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Corner percolation with preferential directions

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Abstract. Corner percolation is a dependent bond percolation model on $\mathbb{Z}^2$ introduced by Bálint Tóth, in which each vertex has exactly two incident edges, perpendicular to each other. In this model, connected components are either finite cycles or bi-infinite simple paths. Gábor Pete proved in 2008 that under the maximal entropy probability measure, all connected components are finite cycles almost surely. We consider here a variation where West and North directions are preferred with probability p and q respectively, with $(p,q) \neq (\frac{1}{2}, \frac{1}{2})$. We prove that almost surely, there exist infinitely many bi-infinite simple paths. Furthermore, they all have the same asymptotic slope.

1. Introduction

Various constrained percolation models on $\mathbb{Z}^2$ have been studied, including models with restrictions on the degree of each vertex, see for example Garet et al. (2018) or de Lima et al. (2020). In the present work, we study an even more constrained percolation model, called *corner percolation*, where each vertex has exactly two incident open edges, perpendicular to each other. It was introduced by Bálint Tóth as a four-vertex model, since the constraint only allows four configurations on a given vertex. Corner percolation can also be seen as a degenerated and simpler version of the six-vertex model, containing only four of the six allowed configurations, see Figures 1.1 and 1.2. We refer to Zinn-Justin (2009) for more details on the six-vertex model.

Corner percolation configurations are also known under the name of *hitomezashi* design, by analogy with a Japanese style of embroidery called *sashiko*, that creates patterns satisfying the same constraints, see Figure 1.3. Since at each vertex, there is exactly one horizontal and one vertical open edge, the components are either finite cycles or bi-infinite paths, made in both case of a perfect alternation of horizontal and vertical edges. The mathematical properties of hitomezashi loops have been recently investigated and still raise many questions. In particular, Defant and Kravitz (2024) showed that their length is always congruent to 4 modulo 8 (a shorter proof has

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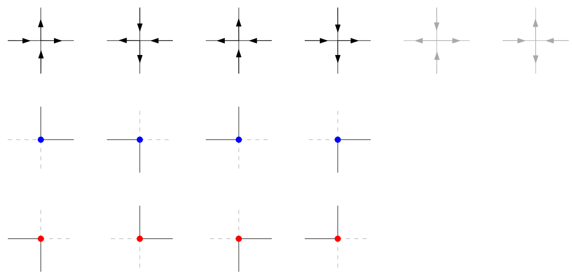

Figure 1.1. Correspondence between a degenerate six-vertex model (first line), which does not allow gray configurations, and a corner percolation. Blue (resp. red) points are points of $\mathbb{Z}_o^2 = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 \in 1 + 2\mathbb{Z}\}$ (resp. $\mathbb{Z}_e^2 = \mathbb{Z}^2 \setminus \mathbb{Z}_o^2$).

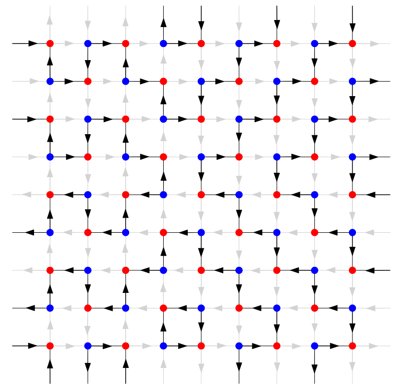

Figure 1.2. A configuration of corner percolation (black edges) and its associated six vertex model configuration (black and gray arrows).

been found very recently by Ren and Zhang (2024)), and the enumeration of hitomezashi loops according to their length is still an open question.

Although they exhibit a rich combinatorial structure, corner percolation configurations can be created by a very simple procedure. Indeed, in order to satisfy the constraint, there must be on each horizontal line a perfect alternation of open and closed edges, and the same holds for vertical lines. Consequently, a configuration can be parametrized by two binary sequences specifying, for each horizontal and vertical line, which edges are open.

A natural question is then to ask whether there exists an infinite connected component, depending on the probability distributions of the two binary sequences. Pete (2008) studied the model under the maximal entropy probability measure. In this case, the bits of the two sequences encoding the corner percolation are chosen independently and uniformly at random. The key object introduced by Pete is the *height function*, which is a simple function of the two random walks naturally associated to the two binary sequences. The connected components are in fact the level lines of this height function. Using this height function, Pete (2008) proved that all connected components are finite cycles almost surely.

Let us give an alternative interpretation of the maximal entropy probability measure. We consider that each horizontal line of the grid $\mathbb{Z}^2$ is a one-way road that can be either oriented to the East or to the West with probability $\frac{1}{2}$, and the same for vertical lines with North and South. This orientation is fixed once and for all. We start from the origin of the grid and follow the horizontal road according to its orientation. At each corner, we turn left or right according to the direction of the road encountered, see Figure 1.4. This process describes a deterministic walk in a random environment, and it can be seen that the path taken follows the component of the origin in a corner percolation configuration distributed under the maximal entropy probability measure. Consequently, Pete's result ensures that the trajectory necessarily comes back where it started.

This point of view leads us to consider a case where some directions can be preferred: we extend this model by considering that each vertical road is oriented to the North with probability q, and that each horizontal road is oriented to the West with probability p. This amounts to take, for the two binary sequences encoding the corner percolation, independent Bernoulli variables, with parameters p for the horizontal lines, and q for the vertical lines, rather than parameter $\frac{1}{2}$ for both. Using the height function of Pete, we prove the following:

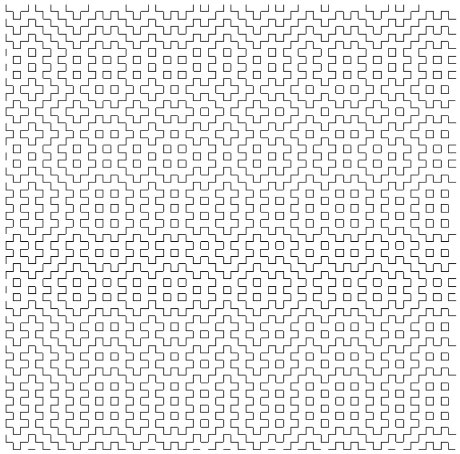

FIGURE 1.3. A configuration of corner percolation.

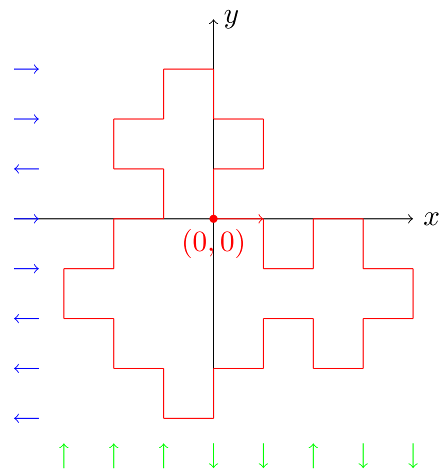

FIGURE 1.4. An instance of a finite trajectory for the deterministic walk starting from the origin, in an environment giving the directions of the roads.

Theorem 1.1. *If $(p,q) \neq (\frac{1}{2},\frac{1}{2})$, then with positive probability the path starting from the origin never goes back to its starting point. Moreover, for all $h \in \mathbb{Z}$, there almost surely exists a unique bi-infinite simple path of level h, and all these paths have a same asymptotic slope, which is equal to $\frac{2q-1}{1-2p}$ (if $p = 1/2$, the asymptotic slope is infinite). In particular, there is almost surely an infinite number of bi-infinite simple paths, and they all have the same direction.*

In our case, the two random walks encoding corner percolation are biased, and the strong law of large numbers is sufficient to obtain informations on the level lines of the height function. In the maximal entropy case studied by Pete, the random walks are symmetric, thus the analysis is much more involved.

Burton and Keane (1991) studied the topological form and arrangement of infinite clusters in percolation on $\mathbb{Z}^2$, in a very general case. With their classification, thus corner percolation is of "two-sided infinite order type".

Theorem 1.1 will be proved in two steps, Proposition 3.3 and Proposition 3.6. In the next section, we define the model more formally and introduce some notations.

2. Definition of the model

2.1. *Definition and notations.* Let us denote by $(\mathbb{Z}^2, \mathbb{E}^2)$ the two-dimensional grid, that is, the graph whose set of vertices is $\mathbb{Z}^2$ and whose set of edges is $\mathbb{E}^2 = \{\{x, x+e_1\} : x \in \mathbb{Z}^2\} \cup \{\{x, x+e_2\} : x \in \mathbb{Z}^2\}$, where $e_1 = (1,0)$ and $e_2 = (0,1)$.

Let $\mathcal{G}$ be the set of subgraphs $G = (\mathbb{Z}^2, \mathcal{E})$ of $(\mathbb{Z}^2, \mathbb{E}^2)$ such that each vertex has exactly one horizontal and one vertical adjacent edge, that is: for any $x \in \mathbb{Z}^2$, exactly one of the edges among $\{x, x+e_1\}$ and $\{x, x-e_1\}$ belongs to $\mathcal{E}$, and exactly one of the edges among $\{x, x+e_2\}$ and $\{x, x-e_2\}$ belongs to $\mathcal{E}$.

Let us introduce the following sets of *even* and *odd* elements of $\mathbb{Z}^2$:

$$\mathbb{Z}^2_e = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 \in 2\mathbb{Z}\},$$
$$\mathbb{Z}^2_o = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 \in 1 + 2\mathbb{Z}\}.$$

Let $\Omega = \{-1,1\}^{\mathbb{Z}} \times \{-1,1\}^{\mathbb{Z}}$. Our notations are inspired by those of Pete. For a configuration $\omega = (\xi, \eta) \in \Omega$, we define the set $\mathcal{E}(\omega)$ of open edges as follows. For all $i \in \mathbb{Z}$,

- if $\xi(i) = 1$, then for all $k \in \mathbb{Z}$, the vertical edge $\{(i,k),(i,k)+e_2\}$ is open if and only if $(i,k) \in \mathbb{Z}_o^2$. If $\xi(i) = -1$, then for all $k \in \mathbb{Z}$, the vertical edge $\{(i,k),(i,k)+e_2\}$ is open if and only if $(i,k) \in \mathbb{Z}_e^2$.
- if $\eta(i) = 1$, then for all $k \in \mathbb{Z}$, the horizontal edge $\{(k,i),(k,i)+e_1\}$ is open if and only if $(k,i) \in \mathbb{Z}_o^2$. If $\eta(i) = -1$, then for all $k \in \mathbb{Z}$, the horizontal edge $\{(k,i),(k,i)+e_1\}$ is open if and only if $(k,i) \in \mathbb{Z}_e^2$.

We then denote by $X(\omega)$ the graph in $\mathcal{G}$ with vertices set $\mathbb{Z}^2$ and set of edges $\mathcal{E}(\omega)$. Note that the

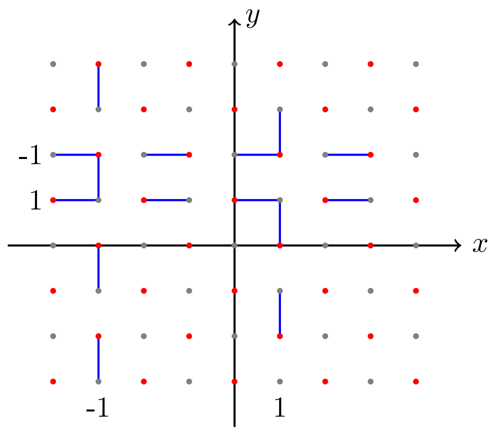

FIGURE 2.5. Construction of some open edges knowing that $\xi(1) = 1$, $\xi(-3) = -1$, $\eta(1) = 1$ and $\eta(2) = -1$. Points of $\mathbb{Z}_e^2$ are colored in gray, points of $\mathbb{Z}_o^2$ in red.

map $X : \Omega \to \mathcal{G}$ is bijective. For $p, q \in]0,1[$, we introduce the probability

$$\mu^{p,q} = (q\delta_1 + (1-q)\delta_{-1})^{\otimes \mathbb{Z}} \otimes (p\delta_1 + (1-p)\delta_{-1})^{\otimes \mathbb{Z}}$$

on Ω (with the product σ-algebra). We denote by $\mathbb{P}^{p,q}$ the image of $\mu^{p,q}$ by X, and we call $\mathbb{P}^{p,q}$ the *corner percolation model* with parameters (p,q). When there is no ambiguity on the parameters, we write μ (respectively $\mathbb{P}$) instead of $\mu^{p,q}$ (respectively $\mathbb{P}^{p,q}$). In the special case $q = p = \frac{1}{2}$, this distribution can be interpreted as the uniform distribution on $\mathcal{G}$. In this work, we are interested in the properties of random graphs of $\mathcal{G}$ distributed according to $\mathbb{P}^{p,q}$, and more specifically in the existence and properties of bi-infinite simple paths.

2.2. *Infinite simple paths.* Let $z \in \mathbb{Z}_e^2$. We encode the connected component which contains z by a bi-infinite, possibly periodic, path $\Gamma^z = (\Gamma_n^z)_{n \in \mathbb{Z}}$ of vertices. The origin of the path is $\Gamma_0^z = z$. We choose the orientation of Γ^z by taking Γ_1^z such that $\{\Gamma_0^z, \Gamma_1^z\}$ is a horizontal edge. Note that two points of $\Gamma \cap \mathbb{Z}_e^2$ induce the same orientation on Γ. If the connected component of z is infinite, then Γ^z is simple, while if the connected component of z is a circuit, then Γ^z is periodic. Also, we define the *forward path* Γ_z^+ (resp. the *backward path* Γ_z^-) by

$$\Gamma_z^+ = (\Gamma_n^z)_{n \in \mathbb{Z}_+} \quad (\text{resp. } \Gamma_z^- = (\Gamma_{-n}^z)_{n \in \mathbb{Z}_+}).$$

If Γ^z is a bi-infinite simple path, then both Γ_z^+ and Γ_z^- are infinite simple paths.

We also define a notion of neighborhood between bi-infinite simple paths. We denote by O the complement set in $\mathbb{R}^2$ of the set of open edges. We say that two bi-infinite simple paths are *neighbors* if they are on the border of the same connected component of O.

3. Height function and infinite simple paths

Now, we recall the definition and properties of the *height function* $H : \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2}) \to \mathbb{Z}$, which was the key ingredient introduced by Pete (2008) to study the case $(\frac{1}{2}, \frac{1}{2})$. We define the *faces of* $\mathbb{Z}^2$ to be the unit squares of center $(n+\frac{1}{2}, m+\frac{1}{2})$ for $n, m \in \mathbb{Z}$. We color the faces of $\mathbb{Z}^2$ in a chessboard manner: a face of center $(n+\frac{1}{2}, m+\frac{1}{2})$ is black if $n+m \in 2\mathbb{Z}$, otherwise it is white. Note that a path has only black faces along one side and only white faces along the other. We set $H\left(\frac{1}{2}, \frac{1}{2}\right) = 0$. To define $H\left(n+\frac{1}{2}, m+\frac{1}{2}\right)$, consider a path in $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$, from $(\frac{1}{2}, \frac{1}{2})$ to $(n+\frac{1}{2}, m+\frac{1}{2})$. We follow this path starting from $(\frac{1}{2}, \frac{1}{2})$, and each time the path crosses an open edge from a black face to a white face (respectively from a white face from a black face), we add (respectively substract) 1 to the height, until we reaches $(n+\frac{1}{2}, m+\frac{1}{2})$. Note that the definition of $H\left(n+\frac{1}{2}, m+\frac{1}{2}\right)$ does not depend on the choice of the path, and all black faces (respectively all white faces) have the same height along a path. The *level of a path* is the height of a black face along this path.

Despite the strong dependencies of the model, the height function can be expressed as a very simple function of two independent simple random walks on $\mathbb{Z}$. Remember that we denote by $(\xi(n))_{n\in\mathbb{Z}}$ (resp. $(\eta(m))_{m\in\mathbb{Z}}$) the values of the vertical (resp. horizontal) lines, and that its law is $(q\delta_1+(1-q)\delta_{-1})^{\otimes\mathbb{Z}}$ (resp. $(p\delta_1+(1-p)\delta_{-1})^{\otimes\mathbb{Z}}$). We define two independent random walks $(X_n)_{n\in\mathbb{Z}}$ and $(Y_m)_{m\in\mathbb{Z}}$ by setting:

$$X_0 = 0, \quad \text{for } n > 0,\ X_n = \sum_{i=1}^{n} \xi(i), \quad \text{and for } n < 0,\ X_n = -\sum_{i=n+1}^{0} \xi(i);$$

$$Y_0 = 0, \quad \text{for } m > 0,\ Y_m = \sum_{i=1}^{m} \eta(i), \quad \text{and for } m < 0,\ Y_m = -\sum_{i=m+1}^{0} \eta(i).$$

It is not difficult to check that:

$$H\left(n+\frac{1}{2}, m+\frac{1}{2}\right) = \left\lceil \frac{X_n+Y_m}{2} \right\rceil.$$

From the strong law of large numbers, we obtain directly the following lemma:

Lemma 3.1. *For all $\theta \in [0, 2\pi[$, we have almost surely, uniformly in θ:*

$$\frac{X_{r\cos(\theta)} + Y_{r\sin(\theta)}}{r} \underset{r\to+\infty}{\longrightarrow} (2q-1)\cos(\theta) + (2p-1)\sin(\theta).$$

Proof: Let $\varepsilon > 0$. By the strong law of large numbers, there almost surely exists $N \in \mathbb{Z}_+^*$ such that, for all $n \geq N$,

$$\max\left(\left|\frac{X_n}{n} - (2q-1)\right|, \left|\frac{X_{-n}}{n} - (1-2q)\right|, \left|\frac{Y_n}{n} - (2p-1)\right|, \left|\frac{Y_{-n}}{n} - (1-2p)\right|\right) \leq \frac{\varepsilon}{2}. \tag{3.1}$$

We set $R = \frac{4N}{\varepsilon}$. For $\theta \in [0, 2\pi[$ and $r \geq R$ such that $(r\cos(\theta), r\sin(\theta)) \in \mathbb{Z}^2$, we have

$$\begin{aligned}&\left|\frac{X_{r\cos(\theta)} + Y_{r\sin(\theta)}}{r} - [(2q-1)\cos(\theta) + (2p-1)\sin(\theta)]\right| \\ &\leq \left|\frac{X_{r\cos(\theta)}}{r} - (2q-1)\cos(\theta)\right| + \left|\frac{Y_{r\sin(\theta)}}{r} - (2p-1)\sin(\theta)\right|.\end{aligned} \tag{3.2}$$

We now bound the first term from above. We have two cases:

Case 1: $|\cos(\theta)| \leq \frac{\varepsilon}{4}$.
In this case, we bound from above $X_{r\cos(\theta)}$ with the triangular inequality:

$$\left|\frac{X_{r\cos(\theta)}}{r}\right| \leq |\cos(\theta)|.$$

Since $|(2q-1)\cos(\theta)| \leq |\cos(\theta)|$, we have

$$\left|\frac{X_{r\cos(\theta)}}{r} - (2q-1)\cos(\theta)\right| \leq 2\cos(\theta) \leq \frac{\varepsilon}{2}.$$

Case 2: $|\cos(\theta)| > \frac{\varepsilon}{4}$.
This time we have $r|\cos(\theta)| \geq N$, so by (3.1), we have

$$\left|\frac{X_{r\cos(\theta)}}{r\cos(\theta)} - (2q-1)\right| \leq \frac{\varepsilon}{2}.$$

Then we have

$$\begin{aligned}\left|\frac{X_{r\cos(\theta)}}{r} - (2q-1)\cos(\theta)\right| &\leq \left|\frac{X_{r\cos(\theta)}}{r\cos(\theta)}\cos(\theta) - (2q-1)\cos(\theta)\right| \\ &\leq |\cos(\theta)|\left|\frac{X_{r\cos(\theta)}}{r\cos(\theta)} - (2q-1)\right| \leq \frac{\varepsilon}{2}.\end{aligned}$$

Working in the same manner for the second term in (3.2), we finally obtain, for all $r \geq R$, with $R = \frac{4N}{\varepsilon}$ independent of θ,

$$\left|\frac{X_{r\cos(\theta)} + Y_{r\sin(\theta)}}{r} - \langle z_{p,q}, e^{i\theta}\rangle_{\mathbb{R}^2}\right| \leq \varepsilon.$$

□

Remark 3.2. We can easily check that the conclusion of Lemma 3.1 is equivalent to the following: the graph of the map

$$\begin{array}{ccc}[-1,1]^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & \frac{1}{n}H(\lceil nx\rceil + \frac{1}{2}, \lceil ny\rceil + \frac{1}{2})\end{array}$$

converges almost surely to the graph of the map

$$\begin{array}{ccc}[-1,1]^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & \frac{x}{2}(2q-1) + \frac{y}{2}(2p-1)\end{array}$$

for the Hausdorff metric on the subsets of $\mathbb{R}^2 \times \mathbb{R}$.

With Lemma 3.1 we are going to localize, for all $h \in \mathbb{Z}$, the level set

$$L_h = \left\{z \in \mathbb{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right) \ : \ H(z) = h\right\}$$

of the height function. In order to do that, we introduce some notations. For all $R \in \mathbb{R}_+^*$, we denote by D_R the disk of radius R, centered at the origin, and by $\mathbb{Z}[i]$ the set $\{a+ib \ : \ a,b \in \mathbb{Z}\} \subset \mathbb{C}$. Fix $p,q \in (0,1)$. Using the natural symmetry properties of the model, we can suppose that, without loss of generality, $p \geq q \geq \frac{1}{2}$ and $p \neq \frac{1}{2}$. We set $z_{p,q} = 2q-1+i(2p-1)$. We denote by $\theta_{p,q}$ the argument of $z_{p,q}$ in $[0, 2\pi[$. Then we define, for $\varepsilon > 0$, the following discrete cone

$$C_{p,q}^{\varepsilon} = \left\{z \in \mathbb{Z}[i] \ : \ \langle \frac{z}{|z|}, e^{i\theta_{p,q}}\rangle_{\mathbb{R}^2} \leq \varepsilon\right\}.$$

Note that $\tan\left(\theta_{p,q} + \frac{\pi}{2}\right) = \frac{2q-1}{1-2p}$. Therefore, $C_{p,q}^{\varepsilon}$ is centered around the line of equation $y = \frac{2q-1}{1-2p}x$.

Proposition 3.3. *Let $(p,q) \neq (\frac{1}{2}, \frac{1}{2})$. Then we have, for all $h \in \mathbb{Z}$ and $\varepsilon > 0$,*

$$\mathbb{P}(\exists R \in \mathbb{R}_+^* \ : \ L_h \subset D_R \cup C_{p,q}^{\varepsilon}) = 1.$$

It follows directly that we have, for all $z \in \mathbb{Z}^2$ and $\varepsilon > 0$,

$$\mathbb{P}(\exists R \in \mathbb{R}_+^* \ : \ \Gamma^z \subset D_R \cup C_{p,q}^{\varepsilon}) = 1.$$

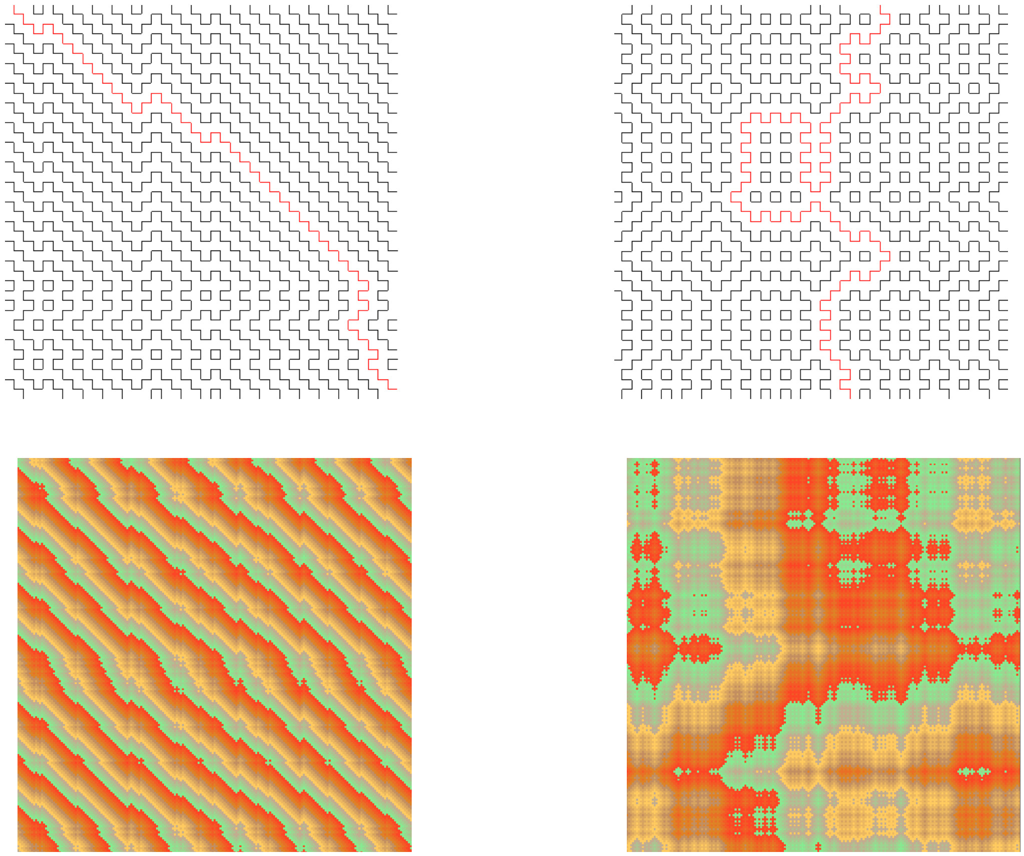

FIGURE 3.6. Simulations on the left (resp. right) have parameters $p = q = 0.9$ (resp. $p = 0.5$ and $q = 0.6$). Faces are colored acccording to their heights, with colors that repeat periodically.

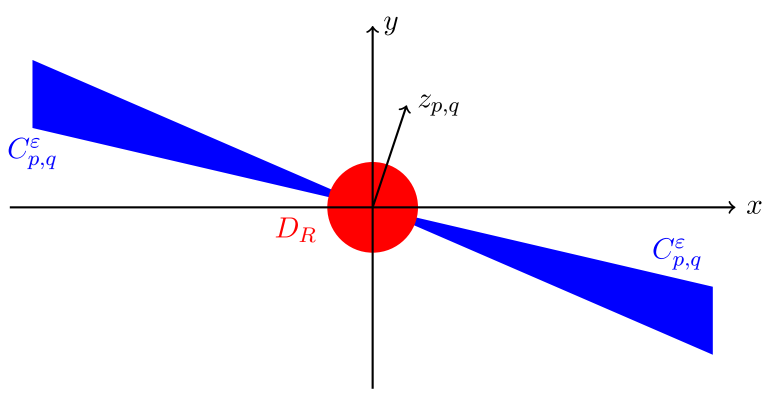

FIGURE 3.7. Representation of the set described in Proposition 3.3. The level set L_h is included in the colored area for some value of $R \in \mathbb{R}_+^*$ depending on h.

Proof: Let $\varepsilon, \varepsilon_0 > 0$ be such that $0 < \varepsilon_0 < |z_{p,q}|\varepsilon$. Lemma 3.1 implies that almost surely, there exists $R_0 \in \mathbb{R}_+^*$ such that, for all $\theta \in [0, 2\pi[$ and $r \geq R_0$ satisfying $(r\cos(\theta), r\sin(\theta)) \in \mathbb{Z}^2$,

$$\left| \frac{X_{r\cos(\theta)} + Y_{r\sin(\theta)}}{r} - \langle z_{p,q}, e^{i\theta} \rangle_{\mathbb{R}^2} \right| \leq \varepsilon_0. \tag{3.3}$$

We set $R = \max\left(R_0, \frac{2(|h|+1)}{|z_{p,q}|\varepsilon - \varepsilon_0}\right)$, and let $(n,m) \in \mathbb{Z}^2 \setminus D_R$ be such that $(n,m) \notin C^{\varepsilon}_{p,q}$. There exist $r \in \mathbb{R}^*_+$ and $\theta \in [0, 2\pi[$ such that $(n,m) = (r\cos(\theta), r\sin(\theta))$, with $r > R$ and $\left|\langle e^{i\theta_{p,q}}, e^{i\theta}\rangle_{\mathbb{R}^2}\right| > \varepsilon$. Then we have $\left|\langle z_{p,q}, e^{i\theta}\rangle_{\mathbb{R}^2}\right| > |z_{p,q}|\varepsilon$, and

$$\begin{aligned}\left|H\left(n+\frac{1}{2}, m+\frac{1}{2}\right)\right| &\geq \frac{\left|X_{r\cos(\theta)} + Y_{r\sin(\theta)}\right|}{2} - 1 \\ &\geq \frac{r}{2}\left|\left|\frac{\left|X_{r\cos(\theta)} + Y_{r\sin(\theta)}\right|}{r} - \langle z_{p,q}, e^{i\theta}\rangle_{\mathbb{R}^2}\right| - \left|\langle z_{p,q}, e^{i\theta}\rangle_{\mathbb{R}^2}\right|\right| - 1.\end{aligned}$$

By (3.3) and the fact that $\left|\langle z_{p,q}, e^{i\theta}\rangle_{\mathbb{R}^2}\right| > |z_{p,q}|\varepsilon > \varepsilon_0$, we deduce that

$$\begin{aligned}\left|H\left(n+\frac{1}{2}, m+\frac{1}{2}\right)\right| &\geq \frac{r}{2}\left(\left|\langle z_{p,q}, e^{i\theta}\rangle_{\mathbb{R}^2}\right| - \varepsilon_0\right) - 1 \\ &\geq \frac{r}{2}\left(|z_{p,q}|\varepsilon - \varepsilon_0\right) - 1.\end{aligned}$$

Finally, the choice of R implies that

$$\left|H\left(n+\frac{1}{2}, m+\frac{1}{2}\right)\right| > |h|,$$

so that

$$L_h \subset D_R \cup C^{\varepsilon}_{p,q}.$$

□

Note that Proposition 3.3 implies that for all $z \in \mathbb{Z}^2$, if the event $\{\Gamma^z \text{ is simple}\}$ has positive probability, then conditionally on this event, we have

$$\lim_{n\to+\infty} \frac{\Gamma^+_n}{|\Gamma^+_n|} \in \{e^{i\tilde{\theta}_{p,q}}, e^{-i\tilde{\theta}_{p,q}}\} \text{ a.s.} \quad \text{and} \quad \lim_{n\to+\infty} \frac{\Gamma^-_n}{|\Gamma^-_n|} \in \{e^{i\tilde{\theta}_{p,q}}, e^{-i\tilde{\theta}_{p,q}}\} \text{ a.s.,} \tag{3.4}$$

where $\tilde{\theta}_{p,q} = \theta_{p,q} + \frac{\pi}{2}$ (modulo 2π). Using one more time the height function, and the following easy lemma, we can have a more precise result in Lemma 3.5.

Lemma 3.4. *Let $(n,m), (n',m') \in \mathbb{Z}^2_e$ be such that $H\left(n+\frac{1}{2}, m+\frac{1}{2}\right) = H\left(n'+\frac{1}{2}, m'+\frac{1}{2}\right)$. We have:*

- *if $n = n'$, then $Y_m = Y_{m'}$,*
- *if $m = m'$, then $X_n = X_{n'}$.*

Proof: Suppose that $n = n'$. Since X_n and Y_m are integers, we have

$$|Y_m - Y_{m'}| \leq 1.$$

Furthermore, since m and m' have the same parity, we have

$$|Y_m - Y_{m'}| \in 2\mathbb{Z},$$

and so finally $Y_m = Y_{m'}$. The case $m = m'$ is treated in the same way. □

Lemma 3.5. *Suppose that $p \geq q \geq \frac{1}{2}$ and $p \neq \frac{1}{2}$. Let Γ be a bi-infinite simple path, and let $z \in \Gamma$. We set $\Gamma^+_z = (x^+_i, y^+_i)_{i\in\mathbb{Z}_+}$ and $\Gamma^-_z = (x^-_i, y^-_i)_{i\in\mathbb{Z}_+}$.*

$$\begin{cases}\lim\limits_{i\to+\infty} x^+_i = \limsup\limits_{i\to+\infty} y^-_i = +\infty \\ \lim\limits_{i\to+\infty} x^-_i = \liminf\limits_{i\to+\infty} y^+_i = -\infty\end{cases} \quad or \quad \begin{cases}\lim\limits_{i\to+\infty} x^+_i = \liminf\limits_{i\to+\infty} y^-_i = -\infty \\ \lim\limits_{i\to+\infty} x^-_i = \limsup\limits_{i\to+\infty} y^+_i = +\infty\end{cases}. \tag{3.5}$$

Consequently, Γ intersects at least one time any vertical line.

This lemma and (3.4) imply that, for all $(p,q)\neq(\frac{1}{2},\frac{1}{2})$ and $z\in\mathbb{Z}^2$, if the event $\{\Gamma^z$ is simple$\}$ has positive probability, then conditionally on this event, we have

$$\lim_{n\to+\infty}\frac{\Gamma_n^+}{|\Gamma_n^+|}=-\lim_{n\to+\infty}\frac{\Gamma_n^-}{|\Gamma_n^-|}\text{ a.s.}\tag{3.6}$$

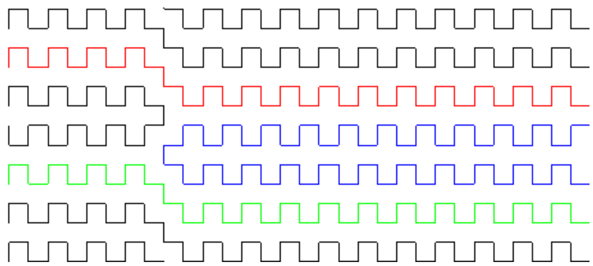

FIGURE 3.8. Lemma 3.5 excludes the possibility of having a bi-infinite simple path whose forward and backward paths go to the same direction (like the blue one here), almost surely. This implies that each bi-infinite simple path has at most two neighbors, one on each side. This could have been proved using Theorem 2 of Burton and Keane (1991). Proposition 3.6 gives a more precise result: in fact, there are exactly two neighbors, one on each side.

Proof: Assume without loss of generality that the level of Γ is 0. Proposition 3.3 ensures that

$$\text{both limits }\lim_{i\to+\infty}x_i^+\text{ and }\lim_{i\to+\infty}x_i^-\text{ exist and are in }\{-\infty,+\infty\}.$$

Now we prove that $\lim_{i\to+\infty}x_i^-=-\lim_{i\to+\infty}x_i^+$ by contradiction. Suppose without loss of generality that

$$\lim_{i\to+\infty}x_i^+=\lim_{i\to+\infty}x_i^-=+\infty.$$

The random walk $(X_i)_{i\in\mathbb{Z}}$ is either recurrent or transient. In both cases, we have $\limsup_{i\to+\infty}X_i=+\infty$. So since $\lim_{i\to+\infty}x_i^+=+\infty$, we have $\limsup_{i\to+\infty}X_{x_i^+}=+\infty$. For all $i\in\mathbb{Z}_+$, we have

$$H\left(x_i^++\frac{1}{2},y_i^++\frac{1}{2}\right)\in\{0,1\},$$

therefore we have $\liminf_{i\to+\infty}y_i^+=-\infty$. Similarly, we have $\liminf_{i\to+\infty}y_i^-=-\infty$.

Now we want to construct a sequence of distinct integers $(m_i)_{i\in\mathbb{Z}_+}$ such that, for all $i\in\mathbb{Z}_+$, $Y_{m_i}=Y_{m_0}$, which will be in contradiction with the transience of the random walk $(Y_i)_{i\in\mathbb{Z}_+}$. Since $\lim_{i\to+\infty}x_i^+=\lim_{i\to+\infty}x_i^-=+\infty$, then we can find $n_0,m_0,m_1\in\mathbb{Z}$ such that

- $(n_0,m_0)\in\Gamma_z^+\cap\mathbb{Z}_e^2$ and $(n_0,m_1)\in\Gamma_z^-\cap\mathbb{Z}_e^2$,
- both $\Gamma^+_{(n_0,m_0)}$ and $\Gamma^-_{(n_0,m_1)}$ do not intersect the vertical line $x=x_0^+$,

see Figure 3.9. Without loss of generality, we suppose that $m_1<m_0$. Since $(n_0,m_0),(n_0,m_1)\in\mathbb{Z}_e^2\cap\Gamma^z$, then the faces $(n_0+\frac{1}{2},m_0+\frac{1}{2})$ and $(n_0+\frac{1}{2},m_1+\frac{1}{2})$ are black and have height 0. Therefore, by Lemma 3.4, we have

$$Y_{m_0}=Y_{m_1}.$$

Now, since $\liminf_{i\to+\infty}y_i^+=-\infty$, then the path $\Gamma^+_{(n_0,m_0)}$ intersects the horizontal line $y=m_1$, so there exists $n_1>n_0$ such that $(n_1,m_1)\in\Gamma^+_{(n_0,m_0)}\cap\mathbb{Z}_e^2$. Since $\Gamma^-_{(n_0,m_1)}$ cannot intersect neither Γ nor the vertical line $x=z_1$, then there exists $m_2\in\mathbb{Z}$ such that $(n_1,m_2)\in\Gamma^-_{(n_0,m_1)}$ and $m_2<m_1$, see

Figure 3.9. By iterating this procedure, we can construct two sequences $(n_i)_{i\in\mathbb{Z}_+}$ and $(m_i)_{i\in\mathbb{Z}_+}$ such that $(m_i)_{i\in\mathbb{Z}_+}$ is decreasing, and for all $i \in \mathbb{Z}_+$:

$$\begin{aligned} H\left(n_i+\frac{1}{2}, m_{2i}+\frac{1}{2}\right) &= H\left(n_i+\frac{1}{2}, m_{2i+1}+\frac{1}{2}\right) \\ = H\left(n_{i+1}+\frac{1}{2}, m_{2i+1}+\frac{1}{2}\right) &= H\left(n_{i+1}+\frac{1}{2}, m_{2i+2}+\frac{1}{2}\right). \end{aligned}$$

By Lemma 3.4, this implies that

$$\forall i \in \mathbb{Z}_+, \quad Y_{m_i} = Y_{m_0}.$$

Since the random walk $(Y_i)_{i\in\mathbb{Z}_+}$ is transient, it is a contradiction. □

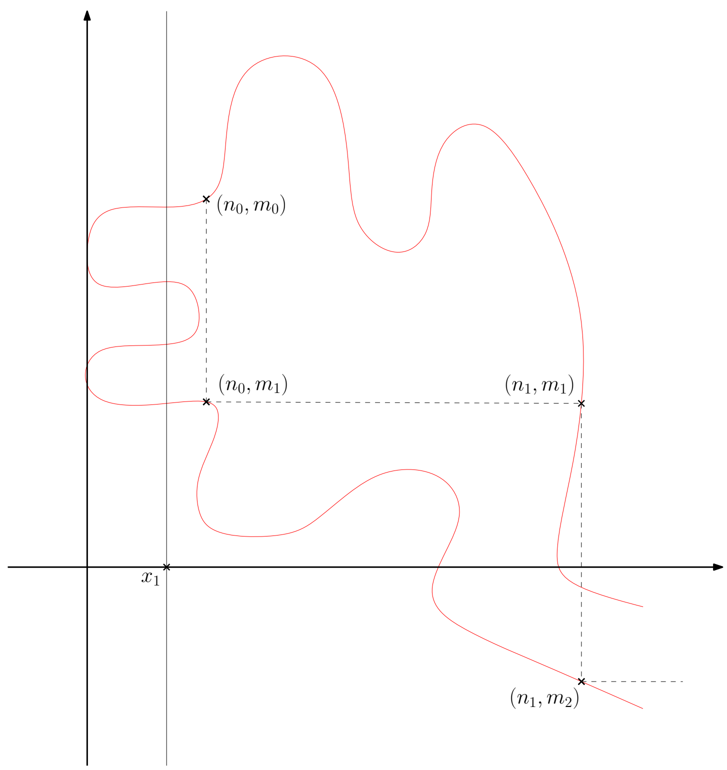

FIGURE 3.9. Illustration of the construction of the sequence $(m_i)_{i\in\mathbb{Z}_+}$. The red curve is a schematic representation of Γ.

Now we can prove the last part of our main result:

Proposition 3.6. *Let $p \geq q \geq \frac{1}{2}$, $p \neq \frac{1}{2}$ and $h \in \mathbb{Z}$. Then there almost surely exists a unique bi-infinite simple path Γ of level h. Moreover, if $z \in \Gamma \cap \mathbb{Z}_e^2$, then we have*

$$\lim_{n\to+\infty} \frac{\Gamma_n^+}{|\Gamma_n^+|} = e^{i\tilde{\theta}_{p,q}} \quad \textit{and} \quad \lim_{n\to+\infty} \frac{\Gamma_n^-}{|\Gamma_n^-|} = e^{-i\tilde{\theta}_{p,q}} \quad \textit{a.s.},$$

where $\Gamma_0^+ = z$.

Proof: Existence. Let l be the vertical line $\{\frac{1}{2}\}\times\mathbb{R}$. For all $n \in \mathbb{Z}$, we denote by F_n the face of center $(\frac{1}{2}, n+\frac{1}{2})$, and by e_n the edge adjacent to F_n and F_{n+1}. Note that each open edge intersecting l is adjacent to a face of this sequence. Since we have

$$\lim_{i\to+\infty} Y_i = -\lim_{i\to-\infty} Y_i = +\infty,$$

then identifying a face to its center, we have

$$\lim_{n\to+\infty} H(F_n) = -\lim_{n\to-\infty} H(F_n) = +\infty. \tag{3.7}$$

This implies that the line l has only a finite number of intersections with level h paths.

Now let us prove that the line l has an odd number of intersections with level h paths. The height function has a *discrete continuity property*, that is

$$\forall x, y \in \mathbb{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right), \text{ if } ||x-y||_1 \leq 1 \text{ then } |H(x) - H(y)| \leq 1.$$

In addition with (3.7), it implies that the set

$$I_h = \{n \in \mathbb{Z} \ : \ (H(F_n), H(F_{n+1})) \in \{(h, h+1), (h+1, h)\}\}.$$

is non-empty, finite and its cardinal is an odd number. Precisely, if we denote by I_h^+ the set $\{n \in \mathbb{Z} \ : \ (H(F_n), H(F_{n+1})) = (h, h+1)\}$ (resp. by I_h^- the set $\{n \in \mathbb{Z} \ : \ (H(F_n), H(F_{n+1})) = (h+1, h)\}$), then we have $\operatorname{card}(I_h^+) = \operatorname{card}(I_h^-) + 1$, so that $\operatorname{card}(I_h) = 2\operatorname{card}(I_h^-) + 1$. The edges $(e_n)_{n\in I_h}$ are exactly the edges that intersect l and belong to a level h path. Therefore, there is an odd number of intersections between level h paths and l.

Now to conclude, note that finite cycles have an even number of intersections with line l, so there exists at least one level h bi-infinite simple path that intersects l.

Unicity.

Let us prove the unicity by contradiction. Suppose that there exist two level h bi-infinite simple paths Γ^{z_0} and Γ^{z_1}. As they are both in a same "full cone" $D_R + C_{p,q}^{\varepsilon}$ a.s., at least two of the four half-paths $\Gamma_{z_0}^+$, $\Gamma_{z_0}^-$, $\Gamma_{z_1}^+$ and $\Gamma_{z_1}^-$ are in the same "half-cone". Therefore, we can use these two half-paths to construct a sequence $(m_i)_{i\in\mathbb{Z}_+}$ of distinct integers, as in the proof of Lemma 3.5, such that

$$\forall i \in \mathbb{Z}_+, \quad Y_{m_i} = Y_{m_0},$$

see Figure 3.10 for an illustration. Since the random walk $(Y_i)_{i\in\mathbb{Z}_+}$ is transient, it is a contradiction.

Direction.

By the two preceding points, we can consider Γ, the unique level h bi-infinite simple path. We saw that I_h is finite and Γ intersects l, so we can define $M_h = \max(I_h)$. By the discrete continuity property and the fact that $\lim_{n\to+\infty} H(F_n) = +\infty$, we have $M_h \in I_h^+$. Now let us define $N_h = \max\{n \in I_h \ : \ e_n \in \Gamma\}$. Between the edges e_{M_h} and e_{N_h}, all the edges of I_h belong to finite cycles that are above Γ, since they cannot cross it. Since each finite cycle that intersects l is contributing to I_h with the same number of couples $(h, h+1)$ and $(h+1, h)$, we have also $N_h \in I_h^+$.

Let us consider the smallest $n > N_h$ such that e_n belongs to a bi-infinite simple path. All the edges between e_{N_h} and e_n belong to finite cycles, therefore we have $H(F_{N_h+1}) = H(F_n) = h+1$. Moreover, by the discrete continuity property and the unicity of the level h bi-infinite simple path, we have $H(F_{n+1}) = h+2$. More generally, there exist infinitely many bi-infinite simple paths above and below Γ. Moreover, the upper neighbor of Γ has level $h+1$, and its lower neighbor has level $h-1$.

Finally, note that the choice of orientation we made for Γ (see Subsection 2.2) implies that the faces with height $h+1$ are on the right side of the path. This implies that, for any $z \in \Gamma \cap \mathbb{Z}_e^2$, Γ_z^+ ends in the half-plane $\{x \leq 0\}$, which proves the last point. □

We finish by discussing about the distribution of the sequences ξ and η. We assumed these sequences to be i.i.d., but in fact we could take ξ and η as two independent ergodic stationary Markov chains on $\{-1, 1\}$, the first one having $(1-p, p)$ as stationary distribution, the second one having $(1-q, q)$. Birkhoff's ergodic theorem would play the role of the law of large numbers in the proofs.

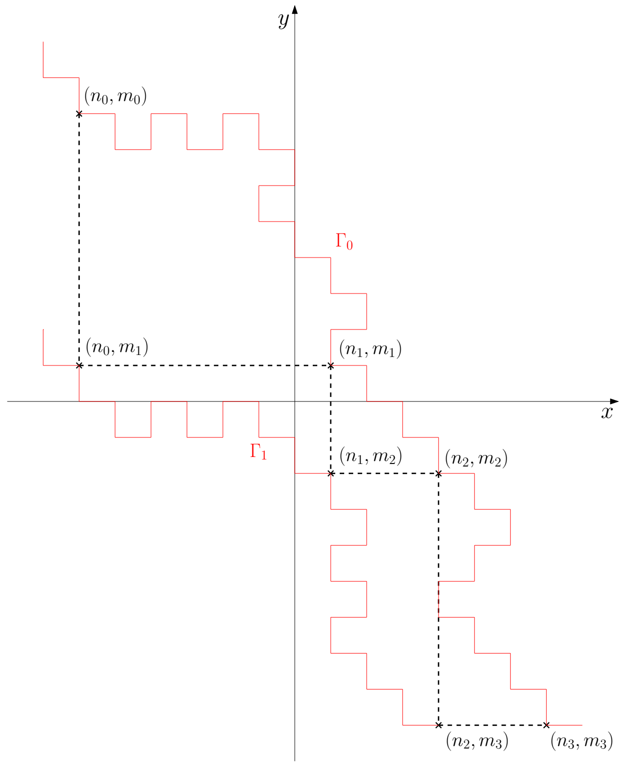

FIGURE 3.10. Construction of the sequence $(m_i)_{i \in \mathbb{Z}_+}$.

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