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Extremes of the 2d scale-inhomogeneous discrete Gaussian free field: Sub-leading order and exponential tails

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Abstract. This is the first of a three paper series in which we present a comprehensive study of the extreme value theory of the scale-inhomogeneous discrete Gaussian free field. Arguin and Ouimet (2016) computed the first order of the maximum. In this first paper we establish tail estimates for the maximum value, which allow to deduce the log-correction to the order of the maximum and tightness of the centred maximum. Our proofs are based on the second moment method and Gaussian comparison techniques.

In recent years, so-called log-correlated (Gaussian) processes have received considerable attention, see e.g. Arguin et al. (2017, 2019); Biskup and Louidor (2018); Bovier and Hartung (2017); Ding et al. (2017a); Fyodorov et al. (2012); Mallein (2015a). One of the reasons for this is that their correlation structure becomes relevant for the properties of the extremes of the processes. Some prominent examples that fall into this class are branching Brownian motion (BBM), the two-dimensional discrete Gaussian free field (2d DGFF), local maxima of the randomised Riemann zeta function on the critical line and cover times of Brownian motion on the torus. The 2d DGFF is one of the well understood non-hierarchical log-correlated fields (see Biskup and Louidor (2016, 2018); Bolthausen et al. (2001); Bramson et al. (2016)). For simplicity, consider the 2d DGFF on a square lattice box of side length N. It turns out that the maximum can be written as a first order term which is proportional to the logarithm of the volume of the box, a second order correction which is proportional to the logarithm of the first order and stochastically bounded fluctuations. If one considers an uncorrelated Gaussian field on the same box with identical variances, a simple computation shows that the first order of the maximum coincides with the one of the DGFF, whereas the constant in front of the second order correction differs. Arguin and Zindy (2015) introduced the scale-inhomogeneous 2d DGFF with two scales. This was generalized in Arguin and Ouimet (2016) to finitely many scales as we consider here, the analogue model of variable speed BBM (Mallein,

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2015a), which allows to consider different variance profiles. Arguin and Ouimet (2016) determine the first order of the maximum. In this paper we continue the study of the maximum, find tail estimates on the maximum value which allow us to deduce the second order correction and tightness of the centred maximum. In the other two papers in this series, we prove, in the regime of weak correlations, convergence of the centred maximum (Fels and Hartung, 2021a) and convergence of the extremal process (Fels and Hartung, 2021b).

1. Introduction

1.1. *The 2d discrete Gaussian free field.* Let $V \subset \mathbb{Z}^2$ be a finite set and define its boundary as

$$\partial V := \{v \in V | \exists w \in \mathbb{Z}^2 \setminus V \text{ such that } \|v-w\|_2 = 1\} \tag{1.1}$$

where $\|\cdot\|_2$ denotes the Euclidean distance on $\mathbb{Z}^2$. We call $V^o := V \setminus \partial V$ the interior of V. Moreover, for points $v, w \in V$ we write $v \sim w$, if and only if $\|v-w\|_2 = 1$. Let $\mathbb{P}_v$ be the law of a SRW $\{W_k\}_{k\in\mathbb{N}}$ starting at $v \in \mathbb{Z}^2$. The normalised Green kernel is given by

$$G_V(v,w) := \frac{\pi}{2}\mathbb{E}_v\left[\sum_{i=0}^{\tau_{\partial V}} \mathbb{1}_{\{W_i = w\}}\right], \text{ for } v, w \in V. \tag{1.2}$$

Here, $\tau_{\partial V} := \inf\{k \geq 0 : W_k \in \partial V\}$ is the first hitting time of the boundary ∂V. In what follows we consider for $N \in \mathbb{N}$, $V_N := ([0,N) \cap \mathbb{Z})^2$, boxes of side length N. For $\delta \in (0,1)$, we set $V_N^\delta := (\delta N, (1-\delta)N)^2 \cap \mathbb{Z}^2$. By Daviaud (2006, Lemma 2.1), we have for all N and $u, v \in V_N^\delta$,

$$G_{V_N}(v,w) = \log N - \log\left(\|v-w\|_2 \vee 1\right) + O(1). \tag{1.3}$$

Definition 1.1. The 2d discrete Gaussian free field (DGFF) on V_N, $\phi^N := \{\phi_v^N\}_{v\in V_N}$, is a centred Gaussian field with covariance matrix G_{V_N} and entries $G_{V_N}(v,w) = \mathbb{E}[\phi_v^N \phi_w^N]$, for $v, w \in V_N$.

From Definition 1.1 it follows that $\phi_v^N = 0$ for $v \in \partial V_N$. For convenience, we set $\phi_v^N \equiv 0$ for $v \in \mathbb{Z}^2 \setminus V_N$.

1.2. *The 2d scale-inhomogeneous discrete Gaussian free field.*

Definition 1.2. (2d scale-inhomogeneous discrete Gaussian free field). Let $\phi^N = \{\phi_v^N\}_{v\in V_N}$ be a 2d DGFF on V_N. For $v = (v_1, v_2) \in V_N$, we consider

$$[v]_\lambda \equiv [v]_\lambda^N := \left(\left[v_1 - \frac{1}{2}N^{1-\lambda}, v_1 + \frac{1}{2}N^{1-\lambda}\right] \times \left[v_2 - \frac{1}{2}N^{1-\lambda}, v_2 + \frac{1}{2}N^{1-\lambda}\right]\right) \cap V_N \tag{1.4}$$

a box centred at v of side length $N^{1-\lambda}$ that is cut-off by ∂V_N, and set $[v]_0^N := V_N$, $[v]_1^N := \{v\}$ and denote by $[v]_\lambda^o$ the interior of $[v]_\lambda$. For fixed N and $A \subset \mathbb{Z}^2$, let

$$\mathcal{F}_{\partial A} \equiv \mathcal{F}_{\partial A \cup A^c} := \sigma\left(\phi_v^N : v \in \partial A \cup A^c\right), \tag{1.5}$$

be the σ-algebra generated by the random variables, ϕ_v^N, for which their index $v \notin A^o$ is outside the interior A^o. In particular, $\mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c} := \sigma\left(\{\phi_v^N, v \notin [v]_\lambda^o\}\right)$ is the $\sigma-$algebra generated by the random variables with index not in $[v]_\lambda^o$. Note that for such fixed N and v, $\left(\mathcal{F}_{\partial[v]_\lambda^N}\right)_{\lambda\in[0,1]}$ is a filtration. We define $\phi_v^N(\lambda)$ by conditioning on the DGFF with indices in the complement of the interior of $[v]_\lambda^N$, i.e.

$$\phi_v^N(\lambda) = \mathbb{E}\left[\phi_v^N | \mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c}\right], \quad \lambda \in [0,1]. \tag{1.6}$$

We denote by $\nabla\phi_v^N(\lambda)$ the derivative $\partial_\lambda \phi_v^N(\lambda)$ of the DGFF at vertex v and scale λ. Further, let $s \mapsto \sigma(s)$ be a non-negative function such that $\mathcal{I}_{\sigma^2}(\lambda) := \int_0^\lambda \sigma^2(x)\mathrm{d}x$ is a non-decreasing function on

$[0,1]$ with $\mathcal{I}_{\sigma^2}(0)=1$ and $\mathcal{I}_{\sigma^2}(1)=1$. Then the 2d scale-inhomogeneous DGFF on V_N is a centred Gaussian field $\psi^N := \{\psi_v^N\}_{v\in V_N}$ defined as

$$\psi_v^N := \int_0^1 \sigma(s)\nabla\phi_v^N(s)\mathrm{d}s. \tag{1.7}$$

In this paper, we consider the case when σ is a right-continuous step function taking $M \in \mathbb{N}$ values. Thus, there are variance parameters $(\sigma_1,\dots,\sigma_M)\in(0,\infty)^M$ and scale parameters $(\lambda_1,\dots,\lambda_M)\in(0,1]^M$ with $0 =: \lambda_0 < \lambda_1 \dots < \lambda_M := 1$, such that

$$\sigma(s) = \sum_{i=1}^M \sigma_i \mathbb{1}_{[\lambda_{i-1},\lambda_i)}(s),\ \ s\in[0,1]. \tag{1.8}$$

In this case, the scale-inhomogeneous 2d DGFF or 2d $(\sigma,\lambda)-$DGFF in (1.7) takes the form

$$\psi_v^N = \sum_{i=1}^M \sigma_i(\phi_v^N(\lambda_i)-\phi_v^N(\lambda_{i-1})). \tag{1.9}$$

The scale-inhomogeneous 2d DGFF with two scales ($M=2$) first appeared in Arguin and Zindy (2015), the case of finitely many scales as in (1.9) was introduced in Arguin and Ouimet (2016, Defintion 1.1). A natural continuous analogue, the continuous scale-inhomogeneous Gaussian Free Field, that takes the form as in (1.7) in which $\phi_v^N(s)$ is being replaced by the so called circle-averages of the continuous Gaussian Free Field was introduced (but not considered any further) in Arguin and Ouimet (2016, (1.6) and following). Similarly to (1.6), we set for $v\in V_N$ and $\lambda\in[0,1]$,

$$\psi_v^N(\lambda) := \mathbb{E}\left[\psi_v^N \middle| \mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c}\right]. \tag{1.10}$$

The following Lemma provides an estimate for the covariances of $\{\psi_v^N\}_{v\in V_N}$. The proof is given in the Appendix B .

Lemma 1.3. *Let $\delta\in(0,1/2)$ and $v,w\in V_N^\delta$. Then,*

$$\mathbb{E}\left[\psi_v^N\psi_w^N\right] = \log N \mathcal{I}_{\sigma^2}\left(\frac{\log N - \log\left(\|v-w\|_2 \vee 1\right)}{\log N}\vee 0\right) + O(\sqrt{\log(N)}). \tag{1.11}$$

Remark 1.4. (1.11) motivates the following tree intuition: Assume that $\log_2 N\in\mathbb{N}$. Let $\{X_v\}_{v\in\mathcal{T}_N}$ be a time-inhomogeneous BRW with centred Gaussian displacements of variance $\log N\mathcal{I}_{\sigma^2}((k-1)/\log_2 N, k/\log_2 N)$ at generation $1\le k\le \log_2 N$, indexed by the leaves of a 4-ary tree $\mathcal{T}_N$ of depth $\log_2 N$. For $v,w\in\mathcal{T}_N$ we denote by $q_N(v,w)$ the tree distance, that is $\log_2 N$ minus the generation of the most recent common ancestor of v,w. Then $\mathbb{E}\left[X_vX_w\right] = \log N\mathcal{I}_{\sigma^2}\left(\frac{\log_2 N - q_N(v,w)}{\log_2 N}\right)$, which is exactly of the form in (1.11) with the logarithm of the Euclidean distance replaced with the tree distance and without the additional $+O(\sqrt{\log(N)})$ term. As both this BRW, whose maximum is well understood (see e.g. Mallein (2015a)), and the scale-inhomogeneous DGFF are Gaussian, this motivates the use of Gaussian comparison (see Corollary A.3). However, the term $O(\sqrt{\log(N)})$ in (1.11) prevents from a precise comparison.

2. Main results

The main results of this paper are tail estimates for $\psi_N^* := \max_{v\in V_N}\psi_v^N$, the maximum of the scale-inhomogeneous 2d DGFF, in the case of finitely many scales. As simple consequences, we deduce the correct second order correction and tightness of the centred maximum. We start with

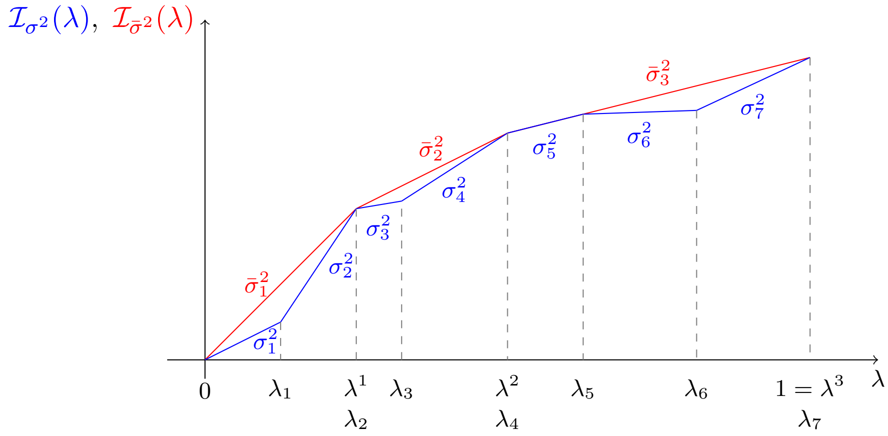

FIGURE 2.1. An example of variance and effective variance.

some notation. Let $\hat{\mathcal{I}}_{\sigma^2}(s)$ be the concave hull of $\mathcal{I}_{\sigma^2}(s)$. There exists a unique non-increasing, right-continuous step function $s \to \bar\sigma(s)$, which we call "effective variance", such that

$$\hat{\mathcal{I}}_{\sigma^2}(s) = \int_0^s \bar\sigma^2(r)\mathrm{d}r =: \mathcal{I}_{\bar\sigma^2}(s) \quad \text{for all } s \in [0,1]. \tag{2.1}$$

The points at which $\bar\sigma$ jumps we call "effective scale" parameters

$$0 =: \lambda^0 < \lambda^1 < \ldots < \lambda^m := 1, \tag{2.2}$$

where $m \le M$. Note that these are indexed with superscripts, whereas the actual scale parameters, $\lambda_1, \ldots, \lambda_M$, are indexed b subscripts. We set as "effective variance" parameters $\bar\sigma_l := \bar\sigma(\lambda^{l-1})$, for $1 \le l \le m$. For any, possibly finite, sequence $\{x_i\}_{i\ge 0}$ of real numbers we denote by $\Delta x_i = x_i - x_{i-1}$ the discrete increment. It turns out that the concave hull of $\mathcal{I}_{\sigma^2}$, denoted $\hat{\mathcal{I}}_{\sigma^2}$, gives the desired control for the first order of the maximum. Arguin and Ouimet (2016, Theorem 1.2) determined the correct first order behaviour, i.e. they showed that in probability,

$$\lim_{N\to\infty} \frac{\psi_N^*}{2\log(N)} = \mathcal{I}_{\bar\sigma}(1) = \sum_{i=1}^m \bar\sigma_i \Delta\lambda^i. \tag{2.3}$$

In the following, the goal is to prove a second order correction and tightness of the maximum around its mean. Let $\pi_j \in [0, M]$ be the unique index for $1 \le j \le m$ such that $\lambda^j = \lambda_{\pi_j}$. Write $t^j = \lambda^j \frac{\log N}{\log 2}$ as well as $t_j = \lambda_j \frac{\log N}{\log 2}$ and set

$$m_N := \sum_{j=1}^m \left[2\log(2)\bar\sigma_j \Delta t^j - \frac{w_j \bar\sigma_j \log\left(\Delta t^j\right)}{4} \right], \tag{2.4}$$

where

$$w_j = \begin{cases} 3, & \mathcal{I}_{\bar\sigma^2}|_{(\lambda^{j-1},\lambda^j]} \equiv \mathcal{I}_{\sigma^2}|_{(\lambda^{j-1},\lambda^j]} \\ 1, & \text{else} \end{cases} \tag{2.5}$$

The following theorem establishes tail estimates of the maximum centred by m_N.

Theorem 2.1. *Let $N \in \mathbb{N}$ and $\{\psi_v^N\}_{v\in V_N}$ be a 2d (σ,λ)-DGFF on V_N with $M \in \mathbb{N}$ scales. Assume that on each interval $(\lambda^{i-1}, \lambda^i)$ and $i = 1, \ldots, m$, we have either $\mathcal{I}_{\sigma^2} \equiv \mathcal{I}_{\bar\sigma^2}$ or $\mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar\sigma^2}$. Then, there exist constants $C, c > 0$ such that for any $x \in [0, \sqrt{\log N}]$,*

$$C^{-1}\left(1 + x\mathbb{1}_{\sigma_1=\bar\sigma_1}\right) e^{-x\frac{2}{\bar\sigma_1}} \le \mathbb{P}\left(\max_{v\in V_N} \psi_v^N \ge m_N + x\right) \le C(1 + x\mathbb{1}_{\sigma_1=\bar\sigma_1})e^{-x\frac{2}{\bar\sigma_1}}, \tag{2.6}$$

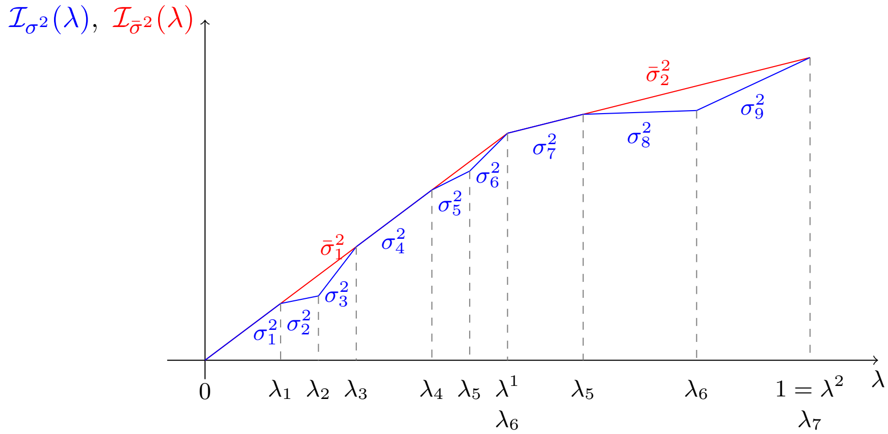

FIGURE 2.2. An example of variance and effective variance that is not covered by the assumptions in Theorem 2.1.

and for any $0 \leq \lambda \leq (\log\log N)^{2/3}$,

$$\mathbb{P}\left(\max_{v\in V_N} \psi_v^N \leq m_N - \lambda\right) \leq Ce^{-c\lambda}. \tag{2.7}$$

Remark 2.2. Note that the result for the right-tail in (2.6) is precise up to a multiplicative constant.

Remark 2.3. By Borell's inequality (see Theorem A.1) and Arguin and Ouimet (2016, Lemma A.3), there is a constant $c_\sigma \in (0,\infty)$, depending only on the variance parameter σ, such that, for any $c > 0$ and $x > c\sqrt{\log N}$,

$$\mathbb{P}\left(|\psi_N^* - m_N| \geq x\right) \leq 2e^{-c_\sigma x^2/\log(N)}. \tag{2.8}$$

As a simple consequence of Theorem 2.1 and Remark 2.3, we obtain the following corollary.

Corollary 2.4. *Under the same assumptions of Theorem 2.1, the sequence of the centred maximum* $\{\psi_N^* - m_N\}_{N\geq 0}$ *is tight. In particular,*

$$\mathbb{E}\left[\psi_N^*\right] = m_N + O(1), \tag{2.9}$$

where the term $O(1)$ *is bounded by a constant which is uniform in* N*.*

An interesting fact is that the profile of the variance matters both for the leading term and the logarithmic correction. This phenomenon was first observed in the context of the GREM by Derrida and Gardner (1986); Bovier and Kurkova (2004a,b), and in the context of the time-inhomogeneous branching Brownian motion/branching random walk by Bovier and Hartung (2014, 2015), Fang and Zeitouni (2012a), Maillard and Zeitouni (2016) and Mallein (2015b).

Remark 2.5. In this remark, we give an example for which the assumption on the variance profile in Theorem 2.1 is not satisfied.

We take variances and effective variances as in Figure 2. In this example there are two effective variances $\bar\sigma_1 > \bar\sigma_2 > 0$ with effective scales $0 < \lambda^1 < \lambda^2 = 1$. On both intervals $[0,\lambda^1]$ and $[\lambda^1,\lambda^2]$ the assumptions of Theorem 2.1 are violated: For instance, on both $[0,\lambda_1] \subset [0,\lambda^1]$ and $[\lambda_3,\lambda_4] \subset [0,\lambda^1]$, $\mathcal{I}_{\sigma^2} \equiv \mathcal{I}_{\bar\sigma^2}$, whereas on both intervals $(\lambda_1,\lambda_3) \subset [0,\lambda^1]$ and $(\lambda_4,\lambda_6) \subset [0,\lambda^1]$, $\mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar\sigma^2}$. In particular, there exist $s,t \in (0,\lambda^1)$ such that both $\mathcal{I}_{\sigma^2}(s) = \mathcal{I}_{\bar\sigma^2}(s)$ and $\mathcal{I}_{\sigma^2}(t) < \mathcal{I}_{\bar\sigma^2}(t)$, which violates the assumption in Theorem 2.1.

In the next remark, we discuss and conjecture the correct centring m_N of the maximum in cases when Theorem 2.1 does not apply. The effective scale and variance parameters in this discussion are as defined in Section 1.

Remark 2.6. In case of finitely many scales without further assumptions, we expect that there are essentially two properties which determine the logarithmic correction: For each "effective interval" $(\lambda^{j-1}, \lambda^j)$, $1 \leq j \leq m$, one checks if there exists $\epsilon > 0$ such that $\mathcal{I}_{\sigma^2}(s) = \mathcal{I}_{\bar{\sigma}^2}(s)$ for any $s \in (\lambda^{j-1}, \lambda^{j-1}+\epsilon)$ or any $s \in (\lambda^j - \epsilon, \lambda^j)$. In the example, Figure 2, $\mathcal{I}_{\sigma^2}(s) = \mathcal{I}_{\bar{\sigma}^2}(s)$ for $s \in [0, \lambda_1]$ and for $s \in [\lambda_6, \lambda_7]$. For each of those intervals on which we have equality, $\mathcal{I}_{\sigma^2}(s) = \mathcal{I}_{\bar{\sigma}^2}(s)$, for all $s \in (\lambda^{j-1}, \lambda^{j-1}+\epsilon)$ or $s \in (\lambda^j - \epsilon, \lambda^j)$, an additional correction factor of $1/2$ needs to be added to the fixed $1/2$ correction which exists in any case as in (2.5). Namely in our example these are $w_1 = 1 + \mathbb{1}_{\sigma_1 = \bar{\sigma}_1} + \mathbb{1}_{\sigma_6 = \bar{\sigma}_1} = 2$ and $w_2 = 1 + \mathbb{1}_{\sigma_7 = \bar{\sigma}_2} + \mathbb{1}_{\sigma_9 = \bar{\sigma}_2} = 2$. In particular, only equality of $\mathcal{I}_{\sigma^2}(s) = \mathcal{I}_{\bar{\sigma}^2}(s)$ at the beginning or end on each effective scale interval, $[\lambda^{i-1}, \lambda^i]$, but not equality for some $s \in [\lambda^{i-1}+\epsilon, \lambda^i - \epsilon]$, as on the interval $[\lambda_3, \lambda_4]$ in our example, affects the logarithmic correction factor w_i.

The reason for this is that on the one-hand side our proof of the upper bound in Theorem 2.1 readily generalizes by Gaussian comparison (see also Corollary A.3) to the maximum of the time-inhomogeneous branching random walk, for which the result is known, see e.g. Mallein (2015a, Theorem 1.4). On the other hand, our approach by using Gaussian comparison to lower bound the maximum of the scale-inhomogeneous DGFF by the maximum of a MIBRW as we define it here applies also in the more general setting. However, one needs to adjust the "path localization" in the second moment computation in Lemma 4.6 and Lemma 4.7, which we refrain from doing here as it would complicate the notation even further. As such "paths" of the MIBRW, $\{S_v^N(t)\}_{0\leq t\leq n}$, are time-inhomogeneous random walks, the corresponding "localizations" should be those that appear in the analysis for a time-inhomogeneous branching random walk with the same scale and variance parameters. The localization restrictions are known to be ballot type events, hence adopting existing generalized ballot-type estimates such as Mallein (2015a, Lemma 3.9) should enable to generalize our proof.

The case of so-called "weak correlations", in which case we have $\mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar{\sigma}^2}$ on the interval $(0,1)$ with $s \mapsto \sigma(s)$ only assumed to be a non-negative function satisfying a regularity condition at 0 and 1, we treat in details in Fels and Hartung (2021a,b). In Fels and Hartung (2021a), we show that the correct order of the maximum of the weakly correlated scale-inhomogeneous DGFF on V_N is identical to the order of the maximum of N^2 independent centred Gaussian random variables with the same variances and moreover, that the centred maximum converges to a randomly shifted Gumbel. In Fels and Hartung (2021b), we prove convergence of the extremal process in this regime.

If one considers the case of strictly decreasing variance σ in (1.7), we expect the second order correction to be proportional to $\log^{1/3}(N)$ as observed in the analogue setting for variable-speed BBM (Fang and Zeitouni, 2012a).

2.1. *Overview of related results.* In the case when $\sigma \equiv 1$, the 2d scale-inhomogeneous DGFF simply is the 2d DGFF. The maximum and more generally the extremal process of the DGFF has been the subject of intense investigations. Let $\phi_N^* := \max_{v\in V_N} \phi_v^N$ be the maximum of the DGFF. Through the works of Bolthausen et al. (2001) as well as Bramson and Zeitouni (2012) one obtains,

$$\phi_N^* = 2\log N - \frac{3}{4}\log\log N + Y, \tag{2.10}$$

where Y is random variable of order $o(\log\log N)$ in probability. Bramson and Zeitouni further deduced that the centred maximum $\phi_N^* - \mathbb{E}[\phi_N^*]$ is tight as a sequence of real random variables. Convergence of the centred maximum was then shown by Bramson et al. (2016). Biskup and Louidor (2016, 2018) proved that the extremal process converges to a cluster Cox process.

Another closely related model is (variable-speed) branching Brownian motion (BBM). It can be considered as the analogue model to the scale-inhomogeneous DGFF in the context of BBM. It first appeared in a paper by Derrida and Spohn (1988). To define variable-speed BBM, fix a Galton Watson tree, a time horizon $t > 0$ and let $A : [0,1] \to [0,1]$, strictly increasing with $A(0) = 0$, $A(1) = 1$

and bounded second derivatives. The overlap $d(v, w)$ for leaves v, w in the tree is the time of their most recent common ancestor. Variable-speed BBM in time t and with time change $tA(\cdot/t)$ can then be defined as a centred Gaussian process x indexed by the leaves of the tree and covariance $tA(d(v, w)/t)$, where v and w are leaves. BBM is the special case when $A(x) = x$ for $x \in [0, 1]$, and coincides with the generalized random energy model (GREM) on the Galton-Watson tree. Compared to the 2d DGFF, its hierarchical structure makes it easier to analyse and the extremes of BBM are particularly well understood (see Aïdékon et al. (2013); Arguin et al. (2013); Bovier and Hartung (2017); Bramson (1978)). The extreme values and more general the extremal process for variable-speed BBM were investigated in Bovier and Hartung (2014, 2015); Fang and Zeitouni (2012a,b); Maillard and Zeitouni (2016). In particular, the first order and second order correction of the maximum in the regime of weak correlations, i.e. when $A(s) < s$ for $s \in (0, 1)$, are identical to the uncorrelated regime. In this regime, convergence of the extremal process was proved by Bovier and Hartung (2014, 2015). In the case of decreasing speed with finitely many changes in speed, the global maximum is a simple concatenation of the maximum at speed change. When the speed is strictly decreasing, i.e. when $A'' < 0$, Bovier and Kurkova (2004a,b) showed that the first order is as in all other cases determined by the concave hull of A. The second order correction is no longer logarithmic but proportional to $t^{1/3}$, which was shown by Maillard and Zeitouni (2016), building upon the work by Fang and Zeitouni (2012b).
In the discrete analogue model of (variable-speed) BBM, the (time-inhomogeneous) branching random walk (BRW) on the Galton Watson tree, there are results on the first and second order correction by Fang and Zeitouni (2012a), Mallein (2015a) and Ouimet (2018). A notable difference in the context of (time-inhomogeneous) BRW is that one does not need to assume that increments are Gaussian (see Mallein (2015a)). For the usual BRW, Aïdékon (2013) proved convergence of the centred maximum and Madaule (2017) of the extremal process.

2.2. *Idea of proof.* The main idea to prove Theorem 2.1 is to use Slepian's Lemma A.3 to compare the distribution of the centred maximum of the scale-inhomogeneous DGFF with the distribution of the maxima of two auxiliary Gaussian fields, of a time-inhomogeneous BRW (IBRW) and of a modified inhomogeneous branching random walk (MIBRW). The time-inhomogeneous BRW is constructed in such a way that it is slightly less correlated than the scale-inhomogeneous DGFF which allows to use an available upper bound on the right tail of the maximum of the time-inhomogeneous BRW. The MIBRW has correlations that differ from those of the scale-inhomogeneous DGFF inside the field only up to a uniformly bounded constant. This allows, in a first step, to use Gaussian comparison to reduce the lower bound on the right tail of the maximum to a corresponding lower bound on the right tail of the maximum of the MIBRW. In a second step, we prove the lower bound on the right tail of the centred maximum of the MIBRW that, together with the so-called "sprinkling method", also allows to deduce the upper bound on the left tail. The lower bound on the right tail itself is achieved by a modified second moment analysis.

The MIBRW is a generalization of the modified branching random walk (MBRW) which was introduced in Bramson and Zeitouni (2012) to prove tightness and later convergence Bramson et al. (2016) of the maximum of the 2d DGFF. A version of it was also used to study the maximum of log-correlated Gaussian fields Ding et al. (2017b); Schweiger and Zeitouni (2024).

Outline of the paper: In the next section we define two auxiliary Gaussian processes, the time-inhomogeneous branching random walk (IBRW) and the modified time-inhomogeneous branching random walk (MIBRW), and estimate their covariance structure. In Section 4 we provide the necessary tail estimates that allow us to deduce Theorem 2.1. In Subsection 4.1 we focus on the right tail of the maximum. This we sub-divide further into an upper bound, Proposition 4.1, and corresponding lower bound, Proposition 4.2. Finally, in Subsection 4.2, we prove the upper bound on the left tail which then completes the proof of Theorem 2.1. In Appendix A we provide the Gaussian comparison theorems we use in the proof, Borell's Gaussian concentration inequality and a standard Gaussian tail bound which is frequently used. In Appendix B we prove Lemma 1.3 and the covariance estimates stated in Section 3.

3. Auxiliary processes and covariance estimates

Consider $N = 2^n$ for $n \in \mathbb{N}$. For $k = 0, 1, \ldots, n$ let $\mathcal{B}_k$ denote the collection of subsets of $\mathbb{Z}^2$ consisting of squares of side length $2^k - 1$ with corners in $\mathbb{Z}^2$ and let $\mathcal{BD}_k$ denote the subset of $\mathcal{B}_k$ consisting of squares of the form $([0, 2^k-1]\cap\mathbb{Z})^2+(i2^k, j2^k)$, where for any $A \subset \mathbb{Z}^2$, we write $v+A := \{v + w : w \in A\} \subset \mathbb{Z}^2$. We remark that the collection $\mathcal{BD}_k$ partitions $\mathbb{Z}^2$ into disjoint squares. For $v \in V_N$, let $\mathcal{B}_k(v) := \{B \in \mathcal{B}_k : v \in B\}$. Likewise define $\mathcal{BD}_k(v) := \{B \in \mathcal{BD}_k : v \in B\}$. Note that for $v \in V_N$, $\mathcal{BD}_k(v)$ contains exactly one element, whereas $\mathcal{B}_k(v)$ contains 2^{2k} elements. One can

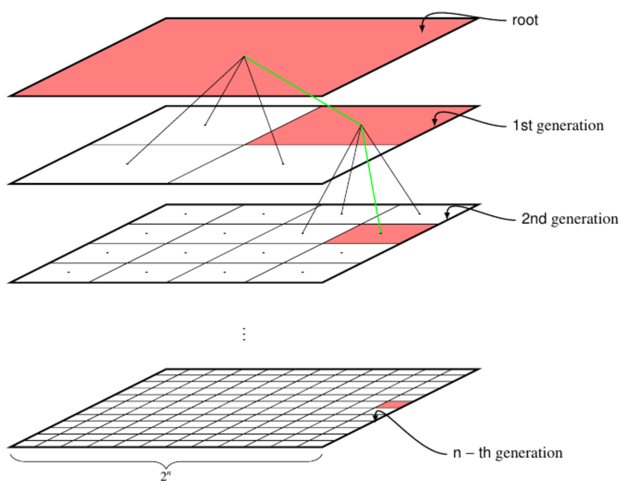

FIGURE 3.3. Tree decomposition

identify a 4-ary tree $\mathcal{T}_N$ of depth n with leaves $v \in V_N$ as follows: Identify $\mathcal{BD}_n \cap V_N$, which has exactly one element, to the root, and each of the four elements in $\mathcal{BD}_{n-1} \cap V_N$ to a child thereof. Treating each "child" as the root of a 4-ary tree of depth $n-1$ and performing this procedure recursively until reaching 0, we obtain a 4-ary tree of depth n and leaves $\mathcal{BD}_0 \cap V_N = V_N$ (see Figure 3). On this tree we define a time-inhomogeneous branching random walk as follows: For $n \in \mathbb{N}$ and $0 \le k \le n$ set

$$\sigma(k,n) := \left(n \int_{k/(n+1)}^{(k+1)/(n+1)} \sigma^2(s)\mathrm{d}s \right)^{1/2}, \tag{3.1}$$

with $s \mapsto \sigma(s)$ defined as in (1.8).

Definition 3.1 (Time-inhomogeneous branching random walk (IBRW))**.** Let $\{a_{k,B}\}_{k\ge 0, B\in\mathcal{BD}_k}$ be an i.i.d. family of standard Gaussian random variables. We define the time-inhomogeneous branching random walk of depth n, $\{R_v^N(t)\}_{v\in V_N, t\in\{0,\ldots,n\}}$, by

$$R_v^N(t) := \sum_{k=n-t}^{n} \sum_{B\in\mathcal{BD}_k(v)} \sqrt{\log(2)}\sigma(n-k,n)a_{k,B}, \tag{3.2}$$

with $\sigma(k,n)$ as in (3.1). We also write $R_v^N = R_v^N(n)$.

For $v, w \in V_N$, $q_N(v,w) := |\{k \geq 0 : \mathcal{BD}_k(v) \cap \mathcal{BD}_k(w) = \emptyset\}|$ denotes the associated "tree distance", which is half the total number of independent (i.e. disjoint) increments, $a_{k,B}$, of $R_v^N(t)$, $R_w^N(t)$ in (3.2). Note that $\log_2(\|v-w\|_2 \vee 1) \leq q_N(v,w)$, for any $v, w \in V_N$. In particular,

$$\mathbb{E}\left[R_v^N R_w^N\right] = \sum_{k=n-q_N(v,w)}^{n} \sum_{B \in \mathcal{BD}_k(v)} \log(2)\sigma^2(n-k,n) = \log(2) n \mathcal{I}_{\sigma^2}\left(\frac{n-q_N(v,w)}{n}\right). \tag{3.3}$$

Next, we introduce another auxiliary process whose covariance structure is much closer to the scale-inhomogeneous DGFF, and is defined by taking uniform averages of IBRWs. For $v \in V_N$, let $\mathcal{B}_k^N(v)$ be the collection of subsets $B \subset \mathbb{Z}^2$ consisting of squares of side length 2^k with lower left corner in V_N and such that $v \in B$. For two sets $B, B' \subset \mathbb{Z}^2$ we write $B \sim_N B'$, if there exist integers i, j such that $B' = B + (iN, jN)$. Let $\{b_{k,B}\}_{k \geq 0, B \in \mathcal{B}_k^N}$ denote an i.i.d. family of centred Gaussian random variables with unit variance and set

$$b_{k,B}^N := \begin{cases} b_{k,B},\ B \in \mathcal{B}_k^N, \\ b_{k,B'},\ B \sim_N B' \in \mathcal{B}_k^N. \end{cases} \tag{3.4}$$

Definition 3.2 (Modified inhomogeneous branching random walk (MIBRW))**.** The modified inhomogeneous branching random walk (MIBRW), $\{S_v^N(t)\}_{v \in V_N, t \in \{0,\dots,n\}}$, is defined by

$$S_v^N(t) := \sum_{k=n-t}^{n} \sum_{B \in \mathcal{B}_k^N(v)} 2^{-k}\sigma(n-k,n) b_{k,B}^N, \tag{3.5}$$

where $0 \leq t \leq n$, $t \in \mathbb{N}$ and $\sigma(k,n)$ as in (3.1). We also write $S_v^N = S_v^N(n)$.

3.1. *Covariance estimates and Gibbs-Markov property.* In order to be able to apply Gaussian comparison, we need to compare the correlations of the processes introduced previously. For $n \in \mathbb{N}$ and any $x \in \mathbb{R}_+$, we write $\log_{+,n}(x) = \max(0, \min(n, \log_2(x)))$ and $\log_+(x) = \max(0, \log x)$. Further, for any $v, w \in \mathbb{Z}^2$, $\|v-w\|_\infty \leq \|v-w\|_2 \leq \sqrt{2}\|v-w\|_\infty$, where $\|\cdot\|_2$ denotes the Euclidean and $\|\cdot\|_\infty$ the maximum distance. In addition, we introduce for $v, w \in V_N$ two distances on the torus induced by V_N,

$$d^N(v,w) := \min_{z:\, z-w \in (N\mathbb{Z})^2} \|v-z\|_2, \qquad d_\infty^N(v,w) := \min_{z:\, z-w \in (N\mathbb{Z})^2} \|v-z\|_\infty. \tag{3.6}$$

Note that for any $v, w \in V_N$, it holds $d^N(v,w) \leq \|v-w\|_2$. However, equality trivially holds if one restricts oneself to a smaller box, e.g. if $v, w \in (N/4, N/4) + V_{N/2} \subset V_N$. In the following we call $\{\tilde{S}_v^N\}_{v \in V_N}$ the homogeneous version of the process $\{S_v^N\}_{v \in V_N}$ which was introduced in Bramson and Zeitouni (2012), and which is the special case of exactly one scale $\lambda_1 = 1$ and single variance parameter $\sigma_1 = 1$.

Lemma 3.3. *There exists a constant C independent of $N = 2^n$ such that for any $v, w \in V_N$,*

i. $\left|\mathbb{E}\left[\tilde{S}_v^N \tilde{S}_w^N\right] - (n - \log_{+,n}(d^N(v,w)))\right| \leq C,$

ii. $\left|\mathbb{E}\left[S_v^N S_w^N\right] - n\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n}\right)\right| \leq C.$

For any $v \in V_N$ set $\tilde{v} := v + (2N, 2N) \in V_{4N}$. Then, for any $v, w \in V_N$

iii. $\left|\mathbb{E}\left[\phi_{\tilde{v}}^{4N} \phi_{\tilde{w}}^{4N}\right] - \log(2)(n - \log_{+,n}(\|v-w\|_2))\right| \leq C,$

iv. $\left|\mathbb{E}\left[\psi_{\tilde{v}}^{4N} \psi_{\tilde{w}}^{4N}\right] - \log(2)\mathbb{E}\left[S_v^N S_w^N\right]\right| \leq C$, *and*

$\left|\mathbb{E}\left[\psi_{\tilde{v}}^{4N} \psi_{\tilde{w}}^{4N}\right] - \log(2) n\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n}(\|v-w\|_2)}{n}\right)\right| \leq C.$

Statements ii. and iv. also hold if there is $i = 1, \dots, M$ with $\sigma_i = 0$.

Proof: See Appendix B. □

Remark 3.4. For what follows we always assume that $N = 2^n$ for some $n \in \mathbb{N}$, as it simplifies notation without removing essential difficulties. In fact, the proof of Lemma 3.3 *iv*. which is based on Lemma B.1 does not make use of this assumption (just replace $S_v^N S_w^N$ by $S_v^{2^{\lfloor \log_2 N \rfloor}} S_w^{2^{\lfloor \log_2 N \rfloor}}$ and n by $\lfloor \log_2(N) \rfloor$). In particular, following the proof of Lemma 3.3 *iv*., there exists a (possibly larger) constant $C > 0$ such that for any $N' \in \mathbb{N}$, $N' \geq 4$ and $N = 2^{\lfloor \log_2 N' \rfloor - 2}$, for any $\tilde{v}, \tilde{w} \in V_{4N} \subset V'_N$ as in *iv*., it holds that $\left| \mathbb{E}\left[\psi_{\tilde{v}}^{N'} \psi_{\tilde{w}}^{N'}\right] - \mathbb{E}\left[[\psi_{\tilde{v}}^{4N} \psi_{\tilde{w}}^{4N}\right] \right| \leq C$. This allows to readily adopt the proof of the main result, Theorem 2.1, as it is based on Gaussian comparison (namely Corollary A.3) which solely uses such covariance estimates with precision of $O(1)$.

An important tool in the analysis of the scale-inhomogeneous DGFF is the Gibbs-Markov property of the DGFF: For two sets $U \subset V \subset \mathbb{Z}^2$ the DGFF on V can be decomposed into a sum of a DGFF on U and an independent Gaussian field, i.e.

$$\phi_v^V \stackrel{d}{=} \phi_v^U + \mathbb{E}\left[\phi_v^V | \sigma\left(\phi_w^V : w \in V \setminus U^o\right)\right], \quad v \in V. \tag{3.7}$$

Further, if $A, B \subset V$ such that $A^o \cap B^o = \emptyset$, then $\{\phi_v^V - \mathbb{E}[\phi_v^V|\mathcal{F}_{\partial A}]\}_{v \in A}$ is a DGFF on A and $\{\phi_v^V - \mathbb{E}[\phi_v^V|\mathcal{F}_{\partial B}]\}_{v \in B}$ is a DGFF on B. Moreover, they are independent. In particular, if for $v, w \in V_N$, $0 \leq \lambda \leq \lambda' \leq 1$ and $0 \leq \mu \leq \mu' \leq 1$, it holds that

$$\left([v]_\lambda^N \cap ([v]_{\lambda'}^N)^c\right) \cap \left([w]_\mu^N \cap ([w]_{\mu'}^N)^c\right) = \emptyset, \tag{3.8}$$

then

$$\mathbb{E}\left[\left(\phi_v^N(\lambda') - \phi_v^N(\lambda)\right)\left((\phi_w^N(\mu') - \phi_w^N(\mu)\right)\right] = 0. \tag{3.9}$$

When setting

$$b_N(v,w) := \sup\{\lambda \in [0,1] : [v]_\lambda^N \cap [w]_\lambda^N \neq \emptyset\}, \tag{3.10}$$

the condition in (3.8) is satisfied if $\lambda, \lambda', \mu, \mu' \in (b_N(v,w), 1]$. With regards to (3.9) we call $b_N(u,v)$ the "branching scale" of $v, w \in V_N$. A simple observation is that for any N and $v, w \in V_N$,

$$\left| b_N(v,w) - \frac{\log N - \log_+(\|v-w\|_2)}{\log N} \right| \leq \frac{4}{\log N}. \tag{3.11}$$

4. Tail estimates and tightness

4.1. *Right tail estimate on the centred maximum.* We need both upper bound and lower bound on the right tail.

Proposition 4.1. *There is a constant $C \in (0, \infty)$, such that for all $N \in \mathbb{N}$ and $x > 0$,*

$$\mathbb{P}\left(\max_{v \in V_N} \psi_v^N \geq m_N + x\right) \leq C(1 + x\mathbb{1}_{\sigma_1 = \bar{\sigma}_1}) e^{-x\frac{2}{\bar{\sigma}_1}}. \tag{4.1}$$

Proposition 4.2. *There is a constant $C \in (0, \infty)$, such that for all $N \in \mathbb{N}$ and $x \in [0, \sqrt{\log N}]$,*

$$\mathbb{P}\left(\max_{v \in V_N} \psi_v^N \geq m_N + x\right) \geq C^{-1}\left(1 + x\mathbb{1}_{\sigma_1 = \bar{\sigma}_1}\right) e^{-x\frac{2}{\bar{\sigma}_1}}. \tag{4.2}$$

The principal idea to prove Proposition 4.1 is to use Gaussian comparison and compare the maximum of the scale-inhomogeneous DGFF to the maximum of the inhomogeneous branching random walk. In order to do so we need to match variances and show that the inhomogeneous branching random walk is less correlated.

Therefore, we sometimes consider so called "up-scaled" elements and sets, denoted $2^\kappa v$ and $2^\kappa A$, for $v \in \mathbb{Z}^2$ and $A \subset \mathbb{Z}^2$. For $N, \kappa \in \mathbb{N}$ and $v = (v_1, v_2) \in A \subset V_N$, we set

$$2^\kappa v := (2^\kappa v_1, 2^\kappa v_2) \in V_{2^\kappa N}, \quad 2^\kappa A := \{2^\kappa v : v \in A\}. \tag{4.3}$$

The idea is that the variances of the Gaussian fields we consider, grows logarithmically with the "up-scaling". On the other hand correlations can be kept approximately constant when considering the Gaussian field values at the corresponding "up-scaled" vertices. This idea is one key in the proof of the following lemma.

Lemma 4.3. *There exists $\kappa > 0$ such that for all $N \in \mathbb{N}$, $\lambda \in \mathbb{R}$,*

$$\mathbb{P}\left(\max_{v \in V_N} \psi_v^N \geq \lambda\right) \leq 4\mathbb{P}\left(\max_{v \in V_N} R_{2^\kappa v}^{2^{\kappa+2}N} \geq \lambda\right), \tag{4.4}$$

where $\left\{R_v^{2^{\kappa+2}N}(t)\right\}_{v \in V_{2^{\kappa+2}N},\, t \in \{0,\dots,n+\kappa+2\}}$ is a time-inhomogeneous BRW of depth $\kappa + 2 + \log_2(N)$ as in (3.2) (with $2^{\kappa+2}N$ in place of N).

Proof: The idea is to use Corollary A.3, hence we verify the assumptions therein and thus need to control the corresponding covariances. To be able to compare covariances on the entirety of V_N, we first reduce the considerations to a modified version of the scale-inhomogeneous DGFF on V_N for which we are able obtain an $O(1)$ control of its covariances on V_N. In a second step we then compare its maximum to the maximum of the time-inhomogeneous BRW using Corollary A.3. As in Lemma 3.3, let $\tilde{v} := v + (2N, 2N) \in V_{4N}$ for any $v \in V_N$ and set $V_N' := \{\tilde{v} : v \in V_N\}$. By the Gibbs-Markov property of the DGFF, for any $v \in V_{4N}$,

$$\phi_v^{4N} = \left(\phi_v^{4N} - \mathbb{E}\left[\phi_v^{4N} \middle| \mathcal{F}_{\partial V_N'}\right]\right) + \mathbb{E}\left[\phi_v^{4N} \middle| \mathcal{F}_{\partial V_N'}\right] =: \tilde{\phi}_v^N + \xi_v^{4N,N}, \tag{4.5}$$

with the summands on the right-hand side being independent Gaussian fields defined through the two terms in the second equation. The first has the same distribution of a DGFF on V_N', with $\tilde{\phi}_v^N = 0$ for $v \notin \left(V_N'\right)^o$, and the second is a mean zero Gaussian field as it is the harmonic average of zero-mean Gaussian random variables. Now, let $\psi_v^N = \sum_{i=1}^M \sigma_i(\tilde{\phi}_v^N(\lambda_i) - \tilde{\phi}_v^N(\lambda_{i-1}))$ for $v \in V_N'$ and note that this has the same distribution as a (σ, λ) scale-inhomogeneous DGFF. Next, we define scale parameters $\tilde{\lambda}_i := (\lambda_i + \frac{\log(4)}{\log N}) \wedge 1$ for $i = 1, \dots, M$ and note that $(4N)^{1-\tilde{\lambda}_i} = N^{1-\lambda_i}$. Hence, by the same martingale transformation as in the definition of the scale-inhomogeneous DGFF in (1.9), for $v \in V_N'$

$$\psi_v^N + \sum_{i=1}^M \sigma_i \left(\mathbb{E}\left[\xi_v^{4N,N} \middle| \mathcal{F}_{\partial [v]_{\tilde{\lambda}_i}^{4N}}\right] - \mathbb{E}\left[\xi_v^{4N,N} \middle| \mathcal{F}_{\partial [v]_{\tilde{\lambda}_{i-1}}^{4N}}\right]\right) = \sum_{i=1}^M \sigma_i \left(\phi_v^{4N}(\tilde{\lambda}_i) - \phi_v^{4N}(\tilde{\lambda}_{i-1})\right). \tag{4.6}$$

Set $\tau := \arg\max_{v \in V_N'} \sum_{i=1}^M \sigma_i \left(\phi_v^{4N}(\tilde{\lambda}_i) - \phi_v^{4N}(\tilde{\lambda}_{i-1})\right)$. Then, almost surely

$$\max_{v \in V_N'} \sum_{i=1}^M \sigma_i \left(\phi_v^{4N}(\tilde{\lambda}_i) - \phi_v^{4N}(\tilde{\lambda}_{i-1})\right) \geq \max_{v \in V_N'} \psi_v^N + \sum_{i=1}^M \sigma_i \left(\mathbb{E}\left[\xi_\tau^{4N,N} \middle| \mathcal{F}_{\partial [\tau]_{\tilde{\lambda}_i}^{4N}}\right] - \mathbb{E}\left[\xi_\tau^{4N,N} \middle| \mathcal{F}_{\partial [\tau]_{\tilde{\lambda}_{i-1}}^{4N}}\right]\right). \tag{4.7}$$

As $\sum_{i=1}^M \sigma_i \left(\mathbb{E}\left[\xi_\tau^{4N,N} \middle| \mathcal{F}_{\partial [\tau]_{\tilde{\lambda}_i}^{4N}}\right] - \mathbb{E}\left[\xi_\tau^{4N,N} \middle| \mathcal{F}_{\partial [\tau]_{\tilde{\lambda}_{i-1}}^{4N}}\right]\right)$ conditionally on τ is an mean zero Gaussian, independent of $\max_{v \in V_N'} \psi_v^N$, it holds that for any $\lambda \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{v \in V_N'} \psi_v^N + \sum_{i=1}^M \sigma_i \left(\mathbb{E}\left[\xi_\tau^{4N,N} \middle| \mathcal{F}_{\partial [\tau]_{\tilde{\lambda}_i}^{4N}}\right] - \mathbb{E}\left[\xi_\tau^{4N,N} \middle| \mathcal{F}_{\partial [\tau]_{\tilde{\lambda}_{i-1}}^{4N}}\right]\right) \geq \lambda\right) \leq 2\mathbb{P}\left(\max_{v \in V_N'} \psi_v^N \geq \lambda\right). \tag{4.8}$$

Noting that the law of a scale-inhomogeneous DGFF is invariant under shifts of its entire domain, it follows that for any $\lambda \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{v \in V_N} \psi_v^N \geq \lambda\right) \leq 2\mathbb{P}\left(\max_{v \in V_N'} \sum_{i=1}^M \sigma_i \left(\phi_v^{4N}(\tilde{\lambda}_i) - \phi_v^{4N}(\tilde{\lambda}_{i-1})\right) \geq \lambda\right). \tag{4.9}$$

Set $\sigma_0 = 0$ and

$$\tilde{\psi}_v^N := \sum_{i=1}^{M} \sigma_i \left(\phi_v^{4N}(\tilde{\lambda}_i) - \phi_v^{4N}(\tilde{\lambda}_{i-1}) \right) + \sigma_0 \left(\phi_v^{4N}(\log(4)/\log(N)) - \phi_v^{4N}(0) \right), \quad v \in V_{4N}. \tag{4.10}$$

In particular, $\left\{ \tilde{\psi}_v^{4N} \right\}_{v \in V_N'}$ has the law of a scale-inhomogeneous DGFF on V_{4N} restricted to V_N' with variance and scale parameters $(0, \sigma_1, \ldots, \sigma_M)$ and $(\log(4)/\log(N), \tilde{\lambda}_1, \ldots, \tilde{\lambda}_M)$ respectively. Moreover, there exists $c > 0$ such that for any $s \in [0,1]$ and N,

$$\left| \log(N)\mathcal{I}_{\sigma^2}(s) - \log(N)\left(\sum_{i:\tilde{\lambda}_i < s} \sigma_i^2 \Delta\tilde{\lambda}_i + \sigma_j(s - \tilde{\lambda}_{j-1}) \mathbb{1}_{\tilde{\lambda}_{j-1} \le s < \tilde{\lambda}_i} \right) \right| \le c. \tag{4.11}$$

Hence, in combination with Lemma 3.3 *iv*. there exists $\alpha_0 > 0$ so that for any N and $v, w \in V_N'$,

$$\left| \mathbb{E}\left[\tilde{\psi}_v^{4N} \tilde{\psi}_w^{4N} \right] - \log(N)\mathcal{I}_{\sigma^2}\left(\frac{n - \log_{+,n}(\|v-w\|_2)}{n} \right) \right| \le \alpha_0, \tag{4.12}$$

which yields the desired control over its covariances. Next, we modify the field $\{\tilde{\psi}_v^{4N}\}_{v \in V_N'}$ in order to match variances and show domination of its covariances over those of the time-inhomogeneous BRW. Note that for any $\kappa, N \in \mathbb{N}$ and $v \in V_{2^\kappa N}$, $\mathrm{Var}\left[R_v^{2^\kappa N}\right] \equiv \log\left(2^\kappa N\right) \mathcal{I}_{\sigma^2}(1)$. Whenever $\kappa \ge \alpha_0$, (4.12) and $\mathcal{I}_{\sigma^2}(1) = \sum_{i=1}^{M} \Delta\lambda_i \sigma_i^2 = 1$ imply that for any $N \in \mathbb{N}$, $\mathrm{Var}\left[\tilde{\psi}_v^{\,4N}\right] \le \mathrm{Var}\left[R_{2^\kappa+v}^{2^{\kappa+2}N}\right]$ for all $v \in V_N'$. In order to use Corollary A.3 we need to match variances. Thus, set

$$a_v^2 := \mathrm{Var}\left[R_{2^\kappa v}^{2^{\kappa+2}N}\right] - \mathrm{Var}\left[\tilde{\psi}_v^{4N}\right] = \log(2^{\kappa+2}N) - \mathrm{Var}\left[\tilde{\psi}_v^{4N}\right], \quad v \in V_N' \tag{4.13}$$

and observe that by (4.12) for any N and $\kappa \ge \alpha_0$,

$$\min_{v \in V_N'} a_v^2 \ge \log(2)(\kappa+2) - \alpha_0 \ge 0, \quad \text{and} \quad \max_{v \in V_N'} a_v^2 \le \log(2)(\kappa+2) + \alpha_0. \tag{4.14}$$

Let X be a standard Gaussian independent of $\left(\tilde{\psi}_v^{4N}\right)_{v \in V_{4N},\, N \in \mathbb{N}}$. Then, for $v \in V_N'$,

$$\mathrm{Var}\left[\tilde{\psi}_v^{4N} + a_v X\right] = \mathrm{Var}\left[R_{2^\kappa v}^{2^{\kappa+2}N}\right]. \tag{4.15}$$

(4.12) together with (4.14) imply that for any $v, w \in V_N'$

$$\begin{aligned} \mathbb{E}\left[(\tilde{\psi}_v^{4N} + a_v X)(\tilde{\psi}_w^{4N} + a_w X)\right] &\ge \log N \mathcal{I}_{\sigma^2}\left(\frac{n - \log_{+,n}\|v-w\|_2}{n} \right) - 2\alpha_0 + \log(2)\kappa \\ &= \log(2^\kappa N)\mathcal{I}_{\sigma^2}\left(\frac{n - \log_{+,n}\|v-w\|_2}{n} \right) + \log(2)\kappa\left(1 - \mathcal{I}_{\sigma^2}\left(\frac{n - \log_{+,n}\|v-w\|_2}{n} \right) \right) - 2\alpha_0. \end{aligned} \tag{4.16}$$

By (3.3) and using for any $v, w \in V_N'$ the fact that $q_{2^{\kappa+2}N}(2^\kappa v, 2^\kappa w) = q_N(v,w) + \kappa + 2$, it holds that $\mathbb{E}[R_{2^\kappa v}^{2^{\kappa+2}N} R_{2^\kappa w}^{2^{\kappa+2}N}] = \log(2^{\kappa+2}N)\mathcal{I}_{\sigma^2}\left(\frac{n - q_N(v,w)}{n+\kappa+2}\right)$. Taking into account that for two vertices $v, w \in V_{N'}$ and $\log_{+,n}\|v-w\|_2 \le q_N(v,w)$, it holds that

$$\begin{aligned} \mathbb{E}\left[R_{2^\kappa v}^{2^{\kappa+2}N} R_{2^\kappa w}^{2^{\kappa+2}N}\right] &\le \log(2^{\kappa+2}N)\mathcal{I}_{\sigma^2}\left(\frac{n - \log_{+,n}\|v-w\|_2}{n+\kappa+2} \right) \\ &\le 2 + \log(2^\kappa N)\mathcal{I}_{\sigma^2}\left(\frac{n - \log_{+,n}\|v-w\|_2}{n+\kappa+2} \right). \end{aligned} \tag{4.17}$$

With regards to (4.16) and (4.17), we wish to show that there exists $\kappa(\alpha_0)$ such that for any $\kappa \geq \kappa(\alpha_0)$, n and $x \in [0, n]$,

$$2(\alpha_0+1) \leq \log(2^\kappa N)\left(\mathcal{I}_{\sigma^2}\left(\frac{n-x}{n}\right) - \mathcal{I}_{\sigma^2}\left(\frac{n-x}{n+\kappa+2}\right)\right) + \log(2)\kappa\left(1-\mathcal{I}_{\sigma^2}\left(\frac{n-x}{n}\right)\right), \quad (4.18)$$

as then it follows that for any n, $\kappa \geq \kappa(\alpha_0)$ and $v, w \in V_N'$,

$$\mathbb{E}\left[R_{2^\kappa v}^{2^{\kappa+2}N} R_{2^\kappa w}^{2^{\kappa+2}N}\right] \leq \mathbb{E}\left[(\tilde{\psi}_v^{4N} + a_v X)(\tilde{\psi}_w^{4N} + a_w X)\right]. \quad (4.19)$$

In order to show (4.18), let $\sigma_{min}^2 := \min_{1\leq i \leq M} \sigma_i^2 > 0$ and observe that

$$\mathcal{I}_{\sigma^2}\left(\frac{n-x}{n}\right) - \mathcal{I}_{\sigma^2}\left(\frac{n-x}{n+\kappa+2}\right) \geq \sigma_{min}^2 \frac{(\kappa+2)(n-x)}{n(n+\kappa+2)} \quad (4.20)$$

as well as

$$1-\mathcal{I}_{\sigma^2}\left(\frac{n-x}{n}\right) \geq \sigma_{min}^2 \frac{x}{n}, \quad (4.21)$$

and so plugging both into the right-hand side in (4.18) it follows that this is at least

$$\log(2)\sigma_{min}^2\left(\frac{n+\kappa}{n+\kappa+2}\frac{(\kappa+2)(n-x)}{n} + \kappa\frac{x}{n}\right) \geq \log(2)\sigma_{min}^2 \kappa \frac{n+\kappa}{n+\kappa+2} \geq \frac{\sigma_{min}^2}{4}\kappa, \quad (4.22)$$

once $\kappa \geq 2$. Choosing $\kappa(\alpha_0) = 8\left\lceil \frac{\alpha_0+1}{\sigma_{min}^2}\right\rceil \vee 2 \vee \alpha_0$ shows that (4.18) and therefore also (4.19) are satisfied. This together with (4.15) allows to apply Corollary A.3, which yields that for any $\lambda \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{v\in V_N'} \tilde{\psi}_v^{4N} + a_v X \geq \lambda\right) \leq \mathbb{P}\left(\max_{v\in V_N'} R_{2^\kappa v}^{2^{\kappa+2}N} \geq \lambda\right). \quad (4.23)$$

Note that the laws of $\max_{v\in V_N'} R_{2^\kappa v}^{2^{\kappa+2}N}$ and $\max_{v\in V_N} R_{2^\kappa v}^{2^{\kappa+2}N}$ coincide. This combined with independence between $\tilde{\psi}^{4N}$ and X as well as using $\mathbb{P}(X \geq 0) = 1/2$ yields

$$\begin{aligned}\mathbb{P}\left(\max_{v\in V_N'} \tilde{\psi}_v^{4N} \geq \lambda\right) &= 2\mathbb{P}\left(\max_{v\in V_N'} \tilde{\psi}_v^{4N} \geq \lambda, X \geq 0\right) \\ &\leq 2\mathbb{P}\left(\max_{v\in V_N'} \tilde{\psi}_v^{4N} + a_v X \geq \lambda\right) \leq 2\mathbb{P}\left(\max_{v\in V_N} R_{2^\kappa v}^{2^{\kappa+2}N} \geq \lambda\right). \quad (4.24)\end{aligned}$$

Plugging this back into (4.9) we obtain (4.4). □

Proof of Proposition 4.1: Mallein (2015a, Theorem 4.1) gives us

$$\mathbb{P}\left(\max_{v\in V_N} R_v^N \geq m_N + x\right) \leq C(1 + x\mathbb{1}_{\sigma_1=\bar{\sigma}_1})e^{-x\frac{2}{\bar{\sigma}_1}}, \quad \forall x \geq 0. \quad (4.25)$$

Combining Lemma 4.3 and (4.25) implies the claim. □

Next, we prepare the proof of Proposition 4.2.

Lemma 4.4. *For $n \in \mathbb{N}$, $n \geq 2$ and $N = 2^n$ let $V_N' = V_{N/2} + (N/4, N/4) \subset V_N$. There is an integer $\kappa > 0$ so that for all such $N \in \mathbb{N}$ and any $\lambda \in \mathbb{R}$,*

$$\frac{1}{2}\mathbb{P}\left(\max_{v\in V_N'} \sqrt{\log(2)} S_v^N \geq \lambda\right) \leq \mathbb{P}\left(\max_{v\in V_{2^{\kappa+2}N}} \psi_v^{2^{\kappa+2}N} \geq \lambda\right). \quad (4.26)$$

Proof: The idea of proof is to use Corollary A.3 for a suitable choice of κ. Similarly as in the proof Lemma 4.3, verifying the conditions of Corollary A.3 is not straightforward and hence we first provide a suitable setup. For $N, \kappa \in \mathbb{N}$ and $v \in V_N$ set $\tilde{v} = 2^\kappa v \in V_{2^{\kappa+2}N}$. By Lemma 3.3 *ii.* and *iv.*, there is a constant $C > 0$ uniformly in $\kappa, N = 2^n$ and all $v, w \in V_N'$ such that

$$\left|\mathrm{Var}\left[\psi_{\tilde{v}}^{2^{\kappa+2}N}\right] - \log(2^\kappa N)\right| \leq C,$$
$$\left|\mathbb{E}\left[\psi_{\tilde{v}}^{2^{\kappa+2}N}\psi_{\tilde{w}}^{2^{\kappa+2}N}\right] - \log(2^\kappa N)\mathcal{I}_{\sigma^2}\left(\frac{n+\kappa-\log_{+,n+\kappa}\|2^k v - 2^k w\|_2}{n+\kappa}\right)\right| \leq C, \tag{4.27}$$

as well as

$$\mathrm{Var}\left[S_v^N\right] = \log_2 N, \quad \left|\mathbb{E}\left[S_v^N S_w^N\right] - \log_2 N \mathcal{I}_{\sigma^2}\left(\frac{n - \log_{+,n} d^N(v,w)}{n}\right)\right| \leq C. \tag{4.28}$$

Note that the variance statement holds simply by construction of the MIBRW and since $\sum_{i=1}^M \Delta\lambda_i \sigma_i^2 = 1$. Using that for any $v \neq w \in V_N$, $\log_{+,n+\kappa}\|2^\kappa v - 2^\kappa w\|_2 = \kappa + \log_{+,n}\|v-w\|_2$, we also have for such v, w,

$$\left|\mathbb{E}\left[\psi_{\tilde{v}}^{2^{\kappa+2}N}\psi_{\tilde{w}}^{2^{\kappa+2}N}\right] - \log(2^\kappa N)\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n}\|v-w\|_2}{n+\kappa}\right)\right| \leq C, \tag{4.29}$$

By (4.27) and (4.28) there exist $\{a_v \in \mathbb{R}_+ : v \in V_N\}$ that satisfy $|a_u - a_v| \leq C$ and $a_v \geq \sqrt{\kappa} - \sqrt{C}$ for any $v, w \in V_N'$ and same constant $C > 0$, such that with an independent standard Gaussian random variable X,

$$\mathrm{Var}\left[\psi_{\tilde{v}}^{2^{\kappa+2}N}\right] = \log(2)\mathrm{Var}\left[S_v^N + a_v X\right], \quad \forall v \in V_N'. \tag{4.30}$$

By (4.27) and (4.28), once κ is sufficiently large, for all N and $v \in V_N'$,

$$\mathrm{Var}\left[\psi_{\tilde{v}}^{2^{\kappa+2}N}\right] \geq \log(2)\mathrm{Var}\left[S_v^N\right]. \tag{4.31}$$

By (4.29) and using $d^N(v,w) \leq \|v-w\|_2$ for $v, w \in V_N$, we have for any $\kappa > 0$ and $v, w \in V_N'$,

$$\mathbb{E}\left[\psi_{\tilde{v}}^{2^{\kappa+2}N}\psi_{\tilde{w}}^{2^{\kappa+2}N}\right] \leq \log(2^\kappa N)\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n+\kappa}\right) + C. \tag{4.32}$$

At the same time by (4.28), (4.30) and as $\mathcal{I}_{\sigma^2}(1) = 1$,

$$\log(2)\big(\mathbb{E}\left[S_v^N S_w^N\right] + a_v a_w\big) \geq \log(N)\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n}\right) + \log(2)\big(\kappa - \sqrt{\kappa C}\big)$$
$$\geq \log(2^\kappa N)\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n}\right) + \log(2)\kappa\left(1 - \mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n}\right)\right) - \log(2)\sqrt{\kappa C}. \tag{4.33}$$

Thus, to conclude that $\mathbb{E}\left[\psi_{\tilde{v}}^{2^{\kappa+2}N}\psi_{\tilde{w}}^{2^{\kappa+2}N}\right] \leq \log(2)\mathbb{E}\left[S_v^N S_w^N\right] + \log(2)a_v a_w$ it suffices to show that there exists $\kappa > 0$ such that for all N and $v \neq w \in V_N'$

$$\log(2^\kappa N)\left(\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n}\right) - \mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n+\kappa}\right)\right)$$
$$+ \log(2)\kappa\left(1 - \mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n}\right)\right) \geq \log(2)\sqrt{\kappa C} + C. \tag{4.34}$$

Setting $\sigma_{min} := \min_{1\le i\le M} \sigma_i > 0$, we have that

$$1-\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n}\right) \ge \sigma_{min}^2 \frac{\log_{+,n} d^N(v,w)}{n}, \tag{4.35}$$

as well as

$$\mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n}\right) - \mathcal{I}_{\sigma^2}\left(\frac{n-\log_{+,n} d^N(v,w)}{n+\kappa}\right) = \int_{\frac{n-\log_{+,n} d^N(v,w)}{n+\kappa}}^{\frac{n-\log_{+,n} d^N(v,w)}{n}} \sigma^2(s)\mathrm{d}s \ge \frac{\kappa}{n+\kappa}\sigma_{min}^2 \frac{n-\log_{+,n} d^N(v,w)}{n}. \tag{4.36}$$

Hence, multiplying (4.36) with $\log(2^\kappa N)$ and (4.35) with $\log(2)\kappa$, then plugging both into (4.34), we obtain that it suffices to show that $\kappa\sigma_{min}^2 \ge \sqrt{\kappa C}+C$. This is obviously true once $\kappa \ge \kappa_0$ for some κ_0. Choosing such κ, we may thus apply Corollary A.3 to conclude that for any $\lambda \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{v\in V_{2^{\kappa+2}N}} \psi_v^{2^{\kappa+2}N} \ge \lambda\right) \ge \mathbb{P}\left(\max_{v\in V_N'} \psi_{\tilde{v}}^{2^{\kappa+2}N} \ge \lambda\right) \ge \mathbb{P}\left(\sqrt{\log(2)}\max_{v\in V_N'}(S_v^N + a_v X) \ge \lambda\right)$$
$$\ge \mathbb{P}(X\ge 0)\mathbb{P}\left(\sqrt{\log(2)}\max_{v\in V_N'} S_v^N \ge \lambda\right) \ge \frac{1}{2}\mathbb{P}\left(\sqrt{\log(2)}\max_{v\in V_N'} S_v^N \ge \lambda\right), \tag{4.37}$$

where in the last line we used that X is an independent standard Gaussian. □

Lemma 4.5. *For any $N, n \in \mathbb{N}$ such that $N = 2^n$, $n \ge 2$ set $M_N^* := m_N/\sqrt{\log(2)}$ and $V_N' = V_{N/2} + (N/4, N/4) \subset V_N$. There is a constant $C > 0$ such that for any such N and $y \in [0, \sqrt{\log N}]$,*

$$\mathbb{P}\left(\max_{v\in V_N'} S_v^N > M_N^* + y\right) \ge C\left(1 + y\mathbb{1}_{\sigma_1=\bar{\sigma}_1}\right) e^{-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}y}. \tag{4.38}$$

Before providing the proof of Lemma 4.5 we give the proof of Proposition 4.2.

Proof of Proposition 4.2: Applying first Lemma 4.4 with $2^{-\kappa-2}N = 2^{n-\kappa-2}$ and $n \ge \kappa+2$ instead of N, we have

$$\mathbb{P}\left(\max_{v\in V_N}\psi_v^N \ge m_N + x\right) \ge \frac{1}{2}\mathbb{P}\left(\max_{v\in V_{2^{-\kappa-2}N}'} \sqrt{\log(2)} S_v^{2^{-\kappa-2}N} \ge m_{2^{-\kappa-2}N} + (m_N - m_{2^{-\kappa-2}N}) + x\right). \tag{4.39}$$

Since $0 \le (m_N - m_{2^{-\kappa-2}N}) \le c(\kappa+2)$, for some $c > 0$ and all N, applying Lemma 4.5 with $y = (m_N - m_{2^{-\kappa-2}N}) + x$ shows that (4.39) is bounded from below by

$$C\left(1 + x/\sqrt{\log 2}\mathbb{1}_{\sigma_1=\bar{\sigma}_1}\right) e^{-\frac{2}{\bar{\sigma}_1}((m_N-m_{2^{-\kappa-2}N})+x)} \ge \tilde{C}\left(1 + x\mathbb{1}_{\sigma_1=\bar{\sigma}_1}\right) e^{-\frac{2}{\bar{\sigma}_1}x}, \tag{4.40}$$

with $\tilde{C} = C\frac{1}{\sqrt{\log 2}}e^{-2c\bar{\sigma}_1^{-1}(\kappa+2)}$ which shows the claim. For $n < \kappa+2$, the claim is trivial and follows choosing the constant $\tilde{C} > 0$ sufficiently small. □

Recall that, for $1 \le j \le m$, $\pi_j \in 0,\dots,M$ is the unique index such that $\lambda^j = \lambda_{\pi_j}$ and that we write $t^j = \lambda^j \frac{\log N}{\log 2}$ as well as $t_j = \lambda_j \frac{\log N}{\log 2}$. Also set

$$M_N^*(t) := \sum_{j=1}^m \frac{t\wedge t^j - t^{j-1}\wedge t}{\Delta t^j}\left[2\sqrt{\log 2}\bar{\sigma}_j \Delta t^j - \frac{(w_j\bar{\sigma}_j \log(\Delta t^j))}{4\sqrt{\log(2)}}\right], \quad t \in \mathbb{R}_+. \tag{4.41}$$

The proof of Lemma 4.5 is based on a second moment computation. We introduce suitable events that control the paths that reach the maximum. Set $V_N' = V_{N/2} + (N/4, N/4) \subset V_N$. For $x \in \mathbb{R}$, $0 \leq k \leq n$ set

$$s_{k,n}(x) := \frac{\mathcal{I}_{\sigma^2}(\lambda^{i-1}, k/n)}{\mathcal{I}_{\sigma^2}(\lambda^{i-1}, \lambda^i)}(x), \qquad \text{if } \lambda^{i-1} < k \leq \lambda^i,\ i = 1, \dots m \tag{4.42}$$

and

$$f_{k,n} := \begin{cases} C_f(\mathcal{I}_{\sigma^2}(k/n)n)^{2/3}, & \text{if } 0 \leq k \leq t_1, \\ C_f(\mathcal{I}_{\sigma^2}(k/n, \lambda^1)n)^{2/3}, & \text{if } t_1 < k \leq t^1, \\ C_f(\mathcal{I}_{\sigma^2}(\lambda^i, k/n)n)^{2/3}, & \text{if } t^i < k \leq t_{\pi_i+1} : i \in \{1, \dots, m-1\} \\ C_f(\mathcal{I}_{\sigma^2}(k/n, \lambda^{i+1})n)^{2/3}, & \text{if } t_{\pi_i+1} < k \leq t^{i+1} : i \in \{1, \dots, m-1\}. \end{cases} \tag{4.43}$$

The constant $C_f > 0$ depends on the parameters (σ, λ) and will be fixed later in the proof (see (4.62)). The idea of $s_{k,n}(x)$ and $f_{k,n}$ is that $|S_v^N(k) - s_{k,n}(x)| \leq f_{k,n}$ with overwhelming probability if we require $|S_v^N - x| \leq 1$. For $v \in V_N$, $x \in \mathbb{R}$, $y \in (0, \infty)$, $i = 1, \dots, m$ and $t^{i-1} < k < t^i$, set

$$I_n^y(1) := [\Delta M_N^*(t^1) + y, \Delta M_N^*(t^1) + y + 1], \tag{4.44}$$

$$I_n^y(i) := [\Delta M_N^*(t^i), \Delta M_N^*(t^i) + 1], \text{ for } 1 < i \leq m \tag{4.45}$$

$$I_{k,n}(x) := \begin{cases} [s_{k,n}(x) - f_{k,n}, s_{k,n}(x) + f_{k,n}], & \text{if } \mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar{\sigma}} \text{ on } (\lambda^{i-1}, \lambda^i) \\ \big(-\infty, s_{k,n}(x) + 1 - C_f \log\big((k - t^{i-1}) \wedge (t^i - k)\big)\big), & \text{if } \mathcal{I}_{\sigma^2} \equiv \mathcal{I}_{\bar{\sigma}} \text{ on } [\lambda^{i-1}, \lambda^i] \end{cases} \tag{4.46}$$

$$\begin{aligned} C_v^{N,y}(r) := \{ &\Delta S_v^N(t^i) \in I_n^y(i),\ S_v^N(k + t^{i-1}) - S_v^N(t^{i-1}) \in I_{k,n}(\Delta S_v^N(t^i)) \\ &\forall 0 < k < t^i - t^{i-1},\ 0 < i \leq m :\ k + t^{i-1} \leq r\}, \end{aligned} \tag{4.47}$$

$$h_N(y) := \sum_{v \in V_N'} \mathbb{1}_{C_v^{N,y}(t^m)}. \tag{4.48}$$

Next, we provide the necessary bounds on first and second moment of $h_N(y)$ that we use in the following proof of Lemma 4.5.

Lemma 4.6. *There are constants $C, c > 0$ such that it holds for all $N, n \in \mathbb{N}$, $N = 2^n$ and $y \in [0, \sqrt{\log N}]$,*

$$c e^{-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1} y} \geq \mathbb{E}[h_N(y)] \geq C e^{-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1} y}. \tag{4.49}$$

Lemma 4.7. *There is a constant $\tilde{C} > 0$ such that for all $N, n \in \mathbb{N}$. $N = 2^n$ and $y \in [0, \sqrt{\log N}]$*

$$\mathbb{E}\left[h_N^2(y)\right] \leq \mathbb{E}[h_N(y)]^2 + (1 + \tilde{C})\mathbb{E}[h_N(y)]. \tag{4.50}$$

Proof of Lemma 4.6: By linearity of expectations, independence of $\{S_v^N(k+1) - S_v^N(k)\}_{0 \leq k \leq t^m}$ and using $|V_{N'}| = N^2/4$,

$$\begin{aligned} \mathbb{E}[h_N] &= |V_{N'}| \mathbb{P}\left(C_v^{N,y}(t^m)\right) \\ &= \frac{1}{4} N^2 \prod_{i=1}^m \mathbb{P}\begin{pmatrix} \Delta S_v^N(t^i) \in I_n^y(i), \\ S_v^N(k + t^{i-1}) - S_v^N(t^{i-1}) \in I_{k,n}(\Delta S_v^N(t^i)) \text{ for } 0 < k < t^i - t^{i-1} \end{pmatrix}. \end{aligned} \tag{4.51}$$

Note that for any k with $t^{i-1} \le k \le t^i$ and $i \in \{1,\dots,m\}$, $\mathrm{Var}\left[S_v^N(k)\right] = n\mathcal{I}_{\sigma^2}(k/n)$. Together with independence of increments, that is $\mathbb{E}\big[\big(S_v^N(t^i - S_v^N(k+t^{i-1}))\big)\big(S_v^N(k+t^{i-1}) - S_v^N(t^{i-1})\big)\big] = 0$,

$$\begin{aligned}\mathbb{E}&\left[\Delta S_v^N(t^i)\left(S_v^N(k+t^{i-1}) - S_v^N(t^{i-1}) - s_{k,n}(\Delta S_v^N(t^i))\right)\right]\\ &= n\mathcal{I}_{\sigma^2}(\lambda^{i-1}, k/n+\lambda^{i-1}) - \frac{\mathcal{I}_{\sigma^2}(\lambda^{i-1}, k/n+\lambda^{i-1})}{\mathcal{I}_{\sigma^2}(\lambda^{i-1},\lambda^i)} n\mathcal{I}_{\sigma^2}(\lambda^{i-1},\lambda^i) = 0. \end{aligned} \tag{4.52}$$

By conditioning the last probability in (4.51) on $(\Delta S_v^N(t^i))_{i=1,\dots,m}$, using that (4.52) implies independence between $\Delta S_v^N(t^i)$ and $\{S_v^N(k+t^{i-1}) - S_v^N(t^{i-1})) - s_{k,n}(\Delta S_v^N(t^i))\}_{k=1}^{t^i-t^{i-1}}$ for $i = 1,\dots,m$, and since $N^2 = \prod_{i=1}^m N^{2\Delta\lambda^i}$, it follows that (4.51) is equal to

$$\frac{1}{4}\prod_{i=1}^m N^{2\Delta\lambda^i}\mathbb{P}\left(\Delta S_v^N(t^i) \in I_n^y(i)\right)\mathbb{P}\left(S_v^N(k+t^{i-1}) - S_v^N(t^{i-1}) \in I_{k,n}(\Delta S_v^N(t^i), 0<k<t^i-t^{i-1}\right). \tag{4.53}$$

We first prove the lower bound and in a second step the upper bound by bounding each probability in (4.53). In the case when $\mathcal{I}_{\sigma^2} \equiv \mathcal{I}_{\bar\sigma}$ on $[\lambda^{i-1},\lambda^i]$ we use the known result for the (homogeneous) MBRW which was used to show the analogue result for the homogeneous DGFF, that is by Bramson and Zeitouni (2012, Lemma 6.2) (with N therein being replaced by $N^{2\Delta\lambda^i}$) there exists a constant $C>0$ such that

$$\begin{aligned}CN^{-2\Delta\lambda^i} \ge \mathbb{P}\left(\Delta S_v^N(t^i) \in I_n^y(i)\right)\mathbb{P}\left(S_v^N(k+t^{i-1}) - S_v^N(t^{i-1}) \in I_{k,n}(\Delta S_v^N(t^i), 0<k<t^i-t^{i-1}\right)&\\ \ge \frac{N^{-2\Delta\lambda^i}}{C}.&\end{aligned} \tag{4.54}$$

Hence, in the remaining estimates on the probabilities in (4.53) we assume $\mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar\sigma}$ on $(\lambda^{i-1},\lambda^i)$ for $i = 1,\dots,m$ and start with estimating the first. Note that, for $i = 1,\dots,m$, $\Delta S_v^N(t^i) \sim \mathcal{N}\left(0,\bar\sigma_i^2\Delta t^i\right)$ and that our assumptions imply $\Delta M_N^*(t^i) = 2\sqrt{\log(2)}\bar\sigma_i\Delta t^i - \frac{1}{4\sqrt{\log(2)}}\log(\Delta t^i)\bar\sigma_i$. Thus,

$$\mathbb{P}\left(S_v^N(t^1) \in I_n^y(1)\right) = \int_{\Delta M_N^*(t^1)+y}^{\Delta M_N^*(t^1)+y+1} \frac{\exp\left[-x^2/(2\bar\sigma_1^2 t^1)\right]}{\sqrt{2\pi\bar\sigma_1^2 t^1}}\mathrm{d}x \ge \frac{\exp\left[-(\Delta M_N^*(t^1)+y+1)^2/(2\bar\sigma_1^2 t^1)\right]}{\sqrt{2\pi\bar\sigma_1^2 t^1}}, \tag{4.55}$$

where we bounded the integral from below by its smallest integrand and used that the volume of the domain of integration is one, By expanding the square in (4.55) and noting that $\frac{(y+1)^2}{2\bar\sigma_1^2 t^1} \to 0$ as $n\to\infty$, we can find a constant $C>0$ such that

$$\mathbb{P}\left(S_v^N(t^1) \in I_n^y(1)\right) \ge CN^{-2\lambda^1}e^{-y\frac{2\sqrt{\log(2)}}{\bar\sigma_1}}. \tag{4.56}$$

Analogously in case when $i = 2,\dots,m$ and taking into account that in all those integrals $y=0$,

$$\mathbb{P}\left(\Delta S_v^N(t^i) \in I_n^y(i)\right) \ge CN^{-2\Delta\lambda^i}. \tag{4.57}$$

Next, we lower bound the second probability in (4.53). For notational simplicity we only treat the case $i=1$, as the case when $i>1$ can be handled analogously. By subadditivity of measures and using (4.60),

$$\mathbb{P}(S_v^N(k) \in I_{k,n}(S_v^N(t^1)), 0<k<t^1) \ge 1 - 2\sum_{k=1}^{t^1-1}\mathbb{P}(S_v^N(k) - s_{k,n}(S_v^N(t^1)) > f_{k,n}) \tag{4.58}$$

To lower bound the probability in (4.58) note that by (4.42) and (4.52), for $0 \le k \le t^1$

$$\mathbb{E}\left[s_{k,n}(S_v^N(t^1))\left(S_v^N(k) - s_{k,n}(S_v^N(t^1))\right)\right] = \frac{\mathcal{I}_{\sigma^2}\left(\frac{k}{n}\right)}{\mathcal{I}_{\sigma^2}(1)}\mathbb{E}\left[S_v^N(t^1)\left(S_v^N(k) - s_{k,n}(S_v^N(t^1))\right)\right] = 0, \tag{4.59}$$

and so

$$\begin{aligned}\operatorname{Var}\left[S_v^N(k) - s_{k,n}(S_v^N(t^1))\right] &= \mathbb{E}\left[(S_v^N(k))^2 - S_v^N(k)s_{k,n}(S_v^N(t^1))\right] \\ &- \mathbb{E}\left[2s_{k,n}(S_v^N(t^1))\left(S_v^N(k) - s_{k,n}(S_v^N(t^1))\right)\right] = n\mathcal{I}_{\sigma^2}\left(\frac{k}{n}\right)\left(1 - \frac{\mathcal{I}_{\sigma^2}(k/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right).\end{aligned} \tag{4.60}$$

Also note that $\mathbb{E}\left[S_v^N(k) - s_{k,n}(S_v^N(t^1))\right] = 0$. Hence, by a standard Gaussian tail bound (cf. Lemma A.4) (4.58) is bounded from below by

$$1 - 2\sum_{k=1}^{t^1-1} C\exp\left[-\frac{1}{2}\frac{f_{k,n}^2}{\mathcal{I}_{\sigma^2}(k/n)n(1 - \frac{\mathcal{I}_{\sigma^2}(k/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)})}\right]. \tag{4.61}$$

By definition of the concave barrier in (4.43), we may split and bound the sum in (4.61) from above by

$$\sum_{k=1}^{t_1} C\exp\left[-\frac{1}{2}C_f^2\sigma_1^{2/3}k^{1/3}\right]\mathbb{1}_{\sigma_1\neq 0} + \sum_{k=t_1+1}^{t^1-1} C\exp\left[-\frac{1}{2}C_f^2\min_{i\in\{2,\dots,\pi_1\}:\sigma_i>0}(\sigma_i)^{2/3}(t^1-k)^{1/3}\right] < \frac{c}{2}, \tag{4.62}$$

where $0 < c < 1$ is a constant independent of n, if C_f is large enough. Inserting (4.62) into (4.58) gives

$$\mathbb{P}(S_v^N(k) \in I_{k,n}(S_v^N(t^1)), \forall 0 < k < t^1) > 1 - c = c_2 > 0. \tag{4.63}$$

Inserting (4.54), (4.63), (4.56) and (4.57) into (4.53) finishes the proof of the lower bound in (4.49). To get an upper bound in (4.53) we bound the second probability therein by 1. For the first probability, similarly as in (4.56) and (4.57), now picking the largest integrand in (4.55) instead of the smallest, yields for $1 \le i \le m$,

$$\mathbb{P}\left(\Delta S_v^N(t^i) \in I_n^y(i)\right) \le \frac{\exp\left[-(\Delta M_N^*(t^i))^2/(2\bar{\sigma}_i^2\Delta t^i)\right]}{\sqrt{2\pi\bar{\sigma}_1^2 t^1}} \le CN^{-2\Delta\lambda^i}\exp\left[-y\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}\mathbb{1}_{i=1}\right]. \tag{4.64}$$

Inserting this into (4.53), we obtain the upper bound in (4.49). □

Proof of Lemma 4.7: For $v, w \in V_N$ set $r(v,w) = n - \min(\lceil \log_2(d_N^\infty(v,w)+1)\rceil, n)$. By decomposing the second moment along $r(\cdot,\cdot)$,

$$\begin{aligned}\mathbb{E}\left[h_N^2(y)\right] &= \sum_{v,w\in V_N'} \mathbb{P}\left(C_v^{N,y}(t^m) \cap C_w^{N,y}(t^m)\right) = \sum_{k=0}^{n}\sum_{\substack{v,w\in V_N' \\ r(v,w)=k}} \mathbb{P}\left(C_v^{N,y}(t^m) \cap C_w^{N,y}(t^m)\right) \\ &\le \mathbb{E}\left[h_N(y)\right]^2 + \mathbb{E}\left[h_N(y)\right] + \sum_{k=1}^{n-1}\sum_{\substack{v,w\in V_N' \\ r(v,w)=k}} \mathbb{P}\left(C_v^{N,y}(t^m) \cap C_w^{N,y}(t^m)\right),\end{aligned} \tag{4.65}$$

where the first summand corresponds to $r(v,w) = n$ which implies independence between $C_v^{N,y}(t^m)$ and $C_w^{N,y}(t^m)$ and the second summand corresponds to the case when $r(v,w) = 0$, which means $v = w$. It remains to show an upper bound on the double sum, which we achieve by bounding each summand from above.

Fix $v, w \in V_N'$ with $r = r(v,w) \in \{1, \dots, n-1\}$. Let $i_0 \in \{1, \dots, m\}$ such that $t^{i_0 - 1} \le r < t^{i_0}$. A crucial observation is that by construction, the process $\left\{S_w^N(k + r(v,w)) - S_w^N\right\}_{k \ge 0}$ is independent of the sigma algebra generated by the processes $\left\{S_v^N(k)\right\}_{k\ge 0}$ and $\left\{S_w^N(k)\right\}_{k \le r(v,w)}$. Thus,

$$\mathbb{P}\left(C_v^{N,y}(t^m) \cap C_w^{N,y}(t^m)\right) \le \mathbb{P}\left(C_v^{N,y}(t^m)\right) \mathbb{P}\left(C_w^N(t^m) \setminus C_w^N(t^{i_0})\right) \\ \times \max_{x \in I_n^y(i_0)} \mathbb{P}\left(S_w^N(t^{i_0}) - S_w^N(r) \in x - I_{r,n}(x)\right). \quad (4.66)$$

The first probability on the right-hand side in (4.66) is equal to $|V_N'|^{-1}\mathbb{E}\left[h_N(y)\right]$, the second is using (4.54) and (4.64) bounded from above by a constant times $2^{-2(n-t^{i_0})}$. For fixed $v \in V_N'$, the number of points $w \in V_N'$ that satisfy $d_N^\infty(v,w) \in [2^k, 2^{k+1}]$ is bounded from above by $c_1 2^{2(k+1)} = c_1 2^{2(n-r)}$ with $c_1 > 0$ a constant which is uniform in N. Using this in (4.66), plugging this then into (4.65), and since $|V_N'|$ is the number of summands in v, there is a constant $C > 0$ uniformly in N, such that the double sum in (4.65) is smaller than

$$C\mathbb{E}\left[h_N(y)\right] \sum_{r=1}^{i_0 - 1} 2^{2(t^{i_0} - r)} \max_{\substack{x \in I_n^y(i_0) \\ v \in V_N}} \mathbb{P}\left(S_v^N(t^{i_0}) - S_v^N(r) \in x - I_{r,n}(x)\right). \quad (4.67)$$

Next, we bound the probability in (4.67). To simplify notation, we assume $i_0 = m = 1$ and $\mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar\sigma^2}$ on $(0,1)$. This means in particular $\lambda^1 = 1$, $t^1 = n$. The case when $\mathcal{I}_{\sigma^2} \equiv \mathcal{I}_{\bar\sigma}$ on $[\lambda^{i-1}, \lambda^i]$ is identical to the homogeneous version of the modified branching random walk with σ being constant, for which we may use the established bound in Bramson and Zeitouni (2012, Lemma 6.3) which in combination with the upper bound in Lemma 4.6 implies that there is a (σ dependent) constant $C > 0$ uniformly in N such that (4.67) is bounded from above by $C\mathbb{E}\left[h_N(y)\right]$. In the case when $\mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar\sigma^2}$, observe that for any $x \in I_n^y(1)$,

$$A_{r,n,x}^y := \mathbb{P}\left(S_v^N(t^1) - S_v^N(r) \in x - I_{r,n}(x)\right) = \int_{x - s_{r,n}(x) - f_{r,n}}^{x - s_{r,n}(x) + f_{r,n}} \frac{\exp\left[-\frac{1}{2}\frac{z^2}{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)n}\right]}{\sqrt{2\pi \mathcal{I}_{\sigma^2}(r/n, \lambda^1)n}} \mathrm{d}z \\ \le \frac{2f_{r,n}}{\sqrt{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)n}} \exp\left[-\frac{1}{2}\frac{(M_N^* + y - s_{r,n}(M_N^* + y) - f_{r,n})^2}{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)n}\right]. \quad (4.68)$$

where we upper bounded the integrand simply by its largest value which is attained when $x = M_N^* + y$ and while recalling (4.42) and (4.46). Using (4.42) and $\lambda^1 = 1$ as well as $\mathcal{I}_{\sigma^2}(1) = 1$, we bound from below the square in the exponential in (4.68) by

$$(M_N^* + y)^2 \left(1 - \mathcal{I}_{\sigma^2}(r/n)\right)^2 - 2f_{r,n}\mathcal{I}_{\sigma^2}(r/n, 1)(M_N^* + y) = (M_N^*)^2 \left(1 - \mathcal{I}_{\sigma^2}(r/n)\right)^2 \\ - 2f_{r,n}M_N^*\mathcal{I}_{\sigma^2}(r/n, 1) + \left(2M_N^* y + y^2\right)\left(1 - \mathcal{I}_{\sigma^2}(r/n)\right)^2 - 2yf_{r,n}\mathcal{I}_{\sigma^2}(r/n, 1). \quad (4.69)$$

Now writing out $M_N^* = M_{t^1}^*$ (see (4.41)), dropping the term involving y^2, and noting that

$$\frac{2yf_{r,n}\mathcal{I}_{\sigma^2}(r/n, 1)}{\mathcal{I}_{\sigma^2}(r/n, 1)n} \le C$$

for some $C > 0$ uniformly in n and y, and furthermore, using (4.41) as well as (4.43), we obtain that the exponent in (4.68) is bounded from above by

$$\begin{aligned}
&-2\log(2)t^1\mathcal{I}_{\sigma^2}(r/n,1) + \frac{\left(1+\frac{y}{4\sqrt{\log(2)t^1}}\right)}{2}\log(t^1)\mathcal{I}_{\sigma^2}(r/n,1) - 2y\sqrt{\log(2)}\mathcal{I}_{\sigma^2}(r/n,1)\\
&-\frac{\log(t^1)^2}{32\log(2)t^1}\mathcal{I}_{\sigma^2}(r/n,1) + \frac{C_f\sigma_1^{4/3}r^{2/3}}{(4\log(2))^{-1/2}\bar{\sigma}_1} = -2\log(2)t^1\mathcal{I}_{\sigma^2}(r/n,1) - 2y\sqrt{\log(2)}\mathcal{I}_{\sigma^2}(r/n,1)\\
&+\frac{1}{2}\log(t^1)\mathcal{I}_{\sigma^2}(r/n,1) - \frac{\log(t^1)}{t^1}\mathcal{I}_{\sigma^2}(r/n,1)\left(\frac{\log(t^1)-4\sqrt{\log(2)}y}{32\log(2)}\right) + \frac{C_f\sigma_1^{4/3}r^{2/3}}{(4\log(2))^{-1/2}\bar{\sigma}_1}. \qquad (4.70)
\end{aligned}$$

Using that $y \in [0, \sqrt{\log(N)}]$, there is a constant $C > 0$ such that

$$\left|\frac{\log(t^1)}{t^1}\mathcal{I}_{\sigma^2}(r/n,1)\left(\frac{\log(t^1)-4\sqrt{\log(2)}y}{32\log(2)}\right)\right| \le C. \qquad (4.71)$$

and hence (4.70) is smaller than

$$-2\log(2)t^1\mathcal{I}_{\sigma^2}(r/n,1) - 2y\sqrt{\log(2)}\mathcal{I}_{\sigma^2}(r/n,1) + \frac{1}{2}\log(t^1)\mathcal{I}_{\sigma^2}(r/n,1) + \frac{C_f\sigma_1^{4/3}r^{2/3}}{(4\log(2))^{-1/2}\bar{\sigma}_1} + C. \qquad (4.72)$$

Since $y >, 0$ (4.72) is bounded from above by

$$-2\log(2)t^1\mathcal{I}_{\sigma^2}(r/n,1) + \frac{1}{2}\log(t^1)\mathcal{I}_{\sigma^2}(r/n,1) + \frac{C_f\sigma_1^{4/3}r^{2/3}}{(4\log(2))^{-1/2}\bar{\sigma}_1} + C. \qquad (4.73)$$

In order to bound (4.73) from below and hence (4.68) from above, we distinguish the cases $0 < r \le t_1$ and $t_1 < k < t^1$.

Case 1: In the case $0 < r \le t_1$, we have $f_{r,n} = C_f(\sigma_1^2 r)^{2/3}$ and since $r \le t_1$, $\frac{1}{\lambda_1}\mathcal{I}_{\sigma^2}(\lambda_1) = \sigma_1^2 \in (0,1)$, where we use the assumption $\mathcal{I}_{\sigma^2}(s) < \hat{\mathcal{I}}_{\sigma^2}(s) = s$ for $s \in (0,1)$. Thus, there is $\eta_1 < 1$, independent of r and $n = t^1$, such that

$$\mathcal{I}_{\sigma^2}(r/n,1)t^1 = t^1 - t^1\mathcal{I}_{\sigma^2}(r/n) = t^1 - r\frac{1}{r/n}\mathcal{I}_{\sigma^2}(r/n) = t^1 - r\frac{1}{\lambda_1}\mathcal{I}_{\sigma^2}(\lambda_1) = t^1 - \eta_1 r. \qquad (4.74)$$

Similarly, we have $\mathcal{I}_{\sigma^2}(r/n,1) \ge 1 - \sigma_1^2\lambda_1$. Using this and (4.74) in (4.73) and plugging this into (4.68), we get possibly different constants $C, \tilde{C} > 0$,

$$A^y_{r,n,x} \le Cr^{2/3}\exp\left(\tilde{C}r^{2/3}\right)2^{-2(t^1-\eta_1 r)}\frac{\exp\left[\log(t^1)\mathcal{I}_{\sigma^2}(r/n,1)/2\right]}{\sqrt{\mathcal{I}_{\sigma^2}(r/n,1)t^1}}. \qquad (4.75)$$

Note that $\mathcal{I}_{\sigma^2}(r/n,1) < 1$ and hence $\exp\left[\log(t^1)\mathcal{I}_{\sigma^2}(r/n,1)/2\right]\Big/\sqrt{\mathcal{I}_{\sigma^2}(r/n,1)t^1} \le 1/\sqrt{\mathcal{I}_{\sigma^2}(r/n,1)}$. Thus,

$$A^y_{r,n,x} \le Cr^{2/3}2^{-2(t^1-\eta_1 r)}e^{\tilde{C}r^{2/3}} \le C2^{-2(t^1-\eta_1 r)+o(r)}. \qquad (4.76)$$

Case 2: In the case $t_1 < r < t^1$, we have $f_{r,n} = C_f(\mathcal{I}_{\sigma^2}(r/n,\lambda^1))^{2/3}$. This together with (4.73) inserted into (4.68) yields

$$A^y_{r,n,x} \le C2^{-2t^1\mathcal{I}_{\sigma^2}(r/n,1)}\frac{(\mathcal{I}_{\sigma^2}(r/n,1)n)^{2/3}}{\sqrt{\mathcal{I}_{\sigma^2}(r/n,1)t^1}}\exp\left[\log(t^1)\mathcal{I}_{\sigma^2}(r/n,1)/2 + \frac{C_f(\mathcal{I}_{\sigma^2}(r/n,1)n)^{2/3}}{(4\log(2))^{-1/2}\bar{\sigma}_1}\right]. \qquad (4.77)$$

As in the first case, $\exp\left[\log(t^1)\mathcal{I}_{\sigma^2}(r/n,1)/2\right] \Big/ \sqrt{\mathcal{I}_{\sigma^2}(r/n,1)t^1} \le 1/\sqrt{\mathcal{I}_{\sigma^2}(r/n,1)}$. Moreover, since $t^1 = \lambda^1 n = n$,

$$t^1\mathcal{I}_{\sigma^2}(r/n,1) = \frac{1}{1-r/n}\mathcal{I}_{\sigma^2}(r/n,1)(t^1-r). \tag{4.78}$$

Note that the latter fraction is a fraction of normalised integrals and note, by assumption, $\mathcal{I}_{\sigma^2}(s) < s$ for $s \in (0,1)$ and $\mathcal{I}_{\sigma^2}(1) = 1$. In combination with $0 < t_1 < r < t^1$, this implies that there exists $\eta_2 > 1$ uniformly in both r and n such that $\frac{1}{1-r/n}\mathcal{I}_{\sigma^2}(r/n,1) \ge \eta_2$. In particular, (4.78) is larger than $\eta_2(t^1-r)$. Plugging this fact into (4.77), we obtain

$$A^y_{r,n,x} \le C2^{-\eta_2(t^1-r)}(\mathcal{I}_{\sigma^2}(r/n,1)n)^{2/3}\exp\left[C_f(\mathcal{I}_{\sigma^2}(r/n,1)n)^{2/3}2\sqrt{\log(2)}/\bar{\sigma}_1\right] \le C2^{-2\eta_2(t^1-r)+o(t^1-r)}. \tag{4.79}$$

Inserting the bounds in (4.76) and (4.79) into (4.67) while observing that $(1-\eta_1) > 0$ as well as $(1-\eta_2) < 0$, it follows that the sum in (4.67) is smaller than

$$\sum_{r=1}^{t^1-1} 2^{2(t^1-r)} \max_{x\in I^y_n(1)} A^y_{r,n,x} \le C\left[\sum_{r=1}^{t_1} 2^{-2r(1-\eta_1)+o(r)} + \sum_{r=t_1+1}^{t^1-1} 2^{2(1-\eta_2)(t^1-r)+o(t^1-r)}\right] \le C_2, \tag{4.80}$$

for a constant $C_2 > 0$. Inserting (4.80) into (4.65) concludes the proof. □

Proof of Lemma 4.5: Since $V_{N'} \subset V_N$ and using the Paley-Zygmund inequality,

$$\mathbb{P}\left(\max_{v\in V_N} S^N_v > M^*_N + y\right) \ge \mathbb{P}\left(\max_{v\in V_{N'}} S^N_v > M^*_N + y\right) \ge \mathbb{P}(h_N(y) \ge 1) \ge \frac{(\mathbb{E}\left[h_N(y)\right])^2}{\mathbb{E}\left[h^2_N(y)\right]}. \tag{4.81}$$

Combining Lemma 4.7 with Lemma 4.6 shows that there are constants, $\tilde{C}, C, c > 0$, such that the above is bounded from below by

$$\frac{\mathbb{E}\left[h_N(y)\right]^2}{\mathbb{E}\left[h_N(y)\right]^2 + (1+\tilde{C})\mathbb{E}\left[h_N(y)\right])} \ge \frac{\mathbb{E}\left[h_N(y)\right]}{1+c} \ge Ce^{-y\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}}, \tag{4.82}$$

which concludes the proof. □

4.2. *Left tail estimate on the centred maximum.* In this subsection we prove an upper bound on the left tail of the centred maximum of the $(\sigma,\lambda)-$DGFF. We start with a bound on the left tail of $S_N - M^*_N$. Recall $V'_N = V_{N/2} + (N/4, N/4) \subset V_N$.

Lemma 4.8. *There exist constants $C, c > 0$, such that for all $N, n \in \mathbb{N}$, $N = 2^n \ge 4$ and $0 \le \lambda \le (\log\log N)^{2/3}$,*

$$\mathbb{P}\left(\max_{v\in V'_N} S^N_v \le M^*_N - \lambda\right) \le Ce^{-c\lambda}. \tag{4.83}$$

Proof: By Lemma 4.5, there are $\beta > 0$ and $\delta_0 \in (0,1)$ such that for all $4 \le N \in \mathbb{N}$,

$$\mathbb{P}\left(\max_{v\in V'_N} S^N_v \ge m_N/\sqrt{\log(2)} - \beta\right) \ge \delta_0. \tag{4.84}$$

Moreover, there is a $\kappa > 0$ such that for all $N, N' \in \mathbb{N}$ with $4 \le N' \le N$,

$$2\sqrt{\log(2)}\mathcal{I}_{\bar{\sigma}}(1)\log\left(\frac{N}{N'}\right) - \frac{3}{4\sqrt{\log(2)}}\sum_{j=1}^{m}\bar{\sigma}_j\log_+\left(\log\left(\frac{N}{N'}\right)\right) - \kappa \le M^*_N - M^*_{N'}$$
$$\le 2\sqrt{\log(2)}\mathcal{I}_{\bar{\sigma}}(1)\log\left(\frac{N}{N'}\right) + \kappa. \tag{4.85}$$

We pick $\lambda' = \frac{\lambda}{2}$, set $n' = \left\lfloor \log_2 N \exp\left[-\frac{1}{2\sqrt{\log(2)\mathcal{I}_{\bar\sigma}(1)}}(\lambda' - \beta - \kappa - 4)\right]\right\rfloor$ and $N' = 2^{n'}$. Without loss of generality we may assume $\lambda' - \beta - \kappa - 4 > 0$, since for smaller λ' we may just increase the constant $C > 0$ in (4.83). With this choice, we deduce from (4.85) that $M_N^* - M_{N'}^* \leq \lambda' - \beta$. Let

$$\mathcal{B} := \left\{(3iN', 3jN') + V'_{N'} \,:\, 1 \leq i, j < \frac{N}{3N'}\right\}, \tag{4.86}$$

be a collection of disjoint boxes $B \in V_N$. One should think of placing a box with lower left corner $V_{N'}$ at each point $(3iN', 3jN')$, for $1 \leq i, j \leq \frac{N}{N'}$, and for each of those a centred sub-box $V'_{N'} = N'/2 + (N'/4, N'/4) \subset V_{N'}$. Note that for two boxes $B, B' \in \mathcal{B}$ with $B \neq B'$, $\inf\{\|v - w\|_2 : v \in B, w \in B'\} \geq 2N'$. Next, set $s := n - n'$ and define, for $v \in V_N$,

$$S_v^{N,s} := S_v^N - S_v^N(s-1). \tag{4.87}$$

We note that the fields $\{S_v^{N,s}\}_{v\in B}$ and $\{S_w^{N,s}\}_{w\in B'}$ are by their definition in (4.87) and (3.5) independent copies of each other (as the boxes indexing their Gaussian increments are disjoint). We may also assume that n is so large that $s/n \leq \lambda_1/2$. For smaller n just increase the constant $C > 0$ in (4.83). In particular, the modified inhomogeneous branching random walks $S^{N,s}$ and S^N differ only in their first effective variance and scale parameters. In fact, the square of the first effective scale of $S^{N,s}$ is computed as

$$\frac{n\big((\mathcal{I}_{\sigma^2}(\lambda^1) - \mathcal{I}_{\sigma^2}(s/n)\big)}{\lambda^1 - s/n} = \bar\sigma_1^2 - \frac{s}{\lambda^1 n - s}(\bar\sigma_1^2 - \sigma_1^2), \tag{4.88}$$

and thus the first effective variance parameter is given by

$$\bar\sigma_1\sqrt{1 - \frac{s}{\lambda^1 n - s}(1 - \sigma_1^2/\bar\sigma_1^2)} = \bar\sigma_1 + o(1), \tag{4.89}$$

where the term $o(1)$ tends to 0 uniformly as $n \to \infty$. The first effective scale parameter of $S^{N,s}$ is simply $\lambda^1 - s/n$, where s/n tends to zero as $n \to \infty$. Lemma 4.5 and (4.84) imply that there is a constant $c_0 \in (0,1)$ such that

$$\mathbb{P}\left(\max_{v\in V'_N} S_v^{N,s} \geq m_N/\sqrt{\log(2)} - \beta\right) \geq c_0\delta_0. \tag{4.90}$$

Moreover, the number of boxes $B \in \mathcal{B}$ is bounded from below by

$$\frac{N}{3N'} - 1 \geq \frac{1}{6}\exp\left[\frac{1}{2\sqrt{\log(2)\mathcal{I}_{\bar\sigma}(1)}}(\lambda' - \beta - \kappa - 4)\right], \tag{4.91}$$

which is itself bounded from below by

$$-\tilde C + \tilde c\lambda', \tag{4.92}$$

for some $\tilde C, \tilde c > 0$, depending on δ_0, β, κ.

Next, by construction $\mathrm{Var}(S_v^N) = \mathrm{Var}(S_v^{N,s}) + n\mathcal{I}_{\sigma^2}\left(\frac{s}{n}\right)$. For $B \in \mathcal{B}$, $v \in B$, and $X \sim \mathcal{N}(0,1)$ being an independent standard Gaussian, define

$$\tilde S_v^N = S_v^{N'} + n\mathcal{I}_{\sigma^2}\left(\frac{s}{n}\right)X, \tag{4.93}$$

and observe that $\mathrm{Var}(S_v^N) = \mathrm{Var}(\tilde S_v^N)$. For $u, v \in \bigcup_{B\in\mathcal{B}} B$, we then have

$$\mathbb{E}\left[(\tilde S_v^N - \tilde S_w^N)^2\right] = \mathbb{E}\left[(S_v^{N,s} - S_w^{N,s})^2\right] \leq \mathbb{E}\left[(S_v^N - S_w^N)^2\right]. \tag{4.94}$$

An application of Corollary A.3 gives that, for any $t \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{v\in V_N'} S_v^N \le t\right) \le \mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} S_v^N \le t\right) \le \mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} \tilde{S}_v^N \le t\right)$$
$$\le \mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} S_v^{N,s} < M_N^* - t\right) + \mathbb{P}\left(X \le -n^{-1}t/\mathcal{I}_{\sigma^2}\left(\frac{s}{n}\right)\right), \quad (4.95)$$

where in the last step we used (4.93) and independence of $\{S_v^{N,s}\}_{v\in B}$ and X. Next, we upper bound the first probability on the right-hand side in (4.95) with $t = M_N^* - \lambda'$. Using $M_N^* - \lambda' \le M_{N'}^* - \beta$ and (4.90), one obtains, for each $B \in \mathcal{B}$,

$$\mathbb{P}\left(\max_{v\in B} S_v^{N,s} \ge M_N^* - \lambda'\right) \ge \mathbb{P}\left(\max_{v\in B} S_v^{N,s} \ge M_N^* - \beta\right) \ge c_0\delta_0. \quad (4.96)$$

By (4.96) and the independence of $\{S_v^{N,s}\}_{v\in B}$ and $\{S_v^{N,s}\}_{v\in B'}$, for $B, B' \in \mathcal{B}$, $B \neq B'$,

$$\mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} S_v^{N,s} < M_N^* - \lambda'\right) \le (1-c_0\delta_0)^{|\mathcal{B}|}. \quad (4.97)$$

As $c_0\delta_0 \in (0,1)$, by (4.92), there are constants, $C, c > 0$, such that

$$(1-c_0\delta_0)^{|\mathcal{B}|} \le Ce^{-c\lambda'}. \quad (4.98)$$

Using (4.95) and $n\mathcal{I}_{\sigma^2}(s/n) \ge s\min_{1\le i\le M}\sigma_i^2 > 0$, we can bound $\mathbb{P}\left(\max_{v\in V_N} S_v^N \le M_N^* - \lambda\right)$ from above by

$$\mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} S_v^{N,s} < M_N^* - \lambda'\right) + \mathbb{P}(X \le -\lambda'/s) \le Ce^{-c\lambda'}, \quad (4.99)$$

with a different constant $C > 0$ and where the last bound follows from (4.98) and since X is centred and has standard Gaussian tails (see also Lemma A.4). □

Lemma 4.8 allows us to deduce the upper bound on the left tail of the centred maximum.

Lemma 4.9. *There exist constants, $C, c > 0$, such that for all $N \in \mathbb{N}$ and $0 \le \lambda \le (\log\log N)^{2/3}$,*

$$\mathbb{P}\left(\max_{v\in V_N} \psi_v^N \le m_N - \lambda\right) \le Ce^{-c\lambda}. \quad (4.100)$$

Proof: Writing $\mathbb{P}(\max_{v\in V_N}\psi_v^N \le m_N - \lambda) = 1 - \mathbb{P}(\max_{v\in V_N}\psi_v^N > m_N - \lambda)$ and using (4.37) shows that there is a finite $\kappa \in \mathbb{N}$ and a collection $\{a_v : v \in V_{2^{-\kappa-2}N}'\}$ satisfying

$$a := \sup_{N\in\mathbb{N}}\sup_{v\in V_N'} a_v \le \alpha\kappa$$

for some $\alpha > 0$ (the latter follows using (4.27), (4.28) and (4.32)) so that for all $N \ge 2^{k+2}$

$$\begin{aligned}
&\mathbb{P}\Big(\max_{v\in V_N}\psi_v^N \le m_N - \lambda\Big) \\
&\quad\le 1 - \mathbb{P}\Big(\max_{v\in V_{2^{-\kappa-2}N}'} \sqrt{\log(2)}S_v^{2^{-\kappa-2}N} + a_vX > m_{2^{-\kappa-2}N} - \lambda'\Big) \\
&\quad= \mathbb{P}\Big(\max_{v\in V_{2^{-\kappa-2}N}'} \sqrt{\log(2)}S_v^{2^{-\kappa-2}N} + a_vX \le m_{2^{-\kappa-2}N} - \lambda'\Big) \\
&\quad\le \mathbb{P}\Big(\max_{v\in V_{2^{-\kappa-2}N}'} \sqrt{\log(2)}S_v^{2^{-\kappa-2}N} \le m_{2^{-\kappa-2}N} - \lambda'/2\Big) + \mathbb{P}\Big(X \le -\frac{\lambda'}{2\sqrt{\log 2a}}\Big), \quad (4.101)
\end{aligned}$$

where X is an independent standard Gaussian, $\lambda' = \lambda - (m_N - m_{2^{-\kappa-2}N})$ and $V_{2^{-\kappa-2}N}' \subset V_{2^{-\kappa-2}N}$ as in the previous Lemma 4.8. Since X is centred and has standard Gaussian tails (see also Lemma

A.4), it remains to bound the first probability in the last line of (4.101). By Lemma 4.8 and assuming $\lambda' \geq 0$, this is bounded from above by

$$Ce^{-\frac{c}{\sqrt{\log 2}}\lambda'} = \tilde{C}e^{-\tilde{c}\lambda}, \tag{4.102}$$

where $C, c > 0$ are the constants in Lemma 4.8 and where $\tilde{c} = \frac{c}{\sqrt{\log 2}}$ as well as $\tilde{C} = Ce^{4\tilde{c}(\kappa+2)}$ (using that $m_N - m_{2^{-\kappa-2}N} < 4\kappa + 8$). If $N \in \{0, \dots, 2^\kappa + 2\}$ we can enlarge the constant C, so that $\tilde{C}e^{-\tilde{c}\lambda} \geq 1$ for all λ as in the statement. In case when $0 \leq \lambda \leq m_N - m_{2^{-\kappa-2}N} \leq 4\kappa + 8$, possibly once more enlarging the constant $\tilde{C}$ allows to conclude the proof.

□

Having all the necessary ingredients, allows to prove Theorem 2.1:

Proof of Theorem 2.1: Lower and upper bound on the right-tail of the centred maximum in (2.6) follow from Proposition 4.2 and Proposition 4.1 respectively. The statement on the upper bound of the left-tail of the centred maximum, that is (2.7), is proved in Lemma 4.9, which allows to conclude the proof. □

Appendix A. Gaussian comparison

Theorem A.1 (Borell's inequality, Ledoux and Talagrand (2011, Lemma 3.1))**.** *Let T be compact and $\{X_t\}_{t\in T}$ a centred Gaussian process on T with continuous covariance. Further assume that almost surely, $X^* := \sup_{t\in T} X_t < \infty$. Then,*

$$\mathbb{E}[X^*] < \infty, \tag{A.1}$$

and

$$\mathbb{P}\left(|X^* - \mathbb{E}[X^*]| > x\right) \leq 2e^{-x^2/2\sigma_T^2}, \tag{A.2}$$

where $\sigma_T^2 := \max_{t\in T} \mathbb{E}[X_t^2]$.

Theorem A.2 (Slepian's Lemma, Ledoux and Talagrand (2011, Theorem 3.11))**.** *Let $T = \{1, \dots, n\}$ and X, Y be two centred Gaussian vectors. Assume that we have two subsets $A, B \subset T \times T$ satisfying*

$$\mathbb{E}[X_iX_j] \leq \mathbb{E}[Y_iY_j], \quad (i,j) \in A \tag{A.3}$$

$$\mathbb{E}[X_iX_j] \geq \mathbb{E}[Y_iY_j], \quad (i,j) \in B \tag{A.4}$$

$$\mathbb{E}[X_iX_j] = \mathbb{E}[Y_iY_j], \quad (i,j) \notin A \cup B. \tag{A.5}$$

Further, suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with at most exponential growth at infinity of f itself, as well as its first and second derivatives, and that

$$\partial_{ij} f \geq 0, \quad (i,j) \in A \tag{A.6}$$

$$\partial_{ij} f \leq 0, \quad (i,j) \in B. \tag{A.7}$$

Then,

$$\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]. \tag{A.8}$$

We use the following corollary of Slepian's Lemma:

Corollary A.3 (Bovier, 2017, Corollary 3.1)**.** *Let $T = \{1, \dots, n\}$ and X, Y be two centred Gaussian vectors. Assume that for all $i, j \in T$*

$$\mathbb{E}\left[X_i^2\right] = \mathbb{E}\left[Y_i^2\right], \quad \mathbb{E}\left[X_iX_j\right] \geq \mathbb{E}\left[Y_iY_j\right]. \tag{A.9}$$

We then have for any $x \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{i\in T} X_i > x\right) \leq \mathbb{P}\left(\max_{i\in T} Y_i > x\right). \tag{A.10}$$

In particular, $\mathbb{E}\left[\max_{i\in T} X_i\right] \leq \mathbb{E}\left[\max_{i\in T} Y_i\right]$.

The following standard Gaussian tail bound we include for convenience, as it is used at many places (see e.g. Gordon (1941, (10))).

Lemma A.4. *Let $X \sim \mathcal{N}(0,1)$ be a standard Gaussian. Then for any $x > 0$,*

$$\frac{x}{x^2+1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \leq \mathbb{P}\left(X \geq x\right) \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{A.11}$$

Appendix B. Covariance estimates

Recall (3.10) that is for $v, w \in V_N$,

$$b_N(v,w) = \sup\{\lambda \in [0,1] : [v]_\lambda^N \cap [w]_\lambda^N \neq \emptyset\} \tag{B.1}$$

and that for $\lambda \in [0,1]$, we write $\phi_v^N(\lambda) = \phi_v^N([v]_\lambda^N)$. We start with the proof of Lemma 1.3:

Proof: Let $\delta \in (0, 1/2)$. We may assume for the proof that $N \in \mathbb{N}$ is so large that $\sqrt{\log N} > 4$. For smaller N the statement is trivial. For any $v, w \in V_N^\delta$, $v \neq w$, by Ouimet (2017, Lemma A.5) with $\alpha = 0$, $\alpha' = 1$ and $\varrho = -\frac{\log \delta}{\log N}$ therein,

$$\mathbb{E}\left[\psi_v^N \psi_w^N\right] = \mathbb{E}\left[\left(\psi_v^N - \psi_v^N(0)\right)\psi_w^N\right] = \mathcal{I}_{\sigma^2}\left(b_N(v,w)\right)\log N + O\big(\sqrt{\log N}\big), \tag{B.2}$$

which concludes the proof. □

For a $B \subset V_N$, we set

$$\phi_v^N(B) := \mathbb{E}\left[\phi_v^N | \sigma\left(\phi_w^N : w \in \partial B \cup B^c\right)\right]. \tag{B.3}$$

Lemma B.1. *Let $\delta \in (0,1/2)$ and $N_0 \in \mathbb{N}$ so that $\min_{1\leq i\leq M} 2^{\frac{2}{\Delta\lambda_i}} \leq N_0$, $8 \leq \min_{1\leq i \leq M} N_0^{\Delta\lambda_i}$, as well as $N_0^{\lambda_1} > \delta^{-1}$. Let $N, M \in \mathbb{N}$, $N \geq N_0$, $0 < \lambda_i < \ldots < \lambda_M = 1$, $v, w \in V_N^\delta$. Then for any $0 \leq i,j \leq M$ with $\lambda_i, \lambda_j \leq b_N(v,w) + \log(2)/\log(N)$,*

$$\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\Delta\phi_w^N(\lambda_j)\right] = \Delta\lambda_i \log(N)\mathbb{1}_{i=j} + O\left(1\right). \tag{B.4}$$

Proof: For $v = w$ the statement is contained in Arguin and Ouimet (2016, Lemma A.2). Let us assume $v \neq w$ throughout the proof. Using symmetry in v, w, it suffices to consider the cases $i = j$, $i \geq j - 2$ and $j = i - 1$.

We start with the case $i = j$. As $\lambda_i \leq b_N(v,w) + \log(2)/\log(N)$,

$$[v]_{\lambda_i - \log(2)/\log(N)} \cap [w]_{\lambda_i - \log(2/\log(N))} \neq \emptyset,$$

which implies that $\|v - w\|_2/2 \leq \sqrt{2}N^{1-\lambda_i}$. We now choose boxes

$$B := \left(\frac{v+w}{2} + \left[-2N^{1-\lambda_i}, 2N^{1-\lambda_i}\right]^2\right) \cap V_N, \text{ and}$$

$$\tilde{B} := \left(\frac{v+w}{2} + \left[-1/4N^{1-\lambda_{i-1}}, 1/4N^{1-\lambda_{i-1}}\right]^2\right) \cap V_N, \tag{B.5}$$

which are centred at $(v+w)/2$ and of side lengths $4N^{1-\lambda_i}$ and $1/2N^{1-\lambda_{i-1}} \geq 2N^{1-b_N(v,w)}$ respectively (see Figure 4). This ensures the inclusions

$$B^c \subset [v]_{\lambda_i}^c,\ B^c \subset [w]_{\lambda_i}^c, \text{ and } \left([v]_{\lambda_{i-1}}^c \cup [w]_{\lambda_{i-1}}^c\right) \subset \tilde{B}^c \subset B^c, \tag{B.6}$$

and hence,

$$\mathcal{F}_{\partial B} \subset \mathcal{F}_{\partial [v]_{\lambda_i}},\ \mathcal{F}_{\partial B} \subset \mathcal{F}_{\partial [w]_{\lambda_i}}, \text{ and } \mathcal{F}_{\partial [v]_{\lambda_{i-1}}} \cup \mathcal{F}_{[w]_{\lambda_{i-1}}} \subset \mathcal{F}_{\partial \tilde{B}} \subset \mathcal{F}_{\partial B}. \tag{B.7}$$

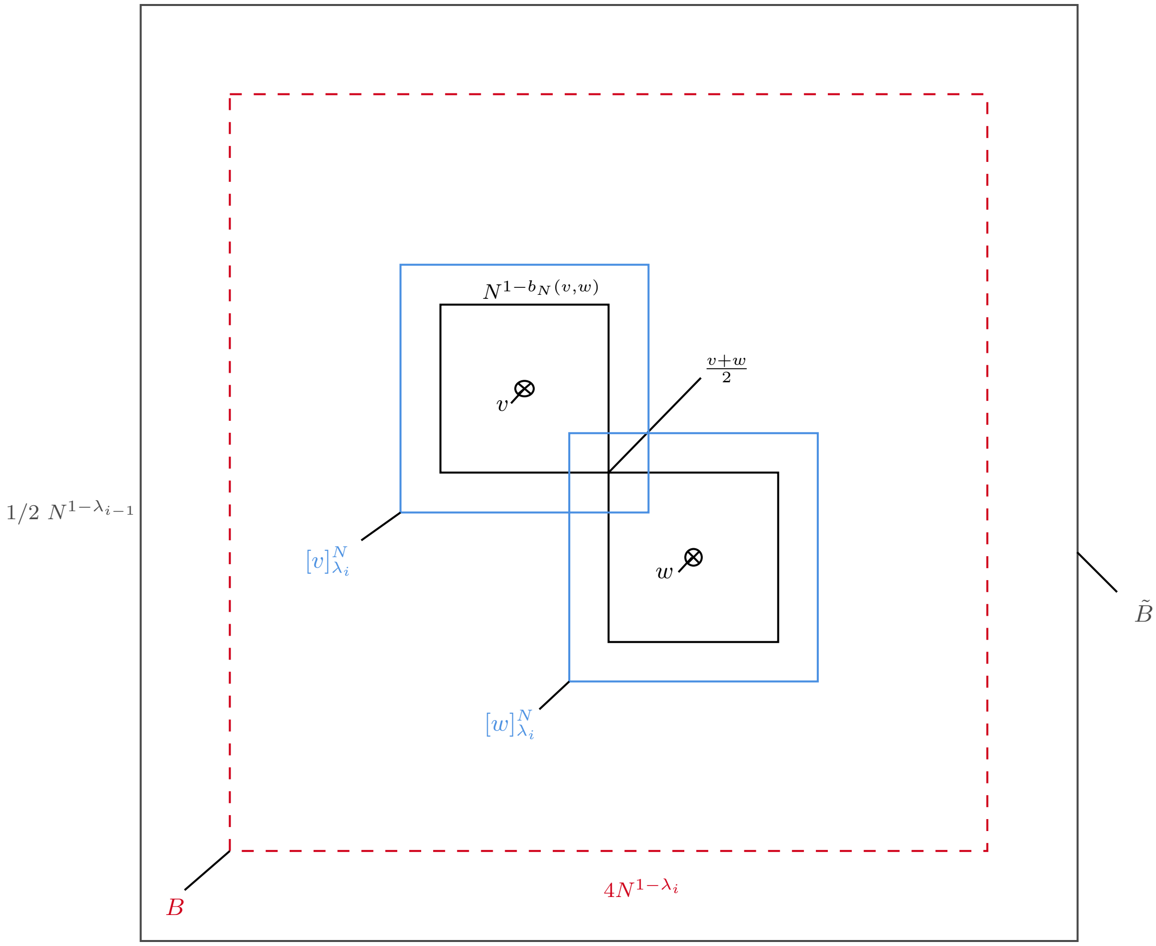

FIGURE B.4. An illustration for the boxes $[v]^N_{\lambda_i}, [w]^N_{\lambda_i}, B$ and $\tilde{B}$.

We write $\Delta\phi^N_v(B) = \phi^N_v(B) - \phi^N_v(\tilde{B})$ and compute,

$$\begin{aligned}\mathbb{E}\left[\Delta\phi^N_v(\lambda_i)\Delta\phi^N_w(\lambda_i)\right] &= \mathbb{E}\Big[\Big(\phi^N_v(\lambda_i) - \phi^N_v(B) + \Delta\phi^N_v(B) + \phi^N_v(\tilde{B}) - \phi^N_v(\lambda_{i-1})\Big) \\ &\quad\times\Big(\phi^N_w(\lambda_i) - \phi^N_w(B) + \Delta\phi^N_w(B) + \phi^N_w(\tilde{B}) - \phi^N_w(\lambda_{i-1})\Big)\Big] \\ &= \mathbb{E}\left[\Delta\phi^N_v(B)\Delta\phi^N_w(B)\right] && \text{(B.8)} \\ &\quad+\mathbb{E}\Big[\Delta\phi^N_v(B)\Big(\phi^N_w(\lambda_i) - \phi^N_w(B) + \phi^N_w(\tilde{B}) - \phi^N_w(\lambda_{i-1})\Big)\Big] && \text{(B.9)} \\ &\quad+\mathbb{E}\Big[\Delta\phi^N_w(B)\Big(\phi^N_v(\lambda_i) - \phi^N_v(B) + \phi^N_v(\tilde{B}) - \phi^N_v(\lambda_{i-1})\Big)\Big] && \text{(B.10)} \\ &\quad+\mathbb{E}\Big[\Big(\phi^N_v(\lambda_i) - \phi^N_v(B)\Big)\Big(\phi^N_w(\lambda_i) - \phi^N_w(B) + \phi^N_w(\tilde{B}) - \phi^N_w(\lambda_{i-1})\Big)\Big] && \text{(B.11)} \\ &\quad-\mathbb{E}\Big[\Big(\phi^N_v(\lambda_{i-1}) - \phi^N_v(\tilde{B})\Big)\Big(\phi^N_w(\lambda_i) - \phi^N_w(B) + \phi^N_w(\tilde{B}) - \phi^N_w(\lambda_{i-1})\Big)\Big]. && \text{(B.12)}\end{aligned}$$

Using the conditional covariance identity

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{A}\right]\mathbb{E}\left[Y|\mathcal{A}\right]\right] = \mathbb{E}\left[XY\right] - \mathbb{E}\left[\left(X - \mathbb{E}\left[X|\mathcal{A}\right]\right)\left(Y - \mathbb{E}\left[Y|\mathcal{A}\right]\right)\right], \tag{B.13}$$

with $X = \phi_v^N(1) - \phi_v^N(\tilde{B})$, $Y = \phi_w^N(1) - \phi_w^N(\tilde{B})$ and $\mathcal{A} = \sigma\left(\phi_u^N : u \notin B^o\right)$, noting that by the tower property of conditional expectation and (B.7) , $\Delta\phi_v^N(B) = \mathbb{E}\left[X|\mathcal{A}\right]$ as well as $\Delta\phi_w^N(B) = \mathbb{E}\left[Y|\mathcal{A}\right]$, along with the fact that by the Gibbs-Markov property (3.7) $\phi_v^N(1) - \phi_v^N(\tilde{B}) \overset{d}{=} \phi_v^{\tilde{B}}$, the term in (B.8) is equal to

$$\mathbb{E}\left[\phi_v^B \phi_w^B\right] - \mathbb{E}\left[\phi_v^{\tilde{B}} \phi_w^{\tilde{B}}\right], \tag{B.14}$$

which by using (1.3) equals

$$\log\left(N^{1-\lambda_i+\log(4)/\log(N)}\right) - \log(\|v-w\|_2 \vee 1) - \log\left(N^{1-\lambda_{i-1}-\log(2)/\log(N)}\right) + \log(\|v-w\|_2 \vee 1) + O(1)$$
$$= \Delta\lambda_i \log(N) + O(1). \tag{B.15}$$

For the remaining terms it suffices to show that each is at most of constant order. As the last two terms (B.11) and (B.12) can be estimated in the same way, we only deal with (B.11). Using the linearity of expectation and then the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left(\phi_v^N(\lambda_i) - \phi_v^N(B)\right)\left(\phi_w^N(\lambda_i) - \phi_w^N(B) - \phi_w^N(\lambda_{i-1}) + \phi_w^N(\tilde{B})\right)\right]$$
$$\leq \mathbb{E}\left[\left(\phi_v^N(\lambda_i) - \phi_v^N(B)\right)^2\right]^{1/2}\left(\mathbb{E}\left[\left(\phi_w^N(\lambda_i) - \phi_w^N(B)\right)^2\right]^{1/2} + \mathbb{E}\left[\left(\phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)^2\right]^{1/2}\right)$$
$$= c_1(c_2 + c_3) = O(1), \tag{B.16}$$

where $c_1, c_2, c_3 > 0$ are some constants that are uniform in N. The last line follows as argued for $(B.8)$ in (B.14), (B.15) (with $B = [w]_{\lambda_i}$, $\tilde{B} = B$ and $v = w$) and using (1.3) for each term in combination with the fact that by (B.7) and the tower property $\mathbb{E}\left[\phi_w^N(\lambda_i \phi_w^N(B))\right] = \mathbb{E}\left[\left(\phi_w^N(B)\right)^2\right]$. Next, we compute (B.9) and to do so, use the relation $\mathcal{F}_{\partial[w]_{\lambda_i}} \subset \mathcal{F}_{\partial B}$ as given in (B.7), along with the tower property for conditional expectations and the law of total expectation. Doing so yields

$$\mathbb{E}\left[\phi_v^N(B)\phi_w^N(\lambda_i)\right] = \mathbb{E}\left[\mathbb{E}\left[\phi_v^N(B)\phi_w^N|\mathcal{F}_{\partial[w]_{\lambda_i}}\right]\right] = \mathbb{E}\left[\phi_v^N(B)\phi_w^N\right]$$
$$= \mathbb{E}\left[\phi_v^N(B)(\phi_w^N - \phi_w^N(B) + \phi_w^N(B)\right] = \mathbb{E}\left[\phi_v^N(B)\phi_w^N(B)\right], \tag{B.17}$$

where in the last step we used independence between $\phi_v^N(B)$ and $\phi_w^N - \phi_w^N(B)$, which holds due to (3.7), the Gibbs-Markov property. Likewise, now instead using $\mathcal{F}_{\partial\tilde{B}} \subset \mathcal{F}_{\partial B} \subset \mathcal{F}_{\partial[w]_{\lambda_i}}$ from (B.7), we obtain $\mathbb{E}\left[\phi_v^N(\tilde{B})\phi_w^N(\lambda_i)\right] = \mathbb{E}\left[\phi_v^N(\tilde{B})\phi_w^N(\tilde{B})\right]$, $\mathbb{E}\left[\phi_v^N(\tilde{B})\phi_w^N(B)\right] = \mathbb{E}\left[\phi_v^N(\tilde{B})\phi_w^N(\tilde{B})\right]$. In combination with (B.17) it follows that $\mathbb{E}\left[\Delta\phi_v^N(B)\left(\phi_w^N(\lambda_i) - \phi_w^N(B)\right)\right] = 0$. Plugging this into (B.9) gives that (B.9) is equal to

$$\mathbb{E}\left[\Delta\phi_v^N(B)\phi_w^N(\tilde{B})\right] - \mathbb{E}\left[\Delta\phi_v^N(B)\phi_w^N(\lambda_{i-1})\right]. \tag{B.18}$$

Arguing as for (B.17), now using instead from (B.7) that $\mathcal{F}_{\partial\tilde{B}} \subset \mathcal{F}_{\partial B}$ for the first and $\mathcal{F}_{[w]_{\lambda_{i-1}}} \subset \mathcal{F}_{\partial\tilde{B}} \subset \mathcal{F}_{\partial B}$ for the second expectation, the last equation is equal to

$$\mathbb{E}\left[(\phi_v^N(\tilde{B}) - \phi_v^N(\tilde{B}))\phi_w^N(\tilde{B})\right] - \mathbb{E}\left[(\phi_v^N - \phi_v^N)\phi_w^N(\lambda_{i-1})\right] = 0. \tag{B.19}$$

In the same way, interchanging the roles of v, w we deduce that (B.10) equals 0.

Next, we deal with the case $i \geq j-2$. By assumption N is so large that $\min_{1\leq l\leq M} N^{\lambda_l - \lambda_{l-1}} \geq 8$. Using $\lambda_i, \lambda_j \leq b_N(v,w) + \log(2)/\log(N)$ and $\|v-w\|_2 \leq \sqrt{2}N^{1-b_N(v,w)}$, we deduce that $\mathcal{F}_{\partial[w]_{\lambda_j}} \cup \mathcal{F}_{\partial[w]_{\lambda_{j-1}}} \subset \mathcal{F}_{\partial[v]_{\lambda_{i-1}}} \subset \mathcal{F}_{\partial[v]_{\lambda_i}}$. In particular, using the tower property of conditional expectation, the law of total expectation and the Gibbs-Markov property as in (B.17), it holds that

$$\mathbb{E}\left[\phi_v^N(\lambda_{i-1})\Delta\phi_w^N(\lambda_j)\right] = \mathbb{E}\left[\mathbb{E}\left[\phi_v^N\Delta\phi_w^N(\lambda_j)\middle|\mathcal{F}_{\partial[v]_{\lambda_{i-1}}}\right]\right] = \mathbb{E}\left[\phi_v^N\Delta\phi_w^N(\lambda_j)\right], \tag{B.20}$$

and likewise

$$\mathbb{E}\left[\phi_v^N(\lambda_i)\Delta\phi_w^N(\lambda_j)\right] = \mathbb{E}\left[\mathbb{E}\left[\phi_v^N\Delta\phi_w^N(\lambda_j)\big|\mathcal{F}_{\partial[v]_{\lambda_i}}\right]\right] = \mathbb{E}\left[\phi_v^N\Delta\phi_w^N(\lambda_j)\right]. \tag{B.21}$$

In particular, $\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\Delta\phi_w^N(\lambda_j)\right] = 0$.

Finally we turn to the remaining case $j = i-1$. Note that by Ouimet (2017, Lemma A.1) and as $\lambda_{i-2} + \log(4)/\log(N) \le \lambda_{i-1} \le b_N(v,w) - \log(4)/\log(N)$,

$$\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\left(\phi_w^N(\lambda_{i-1} - \frac{\log(4)}{\log(N)}) - \phi_w^N(\lambda_{i-2})\right)\right] = 0. \tag{B.22}$$

Hence,

$$\begin{aligned}\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\Delta\phi_w^N(\lambda_{i-1})\right] &= \mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\left(\phi_w^N(\lambda_{i-1}) - \phi_w^N\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)\right] \\ &\quad + \mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\left(\phi_w^N\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right) - \phi_w^N(\lambda_{i-2})\right)\right] \\ &= \mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\left(\phi_w^N(\lambda_{i-1}) - \phi_w^N\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)\right],\end{aligned} \tag{B.23}$$

which by the Cauchy-Schwarz inequality is bounded from above by

$$\mathbb{E}\left[\left(\Delta\phi_v^N(\lambda_i)\right)^2\right]^{1/2}\mathbb{E}\left[\left(\phi_w^N(\lambda_{i-1}) - \phi_w^N\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)^2\right]^{1/2}. \tag{B.24}$$

By Arguin and Ouimet (2016, Lemma A.3) when taking therein the scale-inhomogeneous DGFF with parameter $\sigma \equiv 1$ which simply is the usual DGFF, we obtain that (B.24) is bounded from above by

$$\sqrt{\log N}\sqrt{\log(4)/\log(N)} + C, \tag{B.25}$$

where $C > 0$ is a constant that is uniform in N. Plugging this into (B.23) we obtain

$$\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\Delta\phi_w^N(\lambda_j)\right] \le C, \tag{B.26}$$

where the constant $C > 0$ has changed from line to line. This allows to conclude the proof. □

Proof of Lemma 3.3: For a proof of the statements $i.$ and $iii.$, we refer to Bramson and Zeitouni (2012, Lemma 2.2). We begin with the proof of $ii.$. Note that if $1 \le k < \log_{+,n}(d_\infty^N(v,w)+1)$, there are no boxes of size 2^k that cover both v and w. Thus, if $B, \tilde{B}$ are boxes such that $v \in B, w \in \tilde{B}$ as well as $v \notin \tilde{B}, w \notin B$, then the associated random variables $b_{k,B}, b_{k,\tilde{B}}$ are independent. And so, only random variables $b_{k,B}$ associated to boxes of size 2^k with $k > \lceil \log_2(d_\infty^N(v,w)+1)\rceil$ contribute to the covariance. For $v = (v_1, v_2)$, $w = (w_1, w_2)$ and $i = 1, 2$, we write $r_i(v,w) = \min(|v_i - w_i|, |v_i - w_i - N|, |v_i - w_i + N|)$. Using the fact that the number of common boxes for $v, w \in V_N$, $k > \lceil \log_2(d_\infty^N(v,w)+1)\rceil$, is given by $[2^k - r_1(v,w)][2^k - r_2(v,w)]$,

$$\begin{aligned}\mathbb{E}\left[S_v^N S_w^N\right] &= \sum_{k=\lceil \log_{+,n}(d_\infty^N(v,w))\rceil}^{n} 2^{-2k}\sigma^2(n-k,n)[2^k - r_1(v,w)][2^k - r_2(v,w)] \\ &\le C + \sum_{i=1}^{M}\sum_{k=\lceil \log_{+,n}(d_\infty^N(v,w))\rceil}^{n}\left[\left(1 - \frac{r_1(v,w)}{2^k} - \frac{r_2(v,w)}{2^k} + \frac{r_1(v,w)r_2(v,w)}{2^{2k}}\right)\mathbb{1}_{n-k\in(\lambda_{i-1}n,\lambda_i n]}\sigma_i^2\right],\end{aligned} \tag{B.27}$$

where $C > 0$ is some constant that may depend on the parameters (σ, λ) but is independent of n and v, w. For an upper bound of (B.27), note that since $a + b - ab \ge 0$ for $0 \le a, b \le 1$ and

$a = \frac{r_1(v,w)}{2^k}$, $b = \frac{r_1(v,w)}{2^k}$, $1 - a - b + ab \leq 1$ and thus, when at the same time adding and subtracting the "missing summands" for indices $k = 1 \dots, \lfloor \log_{+,n}(d_\infty^N(v,w)) \rfloor$,

$$\mathbb{E}\left[S_v^N S_w^N\right] \leq C + n\sum_{i=1}^M \sigma_i^2 \Delta\lambda_i - \sum_{i=1}^M \sigma_i^2 \left[n\Delta\lambda_i \mathbb{1}_{n - \lceil \log_{+,n}(d_\infty^N(v,w)) \rceil \leq \lambda_i n}\right] + \sum_{i=1}^M \sigma_i^2 \left[\lambda_i n - \left(n - \lceil \log_{+,n}(d_\infty^N(v,w)) \rceil\right)\right] \mathbb{1}_{\lambda_{i-1} n < n - \lceil \log_{+,n}(d_\infty^N(v,w)) \rceil < \lambda_i n}]. \tag{B.28}$$

Since $\log_{+,n}(d_\infty^N(v,w)) \leq \log_{+,n}(d^N(v,w)) \leq \log_{+,n}(d_\infty^N(v,w)) + 1$ and only summing over what is left after the subtraction in (B.28), this is bounded from above by

$$\begin{aligned} &C + 2\sum_{i=1}^M \sigma_i^2 + \sum_{i=1}^M \sigma_i^2 [n\Delta\lambda_i \mathbb{1}_{n - \lceil \log_{+,n}(d^N(v,w)) \rceil \geq \lambda_i n} \\ &\quad + \left((1 - \lambda_{i-1})n \lceil \log_{+,n}(d^N(v,w)) \rceil\right) \mathbb{1}_{\lambda_{i-1} n < n - \lceil \log_{+,n}(d^N(v,w)) \rceil < \lambda_i n}] \\ &\qquad = 2\sum_{i=1}^M \sigma_i^2 + n\mathcal{I}_{\sigma^2}\left(\frac{n - \lceil \log_{+,n}\left(d^N(v,w)\right) \rceil}{n}\right) + C, \end{aligned} \tag{B.29}$$

where the second sum we rewrote as an integral. For a lower bound in (B.27), note that $a + b - ab \leq a + b$ for $a, b \geq 0$ and a, b as before. Thus, we obtain

$$\begin{aligned} \mathbb{E}\left[S_v^N S_w^N\right] \geq &\sum_{k = \lceil \log_{+,n}(d_\infty^N(v,w)) \rceil}^{n} \sigma^2(n-k, n) - \max_{1 \leq i \leq M} \sigma_i^2 2^{-k+1} d_\infty^N(v,w) \\ &\geq \sum_{i=1}^M \sigma_i^2 \Big[n\Delta\lambda_i \mathbb{1}_{n - \lceil \log_{+,n}(d^N(v,w)) \rceil \geq \lambda_i n} + ((1 - \lambda_{i-1})n \\ &\quad - \lceil \log_{+,n}(d^N(v,w)) \rceil) \mathbb{1}_{\lambda_{i-1} n < n - \lceil \log_{+,n}(d^N(v,w)) \rceil < \lambda_i n}\Big] - C \\ &= n\mathcal{I}_{\sigma^2}\left(\frac{n - \lceil \log_{+,n}\left(d^N(v,w)\right) \rceil}{n}\right) - C, \end{aligned} \tag{B.30}$$

where in the last step we reformulated the sum as an integral and where $C > 0$ in (B.30) is a constant independent of N satisfying $C > 2\max_{1 \leq i \leq M} \sigma_i^2$, possibly changing from line to line and which bounds the second part of the sum in the first line of (B.30).

To prove the last statement *iv.*, we may assume that N is as in Lemma B.1, for smaller N just enlarge the constant C. By Lemma B.1 for $\lambda_i, \lambda_j \in [0, b_N + \log(2)/\log(N)]$ in combination with (3.11) and independence otherwise Arguin and Ouimet (2016, Lemma A.1), we have

$$\begin{aligned} &\mathbb{E}\left[\psi_v^{4N} \psi_w^{4N}\right] = \mathbb{E}\left[\sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j \Delta\phi_v^{4N}(\lambda_i) \Delta\phi_w^{4N}(\lambda_j)\right] \\ &= \sum_{i=1}^M \sigma_i^2 \mathbb{E}\Bigg[(\Delta\phi_v^{4N}(\lambda_i))^2 \mathbb{1}_{n - \lceil \log_{+,n}(\|v-w\|_2) \rceil \geq \lambda_i} + \\ &\qquad \left(\phi_v^{4N}\left(\frac{n - \lceil \log_{+,n}(\|v-w\|_2) \rceil}{n}\right) - \phi_v^{4N}(\lambda_{i-1})\right) \mathbb{1}_{\lambda_{i-1} n < n - \lceil \log_{+,n}(\|v-w\|_2) \rceil < \lambda_i n}\Bigg] + \\ &\quad O(1). \end{aligned} \tag{B.31}$$

Using Lemma B.1 once again, noting that the parameters $\lambda_i, \lambda_j \in [0, b_N(v,w) + \log(2)/\log(N)]$ in the Lemma can be chosen arbitrarily, shows that the latter equals

$$\begin{aligned}\sum_{i=1}^{M} \sigma_i^2 \Big[\log(N)\Delta\lambda_i \mathbb{1}_{n-\lceil \log_{+,n}\|v-w\|_2\rceil \geq \lambda_i n} + ((1-\lambda_{i-1})\log(N) \\ -\lceil \log_+ \|v-w\|_2\rceil \mathbb{1}_{\lambda_{i-1}n < n - \lceil \log_{+,n}\|v-w\|_2\rceil < \lambda_i n})\Big] + O(1) \\ = \log(2) n \mathcal{I}_{\sigma^2}\left(\frac{n - \lceil \log_{+,n}\|v-w\|_2\rceil}{n}\right) + O(1), \quad \text{(B.32)}\end{aligned}$$

where $O(1)$ is uniform in N. To conclude, we note that by the already proven part *ii.* of this Lemma 3.3 and the fact that for $v, w \in (N/4, N/4) + V_{N/2} \subset V_N$, $d^N(v,w) = \|v-w\|_2$, the last line in (B.32) differs from $\log(2)\mathbb{E}\left[S_v^N S_w^N\right]$ at most by an additive constant, which may be chosen uniformly in N. □

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