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Bivariate gamma subordination for a Poisson shock model

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Abstract. The purpose of this paper is to introduce a bivariate Lévy process constructed by subordination of a Poisson process with independent components by a bidimensional subordinator with correlated Gamma margins. Under these assumptions, a new shock model involving competing risks is developed for evaluating the reliability of a system. Indeed, each of the two components of the new process counts the number of shocks of a certain type inflicted to the system, until failure, which occurs when a random threshold level is exceeded by the total number of solicitations. System's lifetime is explicitly computed for specific distributions of the threshold such as geometric and logarithmic distributions.

1. Introduction

Subordination of a multivariate Lévy process by an independent multivariate subordinator is a recent subject of investigation. A seminal paper, which establishes the theoretical set up on the topic, is the one by Barndorff-Nielsen et al. (2001). This extends the concept of subordination by composition of a multidimensional Lévy process with a common stochastic time-change. Models constructed in this way have been studied both theoretically and in applications. For instance, Beghin (2014) showed that an isotropic n-dimensional geometric stable process can be represented as the composition of an isotropic stable vector with an independent gamma subordinator, whilst Beghin and Macci (2016) considered a multivariate version of the space-fractional Poisson process. The multivariate variance gamma process was introduced by Madan and Seneta (1990). See also Luciano and Schoutens (2006) for applications in finance. In this case the external process is a multidimensional Brownian motion and the random clock is the gamma subordinator. The first attempt to construct a multivariate variance gamma process by means of a multivariate, instead of a single, subordinator is due to Semeraro (2008). In this way, the author defines a process such that its margins are variance gamma, but with the right choice of the parameters it is possible to model

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both independence and correlations. Multivariate subordinators generally appear in the financial literature. One of the most recent contributions in this direction is the paper by Semeraro (2022).

The aim of this paper is twofold. First, we contribute to the theory of random time-change with multivariate subordinators by introducing a new Lévy process related to the Poisson process with gamma subordinator (see Orsingher and Toaldo, 2015); the latter follows a negative binomial distribution which is is very popular in applications. Then, on this basis, we construct a competing risks model within the largest class of shock models, thus enriching the literature on survival analysis. For a detailed presentation of the variety of shock models that exist in the literature, the interested reader is referred to Chapters 3 and 4 of Cha and Finkelstein (2018), and references therein. Specifically, we consider a system or a living organism subject to two sequences of mutually exclusive shocks or, equivalently, to two external factors, which arrive according to a bivariate stochastic point process. The failure occurs, due to a single cause, whenever the sum of the shocks inflicted exceeds a certain random threshold. Such competing risks model was pioneered by Di Crescenzo and Longobardi (2008). Two generalizations have been recently proposed by Di Crescenzo and Meoli (2023) and by Soni et al. (2024), where shocks are exerted by two independent Poisson processes subordinated by a common stable and tempered stable process respectively. The process and the stochastic clocks are assumed to be independent in both cases. One reason for studying this type of construction is that it allows time inhomogeneity, stochastically altering the clock on which shocks occur. As a novelty, in our paper we generalize the case of a common random time for all the components by introducing a bivariate Lévy process constructed by subordination of a Poisson process with independent components by an independent bivariate gamma subordinator. In this way, the reliability assessment of the system, or the survival of the organism, depends on two shock patterns, say internal degradation and external environmental conditions, each accumulating according to a certain stochastic clock and each competing in the acceleration or in the slowdown of the degradation process.

The paper is organized as follows. In Section 2 we describe some preliminary results on competing risks within shock models and provide some new related results. Section 3 is split in two subsections. In subsection 3.1 we first introduce the gamma subordinator which we will refer to in the rest of the paper, then we obtain the differential equations satisfied by the density of the subordinator, both with respect to the spatial components and with respect to the temporal component. Interestingly enough, in the first case it is expressed by means of the shift operator. In subsection 3.2 we assess the reliability of the system. In particular, we derive the distribution of the process governing the arrival of the shocks, the hazard rates, the failure subdensities and the probability of failure due to one of the two competing causes. We conclude Section 3 with an example. Section 4 is devoted to the analysis of a special case. We specify the distribution of the treshold describing the handling capacity of the system and obtain an explicit expression for the survival function of the lifetime of the system. Indeed, of particular interest is the case of a geometric treshold, since the resulting distribution turns out to be independent of the specific subordinator.

2. Notation and preliminary results

We denote by T an absolutely continuous non-negative random variable which describes the failure time of a system subject to two types of shock. The arrivals of the shocks are governed by a bivariate counting process $\{\mathbf{N}(t)\}_{t\geq 0}$, where $\mathbf{N}(t) := (\mathscr{N}_1(t), \mathscr{N}_2(t))$, and $\mathscr{N}_i(t)$, $i \in \{1,2\}$, represents the number of shocks of type i occurring in $[0,t]$. The system deteriorates until it fails, due to a single cause, as soon as the sum of the shocks of both types reaches a random threshold M that is independent of $\mathbf{N}$ and that takes values in $\mathbb{N} := \{1, 2, \dots\}$. That is to say, the system can stand exactly $M-1$ deteriorating shocks. We fix notation as follows. Let C be the integer-valued random variable describing the cause of failure. Hence, we set $C = i$ if the shock of type i, $i = 1, 2$, is fatal. Denoting by $f_T(t)$, $t \geq 0$, the probability density function of the failure time (first-hitting time)

$$T = \inf\{t \geq 0 : \mathscr{N}_1(t) + \mathscr{N}_2(t) = M\},$$

we have

$$f_T(t) = f_1(t) + f_2(t), \qquad t \geq 0, \tag{2.1}$$

where $f_i(t)$ is the sub-density defined by

$$f_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}\{T \leq t,\, C = i\}, \qquad t \geq 0, \quad i = 1, 2. \tag{2.2}$$

Furthermore, the probability mass function of C reads

$$\mathrm{P}\{C = i\} = \int_0^{\infty} f_i(t)\,\mathrm{d}t, \qquad i = 1, 2. \tag{2.3}$$

Throughout the paper we use the notation $\boldsymbol{x} := (x_1, x_2)$ for bidimensional vectors of non-negative integers, i.e. $\boldsymbol{x} \in \mathbb{N}_0^2$, with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, we define the state probabilities $\left\{\left\{p_{\boldsymbol{x}}^{\mathbf{N}}(t) : \boldsymbol{x} \in \mathbb{N}_0^2\right\} : t \geq 0\right\}$ as

$$p_{\boldsymbol{x}}^{\mathbf{N}}(t) := \mathrm{P}\left\{\mathbf{N}(t) = \boldsymbol{x}\right\}.$$

In order to express the sub-densities $f_i(t)$, $i = 1, 2$, in terms of the joint probability distribution of $\mathbf{N}(t)$, we now introduce the hazard rates

$$\begin{aligned} r_1(\boldsymbol{x}; t) &= \lim_{\tau \to 0^+} \frac{\mathrm{P}\{\mathbf{N}(t+\tau) = (x_1 + 1,\, x_2) \mid \mathbf{N}(t) = \boldsymbol{x}\}}{\tau}, \\ r_2(\boldsymbol{x}; t) &= \lim_{\tau \to 0^+} \frac{\mathrm{P}\{\mathbf{N}(t+\tau) = (x_1,\, x_2 + 1) \mid \mathbf{N}(t) = \boldsymbol{x}\}}{\tau}, \end{aligned} \tag{2.4}$$

with $(x_1, x_2) \in \mathbb{N}_0^2$ and $t \geq 0$. Given that x_1 shocks of type 1 and x_2 shocks of type 2 occurred in $[0, t]$, the hazard rate $r_i(x_1, x_2; t)$ gives the intensity of the occurrence of a shock of type i immediately after t, with $i = 1, 2$. The probability distribution and the survival probability of M will be respectively denoted by

$$p_k = \mathrm{P}\left\{M = k\right\}, \qquad k \in \mathbb{N}, \tag{2.5}$$

and

$$\overline{P}_k = \mathrm{P}\left\{M > k\right\}, \qquad k \in \mathbb{N}_0. \tag{2.6}$$

Hence, conditioning on M and recalling (2.5) and (2.4), for $t \geq 0$ and $i = 1, 2$, the failure densities defined in (2.2) can be expressed as

$$f_i(t) = \sum_{k=1}^{+\infty} p_k \sum_{x_1 + x_2 = k-1} p_{\boldsymbol{x}}^{\mathbf{N}}(t)\; r_i(x_1, x_2; t). \tag{2.7}$$

A relation similar to (2.7) holds for the survival function of T, denoted by

$$\overline{F}_T(t) = \mathrm{P}\{T > t\}, \qquad t \geq 0.$$

Indeed, conditioning on $(N_1(t), N_2(t))$ and recalling (2.6), we obtain

$$\overline{F}_T(t) = \sum_{k=0}^{+\infty} \overline{P}_k \sum_{x_1 + x_2 = k} p_{\boldsymbol{x}}^{\mathbf{N}}(t), \qquad t \geq 0, \tag{2.8}$$

where $\overline{P}_0 = 1$.

In the following theorem we give an explicit expression for the probabilities in (2.3) and then we prove an interesting property of such failure scheme under the assumption that the hazard rates in Equation (2.4) are constant.

Theorem 2.1. *If the system failure is due to the sum of the shocks, and if both $r_1(x_1, x_2; t)$ and $r_2(x_1, x_2; t)$ do not depend either on the state space or on time, then* *a)* $\mathrm{P}\{C=i\} = \dfrac{r_i}{r_1+r_2}, \quad i=1,2;$

b) the time of failure T and the cause of failure C are independent,

where we have set $r_i(x_1,x_2;t) := r_i$, $i=1,2$.

Proof: Under the assumption of constancy of the hazard rates, the failure subdensities (2.7) become

$$f_i(t) = r_i \sum_{k=1}^{+\infty} p_k \sum_{x_1+x_2=k-1} p_{\boldsymbol{x}}^{\mathbf{N}}(t).$$

Hence, we observe that, from (2.1),

$$\begin{aligned} 1 &= \int_0^{+\infty} f_T(t)\,\mathrm{d}t \\ &= \int_0^{+\infty} (f_1(t) + f_2(t))\,\mathrm{d}t \\ &= (r_1+r_2) \int_0^{+\infty} \sum_{k=1}^{+\infty} p_k \sum_{x_1+x_2=k-1} p_{\boldsymbol{x}}^{\mathbf{N}}(t)\,\mathrm{d}t. \end{aligned}$$

Therefore,

$$\int_0^{+\infty} \sum_{k=1}^{+\infty} p_k \sum_{x_1+x_2=k-1} p_{\boldsymbol{x}}^{\mathbf{N}}(t)\,\mathrm{d}t = \frac{1}{r_1+r_2}$$

and, from (2.3),

$$\mathrm{P}\{C=i\} = r_i \int_0^{+\infty} \sum_{k=1}^{+\infty} p_k \sum_{x_1+x_2=k-1} p_{\boldsymbol{x}}^{\mathbf{N}}(t)\,\mathrm{d}t = \frac{r_i}{r_1+r_2}.$$

Identity *a)* is thus proved. As regards claim *b)*, some straightforward calculations show that the factorization $f_i(t) = \mathrm{P}\{C=i\} f_T(t)$ holds, which yields the desired result. □

We conclude this section by recalling some useful special functions which we will refer to in the rest of the paper. We start with the Gaussian hypergeometric function (cf. Equation (17) in Section 1.2 of Srivastava and Karlsson, 1985) which is defined by

$$_2F_1(a,b;c;z) := \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{2.9}$$

where

$$(x)_k := \begin{cases} x(x+1)\dots(x+k-1) & \text{if } k \geq 1 \\ 1 & \text{if } k=0 \end{cases}$$

is the rising factorial, also called Pochhammer symbol. In terms of gamma functions, we have

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}, \qquad x \neq 0, -1, -2, \dots \tag{2.10}$$

Furthermore, the generalized binomial coefficient may now be expressed, for $k \in \mathbb{N}$ and arbitrary α, as

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} = \frac{(-1)^k (-\alpha)_k}{k!},$$

so that the generalized binomial identity reads

$$(1+x)^\alpha = \sum_{i=0}^{+\infty} \frac{(-\alpha)_i (-x)^i}{i!}. \tag{2.11}$$

A useful summation formula is the following one (cf. 6.8.1.17 of Brychkov et al., 1986):

$$\sum_{k=0}^{+\infty} \frac{(\alpha)_k}{k!} t^k \, {}_2F_1 \left(a, \alpha + k; b; x\right) = (1-t)^{-\alpha} \, {}_2F_1 \left(a, \alpha; b; (1-t)^{-1} x\right), \qquad |t| < 1, \, |x| < 1. \tag{2.12}$$

We also recall the *Appell series* F_1 defined by (see, e.g. Equation (2) in Section 1.3 of Srivastava and Karlsson, 1985)

$$F_1 \left(a, b, b'; c; x, y\right) = \sum_{m,n=0}^{+\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad \max\{|x|, |y|\} < 1. \tag{2.13}$$

3. The subordinator and properties of the model

In this section we specialize the counting process describing the arrival of the shocks and investigate in detail all the reliability characteristics of the associated model. We subordinate a bivariate Poisson process with independent components by an independent bivariate subordinator with gamma margins. To begin with, we focus on the description of the latter.

3.1. *The subordinator.* Let $\{\mathbf{T}(t) : t \geq 0\}$ be the bidimensional gamma subordinator discussed in Example 2.2 of Barndorff-Nielsen et al. (2001). We recall hereafter its construction. Fix $\lambda > 0$ and let $\{Z_1(t) : t \geq 0\}$ and $\{Z_2(t) : t \geq 0\}$ be two independent gamma Lévy processes with $Z_i(t)$ having gamma law $\Gamma(\lambda t, a)$ with shape parameter λt and rate parameter $a > 0$. Define $\{T_i(t) : t \geq 0\}$ by

$$T_i(t) = Z_i\left(t + \lambda^{-1} B^-(\lambda t)\right),$$

where $\{B^-(t) : t \geq 0\}$ is a negative binomial Lévy process with parameter $\theta \in (0,1)$. For a review on the stochastic mechanisms leading to the negative binomial process and other properties see Kozubowski and Podgórski (2009). Then, $\{T_i(t)\}_{t\geq 0}$ is again a gamma process with law $\Gamma(\lambda t, (1-\theta) a)$, and $\mathbf{T}(t) := (T_1(t), T_2(t))$ is a 2-dimensional *gamma subordinator*. The probability density of $\{\mathbf{T}(t)\}_{t\geq 0}$ can be expressed for $x_1, x_2 > 0$ as

$$f_t(x_1, x_2) = (1-\theta)^{\lambda t} \theta^{(-\lambda t+1)/2} a^{\lambda t+1} \frac{1}{\Gamma(\lambda t)} (x_1 x_2)^{(\lambda t-1)/2} e^{-a(x_1+x_2)} I_{\lambda t-1}\left(2a\sqrt{\theta x_1 x_2}\right), \tag{3.1}$$

where $\Gamma(x)$ is the gamma function defined as $\Gamma(x) = \int_0^{+\infty} u^{x-1} e^{-u} \mathrm{d}u$ and, for a real number γ, $I_\gamma(\cdot)$ is the modified Bessel function of the first kind with order γ:

$$I_\gamma(x) = \left(\frac{x}{2}\right)^\gamma \sum_{k=0}^{+\infty} \frac{\left(\frac{x^2}{4}\right)^k}{k!\Gamma(\gamma+k+1)}. \tag{3.2}$$

The correlation coefficient between $T_1(t)$ and $T_2(t)$ turns out to be θ. We remark that the density (3.1) generalises Kibble's bivariate gamma distribution, which has has been used in several research areas. For example, Phatarfod (1976) used this distribution as a model to describe summer and winter streamflows, whilst Chatelain et al. (2007) studied applications to image registration and change detection. The joint Laplace–Stieltjes transform of the bivariate gamma subordinator is given by

$$\mathrm{E}\left[e^{-\eta_1 T_1(t) - \eta_2 T_2(t)}\right] = \left(1 + \frac{a\eta_1 + a\eta_2 + \eta_1\eta_2}{(1-\theta)a^2}\right)^{-\lambda t} \tag{3.3}$$

and, by comparing the latter with Equation (5.11) in Theorem 5.2 of Torricelli et al. (2022), the process $\{\mathbf{T}(t)\}_{t\geq 0}$ turns out to be a special case of a bivariate Lévy process with correlated tempered positive Linnik (tpL) marginals. That same Theorem gives insight into the Lévy structure of the subordinator: it has zero drift and its Lévy measure decomposes in an independent multivariate tpL measure plus a combinatorial expression of one-dimensional tpL Lévy measures depending on the parameter accounting for the dependence across the marginals. From (3.3), it is an easy task to compute the Fourier transform of $\{\mathbf{T}(t)\}_{t\geq 0}$, for $\mathrm{i}=\sqrt{-1}$, as

$$\mathcal{F}[f_t(x_1,x_2);(\alpha,\beta)] = \mathrm{E}[e^{\mathrm{i}(\alpha T_1(t)+\beta T_2(t))}] = \left(1-\frac{a^2-(a-\mathrm{i}\alpha)(a-\mathrm{i}\beta)}{(1-\theta)a^2}\right)^{-\lambda t}. \tag{3.4}$$

In order to derive the governing equation of the transition density (3.1), we resort to the shift operator, defined as

$$e^{-k\partial_t} f(t) := \sum_{n=0}^{+\infty}\frac{(-k\partial_t)^n}{n!} f(t) = f(t-k), \qquad k\in\mathbb{R}, \tag{3.5}$$

where $f:\mathbb{R}\to\mathbb{R}$ is any analytical function.

Theorem 3.1. *The density of the bivariate gamma subordinator (3.1) satisfies the following equation, for x_1, $x_2>0$ and $t\geq 0$:*

$$\frac{\partial^2}{\partial x_1\,\partial x_2} f_t(x_1,x_2) + a\frac{\partial}{\partial x_1} f_t(x_1,x_2) + a\frac{\partial}{\partial x_2} f_t(x_1,x_2) = -(1-\theta)a^2\left(1-e^{-\frac{1}{\lambda}\partial_t}\right) f_t(x_1,x_2), \tag{3.6}$$

where e^{∂_t} is the partial derivative version of the shift operator defined in (3.5) for $k=1/\lambda$. The initial and boundary conditions are the following:

$$\begin{cases} f_0(x_1,x_2) = \delta(x_1,x_2) \\ \lim_{\|\mathbf{x}\|\to+\infty} f_t(x_1,x_2) = 0, \end{cases} \tag{3.7}$$

where $\|\cdot\|$ denotes the Euclidean norm.

Proof: The first condition in (3.7) is a consequence of (3.4) and of the definition of the delta function:

$$\begin{aligned}\delta(x_1,x_2) = \delta(x_1)\delta(x_2) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}\theta_1 x_1}\mathrm{d}\theta_1\cdot\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}\theta_2 x_2}\mathrm{d}\theta_2 \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-\mathrm{i}(\theta_1 x_1+\theta_2 x_2)}\mathrm{d}\theta_1\mathrm{d}\theta_2.\end{aligned}$$

The second one in (3.7) is immediately satisfied by (3.1). Let us now compute the Fourier transform of the left-hand side of Equation (3.6) and show that it coincides with the Fourier transform of the right-hand side. Due to formula 1.3.29 of Kilbas et al. (2006) and to (3.4), we have

$$\begin{aligned}&\mathcal{F}\left[\frac{\partial^2}{\partial x_1\,\partial x_2} f_t(x_1,x_2) + a\frac{\partial}{\partial x_1} f_t(x_1,x_2) + a\frac{\partial}{\partial x_2} f_t(x_1,x_2);(\alpha,\beta)\right] \\ &= [(-\mathrm{i}\alpha)(-\mathrm{i}\beta) + a(-\mathrm{i}\alpha) + a(-\mathrm{i}\beta)]\,\mathcal{F}[f_t(x_1,x_2);(\alpha,\beta)] \\ &= [(-\mathrm{i}\alpha)(-\mathrm{i}\beta) + a(-\mathrm{i}\alpha) + a(-\mathrm{i}\beta)]\left(1-\frac{a^2-(a-\mathrm{i}\alpha)(a-\mathrm{i}\beta)}{(1-\theta)a^2}\right)^{-\lambda t} \\ &= -(\alpha\beta+\mathrm{i}\alpha a+\mathrm{i}\beta a)\left(\frac{(1-\theta)a^2}{(1-\theta)a^2-\mathrm{i}\alpha a-\mathrm{i}\beta a-\alpha\beta}\right)^{\lambda t}.\end{aligned}$$

The above expression coincides with the Fourier transform of the right-hand side of Equation (3.6). Indeed

$$\begin{aligned}-(1-\theta)a^2\left(1-e^{-\frac{1}{\lambda}\partial_t}\right)\mathcal{F}[f_t(x_1,x_2);(\alpha,\beta)] &= -(1-\theta)a^2\left(\frac{(1-\theta)a^2}{(1-\theta)a^2-\mathrm{i}\alpha a-\mathrm{i}\beta a-\alpha\beta}\right)^{\lambda t} \\ &\quad + (1-\theta)a^2\left(\frac{(1-\theta)a^2}{(1-\theta)a^2-\mathrm{i}\alpha a-\mathrm{i}\beta a-\alpha\beta}\right)^{\lambda t-1}\end{aligned}$$

$$= -(1-\theta)\,a^2\left(\frac{(1-\theta)\,a^2}{(1-\theta)\,a^2-\mathrm{i}\alpha a-\mathrm{i}\beta a-\alpha\beta}\right)^{\lambda t} \times\left(1-\frac{(1-\theta)\,a^2-\mathrm{i}\alpha a-\mathrm{i}\beta a-\alpha\beta}{(1-\theta)\,a^2}\right).$$

The proof is thus complete. □

Theorem 3.1 extends Lemma 2.1 in Beghin (2014), where the partial differential equation satisfied by the transition density of the univariate gamma subordinator is established. An alternative differential equation which involves a standard time derivative can be obtained by considering a differential operator $\mathcal{P}_{k,q}$, $k,q>0$, which acts on any given infinitely differentiable function f as follows:

$$\mathcal{P}_{k,q}f\,(x_1,x_2) := \sum_{l=1}^{+\infty}\frac{1}{lk^l}\sum_{h=0}^{l}\binom{l}{h}q^{2h}\,(-1)^{l-h}\sum_{j=0}^{l-h}\binom{l-h}{j}q^j\sum_{k=0}^{l-h}\binom{l-h}{k}q^k D_{x_1,x_2}^{(l-h-j,\,l-h-k)}f\,(x_1,x_2)\,, \tag{3.8}$$

for $x_1,x_2>0$. We point out that $D_{x_1,x_2}^{(m,n)}$ stands for $\partial^{m+n}/\partial x_1^m\partial x_2^n$.

Theorem 3.2. *The density of the bivariate gamma subordinator (3.1) satisfies the following equation, for x_1, $x_2>0$ and $t\geq 0$:*

$$\frac{\partial}{\partial t}f_t\,(x_1,x_2) = \lambda\mathcal{P}_{(1-\theta)a^2,a^2}f_t\,(x_1,x_2)\,, \tag{3.9}$$

where $\mathcal{P}_{k,q}$ is the differential operator defined in (3.8) for $k=(1-\theta)\,a^2$ and $q=a^2$. The initial and boundary conditions are the following:

$$\begin{cases} f_0\,(x_1,x_2) = \delta\,(x_1,x_2) \\ \lim\limits_{\|\mathbf{x}\|\to+\infty} D_{x_1,x_2}^{(m,n)}f_t\,(x_1,x_2) = 0, \quad m,n\in\mathbb{N}_0. \end{cases} \tag{3.10}$$

Proof: Conditions (3.10) are immediately verified by f_t. By taking the Fourier transform of the left-hand side of (3.9), recalling (3.4) and the Taylor expansion of the logarithm, we have

$$\begin{aligned}
\mathcal{F}\left[\frac{\partial}{\partial t}f_t\,(x_1,x_2)\,;(\alpha,\beta)\right] &= \frac{\partial}{\partial t}\left(1-\frac{a^2-(a-\mathrm{i}\alpha)\,(a-\mathrm{i}\beta)}{(1-\theta)\,a^2}\right)^{-\lambda t} \\
&= -\lambda\left(1-\frac{a^2-(a-\mathrm{i}\alpha)\,(a-\mathrm{i}\beta)}{(1-\theta)\,a^2}\right)^{-\lambda t}\ln\left(1-\frac{a^2-(a-\mathrm{i}\alpha)\,(a-\mathrm{i}\beta)}{(1-\theta)\,a^2}\right) \\
&= \lambda\left(1-\frac{a^2-(a-\mathrm{i}\alpha)\,(a-\mathrm{i}\beta)}{(1-\theta)\,a^2}\right)^{-\lambda t}\sum_{l=1}^{+\infty}\left(\frac{a^2-(a-\mathrm{i}\alpha)\,(a-\mathrm{i}\beta)}{(1-\theta)\,a^2}\right)^{l}\frac{1}{l} \\
&= \lambda\left(1-\frac{a^2-(a-\mathrm{i}\alpha)\,(a-\mathrm{i}\beta)}{(1-\pi)\,a^2}\right)^{-\lambda t}\sum_{l=1}^{+\infty}\frac{1}{l\,(1-\pi)^l\,a^{2l}} \\
&\times\sum_{h=0}^{l}\binom{l}{h}a^{2h}\,(-1)^{l-h}\,(a-\mathrm{i}\alpha)^{l-h}\,(a-\mathrm{i}\beta)^{l-h} \\
&= \lambda\left(1-\frac{a^2-(a-\mathrm{i}\alpha)\,(a-\mathrm{i}\beta)}{(1-\pi)\,a^2}\right)^{-\lambda t}\sum_{l=1}^{+\infty}\frac{1}{l\,(1-\pi)^l\,a^{2l}}\sum_{h=0}^{l}\binom{l}{h}a^{2h}\,(-1)^{l-h} \\
&\times\sum_{j=0}^{l-h}\binom{l-h}{j}a^j\,(-\mathrm{i}\alpha)^{l-h-j}\sum_{k=0}^{l-h}\binom{l-h}{k}a^k\,(-\mathrm{i}\beta)^{l-h-k}
\end{aligned}$$

$$= \lambda \sum_{l=1}^{+\infty} \frac{1}{l(1-\pi)^l a^{2l}} \sum_{h=0}^{l} \binom{l}{h} a^{2h} (-1)^{l-h} \sum_{j=0}^{l-h} \binom{l-h}{j} a^j \sum_{k=0}^{l-h} \binom{l-h}{k} a^k$$
$$\times \left(-\mathrm{i}(\alpha,\beta)\right)^{(l-h-j,\, l-h-k)} \mathcal{F}\left[f_t(x_1,x_2);(\alpha,\beta)\right]$$
$$= \lambda \sum_{l=1}^{+\infty} \frac{1}{l(1-\pi)^l a^{2l}} \sum_{h=0}^{l} \binom{l}{h} a^{2h} (-1)^{l-h} \sum_{j=0}^{l-h} \binom{l-h}{j} a^j \sum_{k=0}^{l-h} \binom{l-h}{k} a^k$$
$$\times \mathcal{F}\left[D_{x_1,x_2}^{(l-h-j,\, l-h-k)} f_t(x_1,x_2);(\alpha,\beta)\right].$$

Equation (3.9) thus follows from (3.8). □

Theorem 3.2 provides a particular case of the equation governing bivariate subordinators. Indeed, if $\{H(t)\}_{t\geq 0}$, is a bivariate subordinator such that $H(t) := \{(H_1(t), H_2(t)), t \geq 0\}$, with joint density $q_t(x_1,x_2)$ and Lévy measure $\phi(\mathrm{d}x_1,\mathrm{d}x_2)$, then (cf. Meerschaert and Sikorskii, 2019),

$$\frac{\partial}{\partial t} q_t(x_1,x_2) = -\mathcal{D}_{x_1,x_2} q_t(x_1,x_2)$$

where

$$\mathcal{D}_{x_1,x_2} h(x_1,x_2) := \int\int_{\mathbb{R}_+^2} \left(h(x_1,x_2) - h(x_1-y_1, x_2-y_2)\right) \phi(\mathrm{d}y_1,\mathrm{d}y_2)$$

on a suitable class of functions h. A more recent result regarding the equation for the biparameter process $(H_1(t_1), H_2(t_2))$ can be found in Beghin et al. (2020).

3.2. *Reliability characteristics of the model.* From now on, we focus our attention again on the survival model. To begin with, we define the underlying arrival point process.

Definition 3.3. Let $\{\{N_i(t) : t \geq 0\} : i \in \{1,2\}\}$ be two independent Poisson processes with intensities λ_1, $\lambda_2 > 0$, respectively, and set

$$N(t) := (N_1(t), N_2(t)).$$

Then, we consider the bivariate process $\{\mathbf{Y}(t) : t \geq 0\}$ defined by:

$$\mathbf{Y}(t) = (Y_1(t), Y_2(t)) := (N_1(T_1(t)), N_2(T_2(t))),$$

where $\{T_1(t)\}_{t\geq 0}$ and $\{T_2(t)\}_{t\geq 0}$ are the components of the gamma subordinator $\{\mathbf{T}(t)\}_{t\geq 0}$, which is assumed to be independent of $\{N(t)\}_{t\geq 0}$.

The process $\{\mathbf{Y}(t)\}_{t\geq 0}$ is itself a Lévy process, as outlined in Barndorff-Nielsen et al. (2001). The following result provides the joint distribution of the number of events at a certain time.

Theorem 3.4. *For all integers k_1, $k_2 \geq 0$, the state probabilities of $\{\mathbf{Y}(t)\}_{t\geq 0}$ read*

$$p_{\boldsymbol{k}}^{\mathbf{Y}}(t) = \left(\frac{\lambda_1}{\lambda_1+a}\right)^{k_1} \left(\frac{\lambda_2}{\lambda_2+a}\right)^{k_2} \frac{(\lambda t)_{k_1}}{k_1!} \frac{(\lambda t)_{k_2}}{k_2!} \left(\frac{(1-\theta)a^2}{(\lambda_1+a)(\lambda_2+a)}\right)^{\lambda t}$$
$$\times {}_2F_1\left(k_1+\lambda t, k_2+\lambda t; \lambda t; \frac{\theta\alpha^2}{(\lambda_1+a)(\lambda_2+a)}\right). \tag{3.11}$$

Proof: Fix $t \geq 0$. By construction, we have

$$p_{\boldsymbol{k}}^{\mathbf{Y}}(t) = \int_0^{+\infty}\int_0^{+\infty} \mathrm{P}\{N_1(s) = k_1,\ N_2(y) = k_2\} f_t(s,y)\,\mathrm{d}s\,\mathrm{d}y$$
$$\overset{by\ (3.1)}{=} \frac{\lambda_1^{k_1}}{k_1!} \frac{\lambda_1^{k_2}}{k_2!} \frac{(1-\theta)^{\lambda t} \theta^{\frac{-\lambda t+1}{2}} a^{\lambda t+1}}{\Gamma(\lambda t)}$$

$$\times \int_0^{+\infty} \int_0^{+\infty} e^{-(\lambda_1+a)s-(\lambda_2+a)y} s^{k_1+\frac{\lambda t-1}{2}} y^{k_2+\frac{\lambda t-1}{2}} I_{\lambda t-1}\left(2a\sqrt{\theta sy}\right) \mathrm{d}s\, \mathrm{d}y$$

$$\overset{by\ (3.2)}{=} \frac{\lambda_1^{k_1}}{k_1!} \frac{\lambda_1^{k_2}}{k_2!} \frac{(1-\theta)^{\lambda t} a^{2\lambda t}}{\Gamma(\lambda t)} \sum_{h=0}^{+\infty} \frac{\left(a^2\theta\right)^h}{h!\,\Gamma(\lambda t+h)}$$

$$\times \int_0^{+\infty} s^{k_1+h+\lambda t-1} e^{-(\lambda_1+a)s} \mathrm{d}s \int_0^{+\infty} y^{k_2+h+\lambda t-1} e^{-(\lambda_2+a)s} \mathrm{d}s.$$

Taking into account the definition of the gamma function, we have

$$p_{\boldsymbol{k}}^{\mathbf{Y}}(t) = \frac{1}{k_1!\,k_2!\,\Gamma(\lambda t)} \left(\frac{\lambda_1}{\lambda_1+a}\right)^{k_1} \left(\frac{\lambda_2}{\lambda_2+a}\right)^{k_2} \left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)(\lambda_2+a)}\right)^{\lambda t}$$

$$\times \sum_{h=0}^{+\infty} \frac{\Gamma(k_1+\lambda t+h)\,\Gamma(k_2+\lambda t+h)}{h!\,\Gamma(\lambda t+h)} \left(\frac{\theta a^2}{(\lambda_1+a)(\lambda_2+a)}\right)^h.$$

We conclude the proof recalling the definition of the Gaussian hypergeometric function in Equation (2.9). □

In Fig. 3.1 we show various plots of the state probabilities (3.11) as time varies for some fixed values of the parameters.

Theorem 3.5. *For all* $(x_1, x_2) \in \mathbb{N}_0^2$ *and* $t \geq 0$, *the hazard rates in Equation* (2.4) *read:*

$$r_1(x_1, x_2; t) = \frac{\lambda\lambda_1(\lambda_2+a)}{(\lambda_1+a)(\lambda_2+a)-a^2\theta}; \tag{3.12}$$

$$r_2(x_1, x_2; t) = \frac{\lambda\lambda_2(\lambda_1+a)}{(\lambda_1+a)(\lambda_2+a)-a^2\theta}.$$

Proof: In the following, we show how to compute $r_1(x_1, x_2; t)$. We notice that the first Equation in (2.4) is equivalent to

$$\lim_{\tau\to t} \frac{\mathrm{P}\{Y_1(\tau) = x_1+1,\, Y_2(\tau) = x_2,\, Y_1(t) = x_1,\, Y_2(t) = x_2\}}{(\tau-t)\, p_{\boldsymbol{x}}^{\mathbf{Y}}(t)}. \tag{3.13}$$

We focus now on the numerator. By a conditioning argument, it is:

$$\begin{aligned}
&\mathrm{P}\{Y_1(\tau) = x_1+1,\, Y_2(\tau) = x_2,\, Y_1(t) = x_1,\, Y_2(t) = x_2\} \\
&= \int_0^{+\infty}\int_0^{u}\int_0^{+\infty}\int_0^{v} \mathrm{P}\{N_1(u) = x_1+1,\, N_2(v) = x_2,\, N_1(w) = x_1,\, N_2(z) = x_2\} \\
&\times \mathrm{P}\{T_1(\tau)\in \mathrm{d}u,\, T_2(\tau)\in \mathrm{d}v,\, T_1(t)\in \mathrm{d}w,\, T_2(t)\in \mathrm{d}z\} \\
&= \int_0^{+\infty}\int_0^{u}\int_0^{+\infty}\int_0^{v} \mathrm{P}\{N_1(u) = x_1+1,\, N_2(v) = x_2,\, N_1(w) = x_1,\, N_2(z) = x_2\} \\
&\times f_{\tau-t}(u-w, v-z)\, f_t(w,z)\, \mathrm{d}u\, \mathrm{d}w\, \mathrm{d}v\, \mathrm{d}z,
\end{aligned} \tag{3.14}$$

where the last equality follows from the fact that $\{\mathbf{T}(t)\}$ is a Lévy process. Moreover, since $\{N_1(t)\}$ and $\{N_2(t)\}$ are independent, we have

$$\begin{aligned}
&\mathrm{P}\{N_1(u) = x_1+1,\, N_2(v) = x_2,\, N_1(w) = x_1,\, N_2(z) = x_2\} \\
&= \mathrm{P}\{N_1(u-w) = 1,\, N_2(v-z) = 0\}\, \mathrm{P}\{N_1(w) = x_1,\, N_2(z) = x_2\} \\
&= \frac{e^{-\lambda_1 u-\lambda_2 v}\lambda_1^{x_1+1} w^{x_1}(\lambda_2 z)^{x_2}(u-w)}{x_1!\,x_2!}.
\end{aligned}$$

We put the latter in (3.14) and get

$$\mathrm{P}\{Y_1(\tau) = x_1+1,\, Y_2(\tau) = x_2,\, Y_1(t) = x_1,\, Y_2(t) = x_2\}$$

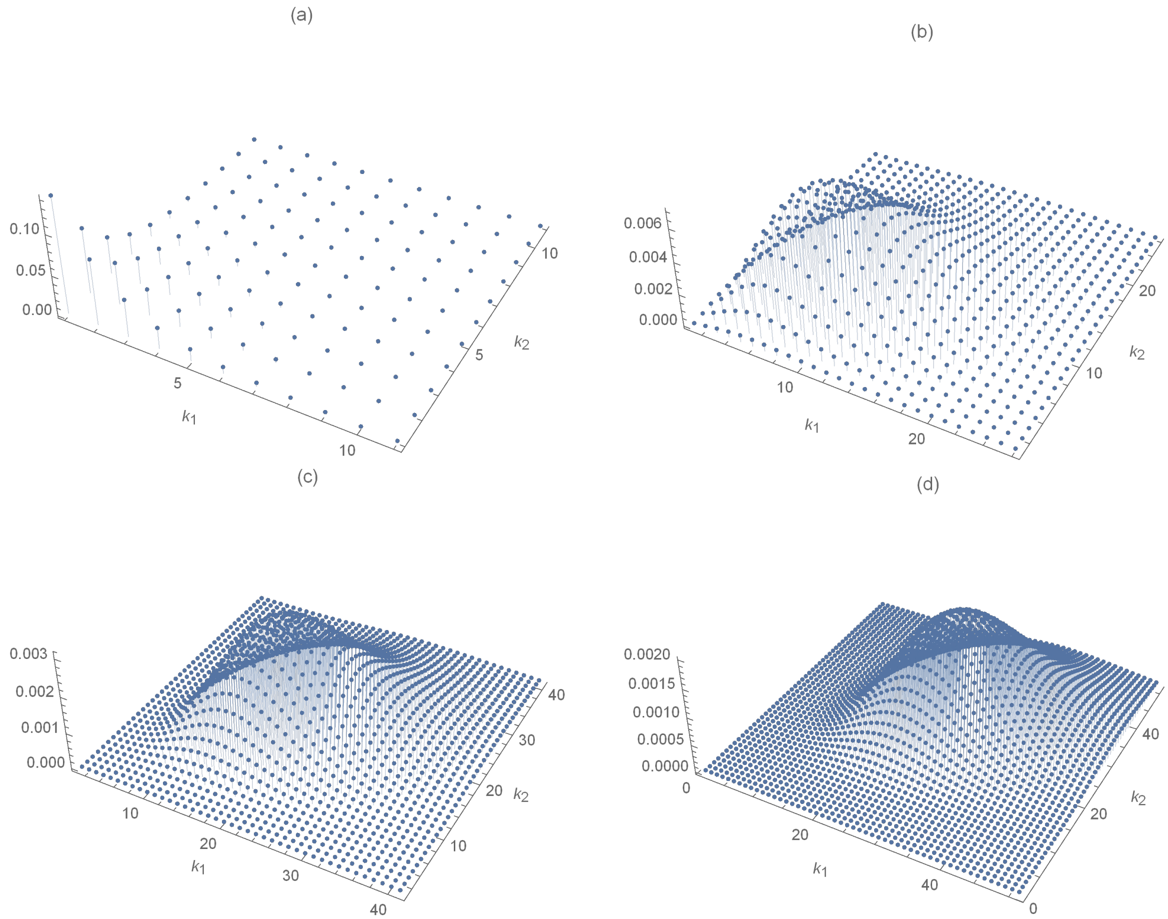

FIGURE 3.1. Plots of the probability mass function of the process $\mathbf{Y}(t)$, with $\lambda_1 = \lambda_2 = \lambda = \alpha = 1$, $\theta = 0.5$ and: (a) $t = 1$, (b) $t = 5$, (c) $t = 10$, (d) $t = 15$. The displayed probability masses are respectively: 0.978375, 0.976804, 0.975814, 0.950374.

$$
\begin{aligned}
&= \frac{\lambda_1^{x_1+1}}{x_1!} \frac{\lambda_2^{x_2}}{x_2!} \frac{\left((1-\theta)\, a^2\right)^{\lambda\tau}}{\Gamma\left(\lambda\left(\tau-t\right)\right)\Gamma\left(\lambda t\right)} \sum_{i=0}^{+\infty} \frac{\left(a^2\theta\right)^i}{i!\,\Gamma\left(\lambda\left(\tau-t\right)+i\right)} \sum_{j=0}^{+\infty} \frac{\left(a^2\theta\right)^j}{j!\,\Gamma\left(\lambda t+j\right)} \\
&\quad \times \int_0^{+\infty} e^{-(\lambda_1+a)u} \mathrm{d}u \int_0^u (u-w)^{\lambda(\tau-t)+i}\, w^{x_1+\lambda t-1+j} \mathrm{d}w \\
&\quad \times \int_0^{+\infty} e^{-(\lambda_2+a)v} \mathrm{d}v \int_0^v (v-z)^{\lambda(\tau-t)-1+i}\, z^{x_2+\lambda t-1+j} \mathrm{d}z.
\end{aligned}
$$

Due to 3.191.1 of Gradshteyn and Ryzhik (2014) and to the definition of the gamma function, the above equality is equivalent to

$$
\begin{aligned}
&\mathrm{P}\left\{Y_1(\tau) = x_1+1,\, Y_2(\tau) = x_2,\, Y_1(t) = x_1,\, Y_2(t) = x_2\right\} \\
&= \frac{\lambda_1^{x_1+1}}{x_1!} \frac{\lambda_2^{x_2}}{x_2!} \frac{\left((1-\theta)\, a^2\right)^{\lambda\tau}}{\Gamma\left(\lambda t\right)} \sum_{i=0}^{+\infty} \frac{\left(a^2\theta\right)^i \left(\lambda\left(\tau-t\right)\right)_{i+1}}{i!} \sum_{j=0}^{+\infty} \frac{\left(a^2\theta\right)^j}{j!\,\Gamma\left(\lambda t+j\right)} \\
&\quad \times \left(\lambda_1+a\right)^{-1-\lambda\tau-i-x_1-j} \Gamma\left(x_1+\lambda t+j\right) \left(\lambda_2+a\right)^{-\lambda\tau-i-x_2-j} \Gamma\left(x_2+\lambda t+j\right).
\end{aligned}
$$

A rearrangement of the terms yields

$$
\begin{aligned}
&\mathrm{P}\left\{Y_1(\tau)=x_1+1,\, Y_2(\tau)=x_2,\, Y_1(t)=x_1,\, Y_2(t)=x_2\right\}\\
&=\frac{\lambda_1^{x_1+1}}{x_1!}\frac{\lambda_2^{x_2}}{x_2!}\frac{\left((1-\theta)\,a^2\right)^{\lambda\tau}(\lambda_1+a)^{-1-\lambda\tau-x_1}(\lambda_2+a)^{-\lambda\tau-x_2}}{\Gamma(\lambda t)}\\
&\quad\times\sum_{i=0}^{+\infty}\left(\frac{a^2\theta}{(\lambda_1+a)(\lambda_2+a)}\right)^i\frac{(\lambda(\tau-t))_{i+1}}{i!}\\
&\quad\times\sum_{j=0}^{+\infty}\left(\frac{a^2\theta}{(\lambda_1+a)(\lambda_2+a)}\right)^j\frac{\Gamma(x_1+\lambda t+j)\,\Gamma(x_2+\lambda t+j)}{j!\,\Gamma(\lambda t+j)}\\
&\overset{\text{by (2.9) and (2.10)}}{=}\frac{\lambda_1^{x_1+1}}{x_1!}\frac{\lambda_2^{x_2}}{x_2!}\left((1-\theta)\,a^2\right)^{\lambda\tau}\frac{(\lambda t)_{x_1}}{(\lambda_1+a)^{1+\lambda\tau+x_1}}\frac{(\lambda t)_{x_2}}{(\lambda_2+a)^{\lambda\tau+x_2}}\lambda(\tau-t)\\
&\quad\times{}_2F_1\left(x_1+\lambda t,x_2+\lambda t;\lambda t;\frac{a^2\theta}{(\lambda_1+a)(\lambda_2+a)}\right)\\
&\quad\times\sum_{i=0}^{+\infty}\left(\frac{a^2\theta}{(\lambda_1+a)(\lambda_2+a)}\right)^i\frac{(\lambda(\tau-t)+1)_i}{i!}\\
&=\frac{\lambda_1^{x_1+1}}{x_1!}\frac{\lambda_2^{x_2}}{x_2!}\left((1-\theta)\,a^2\right)^{\lambda\tau}\frac{(\lambda t)_{x_1}}{(\lambda_1+a)^{1+\lambda\tau+x_1}}\frac{(\lambda t)_{x_2}}{(\lambda_2+a)^{\lambda\tau+x_2}}\lambda(\tau-t)\\
&\quad\times\left(1-\frac{a^2\theta}{(\lambda_1+a)(\lambda_2+a)}\right)^{-\lambda(\tau-t)-1}{}_2F_1\left(x_1+\lambda t,x_2+\lambda t;\lambda t;\frac{a^2\theta}{(\lambda_1+a)(\lambda_2+a)}\right),
\end{aligned}
$$

where the last equality follows from (2.11). By putting the latter and formula (3.11) into (3.13), Equation (3.12) can be easily checked. The expression for $r_2(x_1,x_2;t)$ can be proved analogously. □

Corollary 3.6. *Theorem 3.5 shows, in particular, that, for the proposed model, the hazard rates are constant. This implies that:*

a) *taking into account (3.11), the failure subdensities (2.7) can be expressed, for $i=1,2$ and $t\geq 0$, as*

$$
\begin{aligned}
f_i(t)=r_i&\left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)(\lambda_2+a)}\right)^{\lambda t}\sum_{k=1}^{+\infty}p_k\\
&\times\sum_{x_1=0}^{k-1}\frac{(\lambda t)_{x_1}}{x_1!}\frac{(\lambda t)_{k-x_1-1}}{(k-x_1-1)!}\left(\frac{\lambda_1}{\lambda_1+a}\right)^{x_1}\left(\frac{\lambda_2}{\lambda_2+a}\right)^{k-x_1-1}\\
&\times{}_2F_1\left(x_1+\lambda t,k-x_1-1+\lambda t;\lambda t;\frac{a^2\theta}{(\lambda_1+a)(\lambda_2+a)}\right);
\end{aligned}
$$

b) *due to Theorem 2.1, the cause of failure has the following distribution:*

$$
\mathrm{P}\{C=i\}=\frac{\lambda\lambda_1\lambda_2+\lambda\lambda_i a}{\lambda\lambda_1(\lambda_2+a)+\lambda\lambda_2(\lambda_1+a)},\quad i=1,2;
$$

c) *due to Theorem 2.1, the time and the cause of failure are independent.*

We remark that the issue of independence of time and cause has been recently addressed by Kella (2024).

Example 3.7. Hereafter, we suppose that the threshold M has a logarithmic distribution with parameter p, $0<p<1$. Therefore,

$$\overline{P_k} = -\frac{B\left(p; k+1, 0\right)}{\ln\left(1-p\right)},$$

where $B\left(x; a, b\right) = \int_0^x u^{a-1}\left(1-u\right)^{b-1} \mathrm{d}u$ is the incomplete beta function, and

$$p_k = \frac{-1}{\ln\left(1-p\right)} \frac{p^k}{k}. \tag{3.15}$$

The survival function of T in this case reads

$$\begin{aligned} \overline{F}_T\left(t\right) = & -\frac{\left[\left(1-\theta\right)a^2\right]^{\lambda t}}{\ln\left(1-p\right)} \left\{ -\frac{\left(\lambda_1\lambda_2\right)^{-\lambda t}}{2\lambda t} F_1\left(2\lambda t, \lambda t, \lambda t; 1+2\lambda t; x, y\right) \right. \\ & \left. + \frac{\left[\lambda_1\lambda_2\left(1-p\right)^2\right]^{-\lambda t}}{2\lambda t} F_1\left(2\lambda t, \lambda t, \lambda t; 1+2\lambda t; \frac{x}{1-p}, \frac{y}{1-p}\right) \right\}, \end{aligned} \tag{3.16}$$

where F_1 is the Appell function (2.13) and

$$\begin{aligned} x &:= -\frac{\left(\lambda_1+\lambda_2\right)a + \sqrt{a^2\left(\lambda_1+\lambda_2\right)^2 - 4\lambda_1\lambda_2 a^2\left(1-\theta\right)}}{2\lambda_1\lambda_2}, \\ y &:= \frac{-\left(\lambda_1+\lambda_2\right)a + \sqrt{a^2\left(\lambda_1+\lambda_2\right)^2 - 4\lambda_1\lambda_2 a^2\left(1-\theta\right)}}{2\lambda_1\lambda_2}. \end{aligned}$$

The subdensities $f_i\left(t\right),\ i=1,2$, read

$$\begin{aligned} f_i\left(t\right) = & r_i\left(-\frac{1}{\ln\left(1-p\right)}\right)\left(\left(1-\theta\right)a^2\right)^{\lambda t} \left[\frac{\left(\gamma p^2+\beta p+\alpha\right)^{-\lambda t}}{2\gamma\left(-\lambda t+1\right)\left(2q\right)^{\lambda t}} \right. \\ & \times \frac{\beta-q+2\gamma p}{\left(\beta+q+2\gamma p\right)^{-\lambda t}} {}_2F_1\left(\lambda t, -\lambda t+1; -\lambda t+2; \frac{-\beta+q-2\gamma p}{2q}\right) \\ & \left. - \frac{\alpha^{-\lambda t}}{2\gamma\left(-\lambda t+1\right)\left(2q\right)^{\lambda t}} \frac{\beta-q}{\left(\beta+q\right)^{-\lambda t}} {}_2F_1\left(\lambda t, -\lambda t+1; -\lambda t+2; \frac{-\beta+q}{2q}\right)\right], \end{aligned} \tag{3.17}$$

where r_i are the hazard rates given in Corollary 3.6, and

- $\alpha := \left(1-\theta\right)a^2 + \lambda_1\lambda_2 + \lambda_1 a + \lambda_2 a$;
- $\beta := -\left(2\lambda_1\lambda_2 + \lambda_1 a + \lambda_2 a\right)$;
- $\gamma := \lambda_1\lambda_2$;
- $q := \sqrt{\beta^2 - 4\alpha\gamma}$.

In the Appendix we show how to prove Equation (3.16) and Equation (3.17). We use the recursion formula for the function F_1 in (2.13) about the numerator parameter a (see formula (1) of Theorem 1 of Wang, 2012 for $n=1$),

$$\begin{aligned} F_1\left(a+1, b_1, b_2; c; x, y\right) = & F_1\left(a, b_1, b_2; c; x, y\right) + \frac{b_1 x}{c} F_1\left(a+1, b_1+1, b_2; c+1; x, y\right) \\ & + \frac{b_2 y}{c} F_1\left(a+1, b_1, b_2+1; c+1; x, y\right), \end{aligned}$$

to plot the survival function (3.16) in Fig. 3.2 for different values of the parameters.
We observe that it is possible to express the hazard rate $h_T\left(t\right),\ t \geq 0$, of T in closed form. Indeed,

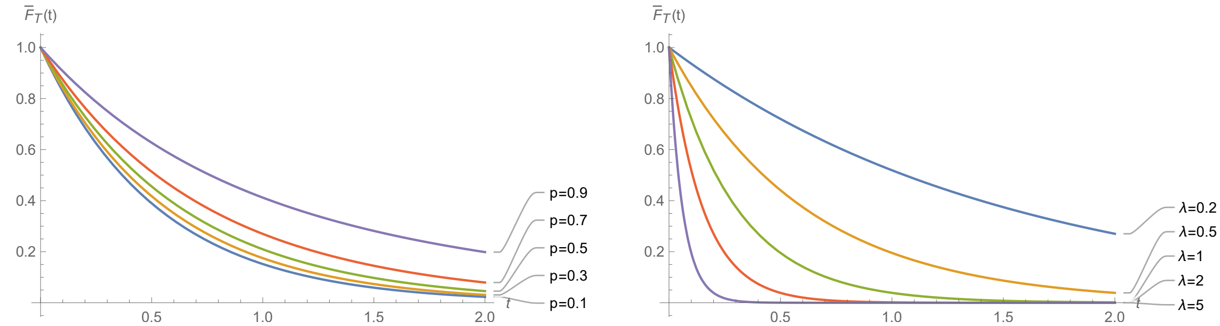

FIGURE 3.2. Plots of the survival function of T when M has logarithmic distribution with $\theta = 0.5$, $a = \lambda_1 = \lambda_2 = 1$ and: $\lambda = 1$ on the left-hand side, and $p = 0.5$ on the right-hand side.

due to Corollary 3.6, the time of failure T and the cause of failure C are independent. Therefore, recalling (2.2), we have

$$f_i(t) = \mathrm{P}\{C = i\} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{P}\{T \le t\} = -\mathrm{P}\{C = i\} \frac{\mathrm{d}}{\mathrm{d}t} \overline{F}_T(t),$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}t} \overline{F}_T(t) = \frac{f_i(t)}{\mathrm{P}\{C = i\}}.$$

Hence, for $t \ge 0$,

$$h_T(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln \overline{F}_T(t) = -\frac{1}{\overline{F}_T(t)} \frac{\mathrm{d}}{\mathrm{d}t} \overline{F}_T(t) = \frac{f_i(t)}{\overline{F}_T(t)\, \mathrm{P}\{C = i\}},$$

where $f_i(t)$, $\overline{F}_T(t)$ and $\mathrm{P}\{C = i\}$ can be recovered from (3.17), (3.16) and Corollary 3.6, item b), respectively. It is a well-known fact that, if the time and the cause of failure are independent, then the hazard rate of T does not depend on the cause of failure (cf. for example Remark 2.1 of Di Crescenzo and Longobardi, 2006).

4. A special case

In this section we provide an explicit expression for the survival function of T under specific assumptions on the distribution of the threshold M. Consider a generic bivariate subordinator $\{H(t)\}_{t \ge 0}$, such that $H(t) := \{(H_1(t), H_2(t)), t \ge 0\}$ with Laplace exponent $S(\eta_1, \eta_2)$, $\eta_1, \eta_2 \ge 0$. This means that the joint Laplace-Stieltjes transform reads

$$\mathrm{E}e^{-\eta_1 H_1(t) - \eta_2 H_2(t)} = e^{-tS(\eta_1, \eta_2)}, \qquad \eta_1, \eta_2 \ge 0,$$

with $S(\eta_1, \eta_2)$ a bivariate Bernstein function in the sense of Bochner (1955), Chapter 4. Moreover, suppose that the shocks to a system arrive according to a process $\{\mathcal{N}(t)\}_{t \ge 0}$, where

$$\mathcal{N}(t) := (N_1(H_1(t)), N_2(H_2(t))),$$

and the components of $\mathcal{N}$ are two time-changed independent homogeneous Poisson processes with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. The time-change is represented by an independent generic bivariate subordinator H. The following theorem generalises to the bidimensional case Theorem 4.1 in Soni et al. (2024), where, instead, subordination with a univariate subordinator is considered.

Theorem 4.1. *Assume that the system fails when the sum of the shocks reaches a geometrically distributed threshold with parameter p. Then, under the above assumptions, the survival function of the lifetime of the system reads*

$$\overline{F}_T(t) = e^{-tS(\lambda_1 p,\, \lambda_2 p)}, \qquad t \geq 0,$$

where $S(\cdot,\cdot)$ *is the Laplace exponent of the subordinator.*

Proof: By a subordination argument we have, for $t \geq 0$ and nonnegative integers x_1, k such that $x_1 \leq k$,

$$\begin{aligned}
\mathrm{P}\{N_1(H_1(t)) = x_1,\, N_2(H_2(t)) = k - x_1\} &= \int_0^{+\infty}\int_0^{+\infty} (\mathrm{P}\{N_1(u) = x_1,\, N_2(v) = k - x_1\} \\
&\quad \times \mathrm{P}\{H_1(t) \in \mathrm{d}u, H_2(t) \in \mathrm{d}v\}) \\
&= \frac{\lambda_1^{x_1}}{x_1!}\frac{\lambda_2^{k-x_1}}{(k-x_1)!}\int_0^{+\infty}\int_0^{+\infty} \Big(e^{-\lambda_1 u - \lambda_2 v} u^{x_1} v^{k-x_1} \\
&\quad \times \mathrm{P}\{H_1(t) \in \mathrm{d}u, H_2(t) \in \mathrm{d}v\}\Big).
\end{aligned}$$

Hence, Equation (2.8) can be simplified as follows:

$$\begin{aligned}
\overline{F}_T(t) &= \sum_{k=0}^{+\infty}(1-p)^k \sum_{x_1=0}^{k}\frac{\lambda_1^{x_1}}{x_1!}\frac{\lambda_2^{k-x_1}}{(k-x_1)!} \\
&\quad \times \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 u - \lambda_2 v} u^{x_1} v^{k-x_1}\mathrm{P}\{H_1(t) \in \mathrm{d}u, H_2(t) \in \mathrm{d}v\} \\
&= \sum_{x_1=0}^{+\infty}\frac{[\lambda_1(1-p)]^{x_1}}{x_1!}\sum_{k=x_1}^{+\infty}\frac{[\lambda_2(1-p)]^{k-x_1}}{(k-x_1)!} \\
&\quad \times \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 u - \lambda_2 v} u^{x_1} v^{k-x_1}\mathrm{P}\{H_1(t) \in \mathrm{d}u, H_2(t) \in \mathrm{d}v\} \\
&= \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda_1 u - \lambda_2 v + (1-p)\lambda_1 u + (1-p)\lambda_2 v}\mathrm{P}\{H_1(t) \in \mathrm{d}u, H_2(t) \in \mathrm{d}v\} \\
&= \int_0^{+\infty}\int_0^{+\infty} e^{-p\lambda_1 u - p\lambda_2 v}\mathrm{P}\{H_1(t) \in \mathrm{d}u, H_2(t) \in \mathrm{d}v\} \\
&= e^{-tS(\lambda_1 p,\, \lambda_2 p)}.
\end{aligned}$$

□

Theorem 4.1 makes it clear that the choice of the subordinator does not affect the distribution of T if the threshold is geometric. Indeed, the lifetime T turns out to be exponentially distributed with mean depending on the Laplace exponent of the subordinator. On the other hand, the expression of the hazard rates (2.4) and, as a consequence, the expression of the subdensities (2.7) and of the probability of the cause of failure (2.3) are not that straightforward. Indeed, reasoning similar to that in Theorem 3.5, shows that, for example,

$$r_1(x_1, x_2; t) = \lim_{\tau \to t}\frac{\mathrm{P}\{N_1(H_1(\tau)) = x_1 + 1, N_2(H_2(\tau)) = x_2, N_1(H_1(t)) = x_1, N_2(H_2(t)) = x_2\}}{(\tau - t)\,\mathrm{P}\{N_1(H_1(t)) = x_1, N_2(H_2(t)) = x_2\}},$$

where the numerator equals

$$\begin{aligned}
&\frac{1}{x_1! x_2!}\int_0^{+\infty}\int_0^{u}\int_0^{+\infty}\int_0^{v}\Big(e^{-\lambda_1 u - \lambda_2 v}\lambda_1^{x_1+1}(u-w)\,w^{x_1}(\lambda_2 z)^{x_2} \\
&\qquad\qquad \times \mathrm{P}\{H_1(\tau) \in \mathrm{d}u, H_2(\tau) \in \mathrm{d}v, H_1(t) \in \mathrm{d}w, H_2(t) \in \mathrm{d}z\}\Big),
\end{aligned}$$

and the denominator equals

$$
\begin{aligned}
&(\tau - t) \frac{\lambda_1^{x_1}}{x_1!} \frac{\lambda_2^{x_2}}{x_2!} \int_0^{+\infty} \int_0^{+\infty} e^{-\lambda_1 u - \lambda_2 v} u^{x_1} v^{x_2} \mathrm{P}\left\{ H_1(t) \in \mathrm{d}u, H_2(t) \in \mathrm{d}v \right\} \\
&= (\tau - t) \frac{\lambda_1^{x_1}}{x_1!} \frac{\lambda_2^{x_2}}{x_2!} (-1)^{x_1 + x_2} \frac{\partial^{x_1 + x_2}}{\partial \lambda_1^{x_1} \lambda_2^{x_2}} e^{-tS(\lambda_1, \lambda_2)}.
\end{aligned}
$$

5. Conclusions and perspectives

We have considered a competing risks model within the more general class of random shock models. Specifically, two sources of shock, which occur as events of a bidimensional counting process with different components using different clocks, damage the system until the reaching of a given threshold for the degradation level which implies failure. We have derived expressions for the survival probability and the hazard rates of the system, and established the independence between the time and the cause of failure under the assumption of constant hazard rates. In addition, we have specified the distribution of the threshold in two special cases and obtained a closed form for the survival function of the lifetime of the system. Further research direction might include, but not limited to: systems whose failure does not occur immediately after a shock, but it appears with a random delay; random time change with inverses of multivariate subordinators, in the spirit of Beghin et al. (2020).

Appendix

Hereafter, we show how to compute Equation (3.16) and Equation (3.17), i.e. the survival function of the random lifetime T and the failure subdensities $f_i(t)$, $i = 1, 2$, when the threshold M is logarithmically distributed. For the sake of simplicity, we set

$$
\begin{aligned}
\alpha &:= (1 - \theta) a^2 + \lambda_1 \lambda_2 + \lambda_1 a + \lambda_2 a \\
\beta &:= -(2\lambda_1 \lambda_2 + \lambda_1 a + \lambda_2 a) \\
\gamma &:= \lambda_1 \lambda_2 \\
q &:= \sqrt{\beta^2 - 4\alpha\gamma}
\end{aligned}
\tag{5.1}
$$

The survival function $\overline{F}_T(t)$. From (2.8) and (3.11), we have

$$
\begin{aligned}
\overline{F}_T(t) = &-\frac{1}{\ln(1-p)} \left(\frac{(1-\theta) a^2}{(\lambda_1 + a)(\lambda_2 + a)} \right)^{\lambda t} \sum_{x_1 = 0}^{+\infty} \left(\frac{\lambda_1}{\lambda_1 + a} \right)^{x_1} \frac{(\lambda t)_{x_1}}{x_1!} \\
&\times \sum_{k = x_1}^{+\infty} B(p; k+1, 0) \frac{(\lambda t)_{k - x_1}}{(k - x_1)!} \left(\frac{\lambda_2}{\lambda_2 + a} \right)^{k - x_1} {}_2F_1\left(x_1 + \lambda t, k - x_1 + \lambda t; \lambda t; \frac{a^2 \theta}{(\lambda_1 + a)(\lambda_2 + a)} \right) \\
= &-\frac{1}{\ln(1-p)} \left(\frac{(1-\theta) a^2}{(\lambda_1 + a)(\lambda_2 + a)} \right)^{\lambda t} \sum_{x_1 = 0}^{+\infty} \left(\frac{\lambda_1}{\lambda_1 + a} \right)^{x_1} \frac{(\lambda t)_{x_1}}{x_1!} \\
&\times \sum_{h = 0}^{+\infty} \frac{(\lambda t)_h}{h!} \left(\frac{\lambda_2}{\lambda_2 + a} \right)^{h} {}_2F_1\left(x_1 + \lambda t, h + \lambda t; \lambda t; \frac{a^2 \theta}{(\lambda_1 + a)(\lambda_2 + a)} \right) \int_0^p u^{x_1 + h} (1-u)^{-1} \mathrm{d}u \\
\overset{\text{by (2.12)}}{=} &-\frac{1}{\ln(1-p)} \left(\frac{(1-\theta) a^2}{(\lambda_1 + a)(\lambda_2 + a)} \right)^{\lambda t} \sum_{x_1 = 0}^{+\infty} \left(\frac{\lambda_1}{\lambda_1 + a} \right)^{x_1} \frac{(\lambda t)_{x_1}}{x_1!} \\
&\times \int_0^p u^{x_1} (1-u)^{-1} \left(1 - \frac{\lambda_2 u}{\lambda_2 + a} \right)^{-\lambda t} {}_2F_1\left(x_1 + \lambda t, \lambda t; \lambda t; \frac{a^2 \theta}{(\lambda_1 + a)(\lambda_2 - \lambda_2 u + a)} \right) \mathrm{d}u
\end{aligned}
$$

$$= -\frac{1}{\ln(1-p)}\left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)(\lambda_2+a)}\right)^{\lambda t}\sum_{x_1=0}^{+\infty}\left(\frac{\lambda_1}{\lambda_1+a}\right)^{x_1}\frac{(\lambda t)_{x_1}}{x_1!}$$
$$\times\int_0^p u^{x_1}(1-u)^{-1}\left(1-\frac{\lambda_2 u}{\lambda_2+a}\right)^{-\lambda t}\left(1-\frac{a^2\theta}{(\lambda_1+a)(\lambda_2+a-\lambda_2 u)}\right)^{-x_1-\lambda t}\mathrm{d}u$$

where the function ${}_2F_1$ has been reduced thanks to (2.11). Again, due to (2.12) and (2.11), we get an integral form for the survival function of T, i.e.

$$\overline{F}_T(t) = -\frac{\left((1-\theta)\,a^2\right)^{\lambda t}}{\ln(1-p)}\int_0^p (1-u)^{-1}\left((\lambda_1+a-\lambda_1 u)(\lambda_2+a-\lambda_2 u)-a^2\theta\right)^{-\lambda t}\mathrm{d}u. \tag{5.2}$$

Let us now simplify the integral in the right-hand side of Equation (5.2). Due to (5.1), integral (5.2) can be rewritten as

$$\int_0^p (1-u)^{-1}\left(\gamma u^2+\beta u+\alpha\right)^{-\lambda t}\mathrm{d}u,$$

and, consequently, as

$$\left(\frac{\gamma}{4}\right)^{-\lambda t}\int_0^p (1-u)^{-1-2\lambda t}\left(\frac{-(\beta-q+2\gamma u)}{\gamma(1-u)}\right)^{-\lambda t}\left(\frac{-(\beta+q+2\gamma u)}{\gamma(1-u)}\right)^{-\lambda t}\mathrm{d}u. \tag{5.3}$$

We observe that the following expression

$$c^{-\lambda t}\int_1^{\frac{1}{1-p}} u^{2\lambda t-1}\left(1-\left(1+\frac{\beta-q}{2\gamma}\right)u\right)^{-\lambda t}\left(1-\left(1+\frac{\beta+q}{2\gamma}\right)u\right)^{-\lambda t}\mathrm{d}u$$

reduces to (5.3), by substituting $u=\frac{1}{1-\tilde{u}}$. The result thus follows by applying the next equality

$$\int (bu)^m (c+du)^n (f+gu)^p\,\mathrm{d}u = \frac{c^n f^p (bu)^{m+1}}{b(m+1)}F_1\left(m+1,-n,-p,m+2,-\frac{du}{c},-\frac{gu}{f}\right),$$

provided $m\notin\mathbb{Z}, n\notin\mathbb{Z}, c>0$ and $(p\in\mathbb{Z}$ or $f>0)$, which can be proved by resorting to the generalized binomial identity (2.11) and to the definition (2.13) of the Appell series F_1.

The subdensities $f_i(t)\,,\ i=1,2.$ First, we consider the following preliminary identity:

$$\int (a+bx)^m (c+dx)^n\,\mathrm{d}x = \frac{(a+bx)^{m+1}}{b(m+1)\left(\frac{b}{bc-ad}\right)^n}\,{}_2F_1\left(-n,m+1;m+2;\frac{-d(a+bx)}{bc-ad}\right), \tag{5.4}$$

provided $bc-ad\neq 0$, $m\neq\mathbb{Z}$ and $\left(n\in\mathbb{Z}\text{ or }\frac{b}{bc-ad}>0\right)$. Indeed, by means of simple computations, we have

$$\int (a+bx)^m (c+dx)^n\,\mathrm{d}x = \frac{1}{\left(\frac{b}{bc-ad}\right)^n}\int (a+bx)^m\left(1+\frac{d(a+bx)}{bc-ad}\right)^n\mathrm{d}x$$
$$= \frac{1}{\left(\frac{b}{bc-ad}\right)^n}\int (a+bx)^m\sum_{k=0}^{+\infty}\frac{(-n)_k}{k!}\left(\frac{-d(a+bx)}{bc-ad}\right)^k\mathrm{d}x$$
$$= \frac{1}{\left(\frac{b}{bc-ad}\right)^n}\sum_{k=0}^{+\infty}\frac{(-n)_k}{k!}\left(\frac{-d}{bc-ad}\right)^k\int (a+bx)^{k+m}\,\mathrm{d}x$$
$$= \frac{1}{\left(\frac{b}{bc-ad}\right)^n}\sum_{k=0}^{+\infty}\frac{(-n)_k}{k!}\left(\frac{-d}{bc-ad}\right)^k\frac{(a+bx)^{m+k+1}}{b(m+k+1)}$$

$$= \frac{(a+bx)^{m+1}}{b\,(m+1)\left(\frac{b}{bc-ad}\right)^{n}} \sum_{k=0}^{+\infty} \frac{(-n)_k\,(m+1)_k}{k!\,(m+2)_k} \left(\frac{-d\,(a+bx)}{bc-ad}\right)^{k},$$

and Equation (5.4) follows from (2.9). Now, consider the failure subdensities expressed in Corollary 3.6, item a), with p_k in (3.15),

$$f_i\,(t) = r_i \left(-\frac{1}{\ln(1-p)}\right) \left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)\,(\lambda_2+a)}\right)^{\lambda t} \sum_{x_1=0}^{+\infty} \left(\frac{\lambda_1}{\lambda_1+a}\right)^{x_1} \frac{(\lambda t)_{x_1}}{x_1!}$$
$$+ \sum_{k=x_1+1}^{+\infty} \frac{p^k}{k} \left(\frac{\lambda_2}{\lambda_2+a}\right)^{k-x_1-1} \frac{(\lambda t)_{k-x_1-1}}{(k-x_1-1)!}\, {}_2F_1\left(x_1+\lambda t, k-x_1-1+\lambda t; \lambda t; \frac{a^2\theta}{(\lambda_1+a)\,(\lambda_2+a)}\right).$$

We observe that

$$\frac{p^k}{k} = \int_0^p y^{k-1}\mathrm{d}y,$$

so that

$$f_i\,(t) = r_i \left(-\frac{1}{\ln(1-p)}\right) \left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)\,(\lambda_2+a)}\right)^{\lambda t} \sum_{x_1=0}^{+\infty} \left(\frac{\lambda_1}{\lambda_1+a}\right)^{x_1} \frac{(\lambda t)_{x_1}}{x_1!}$$
$$\times \int_0^p \left(y^{x_1} \sum_{k=x_1+1}^{+\infty} y^{k-x_1-1} \left(\frac{\lambda_2}{\lambda_2+a}\right)^{k-x_1-1} \frac{(\lambda t)_{k-x_1-1}}{(k-x_1-1)!} \right.$$
$$\left. \times {}_2F_1\left(x_1+\lambda t, k-x_1-1+\lambda t; \lambda t; \frac{a^2\theta}{(\lambda_1+a)\,(\lambda_2+a)}\right)\right) \mathrm{d}y$$
$$= r_i \left(-\frac{1}{\ln(1-p)}\right) \left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)\,(\lambda_2+a)}\right)^{\lambda t} \sum_{x_1=0}^{+\infty} \left(\frac{\lambda_1}{\lambda_1+a}\right)^{x_1} \frac{(\lambda t)_{x_1}}{x_1!}$$
$$\times \int_0^p \left(y^{x_1} \sum_{h=0}^{+\infty} y^{h} \left(\frac{\lambda_2}{\lambda_2+a}\right)^{h} \frac{(\lambda t)_{h}}{h!} \right.$$
$$\left. \times {}_2F_1\left(x_1+\lambda t, h+\lambda t; \lambda t; \frac{a^2\theta}{(\lambda_1+a)\,(\lambda_2+a)}\right)\right) \mathrm{d}y.$$

We apply the summation formula (2.12) and then simplify, so to get

$$f_i\,(t) = r_i \left(-\frac{1}{\ln(1-p)}\right) \left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)\,(\lambda_2+a)}\right)^{\lambda t} \int_0^p \left(\left(1-\frac{\lambda_2 y}{\lambda_2+a}\right)^{-\lambda t} \right.$$
$$\left. \times \sum_{x_1=0}^{+\infty} \left(\frac{\lambda_1}{\lambda_1+a}\right)^{x_1} \frac{(\lambda t)_{x_1}}{x_1!} y^{x_1}\, {}_2F_1\left(x_1+\lambda t, \lambda t; \lambda t; \frac{a^2\theta}{(\lambda_1+a)\,(\lambda_2+a-\lambda_2 y)}\right)\right) \mathrm{d}y.$$

Again,

$$f_i\,(t) \overset{by\ (2.12)}{=} r_i \left(-\frac{1}{\ln(1-p)}\right) \left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)\,(\lambda_2+a)}\right)^{\lambda t}$$
$$\times \int_0^p \left(1-\frac{\lambda_2 y}{\lambda_2+a}\right)^{-\lambda t} \left(1-\frac{\lambda_1 y}{\lambda_1+a}\right)^{-\lambda t} {}_2F_1\left(\lambda t, \lambda t; \lambda t; \frac{a^2\theta}{(\lambda_1+a-\lambda_1 y)\,(\lambda_2+a-\lambda_2 y)}\right) \mathrm{d}y$$
$$\overset{by\ (2.9)\ and\ (2.11)}{=} r_i \left(-\frac{1}{\ln(1-p)}\right) \left(\frac{(1-\theta)\,a^2}{(\lambda_1+a)\,(\lambda_2+a)}\right)^{\lambda t}$$

$$\times \int_0^p \left(1-\frac{\lambda_2 y}{\lambda_2+a}\right)^{-\lambda t}\left(1-\frac{\lambda_1 y}{\lambda_1+a}\right)^{-\lambda t}\left(1-\frac{a^2\theta}{(\lambda_1+a-\lambda_1 y)(\lambda_2+a-\lambda_2 y)}\right)^{-\lambda t} \mathrm{d}y$$
$$= r_i\left(-\frac{1}{\ln(1-p)}\right)\left((1-\theta)a^2\right)^{\lambda t}\int_0^p \left((\lambda_1+a-\lambda_1 y)(\lambda_2+a-\lambda_2 y)-a^2\theta\right)^{-\lambda t}\mathrm{d}y. \tag{5.5}$$

Thanks to (5.1), the integral in the right-hand side of (5.5) can be expressed as

$$\int_0^p \left(\gamma y^2+\beta y+\alpha\right)^{-\lambda t}\mathrm{d}y,$$

and then as

$$\frac{\left(\gamma y^2+\beta y+\alpha\right)^{-\lambda t}}{(\beta+q+2\gamma y)^{-\lambda t}(\beta-q+2\gamma y)^{-\lambda t}}\int_0^p (\beta+q+2\gamma y)^{-\lambda t}(\beta-q+2\gamma y)^{-\lambda t}\mathrm{d}y.$$

The result thus follows by applying (5.4). We have

$$f_i(t) = r_i\left(-\frac{1}{\ln(1-p)}\right)\left((1-\theta)a^2\right)^{\lambda t}\left[\frac{\left(\gamma y^2+\beta y+\alpha\right)^{-\lambda t}}{2\gamma(-\lambda t+1)(2q)^{\lambda t}}\right.$$
$$\left.\times\frac{\beta-q+2\gamma y}{(\beta+q+2\gamma y)^{-\lambda t}}\,{}_2F_1\left(\lambda t,-\lambda t+1;-\lambda t+2;\frac{-\beta+q-2\gamma y}{2q}\right)\right]_0^p$$
$$= r_i\left(-\frac{1}{\ln(1-p)}\right)\left((1-\theta)a^2\right)^{\lambda t}\left[\frac{\left(\gamma p^2+\beta p+\alpha\right)^{-\lambda t}}{2\gamma(-\lambda t+1)(2q)^{\lambda t}}\right.$$
$$\times\frac{\beta-q+2\gamma p}{(\beta+q+2\gamma p)^{-\lambda t}}\,{}_2F_1\left(\lambda t,-\lambda t+1;-\lambda t+2;\frac{-\beta+q-2\gamma p}{2q}\right)$$
$$\left.-\frac{\alpha^{-\lambda t}}{2\gamma(-\lambda t+1)(2q)^{\lambda t}}\frac{\beta-q}{(\beta+q)^{-\lambda t}}\,{}_2F_1\left(\lambda t,-\lambda t+1;-\lambda t+2;\frac{-\beta+q}{2q}\right)\right].$$

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