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A law of the iterated logarithm for iterated random walks, with application to random recursive trees

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Abstract. Consider a Crump-Mode-Jagers process generated by an increasing random walk whose increments have finite second moment. Let $Y_k(t)$ be the number of individuals in generation $k \in \mathbb{N}$ born in the time interval $[0, t]$. We prove a law of the iterated logarithm for $Y_k(t)$ with fixed k, as $t \to +\infty$. As a consequence, we derive a law of the iterated logarithm for the number of vertices at a fixed level k in a random recursive tree, as the number of vertices goes to ∞.

1. A law of the iterated logarithm for iterated random walks

Let $\xi_1, \xi_2, \ldots$ be independent copies of an almost surely (a.s.) positive random variable ξ. Denote by $S := (S_n)_{n \in \mathbb{N}}$ the *standard random walk* with increments ξ_n for $n \in \mathbb{N}$, that is, $S_n := \xi_1 + \ldots + \xi_n$ for $n \in \mathbb{N}$. The corresponding *renewal process* $(Y(t))_{t \geq 0}$ is defined by

$$Y(t) := \sum_{n \geq 1} \mathbb{1}_{\{S_n \leq t\}}, \quad t \geq 0.$$

Put $V(t) := \mathbb{E}Y(t)$ for $t \geq 0$. The function V is called *renewal function*.

Now we recall the construction of a general branching process (a.k.a. Crump-Mode-Jagers process) generated by S. There is a population of individuals initiated at time 0 by one individual, the ancestor. An individual born at time $t \geq 0$ produces offspring whose birth times have the same distribution as $(t + S_n)_{n \in \mathbb{N}}$. All individuals act independently of each other. For $k \in \mathbb{N}$, an individual

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resides in the kth generation if it has exactly k ancestors. For $k \in \mathbb{N}$ and $t \geq 0$, denote by $S^{(k)}$ the collection of the birth times in the kth generation and by $Y_k(t)$ the number of the kth generation individuals with birth times smaller than or equal to t. Put $V_k(t) := \mathbb{E}Y_k(t)$. Plainly, $Y_1(t) = Y(t)$ and $V_1(t) = V(t)$ for $t \geq 0$. Following Bohun et al. (2022) and Iksanov et al. (2023) we call the sequence $(S^{(k)})_{k\geq 2}$ an *iterated standard random walk*.

Here is the definition of an *iterated perturbed random walk*. Even though our main result, Theorem 1.1, is stated for iterated standard random walks, some auxiliary results that we use were obtained for iterated perturbed random walks. Also, Lemma 5.1 is proved here for iterated perturbed random walks.

Let (ξ_1, η_1), $(\xi_2, \eta_2), \ldots$ be independent copies of an $\mathbb{R}^2$-valued random vector (ξ, η) with positive arbitrarily dependent components. Put

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N}.$$

The sequence $T := (T_n)_{n\in\mathbb{N}}$ is called *(globally) perturbed random walk*. The sequence $(T^{(k)})_{k\geq 2}$, a counterpart of $(S^{(k)})_{k\geq 2}$ obtained by replacing in the aforementioned construction S with T, is called an *iterated perturbed random walk*. Observe that $T = S$ and $(T^{(k)})_{k\geq 2} = (S^{(k)})_{k\geq 2}$ a.s. whenever $\eta = \xi$ a.s. Denote by Y_k^* and V_k^* the counterparts of Y_k and V_k for iterated perturbed random walks.

Next we discuss the motivation behind the study of iterated standard and perturbed random walks.

– Let $k \geq 2$ be fixed. The sequence $S^{(k)}$ and the counting process Y_k are a natural generalization of the standard random walk S and the renewal process Y. Thus, any results obtained for $S^{(k)}$ and Y_k can be thought of as a contribution to an extension of renewal theory, which may be called *renewal theory on trees*. In particular, one may wonder to what extent the renewal-theoretic properties of S and the renewal process Y are inherited by $S^{(k)}$ and Y_k.

– A random recursive tree can be constructed as a family tree of a general branching process generated by S and stopped at suitable random time. Details can be found in the proof of Theorem 2.1.

– The sequences $(S^{(k)})_{k\geq 2}$ and $(T^{(k)})_{k\geq 2}$ are particular yet non-trivial instances of a general branching process. Hence, an outcome of their analysis is a contribution to the theory of general branching processes.

– Originally, the iterated perturbed random walks were introduced in Section 3 of Buraczewski et al. (2020). Let T be the particular perturbed random walk defined by

$$T_k := |\log W_1| + \ldots + |\log W_{k-1}| + |\log(1 - W_k)|, \quad k \in \mathbb{N},$$

where $(W_k)_{k\in\mathbb{N}}$ are independent identically distributed random variables taking values in $(0, 1)$. For $k \geq 1$, the elements of $T^{(k)}$, with $T^{(1)} = T$, defined in Buraczewski et al. (2020) random hitting probabilities of boxes in the kth level of the nested balls-in-boxes scheme.

We refer to the introductions of the articles Iksanov et al. (2022) and Iksanov et al. (2023) for more details.

Theorem 1.3 in Iksanov and Kabluchko (2018) is a functional central limit theorem (FCLT) for $(Y_k)_{k\in\mathbb{N}}$, properly scaled, normalized and centered, on $D^{\mathbb{N}}$ equipped with the product J_1-topology, under the assumption $\mathbb{E}\xi^2 < \infty$. Here, D denotes the Skorokhod space of càdlàg functions on $[0, \infty)$. Theorem 4 in Iksanov et al. (2023) derives the asymptotics of $\operatorname{Var} Y_k(t)$ as $t \to \infty$ when $k \in \mathbb{N}$ is fixed under the assumptions that $\mathbb{E}\xi^2 < \infty$ and that the distribution of ξ is nonlattice (see Section 3 for the definition).

Now we state a law of the iterated logarithm (LIL) for Y_k for fixed $k \in \mathbb{N}$. This is a natural complement to the aforementioned distributional limit theorem proved in Iksanov and Kabluchko (2018). For a family (x_t) of real numbers we write $C((x_t))$ for the set of its limit points. Recall that $0! = 1$.

Theorem 1.1. *Assume that* $\sigma^2 := \operatorname{Var}\xi \in (0,\infty)$. *Then, for each fixed* $k \in \mathbb{N}$,

$$C\Bigg(\Bigg(\frac{a_k\big(Y_k(t)-t^k/(k!\mu^k)\big)}{(2t^{2k-1}\log\log t)^{1/2}} : t > \mathrm{e}\Bigg)\Bigg) = [-1,1] \quad \text{a.s.},$$

where

$$a_k := \sigma^{-1}\mu^{k+1/2}(k-1)!(2k-1)^{1/2}$$

and $\mu := \mathbb{E}\xi < \infty$. *In particular,*

$$\limsup_{t\to\infty}\frac{a_k\big(Y_k(t)-t^k/(k!\mu^k)\big)}{(2t^{2k-1}\log\log t)^{1/2}} = 1 \quad \text{a.s.}$$

and

$$\liminf_{t\to\infty}\frac{a_k\big(Y_k(t)-t^k/(k!\mu^k)\big)}{(2t^{2k-1}\log\log t)^{1/2}} = -1 \quad \text{a.s.}$$

The centering $t^k/(k!\mu^k)$ *can be replaced with* $\mathbb{E}Y_k(t)$ *everywhere.*

We close the section by reviewing some previously known LILs for standard random walks and their relatives.

- A LIL for a standard random walk S with $\mathbb{E}\xi^2 < \infty$ follows from the theorem on p. 170 in Hartman and Wintner (1941). Nowadays the result is known as the Hartman-Wintner LIL.
– A converse to the Hartman-Wintner LIL was obtained in Strassen (1966): $\mathbb{E}\xi^2 < \infty$ provided that $\mathbb{P}\{\limsup_{n\to\infty}\frac{|S_n - n\mathbb{E}\xi|}{\sqrt{n\log\log n}} < \infty\} > 0$.
– A LIL for the renewal process (a particular case of Theorem 1.1 with $k = 1$) can be found, for instance, in Proposition 3.5 in Iksanov et al. (2017).
– Under the assumption $\mathbb{E}\xi^2 < \infty$, a LIL for the partial maxima $M_n := \max(0, S_1, \ldots, S_n)$ is stated in Theorem 4.5 on p.140 in Gut (2009).

Our proof of Theorem 1.1 does not appeal to any of these LILs. Its main technical tool is a strong approximation result for renewal processes given in Lemma 3.1.

2. Application to random recursive trees

In this section we state a LIL for the profile of the random recursive tree (RRT) and prove it using Theorem 1.1 in the special case when the random variable ξ has an exponential distribution of unit mean. For our purposes, the following continuous-time construction of the RRT is convenient (see, e.g., Example 6.1 in Holmgren and Janson (2017)). At time 0, the RRT consists of 1 vertex, the root, located at level 0. This vertex generates offspring at arrival times of a unit intensity Poisson process. These offspring are located at level 1. More generally, each vertex of the tree, immediately after its birth, starts to generate offspring at rate 1, and all vertices act independently. If some vertex is located at level k, then its offspring appear at level $k+1$, so that the level of any vertex is its distance to the root. Clearly, one can identify the birth times of the vertices at level $k \in \mathbb{N}$ with the process $S^{(k)}$, as defined in Section 1, with the random variable ξ having an exponential distribution of unit mean. Let $\tau_1 < \tau_2 < \ldots$ be the birth times of the vertices of the RRT, excluding the root born at time $\tau_0 = 0$. For $n \in \mathbb{N}$, at time τ_n, the tree consists of $n+1$ vertices. For $k \in \mathbb{N}$, let $X_n(k) = Y_k(\tau_n)$ be the number of vertices in this tree having distance k to the root at time τ_n. The function $k \mapsto X_n(k)$ is called the profile of the RRT. Its asymptotic behavior as $n \to \infty$ has been much studied. For example, a central limit theorem for $X_n(k)$ with fixed k has been obtained in Fuchs et al. (2006); see also Iksanov and Kabluchko (2018) for a functional version. As a corollary of Theorem 1.1 we shall prove the following law of the iterated logarithm for $X_n(k)$.

Theorem 2.1. *For each fixed* $k \in \mathbb{N}$,

$$C\Bigg(\Bigg(\frac{(k-1)!(2k-1)^{1/2}\big(X_n(k)-(\log n)^k/k!\big)}{(2(\log n)^{2k-1}\log\log\log n)^{1/2}} : n > \mathrm{e}^{\mathrm{e}}\Bigg)\Bigg) = [-1,1] \quad \text{a.s.}$$

For $k=1$, the claim is known (see Theorem 3' in Rényi (1962)) since the sequence $(X_n(1))_{n\in\mathbb{N}}$ has the same joint distribution as $(B_1+\ldots+B_n)_{n\in\mathbb{N}}$, where $B_1, B_2, \ldots$ are independent Bernoulli random variables with $\mathbb{P}\{B_k=1\}=1/k$.

Proof of Theorem 2.1: Let ξ be a random variable having an exponential distribution of unit mean. Then $a_k=(k-1)!(2k-1)^{1/2}$ since $\mu=\sigma^2=1$, and Theorem 1.1 takes the form

$$C\bigg(\bigg(\frac{a_k\big(Y_k(t)-t^k/k!\big)}{(2t^{2k-1}\log\log t)^{1/2}}:t>\mathrm{e}\bigg)\bigg)=[-1,1]\quad\text{a.s.}\tag{2.1}$$

For $t\geq 0$, let $n(t)\in\{0,1,\ldots\}$ be the unique index with $\tau_{n(t)}\leq t<\tau_{n(t)+1}$. Then, $(n(t)+1)_{t\geq 0}$ is the Yule process for which it is known (see Theorems 1 and 2 on pp. 111-112 in Athreya and Ney (1972)) that $\lim_{t\to\infty}\mathrm{e}^{-t}n(t)=W$ a.s., where W is a random variable satisfying $W>0$ a.s. It follows that $\lim_{t\to\infty}(\log n(t)-t)=\log W$ a.s. and $\lim_{t\to\infty}t^{-1}\log n(t)=1$ a.s. Consequently,

$$\frac{t^k-(\log n(t))^k}{t^{k-1}}=(t-\log n(t))\cdot\frac{\sum_{j=0}^{k-1}t^j(\log n(t))^{k-1-j}}{t^{k-1}}\underset{t\to\infty}{\longrightarrow}-k\cdot\log W\quad\text{a.s.}$$

Note that $Y_k(\tau_{n(t)})=Y_k(t)$. The identity

$$\frac{Y_k(\tau_{n(t)})-(\log n(t))^k/k!}{(2(\log n(t))^{2k-1}\log\log\log n(t))^{1/2}}=\bigg(\frac{Y_k(t)-t^k/k!}{(2t^{2k-1}\log\log t)^{1/2}}+\frac{t^k-(\log n(t))^k}{k!(2t^{2k-1}\log\log t)^{1/2}}\bigg)\times\frac{(2t^{2k-1}\log\log t)^{1/2}}{(2(\log n(t))^{2k-1}\log\log\log n(t))^{1/2}},$$

in which

$$\lim_{t\to\infty}\frac{(2t^{2k-1}\log\log t)^{1/2}}{(2(\log n(t))^{2k-1}\log\log\log n(t))^{1/2}}=1\quad\text{and}\quad\lim_{t\to\infty}\frac{t^k-(\log n(t))^k}{k!(2t^{2k-1}\log\log t)^{1/2}}=0\quad\text{a.s.},$$

combined with (2.1) entails that

$$C\bigg(\bigg(\frac{a_k\big(Y_k(\tau_{n(t)})-(\log n(t))^k/k!\big)}{(2(\log n(t))^{2k-1}\log\log\log n(t))^{1/2}}:t\geq\tau_{\lceil\mathrm{e}^{\mathrm{e}}\rceil}\bigg)\bigg)=[-1,1]\quad\text{a.s.}$$

It follows that

$$C\bigg(\bigg(\frac{a_k\big(Y_k(\tau_n)-(\log n)^k/k!\big)}{(2(\log n)^{2k-1}\log\log\log n)^{1/2}}:n>\mathrm{e}^{\mathrm{e}}\bigg)\bigg)=[-1,1]\quad\text{a.s.}$$

This completes the proof of Theorem 2.1 since $X_n(k)=Y_k(\tau_n)$. □

3. Auxiliary results

For the proof of Theorem 1.1 we shall need the following strong approximation result, which follows, for instance, from Theorem 12.13 on p. 227 in Kallenberg (1997).

Lemma 3.1. *Assume that $\sigma^2=\mathrm{Var}\,\xi\in(0,\infty)$. Then there exists a standard Brownian motion W such that*

$$\lim_{t\to\infty}\frac{\sup_{0\leq s\leq t}\big|Y(s)-V(s)-\sigma\mu^{-3/2}W(s)\big|}{(t\log\log t)^{1/2}}=0\quad\text{a.s.},$$

where $\mu=\mathbb{E}\xi<\infty$.

Now we lay down the ground for the subsequent proofs. For $t\geq 0$, $k\geq 2$ and $r\in\mathbb{N}$, let $Y_{k-1}^{(r)}(t)$ be the number of the kth generation individuals who are descendants of the first-generation individual with birth time S_r, and whose birth times fall in $[S_r,\,S_r+t]$. Then

$$Y_k(t)=\sum_{r\geq 1}Y_{k-1}^{(r)}(t-S_r)\,\mathbb{1}_{\{S_r\leq t\}}\,.\tag{3.1}$$

By the branching property, $(Y_{k-1}^{(1)}(t))_{t\geq 0}, (Y_{k-1}^{(2)}(t))_{t\geq 0}, \ldots$ are independent copies of $(Y_{k-1}(t))_{t\geq 0}$ which are also independent of S. Noting that, for $y \geq 0$, $k \geq 2$ and $r \in \mathbb{N}$, $V(y) = \sum_{r\geq 1} \mathbb{P}\{S_r \leq y\}$ and $\mathbb{E}Y_{k-1}^{(r)}(y) = V_{k-1}(y)$ and passing in (3.1) to expectations we obtain, for $k \geq 2$ and $t \geq 0$,

$$V_k(t) = \sum_{r\geq 1} \int_{[0,\,t]} \mathbb{E}Y_{k-1}^{(r)}(t-y)\mathrm{d}\mathbb{P}\{S_r \leq y\} = \int_{[0,\,t]} V_{k-1}(t-y)\mathrm{d}V(y) = \int_{[0,\,t]} V(t-y)\mathrm{d}V_{k-1}(y). \quad (3.2)$$

Thus, V_k is the k-fold Lebesgue-Stieltjes convolution of V with itself.

For fixed $d > 0$, the distribution of a positive random variable is called d-lattice if it is concentrated on the lattice $(nd)_{n\in\mathbb{N}_0}$ and not concentrated on $(nd_1)_{n\in\mathbb{N}_0}$ for any $d_1 > d$. The number d is called span of the corresponding lattice distribution. The distribution of a positive random variable is called nonlattice if it is not d-lattice for any $d > 0$. Lemma 3.2 collects some properties of $V_k = \mathbb{E}Y_k$.

Lemma 3.2. *Fix any* $k \in \mathbb{N}$.
(a) Assume that $\mu = \mathbb{E}\xi < \infty$. *Then*

$$\lim_{t\to\infty} \frac{V_k(t)}{t^k} = \frac{1}{k!\mu^k}.$$

(b) Assume that $\mu = \mathbb{E}\xi < \infty$. *Then*

$$\lim_{t\to\infty} \frac{V_k(t+h) - V_k(t)}{t^{k-1}} = \frac{h}{(k-1)!\mu^k}$$

for each $h > 0$ *if the distribution of* ξ *is nonlattice and* $h = id$, $i \in \mathbb{N}$ *if the distribution of* ξ *is* d*-lattice.*
(c) Assume that $\mathbb{E}\xi^2 < \infty$. *Then*

$$-\infty < \liminf_{t\to\infty} \frac{V_k(t) - t^k/(k!\mu^k)}{t^{k-1}} \leq \limsup_{t\to\infty} \frac{V_k(t) - t^k/(k!\mu^k)}{t^{k-1}} < \infty.$$

(d) For all $x, h \geq 0$,

$$V_k(x+h) - V_k(x) \leq (V(h)+1)(V(x+h))^{k-1}. \quad (3.3)$$

Proof: (a) See, for instance, Theorem 1.16 on p. 38 in Mitov and Omey (2014).
(b) When the distribution of ξ is nonlattice, this is a particular case ($\eta = \xi$) of Theorem 2.4 in Iksanov et al. (2023). Assume now that, for some $d > 0$, the distribution of ξ is d-lattice. Then $\lim_{t\to\infty}(V_1(t+h) - V_1(t)) = \mu^{-1}h$ for $h = id$, $i \in \mathbb{N}$ by Blackwell's theorem, see Theorem 1.10 in Mitov and Omey (2014). With this at hand, the same proof by induction as in Iksanov et al. (2023) also works in the lattice case.
(c) Assume that the distribution of ξ is nonlattice. Using Theorem 2.2 in Iksanov et al. (2023), with $\eta = \xi$ in the notation of that paper, we conclude that, for each fixed $k \in \mathbb{N}$,

$$V_k(t) - \frac{t^k}{k!\mu^k} \sim \frac{bkt^{k-1}}{(k-1)!\mu^{k-1}}, \quad t \to \infty,$$

where $b = \mathbb{E}\xi^2/(2\mu^2) - 1 \in \mathbb{R}$.

Assume now that, for some $d > 0$, the distribution of ξ is d-lattice. Then for each $t \geq 0$ there exists $n \in \mathbb{N}_0$ such that $t \in [nd, (n+1)d)$. Hence, by monotonicity,

$$\begin{aligned}\limsup_{t\to\infty} \frac{V_k(t) - t^k/(k!\mu^k)}{t^{k-1}} &\leq \limsup_{n\to\infty} \frac{V_k((n+1)d) - (nd)^k/(k!\mu^k)}{(nd)^{k-1}} \\ &= \limsup_{n\to\infty} \frac{V_k((n+1)d) - V_k(nd)}{(nd)^{k-1}} + \limsup_{n\to\infty} \frac{V_k(nd) - (nd)^k/(k!\mu^k)}{(nd)^{k-1}}.\end{aligned}$$

The former limit superior (actually, the full limit) is finite according to Lemma 3.2(b), and the latter limit superior (the full limit) is finite according to Lemma 5.1 with $\eta = \xi$. The finiteness of the lower limit follows analogously.

(d) We use mathematical induction in k. If $k = 1$, then (3.3) expresses a known fact stating that the renewal function $V + 1$ is subadditive, see, for instance, Theorem 1.7 on p. 10 in Mitov and Omey (2014). Assume that (3.3) holds for $k = j$ and note that, by monotonicity and (3.2), $V_j(h) \le (V(h))^j \le (V(h)+1)(V(x+h))^{j-1}$ for $x, h \ge 0$. We obtain, by another application of (3.2),

$$V_{j+1}(x+h) - V_{j+1}(x) = \int_{[0,\,x]} (V_j(x+h-y) - V_j(x-y)) \mathrm{d}V(y) + \int_{(x,\,x+h]} V_j(x+h-y) \mathrm{d}V(y)$$

$$\le (V(h)+1) \int_{[0,\,x]} (V(x+h-y))^{j-1} \mathrm{d}V(y) + V_j(h)(V(x+h) - V(x))$$

$$\le (V(h)+1)(V(x+h))^{j-1}V(x) + (V(h)+1)(V(x+h))^{j-1}(V(x+h) - V(x)) = (V(h)+1)(V(x+h))^j.$$

□

Here is another important ingredient for our proof of Theorem 1.1.

Lemma 3.3. *Fix any $k \in \mathbb{N}$. Assume that* $\operatorname{Var} \xi \in (0, \infty)$. *Then*

$$\mathbb{E} \sup_{0 \le s \le t} (Y_k(s) - V_k(s))^2 = O(t^{2k-1}), \quad t \to \infty. \tag{3.4}$$

Proof: In the setting of iterated perturbed random walks a counterpart of (3.4) is a consequence of Lemmas 4.2(b) and 3.1(c) in Gnedin and Iksanov (2020) under the assumptions $\operatorname{Var} \xi \in (0, \infty)$ and $\mathbb{E}\eta < \infty$. Relation (3.4) itself follows by putting $\eta = \xi$. □

To facilitate reading of the proof of Theorem 1.1, we single out its most technical part as a lemma.

Lemma 3.4. *For $t \ge 0$ and $k \ge 2$, put*

$$B_{2,k}(t) := \int_{(0,\,t]} \Big(V_{k-1}(t-x) - \frac{(t-x)^{k-1}}{(k-1)!\mu^{k-1}} \Big) \mathrm{d}W(x),$$

where W is a standard Brownian motion appearing in Lemma 3.1, and the integral is understood as an Itô integral. Then

$$\lim_{t \to \infty} \frac{B_{2,k}(t)}{t^{k-1/2}} = 0 \quad \text{a.s.}$$

Proof: It suffices to show that, for all $\varepsilon > 0$,

$$\sum_{n \ge 1} \mathbb{P}\{ \sup_{t \in [n,\,n+1]} |B_{2,k}(t)| > \varepsilon n^{k-1/2} \} < \infty. \tag{3.5}$$

Indeed, if this is true, then, by the Borel–Cantelli lemma,

$$\sup_{t \in [n,\,n+1]} |B_{2,k}(t)| \le \varepsilon n^{k-1/2}$$

for n large enough a.s. Hence, for all large enough n and $t \in [n,\, n+1]$,

$$|B_{2,k}(t)| \le \sup_{t \in [n,\,n+1]} |B_{2,k}(t)| \le \varepsilon n^{k-1/2} \le \varepsilon t^{k-1/2} \quad \text{a.s.}$$

Thus, $\limsup_{t \to \infty} |B_{2,k}(t)|/t^{k-1/2} \le \varepsilon$ a.s. which entails the claim.

Let us prove (3.5). In what follows C_1, $C_2, \ldots$ will denote positive constants, whose values are of no importance. Put $f_k(t) := V_{k-1}(t) - ((k-1)!\mu^{k-1})^{-1}t^{k-1}$ for $k \geq 2$ and $t \geq 0$. Write

$$\begin{aligned}\sup_{t\in[n,\,n+1]}|B_{2,k}(t)-B_{2,k}(n)| &= \sup_{t\in[0,\,1]}\Big|\int_{(0,\,n+t]}f_k(n+t-x)\mathrm{d}W(x)-\int_{(0,\,n]}f_k(n-x)\mathrm{d}W(x)\Big|\\ &= \sup_{t\in[0,\,1]}\Big|\int_{(n,\,n+t]}f_k(n+t-x)\mathrm{d}W(x)+\int_{(0,\,n]}(f_k(n+t-x)-f_k(n-x))\mathrm{d}W(x)\Big|\\ &\leq \sup_{t\in[0,\,1]}\Big|\int_{(n,\,n+t]}f_k(n+t-x)\mathrm{d}W(x)\Big|+\sup_{t\in[0,\,1]}\Big|\int_{(0,\,n]}(f_k(n+t-x)-f_k(n-x))\mathrm{d}W(x)\Big|.\end{aligned}$$

By the definition of the Itô integral in combination with the fact that a standard Brownian motion has independent increments with zero mean normal distributions, we conclude that the variable $B_{2,k}(n)$ has a normal distribution with zero mean and variance $\int_0^n (f_k(x))^2\mathrm{d}x$. By Lemma 3.2(c), for large enough n, $\int_0^n (f_k(x))^2\mathrm{d}x \leq C_1 n^{2k-3}$. Hence, for all $\varepsilon > 0$ and large n,

$$\mathbb{P}\{|B_{2,k}(n)| > \varepsilon n^{k-1/2}\} \leq \Big(\frac{2}{\pi}\Big)^{1/2}\int_{\varepsilon C_1^{-1/2}n}^{\infty}\mathrm{e}^{-x^2/2}\mathrm{d}x \leq \Big(\frac{2C_1}{\varepsilon^2\pi}\Big)^{1/2}\frac{\mathrm{e}^{-\varepsilon^2 n^2/(2C_1)}}{n}.$$

The right-hand side is the nth term of a summable sequence.

Observe that the process $B_{2,k}$ is a.s. continuous. Indeed,

$$B_{2,k}(t) = \int_{[0,\,t)}W(t-x)\mathrm{d}V_{k-1}(x) - \frac{1}{(k-1)!\mu^{k-1}}\int_{[0,\,t)}W(t-x)\mathrm{d}x^{k-1},$$

and each of the summands is a.s. continuous as the Lebesgue-Stieltjes convolution of an a.s. continuous function and nondecreasing function. In view of the a.s. continuity, which entails the a.s. boundedness on $[0, 1]$, we infer, for all $\varepsilon > 0$,

$$-\log\mathbb{P}\{\sup_{t\in[0,\,1]}|B_{2,k}(t)| > \varepsilon n^{k-1/2}\} \sim \frac{\varepsilon^2 n^{2k-1}}{2\int_0^1 f_k^2(y)\mathrm{d}y}, \quad n\to\infty$$

by a large deviation bound for a.s. bounded Gaussian processes, see formula (1.1) in Marcus and Shepp (1972). Since the variable $\sup_{t\in[0,\,1]}\Big|\int_{(n,\,n+t]}f_k(n+t-x)\mathrm{d}W(x)\Big|$ has the same distribution as $\sup_{t\in[0,\,1]}|B_{2,k}(t)|$ we conclude that the sequence

$$n\mapsto\mathbb{P}\Big\{\sup_{t\in[0,\,1]}\Big|\int_{(n,\,n+t]}f_k(n+t-x)\mathrm{d}W(x)\Big| > \varepsilon n^{k-1/2}\Big\}$$

is summable.

To proceed, we note that the variable $\sup_{t\in[0,\,1]}\Big|\int_{(0,\,n]}(f_k(n+t-x)-f_k(n-x))\mathrm{d}W(x)\Big|$ has the same distribution as $\sup_{t\in[0,\,1]}\Big|\int_{(0,\,n]}(f_k(x+t)-f_k(x))\mathrm{d}W(x)\Big|$. Whenever a Skorokhod integral is well-defined, it coincides with the result of (formal) integration by parts. In particular,

$$\begin{aligned}\int_{(0,\,n]}(f_k(x+t)-f_k(x))\mathrm{d}W(x) &= (f_k(n+t)-f_k(n))W(n)-\int_{(0,\,n]}W(x)\mathrm{d}_x(f_k(x+t)-f_k(x))\\ &= (f_k(n+t)-f_k(n))W(n)+\int_{(0,\,t]}W(x)\mathrm{d}f_k(x)+\int_{(0,\,n-t]}(W(x+t)-W(x))\mathrm{d}f_k(x+t)\\ &\qquad -\int_{(n-t,\,n]}W(x)\mathrm{d}f_k(x+t).\end{aligned}$$

Hence, since the function V_{k-1} is nondecreasing,

$$\sup_{t\in[0,\,1]}\Big|\int_{(0,\,n]}(f_k(x+t)-f_k(x))\mathrm{d}W(x)\Big|\leq (V_{k-1}(n+1)-V_{k-1}(n))|W(n)|$$
$$+((k-1)!\mu^{k-1})^{-1}((n+1)^{k-1}-n^{k-1})|W(n)|+\sup_{t\in[0,\,1]}\Big|\int_{(0,\,t]}W(x)\mathrm{d}V_{k-1}(x)\Big|$$
$$+((k-1)!\mu^{k-1})^{-1}\sup_{t\in[0,\,1]}\Big|\int_{(0,\,t]}W(x)\mathrm{d}x^{k-1}\Big|+\sup_{t\in[0,\,1]}\int_{(0,\,n-t]}|W(x+t)-W(x)|\mathrm{d}_xV_{k-1}(x+t)$$
$$+((k-1)!\mu^{k-1})^{-1}\sup_{t\in[0,\,1]}\int_{(0,\,n-t]}|W(x+t)-W(x)|\mathrm{d}_x(x+t)^{k-1}$$
$$+\sup_{t\in[0,\,1]}\int_{(n-t,\,n]}|W(x)|\mathrm{d}_xV_{k-1}(x+t)+((k-1)!\mu^{k-1})^{-1}\sup_{t\in[0,\,1]}\int_{(n-t,\,n]}|W(x)|\mathrm{d}_x(x+t)^{k-1}. \quad (3.6)$$

We shall only treat the terms involving V_{k-1}, for the analysis of the terms involving $t\mapsto t^{k-1}$ is analogous but easier. We start with the penultimate term in (3.6). Observe that

$$\sup_{t\in[0,\,1]}\int_{(n-t,\,n]}|W(x)|\mathrm{d}_xV_{k-1}(x+t)\leq \sup_{t\in[0,\,1]}(V_{k-1}(n+t)-V_{k-1}(n))\sup_{n-t\leq z\leq n}|W(z)|$$
$$\leq (V_{k-1}(n+1)-V_{k-1}(n))\sup_{n-1\leq z\leq n}|W(z)|.$$

By Lemma 3.2(b), for large n, $V_{k-1}(n+1)-V_{k-1}(n)\leq C_2n^{k-2}$. The random variable $\sup_{z\in[n-1,\,n]}|W(z)|$ has the same distribution as $\sup_{z\in[0,1]}|W(z)+W'(n-1)|$, where $W'(n-1)$ is a copy of $W(n-1)$ which is independent of $\sup_{z\in[0,1]}|W(z)|$. Hence, for all $\varepsilon>0$ and large n,

$$\mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))\sup_{z\in[n-1,\,n]}|W(z)|>\varepsilon n^{k-1/2}\}$$
$$\leq \mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))\sup_{z\in[0,1]}|W(z)|>\varepsilon n^{k-1/2}/2\}$$
$$+\mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))|W'(n-1)|>\varepsilon n^{k-1/2}/2\}=:R_{n,1}+R_{n,2}.$$

Further,

$$R_{n,2}\leq\Big(\frac{2}{\pi}\Big)^{1/2}\int_{2^{-1}C_2^{-1}\varepsilon n}^{\infty}\mathrm{e}^{-x^2/2}\mathrm{d}x\leq\Big(\frac{8C_2^2}{\varepsilon^2\pi}\Big)^{1/2}\frac{\mathrm{e}^{-\varepsilon^2n^2/(8C_2^2)}}{n}.$$

The right-hand side is the nth term of a summable sequence. Using the inequalities

$$\mathbb{P}\{\sup_{t\in[0,\,1]}|W(t)|>x\}\leq 2\mathbb{P}\{\sup_{t\in[0,\,1]}W(t)>x\}=2\mathbb{P}\{|W(1)|>x\},\quad x>0 \quad (3.7)$$

we also conclude that the sequence $(R_{n,1})_{n\in\mathbb{N}}$ is summable. Thus, for all $\varepsilon>0$, the sequence

$$n\mapsto\mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))\sup_{z\in[n-1,\,n]}|W(z)|>\varepsilon n^{k-1/2}\}$$

is summable, and so is

$$n\mapsto\mathbb{P}\{(V_{k-1}(n+1)-V_{k-1}(n))|W(n)|>\varepsilon n^{k-1/2}\}$$

because $|W(n)|\leq\sup_{z\in[n-1,\,n]}|W(z)|$. For all $\varepsilon>0$, the sequence

$$n\mapsto\mathbb{P}\Big\{\sup_{t\in[0,\,1]}\Big|\int_{(0,\,t]}W(x)\mathrm{d}V_{k-1}(x)\Big|>\varepsilon n^{k-1/2}\Big\}$$

is summable in view of the bound

$$\sup_{t\in[0,\,1]}\Big|\int_{(0,\,t]}W(x)\mathrm{d}V_{k-1}(x)\Big|\leq\int_{(0,\,1]}|W(x)|\mathrm{d}V_{k-1}(x)\leq\sup_{t\in[0,\,1]}|W(t)|V_{k-1}(1)$$

and (3.7).

Finally,

$$\sup_{t\in[0,\,1]}\int_{(0,\,n-t]}|W(x+t)-W(x)|\mathrm{d}_xV_{k-1}(x+t)\leq V_{k-1}(n)\sup_{t\in[0,\,1]}\sup_{z\in[0,\,n]}|W(z+t)-W(z)|.$$

By Lemma 3.2(a), $V_{k-1}(n)\leq C_3n^{k-1}$ for large n. By Lemma 1.2.1 on p. 29 in Csörgő and Révész (1981), given $\delta>0$ there exists $C_0=C_0(\delta)>0$ such that, for all $\varepsilon>0$ and $n\geq 2$,

$$\mathbb{P}\{V_{k-1}(n)\sup_{t\in[0,\,1]}\sup_{z\in[0,\,n]}|W(z+t)-W(z)|>\varepsilon n^{k-1/2}\}\leq C_0(n+1)\exp\Big(-\frac{\varepsilon^2n}{C_3^2(2+\delta)}\Big).$$

This proves that the sequence

$$n\mapsto\mathbb{P}\Big\{\sup_{t\in[0,\,1]}\int_{(0,\,n-t]}|W(x+t)-W(x)|\mathrm{d}_xV_{k-1}(x+t)>\varepsilon n^{k-1/2}\Big\}$$

is summable. Combining fragments together we arrive at (3.5). □

4. Proof of Theorem 1.1

The possibility of replacing $t\mapsto t^k/(k!\mu^k)$ with V_k is justified by Lemma 3.2(c).

Since Y_1 is a renewal process, the case $k=1$ of Theorem 1.1 was known, see Proposition 3.5 in Iksanov et al. (2017). Thus, in what follows it is tacitly assumed that $k\geq 2$.

Throughout the proof, for notational simplicity, we assume that if the distribution of ξ is lattice, its lattice span is 1. Using (3.1) we obtain a basic decomposition for the present proof: for $k\geq 2$ and $t\geq 0$:

$$Y_k(t)-V_k(t)=\sum_{r\geq 1}\big(Y_{k-1}^{(r)}(t-S_r)-V_{k-1}(t-S_r)\big)\,\mathbb{1}_{\{S_r\leq t\}}\\ +\Big(\sum_{r\geq 1}V_{k-1}(t-S_r)\,\mathbb{1}_{\{S_r\leq t\}}-V_k(t)\Big)=:I_k(t)+J_k(t).$$

We shall prove that

$$C\bigg(\bigg(\frac{a_kJ_k(t)}{(2t^{2k-1}\log\log t)^{1/2}}:t>\mathrm{e}\bigg)\bigg)=[-1,1]\quad\text{a.s.}\tag{4.1}$$

and that

$$\lim_{t\to\infty}\frac{I_k(t)}{(t^{2k-1}\log\log t)^{1/2}}=0\quad\text{a.s.},\tag{4.2}$$

that is, the term J_k gives the principal contribution, whereas the contribution of I_k is negligible.

Proof of (4.1). Recalling (3.2) write, with the help of integration by parts, for $k\geq 2$ and $t\geq 0$,

$$J_k(t)=\int_{[0,\,t]}V_{k-1}(t-x)\mathrm{d}(Y(x)-V(x))=\int_{[0,\,t)}(Y(t-x)-V(t-x))\mathrm{d}V_{k-1}(x)\\ =\int_{[0,\,t)}(Y(t-x)-V(t-x)-\sigma\mu^{-3/2}W(t-x))\mathrm{d}V_{k-1}(x)+\sigma\mu^{-3/2}\int_{[0,\,t)}W(t-x)\mathrm{d}V_{k-1}(x)\\ =:A_k(t)+\sigma\mu^{-3/2}B_k(t),$$

where W is a standard Brownian motion appearing in Lemma 3.1. By Lemmas 3.1 and 3.2(a),

$$|A_k(t)|\leq\sup_{0\leq u\leq t}|Y(u)-V(u)-\sigma\mu^{-3/2}W(u)|V_{k-1}(t)=o\big((t^{2k-1}\log\log t)^{1/2}\big),\quad t\to\infty\quad\text{a.s.}$$

Further,

$$B_k(t)=\frac{1}{(k-1)!\mu^{k-1}}\int_{(0,\,t]}(t-x)^{k-1}\mathrm{d}W(x)+\int_{(0,\,t]}\Big(V_{k-1}(t-x)-\frac{(t-x)^{k-1}}{(k-1)!\mu^{k-1}}\Big)\mathrm{d}W(x)$$
$$=:((k-1)!\mu^{k-1})^{-1}B_{1,k}(t)+B_{2,k}(t).$$

By Lemma 3.4, the contribution of $B_{2,k}(t)$ is negligible. By Theorem 1 in Lachal (1997),

$$\limsup_{t\to\infty}\frac{(2k-1)^{1/2}B_{1,k}(t)}{(2t^{2k-1}\log\log t)^{1/2}}=1 \quad \text{a.s.}$$

Since $-W$ is also a Brownian motion we infer

$$\liminf_{t\to\infty}\frac{(2k-1)^{1/2}B_{1,k}(t)}{(2t^{2k-1}\log\log t)^{1/2}}=-1 \quad \text{a.s.}$$

and thereupon

$$C\Big(\Big(\frac{(2k-1)^{1/2}B_{1,k}(t)}{(2t^{2k-1}\log\log t)^{1/2}}:t>\mathrm{e}\Big)\Big)=[-1,1] \quad \text{a.s.}$$

because the random function $t\mapsto B_{1,k}(t)t^{1/2-k}(\log\log t)^{-1/2}$ is a.s. continuous on (e,∞). This completes the proof of (4.1).

Proof of (4.2). Recall that $k\geq 2$. Invoking Lemmas 3.2(a) and 3.3 yields

$$\mathbb{E}(I_k(t))^2=\int_{[0,\,t]}\mathbb{E}\big(Y_{k-1}(t-x)-V_{k-1}(t-x)\big)^2\mathrm{d}V(x)$$
$$\leq\mathbb{E}\big(\sup_{s\in[0,\,t]}(Y_{k-1}(s)-V_{k-1}(s))^2\big)\cdot V(t)=O(t^{2k-2}),\quad t\to\infty. \tag{4.3}$$

By Markov's inequality and (4.3), for all $\varepsilon>0$,

$$\sum_{n\geq 1}\mathbb{P}\Big\{\frac{I_k(n^{3/2})}{n^{(3/2)(k-1/2)}}>\varepsilon\Big\}\leq\sum_{n\geq 1}\frac{\mathbb{E}(I_k(n^{3/2}))^2}{\varepsilon^2 n^{3(k-1/2)}}<\infty.$$

Hence,

$$\lim_{n\to\infty}\frac{I_k(n^{3/2})}{n^{(3/2)(k-1/2)}}=0 \quad \text{a.s.} \tag{4.4}$$

It remains to pass from an integer argument to a continuous argument. For any $t\geq 0$, there exists $n\in\mathbb{N}_0$ such that $t\in[n^{3/2},(n+1)^{3/2})$. By monotonicity,

$$\frac{I_k(t)}{t^{k-1/2}}\leq\frac{I_k((n+1)^{3/2})}{n^{(3/2)(k-1/2)}}+\frac{\int_{[0,\,(n+1)^{3/2}]}V_{k-1}((n+1)^{3/2}-x)\mathrm{d}Y(x)-\int_{[0,\,n^{3/2}]}V_{k-1}(n^{3/2}-x)\mathrm{d}Y(x)}{n^{(3/2)(k-1/2)}}.$$

Relation (4.4) implies that the first summand on the right-hand side converges to 0 a.s. as $n\to\infty$. The second summand is equal to

$$\int_{(n^{3/2},\,(n+1)^{3/2}]}V_{k-1}((n+1)^{3/2}-x)\mathrm{d}Y(x)$$
$$+\int_{[0,\,n^{3/2}]}(V_{k-1}((n+1)^{3/2}-x)-V_{k-1}(n^{3/2}-x))\mathrm{d}Y(x)=:X_{k,1}(n)+X_{k,2}(n).$$

By monotonicity, as $n\to\infty$ a.s.

$$X_{k,1}(n)\leq V_{k-1}((n+1)^{3/2}-n^{3/2})\big(Y((n+1)^{3/2})-Y(n^{3/2})\big)=o(n^{k/2+1})=o(n^{(3/2)(k-1/2)}).$$

Here, the penultimate equality is justified by the strong law of large numbers for renewal processes $\lim_{n\to\infty}n^{-1}Y(n)=\mu^{-1}$ a.s. and $V_{k-1}((n+1)^{3/2}-n^{3/2})=O(n^{(k-1)/2})$ as $n\to\infty$ which holds true by Lemma 3.2(a).

Using Lemma 3.2(d) we infer

$$X_{k,2}(n) \le \big(V((n+1)^{3/2}-n^{3/2})+1\big)(V((n+1)^{3/2}))^{k-2}Y(n^{3/2}) = O(n^{(3/2)(k-2/3)}) = o(n^{(3/2)(k-1/2)})$$

a.s. as $n\to\infty$. The penultimate equality is secured by Lemma 3.2(a) and the strong law of large numbers for renewal processes.

We have shown that

$$\limsup_{t\to\infty} t^{-(k-1/2)} I_k(t) \le 0 \quad \text{a.s.}$$

An analogous argument proves the converse inequality for the lower limit. The proof of Theorem 1.1 is complete.

5. Appendix

Lemma 5.1 is a lattice analogue of Theorem 2.2 in Iksanov et al. (2023) dealing with iterated perturbed random walks. In the proof of Lemma 3.2(c) we only need a version of Lemma 5.1 for iterated standard random walks.

Lemma 5.1. *Let $d>0$. Assume that the distributions of ξ and η are d-lattice and that $\mathbb{E}\xi^2<\infty$ and $\mathbb{E}\eta<\infty$. Then, for each fixed $k\in\mathbb{N}$,*

$$V_k^*(nd) - \frac{(nd)^k}{k!\mu^k} \sim \frac{(nd)^{k-1}\Big(\frac{d}{2\mu}+k\big(\frac{\mathbb{E}\xi^2}{2\mu^2}-\frac{\mathbb{E}\eta}{\mu}\big)\Big)}{\mu^{k-1}(k-1)!}, \quad n\to\infty, \tag{5.1}$$

where $\mu=\mathbb{E}\xi<\infty$ and V_k^* is a counterpart of V_k for iterated perturbed random walks.

Proof: We use mathematical induction in k. Let $k=1$. Put $U(t):=V(t)+1$ for $t\ge 0$, so that U is the renewal function. Since $V_1^*(t)=\sum_{j\ge1}\mathbb{P}\{S_{j-1}+\eta_j\le t\} =: V^*(t)$ for $t\ge0$, we infer, for $n\in\mathbb{N}$,

$$\begin{aligned} V^*(nd)-\frac{nd}{\mu} &= \int_{[0,\,nd]}\Big(U(nd-x)-\frac{nd-x}{\mu}\Big)\mathrm{d}\mathbb{P}\{\eta\le x\} - \frac{1}{\mu}\int_0^{nd}\mathbb{P}\{\eta>x\}\mathrm{d}x \\ &= \sum_{r=1}^{n}\Big(U((n-r)d)-\frac{(n-r)d}{\mu}\Big)\mathbb{P}\{\eta=rd\} - \frac{1}{\mu}\int_0^{nd}\mathbb{P}\{\eta>x\}\mathrm{d}x. \end{aligned}$$

According to formula (5.14) on p. 59 in Gut (2009),

$$\lim_{n\to\infty}\Big(U(nd)-\frac{nd}{\mu}\Big) = \frac{d}{2\mu}+\frac{\mathbb{E}\xi^2}{2\mu^2} =: D.$$

Hence, given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $U(nd)-\mu^{-1}nd\le D+\varepsilon$ for all $n\ge n_0$. With this at hand, for $n\ge n_0$,

$$\begin{aligned} \sum_{r=1}^{n}\Big(U((n-r)d)-\frac{(n-r)d}{\mu}\Big)\mathbb{P}\{\eta=rd\} &= \sum_{r=1}^{n-n_0}\ldots + \sum_{r=n-n_0+1}^{n}\ldots \\ &\le D+\varepsilon+\sup_{1\le r\le n_0-1}\Big(U(rd)-\frac{rd}{\mu}\Big)\mathbb{P}\{\eta\ge (n-n_0+1)d\}. \end{aligned}$$

Letting $n\to\infty$ and then $\varepsilon\to 0+$ we conclude that

$$\limsup_{n\to\infty}\sum_{r=1}^{n}\Big(U((n-r)d)-\frac{(n-r)d}{\mu}\Big)\mathbb{P}\{\eta=rd\}\le D.$$

The converse inequality for the lower limit follows analogously. Noting that $\lim_{n\to\infty}\int_0^{nd}\mathbb{P}\{\eta> x\}\mathrm{d}x=\mathbb{E}\eta$ completes the proof of (5.1) with $k=1$.

Assume now that (5.1) holds for $k \le j$. In particular, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$V_k^*(nd) - \frac{(nd)^k}{k!\mu^k} \le (C_k + \varepsilon)\frac{(nd)^{k-1}}{(k-1)!\mu^{k-1}}, \quad 1 \le k \le j,$$

where

$$C_k := \frac{d}{2\mu} + k\Big(\frac{\mathbb{E}\xi^2}{2\mu^2} - \frac{\mathbb{E}\eta}{\mu}\Big), \quad k \in \mathbb{N}.$$

Recalling (3.2) we obtain

$$V_{j+1}^*(nd) - \frac{(nd)^{j+1}}{(j+1)!\mu^{j+1}} = \sum_{r=1}^{n}\Big(V^*((n-r)d) - \frac{(n-r)d}{\mu}\Big)\Big(V_j^*(rd) - V_j^*((r-1)d)\Big)$$
$$+ \frac{d}{\mu}\sum_{r=1}^{n-1}\Big(V_j^*(rd) - \frac{(rd)^j}{j!\mu^j}\Big) + \frac{d^{j+1}}{j!\mu^{j+1}}\Big(\sum_{r=1}^{n-1} r^j - \frac{n^{j+1}}{j+1}\Big).$$

Hence, for $n \ge n_0$,

$$A_j(n) := \sum_{r=1}^{n}\Big(V^*((n-r)d) - \frac{(n-r)d}{\mu}\Big)\Big(V_j^*(rd) - V_j^*((r-1)d)\Big) = \sum_{r=1}^{n-n_0}\ldots + \sum_{r=n-n_0+1}^{n}\ldots$$
$$\le (C_1+\varepsilon)V_j^*((n-n_0)d) + \sup_{1\le r\le n_0-1}\Big(V^*(rd) - \frac{rd}{\mu}\Big)\big(V_j^*(nd) - V_j^*((n-n_0)d)\big).$$

Invoking parts (a) and (b) of Lemma 3.2 and letting $n \to \infty$ and then $\varepsilon \to 0+$ yields

$$\limsup_{n\to\infty}\frac{A_j(n)}{(nd)^j} \le \frac{C_1}{j!\mu^j}.$$

Further,

$$B_j(n) := \sum_{r=1}^{n-1}\Big(V_j^*(rd) - \frac{(rd)^j}{j!\mu^j}\Big) = \sum_{r=1}^{n_0-1}\ldots + \sum_{r=n_0}^{n}\ldots \le n_0 \sup_{1\le r\le n_0-1}\Big(V_j^*(rd) - \frac{(rd)^j}{j!\mu^j}\Big)$$
$$+ (C_j+\varepsilon)\sum_{r=n_0}^{n}\frac{(rd)^{j-1}}{(j-1)!\mu^{j-1}} = (C_j+\varepsilon)\frac{d^{j-1}}{(j-1)!\mu^{j-1}}\frac{n^j}{j} + o(n^j), \quad n \to \infty$$

having utilized Faulhaber's formula for the last equality. Thus,

$$\limsup\nolimits_{n\to\infty}\frac{d}{\mu}\frac{B_j(n)}{(nd)^j} \le \frac{C_j}{j!\mu^j}.$$

Analogous arguments prove the converse inequalities for the lower limits involving both $A_j(n)$ and $B_j(n)$. Finally,

$$\frac{d^{j+1}}{j!\mu^{j+1}}\Big(\sum_{r=1}^{n-1} r^j - \frac{n^{j+1}}{j+1}\Big) \;\sim\; -\frac{d}{2\mu}\frac{(nd)^j}{j!\mu^j}, \quad n \to \infty$$

by another application of Faulhaber's formula.

Combining all the fragments together we conclude that

$$V_{j+1}^*(nd) - \frac{(nd)^{j+1}}{(j+1)!\mu^{j+1}} \;\sim\; \Big(C_1 + C_j - \frac{d}{2\mu}\Big)\frac{(nd)^j}{j!\mu^j} = C_{j+1}\frac{(nd)^j}{j!\mu^j}, \quad n \to \infty.$$

□

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