

A universal scaling limit for diffusive amnesic step-reinforced random walks

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Abstract. We introduce a variation of the step-reinforced random walk with general memory. For the diffusive regime, we establish a functional invariance principle and show that, given suitable conditions on the memory sequence, the arising limiting processes are always the sum of a noise reinforced Brownian motion and a (not independent) Brownian motion.

1. Introduction

Motivated by the study of the effects of memory on the asymptotic behaviour of non-Markovian processes, [Schütz and Trimper \(2004\)](#) introduced around 20 years ago the so-called *elephant random walk*. The elephant random walk can be understood as a fundamental example of a step-reinforced random walk. Additionally, it stands as one of the simplest models that lead to anomalous diffusion.

Anomalous diffusion appears in many physical, biological or social systems such as human travel [Brockmann et al. \(2006\)](#) or heartbeat intervals and DNA sequences [Buldyrev et al. \(1994\)](#). Further examples include telomeres in the nucleus of cells [Bronstein et al. \(2009\)](#), ion channels in the plasma membrane [Weigel et al. \(2011\)](#), diffusion in porous materials [Koch and Brady \(1988\)](#), or diffusion in polymer networks [Wong et al. \(2004\)](#) to name only a few. The phenomena of anomalous diffusion often arises in theoretical models by incorporating memory effects such as modeled by the elephant random walk.

The elephant random walk is a discrete-time nearest neighbour random walk on the integer lattice \mathbb{Z} with infinite memory, in allusion to the traditional saying that *an elephant never forgets*. The dynamics of the elephant random walk are governed by a parameter p between zero and one, commonly referred to as the *memory parameter*, that specifies the probability of repetition of certain steps. Roughly speaking, given an initial step of the elephant, say $X_1 = 1$ a.s., then at each integer time $n \geq 2$, the elephant remembers one of its previous steps chosen uniformly at random; then it decides, either with probability p to repeat this step, or with complementary probability of $1 - p$ to walk in the opposite direction. Notably, the steps of the elephant are either plus or minus one. As a

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consequence of the aforementioned dynamics, for $p > 1/2$, the elephant is more inclined to continue its walk in the average direction it has already taken up to that point. Conversely, for $p < 1/2$, it tends to backtrack. In the borderline case of $p = 1/2$, the elephant does not intend to make a decision and its path follows that of a simple symmetric random walk on \mathbb{Z} . In particular, the elephant random walk is a time-inhomogeneous Markov chain, although some works in the literature improperly assert its non-Markovian character. Indeed, if the elephant is at position $k \in \mathbb{Z}$ at time $n \in \mathbb{N}$, then it performed $(n+k)/2$ steps up and $(n-k)/2$ steps down, more information from the past is irrelevant for predicting the $(n+1)$ th step. The asymptotic behaviour after a proper rescaling of the elephant random walk has in recent years been a topic of interest for many authors and is well understood, see [Baur and Bertoin \(2016\)](#); [Bercu \(2018\)](#); [Coletti et al. \(2017a,b\)](#); [Guevara and Suárez \(2020\)](#); [Kubota and Takei \(2019\)](#) and [Bertenghi \(2022\)](#); [Bercu and Laulin \(2021\)](#); [Baur \(2020\)](#); [Laulin \(2022b\)](#); [Gut and Stadtmüller \(2022\)](#); [Bercu and Vázquez Guevara \(2022\)](#); [Guevara and Suárez \(2020\)](#); [González-Navarrete \(2020\)](#); [Miyazaki and Takei \(2020\)](#) for variations.

A step-reinforced random walk extends the dynamics of the elephant random walk to allow for more diverse steps, rather than restricting to plus or minus one. In essence, the steps can follow an arbitrary distribution, typically on \mathbb{R} . Put simply, we are given a sequence $\mathcal{X}_1, \mathcal{X}_2, \dots$ of i.i.d. copies of a random variable \mathcal{X} on \mathbb{R} and again a memory parameter p between zero and one. We then create a sequence of step-reinforced random variables X_1, X_2, \dots as follows: $X_1 = \mathcal{X}_1$ a.s. and subsequently, at each integer time $n \geq 2$, one of the previous steps is chosen uniformly at random. Then, with probability p the step is repeated; otherwise, with the complementary probability of $1-p$, an independent increment following the same distribution as \mathcal{X} is taken. In this setting, when \mathcal{X} follows a Rademacher distribution, [Kürsten \(2016\)](#) (see also [González-Navarrete and Lambert, 2018](#)) pointed out that the step-reinforced random walk is then just a version of the elephant random walk with memory parameter $q = (p+1)/2 \in (1/2, 1)$ in the present notation. Observe that in the degenerate case $p = 1$, the dynamics of the step-reinforced random walk become essentially deterministic. Indeed, when $p = 1$, then the position of the step-reinforced random walk at time n is just given by $n\mathcal{X}_1$ for all $n \geq 1$, in particular the only remaining randomness for this process stems from the random variable \mathcal{X}_1 . Similarly, when $p = 0$, then the step-reinforced random walk reduces to a random walk with i.i.d. increments. In light of this, we will exclude these degenerate cases in our analysis, that is we will only consider $p \in (0, 1)$.

In this work, we introduce an additional layer of complexity to the step-reinforced random walk model. Specifically, we consider a more general underlying memory mechanism that incorporates recent steps being repeated with a higher likelihood, inspired by diminishing memory effects like amnesia. This dynamic will be governed by yet another parameter $\alpha \in \mathbb{R}_+$ that we shall henceforth refer to as the *amnesia parameter*. In that direction for $\alpha \gg 1$, our amnesic step-reinforced random walk is much more likely to repeat steps from its recent past, whereas for α close (or equal) to 1 it behaves more (or exactly) like a step-reinforced random walk. [Laulin \(2022a\)](#) considered a version of the elephant random walk where the underlying memory process also includes amnesia, see also [Chen and Laulin \(2023\)](#) for the multidimensional extension. In contrast to the aforementioned work, our study encompasses a broader range of underlying memory distributions, ultimately including the one detailed in [Laulin \(2022a\)](#). We contend that the memory distribution employed here is both comprehensive and representative, encapsulating all known cases from the literature as specific instances. Furthermore, it remains sufficiently tractable for a thorough analysis.

It is our main objective in this work to establish a functional invariance principle for the properly rescaled amnesic step-reinforced random walk in the so-called diffusive regime. We will show that the resulting limiting processes comprise the non-independent sum of a noise-reinforced Brownian motion and a Brownian motion. For the properly re-scaled (non-amnesic) step-reinforced random walk, [Bertoin \(2021\)](#) established the noise-reinforced Brownian motion as the universal scaling limit in the diffusive regime. Our work corroborates this result and further indicates the presence of a Brownian motion in the limiting process for all $\alpha \neq 1$, which also agrees with Theorem 2.3 in [Laulin](#)

(2022a). A noise-reinforced Brownian motion is a simple real-valued and centered Gaussian process $\hat{B} = (\hat{B}(t))_{t \geq 0}$ with covariance function given by

$$\mathbb{E} \left(\hat{B}(t) \hat{B}(s) \right) = \frac{1}{1-2p} s \left(\frac{t}{s} \right)^p \quad \text{for } 0 \leq s \leq t \quad \text{and } p \in (0, 1/2).$$

This process has notably appeared as the scaling limit for diffusive regimes of the ERW and other Pólya urn related processes, see [Baur and Bertoin \(2016\)](#); [Coletti et al. \(2017a\)](#); [Bertenghi \(2022\)](#); [Bai et al. \(2002\)](#).

The remainder of the paper is organised as follows: In Section 2 we will give an exact definition of the amnesic step-reinforced random walk. In Section 3 we will present the main results of our work. Section 4 contains a short detour to regularly varying sequences, which are quintessential in the definition (and therefore the analysis) of amnesic step-reinforced random walks. In Section 5 we lay the ground work of our analysis before presenting the proofs of our main results in Section 6. For the readers convenience, and to make this work self-contained, technical lemmas and a non-standard result on martingales are provided in the appendix.

2. The Model

In this section, we formally introduce our model. We start by giving our memory sequence (β_n) which is the main character of our work. To do so, we consider a positive sequence (μ_n) and the associated sequence of partial sum (ν_n) , defined as follows:

$$\nu_0 = 0, \quad \nu_n = \nu_{n-1} + \mu_n \text{ for } n \geq 1.$$

The sole (yet crucial) additional assumption we make regarding the sequence (μ_n) is the following:

(A) Let $\alpha \geq 1$, the sequence (μ_n) is regularly varying (at infinity) with index $\alpha - 1$.

This ensures that the sequence (ν_n) is also regularly varying of index $\alpha \geq 1$. Roughly, this means that this sequences will have a polynomial growth. In Section 4 we provide more details and references on the topic of regularly varying sequences.

Now, our memory sequence $(\beta_n : n \geq 2)$ is distributed as follows:

$$\mathbb{P}(\beta_n = k) = \frac{\mu_k}{\sum_{i=1}^n \mu_i} = \frac{\nu_k - \nu_{k-1}}{\nu_n}, \quad \text{for } 1 \leq k \leq n.$$

The regularly varying assumption ensures that the above memory process tends to prioritize recent times over older ones.

Hereafter, consider a sequence $\mathcal{X}_1, \mathcal{X}_2, \dots$ of i.i.d. copies of a random variables \mathcal{X} on \mathbb{R} with finite second order moment. We define X_1, X_2, \dots recursively as follows: Let $(\varepsilon_n : n \geq 2)$ be an independent sequence of Bernoulli random variables with parameter $p \in (0, 1)$, also independent of (β_n) . Initially, set $X_1 = \mathcal{X}_1$, and next for $n \geq 1$, define

$$X_{n+1} = \begin{cases} \mathcal{X}_{n+1}, & \text{if } \varepsilon_{n+1} = 0, \\ X_{\beta_n}, & \text{if } \varepsilon_{n+1} = 1. \end{cases} \quad (2.1)$$

Finally, the sequence

$$S_n = X_1 + \dots + X_n$$

is referred to as a step-reinforced random walk. The definition of the sequence (X_n) implies that for any bounded and measurable $f : \mathbb{R} \rightarrow \mathbb{R}_+$

$$\mathbb{E}(f(X_{n+1})) = (1-p)\mathbb{E}(f(\mathcal{X}_{n+1})) + \frac{p}{\nu_n} \sum_{k=1}^n \mu_k \mathbb{E}(f(X_k)) \quad (2.2)$$

and it follows by induction that each X_n also has law \mathcal{X} .

Finally, for the rest of the paper, we will assume that we are in the diffusive regime which corresponds in our case to $\frac{\alpha-1}{\alpha} < p < \frac{2\alpha-1}{2\alpha}$ (see Proposition 3.1 for more details). The upper bound condition appears when we study the quadratic variations of our martingales from Section 5 as we want the variations to be regularly varying of positive index.

In precise terms, our work holds true for $\alpha > 1/2$. However, we chose to focus on the case of $\alpha \geq 1$ to underscore the presence of amnesia. Indeed, the tendency shifts for $1/2 < \alpha < 1$, as the process exhibits a preference for moments from the early times of the past.

3. Main results

In this section we introduce our main results, Theorem 3.2 and Theorem 3.3. The following proposition ensures that we can indeed speak of *diffusive regime* in our setting.

Proposition 3.1. *For $\frac{\alpha-1}{\alpha} < p < \frac{2\alpha-1}{2\alpha}$, the mean square displacement of the amnesic step-reinforced random walk is of order n , that is*

$$\mathbb{E}(S_n^2) \propto n.$$

3.1. *Law of large numbers.* The (strong) law of large numbers will be an essential tool in establishing the functional invariance principle for the amnesic step-reinforced random walk. As such, Section 6 will first focus on establishing a proof of Theorem 3.2.

Theorem 3.2 (Strong law of large numbers). *For $\frac{\alpha-1}{\alpha} < p < \frac{2\alpha-1}{2\alpha}$ we have the almost sure convergence*

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = \mathbb{E}(\mathcal{X}).$$

We remark that Theorem 3.2 is trivially true for $p = 0$. Indeed, in said case the step-reinforced random walk is just a sequence of centered i.i.d. increments and therefore the (strong) law of large number applies.

In the setting under which we are in this work, it is sufficient to have the above LLN for $p < \frac{2\alpha-1}{2\alpha}$. However, we strongly believe that this holds for the greater range $\frac{\alpha-1}{\alpha} < p < 1$ (as long as $\alpha > 0$). Proving this convergence goes beyond the purpose of this paper, hence we provide the proof in the diffuse regime only.

3.2. *A functional invariance principle.* The main result of this exposition is the following statement:

Theorem 3.3. *Suppose that $\frac{\alpha-1}{\alpha} < p < \frac{2\alpha-1}{2\alpha}$, then we have the following convergence in distribution in $D([0, \infty))$ as n tends to infinity*

$$\left(\frac{S_{\lfloor nt \rfloor} - nt\mathbb{E}(\mathcal{X})}{\sqrt{n \text{Var}(\mathcal{X})}}, t \geq 0 \right) \implies (W_t, t \geq 0)$$

where $(W_t, t \geq 0)$ is a real-valued, continuous and centered Gaussian process starting from the origin with covariance given for $0 \leq s \leq t$ by -

$$\mathbb{E}(W_s W_t) = c_1(p, \alpha)s + \frac{c_2(p, \alpha)}{2\alpha(1-p) - 1} s \left(\frac{t}{s}\right)^{1-\alpha(1-p)}$$

where $c_1(p, \alpha) = \frac{1-\alpha}{(1-p)(1-\alpha(1-p))}$ and $c_2(p, \alpha) = \frac{p(\alpha(2-p)-1)}{(1-p)(1-\alpha(1-p))}$.

Observe that for $\alpha = 1$, Theorem 3.3 recovers Theorem 3.3 in Bertoin (2021). Moreover, we notice that our scaling coefficients in the covariance of $(W_t, t \geq 0)$ agree with Display (2.4) of Theorem 2.3 in Laulin (2022a). Furthermore, we notice that as the amnesia parameter α increases, the memory parameter p also tends towards one which means that reinforcement of the steps is more likely to

occur. However, as $\alpha \nearrow \infty$ we have $p = 1$ and in said case Theorem 3.3 clearly does not apply. Furthermore, as α gets close to $1/2$, we notice that this forces the memory parameter p to be close to zero.

Let us define the coefficients in the covariance function in Theorem 3.3 for fixed but arbitrary $\frac{\alpha-1}{\alpha} < p < \frac{2\alpha-1}{2\alpha}$ as $c_1(p, \alpha) := c_1(\alpha)$ respectively $c_2(p, \alpha) := c_2(\alpha)$.

$\alpha \in$	$(1/2, 1)$	$\{1\}$	$(1, \infty)$
$\text{sgn}(c_1(\alpha))$	+	0	-
$\text{sgn}(c_2(\alpha))$	+	+	+

TABLE 3.1. Distribution of the signs of the coefficients c_1, c_2 with respect to the amnesia parameter α .

Remark 3.4. The covariance's structure of the process $(W_t, t \geq 0)$ looks a lot like the covariance structure of the sum of two independent Gaussian processes: a Brownian motion with scaling coefficient $\sqrt{c_1(\alpha)}$ and a noise reinforced Brownian motion with reinforcement $1 - \alpha(1 - p)$ and a scaling coefficient $\sqrt{c_2(\alpha)}$. However, as we will see in the proof of the Theorem, the process $(W_t, t \geq 0)$ is indeed the sum of such two processes, but those are in fact not independent. The two "scaling quantities" can be negative (see Table 3.1) and thus cannot be scaling Brownian motions.

4. Regularly varying sequences

To make this exposition self-contained, we recall here some useful results from the theory of regularly varying sequences and functions, see [Bojanic and Seneta \(1973\)](#); [Galambos and Seneta \(1973\)](#) for more details on this subject.

Definition 4.1 (Regularly varying function). A function $R : (0, +\infty) \rightarrow (0, +\infty)$ is called *regularly varying* (at infinity) of index $\alpha \in \mathbb{R}$

$$\frac{R(tx)}{R(x)} \xrightarrow{x \rightarrow \infty} t^\alpha, \quad \text{for all } t > 0.$$

Proposition 4.2 (Regularly and slowly varying function). A function $L : (0, +\infty) \rightarrow (0, +\infty)$ is called *slowly varying* if

$$\frac{L(tx)}{L(x)} \xrightarrow{x \rightarrow \infty} 1, \quad \text{for all } t > 0.$$

Then, a function R is regularly varying of index $\alpha \in \mathbb{R}$ if and only if there exists a slowly varying function L such that

$$R(x) = x^\alpha L(x).$$

Intuitively the meaning of Propostion 4.2 is that the growth rate of the function L does not change drastically as its input becomes large. Observe that Definition 4.1 entails that the index α determines how the function behaves at infinity. Indeed, if $\alpha = 0$, then R is (always) slowly varying. For x large enough, if $\alpha > 0$, then R roughly behaves like an increasing function, whereas if $\alpha < 0$, then R roughly behaves like a decreasing function. Moreover, Definition 4.1 immediately yields that R

Definition 4.3 (Regularly varying sequence). A sequence of positive terms (u_n) is called *regularly varying* of index $\alpha \in \mathbb{R}$ if there exists a regularly varying function R of index α such that $u_n = R(n)$.

We next give two examples of regularly varying sequences related to our work.

- Example 4.4.* (1) Let $\mu_n = 1$, then (μ_n) is regularly varying of index 0 and $\nu_n = n$ so that (ν_n) is regularly varying of index $\alpha = 1$. This setting corresponds to the classical ERW.
- (2) Let $\mu_n = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}$, then (μ_n) is regularly varying of index δ and (ν_n) is regularly varying of index $\alpha = \delta + 1$. This is exactly the memory introduced by [Laulin \(2022a, Display \(1.3\)\)](#).

During our computations, we will often make use of the following results:

Theorem 4.5 (See Display (1.1) and Theorem 6 in [Bojanic and Seneta, 1973](#)). *A sequence of positive numbers (u_n) is regularly varying of index $\alpha > -1$ if and only if*

$$\frac{1}{nu_n} \sum_{k=1}^n u_k \xrightarrow{n \rightarrow \infty} \frac{1}{1 + \alpha}.$$

Theorem 4.6 (Theorem 4 in [Bojanic and Seneta, 1973](#)). *A sequence (u_n) of positive numbers is a regularly varying sequence of index α if and only if there is a sequence of positive numbers (v_n) such that $u_n \sim v_n$ and*

$$\lim_{n \rightarrow \infty} n \left(1 - \frac{v_{n-1}}{v_n} \right) = \alpha. \quad (4.1)$$

In particular, the sequences (v_n) which satisfy condition (4.1) are regularly varying sequences of index α . A direct consequence of Theorem 4.5 is that

Corollary 4.7. *If (u_n) is regularly varying of index α then,*

$$\frac{u_{n+1}}{u_n} = 1 + \frac{\alpha}{n} + o\left(\frac{1}{n}\right).$$

Further, see condition (B) in [Galambos and Seneta \(1973\)](#), we have the following equivalence.

Proposition 4.8. *A sequence (u_n) of positive numbers is regularly varying of index α if and only if $(n^{-\sigma}u_n)$ is eventually (in the sense for n large enough) increasing for each $\sigma < \alpha$ and $(n^{-\tau}u_n)$ is eventually decreasing for every $\tau > \alpha$.*

5. A two dimensional martingale approach

This section is a preliminary section to prove the main theorems. We will present two martingales that will be crucial for our analysis. To do this, we first assume, without the loss of generality, that \mathcal{X} is centred and normalised, i.e. $\mathbb{E}(\mathcal{X}) = 0$ and $\sigma^2 = \text{Var}(\mathcal{X}) = 1$. For what follows, we shall make these two assumptions implicitly.

Lemma 5.1. *For $n \geq 1$, we define the following deterministic sequences*

$$\gamma_n = 1 + p \frac{\mu_{n+1}}{\nu_n}, \quad a_n = \prod_{k=1}^{n-1} \gamma_k^{-1}, \quad \eta_n = \sum_{k=1}^{n-1} \frac{1}{a_k \nu_k}.$$

Further, we set $Y_n = \sum_{k=1}^n \mu_k X_k$. Then (M_n) and (N_n) defined by

$$M_n = a_n Y_n \quad \text{and} \quad N_n = S_n - p \eta_n M_n$$

are square-integrable martingales.

Proof: Since $\mathbb{E}(X_k^2) < \infty$ for all $k \in \mathbb{N}$, the square integrability of M_n and N_n is immediate.

Further, by (2.1) it follows that

$$\begin{aligned} \mathbb{E}(X_{n+1} \mid \mathcal{F}_n) &= p\mathbb{E}(X_{\beta_{n+1}} \mid \mathcal{F}_n) \\ &= p\mathbb{E}\left(\sum_{k=1}^n X_k \mathbf{1}_{\beta_{n+1}=k} \mid \mathcal{F}_n\right) \\ &= \frac{p}{\nu_n} \sum_{k=1}^n \mu_k X_k \\ &= \frac{p}{\nu_n} Y_n. \end{aligned} \tag{5.1}$$

In turn, (5.1) yields

$$\begin{aligned} \mathbb{E}(Y_{n+1} \mid \mathcal{F}_n) &= Y_n + \mu_{n+1}\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \\ &= \left(1 + p\frac{\mu_{n+1}}{\nu_n}\right) Y_n \\ &= \gamma_n Y_n. \end{aligned} \tag{5.2}$$

From (5.2) it is then immediate that $M_n = a_n Y_n$ is a martingale. Furthermore,

$$\begin{aligned} \mathbb{E}(N_{n+1} \mid \mathcal{F}_n) &= S_n + \mathbb{E}(X_{n+1} \mid \mathcal{F}_n) - p\eta_{n+1}M_n \\ &= S_n + p\left(\frac{Y_n}{\nu_n} - \eta_{n+1}M_n\right) \\ &= S_n + p\left(\frac{1}{\nu_n a_n} - \eta_{n+1}\right) M_n \\ &= S_n - p\eta_n M_n = N_n \end{aligned} \tag{5.3}$$

and (5.3) entails that $(N_n)_{n \geq 0}$ is also a martingale. □

Observe that Lemma 5.1 allows us to rewrite S_n as

$$S_n = N_n + p\eta_n M_n \tag{5.4}$$

and equation (5.4) allows us to establish the asymptotic behaviour of S_n via an extensive use of martingale theory. In order to investigate the asymptotic behaviour of (S_n) via (5.4), we introduce the two-dimensional martingale (\mathcal{M}_n) defined by

$$\mathcal{M}_n = \begin{pmatrix} N_n \\ M_n \end{pmatrix}$$

where (M_n) and (N_n) are the two square-integrable martingales introduced in Lemma 5.1.

Lemma 5.2. *The quadratic variation of (\mathcal{M}_n) is given by*

$$\langle \mathcal{M} \rangle_n = \sum_{k=0}^{n-1} \left((1-p) + \frac{p}{\nu_k} U_k \right) A_k - \xi_n, \tag{5.5}$$

where $U_n = \sum_{k=1}^n \mu_k X_k^2$, $\xi_n = \sum_{k=0}^{n-1} \frac{p^2}{\nu_k^2} Y_k^2 A_k$ and

$$A_k = \begin{pmatrix} (1 - pa_{k+1}\eta_{k+1}\mu_{k+1})^2 & a_{k+1}\mu_{k+1} - pa_{k+1}^2\eta_{k+1}\mu_{k+1}^2 \\ a_{k+1}\mu_{k+1} - pa_{k+1}^2\eta_{k+1}\mu_{k+1}^2 & a_{k+1}^2\mu_{k+1}^2 \end{pmatrix}$$

Proof: Denote the martingale increment $t_{n+1} = X_{n+1} - \frac{p}{\nu_n} Y_n$ and observe that indeed

$$\mathbb{E}(t_{n+1} \mid \mathcal{F}_n) = 0.$$

We obtain

$$\begin{aligned}
\Delta\mathcal{M}_{n+1} &= \mathcal{M}_{n+1} - \mathcal{M}_n \\
&= \begin{pmatrix} S_{n+1} - S_n - p(\eta_{n+1}M_{n+1} - \eta_nM_n) \\ a_{n+1}Y_{n+1} - a_nY_n \end{pmatrix} \\
&= \begin{pmatrix} X_{n+1} - p(\eta_{n+1}a_{n+1}Y_{n+1} - \eta_n a_n Y_n) \\ a_{n+1}\mu_{n+1}t_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} (1 - pa_{n+1}\eta_{n+1}\mu_{n+1})t_{n+1} \\ a_{n+1}\mu_{n+1}t_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 - pa_{n+1}\eta_{n+1}\mu_{n+1} \\ a_{n+1}\mu_{n+1} \end{pmatrix} t_{n+1}.
\end{aligned} \tag{5.6}$$

Further we have

$$\begin{aligned}
\mathbb{E}(t_{n+1}^2 \mid \mathcal{F}_n) &= \mathbb{E}(X_{n+1}^2 \mid \mathcal{F}_n) - \frac{p^2}{\nu_n^2} Y_n^2 \\
&= (1 - p) + \frac{p}{\nu_n} \sum_{k=1}^n \mu_k X_k^2 - \frac{p}{\nu_n^2} Y_n^2 \\
&= (1 - p) + \frac{p}{\nu_n} U_n - \frac{p}{\nu_n^2} Y_n^2.
\end{aligned} \tag{5.7}$$

In turn, this yields

$$\mathbb{E}((\Delta\mathcal{M}_{n+1})(\Delta\mathcal{M}_{n+1})^T \mid \mathcal{F}_n) = \left((1 - p) + \frac{p}{\nu_n} U_n - \frac{p}{\nu_n^2} Y_n^2 \right) A_k. \tag{5.8}$$

Thanks to (5.8) we immediately arrive at (5.5). \square

Then, we find that:

Corollary 5.3. *We have*

$$\begin{aligned}
\langle M \rangle_n &= \sum_{k=1}^n \left((1 - p) + \frac{p}{\nu_k} U_k \right) (a_k \mu_k)^2 - \zeta_n, \\
\text{where } \zeta_n &= p^2 \sum_{k=1}^n \frac{a_k^2}{\nu_{k-1}^2} \mu_k^2 Y_{k-1}^2
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
\langle N \rangle_n &= \sum_{k=1}^n \left((1 - p) + \frac{p}{\nu_k} U_k \right) (1 - pa_k \eta_k \mu_k)^2 - \chi_n \\
\text{where } \chi_n &= \sum_{k=1}^n \frac{p^2}{\nu_{k-1}^2} (1 - pa_k \eta_k \mu_k)^2 Y_{k-1}^2.
\end{aligned}$$

The asymptotic behaviour of M_n is closely related to the one of

$$\tilde{w}_n = \sum_{k=1}^n \left((1 - p) + \frac{p}{\nu_k} U_k \right) (a_k \mu_k)^2 \tag{5.10}$$

as one can observe from (5.9) that we always have $\langle M \rangle_n \leq \tilde{w}_n$ and that ζ_n is negligible when compared to \tilde{w}_n . By the same token, the asymptotic behaviour of N_n is closely related to the one of \tilde{z}_n where

$$\tilde{z}_n = \sum_{k=1}^n \left((1 - p) + \frac{p}{\nu_k} U_k \right) (1 - pa_k \eta_k \mu_k)^2. \tag{5.11}$$

6. Proof of main results

In this section we give detailed proofs of our main Proposition and Theorems.

We first show that we are in the diffusive regime by showing the mean square displacement are of order n (Proposition 3.1). Then, we need to establish a proof of the law of large numbers (Theorem 3.2) as this result is a requirement in order to establish a proof of our main result (Theorem 3.3).

6.1. Proof of Proposition 3.1.

Proof of Proposition 3.1: Using the decomposition (5.4), we are left to compute

$$\mathbb{E}(S_n^2) = \mathbb{E}(N_n^2) + 2p\eta_n\mathbb{E}(N_nM_n) + p^2\eta_n^2\mathbb{E}(M_n^2).$$

All the required information can be taken from Lemma 5.2 and Corollary 5.3 respectively. Specifically, we have:

$$\begin{aligned} \mathbb{E}(N_n^2) &= \sum_{k=1}^n (1 - pa_k\eta_k\mu_k)^2 - \sum_{k=1}^n (1 - pa_k\eta_k\mu_k)^2 \mathbb{E}(Y_{k-1}^2), \\ \mathbb{E}(M_n^2) &= \sum_{k=1}^n (a_k\mu_k)^2 - p^2 \sum_{k=1}^n \frac{a_k^2}{\nu_{k-1}^2} \mu_k^2 \mathbb{E}(Y_{k-1}^2), \\ \mathbb{E}(N_nM_n) &= \sum_{k=1}^n a_k\mu_k(1 - pa_k\eta_k\mu_k) - \sum_{k=1}^n a_k\mu_k(1 - pa_k\eta_k\mu_k) \frac{p^2}{\eta_k^2} \mathbb{E}(Y_k^2). \end{aligned}$$

Further, we know that

$$\langle \mathcal{M} \rangle_n = \mathcal{A}_n + o(\mathcal{A}_n),$$

this entails that, as n tends to infinity, it suffices to estimate

$$\mathbb{E}(N_n^2) \sim \sum_{k=1}^n (1 - pa_k\eta_k\mu_k)^2, \quad \mathbb{E}(M_n^2) \sim \sum_{k=1}^n (a_k\mu_k)^2, \quad \mathbb{E}(N_nM_n) \sim \sum_{k=1}^n a_k\mu_k(1 - pa_k\eta_k\mu_k).$$

Thanks to the asymptotic behaviour established in A.3 we have

$$\begin{aligned} \mathbb{E}(N_n^2) &\sim n \left(\frac{(1 - \alpha)^2}{(1 - \alpha(1 - p))^2} \right), \\ p^2\eta_n^2\mathbb{E}(M_n^2) &\sim p^2 \frac{n(a_n\mu_n\eta_n)^2}{2\alpha(1 - p) - 1} \sim n \left(\frac{p^2\alpha^2}{(1 - \alpha(1 - p))^2(2\alpha(1 - p) - 1)} \right), \\ 2p\eta_n\mathbb{E}(N_nM_n) &\sim n \left(\frac{2p\alpha(1 - \alpha)}{(1 - \alpha(1 - p))^2} \right) \end{aligned}$$

and the result readily follows. □

6.2. Proof of Theorem 3.2.

We now give the proof of the strong law of large numbers (Theorem 3.2).

Proof of Theorem 3.2: Recall from (5.4) that we have the decomposition

$$N_n = S_n + p\eta_nM_n.$$

It follows from Corollary 5.3 together with Corollary A.3 that almost surely

$$\left(\frac{\eta_n}{n}\right)^2 a_{n+1}^2 \mu_{n+1}^2 \mathbb{E}(t_{n+1}^2 \mid \mathcal{F}_n) = O\left(\frac{1}{n^2}\right)$$

and

$$\left(\frac{1}{n}\right)^2 (1 - pa_{n+1}\eta_{n+1}\mu_{n+1})^2 \mathbb{E}(t_{n+1}^2 \mid \mathcal{F}_n) = O\left(\frac{1}{n^2}\right).$$

Hence, we have almost surely

$$\sum_{n \geq 1} \left(\frac{\eta_n}{n}\right)^2 \mathbb{E}((\Delta M_{n+1})^2 \mid \mathcal{F}_n) < \infty \quad \text{and} \quad \sum_{n \geq 1} \left(\frac{1}{n}\right)^2 \mathbb{E}((\Delta N_{n+1})^2 \mid \mathcal{F}_n) < \infty.$$

Next observe by the proof of Corollary A.3 we know that $\eta_n^{-1} \sim ca_n\mu_n$ for some positive constant c . Further, the sequence $(a_n\mu_n)$ is regularly varying of index $\alpha(1-p) - 1$. Trivially, it holds that $\rho = \alpha(1-p) - 1 > -1 = \delta$. By Proposition 4.8 we thus know that $(\frac{n}{\eta_n})$ is eventually increasing.

Then, (2.17) from Hall and Heyde (1980, Theorem 2.18) ensures that

$$\lim_{n \rightarrow \infty} \frac{\eta_n M_n}{n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{N_n}{n} = 0$$

and we conclude from the definition of (N_n) that

$$\lim_{n \rightarrow \infty} \left(\frac{S_n - p\eta_n M_n}{n}\right) = 0 \quad \text{a.s.}$$

which achieves the proof. □

6.3. *Proof of Theorem 3.3.* Recall that we are working with the two-dimensional martingale (\mathcal{M}_n) defined by

$$\mathcal{M}_n = \begin{pmatrix} N_n \\ M_n \end{pmatrix},$$

where (M_n) and (N_n) are the two square-integrable martingales introduced in Lemma 5.1. By Corollary 5.3 the main difficulty we face is that the predictable quadratic variation processes of (M_n) and (N_n) increase to infinity at two different rates. Hence we will require a matrix normalisation technique in order to establish the asymptotic behaviour of the amnesic step-reinforced random walk.

To simplify the proofs, they are provided here under the assumption that the steps are bounded, i.e. $\|X_k\|_\infty < \infty$ for any $k \geq 1$. This assumption can be lifted through a truncation argument detailed in Appendix B.

Lemma 6.1. *Let (V_n) be the sequence of positive definite diagonal matrices of order 2 given by*

$$V_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 \\ 0 & p\eta_n \end{pmatrix}, \tag{6.1}$$

then $\|V_n\|_\infty$ converges to zero as n tends to infinity.

Further, let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ such that

$$v^T V_n \mathcal{M}_n = \frac{S_n}{\sqrt{n}}.$$

The quadratic variation $\langle \mathcal{M} \rangle_n$ of (\mathcal{M}_n) satisfies in the diffusive regime (i.e. $p < \frac{2\alpha-1}{2\alpha}$),

$$\lim_{n \rightarrow \infty} V_n \langle \mathcal{M} \rangle_n V_n^T = V \quad \text{a.s.}$$

where the matrix V is given by

$$V = \begin{pmatrix} \frac{(1-\alpha)^2}{(1-\alpha(1-p))^2} & \frac{p(1-\alpha)}{(1-p)(1-\alpha(1-p))^2} \\ \frac{p(1-\alpha)}{(1-p)(1-\alpha(1-p))^2} & \frac{p^2\alpha^2}{(1-\alpha(1-p))^2(2\alpha(1-p)-1)} \end{pmatrix} \tag{6.2}$$

Proof: For two sequence (u_n) and (v_n) we say that $u_n \propto v_n$ if there exists a constant C such that $u_n \sim Cv_n$. By Corollary A.3 we have that

$$\eta_n \propto \frac{1}{a_n \mu_n}$$

and the latter is a regularly varying sequence of index $\rho = 1 - \alpha(1 - p)$. Further, it holds that $\rho < \delta = 1/2$, because $p < \frac{2\alpha-1}{2\alpha}$. It then follows that $(n^{-1/2}\eta_n)$ is eventually decreasing. Hence it follows that indeed $\|V_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$.

Note that by Lemma 5.2 and Lemma A.1, we have for large enough n

$$\langle \mathcal{M} \rangle_n = \mathcal{A}_n - \xi_n,$$

where

$$\mathcal{A}_n = \sum_{k=0}^{n-1} \begin{pmatrix} (1 - pa_{k+1}\eta_{k+1}\mu_{k+1})^2 & a_{k+1}\mu_{k+1} - pa_{k+1}^2\eta_{k+1}\mu_{k+1}^2 \\ a_{k+1}\mu_{k+1} - pa_{k+1}^2\eta_{k+1}\mu_{k+1}^2 & a_{k+1}^2\mu_{k+1}^2 \end{pmatrix} \left((1 - p) + \frac{p}{\nu_k} U_k \right)$$

and

$$\xi_n = \sum_{k=0}^{n-1} \frac{p^2}{\nu_k^2} Y_k^2 \begin{pmatrix} (1 - pa_{k+1}\eta_{k+1}\mu_{k+1})^2 & a_{k+1}\mu_{k+1} - pa_{k+1}^2\eta_{k+1}\mu_{k+1}^2 \\ a_{k+1}\mu_{k+1} - pa_{k+1}^2\eta_{k+1}\mu_{k+1}^2 & a_{k+1}^2\mu_{k+1}^2 \end{pmatrix}.$$

Thanks to Theorem 3.2, we immediately have that $\xi_n = o(\mathcal{A}_n)$ since this is true for each coefficient of the matrix. In particular ξ_n is negligible as n tends to infinity. Hence we only need to consider $V_n \mathcal{A}_n V_n$ and thanks to the asymptotic rates established in Corollary A.3 we arrive at

$$\lim_{n \rightarrow \infty} V_n \langle \mathcal{M} \rangle_n V_n^T = V \quad \text{a.s.}$$

with V given by (6.2). □

Corollary 6.2 ((H.1) of Theorem C.1). *In the diffusive regime, the quadratic variation of (\mathcal{M}_n) satisfies for all $t \geq 0$,*

$$\lim_{n \rightarrow \infty} V_n \langle \mathcal{M} \rangle_{[nt]} V_n^T = V_t \quad \text{a.s.}$$

where the matrix V_t is given by

$$V_t = \frac{1}{(1 - \alpha(1 - p))^2} \begin{pmatrix} (1 - \alpha)^2 t & \frac{p(1 - \alpha)}{1 - p} t^{\alpha(1-p)} \\ \frac{p(1 - \alpha)}{1 - p} t^{\alpha(1-p)} & \frac{p^2 \alpha^2}{2\alpha(1 - p) - 1} t^{2\alpha(1-p)-1} \end{pmatrix}.$$

Lemma 6.3 ((H.2) of Theorem C.1: Lindeberg’s condition). *For all $t \geq 0$ and $\epsilon > 0$*

$$\sum_{k=1}^{\tau(nt)} \mathbb{E} (\|V_n \Delta \mathcal{M}_k\|^2 \mathbf{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \epsilon\}} \mid \mathcal{F}_{k-1}) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0,$$

where $\Delta \mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$.

Proof: First of all, we have from (6.1), (5.6) and (5.7) that

$$\|V_n \Delta \mathcal{M}_k\|^2 = \frac{1}{n} \left((1 - pa_k \eta_k \mu_k)^2 + (p \eta_n a_k \mu_k) \right)^2 t_k^2.$$

Since $t_{k+1} = X_{k+1} - \frac{p}{\nu_k} Y_k$, we immediately have, by the assumption that our underlying steps \mathcal{X} are bounded a.s., that $\sup_k |t_k| \leq \|\mathcal{X}\|_\infty < \infty$, and this ensures that

$$\mathbb{E} [\|V_n \Delta \mathcal{M}_k\|^4] \leq \frac{1}{n^2} \left((1 - pa_k \eta_k \mu_k)^2 + p^2 \eta_n^2 a_k^2 \mu_k^2 \right)^2 \|\mathcal{X}\|_\infty^2.$$

It follows from the regularly varying properties of the sequences η_n , a_n and μ_n , and the fact that $p < \frac{2\alpha-1}{2\alpha}$ that,

$$\eta_n^{-2} \sum_{k=1}^{\lfloor nt \rfloor} (a_k \mu_k)^2 = O(n), \quad \eta_n^{-4} \sum_{k=1}^{\lfloor nt \rfloor} (a_k \mu_k)^4 = O(n)$$

Hence, we find that

$$\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\|V_n \Delta \mathcal{M}_k\|^4] = O\left(\frac{1}{n}\right) \quad \text{a.s.}$$

and we deduce that

$$\sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} | \mathcal{F}_{k-1}] \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[\|V_n \Delta \mathcal{M}_k\|^4] \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0.$$

This concludes the proof. □

The next Lemma establishes how we can decompose the matrix V_t .

Lemma 6.4 ((H.3) of Theorem C.1). *The matrix V_t can be written as $V_t = tK_1 + t^{\alpha_2}K_2 + t^{\alpha_3}K_3$ where $\alpha_2 = \alpha(1-p) > 0$ and $\alpha_3 = 2\alpha(1-p) - 1 > 0$ as $p < \frac{2\alpha-1}{2\alpha}$, and the matrix are symmetric,*

$$K_1 = \frac{(1-\alpha)^2}{(1-\alpha(1-p))^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_2 = \frac{p(1-\alpha)}{(1-p)(1-\alpha(1-p))^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$K_3 = \frac{p^2\alpha^2}{(1-\alpha(1-p))^2(2\alpha(1-p)-1)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof: This is an immediate consequence of Corollary 6.2 and Display (6.2). □

Lemma 6.5. *We have the following convergence in the Skorokhod space of càdlàg functions $D([0, +\infty))$,*

$$(V_n \mathcal{M}_{\lfloor nt \rfloor}, t \geq 0) \implies (\mathcal{G}_t, t \geq 0)$$

where $\mathcal{G} = (\mathcal{G}_t, t \geq 0)$ is a continuous \mathbb{R}^2 -valued centered Gaussian process starting at 0 with covariance, for $0 \leq s \leq t$,

$$\mathbb{E}(\mathcal{G}_s \mathcal{G}_t^T) = V_s. \tag{6.3}$$

Proof: Thanks to Corollary 6.2 and Lemmas 6.3, 6.4, the claim follows immediately with an appeal to Theorem C.1. □

We are now in a position to give a proof of our main result.

Proof of Theorem 3.3: Thanks to Lemma 6.5 we have the distributional convergence in the sense of Skorokhod as n tends to infinity

$$(V_n \mathcal{M}_{\lfloor nt \rfloor}, t \geq 0) \implies (\mathcal{G}_t, t \geq 0) \tag{6.4}$$

Further, thanks to (5.4) we can use the fact that $S_{\lfloor nt \rfloor}$ is asymptotically equivalent to

$$N_{\lfloor nt \rfloor} + \frac{p\alpha}{1-\alpha(1-p)} t^{1-\alpha(1-p)} \eta_n M_{\lfloor nt \rfloor}$$

Let now $u_t = (1, t^{1-\alpha(1-p)})^T$, then by multiplying (6.4) by u_t^T from the left we obtain that

$$\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, t \geq 0 \right) \implies (W_t, t \geq 0),$$

where $W_t = u_t^T \mathcal{G}_t$. In order to fully characterise the Gaussian process $W = (W_t, t \geq 0)$ it suffices to compute its covariance function. With an appeal to Lemma 6.4 and Display (6.3) we obtain for $0 \leq s \leq t$

$$\begin{aligned}
\mathbb{E}(W_s W_t) &= u_s^T \mathbb{E}(\mathcal{G}_s \mathcal{G}_t^T) u_t \\
&= u_s^T V_s u_t \\
&= u_s^T (sK_1 + s^{\alpha(1-p)} K_2 + s^{2\alpha(1-p)-1} K_3) \\
&= \frac{(1-\alpha)^2}{(1-\alpha(1-p))^2} s + \frac{p(1-\alpha)}{(1-p)(1-\alpha(1-p))^2} s \\
&\quad + \frac{p(1-\alpha)}{(1-p)(1-\alpha(1-p))^2} s^{\alpha(1-p)} t^{1-\alpha(1-p)} \\
&\quad + \frac{p^2 \alpha^2}{(1-\alpha(1-p))^2 (2\alpha(1-p)-1)} s^{2\alpha(1-p)-1} (st)^{1-\alpha(1-p)} \\
&= \left(\frac{(1-\alpha)^2(1-p) + p(1-\alpha)}{(1-p)(1-\alpha(1-p))^2} \right) s \\
&\quad + \frac{p(1-\alpha)}{(1-p)(1-\alpha(1-p))^2} s \left(\frac{t}{s} \right)^{1-\alpha(1-p)} \\
&\quad + \left(\frac{p^2 \alpha^2}{(1-\alpha(1-p))^2 (2\alpha(1-p)-1)} \right) s \left(\frac{t}{s} \right)^{1-\alpha(1-p)} \\
&= \left(\frac{1-\alpha}{(1-p)(1-\alpha(1-p))} \right) s + \left(\frac{p(\alpha(2-p)-1)}{(1-p)(2\alpha(1-p)-1)(1-\alpha(1-p))} \right) s \left(\frac{t}{s} \right)^{1-\alpha(1-p)}
\end{aligned}$$

This concludes the proof of Theorem 3.3. \square

Appendix A. Technical Lemmas

We provide here some technical results that are useful for our study but not directly related to the proofs or the martingale approach.

Lemma A.1. *It holds that*

$$\lim_{n \rightarrow \infty} \frac{U_n}{\nu_n} = 1 \quad \text{a.s.}$$

Proof: Thanks to the proof of Theorem 3.2, we know that

$$\lim_{n \rightarrow \infty} \frac{\eta_n M_n}{n} = \lim_{n \rightarrow \infty} \frac{a_n \eta_n Y_n}{n} = 0 \quad \text{a.s.}$$

By Corollary A.3 we know that $(a_n \eta_n / n) \sim 1/\nu_n$, hence it follows that

$$\lim_{n \rightarrow \infty} \frac{Y_n}{\nu_n} = 0 \quad \text{a.s.}$$

By assumption, we require our steps to be centred and of variance one. If this is no longer the case we instead modify the process such that these conditions are satisfied again. For example:

$$S_n = X_1^2 + \dots + X_n^2$$

is a step-reinforced random walk with steps distributed as X^2 , see the robustness in equation (2.2). In order to apply the LLN, we then instead work with

$$\begin{aligned}\tilde{S}_n &= X_1^2 - \mathbb{E}(X_1^2) + \cdots + X_n^2 - \mathbb{E}(X_n^2) \\ &= X_1^2 + \cdots + X_n^2 - n\mathbb{E}(X_1^2) \\ &= X_1^2 + \cdots + X_n^2 - n.\end{aligned}$$

The above, by the robustness of SRRW detailed in (2.2), is again a step-reinforced random walk, this time centred and hence the LLN applies and yields

$$\lim_{n \rightarrow \infty} \frac{\tilde{S}_n}{n} = 0 \quad \text{a.s.}$$

which is equivalent to

$$\frac{X_1^2 + \cdots + X_n^2}{n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 1.$$

The exact same argument now holds for the process U_n which is just a modification of Y_n where instead of working with steps X we work with steps $X^2 - \mathbb{E}(X^2)$. It then follows that

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mu_k (X_k^2 - \mathbb{E}(X_k^2))}{\nu_n} &= \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mu_k X_k^2 - \sum_{k=1}^n \mu_k \mathbb{E}(X_k^2)}{\nu_n} \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mu_k X_k^2 - \sum_{k=1}^n \mu_k}{\nu_n} \\ &= \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \mu_k X_k^2 - \nu_n}{\nu_n} \\ &= 0, \quad \text{a.s.}\end{aligned}$$

Or, equivalently,

$$\lim_{n \rightarrow \infty} \frac{U_n}{\nu_n} = 1 \quad \text{a.s.}$$

□

In light of Lemma A.1, we see that the asymptotic behaviour of (\tilde{w}_n) , (\tilde{z}_n) defined in Eq. (5.10) and (5.11) is fully characterised by

$$w_n = \sum_{k=1}^n (a_k \mu_k)^2$$

and

$$z_n = \sum_{k=1}^n (1 - p a_k \eta_k \mu_k)^2$$

respectively.

We now discuss the relevant asymptotic rates more closely.

Lemma A.2. *The sequence (a_n) is regularly varying of index $-\alpha$.*

Proof: By assumption, we have that (ν_n) is regularly varying of index $\alpha > 0$, it follows that

$$\begin{aligned} n \left(1 - \frac{a_{n-1}}{a_n} \right) &= n(1 - \gamma_{n-1}) \\ &= -pn \left(\frac{\nu_n}{\nu_{n-1}} - 1 \right) \\ &= -pn \left(\frac{\alpha}{n} + o\left(\frac{1}{n}\right) \right) \\ &= -p\alpha + o(1). \end{aligned}$$

By Theorem 4.6 the claim follows. □

Corollary A.3. *We record the following asymptotics:*

- (1) $\lim_{n \rightarrow \infty} a_n \mu_n \eta_n = \frac{\alpha}{1 - \alpha(1 - p)}$.
- (2) $\lim_{n \rightarrow \infty} \frac{1}{n(a_n \mu_n)^2} w_n = \frac{1}{2\alpha(1 - p) - 1}$.
- (3) $\lim_{n \rightarrow \infty} \frac{1}{n} z_n = \frac{(1 - \alpha)^2}{(1 - \alpha(1 - p))^2}$.

Proof: We give the proof for each statement separately.

- (1) We have

$$\eta_n = \sum_{k=1}^{n-1} \frac{1}{a_k \nu_k}.$$

By definition, the sequence (ν_n) is regularly varying of index α and by Lemma A.2 the sequence (a_n) is regularly varying of index $-p\alpha$. It follows that $(a_n \nu_n)^{-1}$ is regularly varying of index $-\alpha(1-p)$ and it holds that $-\alpha(1-p) > -1$ for $\alpha < \frac{1}{1-p}$ or, equivalently, for $p > \frac{\alpha-1}{\alpha}$. By Theorem 4.5 it follows that

$$\lim_{n \rightarrow \infty} \frac{a_n \nu_n}{n} \eta_n = \frac{1}{1 - \alpha(1 - p)}.$$

Further, as $\nu_n \sim \frac{n}{\alpha} \mu_n$, the claim follows.

- (2) Recall that

$$w_n = \sum_{k=1}^n (a_k \mu_k)^2$$

and the sequence (a_n) is regularly varying of index $-p\alpha$, whereas the sequence (μ_n) is regularly varying of index $\alpha - 1$. Hence $(a_n \mu_n)^2$ is regularly varying of index $2\alpha(1 - p) - 2$. We observe that $2\alpha(1 - p) - 2 > -1$ because $p < \frac{2\alpha-1}{2\alpha}$ and hence, by Theorem 4.5, it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n(a_n \mu_n)^2} w_n = \frac{1}{2\alpha(1 - p) - 1}.$$

- (3) Here we have

$$z_n = \sum_{k=1}^n (1 - pa_k \eta_k \mu_k)^2.$$

It is then immediate, from the Stolz–Cesàro theorem and the first item, that

$$\lim_{n \rightarrow \infty} \frac{1}{n} z_n = \frac{(1 - \alpha)^2}{(1 - \alpha(1 - p))^2}.$$

This concludes the proof. □

The reduction argument relies on the following lemma taken from [Jacod and Shiryaev \(2003\)](#), that we state for the reader's convenience:

Lemma A.4 (Lemma 3.31 in Chapter VI of [Jacod and Shiryaev, 2003](#)).

Let (Z^n) be a sequence of d -dimensional rcll (càdlàg) processes and suppose that

$$\forall N > 0, \quad \forall \epsilon > 0 \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\sup_{s \leq N} |Z_s^n| > \epsilon \right) = 0.$$

If (Y^n) is another sequence of d -dimensional rcll processes with $Y^n \Rightarrow Y$ in the sense of Skorokhod, then $Y^n + Z^n \Rightarrow Y$ in the sense of Skorokhod.

Appendix B. Truncation argument for removing the boundness assumption

We have established our main result [Theorem 3.3](#) under the simplifying assumption that the underlying steps are bounded, that is $\|\mathcal{X}_k\|_\infty < \infty$ for all $k \in \mathbb{N}$. In this section we present an argument to lift this restriction, inspired by [Bertoin \(2020\)](#); [Bertenghi and Rosales-Ortiz \(2022\)](#). As such, we only make the assumption that $\mathbb{E}(\mathcal{X}) = 0$ and $0 < \text{Var}(\mathcal{X}) = \sigma^2 < \infty$.

First, we require the following bound:

Lemma B.1. Let $\frac{\alpha-1}{\alpha} < p < \frac{2\alpha-1}{2\alpha}$, then we have the bound

$$\mathbb{E} \left(\sup_{k \leq n} |S_k|^2 \right) \leq \sigma^2 \left(4z_n + 4p^2 \eta_n^2 w_n + 8p \sqrt{w_n z_n \eta_n^2} \right).$$

Proof: Recall the decomposition from [\(5.4\)](#)

$$S_n = N_n + p\eta_n M_n.$$

Since (η_n) is an increasing function and $(M_n), (N_n)$ are martingales, (S_n) is a submartingale.

Thanks to Doob's martingale inequality and the Cauchy-Schwarz inequality, we then have

$$\begin{aligned} \mathbb{E} \left(\sup_{k \leq n} |S_k|^2 \right) &\leq 4\mathbb{E}(|S_n|^2) \\ &= 4 \left(\mathbb{E}(N_n^2) + p^2 \eta_n^2 \mathbb{E}(M_n^2) + 2p\eta_n \mathbb{E}(N_n M_n) \right) \\ &\leq 4\mathbb{E}(N_n^2) + 4p\eta_n^2 \mathbb{E}(M_n^2) + 8p \sqrt{\eta_n^2 \mathbb{E}(M_n^2) \mathbb{E}(N_n^2)}. \end{aligned}$$

Recall from [Corollary 5.3](#), [Display 5.10](#) and [Display 5.11](#) that respectively,

$$\begin{aligned} \mathbb{E}(N_n^2) &= \mathbb{E}(\langle N \rangle_n) \leq \sigma^2 \sum_{k=0}^n (1 - pa_k \eta_k \mu_k)^2 = \sigma^2 z_n \\ \mathbb{E}(M_n^2) &= \mathbb{E}(\langle M \rangle_n) \leq \sigma^2 \sum_{k=0}^n a_k^2 \mu_k^2 = \sigma^2 w_n \end{aligned}$$

Using these bounds the claim follows. □

Corollary B.2. There exists a non-negative constant C such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left(\sup_{k \leq n} |S_k|^2 \right) \leq C\sigma^2.$$

Proof: This is now an immediate consequence of Corollary A.3. Indeed we have

$$\begin{aligned} \frac{1}{n} \mathbb{E} \left(\sup_{k \leq n} |S_k|^2 \right) &\leq \sigma^2 \left(4 \frac{1}{n} z_n + 4p^2 \frac{1}{n} w_n \eta_n^2 + 8p \sqrt{\frac{1}{n} z_n \times \frac{1}{n} w_n \eta_n^2} \right) \\ &\sim \sigma^2 (c_1 + c_2 (a_n \mu_n \eta_n)^2 + c_3) \\ &\sim C \sigma^2. \end{aligned}$$

□

We now split each underlying step X_i for $i \in \mathbb{N}$ as

$$X_i = X_i^{\leq K} + X_i^{>K}$$

where respectively,

$$\begin{aligned} X_i^{\leq K} &:= X_i \mathbf{1}_{\{|X_i| \leq K\}} - \mathbb{E} (X_i \mathbf{1}_{\{|X_i| \leq K\}}) \\ X_i^{>K} &:= X_i \mathbf{1}_{\{|X_i| > K\}} - \mathbb{E} (X_i \mathbf{1}_{\{|X_i| > K\}}), \end{aligned}$$

yields a natural decomposition for (S_n) in terms of two step-reinforced random walks

$$\tilde{S}_n = S_n^{\leq K} + S_n^{>K},$$

where $(S_n^{\leq K}), (S_n^{>K})$ are step-reinforced versions with typical step centred and distributed respectively as

$$X^{\leq K} = X \mathbf{1}_{\{|X| \leq K\}} - \mathbb{E} (X \mathbf{1}_{\{|X| \leq K\}})$$

and

$$X^{>K} = X \mathbf{1}_{\{|X| > K\}} - \mathbb{E} (X \mathbf{1}_{\{|X| > K\}}).$$

Moreover, $X^{\leq K}$ is centred with variance σ_K^2 and $\sigma_K^2 \rightarrow \sigma^2$ as $K \nearrow \infty$. Similarly $X^{>K}$ is centred and we denoted it's variance by ς_K^2 , which converges towards zero as $K \nearrow \infty$. We will also write the respective truncated random walks as

$$\begin{aligned} S_n^{\leq K} &= X_1^{\leq K} + \dots + X_n^{\leq K}, \\ S_n^{>K} &= X_1^{>K} + \dots + X_n^{>K}. \end{aligned}$$

Note, that thanks to Theorem 3.3 we know that

$$\left(\frac{S_{[nt]}^{\leq K}}{\sigma_K \sqrt{n}}, t \geq 0 \right) \implies (W_t, t \geq 0),$$

where $(W_t, t \geq 0)$ is the Gaussian process specified in Theorem 3.3.

In order to apply Lemma A.4, we need the following Lemma:

Lemma B.3. *For any sequence (K_n) increasing towards infinity, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left(\sup_{k \leq nt} |S_k^{>K_n}|^2 \right) = 0.$$

Proof: Recall that we denoted by ς_K^2 the variance of $X^{>K}$ and further that $\varsigma_K^2 \rightarrow 0$ as $K \nearrow \infty$. Thanks to Corollary B.2 we know that there exists some non-negative constant C such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left(\sup_{k \leq nt} |S_k^{>K_n}|^2 \right) \leq C \lim_{n \rightarrow \infty} \varsigma_{K_n}^2 t = 0.$$

□

We can now apply Lemma A.4 to the processes

$$Y_t^n = \frac{S_{\lfloor nt \rfloor}^{\leq K_n}}{\sqrt{n}}, \quad Z_t^n = \frac{S_{\lfloor nt \rfloor}^{> K_n}}{\sqrt{n}}, \quad t \geq 0.$$

We see from Lemma B.3 and the Markov inequality that Lemma A.4 applies to said process. It then follows by the decomposition

$$n^{-1/2} S_{\lfloor nt \rfloor} = Y_t^n + Z_t^n \implies \sigma W(t), \quad t \geq 0, \quad \text{as } n \rightarrow \infty.$$

This shows that Theorem 3.3 holds for general, possibly unbounded, steps as long as $\mathcal{X} \in L^2(\mathbb{P})$.

Appendix C. A non-standard result on martingales

The proof of our main result, Theorem 3.3, relies on a non-standard functional central limit theorem for multi-dimensional martingales. A simplified version of Theorem 1 part2) of Touati Touati (1992) is as follows.

Theorem C.1. *Let (\mathcal{M}_n) be a locally square-integrable martingale of \mathbb{R}^d adapted to a filtration (\mathcal{F}_n) , with predictable quadratic variation $\langle \mathcal{M} \rangle_n$. Let (V_n) be a sequence of non-random square matrices of order d such that $\|V_n\|$ decreases to 0 as n tends to infinity. Moreover, let $\tau : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing function going to infinity at infinity. Assume that there exists a symmetric and positive semi-definite matrix V_t that is deterministic and such that for all $t \geq 0$*

$$V_n \langle \mathcal{M} \rangle_{\tau(nt)} V_n^T \xrightarrow[n \rightarrow \infty]{\mathbb{P}} V_t. \quad (\text{H.1})$$

Moreover, assume that Lindeberg's condition is satisfied, that is for all $t \geq 0$ and $\epsilon > 0$,

$$\sum_{k=1}^{\tau(nt)} \mathbb{E} (\|V_n \Delta \mathcal{M}_k\|^2 \mathbf{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \epsilon\}} \mid \mathcal{F}_{k-1}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \quad (\text{H.2})$$

where $\Delta \mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$.

Finally, assume that for some $q \in \mathbb{N}^*$

$$V_t = \sum_{j=1}^q t^{\alpha_j} K_j \quad (\text{H.3})$$

where $\alpha_j > 0$ and K_j is a symmetric matrix.

Then, we have the distributional convergence in the Skorokhod space $D([0, \infty))$ of right-continuous functions with left-hand limits,

$$(V_n \mathcal{M}_{\tau(nt)}, t \geq 0) \implies (\mathcal{G}_t, t \geq 0)$$

where $\mathcal{G} = (\mathcal{G}_t, t \geq 0)$ is a continuous \mathbb{R}^d -valued centered Gaussian process starting at 0 with covariance function given for $0 \leq s \leq t$ for,

$$\mathbb{E} (\mathcal{G}_s \mathcal{G}_t^T) = V_s.$$

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