tilde{w}_1\Big\},$$

it follows from these three lemmas that uniformly on the triplet $\{t, A_{x,t}, A_{x+1,t}\}$, for $t \geq 0$, we have $P(H_t | A_{x,t}, A_{x+1,t}) \geq \sqrt{1-\epsilon}$ almost surely. The occurrence of H_t guarantees that a path with the desired crossing property can be constructed by suitably hopping between x and $x+1$ at most $2\tilde{K}_0$ times to reach time $t+3$ and that it will always encounter an interval of length at least $\tilde{w} = \tilde{w}_0 \wedge \tilde{w}_1$ without cure marks to execute each hop. Fix λ_v such that $P(Y_{\lambda_v} \leq \tilde{w}) = (1-\epsilon)^{\frac{1}{4\tilde{K}_0}}$, and hence it follows by the Markov property of the Poisson process that for any $\lambda \geq \lambda_v$, uniformly on the triplet $\{t, A_{x,t}, A_{x+1,t}\}$, for $t \geq 0$, we have

$$P(C_v(x,t) | A_{x,t}, A_{x+1,t}) \geq P(H_t | A_{x,t}, A_{x+1,t}) \big[P(Y_{\lambda_v} \leq \tilde{w})\big]^{2\tilde{K}_0} \geq 1-\epsilon, \text{ almost surely.}$$

□

Corollary 3.7. *Let $\mathcal{F}_t$ be the filtration generated by the processes $(\mathcal{R}_x)_{x\in\mathbb{Z}}$ and $(\mathcal{N}_{x,y})_{(x,y)\in\mathcal{E}}$. The uniform bounds obtained in Lemmas 3.4 and 3.5 also hold almost surely if we condition on $\mathcal{F}_t$.*

Proof: *(Corollary 3.7)* The space-time boxes $B_h(x,t)$ and $B_v(x,t)$, related to the events $C_h(x,t)$ and $C_v(x,t)$, are entirely above the timeline t. So, these events are independent of the transmission marks up to time t and depend on the cure marks up to time t only through the position of the last cure mark previous to time t in some vertices. □

Now we are finally ready to present the proof of Theorem 1.2.

Proof: *(Theorem 1.2)* Here we will construct a percolation system related to the RCP(μ) in the same way that we did in the proof of Theorem 1.1, taking $b = 1$. Given $x \in \mathbb{Z}$ and $t > 0$, consider the space-time boxes hereby called blocks, $K(x,t) = [x, x+3] \times [t, t+3]$. Each block $K(x,t)$ is said to be *good* if $C_h(x,t) \cap C_h(x,t+2) \cap C_v(x,t) \cap C_v(x+2,t)$ occurs and *bad* otherwise. See for instance the left side image of Figure 2.3.

We consider a graph G in $\mathbb{Z}^2$ with oriented edges $\langle z, z+(1,0)\rangle$ and $\langle z, z+(0,1)\rangle$, and define a percolation system on G related to the RCP(μ) by stating that the vertex (x,y) is open if $K(2x, 3y)$ is a good block in the RCP(μ) and closed otherwise. Then, each edge is open if both endvertices are open. Note that this construction allows us to concatenate paths of adjacent good blocks, implying that when the oriented edge model percolates, the associated RCP(μ) will survive with positive probability. See Figure 2.3.

This percolation model exhibits long-range dependencies along the vertical direction (since the positions of the cure marks in the associated RCP are non-Markovian) and finite range dependency in the first coordinate (since horizontally adjacent blocks overlap each other). But, given $\epsilon > 0$, Lemmas 3.4, 3.5 (and Corollary 3.7) and union bounds imply that if we take $\lambda \geq \lambda_h(\frac{\epsilon}{4}) \vee \lambda_v(\frac{\epsilon}{4})$, then the probability that a site of G is open, given any possible configurations of the other sites with lower vertical coordinate, is bounded from below by $1 - \epsilon$. Applying Theorem 0.0(i) of Liggett et al. (1997) we can conclude that this percolation model dominates a Bernoulli random field with density ρ which can be arbitrarily close to 1, provided we take ϵ small enough. As a consequence, we have that the RCP(μ) survives with positive probability for any $\lambda \geq \lambda_h(\frac{\epsilon}{4}) \vee \lambda_v(\frac{\epsilon}{4})$.

□

Acknowledgements

The authors would like to thank an anonymous referee for all the comments and suggestions that helped to improve the presentation of our paper.

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