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Covariance identities and variance bounds for infinitely divisible random variables and their applications

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Abstract. In this article, we establish a general covariance identity for infinitely divisible distributions (IDD). Using this result, we derive Cacoullos type variance bounds for the IDD. Applications to some important distributions are discussed, in addition to the computation of variance bounds for certain posterior distributions. As another application, we derive the Stein-type identity for the IDD, which involves the Lévy measure. This result in turn is used to derive the Stein-type identity for the $CGMY$ distributions and the variance-gamma distributions (VGD). This approach, especially for the VGD is new and simpler, compared to the ones available in the literature. Finally, as another nontrivial application, we apply the covariance identity in deriving known and some new formulas for the weighted premium calculation principles ($WPCP$) and Gini coefficient for the IDD.

1. Introduction

In 1972, Charles Stein, while attempting a new proof of central limit theorem, invented a new identity for normal distribution and used it to study the normal approximation problems for the sums of dependent random variables (rvs) (see, Stein, 1972). The invention of similar identities for other probability distributions is well studied in the literature, with applications to limit theorems (Barman and Upadhye, 2024; Čekanavičius and Novak, 2022; Novak, 2021; Upadhye and Barman, 2022), runs (Upadhye et al., 2017; Vellaisamy, 2004), estimation theory (Brown et al., 2006; Zhu, 2022), functional inequalities (Barman and Upadhye, 2022), insurance (Psarrakos, 2022) and various other fields.

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In particular, Stein (1972) proved that for a normal random variable (rv) $X \sim \mathcal{N}(\mu, \sigma^2)$,

$$Cov(X-\mu, g(X)) = \sigma^2 \mathbb{E}[g'(X)], \tag{1.1}$$

where g is an absolutely continuous function with $\mathbb{E}[g'(X)] < \infty$.

Goldstein and Reinert (1997) generalized the Stein identity in (1.1) as follows. Let X be a real-valued rv with $\mathbb{E}(X) = 0$ and finite variance σ^2. Then, for any differentiable function g with $\mathbb{E}[Xg(X)] < \infty$, there exists a rv X^* having the unimodal probability density function (*pdf*)

$$f_{X^*}(x) = \frac{1}{\sigma^2}\mathbb{E}\left[X\mathbf{I}(X > x)\right]$$

such that

$$Cov(X, g(X)) = \sigma^2 \mathbb{E}\left[g'(X^*)\right]. \tag{1.2}$$

This fact is connected to the notion of zero bias distribution. In general, for a rv X with $\mathbb{E}[X] = 0$, and $Var(X) = \sigma^2$, the rv X^* that satisfies (1.2) is said to have X-zero bias distribution. We note that, from Goldstein and Reinert (1997, Lemma 2.1(ii)), X^* is supported on the closed convex hull of the support of X. Also, they applied the covariance identity (1.2) to obtain the rate of convergence in the central limit theorem. Further, they presented a nice application to dependent samples.
Later, Papadatos and Papathanasiou (2001) established a covariance identity, similar to (1.2), for an absolutely continuous rv X with non-zero $\mathbb{E}(X) < \infty$, given by

$$Cov(X, g(X)) = \sigma^2 \mathbb{E}[g'(Z^*)], \tag{1.3}$$

where the rv Z^* has the unimodal density

$$f_{Z^*}(x) = \frac{1}{\sigma^2}\int_x^\infty [t - \mathbb{E}(X)] f_X(t) dt. \tag{1.4}$$

From (1.4), it is clear that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z^* \sim \mathcal{N}(\mu, \sigma^2)$. Also, for an absolutely continuous rv X with density $f_X(x)$ and $\mathbb{E}[X] = 0$, the identities in (1.2) and (1.3) coincide. Moreover, the rv Z^* that satisfies (1.3) is said to have X-non-zero bias distribution, see Döbler (2017). Usually, the covariance identities are for X-zero bias distributions, see Daly et al. (2021). Observe also that the Stein-type covariance identity (1.3) depends on the *pdf* of X. The derivation of the identity becomes difficult, whenever $f_X(x)$ is not in the closed form.

Let X follow an infinitely divisible distribution (IDD), denoted by $IDD(\mu, \sigma^2, \nu)$, with parameters μ, σ^2 and the associated Lévy measure ν (see (2.1)). It is known that the *pdf* of some distributions belonging to IDD family may not be available in a closed form. Therefore, the derivation of Stein-type covariance identity is not possible for all the distributions belonging to IDD family using the approach given in Papadatos and Papathanasiou (2001). In 1998, Houdré et al. (1998) established a covariance representation for IDD via the generator approach. They applied it to study the association problems for IDD and also correlation inequalities. Recently, Arras and Houdré (2019) obtained a Stein-type covariance identity, like (1.1), for IDD with first finite moment using covariance representation given in Houdré et al. (1998, Proposition 2). More recently, Upadhye and Barman (2022, Theorem 3.1) established a Stein-type identity for IDD via the characteristic function (cf) approach.

In this article, we first obtain an extended Stein-type covariance identity in Theorem 3.1 for the IDD. The identity for the positive IDD is illustrated for the first time, in terms of cumulants, to the best of our knowledge. Although covariance identities for an IDD are well studied (see, Houdré, 2002 and Houdré et al., 1998), the covariance identity in Theorem 3.1 is useful. Using this result, we establish Cacoullos type variance bounds which are used to compute variance bounds of the parameters of the posterior distributions. Also, we obtain Stein identity for the variance-gamma distributions (VGD), using covariance identity and the associated Lévy measure. Our approach is new and much simpler. We finally apply the covariance identity in deriving some new formulas of weighted premium calculation principles ($WPCP$), that is, for $\mathbb{E}[Xw(X)]/\mathbb{E}[w(X)]$, in terms of Lévy measure. Also, the Gini coefficient for IDD is derived. The approach based on size-biasing is also discussed. Observe that the density of the IDD is not usually available in explicit form, and in such case the approach based on the Lévy measure is quite useful. Several well chosen examples illustrate our methodology, giving simpler derivation of the known results as well as some new formulas.

The article is organized as follows. In Section 2, we give the Lévy-Khintchine representation of $IDD(\mu, \sigma^2, \nu)$. Note that the class of $IDD(\mu, \sigma^2, \nu)$ is quite large and it includes many subclasses of distributions, such as compound Poisson distributions (CPD) and generalized tempered stable distributions ($GTSD$). Also, we discuss rather in detail some special cases of the $GTSD$, which include variance-gamma distributions (VGD), bilateral-gamma distributions (BGD) and Carr, Geman, Madan and Yor ($CGMY$) distributions. In Section 3, we derive an interesting covariance identity for $IDD(\mu, 0, \nu)$ (see Theorem 3.1), which depends on the Lévy measure ν. Using this result, we obtain new covariance identities for $GTSD$ and VGD. In Section 4, using our covariance identity, derived in Section 3, we derive Cacoullos-type variance bounds for $g(X)$, where $X \sim IDD(\mu, 0, \nu)$. Applications to gamma, Laplace and two-sided exponential distributions are considered. Also, as another application to the Bayesian inference, we find variance bounds for the posterior distributions. In Section 5, we consider two important applications of our results. The first one is concerned with obtaining Stein identities via the covariance identity. We consider the VGD and derive the Stein identity using the covariance identity and the Lévy measure of the VGD. It can be seen that our approach involves much simpler calculations, compared to the one used in Gaunt (2014) where the density approach is used, and it involves lengthy calculations and modified Bessel functions. Our second application is the calculation of $WPCP$. First, we obtain a result for the $WPCP$, when $X \sim IDD(\mu, 0, \nu)$. Using this result, we compute $WPCP$ for gamma, CPD, BGD, VGD, inverse Gaussian and $CGMY$ distributions. Finally, we compute also the Gini coefficient for $IDD(\mu, 0, \nu)$ and its applications to VGD and $CGMY$ distributions are discussed.

2. Notations and preliminary results

First we start with the definition of infinite divisibility. Hereafter, $X \stackrel{d}{=} Y$ means that X and Y are identically distributed.

Definition 2.1 (Sato, 1999, p.31)**.** A rv X is said to have IDD, if for every $n \in \mathbb{N}$,

$$X \stackrel{d}{=} X_{n,1} + X_{n,2} + \ldots + X_{n,n},$$

where $X_{n,1}, \ldots, X_{n,n}$ are independent and identically distributed (IID) rvs. The rv X_n is called n-th factor of X

Let $\mathbf{I}_A(.)$ denote the indicator function of the set A. Let X be an infinitely divisible rv. Then its Lévy-Khintchine representation of cf (see Sato, 1999) is given by

$$\phi_X(t) = \exp\left(it\mu_0 - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}} \left(e^{itu} - 1 - itu\mathbf{I}_{\{|u|\leq 1\}}(u)\right)\nu(du)\right),\ t \in \mathbb{R}, \tag{2.1}$$

where $\mu_0 \in \mathbb{R}$, $\sigma \geq 0$ and ν, the Lévy measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge u^2)\nu(du) < \infty$. Observe that if $\int_{\{|u|>1\}} u\nu(du) < \infty$, then (2.1) can be written as

$$\phi_X(t) = \exp\left(it\mu - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}} \left(e^{itu} - 1 - itu\right)\nu(du)\right),\ t \in \mathbb{R}, \tag{2.2}$$

where $\mu = \mu_0 + \int_{\{|u|>1\}} u\nu(du)$. For a rv X with cf (2.2), we write $X \sim IDD(\mu, \sigma^2, \nu)$.

If $X \sim IDD(\mu, \sigma^2, \nu)$, then its n-th cumulant is

$$C_n(X) := (-i)^n \frac{d^n}{dt^n} \log \phi_X(t)\Big|_{t=0}, \; n \geq 1. \tag{2.3}$$

Let $X \sim IDD(\mu, \sigma^2, \nu)$ have moments of arbitrary order. Then

$$C_1(X) = \mathbb{E}(X) = \mu, \tag{2.4}$$

$$C_2(X) = Var(X) = \sigma^2 + \int_{\mathbb{R}} u^2 \nu(du), \text{ and} \tag{2.5}$$

$$C_k(X) = \int_{\mathbb{R}} u^k \nu(du), \; k \geq 3. \tag{2.6}$$

The class of IDD is quite large and see Sato (1999) for more properties. Next, we discuss some important subclasses of IDD, which are useful in later sections.

2.1. *Compound Poisson distributions.* A rv X is said to have CPD if its cf (see Sato, 1999) is given by

$$\phi_{cp}(t) = \exp\left(\nu(\mathbb{R}) \int_{\mathbb{R}} (e^{itu} - 1)\nu_0(du)\right), \; t \in \mathbb{R}, \tag{2.7}$$

where the Lévy measure ν is finite (i.e., $\nu(\mathbb{R}) < \infty$) and ν_0 is the Borel probability measure on $\mathbb{R}$, defined by $\nu_0(du) = \nu(du)/\nu(\mathbb{R})$. We denote it by $CPD(\nu(\mathbb{R}), \nu_0)$. Let $\stackrel{d}{=}$ denotes equality in distribution. Note that $CPD(\nu(\mathbb{R}), \nu_0) \stackrel{d}{=} IDD(\mu, 0, \nu)$, where $\mu = \int_{\mathbb{R}} u\nu(du)$. Let δ_1 be the Dirac measure concentrated at 1. When $\lambda = \nu(\mathbb{R})$ and $\nu_0(du) = \delta_1(du)$, then $CPD(\lambda, \lambda\delta_1)$ has Poisson distribution with mean λ, denoted by $Poi(\lambda)$.

Next, we introduce $GTSD$ and discuss some of their relevant properties (see Cont and Tankov, 2004).

2.2. *Generalized tempered stable distributions.* A rv X is said to have $GTSD$ if its cf (see Cont and Tankov, 2004, Section 4.5) is given by

$$\phi_{\text{gts}}(t) = \exp\left(it\mu + \int_{\mathbb{R}} \left(e^{itu} - 1 - itu\right) \nu_{gts}(du)\right), \; t \in \mathbb{R}, \tag{2.8}$$

where the Lévy measure ν_{gts} is

$$\nu_{gts}(du) = \left(\frac{\alpha^+}{u^{1+\beta}} e^{-\lambda^+ u} \mathbf{I}_{(0,\infty)}(u) + \frac{\alpha^-}{|u|^{1+\beta}} e^{-\lambda^- |u|} \mathbf{I}_{(-\infty,0)}(u)\right) du, \tag{2.9}$$

and the parameters $\mu \in \mathbb{R}$, $\alpha^+, \lambda^+, \alpha^-, \lambda^- \in (0, \infty)$, and $\beta \in [0, 2)$. We denote it by $GTSD(\mu, \beta, \alpha^+, \lambda^+, \alpha^-, \lambda^-)$. Note that $GTSD(\mu, \beta, \alpha^+, \lambda^+, \alpha^-, \lambda^-) \stackrel{d}{=} IDD(\mu, 0, \nu_{gts})$.

Next, we list some special distributions of the $GTSD$ family.

2.2.1. *Bilateral-gamma distributions.* A rv X is said to have BGD if its cf (see Küchler and Tappe, 2008b) is given by

$$\phi_{bg}(t) = \exp\left\{\int_{\mathbb{R}} (e^{itu} - 1)\nu_{bg}(du)\right\}, \; t \in \mathbb{R}, \tag{2.10}$$

where the Lévy measure ν_{bg} is

$$\nu_{bg}(du) = \left(\frac{\alpha^+}{u}e^{-\lambda^+ u}\mathbf{I}_{(0,\infty)}(u) + \frac{\alpha^-}{|u|}e^{-\lambda^-|u|}\mathbf{I}_{(-\infty,0)}(u)\right)du, \tag{2.11}$$

and the parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^+ \in (0,\infty)$. We denote it by $BGD(\alpha^+, \lambda^+, \alpha^-, \lambda^-)$. Note that $BGD(1, \lambda^+, 1, \lambda^-)$ has two-sided exponential distribution (see Sato, 1999). Note also that $BGD(\alpha^+, \lambda^+, \alpha^-, \lambda^-) \stackrel{d}{=} GTSD(\mu, 0, \alpha^+, \lambda^+, \alpha^-, \lambda^-)$, where $\mu = \int_{\mathbb{R}} u\nu_{bg}(du)$.
Let now X_1 and X_2 be two independent gamma $Ga(\alpha^+, \lambda^+)$ and $Ga(\alpha^-, \lambda^-)$ rvs, where $Ga(\alpha, \lambda)$ has the density

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x},\ x > 0.$$

Then $X_1 - X_2 \sim BGD(\alpha^+, \lambda^+, \alpha^-, \lambda^-)$. It is known that the *pdf* of BGD is symmetric about the origin and its symmetric density (Küchler and Tappe, 2008b, Section 3) is given by

$$f_X(x) = \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\alpha^-}\Gamma(\alpha^+)\Gamma(\alpha^-)}\int_0^\infty v^{\alpha^- -1}\left(x + \frac{v}{\lambda^+ + \lambda^-}\right)^{\alpha^+ -1} e^{-v}dv, \tag{2.12}$$

$x \in \mathbb{R}\setminus\{0\}$. We refer the reader to Küchler and Tappe (2008a) for more properties of BGD.

2.2.2. *Variance-gamma distributions.* A rv X is said to have VGD if its cf (see Finlay and Seneta, 2008) is given by

$$\phi_{vg}(t) = e^{it\mu_0}\left(1 - it\left(\frac{1}{\lambda^+} - \frac{1}{\lambda^-}\right) + \frac{t^2}{\lambda^+\lambda^-}\right)^{-\alpha} \tag{2.13}$$

$$= \exp\left\{it\mu_0 + \int_{\mathbb{R}}(e^{itu} - 1)\nu_{vg}(du)\right\},\ t \in \mathbb{R}, \tag{2.14}$$

where the Lévy measure ν_{vg} is

$$\nu_{vg}(du) = \left(\frac{\alpha}{u}e^{-\lambda^+ u}\mathbf{I}_{(0,\infty)}(u) + \frac{\alpha}{|u|}e^{-\lambda^-|u|}\mathbf{I}_{(-\infty,0)}(u)\right)du, \tag{2.15}$$

and the parameters $\mu_0 \in \mathbb{R}$, $\alpha, \lambda^+, \lambda^- \in (0,\infty)$. We denote it by $VGD_0(\mu_0, \alpha, \lambda^+, \lambda^-)$.
Note that $VGD_0(\mu_0, \alpha, \lambda^+, \lambda^-) \stackrel{d}{=} GTSD(\mu, 0, \alpha, \lambda^+, \alpha, \lambda^-)$, where $\mu - \int_{\mathbb{R}} u\nu_{vg}(du) = \mu_0$. Note also that $VGD_0(\mu_0, 1, \frac{1}{\delta}, \frac{1}{\delta})$ has Laplace $La(\mu_0, \delta^2)$ distribution. If we set $\frac{1}{\lambda^+\lambda^-} = \sigma^2, \left(\frac{1}{\lambda^+} - \frac{1}{\lambda^-}\right) = 2\theta$ and $\alpha = \frac{r}{2}$ in (2.13), we get

$$\phi_{vg}(t) = e^{it\mu_0}\left(1 - i2\theta t + \sigma^2 t^2\right)^{-\frac{r}{2}}, \tag{2.16}$$

where $\mu_0, \theta \in \mathbb{R}$ and $\sigma^2, r \in (0,\infty)$, and it is denoted by $VGD_1(\mu_0, \sigma^2, r, \theta)$ (see Forrester, 2024, eqn. 1.10). It is known (Finlay and Seneta, 2008, Section 1) that the *pdf* of $VGD_1(\mu_0, \sigma^2, r, \theta)$ is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{\pi}\Gamma(\frac{r}{2})}e^{\frac{\theta}{\sigma^2}(x-\mu_0)}\left(\frac{|x-\mu_0|}{2\sqrt{\theta^2+\sigma^2}}\right)^{\frac{r-1}{2}} K_{\frac{r-1}{2}}\left(\frac{\sqrt{\theta^2+\sigma^2}}{\sigma^2}|x-\mu_0|\right), \tag{2.17}$$

where $x \in \mathbb{R}$, and $K_\zeta(x)$ is the modified Bessel function of the second kind, given by

$$K_\zeta(x) = \frac{1}{2}\int_0^\infty z^{\zeta-1}e^{-\frac{x}{2}(z+\frac{1}{z})}dz.$$

2.2.3. *CGMY distributions.* A rv X is said to have $CGMY$ distribution if its cf (see Küchler and Tappe, 2013) is given by

$$\phi_{cgmy}(t) = \exp\left\{\int_{\mathbb{R}} (e^{itu} - 1)\nu_{cgmy}(du)\right\},\ t \in \mathbb{R}, \tag{2.18}$$

where the Lévy measure ν_{cgmy} is

$$\nu_{cgmy}(du) = \left(\frac{\alpha}{u^{1+\beta}} e^{-\lambda^+ u}\mathbf{I}_{(0,\infty)}(u) + \frac{\alpha}{|u|^{1+\beta}} e^{-\lambda^-|u|}\mathbf{I}_{(-\infty,0)}(u)\right) du, \tag{2.19}$$

and the parameters $\alpha, \lambda^+, \lambda^- \in (0,\infty)$ and $\beta \in [0,1)$. We denote it by $CGMY(\alpha, \beta, \lambda^+, \lambda^-)$. Note that

$$CGMY(\alpha, \beta, \lambda^+, \lambda^-) \stackrel{d}{=} GTSD(\mu, \beta, \alpha^+, \lambda^+, \alpha^-) \stackrel{d}{=} IDD(\mu, 0, \nu_{cgmy}),$$

where $\mu = \int_{\mathbb{R}} u\nu_{cgmy}(du)$. It is known (Küchler and Tappe, 2013, Remark 7.11) that the *pdf* of $CGMY$ distributions can not be expressed in closed form. We refer the reader to Carr et al. (2002) for more properties of $CGMY$ distributions.

3. A generalized Stein-type covariance identity

In this section, we establish a covariance identity for infinitely divisible rvs. Before stating our result, let us define, for an integer $k \geq 1$,

$$\eta_k^+(u) = \int_u^\infty y^k \nu(dy),\ u > 0,\ \text{and}\ \eta_k^-(u) = -\int_{-\infty}^u y^k \nu(dy),\ u < 0, \tag{3.1}$$

where ν is the Lévy measure of the IDD (see (2.1)). Let

$$\eta_k(u) := \eta_k^+(u)\mathbf{I}_{(0,\infty)}(u) + \eta_k^-(u)\mathbf{I}_{(-\infty,0)}(u),\ u \in \mathbb{R}. \tag{3.2}$$

Let $X \sim IDD(\mu, 0, \nu)$ with cf $\phi_X(t)$ given in (2.2). Also let (X_s, Y_s) be an infinitely divisible random vector with joint cf

$$\phi_s(t,z) = \phi_X^{1-s}(t)\phi_X^{1-s}(z)\phi_X^s(t+z), \tag{3.3}$$

for all $t, z \in \mathbb{R}$ and $s \in [0,1]$. It can be seen that $\phi_{X_s}(t) = \phi_{Y_s}(t) = \phi_X(t)$, so that X_s and Y_s are identically, but not independently, distributed.

Theorem 3.1. *Let $X \sim IDD(\mu, 0, \nu)$ which has $\mathbb{E}(X^n) < \infty$, $n \in \mathbb{N}$, and the random vector (X_s, Y_s) have the cf given in (3.3). Then*

$$Cov(X^n, g(X)) = \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_{\mathbb{R}} g'(X_s + v)\eta_{n-k}(v)dv\right) ds, \tag{3.4}$$

where g is an absolutely continuous function with $\mathbb{E}\left(Y_s^k \int_{\mathbb{R}} g'(X_s+v)\eta_{n-k}(v)dv\right) < \infty$, for $0 \leq k \leq (n-1)$.

Proof: Recall first (see (2.2)) that the cf of X is

$$\phi_X(t) = \exp\left(it\mu + \int_{\mathbb{R}} \left(e^{itu} - 1 - itu\right)\nu(du)\right),\ t \in \mathbb{R}. \tag{3.5}$$

To prove our result, we use the covariance representation obtained in Houdré et al. (1998, Proposition 2) for infinitely divisible rvs. Let $\mathcal{G}^1 = \{f\}$ be the class of real-valued differentiable functions on $\mathbb{R}$ such that f and f' are bounded on any bounded interval, the set of discontinuity points of f' has zero probability, and with

$$\mathbb{E}\left(f^2(X) + \int_{\mathbb{R}} (f(X+u) - f(X))^2\nu(du)\right) < \infty.$$

Then, for $f, g \in \mathcal{G}^1$,

$$Cov(f(X), g(X)) = \int_0^1 \mathbb{E} \int_{\mathbb{R}} \Big(f(Y_s+u) - f(Y_s)\Big)\Big(g(X_s+u) - g(X_s)\Big)\nu(du)ds, \tag{3.6}$$

where (X_s, Y_s) has the cf given in (3.3). Let now $f(x) = x^n$ in (3.6), then we get

$$\begin{aligned}
Cov(X^n, g(X)) &= \int_0^1 \mathbb{E}\Big(\int_{\mathbb{R}} \Big((Y_s+u)^n - Y_s^n\Big)\Big(g(X_s+u) - g(X_s)\Big)\nu(du)\Big)ds \\
&= \int_0^1 \mathbb{E}\left(\int_{\mathbb{R}} \left(\sum_{k=0}^{n-1} \binom{n}{k} Y_s^k u^{n-k} \right) \Big(g(X_s+u) - g(X_s)\Big)\nu(du)\right) ds \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_0^{\infty} \Big(g(X_s+u) - g(X_s)\Big) u^{n-k}\nu(du)\right) ds \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_{-\infty}^{0} \Big(g(X_s+u) - g(X_s)\Big) u^{n-k}\nu(du)\right) ds \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_0^{\infty} \left(\int_0^u g'(X_s+v)dv\right) u^{n-k}\nu(du)\right) ds \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_{-\infty}^{0} \left(\int_u^0 g'(X_s+v)dv\right) (-u^{n-k})\nu(du)\right) ds \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_0^{\infty} \left(g'(X_s+v) \int_v^{\infty} u^{n-k}\nu(du)\right) dv\right) ds \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_{-\infty}^{0} \left(g'(X_s+v) \int_{-\infty}^{v} (-u^{n-k})\nu(du)\right) dv\right) ds \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\Bigg(Y_s^k \int_{\mathbb{R}} g'(X_s+v) \Big(\eta_{n-k}^{+}(v)\mathbf{I}_{(0,\infty)}(v) \\
&\qquad\qquad + \eta_{n-k}^{-}(v)\mathbf{I}_{(-\infty,0)}(v)\Big) dv\Bigg) ds \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_{\mathbb{R}} g'(X_s+v)\eta_{n-k}(v)dv\right) ds,
\end{aligned}$$

which proves the result. □

We next show that, the Proposition 3.8 of Arras and Houdré (2019) follows for the case $n = 1$.

Corollary 3.2. *Let $X \sim IDD(\mu, 0, \nu)$ have moments up to second order and Y_1, independent of X, have the density (see Arras and Houdré, 2019, p.25 and (2.5))*

$$f_1(y) = \frac{\eta_1(y)}{\int_{\mathbb{R}} y^2\nu(dy)} = \frac{\eta_1(y)}{Var(X)}, \; y \in \mathbb{R}. \tag{3.7}$$

Then

$$Cov(X, g(X)) = Var(X)\mathbb{E}\left(g'(X+Y_1)\right), \tag{3.8}$$

where g is such that $\mathbb{E}\left(g'(X+Y_1)\right) < \infty$.

Proof: From (3.4) and (2.5), we have

$$\begin{aligned} Cov(X, g(X)) &= \int_0^1 \mathbb{E}\left(\int_{\mathbb{R}} g'(X_s+v)\eta_1(v)dv\right) ds \\ &= \int_0^1 \mathbb{E}\left(\int_{\mathbb{R}} g'(X+v)\eta_1(v)dv\right) ds \quad (\text{since } X_s \stackrel{d}{=} X) \\ &= \mathbb{E}\left(\int_{\mathbb{R}} g'(X+v)\eta_1(v)dv\right) \\ &= Var(X)\mathbb{E}\left(g'(X+Y_1)\right), \end{aligned}$$

which proves the result. □

Remark 3.3. Let $Z^* = X + Y_1$. Then (3.8) becomes

$$Cov(X, g(X)) = Var(X)\mathbb{E}\left(g'(Z^*)\right),$$

which shows that Z^* has the X-non-zero bias distribution, see Döbler (2017) for more details for non-zero biasing.

Note that

$$\begin{aligned} \int_0^\infty \eta_k^+(u)du &= \int_0^\infty \int_u^\infty y^k \nu(dy)du \\ &= \int_0^\infty y^k\left(\int_0^y du\right)\nu(dy) \\ &= \int_0^\infty y^{k+1}\nu(dy). \end{aligned} \tag{3.9}$$

If $X \sim IDD(\mu, 0, \nu)$ is non-negative, then the k-th cumulant of X is $C_k(X) = \int_0^\infty u^k \nu(du)$, for some integer $k \geq 1$, and in that case

$$f_k(y) = \frac{\eta_k^+(y)}{C_{k+1}(X)}, \ y \in (0, \infty), \tag{3.10}$$

is a density on $\mathbb{R}_+ = (0, \infty)$.

We now have the following Corollary.

Corollary 3.4. *Let $X \sim IDD(\mu, 0, \nu)$ be non-negative which has $\mathbb{E}(X^n) < \infty$, $n \in \mathbb{N}$, and Y_k $(k \geq 1)$, independent of X, has the density as defined in (3.10). Also, let the random vector (X_s, Y_s) have the cf given in (3.3). Then*

$$Cov(X^n, g(X)) = \sum_{k=0}^{n-1}\binom{n}{k} C_{n-k+1}(X)\int_0^1 \mathbb{E}\left(Y_s^k g'(X_s + Y_{n-k})\right) ds, \tag{3.11}$$

where g is such that $\mathbb{E}\left(Y_s^k g'(X_s+Y_{n-k})\right) < \infty$, for $0 \leq k \leq (n-1)$.

Proof: Since $X \sim IDD(\mu, 0, \nu)$ is non-negative, the support of ν is in $(0, \infty)$. Also, from the assumption $\mathbb{E}(X^n) < \infty$, $n \in \mathbb{N}$, for all $v > 0$, $\eta_k^+(v) = \int_v^\infty y^k \nu(dy)$, $k \geq 1$. Hence from Theorem 3.1, we get

$$\begin{aligned} Cov(X^n, g(X)) &= \sum_{k=0}^{n-1}\binom{n}{k}\int_0^1 \mathbb{E}\left(Y_s^k \int_0^\infty g'(X_s+v)\eta_{n-k}^+(v)dv\right) ds \\ &= \sum_{k=0}^{n-1}\binom{n}{k} C_{n-k+1}(X)\int_0^1 \mathbb{E}\left(Y_s^k g'(X_s+Y_{n-k})\right) ds, \end{aligned}$$

which proves the result. □

Remark 3.5. (i) In Theorem 3.1 and Corollary 3.2, we assume that the rv X has higher order moments, which are satisfied for Poisson, negative binomial, gamma, Laplace, bilateral-gamma and variance-gamma distributions.

(ii) Note that the identity (3.8) does not hold for stable distributions which do not have finite second moment.

Next, we discuss some examples.

Example 3.6. (Generalized tempered stable distributions) Let $X \sim GTSD(\mu, \beta, \alpha^+, \lambda^+, \alpha^-, \lambda^-)$ with cf ϕ_{gts} given in (2.8). By Corollary 3.2, a covariance identity for X is

$$Cov(X, g(X)) = \Gamma(2-\beta)\left(\frac{\alpha^+}{(\lambda^+)^{2-\beta}} + \frac{\alpha^-}{(\lambda^-)^{2-\beta}}\right)\mathbb{E}\left(g'(X+Y_{g_1})\right), \tag{3.12}$$

where the rv Y_{g_1}, independent of X, has the density

$$f_1(y) = \frac{\alpha^+\left[\int_y^\infty u^{-\beta}e^{-\lambda^+ u}du\right]\mathbf{I}_{(0,\infty)}(y) + \alpha^-\left[\int_{-\infty}^y |u|^{-\beta}e^{-\lambda^-|u|}du\right]\mathbf{I}_{(-\infty,0)}(y)}{\Gamma(2-\beta)\left(\frac{\alpha^+}{(\lambda^+)^{2-\beta}} + \frac{\alpha^-}{(\lambda^-)^{2-\beta}}\right)},\ y \in \mathbb{R}. \tag{3.13}$$

Similarly, by Theorem 3.1, a covariance identity for X^2 is

$$\begin{aligned}Cov(X^2, g(X)) =& \sum_{k=0}^{1}\binom{2}{k}\int_0^1 \mathbb{E}\left(Y_s^k\int_{\mathbb{R}} g'(X_s+v)\eta_{2-k}(v)dv\right)ds\\ =&2\Gamma(2-\beta)\left(\frac{\alpha^+}{(\lambda^+)^{2-\beta}} + \frac{\alpha^-}{(\lambda^-)^{2-\beta}}\right)\int_0^1 \mathbb{E}\Big(Y_s g'(X_s+Y_{g_2})\Big)ds\\ &+\mathbb{E}\left(\int_{\mathbb{R}} g'(X+v)\eta_2(v)dv\right)\ \text{(since } X_s \overset{d}{=} X\text{)},\end{aligned} \tag{3.14}$$

where (X_s, Y_s) has the cf $\phi_s(t,z) = \phi_{\text{gts}}^{1-s}(t)\phi_{\text{gts}}^{1-s}(z)\phi_{\text{gts}}^{s}(t+z),\ t, z \in \mathbb{R}$ and Y_{g_2}, independent of (X_s, Y_s), has the density as in (3.13) and η_2 is given by

$$\eta_2(v) = \alpha^+\left[\int_v^\infty u^{-(\beta-1)}e^{-\lambda^+ u}du\right]\mathbf{I}_{(0,\infty)}(v) - \alpha^-\left[\int_{-\infty}^v |u|^{-(\beta-1)}e^{-\lambda^-|u|}du\right]\mathbf{I}_{(-\infty,0)}(v),\ v \in \mathbb{R}.$$

Example 3.7. (Variance-gamma distributions) Let $X \sim VGD_0(\mu_0, \alpha, \lambda^+, \lambda^-)$ with cf ϕ_{vg} given in (2.14). By Corollary 3.2, a covariance identity for X is

$$Cov(X, g(X)) = \frac{\alpha((\lambda^+)^2 + (\lambda^-)^2)}{(\lambda^+)^2(\lambda^-)^2}\mathbb{E}\left(g'(X+Y_{v_1})\right), \tag{3.15}$$

where the rv Y_{v_1} has the density

$$f_1(y) = \frac{\eta_1(y)}{\int_{\mathbb{R}} y^2\nu_{vg}(dy)} = \frac{(\lambda^+)^2(\lambda^-)^2}{(\lambda^+)^2 + (\lambda^-)^2}\left(\frac{e^{-\lambda^+ y}}{\lambda^+}\mathbf{I}_{(0,\infty)}(y) + \frac{e^{\lambda^- y}}{\lambda^-}\mathbf{I}_{(-\infty,0)}(y)\right). \tag{3.16}$$

Similarly, by Theorem 3.1, a covariance identity for X^2 is

$$\begin{aligned}Cov(X^2, g(X)) =& \sum_{k=0}^{1}\binom{2}{k}\int_0^1 \mathbb{E}\left(Y_s^k\int_{\mathbb{R}} g'(X_s+v)\eta_{2-k}(v)dv\right)ds\\ =&\frac{2\alpha\Big((\lambda^+)^2 + (\lambda^-)^2\Big)}{(\lambda^+)^2(\lambda^-)^2}\int_0^1 \mathbb{E}\Big(Y_s g'(X_s+Y_{v_2})\Big)ds\\ &+\mathbb{E}\left(\int_{\mathbb{R}} g'(X+v)\eta_2(v)dv\right)\ \text{(since } X_s \overset{d}{=} X\text{)},\end{aligned} \tag{3.17}$$

where (X_s, Y_s) has the cf $\phi_s(t,z) = \phi_{vg}^{1-s}(t)\phi_{vg}^{1-s}(z)\phi_{vg}^{s}(t+z),\ t,z \in \mathbb{R}$ and Y_{v_2}, independent of (X_s, Y_s), has the density as in (3.16) and η_2 is given by

$$\eta_2(v) = \alpha^+ \left[\int_v^\infty u e^{-\lambda^+ u} du\right] \mathbf{I}_{(0,\infty)}(v) - \alpha^- \left[\int_{-\infty}^v |u| e^{-\lambda^-|u|} du\right] \mathbf{I}_{(-\infty,0)}(v),\ v \in \mathbb{R}.$$

When $\alpha = 1$, $\lambda^+ = \lambda^- = \frac{1}{\delta}$, $\delta > 0$, we get the Lévy measure for $La(\mu_0, \delta^2)$ as

$$\nu_l(du) = \left(\frac{e^{-\frac{u}{\delta}}}{u}\mathbf{I}_{(0,\infty)}(u) + \frac{e^{-\frac{|u|}{\delta}}}{|u|}\mathbf{I}_{(-\infty,0)}(u)\right) du, \tag{3.18}$$

and a covariance identity for a rv $X \sim La(\mu_0, \delta^2)$ is

$$Cov(X, g(X)) = 2\delta^2 \mathbb{E}\left(g'(X + Y_\ell)\right).$$

Here the rv Y_ℓ has density

$$\begin{aligned} f_1(y) &= \frac{\eta_1(y)}{\int_{\mathbb{R}} u^2 \nu_l(du)} \\ &= \frac{1}{2\delta^2}\left(\left[\int_y^\infty u\nu_l(du)\right]\mathbf{I}_{(0,\infty)}(y) + \left[\int_{-\infty}^y (-u)\nu_l(du)\right]\mathbf{I}_{(-\infty,0)}(y)\right) \\ &= \frac{1}{2\delta}\left(e^{-\frac{y}{\delta}}\mathbf{I}_{(0,\infty)}(y) + e^{\frac{y}{\delta}}\mathbf{I}_{(-\infty,0)}(y)\right) \\ &= \frac{1}{2\delta}e^{-\frac{|y|}{\delta}}, \end{aligned} \tag{3.19}$$

which is $La(0, \delta^2)$ density.

Example 3.8. (Poisson distribution) Let $X \sim Poi(\lambda)$. Then $\nu(du) = \lambda\delta_1(du)$ and

$$\eta_1(y) = \int_y^\infty u\nu(du) = \lambda \int_y^\infty u\delta_1(du) = \lambda \mathbf{I}_{(0,1)}(y),$$

and $\int_0^\infty u^2\nu(du) = \lambda$. Using Corollary 3.2, a covariance identity for X is

$$Cov(X, g(X)) = \lambda \mathbb{E}\left(g'(X + Y_p)\right), \tag{3.20}$$

where the rv Y_p is independent of X and has the density

$$f_1(y) = \frac{\eta_1(y)}{\int_{\mathbb{R}} u^2\nu(du)} = \mathbf{I}_{(0,1)}(y),\ y > 0.$$

That is, Y_p follows uniform $U(0,1)$ distribution.
Also, by Corollary 3.4, a covariance identity for X^n, $n \geq 2$, is

$$\begin{aligned} Cov(X^n, g(X)) &= \sum_{k=0}^{n-1}\binom{n}{k} C_{n-k+1}(X) \int_0^1 \mathbb{E}\left(Y_s^k g'(X_s + U_{n-k})\right) ds \\ &= \lambda \sum_{k=0}^{n-1}\binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k g'(X_s + U_{n-k})\right) ds, \end{aligned} \tag{3.21}$$

where the last step follows as $C_k(X) = \lambda$, for all $(1 \leq k \leq n)$. Also, the rvs U_k's $(1 \leq k \leq n)$ are independent of (X_s, Y_s) and have the density

$$f_k(y) = \frac{\int_y^\infty u^k\nu(du)}{C_{k+1}(X)} = \int_y^\infty u^k\delta_1(du) = \mathbf{I}_{(0,1)}(y),\ y > 0,$$

which is uniform $U(0,1)$ density.

Note also that (X_s, Y_s) has the cf

$$\phi_s(t,z) = \exp\left(\lambda(1-s)(e^{it}-1) + \lambda(1-s)(e^{iz}-1) + \lambda s(e^{i(t+z)}-1)\right),\ t,z\in\mathbb{R}, \tag{3.22}$$

since $\phi_X(t) = \exp\left(\lambda(e^{it}-1)\right)$.

For example, when $n=2$ and $g(x)=x^2$, we get from (3.21)

$$\begin{aligned} Cov(X^2,X^2) &= 2\lambda\sum_{k=0}^{1}\binom{2}{k}\int_0^1 \mathbb{E}\left(Y_s^k\left(X_s+U_{2-k}\right)\right)ds \\ &= 2\lambda\int_0^1 \mathbb{E}\left(X_s+U_2\right)ds + 4\lambda\int_0^1 \mathbb{E}\left(Y_s\left(X_s+U_1\right)\right)ds \\ &= 2\lambda\int_0^1 \left(\lambda+\frac{1}{2}\right)ds + 4\lambda\int_0^1\left(\mathbb{E}(X_sY_s)+\mathbb{E}(X_s)\mathbb{E}(U_1)\right)ds \\ &= 4\lambda\int_0^1 \mathbb{E}\left(X_sY_s\right)ds + \lambda(4\lambda+1). \end{aligned} \tag{3.23}$$

From (3.22), we get

$$\mathbb{E}\left(X_sY_s\right) = (i)^{-2}\frac{\partial^2}{\partial t\partial z}\phi_s(t,z)\Big|_{t=z=0} = \lambda^2+\lambda s. \tag{3.24}$$

Using (3.24) in (3.23), we get

$$\begin{aligned} Cov(X^2,X^2) &= 4\lambda\int_0^1\left(\lambda^2+\lambda s\right)ds + \lambda(4\lambda+1) \\ &= \lambda(4\lambda^2+6\lambda+1), \end{aligned} \tag{3.25}$$

which can be checked through direct calculation also.

4. Variance bounds

In this section, we discuss the use of covariance identities to establish bounds on variance of function of infinitely divisible rv.

4.1. *Cacoullos type variance bounds.* Deriving upper bounds on the variance of a function of a rv has a long and rich history, starting in 1981 from the work of Chernoff (1981). Over the years, upper and lower variance bounds have received much interest in the statistics literature; see, for instance Cacoullos (1982); Daly et al. (2021) and the references therein. In the following result, we obtain two-sided bounds (Cacoullos type; see Cacoullos, 1982) on the variance of $g(X)$, where $X\sim IDD(\mu,0,\nu)$. Our approach is new and exploits the identity (3.8), which is based on non-zero-biased distribution of X. Recall first that an upper bound for variance of function of an infinitely divisible rv due to Chen (1985, Theorem 4.1) is as follows.

Lemma 4.1. *Let $X\sim IDD(\mu,0,\nu)$. Then*

$$Var(g(X)) \leq \mathbb{E}\int_{\mathbb{R}}\left(g(X+u)-g(X)\right)^2\nu(du), \tag{4.1}$$

where $g:\mathbb{R}\to\mathbb{R}$ is such that $\mathbb{E}\left(g^2(X)+\int_{\mathbb{R}}\left(g(X+u)-g(X)\right)^2\nu(du)\right)<\infty$.

Next, we establish a Cacoullos type variance bounds for an infinitely divisible rv.

Theorem 4.2. *Let $X \sim IDD(\mu, 0, \nu)$ and Y_1, independent of X, have density defined in (3.7). Then*

$$Var(X)\mathbb{E}^2\left[g'(X+Y_1)\right] \le Var(g(X)) \le Var(X)\mathbb{E}\left[g'(X+Y_1)\right]^2, \tag{4.2}$$

where g is an absolutely continuous function with $\mathbb{E}\left(g'(X+Y_1)\right) < \infty$.

Proof: From (3.8), we have

$$\begin{aligned}(Var(X))^2\mathbb{E}^2\left[g'(X+Y_1)\right] =&(C_2(X))^2\mathbb{E}^2\left[g'(X+Y_1)\right]\\ =&\left[Cov(X,g(X))\right]^2\\ &\le Var(X)Var(g(X)),\end{aligned} \tag{4.3}$$

by Cauchy-Schwarz inequality. That is,

$$Var(g(X)) \ge Var(X)\mathbb{E}^2\left[g'(X+Y_1)\right], \tag{4.4}$$

where Y_1 has the density given in (3.7).

Next, observe that

$$\begin{aligned}Var(X)\mathbb{E}\left[g'(X+Y_1)\right]^2 =&C_2(X)\mathbb{E}\left[g'(X+Y_1)\right]^2\\ =&C_2(X)\mathbb{E}\int_{\mathbb{R}}\left[g'(X+v)\right]^2 f_1(v)dv\\ =&\mathbb{E}\int_{\mathbb{R}}\left[g'(X+v)\right]^2 \eta_1(v)dv\\ =&\mathbb{E}\left[\int_0^\infty \left(g'(X+v)\right)^2\int_v^\infty u\nu(du)dv\right]\\ &+\mathbb{E}\left[\int_{-\infty}^0 \left(g'(X+v)\right)^2\int_{-\infty}^v (-u)\nu(du)dv\right]\\ =&\mathbb{E}\left[\int_0^\infty\left(\int_0^u \left(g'(X+v)\right)^2dv\right)u\nu(du)\right]\\ &+\mathbb{E}\left[\int_{-\infty}^0\left(\int_u^0 \left(g'(X+v)\right)^2dv\right)(-u)\nu(du)\right]\\ =&\mathbb{E}\int_{\mathbb{R}}\left(\int_0^u \left(g'(X+v)\right)^2dv\right)u\nu(du)\\ =&\mathbb{E}\int_{\mathbb{R}}\left(\int_0^u dv\int_0^u \left(g'(X+v)\right)^2dv\right)\nu(du)\\ \ge&\mathbb{E}\int_{\mathbb{R}}\left(\int_0^u g'(X+v)dv\right)^2\nu(du)\\ &\quad\text{(by Cauchy-Schwarz inequality)}\\ =&\mathbb{E}\int_{\mathbb{R}}(g(X+u)-g(X))^2\nu(du)\\ \ge&Var(g(X)) \text{ (using (4.1))}.\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), the desired conclusion follows. ☐

Some examples follow.

Example 4.3. (Gamma distribution) Let $X \sim Ga(a,b)$, $(a,b>0)$ with *pdf*

$$f_X(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, \ x > 0. \tag{4.6}$$

Then $X \sim IDD(\mu, 0, \nu_g)$, where $\nu_g(du) = a\frac{e^{-bu}}{u}\mathbf{I}_{(0,\infty)}(u)du$, $\mu = \int_{\mathbb{R}} u\nu_g(du)$ and $Y_1 \sim Ga(1,b)$. Hence, by (4.2), we get

$$\frac{a}{b^2}\mathbb{E}^2\left(g'(Z)\right) \le Var(g(X)) \le \frac{a}{b^2}\mathbb{E}\left(g'(Z)\right)^2, \tag{4.7}$$

where the rv $Z = X + Y_1 \sim Ga(a+1,b)$.

Observe that, the result in (4.7) coincides with the one given in Psarrakos (2022, equ. (4.1)).

The next two examples are not discussed in the literature.

Example 4.4. (Laplace distribution) Let $X \sim La(0,\delta^2)$, $\delta > 0$ with *pdf*

$$f_X(x) = \frac{1}{2\delta} e^{-\frac{|x|}{\delta}}, \ x \in \mathbb{R}.$$

Then $X \sim IDD(\mu, 0, \nu_l)$, where $\mu = \int_{\mathbb{R}} u\nu_l(du)$, ν_l is given in (3.18), and Y_1 has the density given in (3.19).
Hence, by (4.2), we get

$$2\delta^2\mathbb{E}^2\left(g'(Z)\right) \le Var(g(X)) \le 2\delta^2\mathbb{E}\left(g'(Z)\right)^2, \tag{4.8}$$

where the rv $Z = X + Y_1 \sim VGD_1(0,\delta^2,4,0)$.

Example 4.5. (Two-sided exponential distribution) Let X have a two-sided exponential distribution with parameters $a > 0$ and $b > 0$ with density

$$f_X(x) = \frac{ab}{a+b}\left(e^{-ax}\mathbf{I}_{(0,\infty)}(x) + e^{bx}\mathbf{I}_{(-\infty,0)}(x)\right), \ x \in \mathbb{R}. \tag{4.9}$$

Then $X \sim IDD(\mu, 0, \nu_e)$, where

$$\nu_e(du) = \left(\frac{e^{-au}}{u}\mathbf{I}_{(0,\infty)}(u) - \frac{e^{bu}}{u}\mathbf{I}_{(-\infty,0)}(u)\right), \tag{4.10}$$

$\mu = \int_{\mathbb{R}} u\nu_e(du)$ and Y_1 has density

$$f_1(y) = \frac{a^2b^2}{a^2+b^2}\left(\frac{e^{-ay}}{a}\mathbf{I}_{(0,\infty)}(y) + \frac{e^{by}}{b}\mathbf{I}_{(-\infty,0)}(y)\right), \ y \in \mathbb{R}. \tag{4.11}$$

Hence, by (4.2), we get

$$\frac{a^2+b^2}{a^2b^2}\mathbb{E}^2\left(g'(Z)\right) \le Var(g(X)) \le \frac{a^2+b^2}{a^2b^2}\mathbb{E}\left(g'(Z)\right)^2, \tag{4.12}$$

where the rv $Z = X + Y_1$ has density

$$f_Z(z) = \frac{a^3b^3}{(a+b)(a^2+b^2)}\left(\frac{ze^{-az}}{a}\mathbf{I}_{(0,\infty)}(z) + \frac{ze^{bz}}{b}\mathbf{I}_{(-\infty,0)}(z)\right), \ z \in \mathbb{R}. \tag{4.13}$$

4.2. *Application to posterior distributions.* Here, using Cacoullos type bounds, we demonstrate through examples how to compute variance bounds of the parameters of the posterior distributions. Recently, Daly et al. (2021, Example 2.3) discussed variance bounds within a Bayesian context for the Pearson family by using Stein kernels. In the Bayesian methodology, the parameters are treated as random variables. Let $(X_1, \dots, X_n)$ be a random sample of size n, where the joint density of Θ and X_i, $1 \leq i \leq n$, is $\pi(\theta, x)$. Assume the prior density of Θ is $\pi(\theta)$. Then we update from prior to posterior density as $\pi(\theta|x) = \kappa(x)\pi(\theta, x)$, where $\kappa(x)$ is a normalizing constant that depends only on the data $x = (x_1, \dots, x_n)$.

Example 4.6 (Gamma data, inference on scale, gamma prior)**.** Let $(X|\theta) \sim Ga(k, \theta)$ with $k, \theta > 0$, and $\Theta \sim Ga(a, b)$ with $a, b > 0$, the prior distribution. Then the posterior distribution is $(\Theta|x) \sim Ga(nk + a, n\bar{x} + b)$, where $x = (x_1, \dots, x_n)$. Hence, from (4.7), we get

$$\frac{(nk+a)}{(n\bar{x}+b)^2}\mathbb{E}^2\left(g'(Z)\right) \leq Var(g(\Theta|x)) \leq \frac{(nk+a)}{(n\bar{x}+b)^2}\mathbb{E}\left(g'(Z)\right)^2, \tag{4.14}$$

where the rv $Z \sim Ga(nk + a + 1, n\bar{x} + b)$.

For example, when $g(x) = x$, we get from (4.14),

$$Var(\Theta|x) = \frac{(nk+a)}{(n\bar{x}+b)^2}, \tag{4.15}$$

as expected.

From Example B.4 of Daly et al. (2021), we get

$$\frac{1}{(nk+a)}\mathbb{E}^2\left[(\Theta|x)g'(\Theta|x)\right] \leq Var(g(\Theta|x)) \leq \frac{1}{(n\bar{x}+b)}\mathbb{E}\left[(\Theta|x)g'(\Theta|x)^2\right]. \tag{4.16}$$

Recall that for any non-negative rv X with non-zero mean, let X^s denote the size-biased version of X defined by

$$\mathbb{E}\left[h(X^s)\right] = \frac{\mathbb{E}[Xh(X)]}{\mathbb{E}[X]}, \tag{4.17}$$

for any h for which the expectation on the right-hand side exists, see Arratia et al. (2019). Using Example 11.8 of Arratia et al. (2019), we have that $(\Theta|x)^s$ has the same distribution as Z. Hence, by setting $X = (\Theta|x)$, $X^s = Z$, and $h(x) = g'(x)$, from (4.17), we get

$$\mathbb{E}\left[g'(Z)\right] = \frac{\mathbb{E}[(\Theta|x)g'((\Theta|x))]}{\mathbb{E}[(\Theta|x)]}. \tag{4.18}$$

Using (4.18) in (4.16), we get

$$\frac{(nk+a)}{(n\bar{x}+b)^2}\mathbb{E}^2\left(g'(Z)\right) \leq Var(g(\Theta|x)) \leq \frac{(nk+a)}{(n\bar{x}+b)^2}\mathbb{E}\left(g'(Z)\right)^2, \tag{4.19}$$

which coincides with our upper and lower bounds given in (4.14).

Example 4.7 (Poisson data, inference on scale, gamma prior)**.** Let $(X|\theta) \sim Poi(\theta)$ with $\theta > 0$, and $\Theta \sim Ga(a, b)$, with $a, b > 0$, the prior distribution. Then the posterior distribution is $(\Theta|x) \sim Ga(n\bar{x} + a, n + b)$. Again from (4.7), we get

$$\frac{(n\bar{x}+a)}{(n+b)^2}\mathbb{E}^2\left(g'(Z)\right) \leq Var(g(\Theta|x)) \leq \frac{(n\bar{x}+a)}{(n+b)^2}\mathbb{E}\left(g'(Z)\right)^2, \tag{4.20}$$

where the rv $Z \sim Ga(n\bar{x} + a + 1, n + b)$.

For example, when $g(x) = x$, from (4.20), we get

$$Var(\Theta|x) = \frac{(n\bar{x}+a)}{(n+b)^2}, \tag{4.21}$$

as expected.
From Example B.6 of Daly et al. (2021), we get

$$\frac{1}{(n\bar{x}+a)}\mathbb{E}^2\left[(\Theta|x)g'(\Theta|x)\right] \leq Var(g(\Theta|x)) \leq \frac{1}{(n+b)}\mathbb{E}\left[(\Theta|x)g'(\Theta|x)^2\right]. \tag{4.22}$$

Using a size-biasing argument similar to Example 4.6, we can rewrite (4.22) as,

$$\frac{(n\bar{x}+a)}{(n+b)^2}\mathbb{E}^2\left(g'(Z)\right) \leq Var(g(\Theta|x)) \leq \frac{(n\bar{x}+a)}{(n+b)^2}\mathbb{E}\left(g'(Z)\right)^2,$$

where Z is the size-biased version of X. So, our bound given in (4.20) coincide with the bounds given in (4.22).

5. Applications

In this section, we discuss applications of our results.

5.1. *Stein identities via covariance identities.* Here, we demonstrate the use of covariance identities to provide Stein identities for several probability distributions. One of the important applications of such identities is to obtain explicit approximations, using Stein's method, for sums of independent rvs, but such applications are beyond the scope of this paper. Before stating our next results, we recall the Schwartz space of functions. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space defined by

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) : \lim_{|x|\to\infty} |x^m f^{(n)}(x)| = 0, \text{ for all } m,n \in \mathbb{N}_0 \right\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $f^{(n)}$ denotes the n-th derivative of f and $C^\infty(\mathbb{R})$ is the class of infinitely differentiable functions on $\mathbb{R}$ (see Stein and Shakarchi, 2003 for more details). This function space is useful for choosing an appropriate function space for Stein identities.

Our first example yields a Stein identity for $CGMY$ distributions.

Example 5.1. ($CGMY$ distributions) Let $X \sim CGMY(\alpha,\beta,\lambda^+,\lambda^-)$, whose Lévy measure ν_{cgmy} is given in (2.19). Recall that $CGMY(\alpha,\beta,\lambda^+,\lambda^-) \stackrel{d}{=} IDD(\mu,0,\nu_{cgmy})$, where $\mu = \int_{\mathbb{R}} u\nu_{cgmy}(du)$. Hence, by Corollary 3.2, we have

$$Cov(X,g(X)) = Var(X)\mathbb{E}\bigg(g'(X+Y_1)\bigg),$$

where Y_1 has density

$$f_1(v) = \frac{[\int_v^\infty u\nu_{cgmy}(du)]\mathbf{I}_{(0,\infty)}(v) + [\int_{-\infty}^v (-u)\nu_{cgmy}(du)]\mathbf{I}_{(-\infty,0)}(v)}{Var(X)},\ v \in \mathbb{R}.$$

So,

$$
\begin{aligned}
\mathbb{E}((X-\mu)\,g(X)) =& Cov(X, g(X)) \\
=& Var(X)\mathbb{E}\Big(g'(X+Y_1)\Big) \qquad (5.1)\\
=& \mathbb{E}\Big(\int_0^\infty g'(X+v)\int_v^\infty u\nu_{cgmy}(du)dv\Big) \\
&+ \mathbb{E}\Big(\int_{-\infty}^0 g'(X+v)\int_{-\infty}^v (-u)\nu_{cgmy}(du)dv\Big) \\
=& \mathbb{E}\Big(\int_0^\infty \Big(\int_0^u g'(X+v)dv\Big)\, u\nu_{cgmy}(du)\Big) \\
&+ \mathbb{E}\Big(\int_{-\infty}^0 \Big(\int_u^0 g'(X+v)dv\Big)\,(-u)\nu_{cgmy}(du)\Big) \\
=& \mathbb{E}\Big(\int_0^\infty (g(X+u)-g(X))\, u\nu_{cgmy}(du)\Big) \\
&+ \mathbb{E}\Big(\int_{-\infty}^0 (g(X+u)-g(X))\, u\nu_{cgmy}(du)\Big) \\
=& \mathbb{E}\left(\int_{\mathbb{R}} u(g(X+u)-g(X))\nu_{cgmy}(du)\right). \qquad (5.2)
\end{aligned}
$$

Since $\mu = \int_{\mathbb{R}} u\nu_{cgmy}(du)$, then (5.2) simplifies to

$$
\mathbb{E}\left(Xg(X) - \int_{\mathbb{R}} ug(X+u)\nu_{cgmy}(du)\right) = 0, \qquad (5.3)
$$

which is a Stein identity for $CGMY(\alpha, \beta, \lambda^+, \lambda^-)$ distribution.

Next, we obtain a Stein identity for VGD, using our approach.

Example 5.2. (Variance-gamma distributions) Let $X \sim VGD_0(\mu_0, \alpha, \lambda^+, \lambda^-)$ whose Lévy measure ν_{vg} is defined in (2.15). Let $g \in \mathcal{S}(\mathbb{R})$. Now, using the fact $\mathbb{E}(X) = \mu_0 + \int_{\mathbb{R}} u\nu_{vg}(du)$, and following steps similar to Example 5.1, it can be shown that

$$
\begin{aligned}
\mathbb{E}\Big((X-\mu_0)g(X)\Big) &= \mathbb{E}\left(\int_{\mathbb{R}} ug(X+u)\nu_{vg}(du)\right) \\
&= \alpha\mathbb{E}\left(\int_0^\infty \Big(e^{-\lambda^+ u}g(X+u) - e^{-\lambda^- u}g(X-u)\Big)\, du\right). \qquad (5.4)
\end{aligned}
$$

Applying integration by parts formula two times on the right hand side of (5.4) and rearranging the integrals, we have

$$
\begin{aligned}
\mathbb{E}\left((X-\mu_0)g(X)\right) =& \alpha\left(\frac{1}{\lambda^+} - \frac{1}{\lambda^-}\right)\mathbb{E}(g(X)) + \frac{2\alpha}{\lambda^+\lambda^-}\mathbb{E}(g'(X)) \\
&+ \alpha\left(\frac{1}{\lambda^+} - \frac{1}{\lambda^-}\right)\mathbb{E}\left(\int_0^\infty \Big(e^{-\lambda^+ u}g'(X+u) - e^{-\lambda^- u}g'(X-u)\Big)\, du\right) \\
&+ \frac{\alpha}{\lambda^+\lambda^-}\mathbb{E}\left(\int_0^\infty \Big(e^{-\lambda^+ u}g''(X+u) - e^{-\lambda^- u}g''(X-u)\Big)\, du\right). \qquad (5.5)
\end{aligned}
$$

Next taking g' and g'' in (5.4)

$$(a)\ \mathbb{E}\left((X-\mu_0)g'(X)\right) = \alpha\mathbb{E}\left(\int_0^\infty \left(e^{-\lambda^+ u}g'(X+u) - e^{-\lambda^- u}g'(X-u)\right)du\right), \tag{5.6a}$$

$$(b)\ \mathbb{E}\left((X-\mu_0)g''(X)\right) = \alpha\mathbb{E}\left(\int_0^\infty \left(e^{-\lambda^+ u}g''(X+u) - e^{-\lambda^- u}g''(X-u)\right)du\right), \tag{5.6b}$$

as $g \in \mathcal{S}(\mathbb{R})$. Now, using (5.6a) and (5.6b) on (5.5), we get

$$\begin{aligned}\mathbb{E}\left((X-\mu_0)g(X)\right) =&\mathbb{E}\left(\alpha\left(\frac{1}{\lambda^+}-\frac{1}{\lambda^-}\right)g(X)\right) + \frac{2\alpha}{\lambda^+\lambda^-}\mathbb{E}\left(g'(X)\right)\\ &+\left(\frac{1}{\lambda^+}-\frac{1}{\lambda^-}\right)\mathbb{E}\left((X-\mu_0)\,g'(X)\right) + \mathbb{E}\left(\frac{1}{\lambda^+\lambda^-}(X-\mu_0)g''(X)\right).\end{aligned} \tag{5.7}$$

Setting $\frac{1}{\lambda^+\lambda^-} = \sigma^2$, $\left(\frac{1}{\lambda^+}-\frac{1}{\lambda^-}\right) = 2\theta$, and $\alpha = \frac{r}{2}$, we get $\mathbb{E}(X) = \mu_0 + r\theta$ and $Var(X) = r(\sigma^2 + 2\theta^2)$. Also, for these parameters, (5.7) reduces to

$$\mathbb{E}\Big(\sigma^2(X-\mu_0)g''(X) + \left(\sigma^2 r + 2\theta(X-\mu_0)\right)g'(X) + \left(r\theta - (X-\mu_0)\right)g(X)\Big) = 0, \tag{5.8}$$

which is a Stein identity for $VGD_1(\mu_0, \sigma^2, r, \theta)$.

Observe that the Stein identity given by Gaunt (2014), and our Stein identity given in (5.8) match except for the choice of function space. Note that the derivation of Stein identity given in Gaunt (2014) is by modifying the density approach developed by Stein et al. (2004), and the density of VGD is usually written in terms of modified Bessel functions.

Finally, we obtain a Stein identity for BGD. This identity is, in a sense a generalization of the VGD Stein identity, since $VGD_0(0, \alpha, \lambda^+, \lambda^-) \stackrel{d}{=} BGD(\alpha, \lambda^+, \alpha, \lambda^-)$.

Example 5.3. (Bilateral-gamma distributions) Let $X \sim BGD(\alpha^+, \lambda^+, \alpha^-, \lambda^-)$ whose Lévy measure ν_{bg} is defined in (2.11). Then $BGD(\alpha^+, \lambda^+, \alpha^-, \lambda^-) \stackrel{d}{=} IDD(\mu, 0, \nu_{bg})$, where $\mu = \mathbb{E}(X) = \int_{\mathbb{R}} u\nu_{bg}(du)$ and the rv Y_1 has density

$$f_1(v) = \frac{[\int_v^\infty u\nu_{bg}(du)]\mathbf{I}_{(0,\infty)}(v) + [\int_{-\infty}^v (-u)\nu_{bg}(du)]\mathbf{I}_{(-\infty,0)}(v)}{Var(X)},\ v \in \mathbb{R}.$$

Let $g \in \mathcal{S}(\mathbb{R})$. Then by Corollary 3.2,

$$Cov(X, g(X)) = Var(X)\mathbb{E}\left(g'(X+Y_1)\right) = \mathbb{E}\bigg(\int_{\mathbb{R}} u(g(X+u) - g(X))\nu_{bg}(du)\bigg),$$

which can be written as

$$\begin{aligned}\mathbb{E}\Big(Xg(X)\Big) &= \mathbb{E}\left(\int_{\mathbb{R}} ug(X+u)\nu_{bg}(du)\right)\\ &= \mathbb{E}\int_0^\infty \left(\alpha^+ e^{-\lambda^+ u}g(X+u) - \alpha^- e^{-\lambda^- u}g(X-u)\right)du.\end{aligned} \tag{5.9}$$

Following steps similar to the derivation of Stein identity for $VGD_1(\mu_0, \sigma^2, r, \theta)$ in Example 5.2, we get a Stein identity for $BGD(\alpha^+, \lambda^+, \alpha^-, \lambda^-)$ as

$$\begin{aligned}\mathbb{E}\Big(&Xg''(X) + \left((\alpha^+ + \alpha^-) - (\lambda^+ - \lambda^-)\,X\right)g'(X)\\ &+ \left((\alpha^+\lambda^- - \alpha^-\lambda^+) - \lambda^+\lambda^- X\right)g(X)\Big) = 0,\ g \in \mathcal{S}(\mathbb{R}).\end{aligned} \tag{5.10}$$

The identity in (5.10) has been recently and independently derived by Forrester (2024, Proposition 1), using the density approach, and also for a different function space.

5.2. *Application to weighted premium calculation principles.* In this section, we apply Theorem 3.1 and Corollary 3.2 to premium calculation principles. First, we obtain a new formula of the $WPCP$ given in (5.12) for IDD, in terms of Lévy measure via Stein-type covariance identity given in Corollary 3.2 which is novel in our opinion. Recall first that the $WPCP$ due to Furman and Zitikis (2008) is as follows.

Definition 5.4. Let X be a loss rv of risk and $w : [0,\infty) \to [0,\infty)$ be a function such that $0 < \mathbb{E}(w(X)) < \infty$. Then the $WPCP$ is defined as

$$\mathcal{H}_w(X) = \frac{\mathbb{E}\left(Xw(X)\right)}{\mathbb{E}\left(w(X)\right)}, \tag{5.11}$$

which can also be rewritten as

$$\mathcal{H}_w(X) = \mathbb{E}(X) + \frac{Cov(X, w(X))}{\mathbb{E}(w(X))}. \tag{5.12}$$

Proposition 5.5. *Let $X \sim IDD(\mu, 0, \nu)$ with $\mu = \mathbb{E}(X) = \int_{\mathbb{R}} u\nu(du) < \infty$ and $w : [0,\infty) \to [0,\infty)$ be a function such that $0 < \mathbb{E}(w(X)) < \infty$. Then*

$$\mathcal{H}_w(X) = \frac{\mathbb{E}\big(\int_{\mathbb{R}} w(X+u)u\nu(du)\big)}{\mathbb{E}(w(X))}. \tag{5.13}$$

Proof: From Corollary 3.2, we get

$$Cov(X, g(X)) = Var(X)\mathbb{E}\left(g'(X+Y_1)\right),$$

where the random variables X and Y_1 are independent, with *pdf* $f_1(y)$ of Y_1 given in (3.7). Now

$$\begin{aligned}
Cov(X, g(X)) =& Var(X)\mathbb{E}\left(g'(X+Y_1)\right)\\
=& Var(X)\mathbb{E}\bigg(\int_{\mathbb{R}} g'(X+v)f_1(v)dv\bigg)\\
=& \mathbb{E}\bigg(\int_{\mathbb{R}} g'(X+v)\left(\eta_1^+(v)\mathbf{I}_{(0,\infty)}(v) + \eta_1^-(v)\mathbf{I}_{(-\infty,0)}(v)\right)dv\bigg)\\
=& \mathbb{E}\bigg(\int_0^\infty g'(X+v)\int_v^\infty u\nu(du)dv\bigg)\\
&+ \mathbb{E}\bigg(\int_{-\infty}^0 g'(X+v)\int_{-\infty}^v (-u)\nu(du)dv\bigg)\\
=& \mathbb{E}\bigg(\int_{\mathbb{R}} (g(X+u) - g(X))u\nu(du)\bigg),
\end{aligned} \tag{5.14}$$

where the last equality is followed by Fubini's theorem and adjusting the integrals. Replacing g by w in (5.14) and substituting it in (5.12), the desired conclusion follows.

□

Remark 5.6. Recently, Psarrakos (2022) proposed an alternate formula of $WPCP$ for a non-negative continuous rv using Stein-type covariance identity and its formula involves the *pdf* of X. Hence the derivation of $WPCP$ is not straightforward, whenever the *pdf* is not in closed form (or *pdf* is, in terms of, some special functions, for example, bilateral-gamma (Küchler and Tappe, 2008a) and variance-gamma distributions (Gaunt, 2014)).

Example 5.7. (Compound Poisson distributions). Let $X \sim CPD(\nu(\mathbb{R}), \nu_0)$. That is, $X \sim IDD(\mu, 0, \nu)$, where $\mu = \int_{\mathbb{R}} u\nu(du)$ and $\nu(du) = \nu(\mathbb{R})\nu_0(du)$. Then by (5.13), the $WPCP$ is given by

$$\mathcal{H}_w(X) = \frac{\nu(\mathbb{R})\mathbb{E}\left(\int_{\mathbb{R}} w(X+u)u\nu_0(du)\right)}{\mathbb{E}(w(X))}.$$

In particular, for $w(x) = e^{\kappa x}$, $\kappa > 0$, $\nu(\mathbb{R}) = \lambda$ and $\nu_0(du) = \delta_1(du)$, the $WPCP$ reduces to

$$\mathcal{H}_w(X) = \frac{\lambda\mathbb{E}\left(e^{\kappa(X+u)}u\delta_1(du)\right)}{\mathbb{E}(e^{\kappa X})} = \lambda\int_{\mathbb{R}} e^{\kappa u}u\delta_1(du) = \lambda e^{\kappa},$$

which is the Esscher principle for $Poi(\lambda)$ distribution, see Dickson (2017, p. 51).

Note that the covariance identity obtained in Psarrakos (2022) does not hold for discrete distributions. Therefore the $WPCP$ for Poisson distribution can not be retrieved from the covariance identity given in Psarrakos (2022).

The following examples are new to the literature and also establish the importance of our approach using Lévy measure.

Example 5.8 ($CGMY$ distributions)**.** Let $X \sim CGMY(\alpha, \beta, \lambda^+, \lambda^-)$ with non-zero mean. That is, $X \sim IDD(\mu, 0, \nu_{cgmy})$, where $\mu = \mathbb{E}(X) = \int_{\mathbb{R}} u\nu_{cgmy}(du)$ and the Lévy measure ν_{cgmy} is defined in (2.19). Also, $Var(X) = \int_{\mathbb{R}} u^2\nu_{cgmy}(du)$. Then by (5.13), the $WPCP$ is given by

$$\mathcal{H}_w(X) = \frac{\mathbb{E}\left(\int_{\mathbb{R}} w(X+u)u\nu_{cgmy}(du)\right)}{\mathbb{E}(w(X))}.$$

Let $w(x) = x$ (known as the modified variance principle, see Furman and Zitikis, 2008, Section 2), then

$$\mathcal{H}_w(X) = \alpha\left(\frac{\Gamma(1-\beta)}{(\lambda^+)^{1-\beta}} - \frac{\Gamma(1-\beta)}{(\lambda^-)^{1-\beta}}\right) + \frac{\frac{\Gamma(2-\beta)}{(\lambda^+)^{2-\beta}} + \frac{\Gamma(2-\beta)}{(\lambda^-)^{2-\beta}}}{\frac{\Gamma(1-\beta)}{(\lambda^+)^{1-\beta}} - \frac{\Gamma(1-\beta)}{(\lambda^-)^{1-\beta}}}.$$

Also, for $w(x) = e^{\kappa x}$, $\kappa > 0$, the Esscher principle, we have

$$\begin{aligned}\mathcal{H}_w(X) &= \frac{\mathbb{E}\left(\displaystyle\int_{\mathbb{R}} e^{\kappa(X+u)}u\nu_{cgmy}(du)\right)}{\mathbb{E}(e^{\kappa X})}\\ &= \frac{\mathbb{E}(e^{\kappa X})\left(\displaystyle\int_{\mathbb{R}} ue^{\kappa u}\nu_{cgmy}(du)\right)}{\mathbb{E}(e^{\kappa X})}\\ &= \alpha\int_0^\infty ue^{\kappa u}\frac{e^{-\lambda^+ u}}{u^{1+\beta}}du + \alpha\int_{-\infty}^0 ue^{\kappa u}\frac{e^{-\lambda^-|u|}}{|u|^{1+\beta}}du\\ &= \alpha\int_0^\infty\left(u^{-\beta}e^{-(\lambda^+-\kappa)u} - u^{-\beta}e^{-(\lambda^-+\kappa)u}\right)du\\ &= \frac{\alpha\Gamma(1-\beta)}{(\lambda^+-\kappa)^{1-\beta}} - \frac{\alpha\Gamma(1-\beta)}{(\lambda^-+\kappa)^{1-\beta}},\ 0 < \kappa < \lambda^+.\end{aligned}$$

Example 5.9 (Bilateral-gamma distributions)**.** Let $X \sim BGD(\alpha^+, \lambda^+, \alpha^-, \lambda^-)$ with non-zero mean. That is, $X \sim IDD(\mu, 0, \nu_{bg})$, where $\mu = \mathbb{E}(X) = \int_{\mathbb{R}} u\nu_{bg}(du)$ and the Lévy measure ν_{bg} is defined in (2.11). Also, $Var(X) = \int_{\mathbb{R}} u^2\nu_{bg}(du)$. Then by (5.13), the $WPCP$ is given by

$$\mathcal{H}_w(X) = \frac{\mathbb{E}\left(\int_{\mathbb{R}} w(X+u)u\nu_{bg}(du)\right)du}{\mathbb{E}(w(X))}.$$

For $w(x) = x$, the modified variance principle, we have

$$\mathcal{H}_w(X) = \left(\frac{\alpha^+}{\lambda^+} - \frac{\alpha^-}{\lambda^-}\right) + \frac{\left(\frac{\alpha^+}{(\lambda^+)^2} + \frac{\alpha^-}{(\lambda^-)^2}\right)}{\left(\frac{\alpha^+}{\lambda^+} - \frac{\alpha^-}{\lambda^-}\right)}. \tag{5.15}$$

When $\alpha^+ = \alpha^- = \alpha$, from (5.15), we get

$$\begin{aligned}\mathcal{H}_w(X) &= \alpha\left(\frac{1}{\lambda^+} - \frac{1}{\lambda^-}\right) + \frac{\frac{1}{(\lambda^+)^2} + \frac{1}{(\lambda^-)^2}}{\frac{1}{\lambda^+} - \frac{1}{\lambda^-}} \\ &= \left(\lambda^+\lambda^-\right)^{-1}\left(\lambda^- - \lambda^+\right)^{-1}\left(\alpha(\lambda^- - \lambda^+)^2 + (\lambda^+)^2 + (\lambda^-)^2\right),\end{aligned}$$

which is the modified variance principle of $VGD_0(0, \alpha, \lambda^+, \lambda^-)$ distribution.
Also, for $w(x) = e^{\kappa x},\ \kappa > 0$, the Esscher principle, we have

$$\begin{aligned}\mathcal{H}_w(X) &= \frac{\mathbb{E}\left(\int_{\mathbb{R}} e^{\kappa(X+u)} u\nu_{bg}(du)\right)}{\mathbb{E}(e^{\kappa X})} \qquad (5.16)\\ &= \frac{\mathbb{E}(e^{\kappa X})\left(\int_{\mathbb{R}} ue^{\kappa u}\nu_{bg}(du)\right)}{\mathbb{E}(e^{\kappa X})} \\ &= \alpha^+\int_0^\infty e^{-(\lambda^+-\kappa)u}du - \alpha^-\int_0^\infty e^{-(\lambda^-+\kappa)u}du \\ &= \frac{\alpha^+}{\lambda^+ - \kappa} - \frac{\alpha^-}{\lambda^- + \kappa},\ 0 < \kappa < \lambda^+. \qquad (5.17)\end{aligned}$$

Also, when $\alpha^+ = \alpha^- = \alpha$, from (5.17), we get

$$\mathcal{H}_w(X) = \alpha\left(\frac{1}{\lambda^+ - \kappa} - \frac{1}{\lambda^- + \kappa}\right),\ 0 < \kappa < \lambda^+,$$

which is the Esscher principle of $VGD_0(0, \alpha, \lambda^+, \lambda^-)$ distribution.

Example 5.10 (Inverse Gaussian distribution)**.** Let X have an inverse Gaussian distribution with parameters $\alpha, \lambda > 0$ with *pdf*

$$f_X(x) = \frac{\alpha}{x^{\frac{3}{2}}} e^{2\alpha\sqrt{\pi\lambda} - \lambda x - \pi\alpha^2/x},\ x > 0.$$

That is, $X \sim IDD(\mu, 0, \nu_{ig})$, where $\mu = \int_{\mathbb{R}} u\nu_{ig}(du)$ and the Lévy measure ν_{ig} is given by $\nu_{ig}(du) = \alpha e^{-\lambda u}u^{-\frac{3}{2}}\mathbf{I}_{(0,\infty)}(u)du$ (see Cont and Tankov, 2004, Section 4.4.2). Then by (5.13), the $WPCP$ is given by

$$\mathcal{H}_w(X) = \frac{\alpha\mathbb{E}(\int_0^\infty e^{-\lambda u}u^{-\frac{1}{2}}w(X+u)du)}{\mathbb{E}(w(X))}.$$

For $w(x) = x$, the modified variance principle, we have

$$\mathcal{H}_w(X) = \frac{1 + 2\alpha\sqrt{\pi}}{2\lambda}.$$

Also, for $w(x) = e^{\kappa x},\ \kappa > 0$, the Esscher principle, we have

$$\mathcal{H}_w(X) = \frac{\alpha\sqrt{\pi}}{(\lambda - \kappa)^{\frac{3}{2}}},\ 0 < \kappa < \lambda.$$

5.2.1. *Generalized $WPCP$.* Here, we apply Theorem 3.1 to generalized $WPCP$. We first recall the following definition; see Furman and Zitikis (2008, Section 4) for more details.

Definition 5.11. Let X be a rv of risk and $w : [0, \infty) \to [0, \infty)$ be an increasing function. Then for an increasing non-negative function g, the generalized $WPCP$ is defined as

$$\mathcal{H}_{g,w}(X) = \frac{\mathbb{E}(g(X)w(X))}{\mathbb{E}(w(X))}.$$

In particular, using (3.4) with $g(x) = x^n,\ n \geq 1$, we have the following result.

Proposition 5.12. *Let $X \sim IDD(\mu, 0, \nu)$ with $\mu = \int_{\mathbb{R}} u\nu(du) < \infty$ and $w : [0, \infty) \to [0, \infty)$ be a function such that $0 < \mathbb{E}(w(X)) < \infty$. Suppose that for $x \in [0, \infty)$, $g(x) = x^n$, $n \geq 1$. Then*

$$\mathcal{H}_{x^n,w}(X) = \mathbb{E}(X^n) + \frac{\sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_{\mathbb{R}} w'(X_s + v)\eta_{n-k}(v)dv \right) ds}{\mathbb{E}(w(X))}, \tag{5.18}$$

where the random vector (X_s, Y_s) have the cf given in (3.3) and η_k's are defined in (3.2).

Next, we discuss an example of generalized $WPCP$ using the above formula.

Example 5.13. (Gamma distribution) Let $X \sim Ga(a, b)$ $(a, b > 0)$ with *pdf*

$$f_X(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx},\ x > 0. \tag{5.19}$$

Let $\Gamma(s, x) = \int_s^\infty e^{-t} t^{x-1} dt$ $(s, x > 0)$ be the incomplete gamma function. Let $w : [0, \infty) \to [0, \infty)$ be a function such that $0 < \mathbb{E}(w(X)) < \infty$. Suppose also that for $x \in [0, \infty)$, $g(x) = x^n$, $n \geq 1$. Then by (5.18), the generalized $WPCP$ is given by

$$\mathcal{H}_{x^n,w}(X) = \frac{\Gamma(a+n)}{\Gamma(a)b^n} + \frac{\sum_{k=0}^{n-1} \binom{n}{k} \int_0^1 \mathbb{E}\left(Y_s^k \int_0^\infty w'(X_s + v)\eta_{n-k}(v)dv \right) ds}{\mathbb{E}(w(X))}, \tag{5.20}$$

where η_k's are defined by, for any $y > 0$,

$$\eta_k(y) = \frac{a}{b^k} \Gamma(by, k),\ k \geq 1.$$

Note also that (X_s, Y_s) has the cf

$$\phi_s(t, z) = \left(1 - \frac{it}{b}\right)^{-(1-s)a} \left(1 - \frac{iz}{b}\right)^{-(1-s)a} \left(1 - \frac{i(t+z)}{b}\right)^{-sa},\ t, z \in \mathbb{R}, \tag{5.21}$$

since $\phi_X(t) = \left(1 - \frac{it}{b}\right)^{-a}$.

Also, by Corollary 3.4, the formula in (5.20) can be seen as

$$\mathcal{H}_{x^n,w}(X) = \frac{\Gamma(a+n)}{\Gamma(a)b^n} + \frac{(n!)}{b^{n+1}} \frac{\sum_{k=0}^{n-1} \frac{ab^k}{k!} \int_0^1 \mathbb{E}\left(Y_s^k w'(X_s + Y_{n-k}) \right) ds}{\mathbb{E}(w(X))}, \tag{5.22}$$

where the rv Y_k's are independent of X_s and have the density, for any $y > 0$,

$$f_k(y) = \frac{b}{\Gamma(k+1)} \Gamma(by, k),\ k \geq 1.$$

For example, when $n = 1$ and $w(x) = e^{\kappa x},\ \kappa > 0$, from (5.22), we get

$$\mathcal{H}_{x,w}(X) = \frac{a}{b} + \frac{\kappa a}{b^2} \frac{\int_0^1 \mathbb{E}\left(e^{\kappa(X_s+Y_1)}\right) ds}{\mathbb{E}(e^{\kappa X})}, \tag{5.23}$$

where Y_1, independent of X_s, has the density

$$f_1(y) = be^{-by},\ y > 0.$$

Since $X_s \overset{d}{=} X$, the equation (5.23) reduces to

$$\begin{aligned}
\mathcal{H}_{x,w}(X) &= \frac{a}{b} + \frac{\kappa a}{b^2} \frac{\mathbb{E}\left(e^{\kappa(X+Y_1)}\right)}{\mathbb{E}(e^{\kappa X})} \\
&= \frac{a}{b} + \frac{\kappa a}{b^2}\mathbb{E}(e^{\kappa Y_1}) \\
&= \frac{a}{b} + \frac{\kappa a}{b}\int_0^\infty e^{-(b-\kappa)y} dy \\
&= \frac{a}{b} + \frac{\kappa a}{b(b-\kappa)} \\
&= \frac{a}{b-\kappa},\ 0 < \kappa < b,
\end{aligned} \tag{5.24}$$

which is the Esscher principle of gamma $Ga(a, b)$ distribution.

Observe that, for $a = b = 1$, the $WPCP$ in (5.24) matches exactly with the Esscher principle of $Ga(1, 1)$ distribution (exponential distribution with mean 1) (see Dickson, 2017, Example 3.1), where the author uses density function to derive the premium calculation principle.

Finally, as suggested by the anonymous referee, we discuss an alternate approach to $WPCP$, defined in (5.11), using size-biasing. Using the definition (4.17) we have

$$\mathcal{H}_w(X) = \frac{\mathbb{E}[Xw(X)]}{\mathbb{E}[w(X)]} = \frac{\mathbb{E}X}{\mathbb{E}[w(X)]}\mathbb{E}[w(X^s)].$$

It is shown in Arratia et al. (2019, Section 11) that for any non-negative and infinitely divisible rv X, its size-biased version X^s may be written in the form $X + Y$, where Y is independent of X. Using this fact, we have, for the choice of $w = e^{\kappa x},\ \kappa > 0$ (the Esscher principle),

$$\mathcal{H}_w(X) = \frac{\mathbb{E}[X}{\mathbb{E}[w(X)]}\mathbb{E}[w(X+Y)] = \frac{\mathbb{E}[X}{\mathbb{E}[e^{\kappa X}]}\mathbb{E}[e^{\kappa(X+Y)}] = \mathbb{E}[X]\mathbb{E}[e^{\kappa Y}],$$

since X and Y are independent. Then,

- if $X \sim Poi(\lambda)$, as discussed in Example 5.7, it is well-known that $Y = 1$ almost surely, so $\mathcal{H}_w(X) = \lambda e^{\kappa}$ as obtained there.
- If $X \sim Ga(a, b)$, as discussed towards the end of Example 5.13, then Y has an exponential distribution with rate b; see also Example 11.8 of Arratia et al. (2019). Hence, we again obtain $\mathcal{H}_w(X) = \frac{a}{b-\kappa},\ 0 < \kappa < b$.

5.3. *Application to Gini coefficient.* Here, we obtain an alternate formula of the Gini coefficient for IDD, in terms of Lévy measure via Stein-type covariance identity. Recall that the Gini coefficient is defined as follows (see Tziafetas, 1989).

Definition 5.14. Let X have the distribution $F_X(x)$. The Gini coefficient of the distribution function $F_X(x)$ is defined as

$$G = \frac{2}{\mathbb{E}(X)}Cov(X, F_X(X)). \tag{5.25}$$

Proposition 5.15. *Let $X \sim IDD(\mu, 0, \nu)$ be an absolutely continuous rv and $f_X(x)$ be the density of X. Then for $\mu > 0$,*

$$G = \frac{2}{\mu} Var(X)\mathbb{E}\left(f_X(X+Y_1)\right), \tag{5.26}$$

where Y_1 has the density defined in (3.7).

Proof: Let $F_X(x)$ be the distribution function of X. From Corollary 3.2, we get

$$Cov(X, g(X)) = Var(X)\mathbb{E}\left(g'(X+Y_1)\right). \tag{5.27}$$

Replacing g by F_X in (5.27) and using the fact that $F'_X(x) = f_X(x)$, we get

$$Cov(X, F_X(X)) = Var(X)\mathbb{E}\left(f_X(X+Y_1)\right). \tag{5.28}$$

Using (5.28) in (5.25), the desired conclusion follows. □

Next, we discuss some examples.

Example 5.16 (Two-sided exponential distribution)**.** Let X have a two-sided exponential distribution with parameters $a, b > 0$ with density $f_X(x)$ defined in (4.9). Then $X \sim IDD(\mu, 0, \nu_e)$, where ν_e is the Lévy measure defined in (4.10). Also,

$$\mathbb{E}(X) = \mu = \frac{b-a}{ab} \text{ and } Var(X) = \frac{a^2+b^2}{a^2b^2}.$$

Hence, by using (5.26), the Gini coefficient is given by

$$G = \frac{2(a^2+b^2)}{ab(b-a)}\mathbb{E}\left(f_X(X+Y_1)\right), \tag{5.29}$$

where Y_1 has density given in (4.11). Moreover, (5.29) can be written as

$$G = \frac{2(a^2+b^2)}{ab(b-a)}\mathbb{E}\left(f_X(Z)\right),$$

where $Z \stackrel{d}{=} X + Y_1$ and Z has density given in (4.13).

Example 5.17 (Bilateral gamma distribution)**.** Let $X \sim BGD(\alpha^+, \lambda^+, \alpha^-, \lambda^-)$ with density $f_X(x)$ given in (2.12). Then

$$\mathbb{E}(X) = \mu = \frac{\alpha^+}{\lambda^+} - \frac{\alpha^-}{\lambda^-} \text{ and } Var(X) = \frac{\alpha^+}{(\lambda^+)^2} + \frac{\alpha^-}{(\lambda^-)^2}.$$

So, using (5.26), the Gini coefficient is given by

$$G = \frac{2\left(\alpha^+(\lambda^-)^2 + \alpha^-(\lambda^+)^2\right)}{\lambda^+\lambda^-\left(\alpha^+\lambda^- - \alpha^-\lambda^+\right)}\mathbb{E}\left(f_X(X+Y_1)\right), \tag{5.30}$$

where Y_1 has density

$$f_1(y) = \frac{(\lambda^+\lambda^-)^2}{\alpha^+(\lambda^-)^2 + \alpha^-(\lambda^+)^2}\left(\alpha^+\frac{e^{-\lambda^+ y}}{\lambda^+}\mathbf{I}_{(0,\infty)}(y) + \alpha^-\frac{e^{\lambda^- y}}{\lambda^+}\mathbf{I}_{(-\infty,0)}(y)\right), \ y \in \mathbb{R}.$$

When $\alpha^+ = \alpha^- = \alpha$, from (5.30), we get

$$G = \frac{2\left((\lambda^-)^2 + (\lambda^+)^2\right)}{\lambda^+\lambda^-\left(\lambda^- - \lambda^+\right)}\mathbb{E}\left(f_X(X+Y_1)\right),$$

which is the Gini coefficient for $VGD_0(0, \alpha, \lambda^+, \lambda^-)$ distribution. Also, the rv Y_1 has density given in (3.16).

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References

Arras, B. and Houdré, C. *On Stein's method for infinitely divisible laws with finite first moment.* SpringerBriefs in Probability and Mathematical Statistics. Springer, Cham (2019). ISBN 978-3-030-15016-7; 978-3-030-15017-4. MR3931309.

Arratia, R., Goldstein, L., and Kochman, F. Size bias for one and all. *Probab. Surv.*, **16**, 1–61 (2019). MR3896143.

Barman, K. and Upadhye, N. S. On Brascamp-Lieb and Poincaré type inequalities for generalized tempered stable distribution. *Statist. Probab. Lett.*, **189**, Paper No. 109600, 8 (2022). MR4449614.

Barman, K. and Upadhye, N. S. On Stein factors for Laplace approximation and their application to random sums. *Statist. Probab. Lett.*, **206**, Paper No. 109996, 8 (2024). MR4671130.

Brown, L., DasGupta, A., Haff, L. R., and Strawderman, W. E. The heat equation and Stein's identity: connections, applications. *J. Statist. Plann. Inference*, **136** (7), 2254–2278 (2006). MR2235057.

Cacoullos, T. On upper and lower bounds for the variance of a function of a random variable. *Ann. Probab.*, **10** (3), 799–809 (1982). MR659549.

Carr, P., Geman, H., Madan, D., and Yor, M. The Fine Structure of Asset Returns: An Empirical Investigation. *J. Bus.*, **75** (2), 305–332 (2002). DOI: 10.1086/338705.

Čekanavičius, V. and Novak, S. Y. Compound Poisson approximation. *Probab. Surv.*, **19**, 271–350 (2022). MR4416129.

Chen, L. H. Y. Poincaré-type inequalities via stochastic integrals. *Z. Wahrsch. Verw. Gebiete*, **69** (2), 251–277 (1985). MR779459.

Chernoff, H. A note on an inequality involving the normal distribution. *Ann. Probab.*, **9** (3), 533–535 (1981). MR614640.

Cont, R. and Tankov, P. *Financial modelling with jump processes.* Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL (2004). ISBN 1-5848-8413-4. MR2042661.

Daly, F., Ghaderinezhad, F., Ley, C., and Swan, Y. Some simple variance bounds from Stein's method. *ALEA Lat. Am. J. Probab. Math. Stat.*, **18** (2), 1845–1858 (2021). MR4332222.

Dickson, D. C. M. *Insurance risk and ruin.* International Series on Actuarial Science. Cambridge University Press, Cambridge, second edition (2017). ISBN 978-1-107-15460-5. MR3444601.

Döbler, C. Distributional transformations without orthogonality relations. *J. Theoret. Probab.*, **30** (1), 85–116 (2017). MR3615083.

Finlay, R. and Seneta, E. Option pricing with VG-like models. *Int. J. Theor. Appl. Finance*, **11** (8), 943–955 (2008). MR2492088.

Forrester, P. J. On the gamma difference distribution. *Statist. Probab. Lett.*, **211**, Paper No. 110136, 7 (2024). MR4742159.

Furman, E. and Zitikis, R. Weighted premium calculation principles. *Insurance Math. Econom.*, **42** (1), 459–465 (2008). MR2392102.

Gaunt, R. E. Variance-gamma approximation via Stein's method. *Electron. J. Probab.*, **19**, no. 38, 33 (2014). MR3194737.

Goldstein, L. and Reinert, G. Stein's method and the zero bias transformation with application to simple random sampling. *Ann. Appl. Probab.*, **7** (4), 935–952 (1997). MR1484792.

Houdré, C. Remarks on deviation inequalities for functions of infinitely divisible random vectors. *Ann. Probab.*, **30** (3), 1223–1237 (2002). MR1920106.

Houdré, C., Pérez-Abreu, V., and Surgailis, D. Interpolation, correlation identities, and inequalities for infinitely divisible variables. *J. Fourier Anal. Appl.*, **4** (6), 651–668 (1998). MR1665993.

Küchler, U. and Tappe, S. Bilateral gamma distributions and processes in financial mathematics. *Stochastic Process. Appl.*, **118** (2), 261–283 (2008a). MR2376902.

Küchler, U. and Tappe, S. On the shapes of bilateral Gamma densities. *Statist. Probab. Lett.*, **78** (15), 2478–2484 (2008b). MR2462682.

Küchler, U. and Tappe, S. Tempered stable distributions and processes. *Stochastic Process. Appl.*, **123** (12), 4256–4293 (2013). MR3096354.

Novak, S. Y. Poisson approximation in terms of the Gini-Kantorovich distance. *Extremes*, **24** (1), 67–84 (2021). MR4218706.

Papadatos, N. and Papathanasiou, V. Unified variance bounds and a Stein-type identity. In Charalambides, C. A., Koutras, M. V., and Balakrishnan, N., editors, *Probability and Statistical Models with Applications*, pp. 87–100 (2001). ISBN 9780367578923.

Psarrakos, G. How a probabilistic analogue of the mean value theorem yields Stein-type covariance identities. *J. Appl. Probab.*, **59** (2), 350–365 (2022). MR4444023.

Sato, K. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (1999). ISBN 0-521-55302-4. MR1739520.

Stein, C. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. II: Probability theory*, pp. 583–602. Univ. California Press, Berkeley, CA (1972). MR402873.

Stein, C., Diaconis, P., Holmes, S., and Reinert, G. Use of exchangeable pairs in the analysis of simulations. In *Stein's method: expository lectures and applications*, volume 46 of *IMS Lecture Notes Monogr. Ser.*, pp. 1–26. Inst. Math. Statist., Beachwood, OH (2004). MR2118600.

Stein, E. M. and Shakarchi, R. *Fourier analysis. An introduction*, volume 1 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ (2003). ISBN 0-691-11384-X. MR1970295.

Tziafetas, G. N. A Formula for the Gini Coefficient and its Decomposition. *Biom. J.*, **31** (8), 961–967 (1989). DOI: 10.1002/bimj.4710310811.

Upadhye, N. S. and Barman, K. A unified approach to Stein's method for stable distributions. *Probab. Surv.*, **19**, 533–589 (2022). MR4507474.

Upadhye, N. S., Čekanavičius, V., and Vellaisamy, P. On Stein operators for discrete approximations. *Bernoulli*, **23** (4A), 2828–2859 (2017). MR3648047.

Vellaisamy, P. Poisson approximation for (k_1, k_2)-events via the Stein-Chen method. *J. Appl. Probab.*, **41** (4), 1081–1092 (2004). MR2122802.

Zhu, J. Hessian Estimation via Stein's Identity in Black-Box Problems. *Proc. Mach. Learn. Res. (PMLR)*, **145**, 1161–1178 (2022). Available at `https://proceedings.mlr.press/v145/zhu22c/zhu22c.pdf`.