ALEA, Lat. Am. J. Probab. Math. Stat. **22**, 493–516 (2025)
DOI: 10.30757/ALEA.v22-18

The acceptance profile of invasion percolation at p_c in two dimensions

Bounghun Bock and Michael Damron

School of Mathematics, Georgia Institute of Technology, 686 Cherry St, Atlanta, GA 30332, USA
E-mail address: `bbock1106@gmail.com`

School of Mathematics, Georgia Institute of Technology, 686 Cherry St., Atlanta, GA 30332, USA
E-mail address: `mdamron@michaeldamron.org`
URL: `https://michaeldamron.org`

Abstract. Invasion percolation is a stochastic growth model that follows a greedy algorithm. After assigning i.i.d. uniform random variables (weights) to all edges of $\mathbb{Z}^d$, the growth starts at the origin. At each step, we adjoin to the current cluster the edge of minimal weight from its boundary. Chayes et al. (1985) studied the "acceptance profile" of the invasion: for a given $p \in [0,1]$, it is the ratio of the expected number of invaded edges until time n with weight in $[p, p + \mathrm{d}p]$ to the expected number of observed edges (those in the cluster or its boundary) with weight in the same interval. They showed that in all dimensions, the acceptance profile $a_n(p)$ converges to one for $p < p_c$ and to zero for $p > p_c$. In this paper, we consider $a_n(p)$ at the critical point $p = p_c$ in two dimensions and show that it is bounded away from zero and one as $n \to \infty$.

1. Introduction

1.1. *The model.* We begin with the definition of invasion percolation. It is a stochastic growth model introduced independently by two groups (Chandler et al., 1982 and Lenormand and Bories, 1980) and is a simple example of self-organized criticality. That is, although the model itself has no parameter, its structure on large scales resembles that of another critical model: critical Bernoulli percolation.

Let $\mathbb{Z}^2$ be the two-dimensional square lattice and $\mathcal{E}^2$ be the set of nearest-neighbor edges. For a subgraph $G = (V, E)$ of $(\mathbb{Z}^2, \mathcal{E}^2)$, we define the outer (edge) boundary of G as

$$\partial G := \{e = \{x, y\} \in \mathcal{E}^d : e \notin E, \text{but } x \in V \text{ or } y \in V\}.$$

Assign i.i.d uniform random $[0,1]$ variables $(\omega(e))$ to all bonds $e \in \mathcal{E}^2$. The *invasion percolation cluster* (IPC) G can be defined as the limit of an increasing sequence of subgraphs (G_n) as follows. The graph G_0 has only the origin and no edges. Once $G_i = (V_i, E_i)$ is defined, we select the edge e_{i+1} that minimizes $\omega(e)$ for $e \in \partial G_i$, take $E_{i+1} = E_i \cup \{e_{i+1}\}$ and let G_{i+1} be the graph induced by

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Received by the editors July 9th, 2020; accepted July 16th, 2022.
2010 *Mathematics Subject Classification.* 60K35, 82B43.
Key words and phrases. invasion percolation, acceptance profile, near-critical percolation.
The research of M.D. was supported by an NSF CAREER grant.

the edge set E_{i+1}. The graph G_i is called the invaded region at time i, and the graph $G = \cup_{i=0}^{\infty} G_i$ is called the *invasion percolation cluster* (IPC).

The first rigorous study of invasion percolation was done by Chayes et al. (1985), who took a dynamical perspective: their questions were related to the evolution of the graph G_n as n increases. In the '90s and '00s, results focused on a more static perspective: properties of the full invaded region. For example, the fractal dimension of G was determined (Zhang, 1995) along with finer properties of G like relations to other critical models (Járai, 2003), analysis of the pond and outlet structure (Damron and Sapozhnikov, 2011; Damron et al., 2009), and scaling limits (Garban et al., 2018).

In this paper, we return to the earlier dynamical perspective and study the "acceptance profile" of the invasion, introduced in Wilkinson and Willemsen (1983). Roughly speaking, the acceptance profile $a_n(p)$ at value p and time n is the ratio

$$\begin{aligned} &a_n(p) \\ &= \frac{\text{expected number of bonds invaded with weight in } [p, p + \mathrm{d}p]}{\text{expected number of bonds observed with weight in } [p, p + \mathrm{d}p]}, \end{aligned}$$

where both the numerator and denominator are computed until time n, and a bond is observed by time n if it is either invaded by time n or is on the boundary of the invasion at time n. In Chayes et al. (1985, Thms 4.2, 4.3), it is shown that for general dimensions, if $p < \pi_c$ (a certain critical threshold for independent percolation), one has $a_n(p) \to 1$ as $n \to \infty$ and if $p > \bar{p}_c$ (another threshold value with $\bar{p}_c \geq \pi_c$), one has $a_n(p) \to 0$ as $n \to \infty$. Since publication of that paper, it has been established that $\bar{p}_c = \pi_c = p_c$, where p_c is the standard critical value for independent percolation. Since $p_c = 1/2$ in dimension 2, we have

$$\lim_{n\to\infty} a_n(p) = \begin{cases} 1 & \text{if } p < 1/2 \\ 0 & \text{if } p > 1/2. \end{cases}$$

This result means that when $p < p_c$, all observed edges with weight near p are invaded relatively quickly, whereas for $p > p_c$, observed edges with weight near p are never invaded (for n large).

The case $p = p_c$ was left open in Chayes et al. (1985), and it is this case we study here. It would be very interesting to establish the existence of $\lim_{n\to\infty} a_n(p_c)$, which by the following main theorem, would be a number in $(0, 1)$.

Theorem 1.1. *In two dimensions, where* $p_c = 1/2$*,*

$$0 < \liminf_{n\to\infty} a_n(p_c) \leq \limsup_{n\to\infty} a_n(p_c) < 1.$$

This theorem roughly states that when n is large, at least $c\epsilon$ fraction of invaded edges have weight in $(p_c, p_c + \epsilon]$, whereas at least $c\epsilon$ fraction of observed edges with weight in this interval are not yet invaded. To prove this result, we will need to study detailed properties of the invaded region at time n, which can be quite different than those of the full invaded region.

In the physics literature, the acceptance profile was considered earlier, in work of Wilkinson and Willemsen (1983). There, it was loosely defined as $a(r)$, the "number of random numbers in the interval $[r, r + \mathrm{d}r]$ which were accepted into the cluster, expressed as a fraction of the number of random numbers in that range which became available." It was noted in that paper that the acceptance profile appears to approach a step function with jump at p_c, and that for values of p near p_c, "there is a transition region in which some numbers are accepted and some rejected." (See Wilkinson and Willemsen, 1983, Fig. 2.) This observation, although for a different version of the acceptance profile (there is no expected value as in the acceptance profile of Chayes et al., 1985 that we work with), is consistent with our main theorem. The step function property of the profile has later been used to estimate numerical values of p_c (see, for example, Wilkinson and Barsony, 1984).

In the next section, we give a rigorous definition of the acceptance profile along with the results of Chayes et al. (1985). To do this, we will also introduce the standard Bernoulli percolation model.

1.2. *Acceptance Profile.* To define the acceptance profile, we use the notations of Chayes et al. (1985). Let $I_n \in \mathcal{E}^2$ be the invaded bond at time $n \geq 1$ and let x_n be the random weight of I_n (the weight $\omega(I_n)$). For any $y \in [0,1]$, define $X_n(y)$ as the indicator that $x_n \leq y$:

$$X_n(y) = \begin{cases} 1 & \text{if } x_n \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Let R_n be the random number of new bonds which must be checked after the invasion of I_n (that is, $R_0 = 4$, $R_1 = 3$, and R_n is the number of boundary edges of G_n that were not boundary edges of G_{n-1}) and define $L_n := \sum_{j=0}^{n} R_j$ to be the total number of checked bonds until the invasion of I_n. Clearly, $n \leq L_n \leq 4n$. Denote by v_n the value of the n^{th} checked bond. (Here we can enumerate the checked edges counted in R_n in any deterministic fashion.) Set $V_n(y)$ to be the indicator that $v_n \leq y$:

$$V_n(y) = \begin{cases} 1 & \text{if } v_n \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Then the acceptance profile at value x by time n is defined as

$$a_n(x) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}\left[\sum_{j=1}^{n} \Big(X_j(x+\epsilon) - X_j(x)\Big)\right]}{\mathbb{E}\left[\sum_{j=1}^{L_n} \Big(V_j(x+\epsilon) - V_j(x)\Big)\right]}. \tag{1.1}$$

It is shown in Chayes et al. (1985, Proposition 4.1) that $a_n(x)$ is an analytic function of x.

An alternative representation for the acceptance profile will be useful for us. Let $\tilde{Q}_n(x) = \sum_{j=1}^{n} X_j(x)$ be the number of invaded edges until time n with weight $\leq x$ and $\tilde{P}_n(x) = \sum_{j=1}^{L_n} V_j(x)$ be the number of checked edges until time n with weight $\leq x$. From Chayes et al. (1985, Eq. (4.3)), one has

$$\mathbb{E}[\tilde{P}_n(x)] = x\mathbb{E}[L_n],$$

and so we can rewrite (1.1) as

$$a_n(x) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{E}[\tilde{Q}_n(x+\epsilon) - \tilde{Q}_n(x)]}{\epsilon \mathbb{E}[L_n]}. \tag{1.2}$$

Analysis of the IPC and the acceptance profile heavily involves tools from Bernoulli percolation, whose definition depends on a parameter $p \in [0,1]$. We will couple the percolation model to the IPC in the following standard way. For every $e \in \mathcal{E}^2$ and any $p \in [0,1]$, we say that e is p-open if $\omega(e) \leq p$; otherwise, we say that e is p-closed. Note that the variables $(\mathbf{1}_{\{e \text{ is } p\text{-open}\}})_{e \in \mathcal{E}^2}$ are i.i.d. Bernoulli random variables with parameter p. The main object of study in percolation is the connectivity properties of the graph whose edges consist of the p-open edges. If p is large, we expect this graph to contain very large (even infinite) components and if p is small we expect it to contain only small components. To formulate these ideas precisely, we say that a path (a finite or infinite sequence of edges $e_1, e_2, \ldots$ such that e_i and e_{i+1} share at least one endpoint) is p-open if all its edges are p-open, and we write $A \xleftrightarrow{p} B$ for two sets of vertices A and B if there is a p-open path starting at a vertex in A and ending at a vertex in B. We also write $u \xleftrightarrow{p} v$ for vertices u, v when

$A = \{u\}$ and $B = \{v\}$, and we use the term "p-open cluster of u" to refer to the set of vertices v such that $u \overset{p}{\longleftrightarrow} v$. Last, we write $u \overset{p}{\longleftrightarrow} \infty$ to mean that the p-open cluster of u is infinite. Given this setup, we define the critical threshold for percolation as

$$p_c = \sup\{p \in [0,1] : \theta(p) = 0\},$$

where

$$\theta(p) = \mathbb{P}(0 \overset{p}{\longleftrightarrow} \infty).$$

It is known that for all dimensions $d \geq 2$, one has $p_c \in (0,1)$, and for $d = 2$, $p_c = 1/2$. These facts and more can be seen in the standard reference by Grimmett (1989).

In addition to p_c, there are other critical values that have been used in the past, and these have mostly been shown to be equal to p_c. The two that were used in Chayes et al. (1985) are

$$\pi_c = \sup\{p \in [0,1] : \mathbb{E}\#\{v : v \text{ is in the } p\text{-open cluster of } 0\} < \infty)\}, \text{ and}$$
$$\bar{p}_c = \sup\{p \in [0,1] : \mathbb{P}(\exists \text{ infinite } p\text{-open path in a half-space}) = 0\}.$$

In this language, and for general dimensions, the theorems of Chayes et al. (1985) state that

$$\lim_{n\to\infty} a_n(p) = \begin{cases} 1 & \text{if } p < \pi_c \\ 0 & \text{if } p > \bar{p}_c. \end{cases}$$

Because π_c and $\bar{p}_c$ are both known to be equal to p_c (see Grimmett and Marstrand, 1990; Harris, 1960; Menshikov and Sidorenko, 1987), this result specifies the limiting behavior of the acceptance profile at all values of $p \neq p_c$. Our main result, Theorem 1.1, shows that in two dimensions, the limiting behavior of $a_n(p_c)$ is different than that of $a_n(p)$ for any other value of p: it remains bounded away from zero and one.

1.3. *Notation and outline of the paper.* First we gather some notation used in the paper. For $n \geq 1$ let $B(n) = [-n,n]^2$ be the box of sidelength $2n$, and for $m < n$, let $\mathrm{Ann}(m,n)$ be the annulus $B(n) \setminus B(m)$. We will be interested in connection probabilities from points to boundaries of boxes, so we set

$$\pi(p,n) = \mathbb{P}(0 \overset{p}{\longleftrightarrow} \partial B(n)) \text{ and } \pi(n) = \pi(p_c, n).$$

Many connection probabilities (or their complements) can be expressed in terms of connections on the dual graph $(\mathbb{Z}^2)^*$. To define it, let $(\mathbb{Z}^2)^* = \left(\frac{1}{2},\frac{1}{2}\right) + \mathbb{Z}^2$ be the set of dual vertices and let $(\mathcal{E}^2)^*$ be the edges between nearest-neighbor dual vertices. For $x \in \mathbb{Z}^2$ we write $x^* = x + \left(\frac{1}{2},\frac{1}{2}\right)$ for its dual vertex. For an edge $e \in \mathcal{E}^2$, we denote its endpoints (left, respectively right or bottom, respectively top) by $e_x, e_y \in \mathbb{Z}^2$. The edge $e^* = \left\{e_x + \left(\frac{1}{2},\frac{1}{2}\right), e_y - \left(\frac{1}{2},\frac{1}{2}\right)\right\}$ is called the edge dual to e. (It is the unique dual edge that bisects e.) A dual edge e^* is called p-open if e is p-open, and is p-closed otherwise. A dual path is a finite or infinite sequence of dual edges such that consecutive edges share at least one endpoint. A circuit (or dual circuit) is a finite path (or dual path) which has the same initial and final vertices.

For two functions $f(x)$ and $g(x)$ from a set $\mathcal{X}$ to $\mathbb{R}$, the notation $f(x) \asymp g(x)$ means $\frac{f(x)}{g(x)}$ is bounded away from 0 and ∞, uniformly in $x \in \mathcal{X}$.

In the next section, we give the proof of Theorem 1.1. It is split into three subsections. In Section 2.1, we introduce correlation length and results which are frequently used in two-dimensional percolation. In Section 2.2, we prove the lower bound of Theorem 1.1 and in Section 2.3, we prove the upper bound of Theorem 1.1.

2. Proof of Theorem 1.1

2.1. *Preliminaries.* We first introduce the finite-size scaling correlation length (see a more detailed survey in Nolin, 2008). Let

$$\sigma(n,m,p)=\mathbb{P}(\exists \text{ a } p\text{-open horizontal crossing of } [0,n]\times[0,m]).$$

Here, a horizontal crossing is a path which remains in $[0,n]\times[0,m]$, with initial vertex in $\{0\}\times[0,m]$ and final vertex in $\{n\}\times[0,m]$. For any $\epsilon>0$, we set

$$L(p,\epsilon):=\begin{cases}\min\{n:\sigma(n,n,p)\le\epsilon\} & \text{if } p<p_c\\ \min\{n:\sigma(n,n,p)\ge 1-\epsilon\} & \text{if } p>p_c\end{cases}$$

$L(p,\epsilon)$ is called the finite-size scaling correlation length and its scaling as $p\to p_c$ does not depend on ϵ, so long as ϵ is small enough. That is, there exists an $\epsilon_0>0$ such that for $\epsilon_1,\epsilon_2\in(0,\epsilon_0]$, $L(p,\epsilon_1)\asymp L(p,\epsilon_2)$ as $p\to p_c$ (Kesten, 1987, Eq. (1.24)). For this reason, we set

$$L(p)=L(p,\epsilon_0).$$

Because $L(p)\to\infty$ as $p\to p_c$ (Nolin, 2008, Prop. 4) and $L(p)\to 0$ as $p\to 0$ or $p\to 1$, the approximate inverses

$$p_n=\sup\{p>p_c:L(p)>n\}$$
$$q_n=\inf\{q<p_c:L(q)>n\}$$

are well-defined.

Next we list relevant and now standard properties of the correlation length with references to their proofs.

(1) Kesten (1987, Thm. 1) For $n\le L(p)$ and $p\neq p_c$,

$$\pi(p,n)\asymp\pi(n). \tag{2.1}$$

(2) Kesten (1987, Thm. 2) There are positive constants C_1 and C_2 such that for all $p>p_c$

$$\pi(L(p))\le\pi(p,L(p))\le C_1\theta(p)\le C_1\pi(p,L(p))\le C_2\pi(L(p)). \tag{2.2}$$

(3) Járai (2003, Eq. (2.8)) There are positive constants C_3, C_4 such that for all $p>p_c$,

$$\sigma(2mL(p),mL(p),p)\ge 1-C_3\exp(-C_4 m),\quad \text{for } m>1. \tag{2.3}$$

(4) Járai (2003, Eq. (2.10)) There is a constant D such that

$$\lim_{\delta\downarrow 0}\frac{L(p-\delta)}{L(p)}\le D \quad \text{for } p>p_c. \tag{2.4}$$

(5) van den Berg and Kesten (1985, Cor. 3.15) There exists a constant $D_1>0$ such that

$$\frac{\pi(m)}{\pi(n)}\ge D_1\sqrt{\frac{n}{m}} \quad \text{for } m\ge n\ge 1. \tag{2.5}$$

(6) Nolin (2008, Prop. 34) (Arm events). Fix $e=\{e_x,e_y\}$ and let $A_n^{2,2}$ be the event that e_x and e_y are connected to $\partial B(e,n)$ by p_c-open paths not containing e, and $e_x{}^*$ and $e_y{}^*$ are connected to $\partial B(e,n)^*$ by p_c-closed dual paths not containing e^*. Here, $B(e,n)$ is the box of sidelength $2n$ centered at the lower-left endpoint of e. Note that these four paths are disjoint and alternate. For $n\ge 1$,

$$\begin{aligned}(p_n-p_c)n^2\mathbb{P}(A_n^{2,2})&\asymp 1\\ (p_c-q_n)n^2\mathbb{P}(A_n^{2,2})&\asymp 1.\end{aligned} \tag{2.6}$$

(7) Nolin (2008, Sec. 3.2) (Russo-Seymour-Welsh: RSW) For every $k, l \geq 1$, there exists $\delta_{k,l} > 0$ such that for all $p \in [p_c, p_n]$ (respectively $q \in [q_n, p_c]$),

$$\mathbb{P}\left(\begin{array}{c}\exists \text{ a } p\text{-open (respectively } q\text{-open) horizontal} \\ \text{crossing of } [0, kn] \times [0, ln]\end{array}\right) > \delta_{k,l}$$

$$\mathbb{P}\left(\begin{array}{c}\exists \text{ a } p\text{-closed (respectively } q\text{-closed) horizontal} \\ \text{dual crossing of } ([0, kn] \times [0, ln])^*\end{array}\right) > \delta_{k,l}.$$

In addition, applying the FKG inequality (Grimmett, 1989, Ch. 2), for all $p \in [p_c, p_n]$ (resp. $q \in [q_n, p_c]$),

$$\mathbb{P}\left(\begin{array}{c}\text{Ann}(n, kn) \text{ contains a } p\text{-open (resp.} \\ q\text{-open) circuit around the origin}\end{array}\right) > (\delta_{2k,k-1})^4$$

$$\mathbb{P}\left(\begin{array}{c}\text{Ann}(n, kn)^* \text{ contains a } p\text{-closed (resp.} \\ q\text{-closed) dual circuit around the origin}\end{array}\right) > (\delta_{2k,k-1})^4.$$

(8) Járai (2003); Zhang (1995) Let $|\mathcal{S}_n|$ be the number of invaded edges (edges in G) inside $B(n)$. Then,

$$\mathbb{E}|\mathcal{S}_n| \asymp n^2\pi(n). \tag{2.7}$$

Last, we prove some lemmas that will be helpful in the proof of the main theorem. These lemmas will bound the random variables

$$R_n := \min\{k : I_i \subset B(k) \text{ for } i = 1, 2, \cdots, n\}$$
$$r_n := \max\{k : I_i \subset B(k)^c \text{ for all } i > n\}.$$

R_n is a radius of the invaded region at time n, and r_n is the largest size of box such that the invasion does not change in this box after time n.

Lemma 2.1. *There exists a constant* $\mathsf{C}_1 > 0$ *such that for all* $n \geq 1$ *and* $C > 0$,

$$\mathbb{P}(R_{\lfloor Cn^2\pi(n)\rfloor} < n) \leq \frac{\mathsf{C}_1}{C}.$$

Proof: The event $\{R_{\lfloor Cn^2\pi(n)\rfloor} < n\}$ implies that $|\mathcal{S}_n| \geq \lfloor Cn^2\pi(n)\rfloor$. By Markov's inequality and (2.7),

$$\mathbb{P}(R_{\lfloor Cn^2\pi(n)\rfloor} < n) \leq \mathbb{P}(|\mathcal{S}_n| \geq \lfloor Cn^2\pi(n)\rfloor) \leq \frac{\mathbb{E}|\mathcal{S}_n|}{\lfloor Cn^2\pi(n)\rfloor} \leq \frac{\mathsf{C}_1}{C}.$$

□

Lemma 2.2. *For any* $\eta_0 > 0$, *there exists* $\mathsf{C}_2 > 0$ *such that for any* $C \geq \mathsf{C}_2$ *and* $n \geq 1$,

$$\mathbb{P}(r_{\lfloor Cn^2\pi(n)\rfloor} < 2n) \leq \eta_0$$

Proof: For $k, m \geq 1$, we consider the event $D_{k,m}$ defined by the following conditions:

(i) There is a p_c-open circuit around the origin in the annulus $\text{Ann}(2^{k+1}, 2^{k+1+\frac{m}{8}})$.
(ii) There is a $p_{2^{k+1+\frac{m}{4}}}$-closed dual circuit around the origin in the annulus $\text{Ann}(2^{k+1+\frac{m}{8}}, 2^{k+1+\frac{m}{4}})^*$.
(iii) There is a p_c-open circuit around the origin in the annulus $\text{Ann}(2^{k+1+\frac{m}{2}}, 2^{k+1+m})$.
(iv) The circuit from (iii) is connected to infinity by a $p_{2^{k+1+\frac{m}{4}}}$-open path.

(See Fig. 2.1 for an illustration of $D_{k,m}$.)

For $j, k, m \geq 1$, we claim that

$$\left(\{R_j \geq 2^{k+1+m}\} \cap D_{k,m}\right) \subset \{r_j \geq 2^{k+1}\}. \tag{2.8}$$

To see why, suppose the left side occurs, and choose $\mathfrak{C}_1$ as a circuit from (i) in the definition of $D_{k,m}$, $\mathfrak{C}_2$ as a circuit from (ii), and $\mathfrak{C}_3$ as a circuit from (iii). Let n_1 be the time at which the invasion

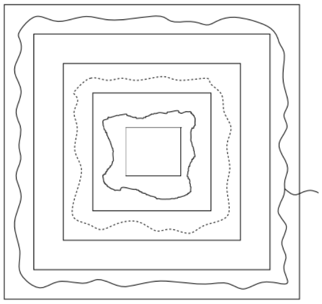

FIGURE 2.1. Illustration of the event $D_{k,m}$. The boxes, in order from smallest to largest, are $B(2^{k+1})$, $B(2^{k+1+\frac{m}{8}})$, $B(2^{k+1+\frac{m}{4}})$, $B(2^{k+1+\frac{m}{2}})$ and $B(2^{k+1+m})$. The solid circuit is p_c-open, the path to infinity is $p_{2^{k+1+\frac{m}{4}}}$-open, and the dotted path is $p_{2^{k+1+\frac{m}{4}}}$-closed.

invades all of $\mathfrak{C}_1$ and for $i = 2, 3$, let n_i be the first time that the invasion invades an edge from $\mathfrak{C}_i$. Note that $n_1 \le n_2 \le n_3 \le j$. (The last inequality holds because $R_j \ge 2^{k+1+m}$.)

After time n_3, the invasion has an unending supply of edges with weight $< p_{2^{k+1+\frac{m}{4}}}$ to invade, so it will never again take an edge with weight larger than that. Furthermore, at time n_2, the invasion must take an edge with weight larger than $p_{2^{k+1+\frac{m}{4}}}$. This implies that at some time $n_4 \in [n_2, n_3)$, the invasion invades an outlet: an edge $\hat{e}$ such that all edges invaded after time n_4 have weight $< \omega(\hat{e})$. Furthermore, this outlet can be chosen to have weight $\omega(\hat{e}) > p_{2^{k+1+\frac{m}{4}}} > p_c$.

Directly before time n_4, the entire boundary of the invasion (excluding $\hat{e}$ itself) consists of edges with weight $> \omega(\hat{e})$. Since invaded weights beyond time n_4 are $< \omega(\hat{e})$, none of these boundary edges will ever be invaded. Therefore all invaded edges after time n_4 are invaded through $\hat{e}$. In other words, if e is any edge invaded after time n_4, there is a path $P(e)$ connecting $\hat{e}$ to e consisting of edges with weight $< \omega(\hat{e})$ and which are invaded after time n_4. It is important to note that $P(e)$ cannot touch $\mathfrak{C}_1$. Indeed, if were to contain an edge f which shared an endpoint with an edge on $\mathfrak{C}_1$ (including the possibility that $f \in \mathfrak{C}_1$), then f would be accessible to the invasion at time n_1, and so f would be invaded before time n_4, a contradiction.

Finally, to prove (2.8), assume that $r_j < 2^{k+1}$. Then there is some time $j' > j$ at which the invasion invades an edge e in $B(2^{k+1})$. Since $j' > n_4$, there is a path $P(e)$ from $\hat{e}$ to e as in the preceding paragraph which cannot touch $\mathfrak{C}_1$. This means $\hat{e}$ is in the interior of $\mathfrak{C}_1$. On the other hand, if f is any edge of $\mathfrak{C}_3$ (necessarily invaded after time n_4), the path $P(f)$ connecting $\hat{e}$ to f would then touch $\mathfrak{C}_1$, a contradiction. This shows (2.8).

Applying (2.8) for $C > 0$ and $k, m \ge 1$, we obtain

$$\mathbb{P}(r_{\lfloor C2^{2k}\pi(2^k)\rfloor} < 2^{k+1}) \le 2\max\{\mathbb{P}(R_{\lfloor C2^{2k}\pi(2^k)\rfloor} < 2^{k+1+m}), \mathbb{P}(D^c_{k,m})\}. \tag{2.9}$$

As in Damron and Sapozhnikov (2012, proof of Thm. 5), the RSW theorem implies that $\mathbb{P}(D^c_{k,m}) \le e^{-\delta m}$ for some $\delta > 0$ uniformly in k, so we can fix m so that

$$\mathbb{P}(D^c_{k,m}) \le \frac{\eta_0}{2} \text{ for all } k \ge 1. \tag{2.10}$$

From Lemma 2.1 and the fact that $\pi(n)$ is decreasing in n, for any $C \ge (2\mathsf{C}_1 2^{2+2m})/\eta_0 =: \mathsf{C}_2$, we get

$$\begin{aligned}\mathbb{P}\big(R_{\lfloor C2^{2k}\pi(2^k)\rfloor} < 2^{k+1+m}\big) &\le \mathbb{P}\big(R_{\lfloor(2\mathsf{C}_1/\eta_0)2^{2(k+1+m)}\pi(2^{k+1+m})\rfloor} < 2^{k+1+m}\big)\\ &\le \frac{\eta_0}{2}.\end{aligned}$$

Combining this with (2.9) and (2.10), we find that for $C \geq \mathsf{C}_2$,

$$\mathbb{P}\big(r_{\lfloor C2^{2k}\pi(2^k)\rfloor} < 2^{k+1}\big) \leq \eta_0,$$

and this completes the proof for n of the form 2^k.

For general n, we let $k = k(n) := \lfloor \log_2 n \rfloor$, so that for any $C \geq 4\mathsf{C}_2$,

$$\mathbb{P}(r_{\lfloor Cn^2\pi(n)\rfloor} < 2n) \leq \mathbb{P}(r_{\lfloor \mathsf{C}_2 2^{2(k+1)}\pi(2^{k+1})\rfloor} < 2^{k+2}) \leq \eta_0.$$

□

2.2. *Lower bound.* In this section, we show that

$$\liminf_{n\to\infty} a_n(p_c) > 0. \tag{2.11}$$

The first step is to show that it suffices to prove this result for only a certain subsequence of values of n. Namely, we first prove that if there exists $\mathsf{C}_3 > 0$ such that

$$\liminf_{n\to\infty} a_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}(p_c) > 0, \tag{2.12}$$

then (2.11) follows.

So assume that (2.12) holds, and let

$$k = k(n) := \max\{\ell : \mathsf{C}_3\ell^2\pi(\ell) \leq n\}.$$

(Note that this k actually exists for large n since $\pi(\ell) \geq D_1/\sqrt{\ell}$ by (2.5).) Since $\tilde{Q}_n(p_c+\epsilon) - \tilde{Q}_n(p_c)$ is increasing in n,

$$\tilde{Q}_n(p_c+\epsilon) - \tilde{Q}_n(p_c) \geq \tilde{Q}_{\lfloor \mathsf{C}_3k^2\pi(k)\rfloor}(p_c+\epsilon) - \tilde{Q}_{\lfloor \mathsf{C}_3k^2\pi(k)\rfloor}(p_c).$$

So using $n \leq L_n \leq 4n$, we obtain

$$\begin{aligned} a_n(p_c) &= \lim_{\epsilon\downarrow 0} \frac{\mathbb{E}[\tilde{Q}_n(p_c+\epsilon) - \tilde{Q}_n(p_c)]}{\epsilon\mathbb{E}[L_n]} \\ &\geq \lim_{\epsilon\downarrow 0} \frac{\mathbb{E}[\tilde{Q}_{\lfloor \mathsf{C}_3k^2\pi(k)\rfloor}(p_c+\epsilon) - \tilde{Q}_{\lfloor \mathsf{C}_3k^2\pi(k)\rfloor}(p_c)]}{\epsilon\mathbb{E}[L_{\lfloor \mathsf{C}_3k^2\pi(k)\rfloor}]} \frac{\mathbb{E}[L_{\lfloor \mathsf{C}_3k^2\pi(k)\rfloor}]}{\mathbb{E}[L_n]} \\ &\geq a_{\lfloor \mathsf{C}_3k^2\pi(k)\rfloor}(p_c)\frac{\mathsf{C}_3k^2\pi(k)}{8n} \end{aligned}$$

Thus to conclude (2.11) from (2.12), it suffices to show that the quantity $\liminf_{n\to\infty} k^2\pi(k)/n$ is positive. For large n, $k(n)$ is greater than 1; therefore,

$$\frac{k^2\pi(k)}{n} \geq \frac{k^2\pi(k)}{\mathsf{C}_3(k+1)^2\pi(k+1)} \geq \mathsf{C}_3^{-1}\left(\frac{k}{k+1}\right)^2 \geq \frac{1}{4\mathsf{C}_3} > 0.$$

To prove (2.12), we use the following lemma, which bounds the k^{th} moment of the number of edges of the IPC with $(p_c, p_c+\epsilon]$ in $B(n)$.

Lemma 2.3. *Let $\mathcal{Y}_n(\epsilon)$ be the number of invaded edges in $B(n)$ with weight in $(p_c, p_c+\epsilon]$ for $\epsilon > 0$. There exist positive constants C_4 and $\mathsf{C}_5 = \mathsf{C}_5(t)$ such that for all $n \geq 1$,*

$$\liminf_{\epsilon\downarrow 0} \frac{\mathbb{E}|\mathcal{Y}_n(\epsilon)|}{\epsilon} \geq \mathsf{C}_4 n^2\pi(n)$$

and

$$\mathbb{E}|\mathcal{Y}_n(\epsilon)|^t \leq \mathsf{C}_5\left(\epsilon n^2\pi(n)\right)^t \text{ for all } t \geq 1 \text{ and } \epsilon > 0.$$

Assuming this lemma for the moment, we can derive (2.12). From Lemma 2.2, we can choose C_3 so that

$$\mathbb{P}(r_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor} < 2n) \le \frac{\mathsf{C}_4^2}{16\mathsf{C}_5(2)} \text{ for all } n \ge 1.$$

On the event $\{r_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor} \ge 2n\}$, the IPC in $B(2n)$ does not change after time $\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor$. It follows that the number of invaded edges with $(p_c, p_c+\epsilon]$ until time $\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor$ is at least $\mathcal{Y}_{2n}(\epsilon)$, which is the number of invaded edges with $(p_c, p_c+\epsilon]$ in $B(2n)$. By Lemma 2.2, Lemma 2.3 and the Cauchy-Schwarz inequality, if ϵ is sufficiently small,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=1}^{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}\Big(X_j(p_c+\epsilon)-X_j(p_c)\Big)\right]\\
&\ge \mathbb{E}\left[\sum_{j=1}^{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}\Big(X_j(p_c+\epsilon)-X_j(p_c)\Big)\cdot \mathbf{1}_{\{r_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}\ge 2n\}}\right]\\
&\ge \mathbb{E}\left[\mathcal{Y}_{2n}(\epsilon)\cdot \mathbf{1}_{\{r_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}\ge 2n\}}\right]\\
&\ge \frac{\mathsf{C}_4}{2}\epsilon(2n)^2\pi(2n) - \mathbb{E}\left[\mathcal{Y}_{2n}(\epsilon)\cdot \mathbf{1}_{\{r_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}< 2n\}}\right]\\
&\ge \frac{\mathsf{C}_4}{2}\epsilon(2n)^2\pi(2n) - \sqrt{\mathsf{C}_5(2)\left(\epsilon(2n)^2\pi(2n)\right)^2\frac{\mathsf{C}_4^2}{16\mathsf{C}_5(2)}}\\
&\ge \frac{\mathsf{C}_4\epsilon(2n)^2\pi(2n)}{4}.
\end{aligned}$$

Combining this with (1.2), (2.5), and the fact that $n \le \mathbb{E}[L_n] \le 4n$, we obtain

$$\begin{aligned}
a_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}(p_c) &= \lim_{\epsilon\downarrow 0}\frac{\mathbb{E}[\tilde{Q}_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}(p_c+\epsilon)-\tilde{Q}_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}(p_c)]}{\epsilon\mathbb{E}[L_{\lfloor \mathsf{C}_3 n^2\pi(n)\rfloor}]}\\
&\ge \lim_{\epsilon\downarrow 0}\frac{\mathsf{C}_4\epsilon(2n)^2\pi(2n)/4}{4\epsilon\mathsf{C}_3 n^2\pi(n)}\\
&= \frac{\mathsf{C}_4 D_1}{4\mathsf{C}_3\sqrt{2}},
\end{aligned}$$

which is positive uniformly in n. This shows (2.12).

The last step is to prove Lemma 2.3.

Proof of Lemma 2.3: The proof of the upper bound is similar to that of Járai (2003, Thm. 1), which shows an upper bound for $|\mathcal{S}_n|$ (that result does not involve a condition on the weight $\omega(e)$) so we will omit some details. We will follow that proof, but make the events independent of $\omega(e)$ so that we can insert the condition $\omega(e) \in (p_c, p_c+\epsilon]$.

We will restrict to n of the form 2^K, as the general result follows from this and monotonicity of $\pi(n)$. Let A_k be $\mathrm{Ann}(2^k, 2^{k+1})$, and $\mathcal{Y}_{A_k}$ be the number of IPC edges in $\mathrm{Ann}(2^k, 2^{k+1})$ with the weight in $(p_c, p_c+\epsilon]$. Then, $B(n) = \cup_{k=1}^{K-1} A_k$ and $\mathcal{Y}_n(\epsilon) = \sum_{k=1}^{K-1}\mathcal{Y}_{A_k}$. Define a sequence $p_k(0) > p_k(1) > \cdots > p_c$ as follows. Let $\log^{(0)} k = k$, and let $\log^{(j)} k = \log(\log^{(j-1)} k)$ for $j \ge 1$ if the right-hand side is defined. For $k > 10$, we define

$$\log^* k = \min\{j > 0 : \log^{(j)} k \text{ is defined and } \log^{(j)} k \le 10\}.$$

Then $\log^{(j)} k > 2$, for $j = 0, 1, \cdots, \log^* k$ and $k > 10$. Let

$$p_k(j) = \inf\left\{p > p_c : L(p) \le \frac{2^k}{C_5 \log^{(j)} k}\right\}, \quad j = 0, 1, \cdots, \log^* k,$$

where the constant C_5 will be chosen later. With (2.4) and Járai (2003, Eq. (2.15)), we get

$$C_5 \log^{(j)} k \le \frac{2^k}{L(p_k(j))} \le DC_5 \log^{(j)} k \tag{2.13}$$

For any fixed $e \subset A_k$ we define

$$\begin{aligned} H_k(j) &= \left\{ \begin{array}{c} \exists \text{ a } p_k(j)\text{-open circuit } \mathcal{D} \text{ around the} \\ \text{origin in } A_{k-1} \text{ and } \mathcal{D} \xleftrightarrow{p_k(j)} \infty \end{array} \right\} \\ H_k^e(j) &= \left\{ \begin{array}{c} H_k(j) \text{ occurs and } \mathcal{D} \xleftrightarrow{p_k(j)} \infty \\ \text{without using the edge } e \end{array} \right\}. \end{aligned} \tag{2.14}$$

To give a lower bound for the probability of $H_k(j)$, Járai constructed an infinite $p_k(j)$-open path starting from $\partial B(2^k)$ using standard 2D constructions only to the right of $B(2^k)$. (See Járai, 2003, Fig. 1). Similarly, to lower bound the probability of $H_k(j)^e$, we build, in addition to Járai's path, an infinite $p_k(j)$-open path starting from $\partial B(2^k)$ in the left of $B(2^k)$. The existence of such disjoint two infinite $p_k(j)$-open paths imply the event $\{\mathcal{D} \xleftrightarrow{p_k(j)} \infty$ without using $e\}$ for any fixed edge $e \in A_k$. As in Járai (2003, Eq. (2.17)), we obtain

$$J_k(j) \cap \left(\bigcap_{m=0}^{\infty} J_{k,L}^m(j) \right) \cap \left(\bigcap_{m=0}^{\infty} J_{k,R}^m(j) \right) \subseteq H_k^e(j) \tag{2.15}$$

where for $m \ge 0$,

$$\begin{aligned} J_k &= \{\exists \text{ a } p_k(j)\text{-open circuit in } A_{k-1}\} \\ J_{k,R}^m &= J_{k,R}^{m,h} \cap J_{k,R}^{m,v}, \text{ and } J_{k,L}^m = J_{k,L}^{m,h} \cap J_{k,L}^{m,v} \\ J_{k,R}^{m,h} &= \left\{ \begin{array}{c} \exists \text{ a } p_k(j)\text{-open horizontal crossing of} \\ [2^{k-2+m}, 2^{k+m}] \times [-2^{k-2+m}, 2^{k-2+m}] \end{array} \right\} \\ J_{k,L}^{m,h} &= \left\{ \begin{array}{c} \exists \text{ a } p_k(j)\text{-open horizontal crossing of} \\ [-2^{k+m}, -2^{k-2+m}] \times [-2^{k-2+m}, 2^{k-2+m}] \end{array} \right\} \\ J_{k,R}^{m,v} &= \left\{ \begin{array}{c} \exists \text{ a } p_k(j)\text{-open vertical crossing of} \\ [2^{k-1+m}, 2^{k+m}] \times [-2^{k-1+m}, 2^{k-1+m}] \end{array} \right\} \\ J_{k,L}^{m,v} &= \left\{ \begin{array}{c} \exists \text{ a } p_k(j)\text{-open vertical crossing of} \\ [-2^{k+m}, -2^{k-1+m}] \times [-2^{k-1+m}, 2^{k-1+m}] \end{array} \right\}. \end{aligned}$$

By (2.3) and (2.13), (See Járai, 2003, Eqs. (2.19), (2.20)),

$$\begin{aligned} &\mathbb{P}(J_k(j)^c) \le 16C_3 \exp\left\{-\frac{1}{4}C_4C_5 \log^{(j)} k\right\} \quad \text{and} \\ &\mathbb{P}(J_{k,R}^m(j)^c \cup J_{k,L}^m(j)^c) \le 4C_3 \exp\left\{-\frac{1}{2}C_4C_5 2^m \log^{(j)} k\right\}. \end{aligned}$$

By these inequalities, one gets

$$\begin{aligned} \mathbb{P}(H_k^e(j)^c) &\le \mathbb{P}\left(J_k(j)\right)^c + \sum_{m=0}^{\infty} \mathbb{P}\left((J_{k,R}^m(j)^c \cup J_{k,L}^m(j)^c\right) \\ &\le (16C_3 + C_6) \exp\left\{-\frac{1}{4}C_4C_5 \log^{(j)} k\right\}. \end{aligned}$$

We write C_7 as $16C_3 + C_6$ and c_1 as $\frac{C_4C_5}{4}$ for short. Then,

$$\mathbb{P}(H_k^e(j)^c) \le C_7 \exp\{-c_1 \log^{(j)} k\}. \tag{2.16}$$

The constant c_1 can be made large by choosing C_5 large.

To estimate the mean of $\mathcal{Y}_{A_k}$, we decompose

$$\begin{aligned}\mathbb{E}\mathcal{Y}_{A_k} &= \mathbb{E}\left[\mathcal{Y}_{A_k}; H_k(0)^c\right] \\ &\quad + \left(\sum_{j=1}^{\log^* k} \mathbb{E}\left[\mathcal{Y}_{A_k}; H_k(j-1) \cap H_k(j)^c\right]\right) \\ &\quad + \mathbb{E}\left[\mathcal{Y}_{A_k}; H_k(\log^* k)\right].\end{aligned}$$

By (2.16) and independence,

$$\begin{aligned}\mathbb{E}\left[\mathcal{Y}_{A_k}; H_k(0)^c\right] &\le \mathbb{E}\left[\mathcal{Y}_{A_k}; H_k^e(0)^c\right] \\ &\le \sum_{e \in A_k} \mathbb{P}(\omega(e) \in (p_c, p_c+\epsilon], H_k^e(0)^c) \\ &= |A_k| \mathbb{P}(\omega(e) \in (p_c, p_c+\epsilon]) \mathbb{P}(H_k^e(0)^c) \\ &\le |A_k| \epsilon C_8 e^{-c_1 k}.\end{aligned} \tag{2.17}$$

Next, since $\omega(e)$ is independent of $H_k^e(j) \cap \{e \overset{p_k(j)}{\longleftrightarrow} \infty\}$,

$$\begin{aligned}&\mathbb{E}|\mathcal{Y}_{A_k}; H_k(j-1) \cap H_k(j)^c| \\ &= \sum_{e \in A_k} \mathbb{P}(\omega(e) \in (p_c, p_c+\epsilon] \cap \{e \overset{p_k(j-1)}{\longleftrightarrow} \infty\} \cap H_k^e(j)^c) \\ &= \epsilon \sum_{e \in A_k} \mathbb{P}(e \overset{p_k(j-1)}{\longleftrightarrow} \infty, H_k^e(j)^c).\end{aligned}$$

Applying the FKG inequality and (2.16) to this, we obtain

$$\begin{aligned}&\mathbb{E}[\mathcal{Y}_{A_k}; H_k(j-1) \cap H_k(j)^c] \\ &\le |A_k| \epsilon \theta(p_k(j-1))\, C_7 \exp\{-c_1 \log^{(j)} k\}.\end{aligned} \tag{2.18}$$

The third term of (2.2) is bounded above by

$$|A_k| \epsilon \theta(p_k(\log^* k)). \tag{2.19}$$

Using (2.2), (2.5) and (2.13),

$$\theta(p_k(j)) \le \frac{\pi(2^k)}{D_1} (D C_5 \log^{(j)} k)^{1/2}.$$

Applying this inequality after placing (2.17), (2.18), and (2.19) into (2.2), we obtain

$$\mathbb{E}\mathcal{Y}_{A_k} \le C_9 |A_k| \epsilon \pi(2^k) \left[\frac{\exp\{-c_1 k\}}{\pi(2^k)} + \left\{\sum_{j=1}^{\log^* k} (\log^{j-1} k)^{1/2 - c_1}\right\} + 1\right].$$

Since $\pi(2^k) \ge C_{10} 2^{-k/2}$ from (2.5), we can choose C_5 (and therefore c_1) so large that

$$\frac{\exp\{-c_1 k\}}{\pi(2^k)} + \left\{\sum_{j=1}^{\log^* k} (\log^{j-1} k)^{1/2 - c_1}\right\} + 1 \quad \text{is bounded in } k,$$

and so $\mathbb{E}\mathcal{Y}_{A_k} \leq C_{11}\epsilon 2^{2k}\pi(2^k)$. Recalling $n = 2^K$, we obtain from this and (2.5) that

$$\begin{aligned}\mathbb{E}\mathcal{Y}_n(\epsilon) = \sum_{k=1}^{K} \mathbb{E}\mathcal{Y}_{A_k} &\leq C_{11}\epsilon 2^{2K}\pi(2^K) \sum_{k=1}^{K} \frac{2^{2k}\pi(2^k)}{2^{2K}\pi(2^K)} \\ &\leq \frac{C_{11}}{D_1}\epsilon 2^{2K}\pi(2^K) \sum_{k=1}^{K} 2^{2(k-K)} 2^{-\frac{1}{2}(k-K)} \\ &\leq C_{12}\epsilon n^2 \pi(n),\end{aligned}$$

completing the proof of the upper bound when $t = 1$. The extension to larger t uses the same ideas as in Járai (2003) and Kesten (1986, Sec. 3), so we omit it.

We now turn to the lower bound. For $k \geq 2$, $\epsilon > 0$, and any $e \subset A_k$, we let $L_k(e)$ be the event that the following hold:

(a) There exists a p_c-open circuit $\mathcal{D}$ around the origin in A_{k-2}.
(b) There exists a $(p_c + \epsilon)$-closed dual circuit around the origin in A_{k+2}.
(c) $\mathcal{D}$ is connected to the edge $e \in A_k$ by a p_c-open path in $B(2^k)$.

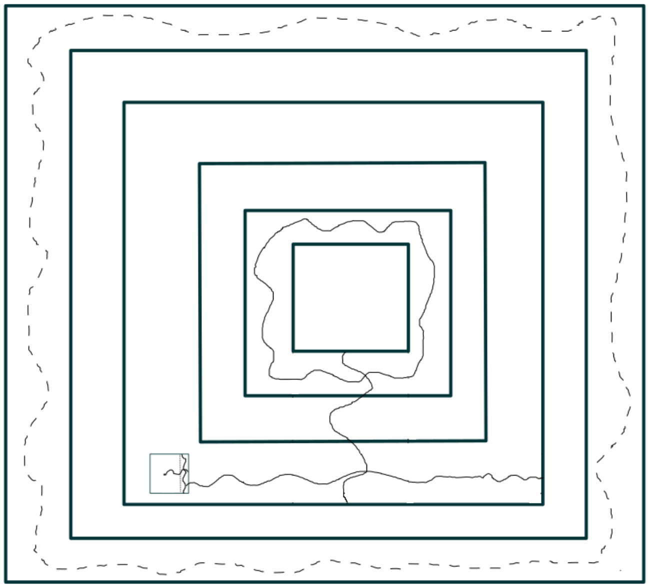

FIGURE 2.2. The event $L_k(e)$. The boxes, in order from smallest to largest, are $B(2^{k-2})$, $B(2^{k-1})$, $B(2^k)$, $B(2^{k+1})$, $B(2^{k+2})$, and $B(2^{k+3})$. The solid curves are p_c-open and the dotted curve is a $(p_c + \epsilon)$-closed dual circuit.

(See Fig. 2.2 for an illustration of $L_k(e)$).

If the events described in (a) and (b) both occur, each $(p_c + \epsilon)$-open edge connected to $\mathcal{D}$ by a $(p_c + \epsilon)$-open path will eventually be invaded. Since the event in (b) depends on edge-variables for edges outside of $B(2^{k+1})$, (b) is independent of both (a) and (c). In addition, the events (a) and (c) are increasing. So, by the FKG inequality and the RSW theorem,

$$\mathbb{P}(L_k(e)) \geq \mathbb{P}((a)) \times \mathbb{P}((b)) \times \mathbb{P}((c)) \geq C_{13}\mathbb{P}((b)) \times \mathbb{P}((c)).$$

By a gluing argument (Grimmett, 1989, Ch. 11) using the FKG inequality and the RSW theorem, $\mathbb{P}((c)) \geq C_{14}\pi(2^k)$. Furthermore, as long as ϵ is so small that $p_c + \epsilon < p_{2^{k+2}}$, then the RSW theorem implies that $\mathbb{P}((b)) \geq C_{15}$. This means that for such ϵ, one has $\mathbb{P}(L_k(e)) \geq C_{13}C_{14}C_{15}\pi(2^k)$. Since $\omega(e)$ and the event $L_k(e)$ are independent,

$$\begin{aligned}\mathbb{E}|\mathcal{Y}_{A_k}| &= \sum_{e\in A_k} \mathbb{P}(e \in \mathrm{IPC}, \omega(e) \in (p_c, p_c+\epsilon]) \\ &\geq \sum_{e\in A_k} \mathbb{P}(L_k(e), \omega(e) \in (p_c, p_c+\epsilon]) \\ &\geq C_{16}\epsilon 2^{2k}\pi(2^k).\end{aligned}$$

For a given $n \geq 1$, choose $k = \lfloor \log_2 n \rfloor$ to complete the proof:

$$\mathbb{E}\mathcal{Y}_n(\epsilon) \geq \mathbb{E}\mathcal{Y}_{A_k} \geq C_{16}\epsilon 2^{2k}\pi(2^k) \geq C_{17}\epsilon n^2\pi(n).$$

□

2.3. *Upper bound.* In this section, we show that

$$\limsup_{n\to\infty} a_n(p_c) < 1. \tag{2.20}$$

To prove (2.20), we define

$$\Xi_n(\epsilon) = \left[\tilde{P}_n(p_c+\epsilon) - \tilde{P}_n(p_c)\right] - \left[\tilde{Q}_n(p_c+\epsilon) - \tilde{Q}_n(p_c)\right],$$

as the number of edges with weight in the interval $(p_c, p_c+\epsilon]$ which the invasion observes until time n but does not invade, and we give the following proposition.

Proposition 2.4. *There exists* $\mathsf{C}_6 > 0$ *and a function* G *on* $[0,\infty)$ *with* $\inf_{r\in[0,m]} G(r) > 0$ *for each* $m \geq 0$ *such that for any* $C \geq \mathsf{C}_6$*, any* $n \geq 1$*, and any* $\epsilon > 0$*,*

$$\mathbb{E}\Xi_{\lfloor Cn^2\pi(n)\rfloor}(\epsilon) \geq G(C)\epsilon n^2\pi(n).$$

Assuming Proposition 2.4 for the moment, let $C \geq \mathsf{C}_6$, and use $\mathbb{E}L_n \leq 4n$ for

$$\begin{aligned} a_{\lfloor Cn^2\pi(n)\rfloor}(p_c) &= \lim_{\epsilon\downarrow 0} \frac{\mathbb{E}\left[\tilde{Q}_{\lfloor Cn^2\pi(n)\rfloor}(p_c+\epsilon) - \tilde{Q}_{\lfloor Cn^2\pi(n)\rfloor}(p_c)\right]}{\mathbb{E}\left[\tilde{P}_{\lfloor Cn^2\pi(n)}(p_c+\epsilon) - \tilde{P}_{\lfloor Cn^2\pi(n)\rfloor}(p_c)\right]} \qquad (2.21)\\ &= \lim_{\epsilon\downarrow 0}\left(1 - \frac{\mathbb{E}\Xi_{\lfloor Cn^2\pi(n)\rfloor}(\epsilon)}{\epsilon\mathbb{E}L_{\lfloor Cn^2\pi(n)\rfloor}}\right) \\ &\leq \lim_{\epsilon\downarrow 0}\left(1 - \frac{G(C)\epsilon n^2\pi(n)}{4C\epsilon n^2\pi(n)}\right) \\ &= 1 - \frac{G(C)}{4C}. \qquad (2.22)\end{aligned}$$

Now note that any $n \geq \mathsf{C}_6$ can be written in the form $\lfloor Ch^2\pi(h)\rfloor$ for some integer $h \geq 1$ and some $C \in [\mathsf{C}_6, 4\mathsf{C}_6]$. To see why, observe that any $n \geq \mathsf{C}_6$ is in some interval of the form $[\mathsf{C}_6h^2\pi(h), \mathsf{C}_6(h+1)^2\pi(h+1))$ for some $h \geq 1$ (since $h^2\pi(h) \to \infty$ as $h \to \infty$ by (2.5)). Then because

$$\frac{\mathsf{C}_6(h+1)^2\pi(h+1)}{\mathsf{C}_6h^2\pi(h)} = \left(1+\frac{1}{h}\right)^2\frac{\pi(h+1)}{\pi(h)} \leq 4,$$

we see that $n = \lfloor C_*\mathsf{C}_6h^2\pi(h)\rfloor$ for some $C_* \in [1,4]$. By (2.22), then, we obtain

$$a_n(p_c) \leq 1 - \frac{\inf_{r\in[\mathsf{C}_6,4\mathsf{C}_6]} G(r)}{4\mathsf{C}_6},$$

and this implies (2.20).

In the remainder of this section, we prove Proposition 2.4.

Proof of Proposition 2.4: For notational convenience, define $t_n = \lfloor Cn^2\pi(n)\rfloor$. To prove a lower bound on $\Xi_{t_n}(\epsilon)$, we will construct a large p_c-open cluster such that with positive probability, independent of n, the invasion has intersected this cluster at time t_n and has explored a positive fraction of its boundary edges, but has not yet absorbed the entire cluster. These explored boundary edges will have probability of order ϵ to have weight in the interval $(p_c, p_c + \epsilon]$, so our lower bound on $\mathbb{E}\Xi_{t_n}(\epsilon)$ will be of order ϵ times the size of this explored boundary, which will itself be of order $n^2\pi(n)$.

To construct this cluster, we need several definitions.

Definition 2.5. Define the event $D(n)$ that the following conditions hold:

1. There exists a q_n-open circuit around the origin in $\mathrm{Ann}(n, 2n)$.
2. There exists an edge $f \in \mathrm{Ann}(6n, 7n)$ with $\omega(f) \in (q_n, p_c)$ such that:
   - (a) there exists a p_c-closed dual path P around the origin in $\mathrm{Ann}(4n, 8n)^*\backslash\{f^*\}$ that is connected to the endpoints of f^* so that $P \cup \{f^*\}$ is a dual circuit around the origin, and
   - (b) there exists a p_c-open path connecting an endpoint of f to $B(n)$, and another disjoint p_c-open path connecting the other endpoint of f to $\partial B(16n)$.
3. There exists a p_c-open circuit around the origin in $\mathrm{Ann}(8n, 16n)$.

For $e \subset \mathrm{Ann}(2n, 4n)$, define $D^e(n)$ as the event that $D(n)$ occurs without using the edge e. (That is, $D(n)$ occurs and the first connection listed in 2(b) does not use e.)

See Fig. 2.3 for an illustration of $D(n)$.

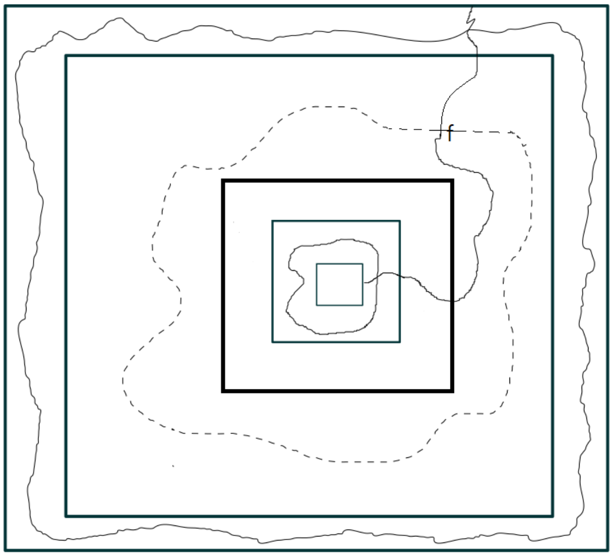

FIGURE 2.3. The event $D(n)$. The boxes, in order from smallest to largest, are $B(n)$, $B(2n)$, $B(4n)$, $B(8n)$ and $B(16n)$. The solid circuit in $\mathrm{Ann}(n, 2n)$ is q_n-open and the path from $\partial B(n)$ to f is p_c-open; the dotted dual path in $\mathrm{Ann}(4n, 8n)$ is p_c-closed, $\omega(f) \in (q_n, p_c)$, and the other solid paths are p_c-open.

When the event $D(n)$ occurs, we can define $\mathcal{C}_*$ as the innermost q_n-open circuit around the origin in $\mathrm{Ann}(n, 2n)$ and D_* as the outermost p_c-open circuit around the origin in $\mathrm{Ann}(8n, 16n)$. Note that on $D(n)$, the circuits $\mathcal{C}_*$ and D_* are part of the same p_c-open cluster; this will form part of our "large cluster" referenced above. We need to make sure that we have started to invade this cluster, but are not yet done at time t_n, so we define times

$$\begin{aligned} t_{D_*} = &\text{ first time at which the invasion invades an edge from } D_* \\ T_{D_*} = &\text{ first time at which the invasion invades the entire} \\ &\quad p_c\text{-open cluster of } D_*. \end{aligned}$$

Note that on $D(n)$, we have $t_{D_*} \le T_{D_*}$ and trivially,

$$\mathbb{E}\Xi_{t_n}(\epsilon) \ge \mathbb{E}\Xi_{t_n}(\epsilon)\mathbf{1}_{D(n)\cap\{t_{D_*}\le t_n < T_{D_*}\}}. \tag{2.23}$$

The next lemma shows that on the events listed on the right, $\Xi_{t_n}(\epsilon)$ is, on average, at least order ϵ times the cardinality of a certain subset of the edge boundary of the p_c-open cluster of D_*. For this we define the size Y_n of this subset:

$$Y_n = \#\{e \subset \mathrm{Ann}(2n, 4n) : \omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\}.$$

Lemma 2.6. *For any $n \ge 1$,*

$$\mathbb{E}\Xi_{t_n}(\epsilon)\mathbf{1}_{D(n)\cap\{t_{D_*}\le t_n < T_{D_*}\}} \ge \frac{\epsilon}{1-p_c}\mathbb{E}Y_n\mathbf{1}_{D(n)\cap\{t_{D_*}\le t_n < T_{D_*}\}}.$$

Proof: First we let

$$\hat{Y}_n = \#\left\{ \begin{array}{c} e \subset \mathrm{Ann}(2n, 4n) : \omega(e) \in (p_c, p_c + \epsilon], \\ e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \end{array} \right\}.$$

On the event $D(n)\cap\{t_{D_*} \le t_n < T_{D_*}\}$, any edge in the set which defines $\hat{Y}_n$ will be observed by the invasion until time t_n but will not be invaded (that is, it is counted in the definition of $\Xi_n(\epsilon)$). To see why, let e be an edge in the set which defines $\hat{Y}_n$. First, we must show that e is not invaded at time t_n. This is because, in order for the invasion to even observe e, it must first pass through the circuit $\mathcal{C}_*$. Since $\omega(e) > p_c$, the invasion will invade the entire p_c-open cluster of $\mathcal{C}_*$ (which equals the p_c-open cluster of D_*) before it invades e. Since $t_n < T_{D_*}$, e cannot be invaded at time t_n. Second, we must show that e is observed by time t_n. The reason is that since $t_{D_*} \le t_n$, at time t_n, the invasion has already invaded an edge from D_*. Since $D(n)$ occurs, the edge f must therefore be invaded before time $t_{D_*} \le t_n$. Before f can be invaded, the entire q_n-open cluster of $\mathcal{C}_*$ must be invaded, so at least one endpoint of e is in the invasion at time t_n. This means that e is observed by time t_n. In conclusion,

$$\begin{aligned} &\mathbb{E}\Xi_{t_n}(\epsilon)\mathbf{1}_{D(n)\cap\{t_{D_*}\le t_n < T_{D_*}\}} \\ &\ge \mathbb{E}\hat{Y}_n\mathbf{1}_{D(n)\cap\{t_{D_*}\le t_n < T_{D_*}\}} \\ &= \sum_{e\subset \mathrm{Ann}(2n,4n)} \mathbb{P}\left(\begin{array}{c} \omega(e) \in (p_c, p_c+\epsilon], e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), \\ D(n), t_{D_*} \le t_n < T_{D_*} \end{array} \right). \end{aligned}$$

The second and final step is to show that for all $e \subset \mathrm{Ann}(2n, 4n)$, we have

$$\begin{aligned} &\mathbb{P}\left(\begin{array}{c} \omega(e) \in (p_c, p_c+\epsilon], e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), \\ D(n), t_{D_*} \le t_n < T_{D_*} \end{array} \right) \\ &= \frac{\epsilon}{1-p_c}\mathbb{P}\left(\begin{array}{c} \omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), \\ D(n), t_{D_*} \le t_n < T_{D_*} \end{array} \right). \end{aligned} \tag{2.24}$$

Once this is done, we can sum the right side and obtain the statement of the lemma.

To argue for (2.24), we need to be able to decouple the value of $\omega(e)$ from the other events. Intuitively this should be possible because when $D(n)$ occurs, after the invasion touches $\mathcal{C}_*$, it does not need to check any weights for edges which are p_c-closed until after time T_{D_*}. To formally prove this, we represent the weights $(\omega(e))$ used for the invasion as functions of three independent variables. This representation is used in the "percolation cluster method" of Chayes et al. (1985), but their method uses them in a dynamic way, whereas ours will be static. For this representation, we assign different variables to the edges: let $(U_e^1, U_e^2, \eta_e)_{e \in \mathcal{E}^2}$ be an i.i.d. family of independent variables, where U_e^1 is uniform on $[0, p_c]$, U_e^2 is uniform on $(p_c, 1]$, and η_e is Bernoulli with parameter p_c. Then we set

$$\omega(e) = \begin{cases} U_e^1 & \text{if } \eta_e = 1 \\ U_e^2 & \text{if } \eta_e = 0. \end{cases}$$

Next, we define another invasion percolation process $(\hat{G}_n)$ (a sequence of growing subgraphs) as follows. If $D(n)$ does not occur, then $\hat{G}_n$ is equal to $(0, \{\})$ for all n (it stays at the origin with no edges). If $D(n)$ does occur, then $\hat{G}_n$ proceeds according to the usual invasion rules (with the weights $(\omega(e))$) until it reaches $\mathcal{C}_*$. After it contains a vertex of $\mathcal{C}_*$, it no longer checks the ω-value of any edge $\hat{e}$ with $\eta_{\hat{e}} = 0$ (it only checks the η-value). When there are no more edges with η-value equal to one for the invasion to invade, it stops (we set $\hat{G}_n$ to be constant after this time). Associated to this new invasion will be times similar to t_{D_*} and T_{D_*}:

$$\begin{aligned} \hat{t}_{D_*} = {} & \text{first time at which the new invasion invades an edge} \\ & \quad \text{from } D_* \\ \hat{T}_{D_*} = {} & \text{first time at which the new invasion invades the entire} \\ & \quad p_c\text{-open cluster of } D_*. \end{aligned}$$

Note that if $D(n)$ does not occur, $\hat{t}_{D_*} = \hat{T}_{D_*} = \infty$, and that if $D(n)$ occurs, $\hat{T}_{D_*}$ equals the first time after which the graphs $\hat{G}_n$ become constant.

Given these definitions, the top equation of (2.24) equals

$$\mathbb{P}\left(\begin{array}{c} U_e^2 \in (p_c, p_c + \epsilon], \eta_e = 0, e \xleftrightarrow{q_n} \partial B(n) \text{ in } B(4n), \\ D(n), t_{D_*} \le t_n < T_{D_*} \end{array} \right).$$

We then claim that

$$\begin{aligned} & \mathbb{P}\left(\begin{array}{c} U_e^2 \in (p_c, p_c + \epsilon], \eta_e = 0, e \xleftrightarrow{q_n} \partial B(n) \text{ in } B(4n), \\ D(n), t_{D_*} \le t_n < T_{D_*} \end{array} \right) \\ = {} & \mathbb{P}\left(\begin{array}{c} U_e^2 \in (p_c, p_c + \epsilon], \eta_e = 0, e \xleftrightarrow{q_n} \partial B(n) \text{ in } B(4n), \\ D(n), \hat{t}_{D_*} \le t_n < \hat{T}_{D_*} \end{array} \right). \end{aligned} \tag{2.25}$$

This equation holds because when $D(n)$ occurs, $t_{D_*} = \hat{t}_{D_*}$ and $T_{D_*} = \hat{T}_{D_*}$. Indeed, if $D(n)$ occurs, then both invasions (G_n) and $(\hat{G}_n)$ are equal until they touch $\mathcal{C}_*$. After this time, the original invasion (G_n) does not invade any p_c-closed edges until time T_{D_*}, and neither does $(\hat{G}_n)$ (by definition). This shows (2.25).

Now that we have (2.25), we simply note that because $(\hat{G}_n)$ does not use any edges in $B(2n)^c$ that are p_c-closed, the times $\hat{t}_{D_*}$ and $\hat{T}_{D_*}$ are independent of $(U_e^2)_{e \in B(2n)^c}$. Furthermore, the events $\{\eta_e = 0\}$, $\{e \xleftrightarrow{q_n} \partial B(n) \text{ in } B(4n)\}$, and $D(n)$ are independent of $(U_e^2)_{e \in B(2n)^c}$, and $U_e^2 \in (p_c, p_c + \epsilon]$ depends only on $(U_e^2)_{e \in B(2n)^c}$. By independence, therefore, the lower equation of (2.25) is equal to

$$\frac{\epsilon}{1 - p_c} \mathbb{P}(\eta_e = 0, e \xleftrightarrow{q_n} \partial B(n) \text{ in } B(4n), D(n), \hat{t}_{D_*} \le t_n < \hat{T}_{D_*}),$$

which equals the bottom equation in (2.24). This shows (2.24). □

Combining Lemma 2.6 with (2.23), and then reducing to the subevent $D^e(n)$ (recall this is the subevent of $D(n)$ on which the paths involved in $D(n)$ do not use the given $e \subset \mathrm{Ann}(2n, 4n)$), we obtain

$$\begin{aligned}
&\mathbb{E}\Xi_{t_n}(\epsilon) \\
&\geq \frac{\epsilon}{1-p_c}\mathbb{E}Y_n\mathbf{1}_{D(n)\cap\{t_{D_*}\leq t_n<T_{D_*}\}} \\
&\geq \frac{\epsilon}{1-p_c}\sum_{e\subset \mathrm{Ann}(2n,4n)}\mathbb{P}\left(\begin{array}{c}\omega(e)>p_c, e\xleftrightarrow{q_n}\partial B(n) \text{ in} \\ B(4n), D^e(n), t_{D_*}\leq t_n<T_{D_*}\end{array}\right).
\end{aligned} \tag{2.26}$$

The most difficult part of the above sum is the term $t_n < T_{D_*}$. To ensure that this occurs, we will construct a large set of vertices in the exterior of D_* which will connect to D_* by p_c-open paths. To do this, we will need to use independence to separate the interior of D_* from its exterior, using the following two events, which comprise pieces of the event $D(n)$.

Definition 2.7. For any circuit $\hat{D}_* \subset \mathrm{Ann}(8n, 16n)$ around the origin, define the event $D^e_{int}(n, \hat{D}_*)$ that the following hold.

1. There exists a q_n-open circuit around the origin in $\mathrm{Ann}(n, 2n)$.
2. There exists an edge $f \subset \mathrm{Ann}(6n, 7n)$ with $\omega(f) \in (q_n, p_c)$ such that:
   - (a) there exists a p_c-closed dual path P around the origin in $\mathrm{Ann}(4n, 8n)^*\backslash\{f^*\}$ that is connected to the endpoints of f^* so that $P \cup \{f^*\}$ is a circuit around the origin, and
   - (b) there exists a p_c-open path connecting an endpoint of f to $B(n)$ (avoiding e), and another disjoint p_c-open path connecting the other endpoint of f to $\hat{D}_*$.

We also define the event $D_{ext}(n, \hat{D}_*)$ that the following hold.

1. There exists a p_c-open path from $\hat{D}_*$ to $\partial B(16n)$.
2. $\hat{D}_*$ is the outermost p_c-open circuit in $\mathrm{Ann}(8n, 16n)$.

Directly from the definitions, we note that for any circuit $\hat{D}_* \subset \mathrm{Ann}(8n, 16n)$, $D^e_{int}(n, \hat{D}_*) \cap D_{ext}(n, \hat{D}_*)$ implies $D^e(n)$ (actually the union over $\hat{D}_*$ of this intersection is equal to $D^e(n)$), and the events $D^e_{int}(n, \hat{D}_*)$ and $D_{ext}(n, \hat{D}_*)$ are independent. Last, for distinct $\hat{D}_*$, the events $(D^e_{int}(n, \hat{D}_*) \cap D_{ext}(n, \hat{D}_*))_{\hat{D}_*}$ are disjoint. Decomposing (2.26) over the choice of the outermost circuit $\hat{D}_*$, we obtain that $\mathbb{E}\Xi_{t_n}(\epsilon)$ is at least equal to

$$\frac{\epsilon}{1-p_c}\sum_{e\subset \mathrm{Ann}(2n,4n)}\sum_{\hat{D}_*}\mathbb{P}\left(\begin{array}{c}\omega(e)>p_c, e\xleftrightarrow{q_n}\partial B(n) \text{ in } B(4n), \\ D^e_{int}(n,\hat{D}_*), D_{ext}(n,\hat{D}_*), \\ t_{\hat{D}_*}\leq t_n<T_{\hat{D}_*}\end{array}\right).$$

(Here $t_{\hat{D}_*}$ and $T_{\hat{D}_*}$ are similar to t_{D_*} and T_{D_*} but defined for the detministic circuit $\hat{D}_*$.) Note that $\{t_{\hat{D}_*} \leq t_n\}$ depends only on the weights in the interior of $\hat{D}_*$, but $\{t_n < T_{\hat{D}_*}\}$ does not depend only on the exterior. To force this dependence, we simply create a large p_c-open cluster in the exterior of $\hat{D}_*$. For our deterministic $\hat{D}_*$, let

$$Z(\hat{D}_*) = \#\{e \subset B(16n)^c : \omega(e) < p_c,\ e \xleftrightarrow{p_c} \hat{D}_*\}.$$

If $Z(\hat{D}_*) > Cn^2\pi(n)$ on $D^e_{int}(n, \hat{D}_*) \cap D_{ext}(n, \hat{D}_*)$, then $t_n < T_{\hat{D}_*}$. Since this event depends on variables for edges in the exterior of $\hat{D}_*$, we can use independence for the lower bound for $\mathbb{E}\Xi_{t_n}(\epsilon)$ of

$$\begin{aligned}
\frac{\epsilon}{1-p_c}\sum_{e\subset \mathrm{Ann}(2n,4n)}\sum_{\hat{D}_*}\Bigg[&\mathbb{P}\left(\begin{array}{c}\omega(e)>p_c, e\xleftrightarrow{q_n}\partial B(n) \text{ in } B(4n), \\ D^e_{int}(n,\hat{D}_*), t_{\hat{D}_*}\leq t_n\end{array}\right) \\
&\times \mathbb{P}\left(D_{ext}(n,\hat{D}_*), Z(\hat{D}_*) > Cn^2\pi(n)\right)\Bigg].
\end{aligned} \tag{2.27}$$

Note that only the first factor inside the double sum depends on e. To bound it, we give the next lemma.

Lemma 2.8. *There exists* C_6 *and* $C_{18} > 0$ *such that for all* $n \geq 1$, *all* $\hat{D}_*$ *around the origin in* $\mathrm{Ann}(8n, 16n)$, *and all* $C \geq \mathsf{C}_6$,

$$\sum_{e \subset Ann(2n,4n)} \mathbb{P}\begin{pmatrix} \omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \textit{ in } B(4n), \\ D^e_{int}(n, \hat{D}_*), t_{\hat{D}_*} \leq t_n \end{pmatrix} \geq C_{18} n^2 \pi(n).$$

Proof: First note that for any $\hat{D}_*$, we have $t_{\hat{D}_*} \leq t_n$ whenever $R_{t_n} \geq 16n$. Therefore it will suffice to show a lower bound for

$$\sum_{e \subset \mathrm{Ann}(2n,4n)} \mathbb{P}\begin{pmatrix} \omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), \\ D^e_{int}(n, \hat{D}_*), R_{t_n} \geq 16n \end{pmatrix}.$$

To do this, we will show both a lower bound

$$\begin{aligned} \sum_{e \subset \mathrm{Ann}(2n,4n)} \mathbb{P}\Big(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\Big) \\ \geq C_{19} n^2 \pi(n). \end{aligned} \tag{2.28}$$

and an upper bound

$$\sum_{e \subset \mathrm{Ann}(2n,4n)} \mathbb{P}\begin{pmatrix} \omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), \\ D^e_{int}(n, \hat{D}_*), R_{t_n} < 16n \end{pmatrix} \leq \frac{C_{19}}{2} n^2 \pi(n), \tag{2.29}$$

for all n, so long as C is larger than some C_6.

Inequality (2.29) is easier, so we start with it. First sum over e and then apply the Cauchy-Schwarz inequality to get the upper bound

$$\begin{aligned} &\left(\mathbb{E}\left(\#\{e \subset \mathrm{Ann}(2n,4n) : e \overset{p_c}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\}\right)^2\right)^{1/2} \\ &\times \Big(\mathbb{P}(R_{t_n} < 16n)\Big)^{1/2} \\ &\leq \left(\sum_{e,f \subset \mathrm{Ann}(2n,4n)} \mathbb{P}(e \overset{p_c}{\longleftrightarrow} \partial B(e,n), f \overset{p_c}{\longleftrightarrow} \partial B(f,n))\right)^{1/2} \\ &\times \Big(\mathbb{P}(R_{t_n} < 16n)\Big)^{1/2}. \end{aligned} \tag{2.30}$$

Here, for example, $B(e,n)$ is the box of sidelength $2n$ centered at the bottom-left endpoint of e. We claim that the sum is bounded above by $(C_{20} n^2 \pi(n))^2$. The argument is similar to those in Kesten (1986, p. 388-391): we first bound the sum from above by

$$\sum_{e \subset \mathrm{Ann}(2n,4n)} \sum_{\ell=0}^{8n} \sum_{f : \mathrm{dist}(e,f) = \ell} \mathbb{P}\begin{pmatrix} e \overset{p_c}{\longleftrightarrow} \partial B\left(e, \frac{\ell}{2}\right), f \overset{p_c}{\longleftrightarrow} \partial B\left(f, \frac{\ell}{2}\right), \\ B\left(\frac{e_x + f_x}{2}, 2\ell\right) \overset{p_c}{\longleftrightarrow} \partial B\left(\frac{e_x + f_x}{2}, \frac{n}{4}\right) \end{pmatrix}.$$

Here, e_x and f_x are the lower-left endpoints of e and f and $\mathrm{dist}(e,f)$ is the ℓ^∞-distance between e_x and f_x. Writing $\pi(m,n)$ for the probability that $B(m) \overset{p_c}{\longleftrightarrow} \partial B(n)$, by independence the above is equal to

$$\sum_{e\subset \mathrm{Ann}(2n,4n)} \sum_{\ell=0}^{8n} \sum_{f:\mathrm{dist}(e,f)=\ell} \pi\left(\frac{\ell}{2}\right)^2 \pi\left(2\ell, \frac{n}{4}\right)$$
$$\le A_1 n^3 \sum_{\ell=0}^{8n} \pi(\ell)^2 \pi(\ell, n)$$

for some constant A_1. In the inequality we have used quasimultiplicativity (Nolin, 2008, Eq. (4.17)) and the RSW theorem to bound $\pi(\ell/2)$ and $\pi(2\ell, n/4)$ above by constant multiples times $\pi(\ell)$ and $\pi(\ell, n)$, respectively. Because $\pi(\ell)\pi(\ell, n)$ is comparable to $\pi(n)$ (also a consequence of quasimultiplicativity), we obtain A_2 such that the above is bounded by

$$A_2 n^3 \pi(n) \sum_{\ell=0}^{8n} \pi(\ell).$$

However, Kesten (1986, Eq. (7)) shows that $\sum_{\ell=0}^{m} \pi(\ell)$ is bounded by a constant multiple of $m\pi(m)$, so this completes the proof of the claim.

Returning to (2.30), this gives us the bound

$$\text{LHS of (2.29)} \le C_{20} n^2 \pi(n) \sqrt{\mathbb{P}(R_{t_n} < 16n)}.$$

Due to Lemma 2.1, given any C_{19} from (2.28) (assuming we show that inequality, which we will in a moment), we can find C_6 such that for $C \ge \mathsf{C}_6$,

$$C_{20} \sqrt{\mathbb{P}(R_{t_n} < 16n)} \le C_{19}/2,$$

and this completes the proof of (2.29).

Turning to the lower bound (2.28), since $\omega(e)$ is independent of both events $\{e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\}$ and $D^e_{int}(n, \hat{D}_*)$,

$$\sum_{e\subset \mathrm{Ann}(2n,4n)} \mathbb{P}\left(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\right)$$
$$= (1 - p_c) \sum_{e\subset \mathrm{Ann}(2n,4n)} \mathbb{P}\left(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\right). \tag{2.31}$$

Estimating each summand from below uses some standard gluing constructions (see Kesten, 1987, Thm. 1 or Damron et al., 2009, Lem. 6.3 for some examples), so we will only indicate the main idea. It will suffice to lower bound the sum over only $e \subset \hat{B}_n := [-4n, -2n] \times [-2n, 2n]$. To construct the event $D^e_{int}(n)$, we build the event $\bar{D}(n)$, defined by the following conditions:

[*a*] There exists a q_n-open circuit around the origin in $\mathrm{Ann}(n, 2n)$.
[*b*] There exists an edge $f \subset B'(n) :=\mathrm{Ann}(6n, 7n) \cap [6n, \infty)^2$ with $\omega(f) \in (q_n, p_c)$ such that:
[*c*] there exists a p_c-closed dual path P around the origin in $\mathrm{Ann}(4n, 8n)^* \backslash \{f^*\}$ that is connected to the endpoints of f^* so that $P \cup \{f^*\}$ is a circuit around the origin, and
[*d*] there exists a p_c-open path connecting one endpoint of f to $B(n)$ and remaining in $[-n, \infty) \times \mathbb{R}$. Also, there exists another disjoint p_c-open path connecting the other endpoint of f to $\partial B(16n)$.

Fig. 2.4 illustrates the event $\bar{D}(n)$.

The event described in [*b*] guarantees item 2(a) in the definition of $D^e_{int}(n, \hat{D}_*)$. Since the event described in [*c*] has a p_c-open path from $\partial B(n)$ to $\partial B(16n)$ containing f without using e, the event

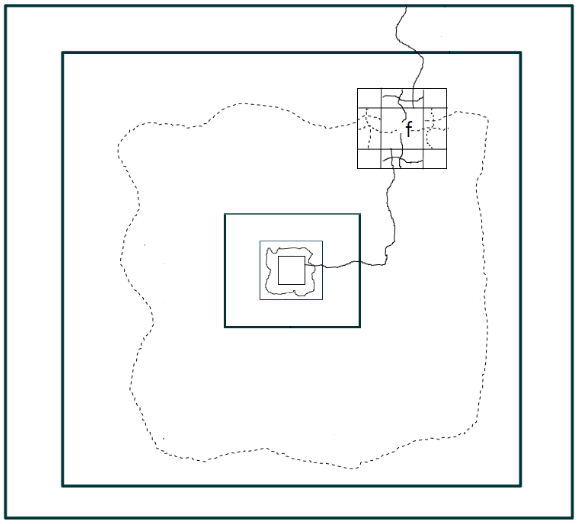

FIGURE 2.4. The event $\bar{D}(n)$. The boxes, in order from smallest to largest, are $B(n)$, $B(2n)$, $B(4n)$, $B(8n)$ and $B(16n)$. The solid circuit in $\mathrm{Ann}(2n,4n)$ is q_n-open. The solid paths from $\partial B(n)$ to f and the solid path from f to $\partial B(16n)$ are p_c-open. The dotted circuit in $\mathrm{Ann}(4n,8n)$ is p_c-closed.

$[c]$ implies item 2(b) in the definition of $D^e_{int}(n,\hat{D}_*)$. Therefore, for any circuit $\hat{D}^e_* \subset \mathrm{Ann}(8n,16n)$, we can estimate the sum in the bottom of (2.31):

$$\sum_{e\subset \mathrm{Ann}(2n,4n)} \mathbb{P}\left(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n,\hat{D}_*)\right)$$

$$\geq \sum_{e\subset \hat{B}_n} \mathbb{P}\left(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\cap(-\infty,-n]\times\mathbb{R}, \bar{D}(n)\right). \tag{2.32}$$

By applying the generalized FKG inequality (positive correlation for certain increasing and decreasing events, so long as they depend on particular regions of space — see Nolin (2008, Lem. 13)), one can decouple the events described in $\bar{D}(n)$ and the event $\{e \overset{p_c}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\cap(-\infty,-n]\times\mathbb{R}\}$ to obtain the lower bound for (2.32) of

$$\sum_{e\subset \hat{B}_n} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\cap(-\infty,-n]\times\mathbb{R})\mathbb{P}([a])\mathbb{P}([b],[c])$$

$$\geq c_2\mathbb{P}([b],[c]) \sum_{e\subset \hat{B}_n} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n)\cap(-\infty,-n]\times\mathbb{R}). \tag{2.33}$$

To give a lower bound for $\mathbb{P}([b],[c])$, let $A(n,f)$ be the event described in [b] and [c] (along with the condition $\omega(f)\in(q_n,p_c)$), so that this probability equals $\mathbb{P}(\cup_f A(n,f))$, and the union is over $f\subset \mathrm{Ann}(6n,7n)\cap[6n,\infty)^2$. Letting $A'(n,f)$ be the same event, but with the p_c-open paths from [c] replaced by q_n-open paths, we obtain

$$\mathbb{P}([b],[c]) = \mathbb{P}(\cup_f A(n,f)) \geq \mathbb{P}(\cup_f A'(n,f)).$$

Note that the events $A'(n,f)$ for distinct f are disjoint. Therefore

$$\mathbb{P}([b],[c]) \geq \sum_f \mathbb{P}(A'(n,f)). \tag{2.34}$$

Let $\Upsilon(n,f)$ be the event that there are two disjoint q_n-open paths connecting f to $\partial B(f,n)$, and two disjoint p_c-closed dual paths connecting f^* to $\partial B(f,n)$. We claim that

$$\mathbb{P}(A'(n,f)) \geq c_3(p_c - q_n)\mathbb{P}(\Upsilon(n,f)). \tag{2.35}$$

The proof of this inequality uses standard gluing methods and is similar to the proof of the lower bound on $\mathbb{P}(C_e(n,N;m))$ in Damron et al. (2009, p. 2324), so we only describe the idea here. Let $\hat{\Upsilon}(n,f)$ be the event that $\Upsilon(n,f)$ occurs, but the q_n-open paths connect f to the top and bottom sides of $\partial B(f,n)$, and the p_c-closed dual paths connect f^* to the left and right sides of $\partial B(f,n)$. Let $\mathfrak{A}(f,n)$ be the event that the left and right sides of $\partial B(f,n)$ are connected by a p_c-closed dual path in $B(f,n)^c \cap \mathrm{Ann}(4n,8n)$ in such a way that the union of this path with $B(f,n)$ contains the origin in its interior, and furthermore that (a) the top side of $B(f,n)$ is connected to $\partial B(16n)$ by a q_n-open path, and (b) the bottom side of $\partial B(f,n)$ is connected by a q_n-open path to $B(n)$ that remains in $[-n,\infty) \times \mathbb{R}$. Then the generalized FKG inequality can be used to glue together the connections from $\hat{\Upsilon}(n,f)$ and $\mathfrak{A}(f,n)$ to show that

$$\begin{aligned}\mathbb{P}(A'(n,f)) &\geq (p_c - q_n)\mathbb{P}(\mathfrak{B}(f,n))\\ &\geq a_1(p_c - q_n)\mathbb{P}(\hat{\Upsilon}(n,f))\mathbb{P}(\mathfrak{A}(f,n)).\end{aligned}$$

Here, a_1 is a positive constant and $\mathfrak{B}(f,n)$ stands for the event that $\hat{\Upsilon}(n,f) \cap \mathfrak{A}(f,n)$ occurs, but the path touching the top (resp. bottom, left, right) side of $\partial B(f,n)$ from the definition of $\hat{\Upsilon}(n,f)$ is connected to the path touching the top (resp. bottom, left, right) side of $\partial B(f,n)$ from the definition of $\mathfrak{A}(f,n)$. Also, the factor $(p_c - q_n)$ appears because the weight of f is independent of the other events involved. (See Fig. 2.4 for an illustration. The extra paths in the small box surrounding f are shown as examples of possible gluing connections.) The event described in $\hat{\Upsilon}(n,f)$ is the same as that in $\Upsilon(n,f)$, except that the paths starting from f or f^* are required to end on specified sides of the boundary $\partial B(f,n)$; that is, the "arms are directed" to these sides. Kesten's arm direction results, summarized and extended in Nolin (2008, Thm. 11), show that for some constant $a_2 > 0$, we have $\mathbb{P}(\hat{\Upsilon}(n,f)) \geq a_2\mathbb{P}(\Upsilon(n,f))$. The RSW theorem and FKG inequality, on the other hand, shows that $\mathbb{P}(\mathfrak{A}(f,n)) \geq a_3 > 0$ for some constant a_3. Putting these in the above inequality gives

$$\mathbb{P}(A'(n,f)) \geq a_2 a_3 (p_c - q_n)\mathbb{P}(\Upsilon(n,f)).$$

This is the claimed inequality (2.35).

Now that we have shown (2.35), we apply a variant of Damron et al. (2009, Lem. 6.3) (instead of taking $p,q \in [p_c,p_n]$, one takes $p,q \in [q_n,p_c]$, with $p = q_n$ and $q = p_c$, and the proof is nearly identical), to deduce that $\mathbb{P}(\Upsilon(n,f)) \asymp \mathbb{P}(A_n^{2,2})$, where $A_n^{2,2}$ is the four-arm event from (2.6). Using this with (2.34) and (2.35) gives

$$\mathbb{P}([b],[c]) \geq c_4(p_c - q_n)\sum_f \mathbb{P}(A_n^{2,2}).$$

By (2.6), we establish $\mathbb{P}([b],[c]) \geq c_5$, and putting this in (2.33),

$$\begin{aligned}&\sum_{e \subset \hat{B}_n} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \cap (-\infty,-n] \times \mathbb{R})\mathbb{P}([a])\mathbb{P}([b],[c])\\ &\geq c_2 c_5 \sum_{e \subset \hat{B}_n} \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n) \cap (-\infty,-n] \times \mathbb{R}).\end{aligned} \tag{2.36}$$

Last, to deal with the summand of (2.36), we can use a gluing construction along with the FKG inequality and the RSW theorem to obtain

$$\mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(n)) \geq c_6 \mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(e, dist(e,\partial B(n))),$$

where *dist* is the ℓ_∞-distance. By (2.1) and (2.5), we have

$$\begin{aligned}&\mathbb{P}(e \overset{q_n}{\longleftrightarrow} \partial B(e, dist(e, \partial B(n))))\\ &\geq c_7 \mathbb{P}(e \overset{p_c}{\longleftrightarrow} \partial B(e, dist(e, \partial B(n))))\\ &\geq c_8 \pi(n).\end{aligned}$$

Placing this in (2.36) and summing over e finally gives

$$\begin{aligned}&\sum_{e \subset \mathrm{Ann}(2n,4n)} \mathbb{P}\left(\omega(e) > p_c, e \overset{q_n}{\longleftrightarrow} \partial B(n) \text{ in } B(4n), D^e_{int}(n, \hat{D}_*)\right)\\ &\geq c_9 n^2 \pi(n),\end{aligned}$$

which finishes the proof of (2.28).

□

Applying the lemma to the lower bound from (2.27), we obtain for all $C \geq \mathsf{C}_6$

$$\begin{aligned}&\mathbb{E}\Xi_{t_n}(\epsilon)\\ &\geq \frac{\epsilon}{1-p_c} C_{21} n^2 \pi(n) \sum_{\hat{D}_*} \mathbb{P}(D_{ext}(n, \hat{D}_*), Z(\hat{D}_*) > Cn^2\pi(n))\\ &= \frac{\epsilon}{1-p_c} C_{21} n^2 \pi(n) \mathbb{P}\left(\bigcup_{\hat{D}_*} \{D_{ext}(n, \hat{D}_*), Z(\hat{D}_*) > Cn^2 \pi(n)\}\right)\\ &\geq \frac{\epsilon}{1-p_c} C_{21} n^2 \pi(n) \mathbb{P}(A_n, B_n(C)),\end{aligned}$$

where A_n is the event that there is a p_c-open circuit around the origin in $\mathrm{Ann}(8n, 16n)$ and $B_n(C)$ is the event that there are more than $Cn^2\pi(n)$ edges g in $B(16n)^c$ with $\omega(g) < p_c$ connected to $B(8n)$ by p_c-open paths. By the FKG inequality and the RSW theorem, for all $n \geq 1$, $\epsilon > 0$ and $C \geq \mathsf{C}_6$,

$$\mathbb{E}\,\Xi_{t_n}(\epsilon) \geq \frac{\epsilon}{1-p_c} C_{21} C_{22} n^2 \pi(n) \mathbb{P}(B_n(C)). \tag{2.37}$$

Last, we argue that there exists a function F on $[0,\infty)$ such that $\inf_{r\in[0,m]} F(r) > 0$ for each $m \geq 0$ and such that

$$\mathbb{P}(B_n(C)) \geq F(C) \text{ for all } n \geq 1 \text{ and } C \geq 0. \tag{2.38}$$

Combining this with (2.37) and setting $G(C) = C_{21}C_{22}F(C)/(1-p_c)$ will complete the proof of Proposition 2.4 and therefore of the proof of the upper bound in Theorem 1.1.

To show (2.38), we use some standard percolation arguments. For $\ell \geq 5$, set

$$Z_n(\ell) := \#\{g \subset \mathrm{Ann}(2^\ell n, 2^{\ell+1}n) : g \overset{p_c}{\longleftrightarrow} \partial B(8n), \omega(g) < p_c\}.$$

By definition of $Z_n(\ell)$ and $B_n(C)$, for any $\ell \geq 5$,

$$\mathbb{P}(B_n(C)) \geq \mathbb{P}(Z_n(\ell) > Cn^2\pi(n)). \tag{2.39}$$

To give a lower bound for the probability of $Z_n(\ell)$, we use the second moment method (Paley-Zygmund inequality):

$$\mathbb{P}\left(Z_n(\ell) \geq \frac{1}{2}\mathbb{E}Z_n(\ell)\right) \geq \frac{1}{4}\frac{(\mathbb{E}Z_n(\ell))^2}{\mathbb{E}Z_n(\ell)^2}. \tag{2.40}$$

Accordingly, we need a lower bound for $\mathbb{E}Z_n(\ell)$ and an upper bound for $\mathbb{E}Z_n(\ell)^2$.

To bound $\mathbb{E}Z_n(\ell)$ from below, note that if there is a p_c-open circuit around the origin in $\mathrm{Ann}(2^{\ell+1}n, 2^{\ell+2}n)$ and a p_c-open path connecting $B(8n)$ to $\partial B(2^{\ell+2}n)$, then any edge g in $\mathrm{Ann}(2^\ell n, 2^{\ell+1}n)$ with weight $< p_c$ that is connected by a p_c-open path to $\partial B(g, 2^{\ell+3}n)$ (the box of sidelength $2 \cdot 2^{\ell+3}n$ centered at the lower-left endpoint of g) contributes to $Z_n(\ell)$. By the FKG inequality and the RSW theorem,

$$\mathbb{E}Z_n(\ell) \geq c_{10} p_c \pi(8n, 2^{\ell+3}n)\pi(2^{\ell+3}n)\#\{g : g \subset \mathrm{Ann}(2^\ell n, 2^{\ell+1}n)\},$$

Here, c_{10} is a lower bound for the probability of existence of the circuit, $\pi(8n, 2^{\ell+3}n)$ is the probability of existence of a p_c-open connection between $B(8n)$ and $\partial B(2^{\ell+2}n)$, and $\pi(2^{\ell+3}n)$ is the probability corresponding to the connection between g and $\partial B(g, 2^{\ell+3}n)$. We then estimate using independence

$$\pi(8n, 2^{\ell+3}n)\pi(2^{\ell+3}n) \geq \frac{\pi(2^{\ell+3}n)}{\pi(8n)} \cdot \frac{\pi(2^{\ell+3}n)}{\pi(n)} \cdot \pi(n)$$

and apply (2.5) for the lower bound

$$\pi(8n, 2^{\ell+3}n)\pi(2^{\ell+3}n) \geq D_1^2 \sqrt{\frac{8n}{2^{\ell+3}n} \cdot \frac{n}{2^{\ell+3}n}}\pi(n) \geq \frac{D_1^2}{2^{\ell+2}}\pi(n).$$

Placing this back in the above equation, we obtain for some $c'_{10} > 0$,

$$\mathbb{E}Z_n(\ell) \geq \left[c'_{10}\frac{D_1^2}{2^{\ell+2}}2^{2\ell}\right] n^2\pi(n).$$

If we fix $\ell = \ell_0$ so large that this is bigger than $2Cn^2\pi(n)$ for all n, we obtain from (2.39) and (2.40) that

$$\mathbb{P}(B_n(C)) \geq \frac{C^2(n^2\pi(n))^2}{\mathbb{E}Z_n(\ell_0)^2}. \tag{2.41}$$

For the upper bound on $\mathbb{E}Z_n(\ell_0)^2$, we follow the strategy of Kesten (1986, p. 388-391). First note that any g counted in $Z_n(\ell_0)$ must have a p_c-open path connecting it to $\partial B(g, 2^{\ell_0-1}n)$. Therefore by independence,

$$\begin{aligned} &\mathbb{E}Z_n(\ell_0)^2 \\ &\leq \sum_{g,h\subset \mathrm{Ann}(2^{\ell_0}n, 2^{\ell_0+1}n)} \mathbb{P}(g \overset{p_c}{\longleftrightarrow} \partial B(g, 2^{\ell_0-1}n), h \overset{p_c}{\longleftrightarrow} \partial B(h, 2^{\ell_0-1}n)) \\ &\leq \sum_{g\subset \mathrm{Ann}(2^{\ell_0}n, 2^{\ell_0+1}n)} \sum_{k=0}^{2^{\ell_0+2}n} \sum_{h:\mathrm{dist}(g,h)=k} \pi(k/2)\pi(k/2)\pi(2k, 2^{\ell_0-1}n). \end{aligned} \tag{2.42}$$

The appearance of π factors in (2.42), and the argument we give below, is directly analogous to that below (2.30). Here, $\pi(2k, 2^{\ell_0-1}n)$ is the probability that there is an open path connecting $B(2k)$ to $\partial B(2^{\ell_0-1}n)$. (If $2k \geq 2^{\ell_0-1}n$, this probability is one.) By quasimultiplicativity (Nolin, 2008, Eq. (4.17)) and the RSW theorem, we have

$$\pi(k/2)\pi(2k, 2^{\ell_0}n) \leq C_{23}\pi(2^{\ell_0}n),$$

which is itself bounded by $C_{23}\pi(n)$, so putting this in (2.42), we have an upper bound

$$\mathbb{E}Z_n(\ell_0)^2 \leq \left[C_{23}2^{2\ell_0}\sum_{k=0}^{2^{\ell_0+2}n}\pi(k)\right] n^2\pi(n).$$

By Kesten (1986, Eq. (7)), we have $\sum_{k=0}^{2^{\ell_0+2}n}\pi(k) \leq C_{24}2^{2(\ell_0+2)}n^2\pi(n)$, and so we finish with $\mathbb{E}Z_n(\ell_0)^2 \leq C_{25}(n^2\pi(n))^2$, where C_{25} depends only on ℓ_0. Putting this into (2.41) finishes the proof of (2.38). □

Acknowledgements

The authors are grateful to two anonymous reviewers for ALEA whose comments helped to improve the article.

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