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Transportation of stationary random measures not charging small sets

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Abstract. Let (ξ, η) be a pair of jointly stationary, ergodic random measures of equal finite intensity. A balancing allocation is a translation-invariant (equivariant) random map $T : \mathbb{R}^d \to \mathbb{R}^d$, defined on the same probability space, such that the image measure of ξ under T is η. We show that as soon as ξ does not charge small sets, i.e. does not give mass to $(d-1)$-rectifiable sets, there is always a balancing allocation T which is measurably depending only on (ξ, η), i.e. T is a factor of (ξ, η).

1. Introduction

Let ξ and η be two random, jointly stationary, and ergodic measures with the same finite intensity, i.e. $\mathbb{E}[\xi(\Lambda_1)] = \mathbb{E}[\eta(\Lambda_1)]$, where $\Lambda_1 = [-1/2, 1/2]^d$. An allocation T is a translation-invariant (equivariant) random mapping $T : \mathbb{R}^d \to \mathbb{R}^d$, defined on the same probability space. It is said to balance ξ and η, if the image measure of ξ under T is equal to η a.s.. Moreover, an allocation T is called a factor allocation if T is a measurable function of the random measures ξ and η, i.e. T is measurable w.r.t. to $\sigma(\xi, \eta)$, the sigma algebra generated by ξ and η. In this article, we are interested in the question of the existence of balancing factor allocations. Note that without the requirement of translation-invariance the existence can be shown via Borel isomorphism theorems as soon as ξ is diffuse, i.e. ξ does not have atoms. However, the requirement of translation-invariance makes the question much harder. Last and Thorisson showed recently the following existence result:

Theorem 1.1 (Last and Thorisson (2023, Theorem 1.1))**.** *Let ξ and η be two random, jointly stationary, and ergodic measures with the same finite intensity. Let ξ be diffuse. Then there exists an allocation balancing ξ and η if one of the following conditions holds:*

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(a) *η has a non-zero discrete component;*
(b) *η is diffuse and there exists a non-zero simple point process χ on $\mathbb{R}^d$ with finite intensity, such that the triple (ξ, η, χ) is jointly stationary and ergodic.*

Moreover, if an extension of the probability space is allowed, then there exists a balancing allocation (cf. Last and Thorisson (2023, Corollary 1.2)). The most interesting applications (see below) of part (b) are in the case when the process χ is derived as a factor of (ξ, η), i.e. if it is measurably dependent on (ξ, η). This raises the question of either characterizing pairs of random measures (ξ, η) admitting a point process factor χ or deriving complementary conditions ensuring the existence of balancing factor allocations, e.g. see Haji-Mirsadeghi and Khezeli (2016); Last and Thorisson (2023). In this article, we concentrate on the latter question and derive conditions on ξ such that for *any* η, such that (ξ, η) are jointly stationary and ergodic, there is a balancing factor allocation. By the example of Last and Thorisson (2023, Section 8), we know that for such an existence result ξ should not give mass to $(d-1)$-dimensional sets. Indeed, Last and Thorisson constructed a pair of jointly stationary, ergodic, diffuse random measures (ξ, η), where ξ is concentrated on a $(d-1)$-dimensional set, such that there is no balancing factor allocation. Our main result gives a general existence result for balancing factor allocations:

Theorem 1.2. *Let ξ and η be two random, jointly stationary, and ergodic measures with the same finite intensity. Assume that ξ does not charge small sets, i.e. does not give mass to $(d-1)$-rectifiable sets. Then there exists a factor allocation balancing ξ and η.*

We note that the assumption on ξ is sharp if one is not allowed to use extra information on or structure of η by the counterexample of Last and Thorisson. We also remark that Theorem 1.2 (just as Theorem 1.1) remains true if one relaxes the assumption of ergodicity and same intensity to the assumption that $\mathbb{E}[\xi(B)|\mathcal{I}] = \mathbb{E}[\eta(B)|\mathcal{I}]$ for some measurable and bounded set $B \subset \mathbb{R}^d$, where $\mathcal{I}$ denotes the σ-algebra of shift invariant events (e.g. see Last and Thorisson (2023, Section 9)).

Parallel and independently to this article, Khezeli and Mellick (2023) have shown Theorem 1.2 by using lacunary sections from descriptive set theory. With this technique, they are able to show much more, namely, they additionally characterise pairs of random measures (ξ, η) that admit a point process factor sharpening Theorem 1.1. It would be interesting to see (and it could well be) whether their result is optimal.

The proof of Theorem 1.2 is based on the optimal transport techniques for random measures, introduced in Huesmann and Sturm (2013) and Huesmann (2016). Let us sketch the argument. By Last and Thorisson (2009, Theorem 5.1), under our assumptions, there exists a (ξ, η)-balancing invariant weighted transport-kernel T or equivalently, in the language of this article, an equivariant coupling q for ξ and η (cf. (2.1)).

By an application of the Lemma of de la Vallée Poussin, we can construct a strictly concave function ϑ, such that q has finite mean transportation cost (cf. (2.2)) w.r.t. $c(x, y) = \vartheta(|x-y|)$. If ξ and η are mutually singular, a variant of Huesmann (2016, Theorem 1.1), Theorem 2.1, implies the existence of an equivariant coupling $q^* = (id, T)_\# \xi$. Here # means the push-forward of the (random measure) ξ under the (random) map $(id, T) : \mathbb{R}^d \to \mathbb{R}^{2d}$. In particular, T is the desired balancing factor allocation. The mutual singularity ensures that the coupling is induced by a transport map T. In general, for measures with common mass, the optimal coupling is not induced by a transport map (see Pegon et al. (2015, Theorem 1.1)).

If instead ξ, η are both absolutely continuous w.r.t. a third measure γ not charging small sets, we first construct an auxiliary factor allocation T between the mutually singular measures $(\xi-\eta)_+$ and $(\eta-\xi)_+$. This can be used to partition $\mathbb{R}^d$ in an equivariant way into sets $\{x : G(T(x)-x) \le t\}$ and $\{x : G(T(x)-x) > t\}$, where $G : \mathbb{R}^d \to \mathbb{R}$ is some arbitrary but fixed bijective deterministic map. It turns out that for a particular choice of $t = t_0$ the measures $\mathbb{1}_{G(T(x)-x)\le t_0}\xi$ and $\mathbb{1}_{G(T(x)-x)>t_0}\eta$ have the same intensity (and are mutually singular). Hence, there exists a balancing factor allocation T_1.

Similarly, we obtain a balancing factor allocation T_2 between $\mathbb{1}_{G(T(x)-x)>t_0}\xi$ and $\mathbb{1}_{G(T(x)-x)\le t_0}\eta$. Combining T_1 and T_2 proves Theorem 1.2 in this case.

For the general case, we decompose $\eta = \eta^a + \eta^s$ into a part $\eta^a \ll \xi$ and a part η^s singular to ξ. Using again an auxiliary allocation to partition $\mathbb{R}^d$ we can prove Theorem 1.2 by a combination of the previous two cases.

The interest in allocations originates from its link to shift couplings of random measures with their Palm version. Let $\mathsf{Leb}_{\mathbb{R}^d}$ be the Lebesgue measure on $\mathbb{R}^d$. If the source is given by $\xi = c\,\mathsf{Leb}_{\mathbb{R}^d}$ for some $c > 0$ and T is an allocation balancing $c\,\mathsf{Leb}_{\mathbb{R}^d}$ and η, then the shifted measure $\eta - T(0)$ is a Palm version of η, i.e. the pair $(\eta, \eta - T(0))$ is a shift coupling of η and a Palm version of η (see Holroyd and Peres (2005)). In particular, if T is a factor allocation, this Palm version of η is a function of η. To the best of our knowledge, the first explicit non-randomized (factor) shift-coupling for point processes was constructed by Liggett (2002). This work together with Holroyd and Peres (2005); Hoffman et al. (2006) initiated a series of constructions of shift couplings by constructing factor allocations, e.g. Holroyd and Peres (2005); Chatterjee et al. (2010); Last et al. (2014); Huesmann and Sturm (2013). Allocations and equivariant couplings or transports between two general random measures ξ and η have been investigated e.g. in Last and Thorisson (2009); Last et al. (2014, 2018); Huesmann (2016). We also refer to Aldous and Thorisson (1993); Thorisson (1996) for the origin of shift-couplings and to Last and Thorisson (2009) and Last and Thorisson (2023, Remark 2.2) for results on shift couplings resulting from allocations between general random measures.

As a particular consequence of the preceding paragraphs and Theorem 1.2, for any η there is always a factor shift coupling of η with a Palm version of η (by Thorisson (1996) we only know that there is some shift coupling on a potentially enlarged probability space).

Corollary 1.3. *Let η be an ergodic random measure with finite intensity. Then there exists a factor shift coupling of η and a Palm version of η.*

Proof: Let $c = \mathbb{E}[\eta(\Lambda_1)]$. Applying Theorem 1.2 to $\xi = c\,\mathsf{Leb}_{\mathbb{R}^d}$ and η yields a balancing factor allocation T. Then Last and Thorisson (2009, Theorem 4.1) shows that $\theta_{T(0)}\eta$ is a Palm version of η. □

2. Setup and Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a measurable flow $\theta_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$. That is, the mapping $(x, \omega) \mapsto \theta_x\omega$ is measurable, $\theta_y \circ \theta_x = \theta_{x+y}$ for all $x, y \in \mathbb{R}^d$ and θ_0 is the identity. Furthermore, let $\mathbb{P}$ be stationary w.r.t. the flow θ, i.e. $\mathbb{P}(A) = \mathbb{P}(\theta_x(A))$ for all $x \in \mathbb{R}^d$. The invariant sigma field $\mathcal{I}$ is defined by $\mathcal{I} = \{A \in \mathcal{F} \mid \forall x \in \mathbb{R}^d : A = \theta_x A\}$ and we assume that $\mathbb{P}$ is ergodic, that is $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{I}$.

In the following, a random measures ξ is a locally finite transition kernel from $(\Omega, \mathcal{F})$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where locally finite means that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the measure $\xi(\omega, \cdot)$ is finite on bounded measurable sets. A random measure ξ is said to be equivariant if for all $\omega \in \Omega$, $x \in \mathbb{R}^d$ and $B \in \mathcal{B}(\mathbb{R}^d)$ it holds that

$$\xi(\omega, B) = \xi(\theta_x\omega, B - x).$$

An allocation is a measurable mapping $T : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ with the following equivariance property

$$T(\theta_x\omega, y) = T(\omega, y + x) - x \quad \forall \omega \in \Omega, x, y \in \mathbb{R}^d.$$

The allocation T balances two random measures ξ and η if for $\mathbb{P}$-a.e. $\omega \in \Omega$ the map T^ω pushes ξ^ω onto η^ω, i.e. $\xi^\omega \circ (T^\omega)^{-1} = \eta^\omega$. For $\omega \in \Omega$ the map $T^\omega : \mathbb{R}^d \to \mathbb{R}^d$ is defined by $T^\omega(x) = T(\omega, x)$, $x \in \mathbb{R}^d$. We say that T is a factor allocation, if T is measurable w.r.t. to $\sigma(\xi, \eta)$, the sigma algebra generated by ξ and η (see Hoffman et al. (2006); Huesmann and Sturm (2013); Last and Thorisson (2023) for further disussion).

A semicoupling q of ξ and η is a transition kernel from $(\Omega, \mathcal{F})$ to $(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d))$ such that for almost every $\omega \in \Omega$ the measure $q(\omega)$ is a semicoupling of $\xi(\omega)$ and $\eta(\omega)$, that is

$$(\pi_1)_\#(q(\omega)) \leq \xi(\omega) \text{ and } (\pi_2)_\#(q(\omega)) = \eta(\omega). \tag{2.1}$$

If the inequality is an equality for almost all $\omega \in \Omega$, q is a coupling of ξ and η. Here π_i denotes the projection onto the i-th coordinate. A semicoupling q is said to be equivariant if

$$q(\omega, A \times B) = q(\theta_x \omega, (A - x) \times (B - x)) \quad \forall \omega \in \Omega, x \in \mathbb{R}^d, A, B \in \mathcal{B}(\mathbb{R}^d).$$

For equivariant random measures ξ and η we denote by $\mathsf{Cpl}_{es}(\xi, \eta)$ the set of all equivariant semicouplings of ξ and η. For a given function $\vartheta : [0, \infty) \to [0, \infty)$ we then define the mean transportation cost by

$$\inf_{q \in \mathsf{Cpl}_{es}(\xi,\eta)} \mathbb{E}\left[\int_{\Lambda_1 \times \mathbb{R}^d} \vartheta(|x - y|) q(dx, dy)\right], \tag{2.2}$$

where $\Lambda_1 = [-1/2, 1/2]^d$.

Optimal transport problems for semicouplings between finite measures have been also investigated under the name of partial optimal transport problems, e.g. Figalli (2010), or incomplete optimal transportation Álvarez Esteban et al. (2011). We will establish a particular uniqueness result for strictly concave cost for a partial optimal transport problem between finite measures in Lemma 2.2 below. It is an important ingredient for the proof of the following theorem, which is a small extension of Huesmann (2016, Theorem 1.1) to the case of strictly concave cost functions.

Theorem 2.1 (Semicoupling)**.** *Let ξ and η be two equivariant random measures, which are a.s. mutually singular. Furthermore, assume that a.s. ξ does not charge small sets, i.e. ξ does not give mass to $(d-1)$-rectifiable sets and that the intensity of ξ is greater than or equal to the intensity of η. Let ϑ be a strictly concave and increasing function $\vartheta : [0, \infty) \to [0, \infty)$ with $\vartheta(0) = 0$ and $\lim_{x\to\infty} \vartheta(x) = \infty$. Assume that the mean transportation cost of ξ and η w.r.t. ϑ is finite.*

Then, (2.2) admits a unique minimizing equivariant semicoupling q of ξ and η. It can be represented as $q = (Id, T)_\#(\mathbb{1}_B \cdot \xi)$, for some allocation $T : \mathrm{supp}(\xi) \to \mathrm{supp}(\eta)$ and random set $B \subset \mathbb{R}^d$[1], measurably only dependent on $\sigma(\xi, \eta)$. Moreover, if ξ and η have equal intensities, then the equivariant semicoupling is in fact a coupling.

Theorem 2.1 can be proven similarly to Huesmann (2016, Theorem 1.1), a sketch of the proof is given below at the end of this section (see page 1908). In Huesmann (2016, Theorem 1.1) the cost functions were chosen such that for bounded sets there is a unique minimizing semicoupling. Since we consider concave cost functions, for which the minimizing semicoupling in general is not unique, we first have to establish the following uniqueness result, where we denote by $\mathsf{Cpl}_s(\mu, \nu)$ the set of semicouplings between μ and ν and by $\mathsf{Cpl}(\mu, \nu)$ the set of couplings between μ and ν.

Lemma 2.2. *Let μ, ν be two finite Borel measures on $\mathbb{R}^d$ such that $\mu(\mathbb{R}^d) \geq \nu(\mathbb{R}^d)$, μ does not charge small sets and μ and ν are mutually singular. Let $c(x, y) = \vartheta(|x - y|)$ for some strictly concave and increasing function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\vartheta(0) = 0$. Then there is a unique optimizer q^* to*

$$\inf_{q \in \mathsf{Cpl}_s(\mu,\nu)} \int c(x, y) dq(x, y). \tag{2.3}$$

Moreover, $q^* = (id, T)(1_B \mu)$ for some measurable set $B \subset \mathbb{R}^d$ and a map T.

Proof: Let $\varepsilon > 0$ and K_1, K_2 be compact sets such that $\mu(K_1^c), \nu(K_2^c) < \varepsilon$. For a semicoupling q between μ and ν we then have $q((K_1 \times K_2)^c) \leq 2\varepsilon$. Hence, the set of all semicouplings between μ and ν is compact. Since c is continuous and bounded from below it follows that the map $q \mapsto$

[1] In the proof of Theorem 3.6 we will show that B can be chosen as an invariant set measurably dependent on $\sigma(\xi, \eta)$.

$\int c(x,y)dq(x,y)$ is lower semicontinuous. Hence, there exists a minimizer q^* with marginals $f\cdot\mu$ and ν. Moreover, since q^* is optimal between its marginals, it follows by mutual singularity of μ and ν that there exists a map T such that $q^* = (id, T)(f\cdot\mu)$, Pegon et al. (2015, Theorem 4.6).

We claim that $f = 1_B$ for some measurable set B. This implies uniqueness. Indeed, if q_1 and q_2 are two potentially different optimizer with densities 1_{B_1} and 1_{B_2} respectively, then also $q_3 = \frac{1}{2}(q_1+q_2)$ is an optimizer by linearity whose density has to satisfy $1_{B_3} = \frac{1}{2}(1_{B_1}+1_{B_2})\ \mu - a.s.$ Hence, we obtain that $B_1 = B_2 = B_3\ \mu - a.s.$ and therefore uniqueness.

To show the claim, we will argue by contradiction. Let us assume that $\mu(\{0<f<1\})>0$ so that there is an $\varepsilon>0$ such that $A=\{0<f\leq 1-\varepsilon\}$ has positive μ measure. Then, $\tilde{q} = q_{|A\times\mathbb{R}^d}$ is optimal between its marginals $\tilde{f}\mu$ and $\tilde{\nu}$. For notational simplicity we can then assume that $\tilde{q}=q$ and $\tilde{f}=f\leq 1-\varepsilon$. Applying the Lebesgue differentiation theorem to the mutually singular measures μ and ν we have for ν-a.e. point $y\in\mathbb{R}^d$ that

$$\lim_{r\to 0}\frac{\mu(B_r(y))}{\nu(B_r(y))} = 0.$$

We fix $y\in\mathbb{R}^d$ such that

i) $\nu(B_r(y))>0$ for all $r>0$, i.e. $y\in\mathsf{supp}(\nu)$

ii) $\lim_{r\to 0}\frac{\mu(B_r(y))}{\nu(B_r(y))} = 0$.

In particular, for any $\delta>0$ there is $r>0$ such that $\nu(B_r(y))>0$ and $\mu(B_r(y))\leq\delta\nu(B_r(y))$ such that $q(B_r(y)^c\times B_r(y))>0$. However, since $f\leq 1-\varepsilon$ we can use the the mass within $T^{-1}(B_r(y))\setminus B_r(y)$ which is transported to $B_r(y)$ more efficiently to produce a coupling with cheaper cost. In the remaining part of the proof, we will explicitly construct such a competitor to q.

Let $(x_0,y_0)\in\mathsf{supp}(q_{|B_r(y)^c\times B_r(y)})$ and choose $r'>0$ sufficiently small, for example $r'<\sqrt{\frac{|x_0-y_0|}{2d}}$ suffices. Since $q(B_{r'}(x_0)\times B_{r'}(y_0))>0$ there exists by Pegon et al. (2015, Lemma 4.1) a point $x_1\in B_{r'}(x_0)$ with the following property

$$\forall\alpha>0,\forall\delta>0,\forall u\in\mathbb{S}^{d-1}: q((C(x_1,u,\delta,\alpha)\cap B_{r'}(x_0))\times B_{r'}(y_0))>0, \tag{2.4}$$

where

$$C(x_1,u,\delta,\alpha) = \{z: u\cdot(z-x_1)\geq(1-\delta)|z-x_1|\}\cap\bar{B}_\alpha(x_1).$$

Let $u_- = \frac{x_1-y_0}{|x_1-y_0|}$ and $u_+ = \frac{y_0-x_1}{|y_0-x_1|}$ and set $C_- = C(x_1,u_-,\delta,\alpha)\setminus\{x_1\}$ and $C_+ = C(x_1,u_+,\delta,\alpha)\setminus\{x_1\}$. Then an elementary geometric argument shows that the following holds. For all $\delta,\alpha>0$ small enough

$$\forall z_-\in C_-,\forall z_+\in C_+,\forall\tilde{y}\in B_{r'}(y_0): |z_+-\tilde{y}|<|z_--\tilde{y}|. \tag{2.5}$$

In the following fix such $r',\delta,\alpha>0$. In particular, let $r'>\alpha$ so that the intersection in (2.4) reduces to

$$C_+\cap B_{r'}(x_0) = C_+,$$

and similiarly for C_-. For $0<s<1$ there exists $t=t(s)>1$ such that

$$sq(C_-\times B_{r'}(y_0)) + tq(C_+\times B_{r'}(y_0)) = q((C_-\cup C_+)\times B_{r'}(y_0)).$$

Let π be an optimal coupling of $(t-1)\mathsf{pr}_1(q_{|C_+\times B_{r'}(y_0)})$ and $(1-s)\mathsf{pr}_2(q_{|C_-\times B_{r'}(y_0)})$. These measures have the same mass since

$$sq(C_-\times B_{r'}(y_0))+tq(C_+\times B_{r'}(y_0)) = q((C_-\cup C_+)\times B_{r'}(y_0)) = q(C_-\times B_{r'}(y_0))+q(C_+\times B_{r'}(y_0)).$$

Then define

$$\hat{q} = sq_{|C_-\times B_{r'}(y_0)} + q_{|C_+\times B_{r'}(y_0)} + \pi.$$

Since $\lim_{s\nearrow 1}t(s)=1$, it follows from $f\leq 1-\epsilon$ that $tf\leq 1$ for $s<1$ large enough. Hence $\hat{q}$ defines a semicoupling of $\mu_{|C_-\cup C_+}$ and $\mathsf{pr}_2(q_{|(C_-\cup C_+)\times B_{r'}(y_0)})$. Thus $\hat{q}$ is an admissible competitor to

$q_{|(C_-\cup C_+)\times B_{r'}(y_0)}$. Disintegration w.r.t. the second marginal of the measures $q_{|(C_-\cup C_+)\times B_{r'}(y_0)}$ and π yields

$$\begin{aligned}&\int_{(C_-\cup C_+)\times B_{r'}(y_0)} \vartheta(|x-y|)dq - \int_{(C_-\cup C_+)\times B_{r'}(y_0)} \vartheta(|x-y|)d\hat{q}\\ &= (1-s)\int_{C_-\times B_{r'}(y_0)} \vartheta(|x-y|)dq - \int_{C_+\times B_{r'}(y_0)} \vartheta(|x-y|)d\pi\\ &= (1-s)\int_{B_{r'}(y_0)} \mathsf{pr}_2(q_{|C_-\times B_{r'}(y_0)})(d\tilde{y})\left[\int_{C_-} dq_{\tilde{y}}(dx)\vartheta(|x-y|) - \int_{C_+} d\pi_{\tilde{y}}(dx)\vartheta(|x-y|)\right].\end{aligned}$$

Inequality (2.5) implies that the last line is strictly positive. That is,

$$\int_{(C_-\cup C_+)\times B_{r'}(y_0)} \vartheta(|x-y|)dq > \int_{(C_-\cup C_+)\times B_{r'}(y_0)} \vartheta(|x-y|)d\hat{q}.$$

This, however, contradicts the optimality of q. □

We give a very short sketch of the proof of Theorem 2.1.

Sketch of proof of Theorem 2.1: The existence of an optimal semicoupling can be proven exactly as in the proof of Huesmann (2016, Proposition 3.18). In order to establish uniqueness, we recall the notion of local optimality (see Huesmann (2016, Definition 5.3) for further discussion). An equivariant semicoupling q is locally optimal iff the following holds for $\mathbb{P}$-a.e. $\omega \in \Omega$:

There exists a nonnegative density ρ^ω and a c-cyclically monotone map $T^\omega : \{\rho^\omega > 0\} \to \mathbb{R}^d$ such that on $\{\rho^\omega > 0\} \times \mathbb{R}^d$

$$q^\omega = (Id, T^\omega)_\#(\rho^\omega \xi^\omega).$$

Local optimality of optimal semicouplings can be shown as in Proposition 3.1 and Theorem 3.6 in Huesmann and Sturm (2013). The proof of Huesmann and Sturm (2013, Proposition 3.1) relies on uniqueness of optimal semicouplings on bounded sets, a fact which in our setting is provided by Lemma 2.2.

Now we can prove uniqueness of optimal semicouplings. Let q_1, q_2 be two optimal semicouplings. By local optimality there exist maps T_i and densities ρ_i, $i = 1, 2$, such that $q_i^\omega = (Id, T_i^\omega)_\#(\rho_i^\omega \xi^\omega)$. Restricting the q_i to bounded sets, it follows from optimality and from Lemma 2.2, that we can assume $\rho_i^\omega = \mathbb{1}_{A_i^\omega}$, for some measurable set $A_i^\omega \subset \mathbb{R}^d$. Applying the same reasoning to the optimal semicoupling $q = \frac{1}{2}(q_1 + q_2)$ proves that $q_1 = q_2$. □

We will need the following version of the classical Lemma of de la Vallée Poussin:

Lemma 2.3. *Let $f : [0,\infty) \to [0,\infty)$ be a Lebesgue integrable function. Then there exists a continuous and strictly concave function $\vartheta : [0,\infty) \to [0,\infty)$ with $\vartheta(0) = 0$ such that*

$$\int_0^\infty f(x)\vartheta(x)dx < \infty.$$

Moreover, ϑ can be chosen to be strictly increasing, smooth on $(0,\infty)$ and such that $\lim_{x\to\infty} \vartheta(x) = \infty$.

Proof: Combine Laurençot (2015, Theorem 2.8) and Laurençot (2015, Proposition 2.14). □

3. Proof of Theorem 1.2

From now on, we will assume that (ξ, η) are jointly stationary and ergodic random measures with the same finite intensities. We start by showing that there is a strictly concave, increasing and diverging function ϑ such that the mean transportation cost (2.2) w.r.t. $c(x,y) = \vartheta(|x-y|)$ is finite. Combining this with Theorem 2.1 implies the existence of allocations in the case that ξ and η are mutually singular, see Subsection 3.2. In a next step we will prove our main result in the case that both ξ and η are absolutely continuous w.r.t. a measure γ not charging small sets, see Subsection 3.3. Finally, we will show the general statement in Subsection 3.4.

3.1. *Existence of an equivariant coupling with finite cost.*

Lemma 3.1. *Let ξ and η be two jointly stationary and ergodic random measures with the same finite intensity. There exists an equivariant coupling q of ξ and η and a strictly concave function $\vartheta : [0,\infty) \to [0,\infty)$ such that*

$$\mathbb{E}\left[\int_{\Lambda_1\times\mathbb{R}^d} \vartheta(|x-y|)q(dx,dy)\right] < \infty.$$

Furthermore, the function ϑ can be chosen to be continuous and strictly increasing and such that $\vartheta(0)=0$ and $\lim_{x\to\infty}\vartheta(x)=\infty$.

Proof: By Last and Thorisson (2009, Theorem 5.1), there exists an equivariant coupling q of ξ and η, since their intensities coincide. We are going to construct the desired function ϑ. Since ξ has finite intensity, we can write

$$\infty > \mathbb{E}\left[\xi\left(\Lambda_1\right)\right] = \mathbb{E}\left[\int_{\Lambda_1\times\mathbb{R}^d} q(dx,dy)\right] = \sum_{n\geq 0}\mathbb{E}\left[\int_{\Lambda_1\times\mathbb{R}^d} \mathbb{1}_{n\leq|x-y|<n+1}q(dx,dy)\right] = \sum_{n\geq 0} a_n,$$

with $a_n = \mathbb{E}\left[\int_{\Lambda_1\times\mathbb{R}^d} \mathbb{1}_{n\leq|x-y|<n+1}q(dx,dy)\right]$. Define the function $f:[0,\infty)\to[0,\infty)$ by $f=\sum_{n\geq 0}\mathbb{1}_{[n+1,n+2)}a_n$. By construction, f is integrable. Hence, from Lemma 2.3 it follows that there exists a function ϑ with the properties listed in the statement of this lemma, which is smooth on $(0,\infty)$ and satifies

$$\sum_{n\geq 0} a_n \int_{n+1}^{n+2}\vartheta(x)dx = \int_0^\infty \vartheta(x)f(x)dx < \infty.$$

Then

$$\begin{aligned}
\mathbb{E}\left[\int_{\Lambda_1\times\mathbb{R}^d} \vartheta(|x-y|)q(dx,dy)\right] &= \sum_{n\geq 0}\mathbb{E}\left[\int_{\Lambda_1\times\mathbb{R}^d} \vartheta(|x-y|)\mathbb{1}_{n\leq|x-y|<n+1}q(dx,dy)\right] \\
&\leq \sum_{n\geq 0}\vartheta(n+1)\mathbb{E}\left[\int_{\Lambda_1\times\mathbb{R}^d} \mathbb{1}_{n\leq|x-y|<n+1}q(dx,dy)\right] \\
&\leq \sum_{n\geq 0} a_n\int_{n+1}^{n+2}\vartheta(x)dx < \infty.
\end{aligned}$$

□

3.2. *Mutually singular measures.*

Corollary 3.2. *Let ξ and η be two jointly stationary and ergodic random measures with the same finite intensity, which are a.s. mutually singular. Furthermore, assume that ξ does not charge $(d-1)$-rectifiable sets. Then there exists a factor allocation.*

Proof: From Lemma 3.1 we obtain a function ϑ, which yields finite mean transportation cost and satisfies the properties listed in Theorem 2.1. The other assumptions of Theorem 2.1 are also satisfied. Finally note that, since the random measures ξ and η have the same intensity, the optimal semicoupling is a coupling. Hence the random map T^ω is a factor allocation for ξ and η. □

3.3. *Measures that do not charge small sets.* In this subsection, we assume that a.s. both ξ and η are absolutely continuous w.r.t. a measure that does not charge small sets. We consider the decompositions $\xi = (\xi \wedge \eta) + (\xi - \eta)_+$ and $\eta = (\xi \wedge \eta) + (\eta - \xi)_+$. Here $(\xi - \eta)_+$ denotes the positive part of the Jordan decomposition of $\xi - \eta$ and the measure $(\eta - \xi)_+$ is analogously defined. Note that the measures $(\xi - \eta)_+$ and $(\eta - \xi)_+$ are mutually singular, do not charge small sets, i.e. do not give mass to $(d-1)$-rectifiable sets, and have the same intensity. For the proof of Proposition 3.4 we will rely on the following invariant differentiation result from Kallenberg (2017, Corollary 7.25).

Proposition 3.3. *Let $\mathcal{M}(\mathbb{R}^d)$ be the space of locally finite measures on $\mathbb{R}^d$ equipped with the topology of vague convergence generated by the maps $\mathcal{M}(\mathbb{R}^d) \ni \mu \mapsto \int f d\mu$ for continuous, compactly supported $f \in C_c(\mathbb{R}^d)$. There exists a measurable function $\varphi : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}$ such that $\varphi(\cdot, \mu, \nu) = \frac{d\mu}{d\nu}$ for all $\mu \ll \nu$ in $\mathcal{M}(\mathbb{R}^d)$. Moreover, the function φ is invariant in the following sense*

$$\varphi(x, \mu, \nu) = \varphi(x + z, \theta_z^{\mathcal{M}(\mathbb{R}^d)}\mu, \theta_z^{\mathcal{M}(\mathbb{R}^d)}\nu), \quad \forall x, z \in \mathbb{R}^d, \mu, \nu \in \mathcal{M}(\mathbb{R}^d), \tag{3.1}$$

where for $z \in \mathbb{R}^d$ the shift $\theta_z^{\mathcal{M}(\mathbb{R}^d)} : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ is defined by

$$\theta_z^{\mathcal{M}(\mathbb{R}^d)}\mu(A) = \mu(A + z), \quad \forall A \in \mathcal{B}(\mathbb{R}^d). \tag{3.2}$$

Proposition 3.4. *Let ξ, η and γ be jointly stationary and ergodic random measures. Assume that ξ and η have the same finite intensity and assume that $\xi \ll \gamma$ and $\eta \ll \gamma$ almost surely. Furthermore, assume that γ does not charge small sets. Then there exists a factor allocation.*

Proof: Let φ be as in Proposition 3.3. We set $f = \varphi(\cdot, \xi, \gamma)$ and $g = \varphi(\cdot, \eta, \gamma)$, i.e. f and g are the (random) densities of ξ and η respectively. We assume that $\xi \neq \eta$, since the case $\xi = \eta$ is trivial. Let $T : \{x \in \mathbb{R}^d : f(x) > g(x)\} \to \{x \in \mathbb{R}^d : f(x) < g(x)\}$ be the factor allocation for the mutually singular measures $(\xi - \eta)_+$ and $(\eta - \xi)_+$, which exists by Corollary 3.2. Since both measures do not charge small sets, there exists also the inverse allocation $T^{-1} : \{x \in \mathbb{R}^d : f(x) < g(x)\} \to \{x \in \mathbb{R}^d : f(x) > g(x)\}$. (In fact, it follows that $T \circ T^{-1} = id$ and $T^{-1} \circ T = id$ by uniqueness of the optimal coupling between $(\xi - \eta)_+$ and $(\eta - \xi)_+$.) We define the (random) function F on $\mathbb{R}^d$ by

$$F(x) = \begin{cases} T(x) & f(x) > g(x) \\ x & f(x) = g(x) \\ T^{-1}(x) & f(x) < g(x) \end{cases}$$

Let $G : \mathbb{R}^d \to \mathbb{R}$ be a measurable and bijective function such that $G(0) = 0$. Define the function $I : \mathbb{R} \to \mathbb{R}$ by

$$I(t) = \mathbb{E}\left[\int_{\Lambda_1} \mathbb{1}_{G(F(x)-x) \leq t} \mathbb{1}_{F(x)-x \neq 0} \xi(dx)\right].$$

This function satisfies $\lim_{t\to-\infty} I(t) = 0$ and $\lim_{t\to\infty} I(t) = \mathbb{E}\left[\int_{\Lambda_1} \mathbb{1}_{F(x)-x\neq 0}\xi(dx)\right] > 0$, since $\xi \neq \eta$. We prove that it is continuous. Since it is increasing in t, it suffices to prove that for fixed $t \in \mathbb{R}$

$$\mathbb{E}\left[\int_{\Lambda_1} \mathbb{1}_{G(F(x)-x)=t} \mathbb{1}_{F(x)-x\neq 0} \xi(dx)\right] = 0.$$

This is true for $t = 0$ so let $t \neq 0$. Then

$$\begin{aligned} &\mathbb{E}\left[\int_{\Lambda_1} \mathbb{1}_{G(F(x)-x)=t} \mathbb{1}_{F(x)-x\neq 0} \xi(dx)\right] = \mathbb{E}\left[\int_{\Lambda_1} \mathbb{1}_{F(x)-x=G^{-1}(t)} \xi(dx)\right] \\ &= \mathbb{E}\left[\int_{\Lambda_1 \cap \{x\in\mathbb{R}^d : f(x)>g(x)\}} \mathbb{1}_{T(x)-x=G^{-1}(t)} \xi(dx)\right] + \mathbb{E}\left[\int_{\Lambda_1 \cap \{x\in\mathbb{R}^d : f(x)<g(x)\}} \mathbb{1}_{T^{-1}(x)-x=G^{-1}(t)} \xi(dx)\right]. \end{aligned} \tag{3.3}$$

Now consider a fixed realisation of the measure $\mathbb{1}_{\Lambda_1\cap\{x\in\mathbb{R}^d:f(x)>g(x)\}}\xi$ and of the corresponding pushforward $T_\#(\mathbb{1}_{\Lambda_1\cap\{x\in\mathbb{R}^d:f(x)>g(x)\}}\xi)$. From Huesmann (2016, Theorem 5.5) it follows that T is an optimal transport map for the measures $\mathbb{1}_{\Lambda_1\cap\{x\in\mathbb{R}^d:f(x)>g(x)\}}\xi$ and $T_\#(\mathbb{1}_{\Lambda_1\cap\{x\in\mathbb{R}^d:f(x)>g(x)\}}\xi)$ w.r.t. the cost $c(x,y)=\vartheta(|x-y|)$. Applying Pegon et al. (2015, Proposition 5.1) thus yields that a.s.

$$\xi\left(\{x\in\Lambda_1\cap\{x\in\mathbb{R}^d:f(x)>g(x)\}:T(x)-x=G^{-1}(t)\}\right)=0.$$

Hence the first expectation is zero. Since we restrict to the set $\{x\in\mathbb{R}^d:f(x)<g(x)\}$, we can bound the second expectation in (3.3) in the following way from above

$$\mathbb{E}\left[\int_{\Lambda_1\cap\{x\in\mathbb{R}^d:f(x)<g(x)\}}\mathbb{1}_{T^{-1}(x)-x=G^{-1}(t)}\xi(dx)\right]\le\mathbb{E}\left[\int_{\Lambda_1\cap\{x\in\mathbb{R}^d:f(x)<g(x)\}}\mathbb{1}_{T^{-1}(x)-x=G^{-1}(t)}\eta(dx)\right].$$

By the same argument we used for the first expectation, it follows that the upper bound is equal to zero. Hence both terms in (3.3) are equal to zero and the continuity is proved. We define the corresponding function $J(t)$ by

$$J(t)=\mathbb{E}\left[\int_{\Lambda_1}\mathbb{1}_{G(F(x)-x)>t}\mathbb{1}_{F(x)-x\neq 0}\eta(dx)\right].$$

This function is continuous as well and has the limits $\lim_{t\to-\infty}J(t)=\mathbb{E}\left[\int_{\Lambda_1}\mathbb{1}_{F(x)-x\neq0}\eta(dx)\right]>0$ and $\lim_{t\to\infty}J(t)=0$. Hence there exists a t_0 such that $I(t_0)=J(t_0)$.

This means that the random measures $\mathbb{1}_{G(F(x)-x)\le t_0}\mathbb{1}_{F(x)-x\neq0}\xi$ and $\mathbb{1}_{G(F(x)-x)>t_0}\mathbb{1}_{F(x)-x\neq0}\eta$ have the same intensity. Since they are mutually singular, we can apply Corollary 3.2 to obtain a factor allocation

$$S_1:\{x\in\mathbb{R}^d:G(F(x)-x)\le t_0\text{ and }F(x)-x\neq0\}\to\mathbb{R}^d.$$

Similar arguments yield a factor allocation S_2 for the measures

$$\mathbb{1}_{G(F(x)-x)>t_0}\mathbb{1}_{F(x)-x\neq0}\xi\text{ and }\mathbb{1}_{G(F(x)-x)\le t_0}\mathbb{1}_{F(x)-x\neq0}\eta.$$

Defining $S_3:\{x\in\mathbb{R}^d:F(x)-x=0\}\to\{F(x)-x=0\}$ to be the identity map, we see that

$$T^*=\mathbb{1}_{G(F(x)-x)\le t_0}\mathbb{1}_{F(x)-x\neq0}S_1+\mathbb{1}_{G(F(x)-x)>t_0}\mathbb{1}_{F(x)-x\neq0}S_2+\mathbb{1}_{F(x)-x=0}S_3$$

is a factor allocation for the measures ξ and η. □

3.4. *General case.* Combining Corollary 3.2 and Proposition 3.4 we prove the most general case. We will use the following invariant disintegration result Kallenberg (2017, Theorem 7.24).

Proposition 3.5. *There exists a kernel ψ from $\mathbb{R}^d\times\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)\times\mathcal{M}(\mathbb{R}^d)$ to $\mathbb{R}^d$ such that $\psi(y,\rho,\nu)(dx)\nu(dy)=\rho(dx,dy)$ for all $\mu,\nu\in\mathcal{M}(\mathbb{R}^d)$ and $\rho\in\mathsf{Cpl}(\mu,\nu)$. Moreover, the kernel ψ is invariant in the following sense*

$$\psi(y,\rho,\nu)=\psi(y+z,\theta_z^{\mathcal{M}(\mathbb{R}^d)}\rho,\theta_z^{\mathcal{M}(\mathbb{R}^d)}\nu),\quad\forall y,z\in\mathbb{R}^d,\mu,\nu\in\mathcal{M}(\mathbb{R}^d),\rho\in\mathsf{Cpl}(\mu,\nu).\tag{3.4}$$

Here the measure $\theta_z^{\mathcal{M}(\mathbb{R}^d)}\rho$ is defined by the diagonal action

$$\theta_z^{\mathcal{M}(\mathbb{R}^d)}\rho(A\times B)=\rho((A+z)\times(B+z)),\quad\forall A,B\in\mathcal{B}(\mathbb{R}^d).$$

Theorem 3.6. *Let ξ and η be two jointly stationary and ergodic random measures with the same finite intensity. Assume that ξ does not charge small sets. Then, there exists a factor allocation.*

Proof: Via the Lebesgue decomposition theorem we can write $\eta = \eta^a + \eta^s$, where η^a is absolutely continuous w.r.t. ξ and the measures η^s and ξ are mutually singular.

By Lemma 3.1, there exists a function ϑ and an equivariant coupling $\tilde{q}$ of ξ and η s.t.

$$\mathbb{E}\left[\int_{\Lambda_1 \times \mathbb{R}^d} \vartheta(|x-y|)\tilde{q}(dx,dy)\right] < \infty.$$

Note that $\tilde{q}$ induces a semicoupling $\hat{q}$ between ξ and η^s by setting $\hat{q}(dx,dy) = \tilde{q}_y(dx)\eta^s(dy)$, where $\tilde{q}_y = \psi(y,\tilde{q},\eta)$ is the disintegration of $\tilde{q}$ w.r.t. η obtained by Proposition 3.5.

This semicoupling has finite mean transportation cost w.r.t. the function ϑ, since the cost is bounded by the above expectation. Since ξ and η^s are mutually singular, Theorem 2.1 yields an equivariant semicoupling q of ξ and η^s. Denote by $\tilde{\xi}$ the first marginal of q and by $\tilde{f} = \varphi(\cdot,\tilde{\xi},\xi)$ the density of $\tilde{\xi}$ w.r.t. ξ (see Proposition 3.3 for the definition of φ). Moreover, note that Theorem 2.1 shows that $\tilde{f}d\xi = \tilde{\xi} = \mathbb{1}_B d\xi$ for a random set $B \subset \mathbb{R}^d$. Hence, $\mathbb{P}$-a.s. $\tilde{f} = \mathbb{1}_B$ ξ-a.e. such that $\mathbb{P}$-a.s. $\mathbb{1}_{\tilde{f}>0} = \mathbb{1}_B$ ξ-a.e. This implies that setting $A = \{\tilde{f} > 0\}$ we have $1_A\xi = 1_B\xi = \tilde{\xi}$ $\mathbb{P}$-a.s. and 1_A is invariant (in the sense of Proposition 3.3) and measurable w.r.t. $\sigma(\xi,\eta)$.

Theorem 2.1 yields a factor allocation S_1 pushing $\tilde{\xi} = \mathbb{1}_A\xi$ onto η^s. Since necessarily $\mathbb{1}_{A^c}\xi$ and η^a have the same intensity and are both absolutely continuous w.r.t. ξ, by Proposition 3.4, we obtain an allocation S_2 pushing $\mathbb{1}_{A^c}\xi$ to η^a. Finally, setting $T = \mathbb{1}_A S_1 + \mathbb{1}_{A^c} S_2$ yields the desired factor allocation.

□

References

Aldous, D. J. and Thorisson, H. Shift-coupling. *Stochastic Process. Appl.*, **44** (1), 1–14 (1993). MR1198659.

Álvarez Esteban, P. C., del Barrio, E., Cuesta-Albertos, J. A., and Matrán, C. Uniqueness and approximate computation of optimal incomplete transportation plans. *Ann. Inst. Henri Poincaré Probab. Stat.*, **47** (2), 358–375 (2011). MR2814414.

Chatterjee, S., Peled, R., Peres, Y., and Romik, D. Gravitational allocation to Poisson points. *Ann. of Math. (2)*, **172** (1), 617–671 (2010). MR2680428.

Figalli, A. The Optimal Partial Transport Problem. *Archive for Rational Mechanics and Analysis*, **195** (2), 533–560 (2010). DOI: 10.1007/s00205-008-0212-7.

Haji-Mirsadeghi, M.-O. and Khezeli, A. Stable transports between stationary random measures. *Electron. J. Probab.*, **21**, Paper No. 51, 25 (2016). MR3539645.

Hoffman, C., Holroyd, A. E., and Peres, Y. A stable marriage of Poisson and Lebesgue. *Ann. Probab.*, **34** (4), 1241–1272 (2006). MR2257646.

Holroyd, A. E. and Peres, Y. Extra heads and invariant allocations. *Ann. Probab.*, **33** (1), 31–52 (2005). MR2118858.

Huesmann, M. Optimal transport between random measures. *Ann. Inst. Henri Poincaré Probab. Stat.*, **52** (1), 196–232 (2016). MR3449301.

Huesmann, M. and Sturm, K.-T. Optimal transport from Lebesgue to Poisson. *Ann. Probab.*, **41** (4), 2426–2478 (2013). MR3112922.

Kallenberg, O. *Random measures, theory and applications*, volume 77 of *Probability Theory and Stochastic Modelling*. Springer, Cham (2017). ISBN 978-3-319-41596-3; 978-3-319-41598-7. MR3642325.

Khezeli, A. and Mellick, S. On the existence of balancing allocations and factor point processes. *ArXiv Mathematics e-prints* (2023). arXiv: 2303.05137.

Last, G., Mörters, P., and Thorisson, H. Unbiased shifts of Brownian motion. *Ann. Probab.*, **42** (2), 431–463 (2014). MR3178463.

Last, G., Tang, W., and Thorisson, H. Transporting random measures on the line and embedding excursions into Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.*, **54** (4), 2286–2303 (2018). MR3865673.

Last, G. and Thorisson, H. Invariant transports of stationary random measures and mass-stationarity. *Ann. Probab.*, **37** (2), 790–813 (2009). MR2510024.

Last, G. and Thorisson, H. Transportation of diffuse random measures on $\mathbb{R}^d$. *ALEA Lat. Am. J. Probab. Math. Stat.*, **20** (1), 577–592 (2023). MR4567722.

Laurençot, P. Weak compactness techniques and coagulation equations. In *Evolutionary equations with applications in natural sciences*, volume 2126 of *Lecture Notes in Math.*, pp. 199–253. Springer, Cham (2015). MR3329324.

Liggett, T. M. Tagged particle distributions or how to choose a head at random. In *In and out of equilibrium (Mambucaba, 2000)*, volume 51 of *Progr. Probab.*, pp. 133–162. Birkhäuser Boston, Boston, MA (2002). MR1901951.

Pegon, P., Santambrogio, F., and Piazzoli, D. Full characterization of optimal transport plans for concave costs. *Discrete Contin. Dyn. Syst.*, **35** (12), 6113–6132 (2015). MR3393269.

Thorisson, H. Transforming random elements and shifting random fields. *Ann. Probab.*, **24** (4), 2057–2064 (1996). MR1415240.