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Blow up and non-blow up of a reaction-diffusion system with time-dependent Lévy generators and reactions of class H

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Abstract. In this paper we provide sufficient conditions for blow up and non-blow up of non-negative mild solutions for a non-autonomous reaction-diffusion system with Dirichlet boundary conditions, with diffusion terms given by generators of killed Lévy processes multiplied by a time-varying function and the reaction terms being functions of class H with time-varying coefficients.

1. Introduction

Consider the following non-autonomous semilinear model

$$\begin{aligned} \frac{\partial u_i}{\partial t}(t,x) &= k_i(t)\mathcal{A}_i u_i(t,x) + h_i(t)\mathcal{R}_i\big(u_{i'}(t,x)\big), \quad t>0,\ x\in D, \\ u_i(0,x) &= f_i(x), \quad x\in D,\ u_i\,|_{D^c}=0,\ i'=3-i,\ i=1,2, \end{aligned} \tag{1.1}$$

where D is a bounded domain (open and connected), $d \geq 1$, $[0,\infty) \ni t \mapsto k_i(t) \in [0,\infty)$, $[0,\infty) \ni t \mapsto h_i(t) \in [0,\infty)$ are continuous functions, $f_i \in C_0(D)$ is non-negative ($C_0(D)$ is the space of all continuous functions on D that vanish on D^c), $[0,\infty) \ni u \mapsto \mathcal{R}_i(u) \in [0,\infty)$ is locally Lipschitz and $\mathcal{A}_i$ is the generator of a Lévy process killed on D^c, $i = 1,2$. In the case of pure jump processes, they generally land in $\overline{D}^c$ upon exiting the domain D; for this reason, the Dirichlet boundary condition presented in (1.1) takes that form.

In López-Mimbela and Pérez (2015) the global existence vs blow up in finite time of solutions to (1.1) was studied for the case where $k_i(t) \equiv k(t)$, $h_i \equiv 1$, $\mathcal{R}_i(u) = u^{\beta_i}$, $\beta_i > 1$, and $\mathcal{A}_i \equiv \mathcal{A}$, $i = 1,2$, being $\mathcal{A}$ the generator of a killed symmetric pure jump Lévy process such that the associated semigroup is intrinsically ultracontractive. Let φ_0 be the first eigenfunction or ground

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state (normalized in the space of square integrable functions $L^2(D)$ on D), of $\mathcal{A}$. The intrinsic ultracontractive property allowed them to build a strongly continuous semigroup on $C_b(D)$ (the space of all continuous and bounded functions on D), invariant with respect to the probability measure $\varphi_0(x)^2\,dx$, this helped them to get conditions for the blow up in finite time of solutions for their particular system of the form (1.1).

In Pérez (2015) the model proposed in López-Mimbela and Pérez (2015) was generalized by considering that the generator $\mathcal{A}_i$, $i=1,2$ are not necessarily equal, the domain D is $C^{1,1}$, the killed semigroups with generators $\mathcal{A}_i$, $i=1,2$ are intrinsically ultracontractive, and the corresponding first eigenfunctions φ_0^i, $i=1,2$, satisfy the inequalities

$$c_1\varphi_0^1(x) \le \varphi_0^2(x) \le c_2\varphi_0^1(x), \quad x\in D, \tag{1.2}$$

for some positive constants c_i, $i=1,2$. Under these assumptions it was proved that (1.1) has solutions blowing up in finite time if

$$\min_{i=1,2}\langle f_i,\varphi_0^i\rangle > \max_{i=1,2}\left[\left(\frac{\beta_1\beta_2-1}{\beta_i+1}\right)\left(\frac{\beta_i+1}{\beta_{i'}+1}\right)^{\frac{\beta_i}{\beta_i+1}}\int_0^\infty \min_{i=1,2} C_i\frac{\mathrm{e}^{-\lambda_0^{i'}K_{i'}(t,0)\beta_i}\mathrm{e}^{\lambda_0^i K_i(t,0)}}{\|\varphi_0^{i'}\|_1^{\beta_i-1}}\,\mathrm{d}t\right]^{\frac{\beta_i+1}{1-\beta_1\beta_2}}, \tag{1.3}$$

where $K_i(t,0) := \int_0^t k_i(s)\,\mathrm{d}s$, $C_i = c_{i'}^{(-1)^i}(c_1c_2^{-1})^{\beta_i}$, $i=1,2$. The method used in their proof consisted of an appropriate adaptation of the technique used in López-Mimbela and Pérez (2015). Criterion (1.3) extends the one given in López-Mimbela and Pérez (2015) and is consistent by taking $c_i\equiv 1$ when $\varphi_0^i \equiv \varphi_0$, $i=1,2$. As reported in Pérez (2015), there exist many important examples of generators whose first eigenfunctions satisfy (1.2) and whose corresponding killed semigroups are intrinsically ultracontractive.

In the case when each $\mathcal{A}_i$ is a distinct fractional Laplacian, it is well known that (1.2) is not true in general. This situation was examined in Ceballos-Lira and Pérez (2020b) and as for non-blow up of solutions, it was shown there that their particular system of the form (1.1) does not blow up in a finite time if

$$\int_0^\infty \max_{i=1,2} h_i(t)m(t)^{\beta_i-1}\,\mathrm{d}t < \frac{1}{\beta_1\vee\beta_2-1}, \tag{1.4}$$

where $m(t) := \sup\{\|U_D^i(s,0)f_i\|_\infty ; 0<s\le t, i=1,2\}$ and $\big(U_D^i(t,s); t\ge s\ge 0\big)$ is the evolution system associated with the generator family $\big(k_i(t)\mathcal{A}_i; t\ge 0\big)$, $i=1,2$. Intrinsic ultracontractivity was not used in the proof of the above criterion; and as observed in Ceballos-Lira and Pérez (2020b) (see Remark 4.1), (1.4) generalizes the globality conditions for the particular cases of (1.1) considered in López-Mimbela and Pérez (2015); Pérez (2015).

The proofs for the blow up in finite time of the models given in López-Mimbela and Pérez (2015); Pérez (2015) (even Ceballos-Lira and Pérez (2020b)) depend crucially on the intrinsic ultracontractivity property. Moreover, the reaction terms considered are power functions of the form u^{β_i}, $\beta_i>1$, $i=1,2$. It is important to emphasize that the few existing papers with other type of reaction term only consider exponential functions and are restricted to the classical diffusion case (see for example Deng (1996) and references therein). The objective of this paper is to provide sufficient conditions for blow up and non-blow up in finite time of non-negative solutions for the system (1.1) with generators $\mathcal{A}_i$, $i=1,2$, which do not necessarily have the intrinsic ultracontractivity property, and whose nonlinear terms $\mathcal{R}_i$, $i=1,2$ are more general than power functions. For this, in the case of blowing up of solutions for (1.1), we only assume that functions φ_0^i, $i=1,2$ satisfy (1.2), and that $\mathcal{R}_i$, $i=1,2$ are functions of class H (see Dannan, 1985, 1986), i.e., for each $i=1,2$,

- $[H_1]$ $\mathcal{R}_i$ is not decreasing.
- $[H_2]$ $\mathcal{R}_i(u)>0$ if $u>0$.
- $[H_3]$ There exists a continuous function $[0,\infty)\ni u\mapsto \rho_i(u)$ such that $\mathcal{R}_i(uv)\le\rho_i(u)\mathcal{R}_i(v)$, $u,v\ge 0$.

In Section 3 we present many examples of generators whose first eigenfunctions satisfy a relation of the form (1.2). The functions of the form $\mathcal{R}(u) = a_1 u^{\beta_1} + \cdots + a_n u^{\beta_n}$ with $a_1 \vee \cdots \vee a_n > 0$ and $\beta_k \in \mathbb{R}$, $k = 1, \ldots, n$, are of class H with $\rho(u) = u^{\beta_1} + \cdots + u^{\beta_n}$, and so is any submultiplicative function with $\rho(u) = C\mathcal{R}(u)$, $u \geq 0$, $C > 0$. Examples of submultiplicative functions can be found in Maligranda (1985); Gustavsson et al. (1989). Functions of class H were studied in Dannan (1985, 1986) to discuss the asymptotic behavior of some nonlinear differential equations of second order and some types of integral inequalities. These functions are stable with respect to addition, product, and composition of functions (Dannan, 1985, Lemma 1). Another example of class H functions is given by the concave functions on an interval $[u_0, \infty)$ with $u_0 \in [0, 1)$ satisfying $[H_1]$ and $[H_2]$. In this case $[H_3]$ is satisfied with $\rho(u) := 1 + (u-1)1_{[1,\infty)}(u)$. Thus, functions like $\mathcal{R}_1(u) = \ln(1+u)$ and $\mathcal{R}_2(u) = \mathrm{e}^{-1/u} 1_{(0,\infty)}(u)$ are of class H. Furthermore, $\mathcal{R}_1$ and $\mathcal{R}_2$ are examples of functions that satisfy $[H_3]$ and are not submultiplicative. Reaction-diffusion models with reaction terms of the form $\mathrm{e}^{-1/u}$ arise naturally from the thermal explosion theory and, in these cases, the reaction rate is said to obey Arrhenius's law (see Harley, 2010).

Regarding the existence of globally defined solutions for (1.1), we only assume that the nonlinear terms $\mathcal{R}_i$, $i = 1, 2$ are of class H, except for Theorem 5.7 where we additionally assume (1.2) and that the operators $\mathcal{A}_i$, $i = 1, 2$ are intrinsically ultracontractive.

The theorems on blow up in finite time and globality presented in this paper are more general than those given in Ceballos-Lira and Pérez (2020b); López-Mimbela and Pérez (2015); Pérez (2015), furthermore they extend and improve criteria (1.3), (1.4), as can be seen in Examples 5.6, 5.9 and 6.5. We also show how these results can be used to obtain conditions for blow up in finite time or globality for models of the form

$$\begin{aligned} \frac{\partial w_i}{\partial t}(t,x) &= k_i(t)\mathcal{A}_i w_i(t,x) + h_i(t)\mathcal{S}_i\big(w_{i'}(t,x)\big), \quad t > 0,\ x \in D, \\ w_i(0,x) &= g_i(x), \quad x \in D,\ w_i \,|_{D^c} = 0,\ i' = 3 - i,\ i = 1, 2, \end{aligned}$$

depending on whether $\mathcal{S}_i \geq \mathcal{R}_i$, $f_i \geq g_i$ or $\mathcal{S}_i \leq \mathcal{R}_i$, $f_i \leq g_i$, respectively (see Theorem 6.7 and Remark 6.9). In Examples 6.8 and 6.10 the above is illustrated in weakly coupled systems whose scalar versions include classical applied models.

The Gaussian case $\mathcal{A}_i \equiv \Delta|_D$ has been extensively studied. Escobedo and Herrero (1993) were the first to obtain conditions of globality and blow up in finite time for the case $k_i = h_i \equiv 1$, $\mathcal{R}_i(u) = u^{\beta_i}$, $i = 1, 2$. In that paper, they interpreted that the solutions u_i, $i = 1, 2$ as the temperatures of two substances constituting a combustible mixture, where the heat release can be described through power laws. Later, Bai et al. (2013) considered the same study but with $h_i(t) = \mathrm{e}^{\delta_i t}$, $\delta_i \in \mathbb{R}$, $t \geq 0$, $i = 1, 2$. The general case wherein $h_i(t)$ is any non-negative continuous function, $i = 1, 2$, was analyzed by Castillo and Loayza (2015).

Currently, there are few papers that consider semilinear models of the form (1.1) with our more general diffusions $\mathcal{A}_i$, $i = 1, 2$ and with time-varying diffusion coefficients k_i, $i = 1, 2$; however, we emphasize that it is important to consider models containing such generalities. Indeed, Weitsman (1976) was one of the first to point out that the moisture sorption phenomenon of composite materials can be reasonably described by the classical diffusion equation and that, in many cases, it is necessary to consider a time-varying diffusivity coefficient. Recently, Tawfik and Abdelhamid (2021) reported that many physical, biological, and geological systems, which exhibit anomalous diffusions, can be adequately studied if the corresponding diffusion coefficients vary over time. In the aforementioned paper, they present three cases where this variability is applicable. This makes evident the importance of studying (1.1) with more general generators, since it is also known that an adequate description of anomalous diffusion can be obtained if it is considered as the generator of a stable Lévy process. For more details on how this is done and its applications in different fields of science, see Abe and Thurner (2005); Bouchaud and Georges (1990); Vázquez (2014) and references therein.

This paper is organized as follows. In Section 2, some preliminary results on killed processes are shown. In Section 3, several examples of generators satisfying (1.2) are presented. In Section 4, we show the local existence of solutions in an even more general context than required in the rest of the paper. Section 5 exhibits sufficient conditions that guarantee non-blow up in finite time and gives some upper and lower bounds for the solutions by means of globally defined functions. Our criteria for blow up in finite time are presented in Section 6, as well as an upper bound for the explosion time.

In this paper $\mathcal{B}(D)$ is the Borel σ-algebra on D, and we denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$, by $\|\cdot\|_p$ the norm on $\mathrm{L}^p(D)$ and consider the inner product $\langle f,g\rangle = \int_D f(x)g(x)\,\mathrm{d}x$, where $\mathrm{d}x$ is the Lebesgue measure.

2. Killed process and mild solution

Let $Z^i = \big(Z^i(t); t\geq 0\big)$ be a symmetric Lévy process on $\mathbb{R}^d$, $d\geq 1$, with characteristics (A^i, b^i, ν^i), $i=1,2$. In this paper we assume that the Lévy measure ν^i is not identically zero and that Z^i possesses a family of transition densities $p^i(t,x,y) = p^i(t,|x-y|)$ that are continuous for $t>0$ and satisfy the property: $\sup\{p^i(t,x); t>0, |x|>o_i\}<\infty$, for each $o_i>0$, $i=1,2$.

Thorough this paper $\mathbb{P}^i_x$ stands for the law of $Z^i(t)$ starting from $Z^i(0)=x$ and $\mathbb{E}^i_x$ stands for the corresponding expectation.

Define $\tau^i_D := \inf\{t>0; Z^i(t)\notin D\}$, $i=1,2$. Under the above assumptions, it is known that (see Grzywny, 2008, pp. 92-94) for each $t>0$, the function

$$p^i_D(t,x,y) := \begin{cases} p^i(t,x,y) - r^i_D(t,x,y), & x,y\in D,\\ 0, & x\notin D \,\text{or}\, y\notin D, \end{cases}$$

where $r^i_D(t,x,y) := \mathbb{E}^i_x\{p^i\big(t-\tau^i_D, Z^i(\tau^i_D), y\big); t>\tau^i_D\}$, is a transition density of the probability transition $P^i_D(t,x,\Gamma)$, $t>0$, $x\in D$, $\Gamma\in\mathcal{B}(D)$ of the killed process $Z^i_D = \big(Z^i_D(t); t\geq 0\big)$ defined by

$$Z^i_D(t) := \begin{cases} Z^i(t), & t<\tau^i_D,\\ \partial, & t\geq \tau^i_D, \end{cases}$$

$i=1,2$. Here ∂ denotes a cemetery point.

It is well known that the killed semigroup $\big(S^i_D(t); t\geq 0\big)$ associated to the process Z^i_D is contractive and strongly continuous on $\mathrm{L}^2(D)$, and that each operator $S^i_D(t)$ is self-adjoint and compact. Also, there exists an orthonormal basis of continuous eigenfunctions (φ^i_n) of the operator $S^i_D(t)$ with corresponding eigenvalues $(\mathrm{e}^{-\lambda^i_n t})$ satisfying $0<\lambda^i_0\leq\lambda^i_1\leq\lambda^i_2\leq\cdots$ and $\lim_{n\to\infty}\lambda^i_n=\infty$, $i=1,2$. The first eigenfunction φ^i_0 is strictly positive on D and satisfies $\mathcal{A}_i\varphi^i_0 = -\lambda^i_0\varphi^i_0$ if $\mathcal{A}_i$ is the generator of Z^i_D, $i=1,2$.

Remark 2.1. If we define $f(\partial)=0$ for each $f\in\mathrm{L}^\infty(D)$, then $f\big(Z^i_D(t)\big) = 1_{\{t<\tau^i_D\}} f\big(Z^i(t)\big)$, $t\geq 0$, $i=1,2$. Using this convention and the definition of conditional expectation we obtain the following equality

$$\mathbb{E}^i_x\{f\big(Z^i_D(t)\big)\big|Z^i_D(s)\} = \mathbb{E}^i_x\{1_{\{t<\tau^i_D\}} f\big(Z^i(t)\big)\big|Z^i(s)\},\quad t\geq s\geq 0,\, i=1,2.$$

In the context of Remark 2.1, we have that

$$S^i_D(t)f(x) = \langle f, p^i_D(t,x,\cdot)\rangle = \mathbb{E}^i_x\{f\big(Z^i_D(t)\big)\} = \mathbb{E}^i_x\{f\big(Z^i(t)\big); t<\tau^i_D\},$$

for all $f\in\mathrm{L}^\infty(D)$, $t\geq 0$, $i=1,2$. Let us define $K_i(t,s) := \int_s^t k_i(r)\,\mathrm{d}r$, $t\geq s\geq 0$ and $\gamma^i_t := \mathbb{P}\{Z^i\big(K_i(t,0)\big)\in\cdot\}$, $t\geq 0$, $i=1,2$. Clearly each γ^i_t is infinitely divisible with characteristics (A^i_t, b^i_t, ν^i_t) given by $A^i_t = K_i(t,0)A^i$, $b^i_t = K_i(t,0)b^i$ and $\nu^i_t = K_i(t,0)\nu^i$, $t\geq 0$, $i=1,2$. The family

$\big((A^i_t, b^i_t, \nu^i_t); t \geq 0\big)$ verifies the conditions $i)$, $ii)$, and $iii)$ of the Theorem 9.8 of Sato (1999, p. 52) for all $i = 1, 2$. Thus, clearly, the process $W^i = \big(W^i(t); t \geq 0\big)$ defined by

$$W^i(t) = Z^i\big(K_i(t, 0)\big), \text{ a.s.}, \quad t \geq 0,\, i = 1, 2, \tag{2.1}$$

is an additive process. Define $\zeta^i_D := \inf\{t > 0; W^i(t) \notin D\}$ and let $W^i_D = \big(W^i_D(t); t \geq 0\big)$ the killed additive process

$$W^i_D(t) := \begin{cases} W^i(t), & t < \zeta^i_D, \\ \partial, & t \geq \zeta^i_D, \end{cases}$$

$i = 1, 2$. From (2.1) it immediately follows that

$$\tau^i_D = K_i(\zeta^i_D, 0), \text{ a.s.}, \, i = 1, 2. \tag{2.2}$$

Let us denote by $P^i_D(s, x, t, \Gamma)$, $t \geq s \geq 0$, $x \in D$, $\Gamma \in \mathcal{B}(D)$ the transition probabilities of the killed additive process W^i_D, and by $\big(U^i_D(t, s); t \geq s \geq 0\big)$ the evolution system associated to W^i_D, $i = 1, 2$. The following theorem provides a transition density of $P^i_D(s, x, t, \Gamma)$ that is closely related to the transition density associated to the killed symmetric process Z^i_D, $i = 1, 2$.

Theorem 2.2. *Let*

$$p^i_D(s, x, t, y) := p^i_D\big(K_i(t, s), x, y\big), \quad t \geq s \geq 0,\, x, y \in D,\, i = 1, 2.$$

Then $p^i_D(s, x, t, y)$ is the transition density for $P^i_D(s, x, t, \Gamma)$, which is symmetric, continuous and strictly positive on $D \times D$.

Proof: Using (2.1), (2.2), Remark 2.1 and the strong Markov property of each Z^i we obtain,

$$\begin{aligned} P^i_D(s, x, t, \Gamma) &:= \mathbb{P}^i\big\{W^i_D(t) \in \Gamma \big| W^i_D(s) = x\big\} \\ &= \mathbb{P}^i\big\{W^i(t) \in \Gamma,\, t < \zeta^i_D \big| W^i(s) = x\big\} \\ &= \mathbb{P}^i_x\big\{Z^i_D\big(K_i(t, s)\big) \in \Gamma\big\}, \end{aligned}$$

which proves our first statement. The other statements are a consequence of the fact that $p^i_D(t, \cdot, \cdot)$ is symmetric, continuous, and strictly positive on $D \times D$. □

From the above theorem it follows that, for each $f \in \mathrm{L}^\infty(D)$, $t \geq s \geq 0$, $x \in D$, $i = 1, 2$,

$$U^i_D(t, s)f(x) := \mathbb{E}^i\big\{f\big(W^i_D(t)\big)\big| W^i_D(s) = x\big\} = \langle f, p^i_D(s, x, t, \cdot)\rangle \; = S^i_D\big(K_i(t, s)\big)f(x). \tag{2.3}$$

Thus, $\big(U^i_D(t, s); t \geq s \geq 0\big)$ is a strongly continuous evolution system of contractions on $\mathrm{L}^2(D)$. Moreover, Theorem 2.2 and the second equality of (2.3) imply that each operator $U^i_D(t, s)$ is strictly positivity-preserving. Since $\mathcal{A}_i$ is the infinitesimal generator of the semigroup $\big(S^i_D(t); t \geq 0\big)$ then, from (2.3) it follows that $\big(U^i_D(t, s); t \geq s \geq 0\big)$ is the evolution system associated to the family of generators $\big(k_i(t)\mathcal{A}_i; t \geq 0\big)$, $i = 1, 2$. It is well known that if (u_1, u_2) is a classical solution of (1.1), then (u_1, u_2) satisfies the integral system

$$u_i(t, x) = U^i_D(t, 0)f_i(x) + \int_0^t h_i(s)U^i_D(t, s)\mathcal{R}_i\big(u_{i'}(s, x)\big)\, \mathrm{d}s, \tag{2.4}$$

for all $t \geq 0$, $x \in D$, $i = 1, 2$. Any solution of the integral system (2.4) is called a **mild solution** of (1.1).

Let us denote by $\tilde{\mathrm{L}}^2_i(D)$ the space of all square integrable functions with respect to the measure $\tilde{\mathbb{P}}^i$ with density $\varphi^i_0(x)^2$, and by $\|\cdot\|_{2,i}$ the norm on $\tilde{\mathrm{L}}^2_i(D)$, $i = 1, 2$. Clearly

$$\|g_i\|_{2,i} = \|g_i\varphi^i_0\|_2, \quad g_i \in \tilde{\mathrm{L}}^2_i(D),\, i = 1, 2. \tag{2.5}$$

This relation between norms implies that $\tilde{\mathrm{L}}^2_i(D) \supseteq \mathrm{L}^2(D)$, $i = 1,2$. Let us define the linear operator $\tilde{S}^i_D(t)$ by

$$\tilde{S}^i_D(t)g_i = \frac{\mathrm{e}^{\lambda^i_0 t} S^i_D(t)(g_i\varphi^i_0)}{\varphi^i_0}, \quad t \geq 0,\ g_i \in \tilde{\mathrm{L}}^2_i(D),\ i = 1,2. \tag{2.6}$$

From (2.5) we obtain that

$$\|\tilde{S}^i_D(t)g_i\|_{2,i} \leq \mathrm{e}^{\lambda^i_0 t}\|g_i\|_{2,i}, \quad g_i \in \tilde{\mathrm{L}}^2_i(D),\ t \geq 0,\ i = 1,2. \tag{2.7}$$

Using (2.5) and (2.7) it can be shown that $\big(\tilde{S}^i_D(t); t \geq 0\big)$ is a strongly continuous semigroup on $\tilde{\mathrm{L}}^2_i(D)$.

Let $\tilde{\mathbb{E}}^i$ be the expected value with respect to the measure of probability $\tilde{\mathbb{P}}^i$. The proof of the following proposition, which states that the semigroup $\big(\tilde{S}^i_D(t); t \geq 0\big)$ is invariant under the probability measure $\tilde{\mathbb{P}}^i$, follows easily from (2.5), (2.6), (2.7) and the fact that every operator $S^i_D(t)$ is self-adjoint. This property will be fundamental in the study of the blow up of the mild solution of (1.1).

Proposition 2.3. *For each $g_i \in \tilde{\mathrm{L}}^2_i(D)$ we have that*

$$\tilde{\mathbb{E}}^i\{\tilde{S}^i_D(t)g_i\} = \tilde{\mathbb{E}}^i\{g_i\}, \quad t \geq 0,\ i = 1,2.$$

3. Comparable operators

In this section, we present examples of generators of killed symmetric Lévy processes that verify a relation like (1.2). Let us first establish the notation that we will require. We denote by $\mathrm{diam}(D)$ and $\delta_D(x)$, the diameter of D and the Euclidian distance from x to D^c, respectively. Let X be a non-empty set and let $\mathrm{X} \ni x \mapsto f(x) \in [0,\infty)$, $\mathrm{X} \ni x \mapsto g(x) \in [0,\infty)$ be functions. The notation $f(x) \asymp g(x)$, $x \in \mathrm{X}$ means that there exist $m, M > 0$ such that $mg(x) \leq f(x) \leq Mg(x)$, for all $x \in \mathrm{X}$. Clearly $f(x) \asymp g(x)$ is an equivalence relation on the set of all non-negative functions defined on X. Given $\eta \in (0,\infty]$, let $(0,\eta) \ni x \mapsto f(x) \in (0,\infty)$, $(0,\eta) \ni x \mapsto g(x) \in (0,\infty)$ be functions. The notation $f(x) \asymp g(x)$, $x \to 0^+$ means that $(f/g)(0+) \in (0,\infty)$. Note that $f(x) \asymp g(x)$, $x \to 0^+$ is also an equivalence relation but on the set of all positive functions defined on a set of the form $(0,\eta)$, $\eta \in (0,\infty]$.

Let $\mathcal{A}_i$ be the generator of the killed process Z^i_D, $i = 1,2$. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are **comparable**, if their corresponding first eigenfunctions φ^1_0 and φ^2_0 satisfy

$$\varphi^1_0(x) \asymp \varphi^2_0(x), \quad x \in D. \tag{3.1}$$

This comparison property will be transcendental for obtaining the blow up conditions of the mild solution of (1.1).

Remark 3.1. Note that if $\eta \in (0,\infty]$, $(0,\eta) \ni r \mapsto f(r) \in (0,\infty)$, $(0,\eta) \ni r \mapsto g(r) \in (0,\infty)$ are continuous functions such that $f(r) \asymp g(r)$, $r \to 0^+$, and $\mathrm{diam}(D) < \eta$, then $(f\circ\delta_D)(x) \asymp (g\circ\delta_D)(x)$, $x \in D$.

Example 3.2. Let $\eta \in (0,\infty]$ be such that $\mathrm{diam}(D) < \eta$. Suppose that D is a $C^{1,1}$ set and that $Z^i = \big(Z^i(t); t \geq 0\big)$ is a subordinate Brownian motion, namely, $Z^i(t) = W^i\big(S^i(t)\big)$, $t \geq 0$, where $S^i = \big(S^i(t); t \geq 0\big)$ is a subordinator independent of the Brownian motion $W^i = \big(W^i(t); t \geq 0\big)$. If ϕ_i is the Laplace exponent of S^i, then it is known that $-\phi_i(-\Delta)$ is the generator of Z^i. Moreover, ϕ_i is a Bernstein function of the form

$$\phi_i(r) = b^i r + \psi_i(r), \quad r \geq 0,\ i = 1,2, \tag{3.2}$$

with

$$\psi_i(r) = \int_0^\infty (1 - \mathrm{e}^{-rt})\mu_i(\mathrm{d}t), \quad r \geq 0,\ i = 1,2, \tag{3.3}$$

where μ_i is the Lévy measure of the subordinator S^i, $i = 1, 2$ and satisfy $\int_0^\infty (1 \wedge t)\mu_i(\mathrm{d}t) < \infty$, $i = 1, 2$. The density j_i of the Lévy measure ν^i of the process Z^i is radial. It is easy to see that Z^i is a symmetric Lévy process that satisfies our initial hypothesis (see Section 2). Let us define $(0, \infty) \ni r \mapsto H_i(r)$ and $(0, \eta) \ni r \mapsto \Phi_i(r)$ by $H_i(r) := \phi_i(r) - r\phi_i'(r)$ and $\Phi_i(r) := 1/\phi_i(1/r^2)$, respectively, $i = 1, 2$.

Consider the following conditions

- $[A_1]$ $b^i = 0$, $i = 1, 2$ (hence $\phi_i \equiv \psi_i$).
- $[A_2]$ There exists $c_i > 0$ such that $j_i(r) \leq c_i j_i(r + 1)$, $r > 1$, $i = 1, 2$.
- $[A_3]$ There exist $\sigma_i \in (0, 2)$, $\varepsilon_i \in \big(1_{[1,\infty)}(\sigma_i)/2, \infty\big)$, $c_l^i \in (0, 1]$, $c_u^i \in [1, \infty)$ and $r_0^i > 0$ such that $c_l^i \lambda^{\varepsilon_i} H_i(r) \leq H_i(\lambda r) \leq c_u^i \lambda^{\sigma_i} H_i(r)$, $\lambda \geq 1$, $r \geq r_0^i$, $i = 1, 2$.
- $[A_4]$ There exists $(0, \eta) \ni r \mapsto \xi(r) \in (0, \infty)$ continuous such that $\sqrt{\Phi_i(r)} \asymp \xi(r)$, $r \to 0^+$, $i = 1, 2$.

It is known (Kim and Mimica, 2018, Theorem 1.3(c)) that if conditions $[A_1]$, $[A_2]$, $[A_3]$ hold, then

$$\mathrm{e}^{\lambda_0^i} p_D^i(1, x, y) \asymp \sqrt{\Phi_i\big(\delta_D(x)\big)}\sqrt{\Phi_i\big(\delta_D(y)\big)}, \quad (x, y) \in D \times D,\, i = 1, 2.$$

The validity of $[A_4]$ and Remark 3.1 imply that

$$\mathrm{e}^{\lambda_0^i} p_D^i(1, x, y) \asymp \xi\big(\delta_D(x)\big)\xi\big(\delta_D(y)\big), \quad (x, y) \in D \times D,\, i = 1, 2. \tag{3.4}$$

From the fact that ξ is continuous and D is a bounded domain, it follows that $\xi \circ \delta_D \in \mathrm{L}^1(D)$. Multiplying (3.4) by $\varphi_0^i(y)$ and integrating with respect to y, we obtain that

$$\varphi_0^i(x) \asymp \xi\big(\delta_D(x)\big), \quad x \in D\,, i = 1, 2.$$

Thus, in this case, the validity of (3.1) is a consequence of the transitivity of $\asymp$.

Now, we present some Bernstein functions satisfying $[A_1]$, $[A_2]$, $[A_3]$ and the associated subordinate processes. In addition, we will write a function ξ satisfying $[A_4]$. We divide our examples in two groups, one with $\eta = \infty$ and the other with $\eta = 1$.

I. $\eta = \infty$.
1. $\psi_\alpha(r) = r^{\alpha/2}$, $\alpha \in (0, 2)$, (*symmetric α-stable process*); $\xi_\alpha(r) = r^{\alpha/2}$.
2. $\psi_{\alpha,m}(r) = (c^{2/\alpha} r + m^{2/\alpha} c^{4/\alpha})^{\alpha/2} - mc^2$, $c, m > 0$ $\alpha \in (0, 2)$, (*relativistic α-stable process*); ξ_α.
3. $\psi_{\alpha,\beta}(r) = r^{\alpha/2} + r^{\beta/2}$, $0 < \beta < \alpha < 2$ (*mixed symmetric α and β stable process*); ξ_α.
4. $\psi_{\delta,\gamma}(r) = \delta(\sqrt{\gamma^2 + 2r} - \gamma)$, $\delta, \gamma > 0$, (*normal inverse Gaussian process without drift*); $\xi_{1/2}$.

For example, if $\alpha, m > 0$ and $0 < \beta < \alpha < 2$ are fixed constants and we take $\xi \equiv \xi_\alpha$, then the generators $\mathcal{A}_1 = -\psi_{\alpha,m}(-\Delta)|_D$, $\mathcal{A}_2 = -\psi_{\alpha,\beta}(-\Delta)|_D$ satisfy (3.1). Furthermore, (3.1) holds for any pair of generators of the same type if they differ only in parameters other than the parameter of ξ. Note that if $\alpha = 1$ in each example above, it is possible to obtain (3.1) for any pair of generators by taking $\xi \equiv \xi_{1/2}$.

Remark 3.3. The Bernstein functions ψ given above satisfy $[A_1]$ and the following properties (see Chen et al., 2014).

- $[A_2']$ ψ is a complete Bernstein function.
- $[A_3']$ There exist $\sigma_i, \varepsilon_i \in (0, 1)$, $c_l^i \in (0, 1]$, $c_u^i \in [1, \infty)$ and $r_0^i > 0$ such that $c_l^i \lambda^{\varepsilon_i} \psi_i(r) \leq \psi_i(\lambda r) \leq c_u^i \lambda^{\sigma_i} \psi_i(r)$, $\quad \lambda \geq 1$, $r \geq r_0^i$, $i = 1, 2$.

It can be shown that $[A_2']$ and $[A_3']$ imply $[A_2]$ (Kim and Mimica, 2018, Remark 1.2). Moreover, it follows from the right-hand side of the inequality in $[A_3']$ it follows that $H_i(r) \asymp \psi_i(r)$, $r \in [r_0^i, \infty)$ (Mimica, 2016, Proposition 29), and using this and the inequality in $[A_3']$ we conclude that H_i satisfies $[A_3]$.

Using the above remark, it is possible to build more examples of subordinate Brownian motion generators satisfying (3.1). In fact, a long table of examples of Bernstein functions satisfying $[A_1]$ and $[A_2']$ is given in Schilling et al. (2010, pp. 299-365). Next, we present two examples from that table satisfying $[A_3']$ and $[A_4]$.

5. $\psi_a(r) = \sqrt{r}(1 + \mathrm{e}^{-2a\sqrt{r}})$, $a > 0$; $\xi_{1/2}$.
6. $\psi_{\alpha,a}(r) = r(a + r)^{-\alpha/2}$, $a > 0$, $\alpha \in (0, 2)$; $\xi_{\bar{\alpha}}(r) = r^{1-\alpha/2}$.

Note that when $\alpha = 1$, any pair of generators associated to examples 1, 2, 3, 4, 5, 6 are comparable.

II. $\eta = 1$.

The following Bernstein functions have already been studied in Kim and Mimica (2018) and have been shown to satisfy $[A_1]$, $[A_2]$, $[A_3]$. Together with them, we write a function ξ that satisfies $[A_4]$.

7. $\psi_\beta(r) = r[\ln(1 + r^{\beta/2})]^{-1}$, $\beta \in (0, 2)$; $\xi(r) = r\sqrt{\ln(1/r)}$.
8. $\psi(r) = r[\ln(1 + r)]^{-1} - 1$; ξ.

A subordinate Brownian motion with Laplace exponent of the form $\phi_\beta(r) = \ln(1 + r^{\beta/2})$, $\beta \in (0, 2]$ is known in the literature as a geometric β-stable process. The geometric 2-stable process is called gamma process. We can then think of the generators associated with examples 7 and 8 as examples of generators of reciprocal geometric β-stable processes.

Example 3.4. Suppose D is a $C^{1,1}$ set and Z^i is a subordinate Brownian motion, $i = 1, 2$ (see the above example). Let ϕ_i and μ_i be the Laplace exponent and the Lévy measure associated to ϕ_i, respectively. As an abuse of notation, let us denote the density μ_i by $\mu_i(t)$. Consider the following conditions

- $[A_1'']$ $b^i > 0$, $i = 1, 2$.
- $[A_2'']$ ϕ_i is a complete Bernstein function.
- $[A_3'']$ There exist $\sigma_i \in (0, 1)$ and c_u^i such that $\psi_i(\lambda r) \le c_u^i \lambda^{\sigma_i} \psi_i(r)$, $\lambda \ge 1$, $r \ge 1$, $i = 1, 2$.
- $[A_4'']$ For each $\kappa_i > 0$, there exists $c^i = c^i(\kappa_i) > 1$ such that $\mu_i(r) \le c^i \mu_i(2r)$, $r \in (0, \kappa_i)$, $i = 1, 2$.

Note that $[A_1'']$ implies that the generator of Z^i has the form $b^i\Delta - \psi_i(-\Delta)$, $i = 1, 2$. If the conditions $[A_1'']$, $[A_2'']$, $[A_3'']$, $[A_4'']$ hold, then (see Chen et al., 2016, Theorem 1.2(ii))

$$\mathrm{e}^{\lambda_0^i} p_D^i(1, x, y) \asymp \delta_D(x)\delta_D(y), \quad (x, y) \in D \times D,\ i = 1, 2. \tag{3.5}$$

Proceeding as in Example 3.2, it is easily shown that the generators $\mathcal{A}_i = \big(b^i\Delta - \psi_i(-\Delta)\big)\big|_D$, $i = 1, 2$, are comparable.

Now, we present some complete Bernstein functions satisfying $[A_3'']$. It follows from equality (3.2) that it is sufficient to provide complete Bernstein functions of the form (3.3). Next to each function ψ we write the density μ of the Lévy measure. We use the notation of Example 3.2.

1. ψ_α, $\alpha \in (0, 2)$; $\mu_\alpha(r) = \alpha r^{-(1+\alpha/2)}/[2\Gamma(1 - \alpha/2)]$.
2. $\psi_{\alpha,m}$, $\alpha \in (0, 2)$, $c, m > 0$; $\mu_{\alpha,m}(r) = \alpha c\, \mathrm{e}^{-(mc)^{2/\alpha} r} r^{-(1+\alpha/2)}/[2\Gamma(1 - \alpha/2)]$.
3. $\psi_{\alpha,\beta}$, $0 < \beta < \alpha < 2$; $\mu_{\alpha,\beta}(r) = \alpha[r^{-(1+\alpha/2)} + r^{-(1+\beta/2)}]/\Gamma(1 - \alpha/2)$.
4. $\psi_{\delta,\gamma}$, $\delta, \gamma > 0$; $\mu_{\delta,\gamma}(r) = \delta \mathrm{e}^{-\gamma^2/2} r^{-3/2}/\sqrt{2\pi}$.
5. ψ_a, $a > 0$; $\mu_a(r) = [(2a^2 - r)\mathrm{e}^{-a^2/r} + r]/(2\sqrt{\pi} r^{5/2})$.
6. $\psi_{a,\alpha}$, $\alpha \in (0, 2)$, $a > 0$; $\mu_{\alpha,a}(r) = \sin\left(\frac{\alpha\pi}{2}\right)\Gamma(1 - \alpha/2)\mathrm{e}^{-ar} r^{-(2-\alpha/2)}(ar + 1 - \alpha/2)$.

For more examples see Schilling et al. (2010, pp. 299-365). We write two more examples that are well known for their applications.

7. $\psi_\gamma(r) = r(r + \gamma)^{-1}$, $\gamma > 0$; $\mu_\gamma(r) = \gamma \mathrm{e}^{-\gamma r}$ (*compound Poisson process with rate* $\gamma > 0$ *and exponential jumps*).

8. $\psi_{\alpha,\mathrm{geo}}(r) = \ln(1+r^{\alpha/2})$, $\alpha \in (0,2)$; $\mu_{\alpha,\mathrm{geo}}(r) = \alpha \mathrm{E}_{\alpha/2}(r)/2r$, where $\mathrm{E}_{\alpha/2}(r) = \mathrm{E}_{\alpha/2,1}(-r)^{\alpha/2}$ and $\mathrm{E}_{\alpha/2,1}$ is the Mittag-Leffler function (*geometric α-stable process*).

Example 3.5. It is well known (see Davies, 1987; Zhang, 2002) that the Dirichlet Laplacian $\Delta|_D$ satisfies a relation as stated in (3.5). The procedure to obtain it is the same as that used in Example 3.4. Therefore, if any of the generators $\mathcal{A}_i$ is a Laplacian and $\mathcal{A}_{i'}$ is the generator of any of the processes in Example 3.4, then both are comparable.

4. Existence and uniqueness of a mild solution

Let V be the set of all real functions defined on $[0,\infty)\times D$ and define $\mathcal{V} := V \times V$. Each element $\boldsymbol{v} \in \mathcal{V}$ will be denoted by (v_1, v_2). Let us define the function $\mathcal{V} \ni \boldsymbol{v} \mapsto \mathrm{A}\boldsymbol{v} = (\mathrm{A}_1 v_1, \mathrm{A}_2 v_2)$ by

$$\mathrm{A}_i v_i(t,x) := U_D^i(t,0) f_i(x) + \int_0^t h_i(s) U_D^i(t,s) \mathcal{R}_i\big(v_{i'}(s,x)\big)\,\mathrm{d}s, \tag{4.1}$$

for any $t \geq 0$, $x \in D$, $i = 1,2$. Next, we prove a theorem on the existence and local uniqueness of a mild solution of (1.1). As is well known, it is sufficient to prove the existence and uniqueness on an interval $[0,\tau]$, because if $[0,\tau_{\max})$ denotes the maximal interval of existence, then $\tau_{\max} = \sup\{\tau \geq 0;\, (2.4)$ has a solution on $[0,\tau]\}$. In order to give the result in a more general context, we consider in this section more general data than those of our initial hypothesis. The functions k_i, $i = 1,2$ are still considered non-negative and continuous.

Theorem 4.1. *Suppose that $f_i \in \mathrm{L}^\infty(D)$, $[0,\infty) \ni t \mapsto h_i(t) \in \mathbb{R}$ is a locally integrable function and $\mathbb{R} \ni u \mapsto \mathcal{R}_i(u) \in \mathbb{R}$ is locally Lipschitz, i.e., for each $r > 0$ there exists $L_r^i > 0$ such that*

$$|\mathcal{R}_i(v) - \mathcal{R}_i(w)| \leq L_r^i |v - w|, \quad |v|, |w| \leq r,\ i = 1,2. \tag{4.2}$$

Then there exists $\tau > 0$ such that the integral system (2.4) has a unique solution on $[0,\tau]$.

Proof: Fix $r > \|f_1\|_\infty \vee \|f_2\|_\infty$ and let L_r^i be the constant in (4.2), $i = 1,2$. Define $B_i := \sup|\mathcal{R}_i|([0,r])$ and choose $\tau > 0$ small enough so that

$$\max_{1=1,2} B_i \int_0^\tau |h_i(t)|\,\mathrm{d}t \leq r - \max_{i=1,2} \|f_i\|_\infty$$

and

$$\mathfrak{c} := \max_{i=1,2} L_r^i \int_0^\tau |h_i(t)|\,\mathrm{d}t \in (0,1).$$

Let $\mathcal{X} := \{\boldsymbol{v} \in \mathcal{V};\, v_i(t,\cdot) \in \mathrm{L}^\infty(D),\ t \geq 0,\ i = 1,2\}$ and $\mathcal{E} := \{\boldsymbol{v} \in \mathcal{X};\, |||\boldsymbol{v}||| < \infty\}$, wherein

$$|||\boldsymbol{v}||| := \sup_{0 \leq t \leq \tau} \max_{i=1,2} \|v_i(t,\cdot)\|_\infty.$$

Clearly $\mathcal{E}$ is a Banach space with respect to the norm $|||\cdot|||$. Given $r > 0$, let $\mathcal{M}$ be the closed ball in $\mathcal{E}$ with center at the origin and radius r. Note that if $\boldsymbol{v} \in \mathcal{M}$, then

$$\|\mathrm{A}_i v_i(t,\cdot)\|_\infty \leq \|f_i\|_\infty + \int_0^t |h_i(s)| \cdot \|\mathcal{R}_i\big(v_{i'}(s,\cdot)\big)\|_\infty\,\mathrm{d}s \leq \|f_i\|_\infty + B_i \int_0^t |h_i(s)|\,\mathrm{d}s,$$

for all $t \geq 0$, $i = 1,2$. Therefore $|||\mathrm{A}\boldsymbol{v}||| \leq r$, and thus A takes values in $\mathcal{M}$ provided it is defined on $\mathcal{M}$. On the other hand, using (4.2), we have that if $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{M}$,

$$\begin{aligned}\|\mathrm{A}_i v_i(t,\cdot) - \mathrm{A}_i w_i(t,\cdot)\|_\infty &\leq \int_0^t |h_i(s)| \cdot \|\mathcal{R}_i\big(v_{i'}(s,\cdot)\big) - \mathcal{R}_i\big(w_{i'}(s,\cdot)\big)\|_\infty\,\mathrm{d}s \\ &\leq L_r^i \int_0^t |h_i(s)| \cdot \|v_{i'}(s,\cdot) - w_{i'}(s,\cdot)\|_\infty\,\mathrm{d}s.\end{aligned}$$

The last inequality implies that $|||\mathrm{A}\boldsymbol{v} - \mathrm{A}\boldsymbol{w}||| \leq \mathfrak{c}|||\boldsymbol{v} - \boldsymbol{w}|||$. We have thus shown that $\mathcal{M} \ni \boldsymbol{v} \mapsto \mathrm{A}\boldsymbol{v} \in \mathcal{M}$ is a contractive function on the complete metric space $\mathcal{M}$. By Banach fixed-point theorem we conclude that there exists a unique $\boldsymbol{u} \in \mathcal{M}$ such that $\mathrm{A}\boldsymbol{u} = \boldsymbol{u}$. Equating the corresponding components, we conclude that $\boldsymbol{u} = (u_1, u_2)$ is the unique solution of (2.4) in $[0, \tau]$. $\square$

On the set $\mathcal{V}$ we define the following partial order: For any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,

$$\boldsymbol{u} \le \boldsymbol{v} \quad \text{iff} \quad u_i \le v_i,\ i = 1, 2. \tag{4.3}$$

Naturally $\boldsymbol{u} \ge \boldsymbol{v}$ means $\boldsymbol{v} \le \boldsymbol{u}$.

Lemma 4.2. *If in the function* A *defined in* (4.1)*,* h_i *is non-negative, and* $\mathcal{R}_i$ *is non-decreasing and non-negative, then* A *is non-decreasing with respect to the partial order defined in* (4.3)*.*

Proof: It follows directly from the fact that $\big(U_D^i(t,s); t \ge s \ge 0\big)$ is a family of positive-preserving operators. $\square$

We use the previous results in the proof of the following proposition.

Proposition 4.3. *Suppose that* h_i *and* $\mathcal{R}_i$ *are as in Lemma 4.2. Let* $(\boldsymbol{u}_n)$ *be the sequence in* $\mathcal{V}$ *defined by*

$$\boldsymbol{u}_n := \begin{cases} \big(U_D^1(t,0)f_1, U_D^2(t,0)f_2\big), & n = 0, \\ \mathrm{A}\boldsymbol{u}_{n-1}, & n \ge 1, \end{cases} \tag{4.4}$$

and $\boldsymbol{u}$ *the mild solution of* (1.1) *on* $[0, \tau_{\max})$*. Then* $\lim_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{u}$*, that is, if* $\boldsymbol{u}_n = (u_{n1}, u_{n2})$ *and* $\boldsymbol{u} = (u_1, u_2)$*,*

$$u_i(t,x) = \lim_{n\to\infty} u_{ni}(t,x), \quad 0 \le t < \tau_{\max},\ x \in D,\ i = 1, 2. \tag{4.5}$$

Proof: From the fact that $\big(U_D^i(t,s); t \ge s \ge 0\big)$ is positive-preserving, it follows that $\boldsymbol{u}_0 \le \boldsymbol{u}_1$. By mathematical induction and Lemma 4.2, we obtain that $(\boldsymbol{u}_n)$ is a non-decreasing sequence. Hence, for each $t \in [0, \tau_{\max})$, $x \in D$, $i = 1, 2$, we have that $\lim_{n\to\infty} u_{ni}(t,x)$ exists. Equating the corresponding components in $\boldsymbol{u}_n = \mathrm{A}\boldsymbol{u}_{n-1}$, $n \ge 1$, and applying the monotone convergence theorem we obtain (4.5). $\square$

As we will see in the next section, the limit (4.5) will allow us to obtain upper bounds for the mild solution $\boldsymbol{u} = (u_1, u_2)$ of (1.1); therefore, the globality of $\boldsymbol{u}$ will depend on the non-blow up of such upper bounds.

5. Globally defined solutions

In this section, we present conditions that guarantee the non-blow up of the mild solution of (1.1). We begin by fixing the notation that we will use in the rest of this paper. We define the bijective function $[0, \infty) \ni y \mapsto \mathcal{P}_i(y)$ by

$$\mathcal{P}_i(y) := \int_0^y \mathcal{R}_i(v)\,\mathrm{d}v, \quad i = 1, 2,$$

and let $\mathbf{R}_i$, $i = 1, 2$, be the function that makes the following diagram commute

$$\begin{array}{ccc} [0,\infty) & \xrightarrow{\ \mathcal{P}_i^{-1}\ } & [0,\infty) \\ \Big\downarrow{\scriptstyle \mathcal{P}_{i'}^{-1}} & & \Big\downarrow{\scriptstyle \mathcal{R}_i} \\ [0,\infty) & \xrightarrow[\ \mathbf{R}_i\ ]{} & [0,\infty) \end{array}$$

Finally, let $[0,\infty) \ni y \mapsto \mathbf{P}_i(y)$ be the function defined by

$$\mathbf{P}_i(y) := \int_y^\infty \frac{\mathrm{d}v}{\mathbf{R}_i(v)}, \quad i = 1,2. \tag{5.1}$$

Note that $\mathbf{R}_i(v) > 0$ if $v > 0$, due to the fact that $\mathcal{R}_i$ satisfies $[H_2]$, $i = 1, 2$.

Lemma 5.1. *Let $[0,\infty) \ni t \mapsto a_i(t) \in [0,\infty)$ be continuous, $y_0^i \geq 0$, and (y_1, y_2) be the solution of the integral system*

$$y_i(t) = y_0^i + \int_0^t a_i(s)\mathcal{R}_i\big(y_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\ i = 1,2. \tag{5.2}$$

Suppose that

$$\int_0^\infty \max_{i=1,2} a_i(t)\,\mathrm{d}t \leq \min_{i=1,2} \mathbf{P}_i(y_0^i), \tag{5.3}$$

where $\mathbf{P}_i$ is the function definited by (5.1). Then, the integral system (5.2) is globally defined.

Proof: Note that

$$y_i(t) \leq y_0^i + \int_0^t a(s)\mathcal{R}_i\big(y_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\ i = 1,2,$$

where $a(s) := (a_1 \vee a_2)(s)$, $s \geq 0$. Let (z_1, z_2) be the solution of the integral system

$$z_i(t) = y_0^i + \int_0^t a(s)\mathcal{R}_i\big(z_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\ i = 1,2.$$

Then by comparison, $y_i(t) \leq z_i(t)$, $t \in [0,\infty)$, $i = 1, 2$. Hence, (z_1, z_2) satisfies

$$\begin{aligned} z_i'(t) &= a(t)\mathcal{R}_i\big(z_{i'}(t)\big), \quad t > 0,\\ z_i(0) &= y_0^i, \quad i = 1,2. \end{aligned}$$

Consequently, the derivative of the function $\mathcal{P}_1 \circ z_2$ coincides with the derivative of the function $\mathcal{P}_2 \circ z_1$, which implies that $\mathcal{P}_i \circ z_{i'}(t) \leq \mathcal{P}_{i'} \circ z_i(t)$, $t \geq 0$, for some $i \in \{1, 2\}$. Using the fact that $\mathcal{P}_i$ is invertible and property $[H_1]$, we obtain that $z_i'(t) \leq a(t)\mathbf{R}_i\big(z_i(t)\big)$, $t \geq 0$. Let z be the solution of

$$\begin{aligned} z'(t) &= a(t)\mathbf{R}_i\big(z(t)\big), \quad t > 0,\\ z(0) &= y_0^i. \end{aligned}$$

Due to (5.3) and Ceballos-Lira et al. (2011, Lemma 2.2), z is globally defined and, by comparison, $z(t) \geq z_i(t) \geq y_i(t)$, $t \geq 0$. If (y_1, y_2) blows up in finite time, then these inequalities imply that z blows up in finite time. Therefore, (y_1, y_2) is defined for all $t \geq 0$. □

Theorem 5.2. *If $f_1, f_2 \in \mathrm{L}^\infty(D)$ are non-negative and*

$$\int_0^\infty \max_{i=1,2} h_i(t)\,\mathrm{d}t \leq \min_{i=1,2} \mathbf{P}_i(\|f_i\|_\infty), \tag{5.4}$$

then the mild solution of (1.1) is globally defined.

Proof: If we define $z_i(t,x) := \int_0^t h_i(s)U_D^i(t,s)\mathcal{R}_i\big(u_{i'}(s,x)\big)\,\mathrm{d}s$, then $u_i(t,x) = U_D^i(t,0)f_i(x)+z_i(t,x)$, for all $t \geq 0$, $x \in D$, $i = 1, 2$. Using $[H_1]$ and the fact that $U_D^i(t,s)$ is positive-preserving, we have that

$$\|z_i(t,\cdot)\|_\infty \leq \int_0^t h_i(t)\mathcal{R}_i\big(\|f_{i'}\|_\infty + \|z_{i'}(s,\cdot)\|_\infty\big)\,\mathrm{d}s, \quad t \geq 0,\ i = 1,2.$$

Defining $\bar{z}_i(t) := \|f_i\|_\infty + \|z_i(t,\dot{)}\|_\infty$, we get

$$\bar{z}_i(t) \leq \|f_i\|_\infty + \int_0^t h_i(s)\mathcal{R}_i\big(\bar{z}_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\ i = 1,2.$$

Let (v_1, v_2) be the solution of the integral system

$$v_i(t) = \|f_i\|_\infty + \int_0^t h_i(s)\mathcal{R}_i\big(v_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\ i = 1, 2. \tag{5.5}$$

By (5.4) and Lemma 5.1, (v_1, v_2) is globally defined. The conclusion of the theorem follows from the fact that $u_i(t, x) \leq v_i(t)$, $t \geq 0$, $x \in D$, $i = 1, 2$. □

Corollary 5.3.

(1) *If the conditions of Theorem 5.2 hold, then*

$$U^i_D(t, 0)f_i(x) \leq u_i(t, x) \leq U^i_D(t, 0)f_i(x) + v_i(t),$$

for all $t \geq 0$, $x \in D$, $i = 1, 2$, *where* (v_1, v_2) *is the solution of* (5.5).

(2) *If coefficients* h_1, h_2 *are constant functions, then a sufficient condition for the global existence of mild solutions of* (1.1) *is* $\mathbf{P}_i(\|f_i\|_\infty) = \infty$, *for all* $i = 1, 2$.

In each previous result, only properties $[H_1]$ and $[H_2]$ of each $\mathcal{R}_i$ were used. Property $[H_3]$ will be used in the following results.

Theorem 5.4. *Let* $f_1, f_2 \in \mathrm{L}^\infty(D)$ *be non-negative,* $[0, \infty) \ni u \mapsto \bar{\rho}_i(u) := \rho_i(u)/u$, *and* $[0, \infty) \ni t \mapsto m(t) \in (0, \infty)$ *non-decreasing such that* $m(t) \geq \|U^i_D(t, 0)f_i\|_\infty$, *for all* $t \geq 0$, $i = 1, 2$. *If*

$$\int_0^\infty \max_{i=1,2} h_i(t)\bar{\rho}_i\big(m(t)\big)\,\mathrm{d}t \leq \min_{i=1,2} \mathbf{P}_i(1), \tag{5.6}$$

then the mild solution of (1.1) *does not blow up in finite time.*

Proof: There exists at least one non-decreasing function that satisfies $m(t) \geq \|U^i_D(t, 0)f_i\|_\infty$, $t \geq 0$, $i = 1, 2$. Indeed, define $m(t) := \|M_1(t, \cdot)\|_\infty \vee \|M_2(t, \cdot)\|_\infty$, $t \geq 0$, wherein $M_i(t, x) := \langle f_i, \bar{p}_i(t, x, \cdot)\rangle$ and $\bar{p}_i(t, x, y) := \sup\{p^i_D(0, x, s, y); 0 \leq s \leq t\}$, $t \geq 0$, $x, y \in D$, $i = 1, 2$.

Let (V_1, V_2) be the solution of the integral system

$$V_i(t) = 1 + \int_0^t h_i(s)\bar{\rho}_i\big(m(s)\big)\mathcal{R}_i\big(V_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\ i = 1, 2, \tag{5.7}$$

which is globally defined due to (5.6) and Lemma 5.1. Let us define $v_i(t, x) := V_i(t)m(t)$, $t \geq 0$, $x \in D$, $i = 1, 2$, and consider the function A defined by (4.1). Using property $[H_3]$ and the fact that m is non-decreasing, it follows that for all $t \geq 0$, $x \in D$, $i = 1, 2$,

$$\begin{aligned}
\mathrm{A}_i v_i(t, x) &\leq m(t) + \int_0^t h_i(s)\rho_i\big(m(s)\big)\mathcal{R}_i\big(V_{i'}(s)\big)\,\mathrm{d}s \\
&\leq m(t) + \int_0^t h_i(s)m(t)\bar{\rho}_i\big(m(s)\big)\mathcal{R}_i\big(V_{i'}(s)\big)\,\mathrm{d}s \\
&= v_i(t, x).
\end{aligned}$$

Thus, if $\boldsymbol{v} := (v_1, v_2)$, we have shown that

$$\mathrm{A}\boldsymbol{v} \leq \boldsymbol{v}, \tag{5.8}$$

where $\leq$ is the partial order defined in (4.3). Since f_1 and f_2 are non-negative, it is immediate that $\mathbf{0} := (0, 0) \leq \mathrm{A}\mathbf{0}$. Using this, (5.8) and Lemma 4.2 it can be shown by mathematical induction that $\mathbf{0} \leq \boldsymbol{u}_n \leq \boldsymbol{v}$, $n = 0, 1, 2, \ldots$, where $(\boldsymbol{u}_n)$ is the sequence defined in (4.4) (see Proposition 4.3). Letting $n \to \infty$, (4.5) implies that $\mathbf{0} \leq \boldsymbol{u} \leq \boldsymbol{v}$, where $\boldsymbol{u} = (u_1, u_2)$ is the mild solution of (1.1). Hence the conclusion follows from the globallity of each v_i, $i = 1, 2$. □

Corollary 5.5. *If the conditions of Theorem 5.4 hold and* (V_1, V_2) *is the solution of the integral system* (5.7), *then*

$$U^i_D(t, 0)f_i(x) \leq u_i(t, x) \leq V_i(t)m(t), \quad t \geq 0,\ x \in D,\ i = 1, 2.$$

Example 5.6. Suppose that $\mathcal{R}_i(u) = \rho_i(u) = u^{\beta_i}$, $\beta_i > 1$, $i = 1, 2$, and consider the non-decreasing function

$$m(t) := \max_{i=1,2} \left\| \sup_{0<s\leq t} U_D^i(s,0) f_i \right\|_\infty .$$

Then, it can be shown that in this case (5.6) reduces to

$$\int_0^\infty \max_{i=1,2} h_i(t) m(t)^{\beta_i - 1} \, \mathrm{d}t \leq \min_{i=1,2} \left(\frac{\beta_{i'}+1}{\beta_i+1} \right)^{\frac{\beta_i}{\beta_i+1}} \frac{\beta_i+1}{\beta_1\beta_2 - 1}. \tag{5.9}$$

In Ceballos-Lira and Pérez (2020b) the globality of the solutions of model (1.1) was studied in the case $\mathcal{A}_i = \Delta_{\alpha_i}|_D$ and $\mathcal{R}_i(u) = u^{\beta_i}$, $\beta_i > 1$, $i = 1, 2$. In that paper, it was proved (see Theorem 4.1) that a sufficient condition for the non-blow up is

$$\int_0^\infty \max_{i=1,2} h_i(t) m(t)^{\beta_i - 1} \, \mathrm{d}t < \frac{1}{\beta_1 \vee \beta_2 - 1}. \tag{5.10}$$

Taking into account that

$$\frac{\beta_1\beta_2 - 1}{\beta_1 \vee \beta_2 - 1} \leq \min_{i=1,2} (\beta_i + 1)^{\frac{1}{\beta_i+1}} (\beta_{i'}+1)^{\frac{\beta_i}{\beta_i+1}}, \tag{5.11}$$

it can be shown that (5.10) implies (5.9); therefore, condition (5.9) is better than the one obtained in Ceballos-Lira and Pérez (2020b, Theorem 4.1).

Suppose now that the generators $\mathcal{A}_1$ and $\mathcal{A}_2$ are intrinsically ultracontractive. Then for each $t > 0$, there exist positive constants m_t^i, M_t^i such that

$$m_t^i \varphi_0^i(x) \varphi_0^i(y) \leq p_D^i(t,x,y) \leq M_t^i \varphi_0^i(x)\varphi_0^i(y), \quad x, y \in D, \; i = 1, 2. \tag{5.12}$$

In terms of the notation introduced at the beginning of Section 5, we can write that: For each $t > 0$, $p_D^i(t,x,y) \asymp \varphi_0^i(x)\varphi_0^i(y)$, $(x,y) \in D \times D$, $i = 1, 2$. If additionally operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are comparable, and the initial conditions f_1, f_2 satisfy $0 \leq f_1 \wedge f_2 \leq f := f_1 \vee f_2$, then from (5.12) and (3.1) we conclude that: For each $t > 0$,

$$U_D^1(t,0) f(x) \asymp U_D^2(t,0) f(x), \quad x \in D. \tag{5.13}$$

We will use this relation in the following theorem.

Theorem 5.7. *Let $f_1, f_2 \in \mathrm{L}^\infty(D)$ be non-negative and non-identically zero, $f := f_1 \vee f_2$ and $[0,\infty) \ni t \mapsto \eta_i(t)$ defined by*

$$\eta_i(t) := \left\| \frac{\rho_i\big(U_D^{i'}(t,0)f\big)}{U_D^i(t,0)f} \right\|_\infty, \quad t \geq 0, \; i = 1, 2.$$

Suppose that ρ_i is non-decreasing, $\bar{\rho}_i(0+) \in [0,\infty)$, $i = 1, 2$, and that the generators $\mathcal{A}_1$, $\mathcal{A}_2$ are comparable and intrinsically ultracontractive. If

$$\int_0^\infty \max_{i=1,2} h_i(t) \eta_i(t) \, \mathrm{d}t \leq \min_{i=1,2} \mathbf{P}_i(1), \tag{5.14}$$

then the mild solution of (1.1) is globally defined.

Proof: The fact that f_1 and f_2 are not identically zero guarantees that f is not identically zero, then the denominator of each η_i is positive. Previous to this theorem, it was shown that the intrinsic ultracontractivity property together with the comparability of the generators $\mathcal{A}_1$ and $\mathcal{A}_2$ imply (5.13). From this relation and the fact that ρ_i is non-decreasing with $\bar{\rho}_i(0+) \in [0,\infty)$, $i = 1, 2$, it follows that $\eta_i(t) < \infty$ for all $t \geq 0$, $i = 1, 2$.

Let $(\hat{V}_1, \hat{V}_2)$ be the solution of the integral system

$$\hat{V}_i(t) = 1 + \int_0^t h_i(s) \eta_i(t) \mathcal{R}_i\big(\hat{V}_{i'}(s)\big) \, \mathrm{d}s, \quad t \geq 0, \; i = 1, 2, \tag{5.15}$$

The condition (5.14) and Lemma 5.1 ensure the non-blow up of $(\hat{V}_1, \hat{V}_2)$. Let us define $\hat{v}_i(t,x) := \hat{V}_i(t) U_D^i(t,0) f(x)$, $t \geq 0$, $x \in D$, $i = 1,2$. From $[H_3]$, the definition of η_i and (5.15) it follows that

$$\begin{aligned} \mathrm{A}_i \hat{v}_i(t,x) &\leq U_D^i(t,0) f_i(x) + \int_0^t h_i(s) U_D^i(t,s) \mathcal{R}_i\big(\hat{V}_{i'}(s)\big) \rho_i\big(U_D^{i'}(s,0) f(x)\big)\, \mathrm{d}s \\ &\leq U_D^i(t,0) f(x) + \int_0^t h_i(s) \eta_i(s) \mathcal{R}_i\big(\hat{V}_{i'}(s)\big) U_D^i(t,0) f(x)\, \mathrm{d}s \\ &= \hat{v}_i(t,x), \end{aligned}$$

for all $t \geq 0$, $x \in D$, $i = 1,2$. From here, the result follows in the same form as in the proof of Theorem 5.4. □

Corollary 5.8. *If the conditions of Theorem 5.7 hold, then*

$$U_D^i(t,0) f_i(x) \leq u_i(t,x) \leq \hat{V}_i(t) U_D^i(t,0) f(x), \quad t \geq 0,\, x \in D,\, i = 1,2,$$

where $(\hat{V}_1, \hat{V}_2)$ is the solution of (5.15).

It is not easy to compare the conditions (5.6) and (5.14) since it is not always trivial to know the constants m_t^i, M_t^i in (5.12). However, in the model proposed in the following example, it is possible to relate criteria (5.6) and (5.14). That model was already studied in López-Mimbela and Pérez (2015).

Example 5.9. Suppose that $\mathcal{A}_1 = \mathcal{A}_2 \equiv \mathcal{A}$, where $\mathcal{A}$ is the generator of a killed symmetric pure jump Lévy process, such that the associated semigroup is intrinsically ultracontractive. Suppose also that $h_1 = h_2 \equiv 1$, $\mathcal{R}_i(u) = \rho_i(u) = u^{\beta_i}$, $\beta_i > 1$, $i = 1,2$ and $f := f_1 \vee f_2$. In López-Mimbela and Pérez (2015) the globality of the solutions of (1.1) for this case was studied, and as proved in Ceballos-Lira and Pérez (2020b, Remark 4.1), a sufficient condition for the non-blow up of solutions is

$$\int_0^\infty \max_{i=1,2} \Big\| \sup_{0<s\leq t} U_D(s,0) f \Big\|_\infty^{\beta_i - 1} \mathrm{d}t < \frac{1}{\beta_1 \vee \beta_2 - 1}. \tag{5.16}$$

Note that $\hat{m}(t) := \| \sup\{U_D(s,0)f; 0 < s \leq t\} \|_\infty$ is a non-decreasing function such that $\hat{m}(t) \geq \|U_D(t,0) f_i\|_\infty$ for all $t \geq 0$, $i = 1,2$. Then (see Example 5.6), (5.6) becomes

$$\int_0^\infty \max_{i=1,2} \Big\| \sup_{0<s\leq t} U_D(s,0) f \Big\|_\infty^{\beta_i - 1} \mathrm{d}t \leq \min_{i=1,2} \left(\frac{\beta_{i'} + 1}{\beta_i + 1} \right)^{\frac{\beta_i}{\beta_i + 1}} \frac{\beta_i + 1}{\beta_1 \beta_2 - 1}. \tag{5.17}$$

Moreover, by Theorem 5.7, (5.14) give us

$$\int_0^\infty \max_{i=1,2} \Big\| \big(U_D(t,0) f\big)^{\beta_i - 1} \Big\|_\infty \mathrm{d}t \leq \min_{i=1,2} \left(\frac{\beta_{i'} + 1}{\beta_i + 1} \right)^{\frac{\beta_i}{\beta_i + 1}} \frac{\beta_i + 1}{\beta_1 \beta_2 - 1}. \tag{5.18}$$

Due to the inequality (5.11), (5.16) implies (5.17), and from the fact that $\eta_i(t) \leq \hat{m}(t)$, $t \geq 0$, $i = 1,2$, (5.18) follows from (5.17); hence criterion (5.18) is better than criteria (5.16) and (5.17). Furthermore, from Corollaries 5.5, 5.8 and from the relation between η_i, $\hat{m}$, it follows that the mild solution (u_1, u_2) satisfies

$$U_D^i(t,0) f_i(x) \leq u_i(t,x) \leq \hat{V}_i(t) U_D^i(t,0) f(x) \leq V_i(t) \hat{m}(t),$$

for all $t \geq 0$, $x \in D$, $i = 1,2$. Consequently $\hat{V}_i(t) U_D^i(t,0) f(x)$ is a better approximation to $u_i(t,x)$ than $V_i(t) \hat{m}(t)$.

6. Blowing up solutions in finite time

In this section, we will give conditions that ensure the blow up in finite time of the mild solutions of (1.1). We will start by giving the proof of a Lemma that will allow us to obtain such conditions. As in Lemma 5.1, we will only use the properties $[H_1]$ and $[H_2]$ of each $\mathcal{R}_i$. The proof is based on Ceballos-Lira and Pérez (2020a, Theorem 5.1).

Lemma 6.1. *Let $[0,\infty) \ni t \mapsto a_i(t) \in [0,\infty)$ be such that $a_i(t) > 0$ if $t > 0$, $y_0^i \geq 0$, and let (y_1, y_2) be the solution of the integral system*

$$y_i(t) = y_0^i + \int_0^t a_i(s)\mathcal{R}_i\big(y_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\, i = 1, 2.$$

Define the bijective function $A(t) := \int_0^t (a_1 \wedge a_2)(s)\,\mathrm{d}s$, $t \geq 0$. Suppose that

$$\max_{i=1,2} \mathbf{P}_i(y_0^i) < A(\infty), \tag{6.1}$$

where $\mathbf{P}_i$ denotes the function defined by (5.1) (see Section 5). Then, the solution (y_1, y_2) blows up in finite time and the explosion time t_e satisfies

$$0 \leq t_e \leq \min_{i=1,2} A^{-1} \circ \mathbf{P}_i(y_0^i). \tag{6.2}$$

Proof: The proof is similar to that in Lemma 5.1:Consider the integral system

$$z_i(t) = y_0^i + \int_0^t a(s)\mathcal{R}_i\big(z_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\, i = 1, 2,$$

where $a(s) := (a_1 \wedge a_2)(s)$, $s \geq 0$. By comparison, the solution (z_1, z_2) satisfies $y_i(t) \geq z_i(t)$, $t \geq 0$, $i = 1, 2$. Using that (z_1, z_2) satisfies

$$z_i'(t) = a(t)\mathcal{R}_i\big(z_{i'}(t)\big), \quad t > 0,$$
$$z_i(0) = y_0^i, \quad i = 1, 2,$$

it can be shown that for some $i \in \{1, 2\}$, $z_i(t) \geq z(t)$, $t \geq 0$, where z is the solution of

$$z'(t) = a(t)\mathbf{R}_i\big(z(t)\big), \quad t > 0,$$
$$z(0) = y_0^i,$$

which blows up in finite time due to (6.1) and Ceballos-Lira et al. (2011, Lemma 2.2). Finally, (6.2) is a consequence of Ceballos-Lira and Pérez (2020a, Remark 5.1). □

Remember that we denote by $\tilde{\mathrm{L}}_i^2(D)$ the space of all square integrable functions with respect to the probability measure $\mathrm{d}\tilde{\mathbb{P}}^i(x) = \varphi_0^i(x)^2\,\mathrm{d}x$, and by $\big(\tilde{S}_D^i(t); t \geq 0\big)$ the strongly continuous semigroup given by (2.6) (see Section 2).Let us define the evolution system $\big(\tilde{U}_D^i(t,s); t \geq s \geq 0\big)$ on $\tilde{\mathrm{L}}_i^2(D)$ by

$$\tilde{U}_D^i(t,s)g_i := \tilde{S}_D^i\big(K_i(t,s)\big)g_i, \quad t \geq s \geq 0,\, g_i \in \tilde{\mathrm{L}}_i^2(D),\, i = 1, 2. \tag{6.3}$$

The following lemma is an immediate consequence of Proposition 2.3.

Lemma 6.2. *For any $g_i \in \tilde{\mathrm{L}}_i^2(D)$, $t \geq s \geq 0$, $i = 1, 2$,*

$$\tilde{\mathbb{E}}^i\{\tilde{U}_D^i(t,s)g_i\} = \tilde{\mathbb{E}}^i\{g_i\}.$$

Theorem 6.3. *Let $f_1, f_2 \in C_0(D)$ be non-negative, and h_i such that $h_i(t) > 0$ if $t > 0$, for all $i = 1, 2$. Define the function*

$$m_i(t) := \frac{\|\varphi_0^i\|_1 \mathrm{e}^{\lambda_0^i K_i(t,0)} h_i(t)}{\rho_i(\|\varphi_0^{i'}/\varphi_0^i\|_\infty)\rho_i(\|\varphi_0^i\|_1)\rho_i(\mathrm{e}^{\lambda_0^{i'} K_{i'}(t,0)})}, \quad t \geq 0,\, i = 1, 2,$$

and the bijective function $M(t) := \int_0^t (m_1 \wedge m_2)(s)\, \mathrm{d}s$, $t \geq 0$. *If the generators* $\mathcal{A}_i$ *are comparable,* $\mathcal{R}_i$ *is convex ,* $i = 1, 2$, *and*

$$\max_{i=1,2} \mathbf{P}_i(\langle f_i, \varphi_0^i \rangle) < M(\infty), \tag{6.4}$$

then the mild solution of (1.1) *blows up in a finite time* t_e *such that*

$$0 \leq t_e \leq \min_{i=1,2} M^{-1} \circ \mathbf{P}_i(\langle f_i, \varphi_0^i \rangle). \tag{6.5}$$

Proof: First, note that relation (3.1) implies that the function $\theta_i := \varphi_0^{i'}/\varphi_0^i \in C_b(D)$, therefore the function m_i is well defined.

Now, proceeding by contradiction, suppose that the mild solution does not blow up in finite time. Define

$$w_i(t,x) := \frac{\mathrm{e}^{\lambda_0^i K_i(t,0)} u_i(t,x)}{\varphi_0^i(x)}, \quad t \geq 0,\, x \in D,\, i = 1, 2.$$

Solving for u_i and substituting in (2.4), from (6.3) we obtain that

$$w_i(t,x) = \tilde{U}_D^i(t,0) g_i(x) + \int_0^t h_i(s) \mathrm{e}^{\lambda_0^i K_i(s,0)} \tilde{U}_D^i(t,s) \frac{\mathcal{R}_i\big(\mathrm{e}^{-\lambda_0^{i'} K_{i'}(s,0)} \varphi_0^{i'}(x) w_{i'}(s,x)\big)}{\varphi_0^i(x)}\, \mathrm{d}s, \tag{6.6}$$

for all $t \geq 0$, $x \in D$, $i = 1, 2$, where we have taken $g_i := f_i/\varphi_0^i$, $i = 1, 2$. In general, any function of the form G/φ_0^i belongs to $\tilde{\mathrm{L}}_i^2(D)$, $i = 1, 2$, provided that $G \in \mathrm{L}^2(D)$ (see (2.5)). Thus, $g_i \in \tilde{\mathrm{L}}_i^2(D)$ since $f_i \in C_0(D)$. On the other hand, properties $[H_1]$ and $[H_3]$ imply that

$$\begin{aligned} \frac{\mathcal{R}_i(\|u_{i'}(s,\cdot)\|_\infty)}{\varphi_0^i(x)} &\geq \frac{\mathcal{R}_i\big(\mathrm{e}^{-\lambda_0^{i'} K_{i'}(s,0)} \varphi_0^{i'}(x) w_{i'}(s,x)\big)}{\varphi_0^i(x)} \\ &\geq \frac{\mathcal{R}_i\big(\theta_i(x) \varphi_0^{i'}(x) w_{i'}(s,x) \|\varphi_0^i\|_1\big)}{\rho_i(\|\theta_i\|_\infty) \rho_i(\mathrm{e}^{\lambda_0^{i'} K_{i'}(s,0)}) \rho_i(\|\varphi_0^i\|_1) \varphi_0^i(x)}, \end{aligned} \tag{6.7}$$

for all $s \in [0,t]$, $x \in D$, $i = 1, 2$. The non-blowing up assumption implies $\mathcal{R}_i(\|u_{i'}(s,\cdot)\|_\infty) \in \mathrm{L}^2(D)$ for each $s \in [0,t]$, $i = 1, 2$. Therefore, the function inside the operator $\tilde{U}_D^i(t,s)$ in the integral term of (6.6) belongs to $\tilde{\mathrm{L}}_i^2(D)$. Taking the expected value $\tilde{\mathbb{E}}^i$ in both members of (6.6), applying Lemma 6.2, and using the second inequality of (6.7), we get that for all $t \geq 0$ and $i = 1, 2$,

$$\tilde{\mathbb{E}}^i\{w_i(t,\cdot)\} \geq \langle f_i, \varphi_0^i \rangle + \int_0^t m_i(s) \tilde{\mathbb{E}}^i \left\{ \frac{\mathcal{R}_i\big(\theta_i \varphi_0^{i'} w_{i'}(s,\cdot) \|\varphi_0^i\|_1\big)}{\|\varphi_0^i\|_1 \varphi_0^i} \right\} \mathrm{d}s.$$

Applying the definition of $\tilde{\mathbb{E}}^i$ on the right hand side of the above inequality, and then using Jensen's inequality with respect to the probability measure $\|\varphi_0^i\|_1^{-1} \varphi_0^i(x)\, \mathrm{d}x$ and the convex function $\mathcal{R}_i$, it follows that

$$\tilde{\mathbb{E}}^i\{w_i(t,\cdot)\} \geq \langle f_i, \varphi_0^i \rangle + \int_0^t m_i(s) \mathcal{R}_i\big(\tilde{\mathbb{E}}^{i'}\{w_{i'}(s,\cdot)\}\big)\, \mathrm{d}s, \quad t \geq 0,\, i = 1, 2.$$

If (p_1, p_2) is the solution of the integral system

$$p_i(t) = \langle f_i, \varphi_0^i \rangle + \int_0^t m_i(s) \mathcal{R}_i\big(p_{i'}(s)\big)\, \mathrm{d}s, \quad t \geq 0,\, i = 1, 2, \tag{6.8}$$

then by comparison $\tilde{\mathbb{E}}^i\{w_i(t,\cdot)\} \geq p_i(t)$, $t \geq 0$, $i = 1, 2$. However, the solution (p_1, p_2) blows up in finite time due to (6.4) and Lemma 6.1. Furthermore, from the fact that

$$\|u_i(t,\cdot)\|_\infty \geq \frac{\mathrm{e}^{-\lambda_0^i K_i(t,0)}}{\|\varphi_0^i\|_1} \tilde{\mathbb{E}}^i\{w_i(t,\cdot)\}, \quad t \geq 0,\, i = 1, 2,$$

it follows that (u_1, u_2) blows up in finite time, which contradicts our non-blowing up assumption.

The second part of our theorem is a consequence of the inequality (6.2) in Lemma 6.1. This concludes our proof. □

Corollary 6.4. *If the conditions of Theorem 6.3 hold, then the mild solution of* (1.1) *satisfies*

$$\|u_i(t,\cdot)\|_\infty \geq \frac{\mathrm{e}^{-\lambda_0^i K_i(t,0)}}{\|\varphi_0^i\|_1} p_i(t), \quad 0 \leq t < t_e,\ i = 1,2,$$

wherein (p_1, p_2) *is the solution of the integral system* (6.8).

Example 6.5. Assume that in (1.1), $\mathcal{R}_i(u) = \rho_i(u) = u^{\beta_i}$, $\beta_i > 1$, $h_i \equiv 1$, $f_i \in C_0(D)$ is non-negative and non-identically zero, and $\mathcal{A}_i$, $i = 1,2$ are comparable. The blow up in finite time condition for this case becomes (see Theorem 6.3)

$$\max_{i=1,2}\left(\frac{\beta_{i'}+1}{\beta_i+1}\right)^{\frac{\beta_i}{\beta_i+1}}\left(\frac{\beta_i+1}{\beta_1\beta_2-1}\right)\langle f_i,\varphi_0^i\rangle^{-\frac{\beta_1\beta_2-1}{\beta_i+1}} < \min_{i=1,2}\int_0^\infty \frac{E_i(t)}{\|\theta_i\|_\infty^{\beta_i}\|\varphi_0^i\|_1^{\beta_i-1}}\,\mathrm{d}t, \tag{6.9}$$

where $E_i(t) := \mathrm{e}^{-\lambda_0^{i'}K_{i'}(t,0)\beta_i}\mathrm{e}^{-\lambda_0^i K_i(t,0)}$, $t \geq 0$, $i = 1,2$.

In Pérez (2015) the blow up in finite time of this model was studied assuming that $\mathcal{A}_i$ is the generator of a symmetric pure jump Lévy process killed in D^c, whose associated semigroup is intrinsically ultracontractive, $i = 1,2$. Under the additional assumption that D is $C^{1,1}$, it was shown that the generators $\mathcal{A}_i$, $i = 1,2$ are comparables. In that work, the blow up criterion obtained was

$$\min_{i=1,2}\langle f_i,\varphi_0^i\rangle > \max_{i=1,2}\left[\left(\frac{\beta_i+1}{\beta_{i'}+1}\right)^{\frac{\beta_i}{\beta_i+1}}\left(\frac{\beta_1\beta_2-1}{\beta_i+1}\right)\cdot\int_0^\infty \min_{i=1,2}\frac{C_i}{\|\varphi_0^{i'}\|_1^{\beta_i-1}}E_i(t)\,\mathrm{d}t\right]^{-\frac{\beta_i+1}{\beta_1\beta_2-1}}, \tag{6.10}$$

where $C_i = c_{i'}^{(-1)^i}(c_1c_2^{-1})^{\beta_i}$, $i = 1,2$, and the positive constants c_1,c_2 satisfy

$$c_1\varphi_0^1(x) \leq \varphi_0^2(x) \leq c_2\varphi_0^1(x), \quad x \in D, \tag{6.11}$$

and $c_1 \leq c_2$. Inequalities in (6.11) follows from (3.1). It can be shown that (6.10) is equivalent to

$$\max_{i=1,2}\left(\frac{\beta_{i'}+1}{\beta_i+1}\right)^{\frac{\beta_i}{\beta_i+1}}\left(\frac{\beta_i+1}{\beta_1\beta_2-1}\right)\left(\min_{i=1,2}\langle f_i,\varphi_0^i\rangle\right)^{-\frac{\beta_1\beta_2-1}{\beta_i+1}} < \int_0^\infty \min_{i=1,2}\frac{C_i}{\|\varphi_0^{i'}\|_1^{\beta_i-1}}E_i(t)\,\mathrm{d}t. \tag{6.12}$$

From (6.11), we can obtain the following estimations

$$\|\theta_i\|_\infty^{\beta_i} \leq c_{i'}^{(-1)^{i'}\beta_i} \quad\text{and}\quad \|\varphi_0^i\|_1^{\beta_i-1} \leq c_i^{(-1)^i(\beta_i-1)}\|\varphi_0^{i'}\|_1^{\beta_i-1}, \quad i = 1,2.$$

Thus,

$$\frac{C_i}{\|\varphi_0^{i'}\|_1^{\beta_i-1}}\cdot\|\theta_i\|_\infty^{\beta_i}\|\varphi_0^i\|_1^{\beta_i-1} \leq (c_ic_{i'}^{-1})^{(-1)^i(\beta_i-1)}(c_1c_2^{-1})^{\beta_i} = \frac{c_1}{c_2} \leq 1.$$

Substituting this in (6.12) we observe that (6.10) implies (6.9). This proves that our condition (6.9) is better than (6.10). Furthermore, (6.9) can be used in models with comparable symmetric Lévy generators whose associated killed semigroups do not have the intrinsic contractivity property.

Finally, we note that if additionally $\mathcal{A}_i \equiv \mathcal{A}$ and $k_i \equiv k$, $i = 1,2$, then $\theta_i \equiv 1$, $i = 1,2$, and (6.9) coincides with (6.10) whenever $c_i \equiv 1$, for $\varphi_0^i \equiv \varphi_0$, $i = 1,2$. The blow up in finite time of this particular case was already studied in López-Mimbela and Pérez (2015), and the additional hypothesis that D is of class $C^{1,1}$ is not required due to the validity of inequalities (6.11). Our blow up criterion is consistent with the criterion obtained in López-Mimbela and Pérez (2015, Proposition 5, p. 728) and applicable to models with symmetric Lévy generators whose associated killed semigroups are not intrinsically ultracontractive.

In the following theorem we improve the blow up criterion (6.4) when generators $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal.

Theorem 6.6. *Let f_i, h_i, $\mathcal{R}_i$ be as in Theorem 6.3. Suppose that $\mathcal{A}_i \equiv \mathcal{A}$, $i = 1, 2$, and take $\lambda_0^i \equiv \lambda_0$, $\varphi_0^i \equiv \varphi_0$. Define*

$$n_i(t) := \frac{\|\varphi_0\|_1 \mathrm{e}^{\lambda_0 K_i(t,0)} h_i(t)}{\rho_i(\|\varphi_0\|_1)\rho_i(\mathrm{e}^{\lambda_0 K_{i'}(t,0)})}, \quad t \geq 0,\ i = 1, 2,$$

and $N(t) := \int_0^t (n_1 \wedge n_2)(s)\,\mathrm{d}s$, $t \geq 0$. If

$$\max_{i=1,2} \mathbf{P}_i(\langle f_i, \varphi_0^i \rangle) < N(\infty), \tag{6.13}$$

then the mild solution of (1.1) blows up in a finite time t_e such that

$$0 \leq t_e \leq \min_{i=1,2} N^{-1} \circ \mathbf{P}_i(\langle f_i, \varphi_0^i \rangle). \tag{6.14}$$

Moreover, the mild solution (u_1, u_2) satisfies

$$\|u_i(t, \cdot)\|_\infty \geq \frac{\mathrm{e}^{-\lambda_0 K_i(t,0)}}{\|\varphi_0\|_1} q_i(t), \quad 0 \leq t < t_e,\ i = 1, 2,$$

wherein (q_1, q_2) is the solution of the integral system

$$q_i(t) = \langle f_i, \varphi_0 \rangle + \int_0^t n_i(s) \mathcal{R}_i\big(q_{i'}(s)\big)\,\mathrm{d}s, \quad t \geq 0,\ i = 1, 2.$$

Proof: The proof is completely similar to the one given in Theorem 6.3. Simply write φ_0, λ_0, $\tilde{\mathbb{E}}$, $\tilde{\mathrm{L}}^2(D)$, instead of φ_0^i, λ_0^i, $\tilde{\mathbb{E}}^i$, $\tilde{\mathrm{L}}_i^2(D)$ respectively, remove θ_i and $\rho(\|\theta_i\|_\infty)$ from each of the expressions that appear, and delete any references to them. Here $\tilde{\mathbb{E}}$ denotes the expected value with respect to the probability measure $\mathrm{d}\tilde{\mathbb{P}}(x) = \varphi_0(x)^2\,\mathrm{d}x$, and $\tilde{\mathrm{L}}^2(D)$ is the space of all square integrable functions with respect to $\mathrm{d}\tilde{\mathbb{P}}$. ☐

Property $[H_3]$ implies that $\rho_i(1) \geq 1$, $i = 1, 2$, consequently the condition (6.13) is better than (6.4), and (6.14) is a better approximation to the blow up time t_e than that given in (6.5). In the following theorem we use the notation introduced in Section 4.

Theorem 6.7. *Let $\boldsymbol{w} = (w_1, w_2)$ be the mild solution of the problem*

$$\begin{aligned} &\frac{\partial w_i}{\partial t}(t,x) = k_i(t)\mathcal{A}_i w_i(t,x) + h_i(t)\mathcal{S}_i\big(w_{i'}(t,x)\big), \quad t > 0,\ x \in D, \\ &w_i(0,x) = g_i(x), \quad x \in D,\ w_i\,|_{D^c} = 0,\ i' = 3 - i,\ i = 1, 2, \end{aligned} \tag{6.15}$$

where k_i, h_i, D, $\mathcal{A}_i$ satisfy our initial hypothesis (see Section 1), and $g_i \in \mathrm{L}^\infty(D)$, $[0,\infty) \ni v \mapsto \mathcal{S}_i(v) \in [0,\infty)$ is locally Lipschitz and non-decreasing, $i = 1, 2$. Let $\boldsymbol{u} = (u_1, u_2)$ be the mild solution of (1.1) where $f_i \in \mathrm{L}^\infty(D)$, and $\mathcal{R}_i$ is locally Lipschitz non-decreasing and non-negative, $i = 1, 2$. Suppose that $\mathcal{S}_i(v) \geq \mathcal{R}_i(v)$ for all $v \geq 0$, $i = 1, 2$, and that $\boldsymbol{u}$, $\boldsymbol{w}$ are defined on some interval $[0, \tau_{\max})$. If $\boldsymbol{g} := (g_1, g_2)$, $\boldsymbol{f} := (f_1, f_2)$ and $\boldsymbol{g} \geq \boldsymbol{f}$, then $\boldsymbol{w} \geq \boldsymbol{u}$.

Proof: As in (4.1), let us define $\mathcal{V} \ni \boldsymbol{v} \mapsto \mathrm{T}\boldsymbol{v} = (\mathrm{T}v_1, \mathrm{T}v_2)$ by

$$\mathrm{T}_i v_i(t,x) := U_D^i(t,0) g_i(x) + \int_0^t h_i(s) U_D^i(t,s) \mathcal{S}_i\big(v_{i'}(s,x)\big)\,\mathrm{d}s,$$

for any $t \geq 0$, $x \in D$, $i = 1, 2$. Since h_i is non-negative, and $\mathcal{S}_i$ is non-decreasing and non-negative, it can be shown (arguing as in Lemma 4.2) that T is non-decreasing with respect to the partial order (4.3). As in Proposition 4.3, we define the sequence $(\boldsymbol{w}_n)$ in $\mathcal{V}$ by

$$\boldsymbol{w}_n := \begin{cases} \big(U_D^1(t,0)g_1, U_D^2(t,0)g_2\big), & n = 0, \\ \mathrm{T}\boldsymbol{w}_{n-1}, & n \geq 1. \end{cases}$$

Following the same reasoning as in the proof of Proposition 4.3, it follows that if $\boldsymbol{w}_n = (w_{n1}, w_{n2})$, then

$$w_i(t,x) = \lim_{n\to\infty} w_{ni}(t,x), \quad 0 \le t < \tau_{\max},\, x \in D,\, i = 1,2. \tag{6.16}$$

Since $U_D^i(t,s)$ is positive-preserving, $\boldsymbol{g} \ge \boldsymbol{f}$ and $\mathcal{S}_i(v) \ge \mathcal{R}_i(v)$, $v \ge 0$, $i = 1,2$, we have that $\boldsymbol{w}_0 \ge \boldsymbol{u}_0$ and $\mathrm{T}\boldsymbol{v} \ge \mathrm{A}\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$. Using mathematical induction we can conclude $\boldsymbol{w}_n \ge \boldsymbol{u}_n$ for all $n = 0,1,2,\ldots$, and hence the inequality $n \to \infty$, $\boldsymbol{w} \ge \boldsymbol{u}$ follows from (6.16) and (4.5). □

Example 6.8. Let $\boldsymbol{w} = (w_1, w_2)$ be the mild solution of the problem

$$\begin{aligned} \frac{\partial w_i}{\partial t}(t,x) &= \mathcal{A} w_i(t,x) + \delta_i \mathrm{e}^{w_{i'}(t,x)}, \quad t > 0,\, x \in D, \\ w_i(0,x) &= f_i(x), \quad x \in D,\, w_i\,|_{D^c} = 0,\, i' = 3-i,\, i = 1,2, \end{aligned} \tag{6.17}$$

where $\mathcal{A}$ is the generator of a symmetric Lévy process, $f_i \in C_0(D)$ is non-negative and $\delta_i > 0$, $i = 1,2$. If

$$\min_{i=1,2} \delta_i > \lambda_0, \tag{6.18}$$

then $\boldsymbol{w}$ blows up in finite time. Indeed, let us define $\delta := \delta_1 \wedge \delta_2$, $m_0 := \langle f_1, \varphi_0\rangle \wedge \langle f_2, \varphi_0\rangle$ and consider the following cases.

I. $\|\varphi_0\|_1 \ge m_0$. Due to (6.18) we can choose $\mu > 0$ such that

$$0 < \mu < \Big(\frac{\delta}{\lambda_0} - 1\Big)\|\varphi_0\|_1 \wedge 1. \tag{6.19}$$

Note that the function $(1,\mathrm{e}) \ni \beta \mapsto \beta(1 - \ln\beta) \in (0,1)$ is strictly decreasing. Using this, (6.19) and the fact that $\|\varphi_0\|_1 + \mu \ge m_0 + \mu$, we can take $\beta \in (1,\mathrm{e})$ such that

$$\left[(1+\mu)\left(\frac{\|\varphi_0\|_1 + \mu}{m_0 + \mu}\right)\right]^\beta < \frac{\delta}{\lambda_0}\|\varphi_0\|_1 \left(\frac{1+\mu}{m_0+\mu}\right) \tag{6.20}$$

and

$$\beta(1 - \ln\beta) > \mu. \tag{6.21}$$

Let $\boldsymbol{u} = (u_1, u_2)$ be the mild solution of

$$\begin{aligned} \frac{\partial u_i}{\partial t}(t,x) &= \mathcal{A} u_i(t,x) + \delta_i [u_{i'}(t,x) + \mu]^\beta, \quad t > 0,\, x \in D, \\ u_i(0,x) &= f_i(x), \quad x \in D,\, u_i\,|_{D^c} = 0,\, i' = 3-i,\, i = 1,2. \end{aligned}$$

Using (6.19) and (6.21), it can be shown that $\mathrm{e}^v \ge (v+\mu)^\beta$ for all $v \ge 0$. Then by Theorem 6.7 it follows that $\boldsymbol{w} \ge \boldsymbol{u}$. But (6.20) and Theorem 6.6 imply that $\boldsymbol{u}$ blows up in finite time, which proves our affirmation for this case.

II. $\|\varphi_0\|_1 < m_0$. From (6.18) we obtain $\mathrm{e} - 1 > \ln(\delta/\lambda_0)/\ln(\|\varphi_0\|_1/m_0)$. Fix $\beta \in (1,\mathrm{e})$ such that

$$\left(\frac{\|\varphi_0\|_1}{m_0}\right)^{\beta-1} < \frac{\delta}{\lambda_0}. \tag{6.22}$$

Let us now assume that $\boldsymbol{u} = (u_1, u_2)$ is the mild solution of

$$\begin{aligned} \frac{\partial u_i}{\partial t}(t,x) &= \mathcal{A} u_i(t,x) + \delta_i u_{i'}(t,x)^\beta, \quad t > 0,\, x \in D, \\ u_i(0,x) &= f_i(x), \quad x \in D,\, u_i\,|_{D^c} = 0,\, i' = 3-i,\, i = 1,2. \end{aligned}$$

Since $\beta \in (1,\mathrm{e})$, it follows that $\mathrm{e}^v \ge v^\beta$, $v \ge 0$, hence $\boldsymbol{w} \ge \boldsymbol{u}$ by Theorem 6.7. Finally, our affirmation follows from (6.22) and Theorem 6.6.

The scalar version of (6.17), with zero initial condition and Gaussian diffusion, has the form

$$\begin{aligned} &\frac{\partial w}{\partial t}(t,x) = \Delta w(t,x) + \delta \mathrm{e}^{w(t,x)}, \quad t>0,\ x\in D,\\ &w(0,x)=0, \quad x\in D,\ w\mid_{D^c}=0. \end{aligned} \tag{6.23}$$

The equation with exponential reaction $\partial_t w = \Delta w + \delta \mathrm{e}^w$ is known in combustion theory as the Frank-Kamenetzky equation (non-dimensionalized). The dimensionless constant δ (called the Frank-Kamenetzky constant) is important because it depends on many parameters, for example, the density of the material and its thermic diffusivity.

The original model proposed by Frank-Kamenetzky describes how the temperature of a combustible material evolves over time, subject to an initial ignition process, and whose external temperature is constant. Through additional assumptions and a change of variables, that original model becomes (6.23). In this case, the explosion phenomenon is known as a thermal explosion or spontaneous ignition (Bebernes and Eberly, 1989; Harley, 2010).

In Bebernes and Kassoy (1981) it is shown that (6.23) has blowing up solutions when

$$\delta > \frac{\lambda_0}{\mathrm{e}}. \tag{6.24}$$

From Example 6.8 it follows that

$$\begin{aligned} &\frac{\partial w}{\partial t}(t,x) = \mathcal{A} w(t,x) + \delta \mathrm{e}^{w(t,x)}, \quad t>0,\ x\in D,\\ &w(0,x)=0, \quad x\in D,\ w\mid_{D^c}=0, \end{aligned} \tag{6.25}$$

blows up in finite time if

$$\delta > \lambda_0. \tag{6.26}$$

Note that model (6.25) includes the Frank-Kamenetzky model (6.23), and that condition (6.26) is consistent with condition (6.24) in the classical diffusion case. Although (6.24) is better than (6.26) in the Gaussian case, model (6.25) allows the use of other diffusions, for example, Laplacian fractional exponents. This approach is known to be useful for the dynamic description of diffusion in disorder or complex media. This phenomenon is called anomalous diffusion and is currently widely studied in other fields of science. For more details, see Vázquez (2014) and references therein.

The equation $\partial_t w = \Delta w + \delta \mathrm{e}^w$ is also important in other fields of science, even in purely mathematics fields such as differential geometry (see Galaktionov and Vázquez, 2002, p. 402 and references therein).

Remark 6.9. Theorem 6.7 also holds when every inequality $\geq$ is replaced by inequality $\leq$. The resulting theorem can be used to study the globality of a given model by comparing it with other model that is globally defined.

Example 6.10. Suppose that $\boldsymbol{u} = (u_1, u_2)$ is the mild solution of the problem

$$\begin{aligned} &\frac{\partial u_i}{\partial t}(t,x) = \mathcal{A} u_i(t,x) + u_{i'}(t,x)\big[\ln\big(1+u_{i'}(t,x)\big)\big]^{\beta}, \quad t>0,\ x\in D,\\ &u_i(0,x) = f_i(x), \quad x\in D,\ u_i\mid_{D^c}=0,\ i'=3-i,\ i=1,2, \end{aligned}$$

where $\mathcal{A}$ is the generator of a killed symmetric Lévy process, $f_i \in C_0(D)$ is non-negative, $i=1,2$, and $\beta \leq 1$. Define $\mathcal{R}_i(v) := (1+v)[\ln(1+v)]^{\beta}$, $v \geq 0$, $i=1,2$, and let $\boldsymbol{w} = (w_1, w_2)$ be the mild solution of

$$\begin{aligned} &\frac{\partial w_i}{\partial t}(t,x) = \mathcal{A} w_i(t,x) + \big[1+w_{i'}(t,x)\big]\big[\ln\big(1+w_{i'}(t,x)\big)\big]^{\beta}, \quad t>0,\ x\in D,\\ &w_i(0,x) = f_i(x), \quad x\in D,\ w_i\mid_{D^c}=0,\ i'=3-i,\ i=1,2. \end{aligned} \tag{6.27}$$

Since $\mathcal{S}_i(v) = v[\ln(1+v)]^{\beta} \leq \mathcal{R}_i(v)$ for all $v\geq 0$, $i=1,2$, it follows from Remark 6.9 that $\boldsymbol{u} \leq \boldsymbol{w}$. But since $\mathbf{P}_i(\|f_i\|_\infty) = \infty$, $i=1,2$, Corollary 5.3 implies that $\boldsymbol{w}$ is globally defined on $[0,\infty)\times D$.

The semilinear heat equation $\partial_t w = \Delta w + (1+w)[\ln(1+w)]^\beta$ (scalar version of (6.27) in the Gaussian case) was introduced in 1979 (with $d = 1$) and is a particular case of more general quasilinear models. The asymptotic behavior of its solutions (with $\beta \leq 1$) is described through solutions (called approximate self-similar invariant solutions) of equations of the Hamilton-Jacobi type (see Galaktionov and Vázquez, 2002, p. 411 and references therein).

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