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Two-point local time penalizations with various clocks for Lévy processes

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Abstract. Long-time limit of one-dimensional Lévy processes weighted and normalized with respect to the exponential functional of two-point local times are studied. The limit processes may vary according to the choice of random clocks.

1. Introduction

A penalization problem is to study the long-time limit of the form

$$\lim_{\tau\to\infty} \frac{\mathbb{P}_x[F_s \cdot \Gamma_\tau]}{\mathbb{P}_x[\Gamma_\tau]}, \tag{1.1}$$

where $(X = (X_t)_{t\geq 0}, (\mathscr{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}})$ is a Markov process, $(\Gamma_t)_{t\geq 0}$ is a non-negative process called a *weight*, $(F_s)_{s\geq 0}$ is a process of test functions adapted to $(\mathscr{F}_s)_{s\geq 0}$, and τ is a net of parametrized random times tending to infinity, called a *clock*.

To solve this problem, we want to find a $(\mathscr{F}_s)$-martingale $(M_s^\Gamma)_{s\geq 0}$ and a function $\rho(\tau)$ of the clock τ such that

$$\lim_{\tau\to\infty} \rho(\tau)\mathbb{P}_x[F_s \cdot \Gamma_\tau] = \mathbb{P}_x[F_s \cdot M_s^\Gamma] \tag{1.2}$$

holds for $s \geq 0$, $x \in \mathbb{R}$, and bounded $\mathscr{F}_s$-measurable functions F_s. If $M_0^\Gamma > 0$ under $\mathbb{P}_x$, the convergence (1.2) implies

$$\lim_{\tau\to\infty} \frac{\mathbb{P}_x[F_s \cdot \Gamma_\tau]}{\mathbb{P}_x[\Gamma_\tau]} = \mathbb{P}_x\left[F_s \cdot \frac{M_s^\Gamma}{M_0^\Gamma}\right],$$

which solves the penalization problem (1.1).

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We consider a one-dimensional Lévy process X which is recurrent and for which every point is regular for itself. Let L_t^x denote the local time of x up to time t for X (subject to a suitable normalization). The weight process we consider is given as

$$\Gamma_{a,b,t}^{\lambda_a,\lambda_b} := e^{-\lambda_a L_t^a - \lambda_b L_t^b} \tag{1.3}$$

for two distinct real points a and b, and two positive constants λ_a and λ_b.

For the characteristic exponent $\Psi(\lambda)$ of X, i.e., $\mathbb{P}_0[e^{i\lambda X_t}] = e^{-t\Psi(\lambda)}$, we always assume the condition

$$\textbf{(A)} \quad \int_0^\infty \left| \frac{1}{q+\Psi(\lambda)} \right| d\lambda < \infty \qquad \text{for each } q > 0.$$

Then, the following conditions hold (Bretagnolle (1971) and Kesten (1969)):

(1) The process X is not a compound Poisson process;
(2) 0 is regular for itself, i.e., $\mathbb{P}_0(T_0 = 0) = 1$;
(3) We have either $\sigma > 0$ or $\int_{(-1,1)} |x|\nu(dx) = \infty$ (or, equivalently, the process X has unbounded variation paths.)

Let $T_x := \inf\{t > 0;\ X_t = x\}$ denote the hitting time of a point $x \in \mathbb{R}$. For $-1 \leq \gamma \leq 1$, we say[1]

$$(c,d) \overset{(\gamma)}{\to} \infty \text{ when } c \to \infty,\ d \to \infty, \text{ and } \frac{d-c}{c+d} \to \gamma.$$

Here for the random clock $\tau = (\tau_\lambda)$, we adopt one of the following:

(1) exponential clock: $\tau = (\boldsymbol{e}_q)$ as $q \to 0+$, where $\boldsymbol{e}_q$ has the exponential distribution with its parameter $q > 0$ and is independent of X;
(2) hitting time clock: $\tau = (T_c)$ as $c \to \pm\infty$, where T_c is the first hitting time at c;
(3) two-point hitting time clock: $\tau = (T_c \wedge T_{-d})$ as $(c,d) \overset{(\gamma)}{\to} \infty$;
(4) inverse local time clock: $\tau = (\eta_u^c)$ as $c \to \pm\infty$, where $\eta^c = (\eta_u^c)_{u \geq 0}$ is an inverse local time.

Then, our main theorem is as follows (see Theorems 3.4, 4.4, 5.2, and 6.3 for the details):

Theorem 1.1. *For distinct points $a, b \in \mathbb{R}$, for constants $\lambda_a, \lambda_b > 0$, and for a constant $-1 \leq \gamma \leq 1$, there exists a positive function $\varphi_{a,b}^{(\gamma),\lambda_a,\lambda_b}(x)$ such that the process*

$$\left(M_{a,b,s}^{(\gamma),\lambda_a,\lambda_b} := \varphi_{a,b}^{(\gamma),\lambda_a,\lambda_b}(X_s) e^{-\lambda_a L_s^a - \lambda_b L_s^b} \right)_{s \geq 0} \tag{1.4}$$

is a martingale, and the following assertions hold:

(1) exponential clock:

$$\lim_{q \to 0+} r_q(0) \mathbb{P}_x\left[F_s \cdot \Gamma_{a,b,\boldsymbol{e}_q}^{\lambda_a,\lambda_b} \right] = \mathbb{P}_x\left[F_s \cdot M_{a,b,s}^{(0),\lambda_a,\lambda_b} \right],$$

(2) hitting time clock:

$$\lim_{c \to \pm\infty} h^B(c) \mathbb{P}_x\left[F_s \cdot \Gamma_{a,b,T_c}^{\lambda_a,\lambda_b} \right] = \mathbb{P}_x\left[F_s \cdot M_{a,b,s}^{(\pm 1),\lambda_a,\lambda_b} \right],$$

(3) two-point hitting time clock:

$$\lim_{(c,d) \overset{(\gamma)}{\to} \infty} h^C(c,-d) \mathbb{P}_x\left[F_s \cdot \Gamma_{a,b,T_c \wedge T_{-d}}^{\lambda_a,\lambda_b} \right] = \mathbb{P}_x\left[F_s \cdot M_{a,b,s}^{(\gamma),\lambda_a,\lambda_b} \right],$$

[1] To describe the penalization limits, our limit $(c,d) \overset{(\gamma)}{\to} \infty$ is more suitable than the limit $(c,d) \overset{\gamma}{\to} \infty$ of the equation (1.9) of Takeda and Yano (2023).

(4) inverse local time clock:

$$\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_x\left[F_s\cdot\Gamma^{\lambda_a,\lambda_b}_{a,b,\eta^c_u}\right]=\mathbb{P}_x\left[F_s\cdot M^{(\pm 1),\lambda_a,\lambda_b}_{a,b,s}\right],$$

where r_q is the q-resolvent density, i.e., $r_q(0)=\mathbb{P}_0[\int_0^\infty e^{-qt}dL^0_t]$ (see (2.1)), $h^B(c)=\mathbb{P}_0[L^0_{T_c}]$ (see (2.9)), and $h^C(c,-d)=\mathbb{P}_0[L^0_{T_c\wedge T_{-d}}]$ (see (2.15)).

Let us explain the backgrounds of our results. Roynette et al. (2006a) and Roynette et al. (2006b) have studied the penalization with *constant clock* $\tau=t$ for a Brownian motion. Let $B=(B_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion and $(\mathscr{F}_t)_{t\geq 0}$ denote the natural filtration of B. In particular, when

$$\Gamma_s=f(L^0_s)$$

for a non-negative integrable function f and with $L=(L^0_s)_{s\geq 0}$ being the local time at 0 of B, this problem is called the *local time penalization*. In this case, the martingale M^Γ in (1.2) is given by

$$M^\Gamma_s=f(L^0_s)|B_s|+\int_0^\infty f(L^0_s+u)du.$$

This result for a Brownian motion has been generalized to other processes. For example, we refer to Debs (2009) for simple random walks, Najnudel et al. (2009) for Bessel processes and Markov chains, and Yano et al. (2009) for symmetric stable processes.

The local time penalization problem using random clocks date back to the *taboo process* of Knight (1969). It was the problem of conditioning to avoid zero, which is considered to be a special case of the local time penalization with weight $\Gamma_s=1_{\{L^0_s=0\}}$. This problem has been generalized to one-dimensional Lévy processes by Chaumont and Doney (2005) for conditioning to stay positive, by Pantí (2017) for conditioning to avoid zero, and to one-dimensional diffusions by Yano and Yano (2015) for conditioning to avoid zero and Profeta et al. (2019) for local time penalization.

Our penalizations with weight (1.3) are an extension of the "one-point" local time penalization in Takeda and Yano (2023) to the "two-point" one. The results of Takeda and Yano (2023) were as follows:

Theorem 1.2 (Takeda and Yano, 2023)**.** *Suppose that the condition **(A)** is satisfied. For $x\in\mathbb{R}$ and $-1\leq\gamma\leq 1$, we define*

$$h^{(\gamma)}(x):=h(x)+\frac{\gamma}{m^2}x,$$

where $m^2=\mathbb{P}_0[X_1^2]$ and the function h is a renormalized zero resolvent (see Proposition 2.1). Then, for a non-negative integrable function f, the process

$$\left(M^{(\gamma,f)}_s:=h^{(\gamma)}(X_s)f(L^0_s)+\int_0^\infty f(L^0_s+u)du\right)_{s\geq 0} \tag{1.5}$$

is a martingale. Moreover if $M^{(\gamma)}_0>0$ under $\mathbb{P}_x$, for any $\mathscr{F}_s$-measurable bounded function F_s, it holds the following[2]:

(1) exponential clock:

$$\lim_{q\to 0+} r_q(0)\mathbb{P}_x\left[F_s\cdot f(L^0_{\boldsymbol{e}_q})\right]=\mathbb{P}_x\left[F_s\cdot M^{(0,f)}_s\right],$$

[2] The results of Takeda and Yano (2023) about the two-point hitting time clocks involve several minor computational errors, all of which we correct in this paper; see Iba and Yano (2025+) for the details.

(2) hitting time clock:

$$\lim_{a\to\pm\infty} h^B(a)\mathbb{P}_x\Big[F_s\cdot f(L^0_{T_a})\Big]=\mathbb{P}_x\Big[F_s\cdot M_s^{(\pm 1,f)}\Big],$$

(3) two-point hitting time clock:

$$\lim_{(a,b)\overset{(\gamma)}{\to}\infty} h^C(a,-b)\mathbb{P}_x\Big[F_s\cdot f(L^0_{T_a\wedge T_{-b}})\Big]=\mathbb{P}_x\Big[F_s\cdot M_s^{(\gamma,f)}\Big],$$

(4) inverse local time clock:

$$\lim_{a\to\pm\infty} h^B(a)\mathbb{P}_x\Big[F_s\cdot f(L^0_{\eta^a_u})\Big]=\mathbb{P}_x\Big[F_s\cdot M_s^{(\pm 1,f)}\Big].$$

Remark 1.3. Note that when $m^2=\infty$, we have

$$h^{(\gamma)}(x)\equiv h^{(0)}(x)\equiv h(x) \tag{1.6}$$

for any $-1\le\gamma\le 1$.

Remark 1.4. The function $\varphi_{a,b}^{(\gamma),\lambda_a,\lambda_b}$ is given explicitly as follows (see Proposition 5.1):

$$\begin{aligned}\varphi_{a,b}^{(\gamma),\lambda_a,\lambda_b}(x)=h^{(\gamma)}(x-a)&-\mathbb{P}_x(T_b<T_a)h^{(\gamma)}(b-a)\\&+\mathbb{P}_x(T_a<T_b)\cdot\frac{h^{(\gamma)}(a-b)}{1+\lambda_a h^B(a-b)}\\&+\mathbb{P}_x(T_a<T_b)\cdot\frac{1}{1+\lambda_a h^B(b-a)}\cdot\frac{1+\lambda_a h^{(\gamma)}(b-a)}{\lambda_a+\lambda_b+\lambda_a\lambda_b h^B(a-b)}\\&+\mathbb{P}_x(T_b<T_a)\cdot\frac{h^{(\gamma)}(b-a)}{1+\lambda_b h^B(a-b)}\\&+\mathbb{P}_x(T_b<T_a)\cdot\frac{1}{1+\lambda_b h^B(b-a)}\cdot\frac{1+\lambda_b h^{(\gamma)}(a-b)}{\lambda_a+\lambda_b+\lambda_a\lambda_b h^B(a-b)}.\end{aligned}$$

Note that $\varphi_{a,b}^{(\gamma),\lambda_a,\lambda_b}(x)$ is symmetric with respect to a and b, i.e., for $x\in\mathbb{R}$,

$$\varphi_{a,b}^{(\gamma),\lambda_a,\lambda_b}(x)=\varphi_{b,a}^{(\gamma),\lambda_b,\lambda_a}(x).$$

When $m^2=\infty$, by (1.6), we have $\varphi_{a,b}^{(\gamma),\lambda_a,\lambda_b}(x)\equiv\varphi_{a,b}^{(0),\lambda_a,\lambda_b}(x)$ and hence $M_{a,b,t}^{(\gamma),\lambda_a,\lambda_b}=M_{a,b,t}^{(0),\lambda_a,\lambda_b}$ for any $-1\le\gamma\le 1$.

Remark 1.5. By letting $a=0,\ \lambda_a=\lambda>0$, and letting $\lambda_b\to 0+$ in the definition (1.4), we have

$$M_{0,b,t}^{(\gamma),\lambda,0}:=\lim_{\lambda_b\to 0+}M_{0,b,t}^{(\gamma),\lambda,\lambda_b}=\left(h^{(\gamma)}(X_t)+\frac{1}{\lambda}\right)e^{-\lambda L_t}.$$

This coincides with $M_t^{(\gamma,f)}$ in (1.5) with $f(x)=e^{-\lambda x}$.

Remark 1.6. In Takeda and Yano (2023), they also state that the process is transient under the limit measure

$$\mathbb{Q}_x^{(\gamma,f)}|_{\mathscr{F}_t}:=\frac{M_t^{(\gamma,f)}}{M_0^{(\gamma,f)}}\cdot\mathbb{P}_x|_{\mathscr{F}_t}.$$

Furthermore, Takeda (2024) studied in detail the behavior of $\mathbb{Q}_x^{(\gamma,f)}$ with $f(x) = 1_{\{0\}}(x)$. Our limit measure

$$\mathbb{Q}^{(\gamma),\lambda_a,\lambda_b}_{x;a,b}|_{\mathscr{F}_s} := \frac{M^{(\gamma),\lambda_a,\lambda_b}_{s;a,b}}{M^{(\gamma),\lambda_a,\lambda_b}_{0;a,b}} \cdot \mathbb{P}_x|_{\mathscr{F}_s}.$$

is also transient by the same proof as Theorem 1.4 of Takeda (2024).

In Roynette et al. (2006b), they consider the penalization problem when the weight process is given as

$$\Gamma_t = \exp\left\{ - \int_{\mathbb{R}} L_t^x q(dx)\right\}$$

for a positive Radon measure $q(dx)$ on $\mathbb{R}$, which satisfies $0 < \int_{\mathbb{R}} (1+|x|)q(dx) < \infty$. This problem is called the *Kac killing penalization*. Our weight process (1.3) can be considered to be the case

$$q = \lambda_a \delta_a + \lambda_b \delta_b.$$

The case $\lambda_a > 0$ and $\lambda_b = \infty$ is formally considered a one-point local time penalization with conditioning to avoid another point. The case $\lambda_a = \lambda_b = \infty$ is formally considered a conditioning to avoid two points. These cases are discussed in Iba and Yano (2025+).

Organization. This paper is organized as follows. In Section 2, we prepare some general results of Lévy processes. In Sections 3, 4, 5, and 6, we discuss the penalization results with exponential clock, hitting time clock, two-point hitting time clock, and inverse local time clock, respectively. The structure of Sections 3, 4, 5, and 6 are similar: (after some preliminary computations,) we study the limit of $\rho(\tau)\mathbb{P}_x[\Gamma_\tau]$ as $\tau \to \infty$, and then state and prove the main theorem.

2. Preliminaries

2.1. *Lévy process and resolvent density.* Let $(X, \mathbb{P}_x)$ be the canonical representation of a real valued Lévy process starting from $x \in \mathbb{R}$ on the càdlàg path space. For $t > 0$, we denote by $\mathscr{F}_t^X = \sigma(X_s,\ 0 \le s \le t)$ the natural filtration of X and write $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s^X$. For $a \in \mathbb{R}$, let T_a be the hitting time of $\{a\}$ for X, i.e.,

$$T_a = \inf\Big\{t > 0;\ X_t = a\Big\}.$$

For $\lambda \in \mathbb{R}$, we denote by $\Psi(\lambda)$ the characteristic exponent of X, which satisfies

$$\mathbb{P}_0\Big[e^{i\lambda X_t}\Big] = e^{-t\Psi(\lambda)}$$

for $t \ge 0$. By Lévy–Khinchine formula, Ψ admits the representation

$$\Psi(\lambda) = iv\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \Big(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}}\Big)\nu(dx),$$

where $v \in \mathbb{R}$, $\sigma \ge 0$, and ν is a measure on $\mathbb{R}$, called a *Lévy measure*, with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1)\nu(dx) < \infty$.

Throughout this paper, we always assume $(X, \mathbb{P}_0)$ is recurrent, i.e.,

$$\mathbb{P}_0\left[\int_0^\infty 1_{\{|X_t - a| < \varepsilon\}} dt\right] = \infty$$

for all $a \in \mathbb{R}$ and $\varepsilon > 0$, and always assume the condition

$$\textbf{(A)} \quad \int_0^\infty \left|\frac{1}{q + \Psi(\lambda)}\right| d\lambda < \infty \qquad \text{for each } q > 0.$$

It is known that X has a bounded continuous resolvent density r_q:

$$\int_{\mathbb{R}} f(x) r_q(x) dx = \mathbb{P}_0\left[\int_0^\infty e^{-qt} f(X_t) dt\right] \tag{2.1}$$

holds for $q > 0$ and non-negative measurable functions f. See, e.g., Theorems II.16 and II.19 of Bertoin (1996). Moreover, there exists an equality that connects the hitting time of 0 and the resolvent density:

$$\mathbb{P}_x\left[e^{-qT_0}\right] = \frac{r_q(-x)}{r_q(0)} \tag{2.2}$$

for $q > 0$ and $x \in \mathbb{R}$. See, e.g., Corollary II.18 of Bertoin (1996). If condition **(A)** holds, then Corollary 15.1 of Tsukada (2018) showed that the resolvent density can be expressed as

$$r_q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{q + \Psi(\lambda)} d\lambda = \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-i\lambda x}}{q + \Psi(\lambda)}\right) d\lambda \tag{2.3}$$

for all $q > 0$ and $x \in \mathbb{R}$.

2.2. *Local time and excursion.* We denote by $\mathcal{D}$ the set of càdlàg paths $e : [0, \infty) \to \mathbb{R} \cup \{\Delta\}$ such that

$$\begin{cases} e(t) \in \mathbb{R} \setminus \{0\} & \text{for } 0 < t < \zeta(e), \\ e(t) = \Delta & \text{for } t \geq \zeta(e), \end{cases}$$

where the point Δ is an isolated point and ζ is the excursion length, i.e.,

$$\zeta = \zeta(e) := \inf\left\{t > 0;\ e(t) = \Delta\right\}.$$

Let Σ denote the σ-algebra on $\mathcal{D}$ generated by cylinder sets.

Assume the condition **(A)** holds. Then, we can define a local time at $a \in \mathbb{R}$, which we denote by $L^a = (L^a_t)_{t \geq 0}$. It is defined by

$$L^a_t := \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t 1_{\{|X_s - a| < \varepsilon\}} ds.$$

It is known that L^a is continuous in t and satisfies

$$\mathbb{P}_x\left[\int_0^\infty e^{-qt} dL^a_t\right] = r_q(a - x)$$

for $q > 0$ and $x \in \mathbb{R}$. See, e.g., Section V of Bertoin (1996). In particular, from this expression, $r_q(x)$ is non-decreasing as $q \to 0+$.

Let $\eta^a = (\eta^a_l)_{l \geq 0}$ be an inverse local time, i.e.,

$$\eta^a_l := \inf\left\{t > 0 :\ L^a_t > l\right\}.$$

It is known that the process $(\eta^a, \mathbb{P}_a)$ is a possibly killed subordinator which has the Laplace exponent

$$\mathbb{P}_a\left[e^{-q\eta^a_l}\right] = e^{-\frac{l}{r_q(0)}}$$

for $l > 0$ and $q > 0$. See, e.g., Proposition V.4 of Bertoin (1996).

We denote ϵ^a_l for an excursion away from $a \in \mathbb{R}$ which starts at local time $l \geq 0$, i.e.,

$$\epsilon^a_l(t) := \begin{cases} X_{t+\eta^a_{l-}} & \text{for } 0 \leq t < \eta^a_l - \eta^a_{l-}, \\ \Delta & \text{for } t \geq \eta^a_l - \eta^a_{l-}. \end{cases}$$

Then, $(\epsilon^a_l)_{l\geq 0}$ is a Poisson point process, and we write n^a for the characteristic measure of ϵ^a. It is known that $(\mathcal{D}, \Sigma, n^a)$ is a σ-finite measure space. See, e.g., Section IV of Bertoin (1996). For $B \in \mathscr{B}(0,\infty) \otimes \Sigma$, we define

$$N^a(B) := \#\Big\{(l,e) \in B : \ \epsilon^a_l = e\Big\}.$$

Then, N^a is a Poisson random measure with its intensity measure $ds \times n^a(de)$. It is known that the subordinator η^0 has no drift and its Lévy measure is $n^0(T_0 \in dx)$. Thus, we have

$$e^{-\frac{l}{r_q(0)}} = \mathbb{P}_0\left[e^{-q\eta^0_l}\right] = e^{-ln^0[1-e^{-qT_0}]}$$

for $l \geq 0$. This implies that

$$n^0\left[1 - e^{-qT_0}\right] = \frac{1}{r_q(0)}. \tag{2.4}$$

Now, we set

$$\kappa := \lim_{q\to 0+} \frac{1}{r_q(0)} = n^0(T_0 = \infty). \tag{2.5}$$

It is known that $\kappa = 0$ if and only if X is recurrent. See, e.g., Theorem I.17 of Bertoin (1996) and Theorem 37.5 of Sato (1999).

For the excursion measure, the following famous equality, called the compensation formula, is known: if a function $F : (0,\infty) \times \mathcal{D} \times \mathcal{D} \to [0,\infty)$ satisfies the following:

(1) F is a measurable function;
(2) For each $t > 0$, $(\omega, e) \mapsto F(t,\omega,e)$ is $\mathscr{F}_t \otimes \Sigma$-measurable;
(3) For each $e \in \mathcal{D}$, $(F(t,\cdot,e),\ t \geq 0)$ is almost surely a left-continuous process.

Then, for $t \geq 0$, it holds that

$$\mathbb{P}_0\left[\int_{(0,t]}\int_{\mathcal{D}} F(s,X,e)N^0(ds\otimes de)\right] = \mathbb{P}_0\left[\int_0^t\int_{\mathcal{D}} F(s,X,e)n^0(de)ds\right].$$

See, e.g., Theorem 4.4 of Kyprianou (2014). Moreover, the excursion measure n^0 has the following form of the Markov property: it holds that for any stopping time $T < \infty$, any non-negative $\mathscr{F}_t$-measurable functional Z_t, and any non-negative measurable functional F on $\mathcal{D}$,

$$n^0\Big[Z_tF(X\circ\theta_T)\Big] = \int n^0\Big[Z_t;\ X_T \in dx\Big]\mathbb{P}^0_x[F(X)],$$

where $\mathbb{P}^0_x$ is the distribution of the killed process upon T_0. See, e.g., Theorem III.3.28 of Blumenthal (1992).

2.3. *The renormalized zero resolvent.* We define

$$h_q(x) := r_q(0) - r_q(-x)$$

for $q > 0$ and $x \in \mathbb{R}$. By (2.3), we have

$$h_q(x) = \frac{1}{\pi}\int_0^\infty \Re\left(\frac{1 - e^{i\lambda x}}{q + \Psi(\lambda)}\right)d\lambda.$$

It is clear that $h_q(0) = 0$, and by (2.2), we have $h_q(x) \geq 0$. The following theorem plays a key role in our penalization results. Recall that X is assumed recurrent.

Proposition 2.1 (Theorem 1.1 of Takeda and Yano, 2023)**.** *If the condition **(A)** is satisfied, then the following assertions hold:*

(i) For any $x \in \mathbb{R}$, $h(x) := \lim_{q\to 0+} h_q(x)$ exists and is finite;
(ii) h is continuous;

(iii) h is subadditive, that is, for $x, y \in \mathbb{R}$, it holds that

$$h(x+y) \le h(x) + h(y).$$

We call h the *renormalized zero resolvent.*

Proposition 2.2 (Theorem 1.2 of Takeda and Yano, 2023)**.** *If the condition **(A)** is satisfied, then the following assertions hold:*

(i) It holds that

$$\lim_{x \to \pm\infty} \frac{h(x)}{|x|} = \frac{1}{m^2} \in [0, \infty); \tag{2.6}$$

(ii) For all $x \in \mathbb{R}$, it holds that

$$\lim_{y \to \pm\infty} \Big\{ h(x+y) - h(y) \Big\} = \pm \frac{x}{m^2}, \tag{2.7}$$

where $m^2 = \mathbb{P}_0[X_1^2] \in (0, \infty]$.

Proposition 2.3 (Theorem 15.2 of Tsukada, 2018)**.** *For $t \ge 0$, it hold that*

$$h_q(X_t) \to h(X_t) \text{ in } L^1(\mathbb{P}_x) \tag{2.8}$$

as $q \to 0+$.

2.4. *The function h^B.* We will introduce the functions h_q^B and h^B.

Proposition 2.4 (Lemmas 3.5(i), 3.7, and 3.8 of Takeda and Yano, 2023)**.** *The following assertions hold:*

(i) For $a \in \mathbb{R}$, it holds that

$$h_q^B(a) := \mathbb{P}_0\left[\int_0^{T_a} e^{-qt} dL_t^0 \right] = h_q(a) + h_q(-a) - \frac{h_q(a) h_q(-a)}{r_q(0)}.$$

Consequently, it holds that

$$h^B(a) := \lim_{q \to 0+} h_q^B(a) = \mathbb{P}_0[L_{T_a}] = h(a) + h(-a); \tag{2.9}$$

(ii) It holds that

$$\lim_{x \to \infty} h^B(x) = \infty. \tag{2.10}$$

(iii) For $a \in \mathbb{R} \setminus \{0\}$, it holds that

$$n^0(T_a < T_0) = \frac{1}{h^B(a)}. \tag{2.11}$$

Next, we introduce the following useful formulas:

Proposition 2.5 (Lemma 3.5(ii) of Takeda and Yano, 2023)**.** *For $x \in \mathbb{R}$, $q > 0$, and distinct points $a, b \in \mathbb{R}$, it holds that*

$$\mathbb{P}_x\Big[e^{-qT_a};\ T_a < T_b \Big] = \frac{h_q(b-a) + h_q(x-b) - h_q(x-a) - \frac{h_q(x-b) h_q(b-a)}{r_q(0)}}{h_q^B(a-b)}. \tag{2.12}$$

Consequently, it holds that

$$\mathbb{P}_x(T_a < T_b) = \frac{h(b-a) + h(x-b) - h(x-a)}{h^B(a-b)}. \tag{2.13}$$

2.5. *Various expectations of local time at random times.* In this subsection, we present the results of various expectations of local time clock that we will use frequently later.

Proposition 2.6 (Lemma 4.1 of Takeda and Yano, 2023)**.** *Let f be a non-negative measurable function. Then, it holds that for $q > 0$ and $x \in \mathbb{R}$,*

$$\mathbb{P}_x\left[f(L^0_{\boldsymbol{e}_q})\right] = \frac{1}{r_q(0)}\left\{h_q(x)f(0) + \left(1 - \frac{h_q(x)}{r_q(0)}\right)\int_0^\infty e^{-\frac{u}{r_q(0)}} f(u)du\right\}.$$

Proposition 2.7 (Lemma 5.1 of Takeda and Yano, 2023)**.** *For $x \in \mathbb{R},\ a \in \mathbb{R}\backslash\{0\}$, and non-negative measurable function f, we have*

$$\mathbb{P}_x\left[f(L^0_{T_a})\right] = \mathbb{P}_x(T_0 > T_a)f(0) + \frac{\mathbb{P}_x(T_0 < T_a)}{h^B(a)}\int_0^\infty e^{-\frac{u}{h^B(a)}} f(u)du. \tag{2.14}$$

Proposition 2.8 (Lemma 6.1 of Takeda and Yano, 2023)**.** *For distinct points $a, b \in \mathbb{R}$, it holds that*

$$\begin{aligned} h^C(a,b) :&= \mathbb{P}_0\left[L^0_{T_a \wedge T_b}\right] \\ &= \frac{1}{h^B(a-b)}\left\{\begin{aligned}&(h(b) + h(-a))h(a-b) + (h(a) + h(-b))h(b-a) \\ &\qquad + (h(a) - h(b))(h(-b) - h(-a)) - h(a-b)h(b-a)\end{aligned}\right\}. \end{aligned} \tag{2.15}$$

Proposition 2.9 (Lemma 6.2 of Takeda and Yano, 2023)**.** *For $x \in \mathbb{R}$ and distinct points $a, b, c \in \mathbb{R}$, it hold that*

$$\begin{aligned} &\mathbb{P}_x\left[e^{-qT_a};\ T_a < T_b \wedge T_c\right] \\ &= \frac{\mathbb{P}_x\left[e^{-qT_a};\ T_a < T_b\right] - \mathbb{P}_x\left[e^{-qT_c};\ T_c < T_b\right]\mathbb{P}_c\left[e^{-qT_a};\ T_a < T_b\right]}{1 - \mathbb{P}_a\left[e^{-qT_c};\ T_c < T_b\right]\mathbb{P}_c\left[e^{-qT_a};\ T_a < T_b\right]}. \end{aligned}$$

Consequently, it holds that

$$\mathbb{P}_x(T_a < T_b \wedge T_c) = \frac{\mathbb{P}_x(T_a < T_b) - \mathbb{P}_x(T_c < T_b)\mathbb{P}_c(T_a < T_b)}{1 - \mathbb{P}_a(T_c < T_b)\mathbb{P}_c(T_a < T_b)}. \tag{2.16}$$

3. Exponential clock

From this section to the end, we write simply

$$\Gamma_t := e^{-\lambda_a L^a_t - \lambda_b L^b_t}.$$

Let us find the limit of $r_q(0)\mathbb{P}_x[\Gamma_{\boldsymbol{e}_q}]$ as $q \to 0+$.

Proposition 3.1. *For distinct points $a, b \in \mathbb{R}$ and for constants $\lambda_a, \lambda_b > 0$ it holds that*

$$\lim_{q \to 0+} r_q(0)\mathbb{P}_x[\Gamma_{\boldsymbol{e}_q}] = \varphi^{(0),\lambda_a,\lambda_b}_{a,b}(x) \tag{3.1}$$

for all $x \in \mathbb{R}$, and

$$\lim_{q \to 0+} r_q(0)\mathbb{P}_{X_t}[\Gamma_{\boldsymbol{e}_q}] = \varphi^{(0),\lambda_a,\lambda_b}_{a,b}(X_t) \text{ a.s. and in } L^1(\mathbb{P}_x). \tag{3.2}$$

3.1. *Preliminary computations.* Before proving Proposition 3.1, we prove the following lemmas.

Lemma 3.2. *For distinct points $a, b \in \mathbb{R}$, for constants $\lambda_a, \lambda_b > 0$, and for $q > 0$, it holds that*

$$\mathbb{P}_a[\Gamma_{\boldsymbol{e}_q}] = \frac{I^q_{a,b} + J^q_{a,b} I^q_{b,a}}{1 - J^q_{a,b} J^q_{b,a}}, \tag{3.3}$$

where

$$J^q_{a,b} := \mathbb{P}_a\left[e^{-qT_b} e^{-\lambda_a L^a_{T_b}}\right]$$

and

$$I^q_{a,b} := \mathbb{P}_a\left[\int_0^{T_b} e^{-\lambda_a L^a_s} q e^{-qs} ds\right].$$

Proof: By the strong Markov property at T_b, we have

$$\mathbb{P}_a[\Gamma_{\boldsymbol{e}_q}] = \mathbb{P}_a\left[\int_0^{T_b} e^{-\lambda_a L^a_s} q e^{-qs} ds\right] + \mathbb{P}_a\left[\int_{T_b}^{\infty} e^{-\lambda_a L^a_s - \lambda_b L^b_s} q e^{-qs} ds\right] = I^q_{a,b} + J^q_{a,b} \mathbb{P}_b[\Gamma_{\boldsymbol{e}_q}]. \tag{3.4}$$

Similarly, we have

$$\mathbb{P}_b[\Gamma_{\boldsymbol{e}_q}] = I^q_{b,a} + J^q_{b,a} \mathbb{P}_a[\Gamma_{\boldsymbol{e}_q}]. \tag{3.5}$$

Thus, by solving equations (3.4) and (3.5), we obtain (3.3). □

Lemma 3.3. *For distinct points $a, b \in \mathbb{R}$, for constants $\lambda_a, \lambda_b > 0$, and for $q > 0$, it holds that*

$$I^q_{a,b} = \frac{1 - n^a[e^{-qT_b}] h_q(b-a)}{1 + \lambda_a r_q(0) + n^a[e^{-qT_b}] r_q(a-b)}. \tag{3.6}$$

Proof: By dividing the range of integration into excursion intervals, we have

$$I^q_{a,b} = \mathbb{P}_a\left[\sum_{l \le \sigma^a_{\{T_b < \infty\}}} \int_{\eta^a_{l-}}^{\eta^a_l \wedge T_b} e^{-\lambda_a l} q e^{-qs} ds\right], \tag{3.7}$$

where for $A \in \Sigma$, σ^a_A is the first hitting time for the Poisson point process $(\epsilon^a_t)_{t \ge 0}$:

$$\sigma^a_A := \inf\left\{t \ge 0 : \ \epsilon^a_t \in A\right\}.$$

Then, we have

$$\begin{aligned} &\mathbb{P}_a\left[\int_{s \le \sigma^a_A} \int_{\mathcal{D}} F(\eta^a_{s-}, X, e) N^a(ds \otimes de)\right] \\ &= \mathbb{P}_a\left[\left(\int_{s < \sigma^a_A} + \int_{s = \sigma^a_A}\right) \int_{\mathcal{D}} F(\eta^a_{s-}, X, e) N^a(ds \otimes de)\right] =: (a) + (b). \end{aligned} \tag{3.8}$$

Now, we let

$$N^{a,A}(\cdot) := N^a\Big(\cdot \cap \{(0,\infty) \times A\}\Big)$$

and let

$$\eta^{a,A}_s := \int_{(0,s]} \int_{\mathcal{D}} \zeta(e) N^{a,A}(ds \otimes de),$$

where $\zeta(e)$ is the excursion length of e. Since σ_A^a and N^{a,A^c} are independent, we have by the compensation formula,

$$\begin{aligned}
(a) &= \mathbb{P}_a\left[\int_{s<\sigma_A^a}\int_{\mathcal{D}} F\big(\eta_{s-}^{a,A^c}, X, e\big) N^{a,A^c}(ds\otimes de)\right] \\
&= \mathbb{P}_a\left[\mathbb{P}_a\left[\int_{s\le t}\int_{\mathcal{D}} F\Big(\eta_{s-}^{a,A^c}, X, e\Big) N^{a,A^c}(ds\otimes de)\right]\Bigg|_{t=\sigma_A^a}\right] \\
&= \mathbb{P}_a\left[\int_0^{\sigma_A^a}\int_{\mathcal{D}} F\Big(\eta_{s-}^{a,A^c}, X, e\Big) n^a(de\cap A^c)ds\right]. \qquad (3.9)
\end{aligned}$$

Moreover, since σ_A^a has the exponential distribution with its parameter $n^a(A)$, and since $\epsilon_{\sigma_A^a}^a$ has the distribution $n^a(\cdot|A)$, we have

$$\begin{aligned}
(b) &= \mathbb{P}_a\left[\int_{s=\sigma_A^a}\int_{\mathcal{D}} F\Big(\eta_{s-}^{a,A^c}, X, e\Big) N^{a,A}(ds\otimes de)\right] \\
&= \mathbb{P}_a\left[F\Big(\eta_{\sigma_A^a-}^{a,A^c}, X, \epsilon_{\sigma_A^a}^a\Big)\right] \\
&= \mathbb{P}_a\left[\int_0^\infty ds\int_{\mathcal{D}} F\Big(\eta_{s-}^{a,A^c}, X, e\Big) n^a(A)e^{-n^a(A)s}n^a(de|A)\right] \\
&= \mathbb{P}_a\left[\int_0^\infty ds\int_{\mathcal{D}} F\Big(\eta_{s-}^{a,A^c}, X, e\Big) e^{-n^a(A)s}n^a(de\cap A)\right]. \qquad (3.10)
\end{aligned}$$

Thus, by (3.7), (3.8), (3.9), and (3.10), we have

$$\begin{aligned}
I_{a,b}^q &= \mathbb{P}_a\left[\sum_{l\le\sigma_{\{T_b<\infty\}}^a}\int_0^{\eta_l^a\wedge T_b-\eta_{l-}^a} e^{-\lambda_a l}qe^{-q(s+\eta_{l-}^a)}ds\right] \\
&= \mathbb{P}_a\left[\int_{l\le\sigma_{\{T_b<\infty\}}^a}\int_{\mathcal{D}}\left(\int_0^{\eta_l^a\wedge T_b-\eta_{l-}^a} e^{-\lambda_a l}qe^{-q(s+\eta_{l-}^a)}ds\right)N^a(dl\otimes de)\right] \\
&= \mathbb{P}_a\left[\int_0^{\sigma_{\{T_b<\infty\}}^a}\int_{\mathcal{D}}\left(\int_0^{T_a} e^{-\lambda_a l}qe^{-q(s+\eta_{l-}^{a,\{T_b=\infty\}})}ds\right)n^a\Big(de\cap\{T_b=\infty\}\Big)dl\right] \\
&\quad + \mathbb{P}_a\left[\int_0^\infty dl\int_{\mathcal{D}}\left(\int_0^{T_b} e^{-\lambda_a l}qe^{-q(s+\eta_{l-}^{a,\{T_b=\infty\}})}ds\right)e^{-n^a(T_b<\infty)l}n^a\Big(de\cap\{T_b<\infty\}\Big)\right] \\
&= \mathbb{P}_a\left[\int_0^{\sigma_{\{T_b<\infty\}}^a} e^{-\lambda_a l}e^{-q\eta_{l-}^{a,\{T_b=\infty\}}}dl\right]n^a\Big(1-e^{-qT_a},\ T_b=\infty\Big) \\
&\quad + \mathbb{P}_a\left[\int_0^\infty e^{-\lambda_a l}e^{-q\eta_{l-}^{a,\{T_b=\infty\}}}e^{-n^a(T_b<\infty)l}dl\right]n^a\Big(1-e^{-qT_b}, T_b<\infty\Big) \\
&= \left(\int_0^\infty dt\int_0^t e^{-\lambda_a l}\mathbb{P}_a\left[e^{-q\eta_{l-}^{a,\{T_b=\infty\}}}\right]n^a(T_b<\infty)e^{-n^a(T_b<\infty)t}dl\right)n^a\Big(1-e^{-qT_a},\ T_b=\infty\Big) \\
&\quad + \left(\int_0^\infty e^{-\lambda_a l}\mathbb{P}_a\left[e^{-q\eta_{l-}^{a,\{T_b=\infty\}}}\right]e^{-n^a(T_b<\infty)l}dl\right)n^a\Big(1-e^{-qT_b}, T_b<\infty\Big). \qquad (3.11)
\end{aligned}$$

Since $(\eta_l^a)_{l\geq 0}$ is a subordinator with its Lévy measure $n^a(T_a \in dx)$ and no drift, by Lévy–Khinchin formula, we have

$$\begin{aligned}
\mathbb{P}_a\Big[e^{-q\eta_{l-}^{a,\{T_b=\infty\}}}\Big] &= \mathbb{P}_a\Big[e^{-q\eta_l^{a,\{T_b=\infty\}}}\Big] \\
&= \exp\Big\{-l\int_{(0,\infty)}\Big(1-e^{-qx}\Big)n^a\Big(T_a \in dx \cap \{T_b=\infty\}\Big)\Big\} \\
&= \exp\Big\{-ln^a\Big[1-e^{-qT_a};\ T_b=\infty\Big]\Big\}. \qquad (3.12)
\end{aligned}$$

By the strong Markov property of the excursion measure n^a and by (2.2), we have

$$\begin{aligned}
n^a\Big[e^{-qT_a};\ T_b<\infty\Big] &= \int n^a\Big[e^{-qT_b}1_{\{T_b<T_a\}};\ X_{T_b}\in dx\Big]\mathbb{P}_x^a\Big[e^{-qT_a}\Big] \\
&= n^a\Big[e^{-qT_b};\ T_b<\infty\Big]\mathbb{P}_b\Big[e^{-qT_a}\Big] \\
&= n^a\Big[e^{-qT_b}\Big]\cdot\frac{r_q(a-b)}{r_q(0)}, \qquad (3.13)
\end{aligned}$$

where $\mathbb{P}_x^a$ is the distribution of the killed process upon T_a. Thus, by (2.4), (2.11), and (3.13), we have

$$\begin{aligned}
n^a\Big[1-e^{-qT_a};\ T_b=\infty\Big] &= n^a\Big[1-e^{-qT_a}\Big]-n^a\Big[T_b<\infty\Big]+n^a\Big[e^{-qT_a};\ T_b<\infty\Big] \\
&= \frac{1}{r_q(0)}-\frac{1}{h^B(b-a)}+n^a\Big[e^{-qT_b}\Big]\cdot\frac{r_q(a-b)}{r_q(0)}. \qquad (3.14)
\end{aligned}$$

Consequently, by (3.11), (3.12), and (3.14), we obtain

$$\begin{aligned}
I_{a,b}^q &= \Big(\int_0^\infty\int_0^t e^{-\lambda_a l}e^{-ln^a[1-e^{-qT_a};\ T_b=\infty]}dl\cdot\frac{1}{h^B(b-a)}\cdot e^{-\frac{t}{h^B(b-a)}}dt\Big)n^a\Big[1-e^{-qT_a};\ T_b=\infty\Big] \\
&\quad+\Big(\int_0^\infty e^{-\lambda_a l}e^{-ln^a[1-e^{-qT_a};\ T_b=\infty]}e^{-\frac{l}{h^B(b-a)}}dl\Big)n^a\Big[1-e^{-qT_b};\ T_b<\infty\Big] \\
&= \frac{1-n^a[e^{-qT_b}]h_q(b-a)}{1+\lambda_a r_q(0)+n^a[e^{-qT_b}]r_q(a-b)}.
\end{aligned}$$

The proof is complete. □

3.2. *Proof of Proposition 3.1.*

Proof of Proposition 3.1: First, by dividing the range of integration, we have

$$\begin{aligned}
\mathbb{P}_x[\Gamma_{\boldsymbol{e}_q}] &= \mathbb{P}_x\Big[\int_0^\infty e^{-\lambda_a L_s^a-\lambda_b L_s^b}qe^{-qs}ds\Big] \\
&= \mathbb{P}_x\Big[\int_0^{T_a\wedge T_b} qe^{-qs}ds\Big] \\
&\quad+\mathbb{P}_x\Big[\int_{T_a}^\infty e^{-\lambda_a L_s^a-\lambda_b L_s^b}qe^{-qs}ds,\ T_a<T_b\Big] \\
&\quad+\mathbb{P}_x\Big[\int_{T_b}^\infty e^{-\lambda_a L_s^a-\lambda_b L_s^b}qe^{-qs}ds,\ T_b<T_a\Big]. \qquad (3.15)
\end{aligned}$$

In the first term of (3.15), by a simple calculation, we have

$$\mathbb{P}_x\left[\int_0^{T_a\wedge T_b} qe^{-qs}ds\right] = 1-\mathbb{P}_x\left[e^{-q(T_a\wedge T_b)}\right]. \tag{3.16}$$

Next, we consider the second term of (3.15). By the strong Markov property, we have

$$\begin{aligned}
&\mathbb{P}_x\left[\int_{T_a}^{\infty} e^{-\lambda_a L_s^a-\lambda_b L_s^b} qe^{-qs}ds,\ T_a<T_b\right] \\
&= \mathbb{P}_x\left[\int_{T_a}^{T_b} e^{-\lambda_a L_s^a} qe^{-qs}ds,\ T_a<T_b\right] + \mathbb{P}_x\left[\int_{T_b}^{\infty} e^{-\lambda_a L_s^a-\lambda_b L_s^b} qe^{-qs}ds,\ T_a<T_b\right] \\
&= \mathbb{P}_x\left[\int_0^{T_b-T_a} e^{-\lambda_a (L_{s+T_a}^a - L_{T_a}^a)} qe^{-q(s+T_a)}ds,\ T_a<T_b\right] \\
&\quad + \mathbb{P}_x\left[\int_0^{\infty} e^{-\lambda_a (L_{s+T_b}^a - L_{T_b}^a + L_{T_b}^a) - \lambda_b (L_{s+T_b}^b - L_{T_b}^b)} qe^{-q(s+T_b)}ds,\ T_a<T_b\right] \\
&= \mathbb{P}_x\left[e^{-qT_a},\ T_a<T_b\right]\mathbb{P}_a\left[\int_0^{T_b} e^{-\lambda_a L_s^a} qe^{-qs}ds\right] + \mathbb{P}_x\left[e^{-\lambda_a L_{T_b}^a} e^{-qT_b},\ T_a<T_b\right]\mathbb{P}_b[\Gamma_{\boldsymbol{e}_q}].
\end{aligned} \tag{3.17}$$

Therefore, from (3.6), (3.3), (3.15), (3.16), and (3.17), we complete the calculation of the expectation $\mathbb{P}_x[\Gamma_{\boldsymbol{e}_q}]$.

Next, we consider the limit of $r_q(0)\mathbb{P}_x[\Gamma_{\boldsymbol{e}_q}]$ as $q\to 0+$. By Theorem 2.1 and by (3.16), (2.12), and (2.13), we obtain

$$\begin{aligned}
&\lim_{q\to 0+} r_q(0)\mathbb{P}_x\left[\int_0^{T_a\wedge T_b} qe^{-qs}ds\right] \\
&= \lim_{q\to 0+} \frac{h_q(x-b)h_q(b-a)+h_q(x-a)h_q(a-b)-h_q(a-b)h_q(b-a)}{h_q^B(a-b)} \\
&= \frac{h(x-b)h(b-a)+h(x-a)h(a-b)-h(a-b)h(b-a)}{h^B(a-b)} \\
&= h(x-a)-\mathbb{P}_x(T_b<T_a)h(b-a).
\end{aligned} \tag{3.18}$$

By the monotone convergence theorem and by (2.11), we have

$$\lim_{q\to 0+} n^a\left[e^{-qT_b}\right] = n^a(T_b<\infty) = \frac{1}{h^B(a-b)}, \tag{3.19}$$

and by (2.2),

$$\lim_{q\to 0+} \frac{r_q(a-b)}{r_q(0)} = \lim_{q\to 0+} \mathbb{P}_{b-a}\left[e^{-qT_0}\right] = \mathbb{P}_{b-a}(T_0<\infty)=1. \tag{3.20}$$

Thus, by (3.6), (2.5), (3.19), and (3.20), we have

$$\lim_{q\to 0+} r_q(0)I_{a,b}^q = \lim_{q\to 0+} \frac{1-n^a[e^{-qT_b}]h_q(b-a)}{\frac{1}{r_q(0)}+\lambda_a+n^a[e^{-qT_b}]\cdot\frac{r_q(a-b)}{r_q(0)}} = \frac{h(a-b)}{1+\lambda_a h^B(a-b)}. \tag{3.21}$$

By the monotone convergence theorem and by (2.14), we have

$$\lim_{q\to 0+} J_{a,b}^q = \mathbb{P}_a\left[e^{-\lambda_a L_{T_b}^a}\right] = \frac{1}{h^B(a-b)}\int_0^{\infty} e^{-\frac{u}{h^B(a-b)}} e^{-\lambda_a u}du = \frac{1}{1+\lambda_a h^B(a-b)}. \tag{3.22}$$

Thus, by (3.3), (3.21), and (3.22), we obtain

$$\lim_{q\to 0+} r_q(0)\mathbb{P}_a[\Gamma_{\boldsymbol{e}_q}] = \lim_{q\to 0+} \frac{r_q(0)I_{a,b}^q + J_{a,b}^q\cdot r_q(0)I_{b,a}^q}{1-J_{a,b}^q J_{b,a}^q} = \frac{1+\lambda_b h(a-b)}{\lambda_a+\lambda_b+\lambda_a\lambda_b h^B(a-b)}. \tag{3.23}$$

By the monotone convergence theorem and by (2.14), we have

$$\begin{aligned}\lim_{q\to 0+} \mathbb{P}_x\left[e^{-\lambda_a L^a_{T_b}} e^{-qT_b},\ T_a < T_b\right] &= \mathbb{P}_x\left[e^{-\lambda_a L^a_{T_b}},\ T_a < T_b\right] \\ &= \mathbb{P}_x\left[e^{-\lambda_a L^a_{T_b}}\right] - \mathbb{P}_x(T_b < T_a) \\ &= \mathbb{P}_x(T_a < T_b)\cdot \frac{1}{1+\lambda_a h^B(b-a)}. \end{aligned} \tag{3.24}$$

Therefore, the assertion (3.1) holds by (3.15), (3.17), (3.18), (3.21), (3.23), and (3.24). Moreover, from (2.8), we also obtain (3.2). □

3.3. *Main theorem in the exponential case.* We define

$$\begin{aligned} N^{q,\lambda_a,\lambda_b}_{a,b,t} &:= r_q(0)\mathbb{P}_x\left[\Gamma_{\boldsymbol{e}_q};\ t<\boldsymbol{e}_q \middle| \mathscr{F}_t\right], \\ M^{q,\lambda_a,\lambda_b}_{a,b,t} &:= r_q(0)\mathbb{P}_x\left[\Gamma_{\boldsymbol{e}_q} \middle| \mathscr{F}_t\right] \end{aligned}$$

for $q>0$.

Theorem 3.4. *Let $x\in\mathbb{R}$. Then, $(M^{(0),\lambda_a,\lambda_b}_{a,b,t},\ t\ge 0)$ is a non-negative $((\mathscr{F}_t),\mathbb{P}_x)$-martingale, and it holds that*

$$\lim_{q\to 0+} N^{q,\lambda_a,\lambda_b}_{a,b,t} = \lim_{q\to 0+} M^{q,\lambda_a,\lambda_b}_{a,b,t} = M^{(0),\lambda_a,\lambda_b}_{a,b,t} \quad \text{a.s. and in } L^1(\mathbb{P}_x).$$

Consequently, if $M^{(0),\lambda_a,\lambda_b}_{a,b,0} > 0$ under $\mathbb{P}_x$, it holds that

$$\lim_{q\to 0+} \frac{\mathbb{P}_x[F_s\cdot \Gamma_{\boldsymbol{e}_q}]}{\mathbb{P}_x[\Gamma_{\boldsymbol{e}_q}]} = \mathbb{P}_x\left[F_s\cdot \frac{M^{(0),\lambda_a,\lambda_b}_{a,b,s}}{M^{(0),\lambda_a,\lambda_b}_{a,b,0}}\right]$$

for all bounded $\mathscr{F}_s$-measurable functionals F_s.

Proof: We denote by $\mathbb{P}^{\mathscr{G}}_x[\cdot]$ the conditional expectation for $\mathbb{P}_x$ with respect to a σ-algebra $\mathscr{G}$. By the lack of memory property of an exponential distribution and by the Markov property, we have

$$\begin{aligned} N^{q,\lambda_a,\lambda_b}_{a,b,t} &= r_q(0)\mathbb{P}^{\mathscr{F}_t}_x\left[e^{-\lambda_a L^a_{\boldsymbol{e}_q - t + t} - \lambda_b L^b_{\boldsymbol{e}_q - t+t}} \middle|\ t<\boldsymbol{e}_q\right]\mathbb{P}_x(t<\boldsymbol{e}_q) \\ &= r_q(0)e^{-qt}\mathbb{P}_x\left[e^{-\lambda_a(L^a_{\boldsymbol{e}_q}\circ\theta_t + L^a_t) - \lambda_b(L^b_{\boldsymbol{e}_q}\circ\theta_t + L^b_t)} \middle| \mathscr{F}_t\right] \\ &= r_q(0)e^{-qt}e^{-\lambda_a L^a_t - \lambda_b L^b_t}\mathbb{P}_{X_t}\left[e^{-\lambda_a L^a_{\boldsymbol{e}_q} - \lambda_b L^b_{\boldsymbol{e}_q}}\right]. \end{aligned}$$

By (3.2) of Proposition 3.1, we obtain

$$\lim_{q\to 0+} N^{q,\lambda_a,\lambda_b}_{a,b,t} = M^{(0),\lambda_a,\lambda_b}_{a,b,t} \quad \text{a.s. and in } L^1(\mathbb{P}_x).$$

Since $qr_q(0)\to 0$ as $q\to 0+$ by e.g., Lemma 15.5 of Tsukada (2018), we have

$$\begin{aligned} M^{q,\lambda_a,\lambda_b}_{a,b,t} - N^{q,\lambda_a,\lambda_b}_{a,b,t} &= r_q(0)\mathbb{P}_{X_t}\left[\Gamma_{\boldsymbol{e}_q};\ \boldsymbol{e}_q \le t \middle| \mathscr{F}_t\right] \\ &= qr_q(0)\int_0^t e^{-\lambda_a L^a_u - \lambda_b L^b_u} e^{-qu}du \\ &\le qr_q(0)\cdot t \to 0 \end{aligned}$$

as $q \to 0+$. Thus, we obtain

$$\lim_{q \to 0+} M^{q,\lambda_a,\lambda_b}_{a,b,t} = M^{(0),\lambda_a,\lambda_b}_{a,b,t} \text{ a.s. and in } L^1(\mathbb{P}_x).$$

Therefore, the proof is complete. □

4. Hitting time clock

Let us find the limit of $h^B(c)\mathbb{P}_x[\Gamma_{T_c}]$ as $c \to \pm\infty$.

Proposition 4.1. *For distinct points $a, b \in \mathbb{R}$ and for constants $\lambda_a, \lambda_b > 0$, it holds that*

$$\lim_{c \to \pm\infty} h^B(c)\mathbb{P}_x[\Gamma_{T_c}] = \varphi^{(\pm 1),\lambda_a,\lambda_b}_{a,b}(x) \tag{4.1}$$

for all $x \in \mathbb{R}$, and

$$\lim_{c \to \pm\infty} h^B(c)\mathbb{P}_{X_t}[\Gamma_{T_c}] = \varphi^{(\pm 1),\lambda_a,\lambda_b}_{a,b}(X_t) \text{ a.s. and in } L^1(\mathbb{P}_x). \tag{4.2}$$

4.1. *Preliminary computations.* Before proving Proposition 4.1, we prove the following lemmas.

Lemma 4.2. *For $x \in \mathbb{R}$, for distinct points $a, b, c \in \mathbb{R}$, and for $\lambda_a > 0$, it holds that*

$$\mathbb{P}_x\left[e^{-\lambda_a L^a_{T_c}};\ T_a < T_c < T_b\right] = \mathbb{P}_x(T_a < T_b \wedge T_c) \cdot \frac{\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]}{1 - \mathbb{P}_c\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]}. \tag{4.3}$$

In particular, it can be represented only by h.

Proof: By the strong Markov property at T_a, we have

$$\mathbb{P}_x\left[e^{-\lambda_a L^a_{T_c}};\ T_a < T_c < T_b\right] = \mathbb{P}_x(T_a < T_b \wedge T_c)\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right]. \tag{4.4}$$

By the strong Markov property at T_b, we have

$$\begin{aligned}
\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_b < T_c\right] &= \mathbb{P}_a\left[1_{\{T_b<T_c\}}\mathbb{P}_a\left[e^{-\lambda_a (L^a_{T_c - T_b + T_b} - L^a_{T_b} + L^a_{T_b})}\middle|\mathscr{F}_{T_b}\right]\right] \\
&= \mathbb{P}_a\left[1_{\{T_b<T_c\}}e^{-\lambda_a L^a_{T_b}}\mathbb{P}_a\left[e^{-\lambda_a L^a_t \circ \theta_s}\middle|\mathscr{F}_s\right]_{t = T_c\circ\theta_s,\ s = T_b}\right] \\
&= \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right].
\end{aligned} \tag{4.5}$$

Similarly, we have

$$\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_c < T_b\right] = \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right]\mathbb{P}_c\left[e^{-\lambda_a L^a_{T_b}}\right]. \tag{4.6}$$

Thus, we have by (4.5) and (4.6),

$$\begin{aligned}
&\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right]\\
&= \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_b < T_c\right]\\
&= \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]\\
&= \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \left(\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_c < T_b\right]\right)\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]\\
&= \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right] + \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right]\mathbb{P}_c\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right].
\end{aligned} \tag{4.7}$$

Therefore, solving the equation (4.7), we have

$$\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right] = \frac{\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]}{1 - \mathbb{P}_c\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]}. \tag{4.8}$$

Plugging (4.8) into (4.4), we obtain the equation (4.3).

Finally, since

$$\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] = \frac{1}{1 + \lambda_a h^B(c-a)} \tag{4.9}$$

by (2.14) and (2.13), and since

$$\begin{aligned}
\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right] &= \mathbb{P}_{b-a}(T_0 > T_{c-a}) + \frac{\mathbb{P}_{b-a}(T_0 < T_{c-a})}{h^B(c-a)}\int_0^\infty e^{-\frac{u}{h^B(c-a)}} e^{-\lambda_a u}\,du\\
&= \mathbb{P}_b(T_a > T_c) + \frac{\mathbb{P}_b(T_a < T_c)}{1 + \lambda_a h^B(c-a)}\\
&= \frac{1 + \lambda_a\{h(a-c) + h(b-a) - h(b-c)\}}{1 + \lambda_a h^B(c-a)},
\end{aligned} \tag{4.10}$$

by (2.14) and (2.13), (4.3) can be represented only by h. □

Lemma 4.3. *For $x \in \mathbb{R}$, for distinct points $a, b, c \in \mathbb{R}$, and for constants $\lambda_a, \lambda_b > 0$, it holds that*

$$\begin{aligned}
&\mathbb{P}_x\left[\Gamma_{T_c};\ T_a < T_b < T_c\right]\\
&= \mathbb{P}_x(T_a < T_b \wedge T_c)\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]\\
&\quad\times \frac{\mathbb{P}_b\left[e^{-\lambda_b L^b_{T_c}};\ T_c < T_a\right] + \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right]}{1 - \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]}.
\end{aligned} \tag{4.11}$$

In particular, it can be represented only by h.

Proof: By the strong Markov property at T_a and T_b, we have

$$\begin{aligned}\mathbb{P}_x\left[\Gamma_{T_c};\ T_a < T_b < T_c\right] &= \mathbb{P}_x(T_a < T_b \wedge T_c)\mathbb{P}_a\left[\Gamma_{T_c};\ T_b < T_c\right] \\ &= \mathbb{P}_x(T_a < T_b \wedge T_c)\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]\mathbb{P}_b[\Gamma_{T_c}].\end{aligned}$$

By the strong Markov property, we have

$$\begin{aligned}\mathbb{P}_b[\Gamma_{T_c}] &= \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_c}};\ T_c < T_a\right] + \mathbb{P}_b\left[\Gamma_{T_c};\ T_a < T_c\right] \\ &= \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_c}};\ T_c < T_a\right] + \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a[\Gamma_{T_c}] \\ &= \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_c}};\ T_c < T_a\right] + \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right] \\ &\quad \times \left(\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right] + \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]\mathbb{P}_b[\Gamma_{T_c}]\right). \end{aligned} \tag{4.12}$$

Thus, solving the equation (4.12), we have

$$\mathbb{P}_b[\Gamma_{T_c}] = \frac{\mathbb{P}_b\left[e^{-\lambda_b L^b_{T_c}};\ T_c < T_a\right] + \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right]}{1 - \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]}.$$

Summarizing the above calculation, we obtain the equation (4.11).

Finally, from (4.8), (4.9), and (4.10), (4.11) can be represented only by h. □

4.2. *Proof of Proposition 4.1.*

Proof of Proposition 4.1: First, we have

$$\begin{aligned}\mathbb{P}_x[\Gamma_{T_c}] &= \mathbb{P}_x(T_c < T_a \wedge T_b) \\ &\quad + \mathbb{P}_x\left[e^{-\lambda_a L^a_{T_c}};\ T_a < T_c < T_b\right] + \mathbb{P}_x\left[e^{-\lambda_b L^b_{T_c}};\ T_b < T_c < T_a\right] \\ &\quad + \mathbb{P}_x\left[\Gamma_{T_c};\ T_a < T_b < T_c\right] + \mathbb{P}_x\left[\Gamma_{T_c};\ T_b < T_a < T_c\right]. \end{aligned} \tag{4.13}$$

Therefore, from (2.16), (4.3), (4.11), and (4.13), we complete the calculation of the expectation $\mathbb{P}_x[\Gamma_{T_c}]$.

Next, we consider the limit of $h^B(c)\mathbb{P}_x[\Gamma_{T_c}]$ as $c \to \pm\infty$. By (2.16), we have

$$\begin{aligned}\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_x(T_c < T_a \wedge T_b) &= \lim_{c\to\pm\infty} h^B(c)\cdot\frac{\mathbb{P}_x(T_c < T_a) - \mathbb{P}_x(T_b < T_a)\mathbb{P}_b(T_c < T_a)}{1 - \mathbb{P}_c(T_b < T_a)\mathbb{P}_b(T_c < T_a)} \\ &= h^{(\pm 1)}(x-a) - \mathbb{P}_x(T_b < T_a)h^{(\pm 1)}(b-a), \end{aligned} \tag{4.14}$$

since

$$\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_b(T_c < T_a) = \lim_{c\to\pm\infty}\frac{\{h(a-c) - h(b-c)\} + h(b-a)}{\frac{h^B(c-a)}{h^B(c)}} = h^{(\pm 1)}(b-a) \tag{4.15}$$

by (2.9), (2.6), (2.13), and (2.7), and

$$\lim_{c\to\pm\infty}\mathbb{P}_c(T_b < T_a) = \lim_{c\to\pm\infty}\frac{h(a-b) + h(c-a) - h(c-b)}{h^B(a-b)} = \frac{h^{(\mp 1)}(a-b)}{h^B(a-b)}$$

by (2.13) and (2.7). By (4.3), we have

$$\begin{aligned}
&\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_x\left[e^{-\lambda_a L^a_{T_c}};\ T_a < T_c < T_b\right] \\
&= \lim_{c\to\pm\infty} h^B(c)\mathbb{P}_x(T_a < T_b \wedge T_c)\cdot \frac{\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]}{1-\mathbb{P}_c\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]} \\
&= \mathbb{P}_x(T_a < T_b)\cdot \frac{h^{(\pm 1)}(a-b)}{1+\lambda_a h^B(b-a)},
\end{aligned} \tag{4.16}$$

since

$$\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] = \lim_{c\to\pm\infty} \frac{1}{\frac{1}{h^B(c)} + \lambda_a \frac{h^B(c-a)}{h^B(c)}} = \frac{1}{\lambda_a}$$

by (4.9) and (2.10),

$$\begin{aligned}
\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right] &= \lim_{c\to\pm\infty} \frac{1+\lambda_a\{[h(a-c)-h(b-c)]+h(b-a)\}}{\frac{1}{h^B(c)} + \lambda_a \frac{h^B(c-a)}{h^B(c)}} \\
&= \frac{1}{\lambda_a} + h^{(\pm 1)}(b-a)
\end{aligned} \tag{4.17}$$

by (4.10), (2.7), and (2.10), and

$$\begin{aligned}
\lim_{c\to\pm\infty} \mathbb{P}_c\left[e^{-\lambda_a L^a_{T_b}}\right] &= \lim_{c\to\pm\infty} \frac{1+\lambda_a\{h(a-b)+[h(c-a)-h(c-b)]\}}{1+\lambda_a h^B(b-a)} \\
&= \frac{1+\lambda_a h^{(\mp 1)}(a-b)}{1+\lambda_a h^B(b-a)}
\end{aligned} \tag{4.18}$$

by (4.10) and (2.7). By (4.11), we have

$$\begin{aligned}
&\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_x\left[\Gamma_{T_c};\ T_a < T_b < T_c\right] \\
&= \lim_{c\to\pm\infty} h^B(c)\mathbb{P}_x(T_a < T_b \wedge T_c)\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right] \\
&\quad \times \frac{\mathbb{P}_b\left[e^{-\lambda_b L^b_{T_c}};\ T_c < T_a\right] + \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right]}{1-\mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]} \\
&= \mathbb{P}_x(T_a < T_b)\cdot \frac{1}{1+\lambda_a h^B(b-a)}\cdot \frac{1+\lambda_a h^{(\pm 1)}(b-a)}{\lambda_a\lambda_b h^B(b-a)+\lambda_a+\lambda_b},
\end{aligned} \tag{4.19}$$

since

$$\begin{aligned}
&\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right] \\
&= \lim_{c\to\pm\infty} \frac{h^B(c)\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}}\right]h^B(c)\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]}{1-\mathbb{P}_c\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]} = \frac{h^{(\pm 1)}(a-b)}{1+\lambda_a h^B(b-a)}
\end{aligned} \tag{4.20}$$

by (4.8), (4.17), (4.18), and (4.19), and

$$\lim_{c\to\pm\infty} \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right] = \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}}\right] = \frac{1}{1+\lambda_a h^B(b-a)}$$

by (4.9). Therefore, the assertion (4.1) follows from (4.13), (4.14), (4.16), and (4.19).

Finally, we show the $L^1(\mathbb{P}_x)$-convergence. By the subadditivity of h, we have

$$0 \le h(a-c) + h(X_t - a) - h(X_t - c) \le h(X_t - a) + h(a - X_t).$$

The right-hand side belongs to $L^1(\mathbb{P}_x)$. See, e.g., proof of Theorem 15.2 of Tsukada (2018). Thus, by the dominated convergence theorem and by (4.15),

$$\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_{X_t}(T_c < T_a) = h^{(\pm1)}(X_t - a) \text{ in } L^1(\mathbb{P}_x).$$

Therefore, from this, we also obtain (4.2). □

4.3. *Main theorem in the hitting time case.* We define

$$N^{c,\lambda_a,\lambda_b}_{a,b,t} := h^B(c)\mathbb{P}_x\left[\Gamma_{T_c};\ t < T_c \middle| \mathscr{F}_t\right],$$
$$M^{c,\lambda_a,\lambda_b}_{a,b,t} := h^B(c)\mathbb{P}_x\left[\Gamma_{T_c} \middle| \mathscr{F}_t\right]$$

for distinct points $a, b, c \in \mathbb{R}$.

Theorem 4.4. *Let $x \in \mathbb{R}$. Then $(M^{(\pm1),\lambda_a,\lambda_b}_{a,b,t},\ t \ge 0)$ is a non-negative $((\mathscr{F}_t), \mathbb{P}_x)$-martingale, and it holds that*

$$\lim_{c\to\pm\infty} N^{c,\lambda_a,\lambda_b}_{a,b,t} = \lim_{c\to\pm\infty} M^{c,\lambda_a,\lambda_b}_{a,b,t} = M^{(\pm1),\lambda_a,\lambda_b}_{a,b,t} \text{ a.s. and in } L^1(\mathbb{P}_x).$$

Consequently, if $M^{(\pm1),\lambda_a,\lambda_b}_{a,b,0} > 0$ under $\mathbb{P}_x$, it holds that

$$\lim_{c\to\pm\infty} \frac{\mathbb{P}_x[F_t \cdot \Gamma_{T_c}]}{\mathbb{P}_x[\Gamma_{T_c}]} = \mathbb{P}_x\left[F_t \cdot \frac{M^{(\pm1),\lambda_a,\lambda_b}_{a,b,t}}{M^{(\pm1),\lambda_a,\lambda_b}_{a,b,0}}\right]$$

for all bounded $\mathscr{F}_t$-measurable functionals F_t.

The proof is almost the same as that of Theorem 3.4, based on (4.2) of Proposition 4.1 and so we omit it.

5. Two-point hitting time clock

5.1. *Limit behavior of the expected weight.*

For $-1 \le \gamma \le 1$, let us find the limit of $h^C(c,-d)\mathbb{P}_x[\Gamma_{T_c\wedge T_{-d}}]$ as $(c,d) \overset{(\gamma)}{\to} \infty$.

Proposition 5.1. *For distinct points $a, b \in \mathbb{R}$, for constants $\lambda_a, \lambda_b > 0$, and for $-1 \le \gamma \le 1$, it holds that*

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_c};\ T_c < T_{-d}\right] = \frac{1+\gamma}{2} \cdot \varphi^{(+1),\lambda_a,\lambda_b}_{a,b}(x), \tag{5.1}$$

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_{-d}};\ T_{-d} < T_c\right] = \frac{1-\gamma}{2} \cdot \varphi^{(-1),\lambda_a,\lambda_b}_{a,b}(x), \tag{5.2}$$

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_c\wedge T_{-d}}\right] = \varphi^{(\gamma),\lambda_a,\lambda_b}_{a,b}(x), \tag{5.3}$$

for all $x \in \mathbb{R}$, *and*

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} h^C(c,-d)\mathbb{P}_{X_t}\Big[\Gamma_{T_c};\ T_c < T_{-d}\Big] = \frac{1+\gamma}{2} \cdot \varphi_{a,b}^{(+1),\lambda_a,\lambda_b}(X_t) \text{ a.s. and in } L^1(\mathbb{P}_x),$$

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} h^C(c,-d)\mathbb{P}_{X_t}\Big[\Gamma_{T_{-d}};\ T_{-d} < T_c\Big] = \frac{1-\gamma}{2} \cdot \varphi_{a,b}^{(-1),\lambda_a,\lambda_b}(X_t) \text{ a.s. and in } L^1(\mathbb{P}_x),$$

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} h^C(c,-d)\mathbb{P}_{X_t}\Big[\Gamma_{T_c\wedge T_{-d}}\Big] = \varphi_{a,b}^{(\gamma),\lambda_a,\lambda_b}(X_t) \text{ a.s. and in } L^1(\mathbb{P}_x).$$

Proof: First, it hold that for distinct points $a, b, c, d \in \mathbb{R}$ with $c, d > 0$ and for constants $\lambda_a, \lambda_b > 0$,

$$\mathbb{P}_x\Big[\Gamma_{T_c};\ T_c < T_{-d}\Big] = \frac{\mathbb{P}_x[\Gamma_{T_c}] - \mathbb{P}_x[\Gamma_{T_{-d}}]\mathbb{P}_{-d}[\Gamma_{T_c}]}{1 - \mathbb{P}_c[\Gamma_{T_{-d}}]\mathbb{P}_{-d}[\Gamma_{T_c}]}, \tag{5.4}$$

since

$$\begin{aligned}\mathbb{P}_x\Big[\Gamma_{T_c};\ T_{-d} < T_c\Big] &= \mathbb{P}_x\Big[\Gamma_{T_{-d}};\ T_{-d} < T_c\Big]\mathbb{P}_{-d}[\Gamma_{T_c}] \\ &= \Big\{\mathbb{P}_x[\Gamma_{T_{-d}}] - \mathbb{P}_x\Big[\Gamma_{T_{-d}};\ T_c < T_{-d}\Big]\Big\}\mathbb{P}_{-d}[\Gamma_{T_c}] \\ &= \mathbb{P}_x[\Gamma_{T_{-d}}]\mathbb{P}_{-d}[\Gamma_{T_c}] - \mathbb{P}_x\Big[\Gamma_{T_c};\ T_c < T_{-d}\Big]\mathbb{P}_c[\Gamma_{T_{-d}}]\mathbb{P}_{-d}[\Gamma_{T_c}],\end{aligned}$$

by the strong Markov property at T_{-d} and T_c.

Note that if $(c,d) \overset{(\gamma)}{\to} \infty$, then $\frac{d}{c} \to \frac{1+\gamma}{1-\gamma}$. Thus, we have by (2.6),

$$\frac{h(c-a)}{h(-c-d)} = \frac{h(c-a)}{|c-a|} \cdot \frac{|-c-d|}{h(-c-d)} \cdot \frac{|1-\frac{a}{c}|}{|-1-\frac{d}{c}|} \to \frac{1-\gamma}{2}, \tag{5.5}$$

$$\frac{h(-d)}{h(c+d)} = \frac{h(-d)}{|-d|} \cdot \frac{|-c-d|}{h(-c-d)} \cdot \frac{|-\frac{d}{c}|}{|-1-\frac{d}{c}|} \to \frac{1+\gamma}{2} \tag{5.6}$$

as $(c,d) \overset{(\gamma)}{\to} \infty$. Thus, we have by (2.15) and (2.9),

$$\frac{h^B(c)}{h^C(c,-d)} \to \frac{2}{1+\gamma},$$

$$\frac{h^B(d)}{h^C(c,-d)} \to \frac{2}{1-\gamma},$$

as $(c,d) \overset{(\gamma)}{\to} \infty$. By (4.13), we have

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} \mathbb{P}_{-d}[\Gamma_{T_c}] = 0,$$

since

$$\begin{aligned}\lim_{(c,d)\overset{(\gamma)}{\to}\infty} \mathbb{P}_{-d}(T_c < T_a \wedge T_b) &= \lim_{(c,d)\overset{(\gamma)}{\to}\infty} \frac{\mathbb{P}_{-d}(T_c < T_a) - \mathbb{P}_{-d}(T_b < T_a)\mathbb{P}_b(T_c < T_a)}{1 - \mathbb{P}_c(T_b < T_a)\mathbb{P}_b(T_c < T_a)} \\ &= \lim_{(c,d)\overset{(\gamma)}{\to}\infty} \mathbb{P}_{-d}(T_c < T_a) \\ &= \lim_{(c,d)\overset{(\gamma)}{\to}\infty} \frac{\frac{h(a-c)}{h(-c-d)} + \frac{h(-d-a)}{h(-c-d)} - 1}{\frac{h(c-a)}{h(-c-d)} + \frac{h(a-c)}{h(-c-d)}} = 0\end{aligned}$$

by (2.16), (2.13), (5.5), and (5.6),

$$\begin{aligned}&\lim_{(c,d)\overset{(\gamma)}{\to}\infty} \mathbb{P}_{-d}\left[e^{-\lambda_a L^a_{T_c}};\ T_a < T_c < T_b\right]\\ &= \lim_{(c,d)\overset{(\gamma)}{\to}\infty} \mathbb{P}_{-d}(T_a < T_b \wedge T_c) \cdot \frac{\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}}\right] - \mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]}{1 - \mathbb{P}_c\left[e^{-\lambda_a L^a_{T_b}}\right]\mathbb{P}_b\left[e^{-\lambda_a L^a_{T_c}}\right]} = 0\end{aligned}$$

by (4.3) and (4.18), and

$$\begin{aligned}&\lim_{(c,d)\overset{(\gamma)}{\to}\infty} \mathbb{P}_{-d}\left[\Gamma_{T_c};\ T_a < T_b < T_c\right]\\ &= \lim_{(c,d)\overset{(\gamma)}{\to}\infty} \mathbb{P}_{-d}(T_a < T_b \wedge T_c)\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]\\ &\quad\times \frac{\mathbb{P}_b\left[e^{-\lambda_b L^b_{T_c}};\ T_c < T_a\right] + \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_c}};\ T_c < T_b\right]}{1 - \mathbb{P}_b\left[e^{-\lambda_b L^b_{T_a}};\ T_a < T_c\right]\mathbb{P}_a\left[e^{-\lambda_a L^a_{T_b}};\ T_b < T_c\right]} = 0\end{aligned}$$

by (4.11), (2.10), and (4.20). Summarizing the above calculation, we obtain by (5.4) and (4.1),

$$\begin{aligned}&\lim_{(c,d)\overset{(\gamma)}{\to}\infty} h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_c};\ T_c < T_{-d}\right]\\ &= \lim_{(c,d)\overset{(\gamma)}{\to}\infty} \frac{\frac{h^C(c,-d)}{h^B(c)} \cdot h^B(c)\mathbb{P}_x[\Gamma_{T_c}] - \frac{h^C(c,-d)}{h^B(-d)} \cdot h^B(-d)\mathbb{P}_x[\Gamma_{T_{-d}}]\mathbb{P}_{-d}[\Gamma_{T_c}]}{1 - \mathbb{P}_c[\Gamma_{T_{-d}}]\mathbb{P}_{-d}[\Gamma_{T_c}]}\\ &= \frac{1+\gamma}{2} \lim_{(c,d)\overset{(\gamma)}{\to}\infty} h^B(c)\mathbb{P}_x[\Gamma_{T_c}] = \frac{1+\gamma}{2} \cdot \varphi^{(+1),\lambda_a,\lambda_b}_{a,b}(x).\end{aligned}$$

Hence, we have proved the first equation. The second equation (5.2) can be shown similarly. The third equation (5.3) can be proved by summing up the first and the second equations. The remaining equations are clear from (4.2) of Proposition 4.1. □

5.2. *Main theorem in the two-point hitting time case.* We define

$$\begin{aligned}N^{1,c,d,\lambda_a,\lambda_b}_{a,b,t} &:= h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_c \wedge T_{-d}};\ t < T_c \wedge T_{-d}\,\middle|\,\mathscr{F}_t\right],\\ N^{2,c,d,\lambda_a,\lambda_b}_{a,b,t} &:= h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_c};\ t < T_c < T_{-d}\,\middle|\,\mathscr{F}_t\right],\\ N^{3,c,d,\lambda_a,\lambda_b}_{a,b,t} &:= h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_{-d}};\ t < T_{-d} < T_c\,\middle|\,\mathscr{F}_t\right],\\ M^{1,c,d,\lambda_a,\lambda_b}_{a,b,t} &:= h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_c \wedge T_{-d}}\,\middle|\,\mathscr{F}_t\right],\\ M^{2,c,d,\lambda_a,\lambda_b}_{a,b,t} &:= h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_c};\ T_c < T_{-d}\,\middle|\,\mathscr{F}_t\right],\\ M^{3,c,d,\lambda_a,\lambda_b}_{a,b,t} &:= h^C(c,-d)\mathbb{P}_x\left[\Gamma_{T_{-d}};\ T_{-d} < T_c\,\middle|\,\mathscr{F}_t\right]\end{aligned}$$

for $c, d > 0$.

Theorem 5.2. *Let $x \in \mathbb{R}$ and $-1 \le \gamma \le 1$. Then, $(M^{(\gamma),\lambda_a,\lambda_b}_{a,b,t},\ t \ge 0)$ is a non-negative $((\mathscr{F}_t), \mathbb{P}_x)$-martingale, and it holds that*

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} N^{1,c,d,\lambda_a,\lambda_b}_{a,b,t} = \lim_{(c,d)\overset{(\gamma)}{\to}\infty} M^{1,c,d,\lambda_a,\lambda_b}_{a,b,t} = M^{(\gamma),\lambda_a,\lambda_b}_{a,b,t} \text{ a.s. and in } L^1(\mathbb{P}_x),$$

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} N^{2,c,d,\lambda_a,\lambda_b}_{a,b,t} = \lim_{(c,d)\overset{(\gamma)}{\to}\infty} M^{2,c,d,\lambda_a,\lambda_b}_{a,b,t} = \frac{1+\gamma}{2} \cdot M^{(+1),\lambda_a,\lambda_b}_{a,b,t} \text{ a.s. and in } L^1(\mathbb{P}_x),$$

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} N^{3,c,d,\lambda_a,\lambda_b}_{a,b,t} = \lim_{(c,d)\overset{(\gamma)}{\to}\infty} M^{3,c,d,\lambda_a,\lambda_b}_{a,b,t} = \frac{1-\gamma}{2} \cdot M^{(-1),\lambda_a,\lambda_b}_{a,b,t} \text{ a.s. and in } L^1(\mathbb{P}_x).$$

Consequently, if $M^{(\gamma),\lambda_a,\lambda_b}_{a,b,0} > 0$ under $\mathbb{P}_x$, it holds that

$$\lim_{(c,d)\overset{(\gamma)}{\to}\infty} \frac{\mathbb{P}_x[F_t \cdot \Gamma_{T_c \wedge T_{-d}}]}{\mathbb{P}_x[\Gamma_{T_c \wedge T_{-d}}]} = \mathbb{P}_x\left[F_t \cdot \frac{M^{(\gamma),\lambda_a,\lambda_b}_{a,b,t}}{M^{(\gamma),\lambda_a,\lambda_b}_{a,b,0}}\right]$$

for all bounded $\mathscr{F}_t$-measurable functionals F_t.

The proof is almost the same as that of Theorem 3.4, based on Proposition 5.1 and so we omit it.

6. Inverse local time clock

6.1. *Limit behavior of the expected weight.* Let us find the limit of $h^B(c)\mathbb{P}_x[\Gamma_{\eta^c_u}]$ as $c \to \pm\infty$.

Proposition 6.1. *For distinct points $a, b \in \mathbb{R}$, for constants $\lambda_a, \lambda_b > 0$, and for $u > 0$, it holds that*

$$\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_x[\Gamma_{\eta^c_u}] = \varphi^{(\pm1),\lambda_a,\lambda_b}_{a,b}(x) \tag{6.1}$$

for all $x \in \mathbb{R}$, and

$$\lim_{c\to\pm\infty} h^B(c)\mathbb{P}_{X_t}[\Gamma_{\eta^c_u}] = \varphi^{(\pm1),\lambda_a,\lambda_b}_{a,b}(X_t) \text{ a.s. and in } L^1(\mathbb{P}_x).$$

Before proving Proposition 6.1, we prove the following lemma.

Lemma 6.2. *For distinct points $a, b \in \mathbb{R}$, for $u > 0$, and for constants $\lambda_a, \lambda_b > 0$, it holds that*

$$\lim_{c\to\pm\infty} \mathbb{P}_c[\Gamma_{\eta^c_u}] = 1. \tag{6.2}$$

Proof: Since L^a may increase during excursion away from c, we can write

$$L^a_{\eta^c_u} = \sum_{v \le u} \left(L^a_{\eta^c_v} - L^a_{\eta^c_{v-}}\right) = \sum_{v \le u} L^a_{T_c}\left(\epsilon^c(v)\right).$$

Thus, using Theorem 2.7 of Kyprianou (2014), we have

$$\begin{aligned}
\mathbb{P}_c\Big[\Gamma_{\eta^c_u}\Big] &= \mathbb{P}_c\bigg[\exp\bigg\{-\sum_{v\le u}\Big(\lambda_a L^a_{T_c}\big(\epsilon^c(v)\big)+\lambda_b L^b_{T_c}\big(\epsilon^c(v)\big)\Big)\bigg\}\bigg] \\
&= \exp\bigg\{-\int_{(0,u]\times\mathcal{D}}\Big(1-e^{-\lambda_a L^a_{T_c}(e)-\lambda_b L^b_{T_c}(e)}\Big)dt\otimes n^c(de)\bigg\} \\
&= \exp\bigg\{-un^c\Big[1-e^{-\lambda_a L^a_{T_c}-\lambda_b L^b_{T_c}}\Big]\bigg\} \\
&= \exp\bigg\{-un^c\Big[1-e^{-\lambda_a L^a_{T_c}};\ T_a<T_c<T_b\Big]\bigg\} \\
&\quad\times\exp\bigg\{-un^c\Big[1-e^{-\lambda_b L^b_{T_c}};\ T_b<T_c<T_a\Big]\bigg\} \\
&\quad\times\exp\bigg\{-un^c\Big[1-e^{-\lambda_a L^a_{T_c}-\lambda_b L^b_{T_c}};\ T_a<T_b<T_c\Big]\bigg\} \\
&\quad\times\exp\bigg\{-un^c\Big[1-e^{-\lambda_a L^a_{T_c}-\lambda_b L^b_{T_c}};\ T_b<T_a<T_c\Big]\bigg\}.
\end{aligned} \tag{6.3}$$

By the strong Markov property for n^c, and by (2.10) and (2.11), we have

$$\begin{aligned}
&n^c\Big[1-e^{-\lambda_a L^a_{T_c}};\ T_a<T_c<T_b\Big] \\
&= \int n^c\Big(1_{\{T_a<T_b\wedge T_c\}};\ X_{T_a}\in dx\Big)\mathbb{P}^c_x\Big[1-e^{-\lambda_a L^a_{T_c}},\ T_c<T_b\Big] \\
&= n^c(T_a<T_b\wedge T_c)\mathbb{P}_a\Big[1-e^{-\lambda_a L^a_{T_c}},\ T_c<T_b\Big] \\
&\le n^c(T_a<T_c)\mathbb{P}_a\Big[1-e^{-\lambda_a L^a_{T_c}},\ T_c<T_b\Big]\to 0
\end{aligned} \tag{6.4}$$

as $c\to\pm\infty$. Moreover, by the strong Markov property for n^a and $\mathbb{P}_a$, we have

$$\begin{aligned}
&n^c\Big[1-e^{-\lambda_a L^a_{T_c}-\lambda_b L^b_{T_c}};\ T_a<T_b<T_c\Big] \\
&= n^c(T_a<T_b\wedge T_c)\Big\{\mathbb{P}_a(T_b<T_c)-\mathbb{P}_a\Big[\Gamma_{T_c};\ T_b<T_c\Big]\Big\} \\
&= n^c(T_a<T_b\wedge T_c)\Big\{\mathbb{P}_a(T_b<T_c)-\mathbb{P}_a\Big[e^{-\lambda_a L^a_{T_b}};\ T_b<T_c\Big]\mathbb{P}_b[\Gamma_{T_c}]\Big\}\to 0
\end{aligned} \tag{6.5}$$

as $c\to\pm\infty$. Therefore, (6.2) follows (6.3), (6.4), and (6.5). □

Let us proceed to the proof of Proposition 6.1.

Proof: By the strong Markov property, we have for distinct points $a,b,c\in\mathbb{R}$,

$$\mathbb{P}_x[\Gamma_{\eta^c_u}] = \mathbb{P}_c[\Gamma_{\eta^c_u}]\cdot\mathbb{P}_x[\Gamma_{T_c}]. \tag{6.6}$$

Therefore, (6.1) follows from (4.1) and (6.6). The remaining equations are clear from (4.2) of Proposition 4.1. □

6.2. *Main theorem in the inverse local time case.* We define

$$N_{a,b,t}^{c,u,\lambda_a,\lambda_b} := h^B(c)\mathbb{P}_x\left[\Gamma_{\eta_u^c};\ t < \eta_u^c \middle| \mathscr{F}_t\right],$$
$$M_{a,b,t}^{c,u,\lambda_a,\lambda_b} := h^B(c)\mathbb{P}_x\left[\Gamma_{\eta_u^c} \middle| \mathscr{F}_t\right]$$

for $c \in \mathbb{R}$ and $u > 0$.

Theorem 6.3. *Let $x \in \mathbb{R}$. Then, it holds that*

$$\lim_{c\to\pm\infty} N_{a,b,t}^{c,u,\lambda_a,\lambda_b} = \lim_{c\to\pm\infty} M_{a,b,t}^{c,u,\lambda_a,\lambda_b} = M_{a,b,t}^{(\pm1),\lambda_a,\lambda_b} \text{ a.s. and in } L^1(\mathbb{P}_x).$$

Consequently, if $M_{a,b,0}^{(\pm1),\lambda_a,\lambda_b} > 0$ under $\mathbb{P}_x$, it holds that

$$\lim_{c\to\pm\infty} \frac{\mathbb{P}_x[F_t \cdot \Gamma_{\eta_u^c}]}{\mathbb{P}_x[\Gamma_{\eta_u^c}]} = \mathbb{P}_x\left[F_t \cdot \frac{M_{a,b,t}^{(\pm1),\lambda_a,\lambda_b}}{M_{a,b,0}^{(\pm1),\lambda_a,\lambda_b}}\right]$$

for all bounded $\mathscr{F}_t$-measurable functionals F_t.

The proof is almost the same as that of Theorem 3.4, based on Proposition 6.1 and so we omit it.

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