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Macroscopic flow out of a segment for Activated Random Walks in dimension 1

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Abstract. Activated Random Walk is a system of interacting particles which presents a phase transition and a conjectured phenomenon of self-organized criticality. In this note, we prove that, in dimension 1, in the supercritical case, when a segment is stabilized with particles being killed when they jump out of the segment, a positive fraction of the particles leaves the segment with positive probability.

This was already known to be a sufficient condition for being in the active phase of the model, and the result of this paper is that this condition is also necessary, except maybe precisely at the critical point. This result can also be seen as a partial answer to some of the many conjectures which connect the different points of view on the phase transition of the model.

1. Introduction

We begin with a brief informal presentation of some aspects of the model. The reader who is familiar with Activated Random Walks may skip the two following subsections, passing directly to Subsection 1.3 where our results are presented. Some illustrated sketches of proofs are given in Subsection 2.

1.1. *Presentation of the model.* The model of Activated Random Walks consists of particles performing independent random walks on a graph, which fall asleep with a certain rate and get reactivated in the presence of other particles on the same site. The model was popularized by Rolla, Sidoravicius and Dickman (Rolla, 2008; Rolla and Sidoravicius, 2012; Dickman et al., 2010), and can be seen as a variant of the frog model (Alves et al., 2002a,b). Its study is motivated by its connection with the concept of self-organized criticality, which was introduced by the physicists Bak, Tang and Wiesenfeld (Bak et al., 1987) to describe physical systems which present a critical-like behaviour but without the need to tune the parameters of the system to particular values (like is the case for an ordinary phase transition). To illustrate this concept, Bak, Tang and Wiesenfeld introduced an interacting particle system called the Abelian sandpile model, which is a close cousin of Activated Random Walks. Yet, one of the key differences between these two models is their mixing behaviour.

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The mixing properties of Activated Random Walks are investigated in Levine and Liang (2024); Bristiel and Salez (2024), and it turns out that this model mixes faster than the Abelian sandpile. This can explain why Activated Random Walks are expected to have a behaviour which is more universal, in that it is less sensitive to microscopic details of the system.

Let us now define informally the Activated Random Walk model on $\mathbb{Z}^d$. A configuration of the model consists of a given number of particles on each site of $\mathbb{Z}^d$, each of these particles being in one of two possible states: active or sleeping.

The model evolves as follows. Each active particle performs a continuous-time random walk on $\mathbb{Z}^d$ with jump rate 1, with a certain jump distribution $P : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$ (such that $\sum_{y\in\mathbb{Z}^d} P(x, y) = 1$ for each $x \in \mathbb{Z}^d$). This means that, after a random time distributed as an exponential with parameter 1, the active particle at x jumps to some other site, the probability of jumping from x to y being $P(x, y)$. This distribution is called translation-invariant if $P(x, y)$ is a function of $y - x$ only.

In parallel, each active particle also carries another exponential clock with a certain parameter $\lambda > 0$ and, when this clock rings, if there are no other particles on the same site, the particle falls asleep (otherwise, if the particle is not alone, nothing happens). A sleeping particle stops moving (its continuous-time random walk is somewhat paused), until it wakes up, which happens when another particle arrives on the same site. Then, the reactivated particle resumes its continuous-time random walk with jump rate 1. Equivalently, one may also consider that a particle can fall asleep even when it is not alone on a site but, whenever this happens, the particle is instantaneously waken up by the presence of the other particles.

Thus, there can never be two sleeping particles at a same site. Hence, at every time $t \geq 0$, the configuration of the model at time t can by encoded into a function $\eta_t : \mathbb{Z}^d \to \mathbb{N} \cup \{\mathfrak{s}\}$, where $\eta_t(x) = k \in \mathbb{N}$ means that there are k active particles at the site x, while $\eta_t(x) = \mathfrak{s}$ means that there is one sleeping particle at x. Note that, with this notation, particles are indistinguishable: we only keep track of the number and states of particles on each site, but not of the individual trajectory of each particle.

Regarding the initial configuration η_0, various setups are interesting to consider. One possibility is to take η_0 which follows a translation-invariant and ergodic probability distribution on the set of all possible configurations, with a finite mean number of particles per site. This initial configuration may have only active particles, or both active and sleeping particles. Another case of interest is that of η_0 with only finitely many particles, for example n particles on the origin. The model can also be defined on different graphs, or with slight modifications of the dynamics, like for example adding a sink vertex where particles get trapped forever.

For a rigorous construction of the process $(\eta_t)_{t\geq 0}$, we refer the reader to Rolla and Sidoravicius (2012), or to the review Rolla (2020). See also Levine and Silvestri (2024) for a presentation of various different settings of interest and fascinating conjectures connecting these different points of view on the model.

1.2. *Phase transition.* Let us consider the model on $\mathbb{Z}^d$ starting with η_0 following a translation-ergodic distribution with mean particle density ζ. Depending on the sleep rate λ, on the jump distribution P and on the particle density ζ, the model can exhibit very different behaviours. A natural question is: if we start with only active particles, do they eventually all fall asleep, or is activity maintained forever? Note that, on the infinite lattice $\mathbb{Z}^d$, almost surely there exists no finite time when all the particles are sleeping. However, we have the following notion of fixation: we say that the system fixates if the origin is visited finitely many times by an active particle during the evolution of the process. Then, up to events of 0 probability, fixation is equivalent to the configuration on every finite set of $\mathbb{Z}^d$ eventually being constant and with only sleeping particles or, if we follow the trajectory of each individual particle, fixation also turns out to be equivalent to each particle walking only a finite number of steps, or to one given particle walking only a finite number of steps (Amir and Gurel-Gurevich, 2010). If the system does not fixate, we say that the system stays active.

Due to the ergodicity assumption on η_0, the probability of fixation can only be 0 or 1 (see Rolla and Sidoravicius, 2012). Thus, we can have two different regimes, depending on the sleep rate λ, on the jump distribution P and on the law of the initial configuration: either the system almost surely fixates (this regime is called the fixating phase, or stable phase), or the system almost surely stays active (this is called the active phase, or exploding phase). Moreover, we have the following key result about this phase transition:

Theorem 1.1 (Rolla et al., 2019)**.** *In any dimension $d \geq 1$, for every sleep rate $\lambda \in (0, \infty]$ and every translation-invariant jump distribution P which generates all $\mathbb{Z}^d$, there exists ζ_c such that, for every translation-ergodic initial distribution with no sleeping particles and an average density of active particles ζ, the Activated Random Walk model on $\mathbb{Z}^d$ with sleep rate λ almost surely fixates if $\zeta < \zeta_c$, whereas it almost surely stays active if $\zeta > \zeta_c$.*

This result shows in particular that the critical density is in some sense universal, in that it depends on the initial configuration only through the mean density of particles ζ. Thus, to study this critical density, it is enough to consider the particular case where the configuration is i.i.d., with a given probability distribution on $\mathbb{N}$ with finite mean.

An important challenge in the study of this phase transition is to relate the property of fixation, which concerns the model on the infinite lattice with infinitely many particles, to some finite counterparts of the model. A key example is the sufficient condition for activity given by Theorem 1.2 below.

On the finite box $V_n = (-n/2,\, n/2\,]^d \cap \mathbb{Z}^d$, let us consider a variant of the model where particles are killed and removed from the system when they jump out of V_n (or equivalently, we can consider the model on $\mathbb{Z}^d$ where particles are frozen outside V_n, so that particles which start out of V_n or which jump out of V_n are frozen forever and cannot move any more). Let M_n count the number of particles that jump out of V_n when we let this system evolve until all the sites of V_n become stable (a site x is called stable if it is either empty or it contains a sleeping particle).

Theorem 1.2 (Rolla and Tournier, 2018)**.** *With the notation defined above, for every sleep rate λ and every translation-invariant jump distribution P which generates all $\mathbb{Z}^d$, if the initial configuration η_0 is i.i.d. and if*

$$\limsup_{n\to\infty} \frac{\mathbb{E}\big[M_n\big]}{|V_n|} \;>\; 0\,, \tag{1.1}$$

then the model on $\mathbb{Z}^d$ with sleep rate λ, jump distribution P and initial configuration η_0 almost surely stays active.

This result relies on the following intuitive idea: if with positive probability a large box looses a positive fraction of its particles during stabilization, then a particle starting at the origin in the model on $\mathbb{Z}^d$ has a positive probability of walking arbitrarily far away, which shows that the system stays active with positive probability, and thus with probability 1 (because we have a 0-1 law).

1.3. *Main results.* The main result of this paper consists in the addition of a reciprocal to the implication of Theorem 1.2, in the particular case of dimension 1. Recall that M_n denotes the number of particles that jump out of $V_n = (-n/2,\, n/2\,] \cap \mathbb{Z}$ during the stabilization of V_n with particles being killed upon leaving V_n.

Theorem 1.3. *In dimension $d = 1$, for every sleep rate $\lambda > 0$ and every nearest-neighbour translation-invariant jump distribution P, if the initial configuration η_0 is i.i.d. with mean ζ and all particles are initially active, then we have the equivalence:*

$$\zeta \;>\; \zeta_c \quad\Longleftrightarrow\quad \liminf_{n\to\infty} \frac{\mathbb{E}[M_n]}{|V_n|} \;>\; 0\,.$$

This shows that the sufficient condition for activity given by Theorem 1.2 is also necessary, except maybe exactly at the critical point. Indeed, very few things are known rigorously about the critical regime $\zeta = \zeta_c$, with the exception of the particular case of directed walks in dimension 1 starting with η_0 i.i.d., for which a proof of non-fixation at criticality, due to Hoffman and Sidoravicius, appears in Cabezas et al. (2014).

Theorem 1.3 answers a conjecture of Levine and Silvestri (2024) (in the particular case of dimension 1), showing that the density ζ_w that they define in Section 5.3, and which corresponds to the infimum of the ζ for which condition (1.1) holds when η_0 is i.i.d. Poisson, is in fact equal to the critical density ζ_c.

Our result is made more precise by the following theorem, which indicates an explicit positive fraction which exits with positive probability, as a function of the sleep rate λ and the density ζ. For every deterministic initial configuration $\eta : V_n \to \mathbb{N}$ (with only active particles), let us denote by $\|\eta\| = \sum_{x\in V_n} \eta(x)$ the total number of particles in the configuration η, and let us write $\mathbb{P}_\eta$ for the probability relative to the system started with deterministic initial configuration equal to η.

Theorem 1.4. *In dimension $d = 1$, for every sleep rate $\lambda > 0$ and every nearest-neighbour translation-invariant jump distribution P, for every $\zeta > \zeta_c$ we have*

$$\forall\, \varepsilon \in \left[0,\, \frac{\lambda(\zeta-\zeta_c)}{4(1+\lambda)\zeta_c}\right) \qquad \liminf_{n\to\infty} \inf_{\substack{\eta: V_n\to\mathbb{N}:\\ \|\eta\|\geq \zeta n}} \mathbb{P}_\eta\big(M_n > \varepsilon n\big) \;\geq\; 1 - \frac{\zeta_c}{\zeta}\left(1 + \frac{4(1+\lambda)\varepsilon}{\lambda}\right) \;>\; 0\,.$$

To show this, as a first step we prove the following result, which gives an explicit upper bound on the probability that no particle exits during stabilization. This bound is not optimal but, as explained later, it allows us to obtain the bound of Theorem 1.4.

Theorem 1.5. *In dimension $d = 1$, for every sleep rate $\lambda > 0$ and every nearest-neighbour translation-invariant jump distribution P, for every $\zeta > \zeta_c$, for every $n \geq 1$ and every initial configuration $\eta : V_n \to \mathbb{N}$ with $\|\eta\| \geq \zeta n$ particles, initially all active, we have*

$$\mathbb{P}_\eta\big(M_n = 0\big) \;\leq\; \frac{\zeta_c}{\zeta}\,.$$

Let us stress that the bounds of Theorems 1.4 and 1.5 hold for any deterministic initial configuration with at least ζn particles, which includes in particular the case of interest where all the particles start from the origin (see comments about this in Section 1.4).

To show that Theorem 1.5 implies Theorem 1.4, we use the following important observation: adding empty intervals around V_n where particles are not allowed to sleep and stabilizing the configuration in this enlarged segment does not increase, in distribution, the number of particles which jump out. This is the content of Lemma 5.1, that we postpone to a later Section but which can be of independent interest.

Lastly, in the course of the proof of Theorem 1.3, to deal with the case $\zeta = \zeta_c$, we establish the following fact, which can also be of independent interest:

Proposition 1.6. *In any dimension $d \geq 1$, for every sleep rate $\lambda > 0$ and every translation-invariant jump distribution P on $\mathbb{Z}^d$ whose support generates all the group $\mathbb{Z}^d$, if η_0 is i.i.d. with mean ζ_c then we have*

$$\lim_{n\to\infty} \frac{\mathbb{E}[M_n]}{|V_n|} \;=\; 0\,.$$

Notably, this last result shows that, at least in the case of directed walks in dimension 1, for which it is known that there is no fixation at criticality (see the remark following the statement of Theorem 1.3), the sufficient condition for activity given by Theorem 1.2 is not necessary: in this case, at $\zeta = \zeta_c$, the system is active despite the condition (1.1) not being satisfied. That is to say, the system is active if and only if $\zeta \geq \zeta_c$, whereas condition (1.1) is equivalent to the strict inequality $\zeta > \zeta_c$.

1.4. *Some perspectives.* Since the seminal works which established general properties of the phase transition, various techniques have been developed to study Activated Random Walks. In particular, a series of works (Rolla and Sidoravicius, 2012; Stauffer and Taggi, 2018; Asselah et al., 2022; Taggi, 2019; Basu et al., 2018; Hoffman et al., 2023; Forien and Gaudillière, 2024; Hu, 2022; Asselah et al., 2024) established that the critical density is always strictly between 0 and 1 and obtained bounds on ζ_c as a function of λ. But many of the techniques used only work far from criticality, when the density is either much larger or much smaller than ζ_c, and few results have been proved to hold up to the critical density. For example, Basu et al. (2019) shows that the model on the torus stabilizes fast when ζ is very small, and slowly when ζ is close to 1, but we lack sharper results about a transition exactly at ζ_c from fast to slow stabilization.

Some exceptions giving insight about the behaviour at or close to ζ_c are the study of the critical regime in the case of directed walks in one dimension (see Cabezas et al., 2014, with an argument due to Hoffman and Sidoravicius, and Cabezas and Rolla, 2021), the continuity of ζ_c as a function of λ (Taggi, 2023), and the recent work Járai et al. (2023) which considers the model on the complete graph and computes the exact value of the critical density.

In this regard, the results of the present article have the merit to hold up to the critical point. However, the bounds presented here are far from being optimal, and there remains a lot of space for improvement. For example, Theorem 1.3 can be seen as a partial answer to the so-called hockey stick conjecture (conjecture 17 in Levine and Silvestri, 2024), which predicts that $M_n/|V_n|$ should converge in probability to $\max(0, \zeta-\zeta_c)$, at least in the particular case when the initial distribution is i.i.d. Poisson.

Similarly, the bound given in Theorem 1.5 is not optimal, and it is expected that when $\zeta > \zeta_c$, the probability that $M_n = 0$ in fact decays exponentially fast with n (see conjecture 20 of Levine and Silvestri, 2024).

Note that Theorem 1.5 can also be seen as a partial answer to the so-called ball conjecture (see conjectures 1 and 12 in Levine and Silvestri, 2024). This conjecture predicts that, when starting with n particles at the origin, if we let these particles stabilize in $\mathbb{Z}^d$, the random set of visited sites A_n is such that, for every $\varepsilon > 0$, with probability tending to 1 as $n \to \infty$, the set A_n contains all the sites of $\mathbb{Z}^d$ that belong to the origin-centred Euclidean ball of volume $(1-\varepsilon)n/\zeta_c$ and is contained in the origin-centred Euclidean ball of volume $(1+\varepsilon)n/\zeta_c$. Theorem 1.5 implies that the probability that A_n is included into the ball of volume $(1-\varepsilon)n/\zeta_c$ is less than $1-\varepsilon$. Note that another partial result was obtained in this direction in Levine and Silvestri (2021), also in dimension 1, showing an inner and an outer bound on A_n.

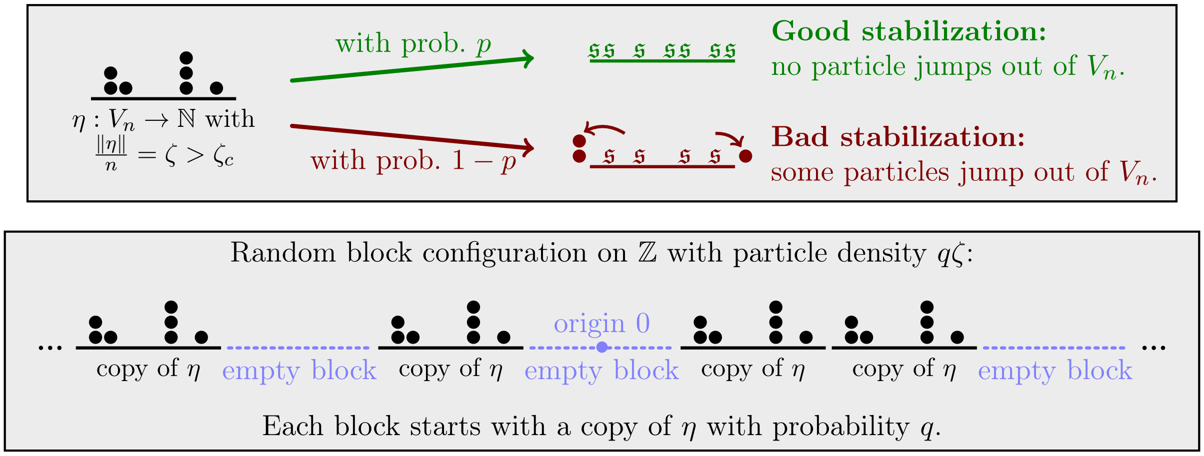
Last but not least, the proofs of the present paper are very specific to the one-dimensional case, and it would be interesting to obtain at least similar results in higher dimension. See more comments on this in Section 2.4.

Update: While this paper was under review, a new result appeared (Hoffman et al., 2024a,b) which proves the hockey stick conjecture and the ball conjecture for activated random walk in dimension 1, showing that the various definitions of the critical density coincide, and strengthening the bounds of Theorems 1.4 and 1.5. Their technique involves a detailed study of the odometers and a comparison with a percolation process, and it is very different from the ideas of the present paper. We thus hope that the ideas that we present here are nevertheless of interest.

2. Sketches of the proofs

Let us now summarize the strategy of the proofs. We start with Theorem 1.5, before explaining how the other results follow.

We will make extensive use of the abelian property of the model, which allows us to choose the order with which the particles act: as long as there is at least one active particle, we pick one with a certain rule, and with probability $\lambda/(1+\lambda)$ it falls asleep if it is alone, whereas with



$\longrightarrow$ If $q < p$, then with positive probability, the origin is never visited, whence $q\zeta \leq \zeta_c$.

FIGURE 2.1. Outline of the proof of Theorem 1.5: assuming that a given deterministic initial configuration $\eta : V_n \to \mathbb{N}$ produces a good stabilization with probability p, we build a block configuration on $\mathbb{Z}$ with density $q\zeta$ which fixates if $q < p$, showing that $q\zeta \leq \zeta_c$. Since this holds for every $q < p$, we get $p\zeta \leq \zeta_c$.

probability $1/(1+\lambda)$ it makes a jump distributed according to the prescribed jump distribution P. The abelian property states that, if the randomness of these sleeps and jumps is quenched into an array with a list of instructions above each site, then once this array is fixed, the final stable configuration and the number of instructions used at each site do not depend on the order with which the moves are performed. The formalism of this quenched array of instructions is presented in Section 3, where the abelian property corresponds to Lemma 3.1.

We also make use of the monotonicity property of the model with respect to enforced activation. This means that, stabilizing the configuration by forcing sometimes sleeping particles to wake up, we obtain upper bounds on the number of instructions needed to stabilize (see Lemma 3.2).

2.1. *Sketch of the proof of Theorem 1.5.* Let $\lambda > 0$, let P be a nearest-neighbour translation-invariant jump distribution, let $\zeta > \zeta_c$ and $n \geq 1$ and consider a fixed deterministic initial configuration $\eta : V_n \to \mathbb{N}$ with $\|\eta\| = \zeta n$. Let us write $p = \mathbb{P}_\eta(M_n = 0)$. This means that, when we stabilize the configuration η in V_n with particles ignored once they jump out of V_n, with probability p all the particles fall asleep inside V_n with no particle jumping out: we call this a good stabilization. The stabilization of η is called bad if at least one particle jumps out of V_n (see first part of Figure 2.1). Our aim is to show that $p \leq \zeta_c/\zeta$.

The strategy is the following: for every $\zeta' < p\zeta$, we construct a random initial configuration on $\mathbb{Z}$ with particle density ζ' and we show that this configuration fixates, which implies that $\zeta' \leq \zeta_c$. This being true for every $\zeta' < p\zeta$, we deduce that $p\zeta \leq \zeta_c$.

The configuration that we consider is a "block configuration" represented in the second part of Figure 2.1 and constructed as follows. Let $q = \zeta'/\zeta$. For each block of the form $V_n + kn$ for $k \in \mathbb{Z}$, with probability q this block starts with a copy of the configuration η, and with probability $1-q$ the block starts empty, independently for different blocks.

Then, our strategy is to stabilize this configuration one block after another, always choosing in priority an unstable block which is as close as possible to the origin. This unstable block, which contains a copy of η, gives a good stabilization with probability p. If this happens, we turn to the next block. If a bad stabilization occurs, that is to say, if some particles jump out of their block, then we force all the particles of this block to walk, forbidding them to fall asleep, until we obtain again a copy of η translated on the following block in the direction of the origin. If this block in the direction of the origin was already occupied by sleeping particles resulting from a previous good stabilization, then we force these sleeping particles to wake up and walk until they form a copy of η on the following block towards the origin, leaving behind them a copy of η. And if again we arrive on a block already occupied by sleeping particles, we repeat this displacement operation until we go back to a situation where, in each block, we have either a copy of η (active block) or only sleeping particles (stable block).

We want to show that, with positive probability, the blocks can be stabilized with this strategy with no particle ever visiting the block of the origin. To study this stabilization block by block, we couple the model with another instance of activated random walk, that we call "coarse-grained model", which has an initial configuration i.i.d. Bernoulli with parameter q (each site corresponds to one block in the original model), a sleep rate λ' such that $\lambda'/(1+\lambda') = p$ (a particle falling asleep corresponds to the good stabilization of a block in the original model) and a jump distribution such that all sites $x \neq 0$ jump towards the origin with probability 1 (because at each bad stabilization we force the particles to move towards the origin).

This coarse-graining coupling is illustrated in Figure 2.2 and formalized in Proposition 4.1, which is postponed to Section 4 because it requires some notation introduced in Section 3. More precisely, this proposition shows that the original model and the coarse-grained model can be coupled in such a way that, if the origin is never visited in the coarse-grained model, then the sites in the block of the origin are never visited in the original model.

Once this coupling is established, we use the following result of Rolla and Sidoravicius:

Theorem 2.1 (Theorem 2 in Rolla and Sidoravicius, 2012)**.** *For every $\lambda > 0$, every nearest-neighbour jump distribution P and every $q < \lambda/(1+\lambda)$, in the model with sleep rate λ, jump distribution P and initial configuration i.i.d. with mean particle density q, with positive probability the origin is never visited.*

In fact this statement is not exactly Theorem 2 in Rolla and Sidoravicius (2012), but it directly follows from the proof (which by the way is stated for a Poisson initial distribution, but holds for any i.i.d. initial distribution).

We apply this result to the coarse-grained model, which has particle density q and sleep rate λ' such that $\lambda'/(1+\lambda') = p$. Since $q < p$, we deduce that with positive probability the origin is never visited in the coarse-grained model, which implies that with positive probability the block of the origin is never visited in the original model, which implies that the particle density in the block configuration, namely $q\zeta = \zeta'$, is at most ζ_c.

A small issue that we just swept under the carpet is that the block configuration is not translation-invariant, but this problem is easily overcome by applying a random translation, as explained in Section 4.4. Interestingly, our technique is one of the rare applications of the universality result of Theorem 1.1 to an initial configuration which is not i.i.d. but only translation-invariant and ergodic.

The proof of Theorem 1.5, with the statement and proof of Proposition 4.1 which establishes the coupling, is the content of Section 4.

2.2. *Proving that Theorem 1.5 implies Theorem 1.4.* To deduce Theorem 1.4 from Theorem 1.5, the idea is that, if less than εn particles jump out of the segment, then, taking a slightly larger segment, with enough empty space around to easily accommodate these particles which jumped out, we can stabilize this larger segment with no particles jumping out of it.

We proceed in two steps, as represented in Figure 2.3. We start with a configuration $\eta : V_n \to \mathbb{N}$ with density $\|\eta\|/n = \zeta > \zeta_c$ and empty strips of length $2\alpha n$ on each side of V_n.

During the first step, we stabilize $V_{n+2\alpha n}$, forbidding particles to fall asleep out of V_n, and freezing particles once they jump out of $V_{n+2\alpha n}$. The number of particles which jump out of $V_{n+2\alpha n}$ during this step, which we denote by M'_n, may differ from M_n, which is the number of particles jumping

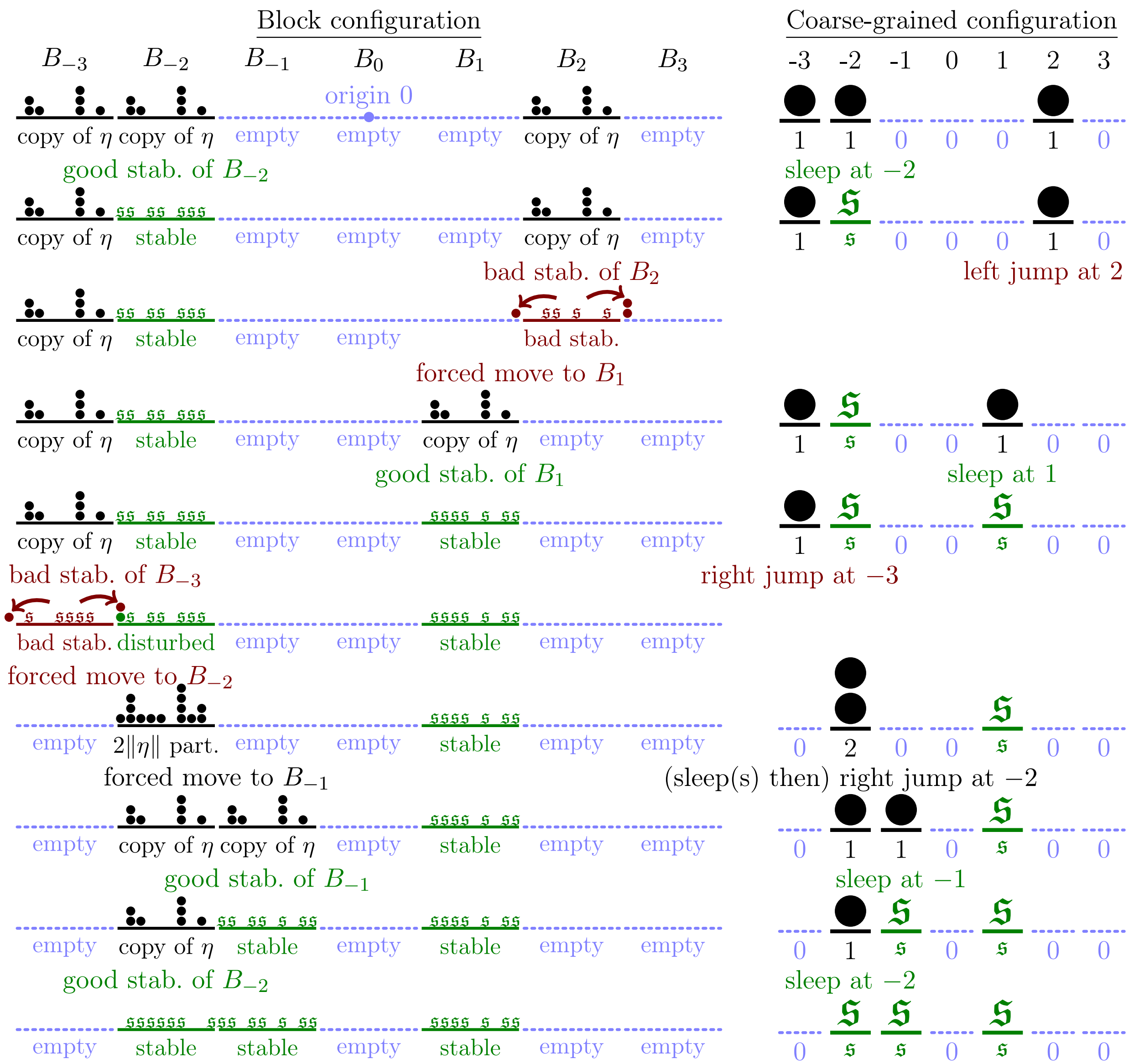


FIGURE 2.2. Coupling between the stabilization of the block configuration, on the left side, and the stabilization of a coarse-grained configuration, on the right side (we refer to Section 2.1 for an informal sketch and to Proposition 4.1 for a more detailed presentation of the coupling).

out of V_n when we just stabilize V_n. Yet, it turns out that M'_n is stochastically dominated by M_n. This is the content of Lemma 5.1: adding these no man's lands around V_n does not increase, in distribution, the number of particles which jump out.

Then, during the second step, we try to stabilize these M'_n particles in $V_{n+4\alpha n} \setminus V_n$. This step is said to be successful if, doing so, no particles jump out of $V_{n+4\alpha n}$ or come back in V_n. In this case, we managed to stabilize the configuration η in $V_{n+4\alpha n}$ with no particle jumping out of $V_{n+4\alpha n}$, forbidding some particles to fall asleep at some stages. By the monotonicity property with respect to enforced activation, this implies that no particle would have jumped out of $V_{n+4\alpha n}$ also if we had not forbidden these particles to fall asleep, that is to say, $M_{n+4\alpha n} = 0$.

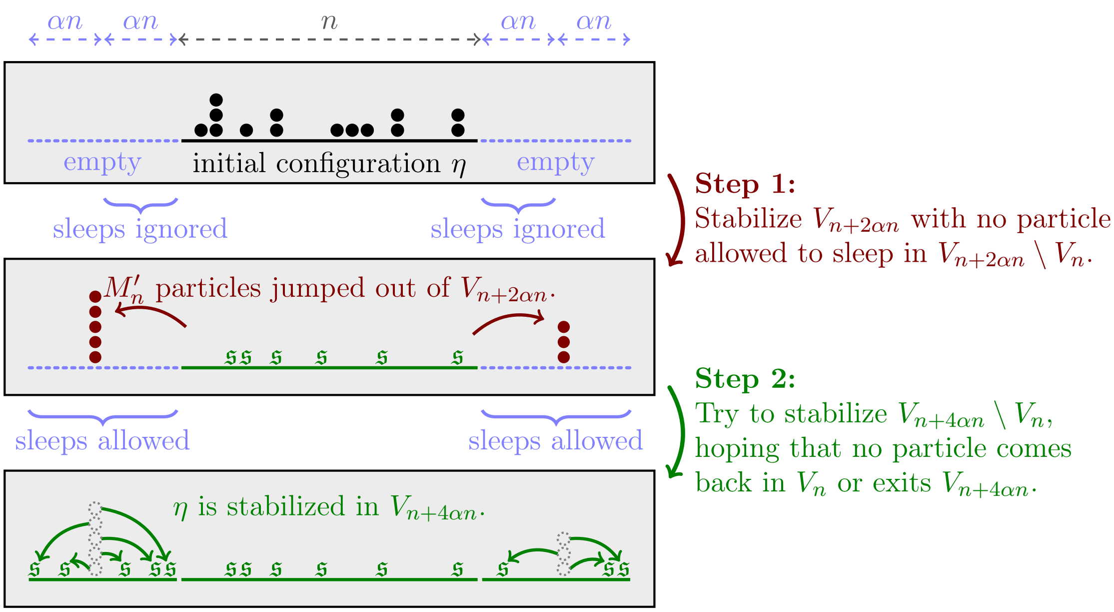


FIGURE 2.3. The two steps of the proof of Theorem 1.4.

Note that the overall density in the enlarged segment with the empty spaces around is $\zeta/(1+4\alpha)$. Thus, if α is chosen small enough so that $\zeta/(1+4\alpha) > \zeta_c$, then Theorem 1.5 gives an upper bound on the probability that $M_{n+4\alpha n} = 0$.

Then, if we consider $\varepsilon > 0$ small enough so that $\varepsilon/\alpha < \lambda/(1+\lambda)$, then we can show that the second stage succeeds with high probability, conditioned on the event that $M'_n \le \varepsilon n$ (to show this we adapt the trapping procedure which is used in Rolla and Sidoravicius (2012) to obtain Theorem 2.1). Thus, we can translate the upper bound on $\mathbb{P}_\eta\big(M_{n+4\alpha n} = 0\big)$ coming from Theorem 1.5 into an upper bound on $\mathbb{P}_\eta(M'_n \le \varepsilon n)$. The stochastic domination given by Lemma 5.1 then allows us to translate this into the claimed upper bound on $\mathbb{P}_\eta(M_n \le \varepsilon n)$.

The proof of Theorem 1.4 is the object of Section 5.

2.3. *Obtaining Theorem 1.3 and Proposition 1.6.* Given Theorem 1.4, our Theorem 1.3 easily follows. The only detail is that, to show that we have indeed an equivalence, there remains to show Proposition 1.6, which states that, at criticality, there is not a positive fraction which jumps out of the box. This is presented in Section 6.

2.4. *Higher dimension or longer jumps.* As explained before, our proof strategy beaks down in higher dimension or when particles can make longer jumps. More precisely, there are two stages where we crucially rely on the fact that we are in dimension 1 and that jumps are limited to the nearest neighbours.

First, we use these two assumptions in the construction of the coarse-graining coupling given by Proposition 4.1 when, after the bad stabilization of a block, we force the particles to move to form a new copy of the configuration η on the following block in the direction of the origin. This can be done without disturbing the sleeping particles which are on other blocks closer to the origin, but only because we are in dimension 1 with nearest-neighbour jumps (otherwise the particles could go around or skip the site where we want to bring them).

These two assumptions are also used in the proof of Lemma 5.1, which shows that adding no man's lands around a segment does not increase, in distribution, the number of particles which jump out. Indeed, the proof of this Lemma uses the fact that, if you stabilize a segment by always toppling the leftmost active site, then each time that a particle jumps out by the left exit, all the other particles must be active. In higher dimension or with longer jumps this would no longer be the case, but it would be interesting to investigate whether or not a result similar to Lemma 5.1 would still hold.

3. The site-wise representation of the model

We now describe the site-wise representation of the model, with an array of sleep and jump instructions above the sites. We refer to the survey Rolla (2020) for a more detailed presentation.

3.1. *Topplings and odometers.* A crucial ingredient in the study of Activated Random Walks is the site-wise representation, also known as Diaconis-Fulton representation (Diaconis and Fulton, 1991; Rolla and Sidoravicius, 2012). Let $\eta : \mathbb{Z}^d \to \mathbb{N} \cup \{\mathfrak{s}\}$ be a fixed initial configuration, and let us consider a fixed array of instructions $\tau = (\tau_{x,i})_{x\in\mathbb{Z}^d,\, i\geq 1}$ where, for every $x \in \mathbb{Z}^d$ and every $i \geq 1$, the instruction $\tau_{x,i}$ can be either a sleep instruction or a jump instruction to some site $y \in \mathbb{Z}^d$.

The idea is that, once this initial configuration and this array of instructions are fixed, the evolution of the system can be constructed by looking at these instructions each time that something happens at some site. As we use instructions of the array, we keep track of which instructions have already been used, with the help of a function called the odometer $h : \mathbb{Z}^d \to \mathbb{N}$, which counts, at each site, how many instructions have already been used.

When we use an instruction at a site x, we say that we topple x. For a given fixed configuration η, we say that it is legal (respectively, acceptable) for η to topple a site x if x contains at least one active particle (respectively, at least one particle) in η.

If a toppling is legal or acceptable, then this toppling consists in using the next instruction $\tau_{x,h(x)+1}$ to update the configuration η: if this instruction is a sleep instruction, then the particle at x falls asleep if it is alone (whereas nothing happens if there are at least two particles at x), and if it is a jump instruction to another site y, one particle at x jumps to site y, waking up the sleeping particle there if there is one. If the toppling was only acceptable but not legal, we first wake up the particle at x before applying the toppling. The resulting configuration is denoted by $\tau_{x,\,h(x)+1}\eta$. Thus, for a fixed realization of the array τ, the toppling at a site x consists of an operator

$$\Phi_x^\tau \;:\; (\eta,\, h) \;\longmapsto\; (\tau_{x,\,h(x)+1}\eta,\, h+\delta_x)\,,$$

which is only defined if the toppling is acceptable.

If $\alpha = (x_1, \ldots, x_k)$ is a sequence of sites of $\mathbb{Z}^d$, we say that the toppling sequence α is τ-legal (resp., τ-acceptable) for (η, h) if for every $i \in \{1, \ldots, k\}$, it is legal (resp., acceptable) for $\Phi_{x_{i-1}}^\tau \circ \cdots \circ \Phi_{x_2}^\tau \circ \Phi_{x_1}^\tau(\eta, h)$ to topple x_i, that is to say, if the configuration resulting from the first $i-1$ topplings has at least one active particle (resp., at least one particle) on the site x. If α is acceptable, applying the toppling sequence α means applying $\Phi_\alpha^\tau = \Phi_{x_k}^\tau \circ \cdots \circ \Phi_{x_1}^\tau$. We define the odometer of a toppling sequence α as $m_\alpha = \delta_{x_1} + \cdots + \delta_{x_k}$, which simply counts how many times each site appears in the sequence α. We also define, for every $V \subset \mathbb{Z}^d$,

$$m_{V,\eta}^\tau \;=\; \sup_{\alpha\subset V,\ \alpha \text{ is } \tau\text{-legal for } \eta} m_\alpha\,, \tag{3.1}$$

where the notation $\alpha \subset V$ means that all the sites appearing in α must belong to V. The total stabilization odometer associated with the configuration η is defined as:

$$m_\eta^\tau \;=\; \sup_{\alpha \text{ is } \tau\text{-legal for } \eta} m_\alpha \;=\; \sup_{V\subset\mathbb{Z}^d} m_{V,\eta}^\tau\,. \tag{3.2}$$

3.2. *Abelian property and monotonicity.* An important advantage of the site-wise construction is the following property, which states that the order with which we perform the topplings is irrelevant, allowing us to use whatever convenient strategy to choose which sites to topple. We say that a sequence of topplings stabilizes η in V if the configuration resulting from the application of the toppling sequence is stable in V, meaning that there are no active particles in V.

Lemma 3.1 (Abelian property, Lemma 2 in Rolla and Sidoravicius, 2012)**.** *If α and β are both legal toppling sequences for η that are contained in V and stabilize η in V, then $m_\alpha = m_\beta = m^\tau_{V,\eta}$ and the resulting configurations are equal, that is to say, $\Phi^\tau_\alpha(\eta, 0) = \Phi^\tau_\beta(\eta, 0)$.*

We also have the following monotonicity property, which shows that acceptable topplings may be used when looking for upper bounds on the legal odometer:

Lemma 3.2 (Lemma 2.1 in Rolla, 2020)**.** *If α is an acceptable sequence of topplings that stabilizes η in V, and $\beta \subset V$ is a legal sequence of topplings for η, then $m_\alpha \geq m_\beta$. Thus, if α is an acceptable sequence of topplings that stabilizes η in V, then $m_\alpha \geq m^\tau_{V,\eta}$.*

3.3. *Probability of fixation.* We now make η and τ become random. Let $\lambda > 0$, let $P : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$ be a jump distribution and let ν be a probability distribution on the set $(\mathbb{N} \cup \{\mathfrak{s}\})^{\mathbb{Z}^d}$ of all possible initial configurations. We then write $\mathbb{P}^\nu_{\lambda,P}$ for the measure relative to the activated random walk model with sleep rate λ, jump distribution P and initial configuration distributed according to ν. Then, the probability of fixation of the model is related to the stabilization odometer through (see Rolla and Sidoravicius, 2012)

$$\mathbb{P}^\nu_{\lambda,P}\big(\text{the system fixates}\big) \;=\; \nu \otimes \mathcal{P}_{\lambda,P}\big(m^\tau_\eta(0) < \infty\big)\,, \tag{3.3}$$

where $\mathcal{P}_{\lambda,P}$ is the measure on all the possible stacks of instructions $(\tau_{x,i})_{x\in\mathbb{Z}^d,\, i\geq 1}$ such that the instructions are independent and for every $x \in \mathbb{Z}^d$ and $i \geq 1$, the instruction $\tau_{x,i}$ is a sleep instruction with probability $\lambda/(1 + \lambda)$ and it is a jump instruction to $y \in \mathbb{Z}^d$ with probability $P(x, y)/(1 + \lambda)$. Thus, to know whether the system fixates or not, it is enough to look at this array of instructions τ and to determine whether the stabilization odometer at the origin, $m^\tau_\eta(0)$, is finite or not.

4. Proof of Theorem 1.5: the probability that no particle exits

This section is devoted to the proof of Theorem 1.5, following the strategy described in Section 2.

4.1. *The block configuration.* We now construct the random block configuration. Let $n \geq 1$, let $\eta : V_n \to \mathbb{N}$ and let ν be a probability distribution on $\mathbb{N}^{\mathbb{Z}}$. For every $k \in \mathbb{Z}$ we define $B_k = V_n + kn$, called the block number k. Let $\bar\eta$ be the configuration which contains a copy of η inside each of these blocks, i.e., for every $k \in \mathbb{Z}$ and every $x \in B_k$ we have $\bar\eta(x) = \eta(x - kn)$. Then, we consider the application $\varphi_\eta : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ defined by

$$\forall \xi \in \mathbb{N}^{\mathbb{Z}} \quad \forall k \in \mathbb{Z} \quad \forall x \in B_k \qquad \varphi_\eta(\xi)(x) \;=\; \xi(k) \times \bar\eta(x)\,.$$

With this notation, we define $\nu_\eta = (\varphi_\eta)_\star \nu$, the push-forward measure of ν through this map φ_η. For example if under ν, the variables $(\xi(k))_{k\in\mathbb{Z}}$ are i.i.d. Bernoulli with a certain parameter q, then ν_η is the measure on the initial configurations such that each of the blocks B_k for $k \in \mathbb{Z}$ contains a translated copy of η with probability q, independently for each block.

4.2. *The coarse-graining coupling.* We now formalize the coupling that will allow us to study the model started with this block configuration. This coupling is illustrated in Figure 2.2.

Proposition 4.1. *Let $\lambda > 0$, let $P : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ be a translation-invariant nearest-neighbour jump distribution, let $n \geq 1$, let $\eta : V_n \to \mathbb{N}$ with $\eta \neq 0$ and let $p = \mathbb{P}_\eta(M_n = 0)$. Let $\lambda' = p/(1-p)$ and let $D : \mathbb{Z} \times \mathbb{Z} \to [0,1]$ be a nearest-neighbour jump distribution with directed jumps towards the origin, that is to say, such that for every $x \geq 1$, we have $D(x,\, x-1) = 1$ and $D(-x,\, -x+1) = 1$.*

Then, for every probability distribution ν on $\mathbb{N}^{\mathbb{Z}}$, there exists a coupling between $\nu_\eta \otimes \mathcal{P}_{\lambda,P}$ (called the block model) and $\nu \otimes \mathcal{P}_{\lambda',D}$ (called the coarse-grained model) such that, if $(\xi,\, \tau)$ and $(\xi',\, \tau')$ are coupled with these two respective distributions then we have the implication

$$\left\{ m_{\xi'}^{\tau'}(0) = 0 \right\} \ \subset\ \left\{ \forall x \in B_0\,,\ m_\xi^\tau(x) = 0 \right\}, \tag{4.1}$$

that is to say, if the origin is never visited in the coarse-grained model then the block of the origin is never visited in the block model.

Proof: Let λ, P, n, η, p, λ', D and ν be as in the statement. Let ξ' be a random initial configuration with distribution ν, and let $\xi = \varphi_\eta(\xi')$ be the block configuration obtained from ξ', as defined in Section 4.1, so that ξ has distribution ν_η. Let τ be an array of instructions with distribution $\mathcal{P}_{\lambda,P}$, independent of ξ' (and hence also independent of ξ).

There now remains to define the array τ'. The idea is to stabilize the configuration ξ using the instructions in τ, one block after another, writing instructions in τ' along the procedure, depending on whether we get good or bad stabilizations. If the block of the origin is visited by a particle then the procedure stops, because we only care about the event that the block of the origin is never visited.

Formally, we construct recursively, for each $j \in \mathbb{N}$, an array of instructions τ'_j and two sequences of topplings α_j and α'_j such that, for every $j \in \mathbb{N}$, if we denote respectively by β_j the concatenation of the sequences $\alpha_0, \ldots, \alpha_j$ and by β'_j the concatenation of the sequences $\alpha'_0, \ldots, \alpha'_j$, the following properties are satisfied for every $j \in \mathbb{N}$ (with (v) and (vi) concerning only $j \geq 1$):

(i) The sequence β_j is τ-acceptable for $(\xi,\, 0)$, allowing us to define $(\xi_j,\, h_j) = \Phi^\tau_{\beta_j}(\xi,\, 0)$.

(ii) The sequence β'_j is τ'_j-legal for $(\xi',\, 0)$, allowing us to define $(\xi'_j,\, h'_j) = \Phi^{\tau'_j}_{\beta'_j}(\xi',\, 0)$.

(iii) The value of the coarse-grained configuration ξ'_j at every site $k \in \mathbb{Z}$ is related to the configuration ξ_j inside the block B_k through:

$$\begin{cases} \xi'_j(k) = 0 \quad \Longrightarrow \quad \xi_j \text{ empty in } B_k, \\ \xi'_j(k) = \mathfrak{s} \quad \Longrightarrow \quad \xi_j \text{ stable in } B_k, \text{ with } \|\eta\| \text{ sleeping particles in } B_k, \\ \xi'_j(k) = \ell \in \mathbb{N} \setminus \{0\} \quad \Longrightarrow \quad \xi_j \text{ contains } \ell \times \|\eta\| \text{ particles in } B_k, \text{ including a copy of } \eta, \end{cases}$$

where in the last case we mean that $\sum_{x \in B_k} |\xi_j(x)| \geq \ell\, \|\eta\|$ and $\xi_j \geq \bar{\eta}$ in B_k, where $\bar{\eta}$ is the configuration defined in Section 4.1, so that in the case $\ell = 1$ this reduces to $\xi_j = \bar{\eta}$ in B_k.

(iv) Active particles remain farther away from the origin than sleeping particles, in the following sense: for every $k,\, \ell \in \mathbb{N}$,

$$\Big[\big(\xi'_j(k) = 1 \ \text{ and } \ \xi'_j(\ell) = \mathfrak{s} \big) \quad \text{or} \quad \big(\xi'_j(-k) = 1 \ \text{ and } \ \xi'_j(-\ell) = \mathfrak{s} \big) \Big] \quad \Longrightarrow \quad \ell \leq k\,.$$

(v) We always topple in priority a block as close as possible to the origin, that is to say, for every $k \geq 1$, if the set $\{-k, \ldots, k\}$ is unstable in ξ'_{j-1} and $\xi'_{j-1}(0) = 0$, then α'_j topples at least one site in this set.

(vi) The instructions in the arrays τ'_j are modified only when used by α'_j, that is to say, for every $x \in \mathbb{Z}$ and every $i \geq 1$, if $(\tau'_j)_{x,i} \neq (\tau'_{j-1})_{x,i}$ then $h'_{j-1}(x) < i \leq h'_j(x)$.

(vii) As long as the origin remains empty in the coarse-grained configuration ξ'_j, the block of the origin B_0 is not toppled by the sequence β_j, that is to say,

$$\xi'_j(0) = 0 \quad \Longrightarrow \quad \big(\forall x \in B_0 \quad h_j(x) = 0 \big)\,.$$

(viii) The array τ'_j has distribution $\mathcal{P}_{\lambda',D}$ and is independent of ξ'.

For $j = 0$, we simply take τ'_0 to be an array of instructions with distribution $\mathcal{P}_{\lambda',D}$, independent of (ξ, ξ', τ), and we let α_0 and α'_0 be empty toppling sequences, so that $\xi_0 = \xi$ and $\xi'_0 = \xi'$. Thus, the items (iii) and (iv) hold by definition of $\xi = \varphi_\eta(\xi')$ and because there is no sleeping particle in ξ', and the other items hold trivially.

Let $j \in \mathbb{N}$, assume that the array τ'_j and the sequences $\alpha_0, \ldots, \alpha_j$ and $\alpha'_0, \ldots, \alpha'_j$ are already constructed and satisfy the eight above properties, and let us construct τ'_{j+1}, α_{j+1} and α'_{j+1} which satisfy the properties at rank $j+1$.

If the configuration ξ'_j is stable or such that $\xi'_j(0) \neq 0$ then we simply take α_{j+1} and α'_{j+1} to be empty toppling sequences, and we let $\tau'_{j+1} = \tau'_j$, and the conditions (i) to (vii) are inherited at rank $j+1$. We will deal with condition (viii) afterwards, once we defined τ'_{j+1} in the different cases.

Assume now that ξ'_j is unstable and $\xi'_j(0) = 0$. Let $k \in \mathbb{Z}$ be such that $\xi'_j(k) \geq 1$, with $|k|$ being minimal among such sites (with an arbitrary deterministic rule to break the tie if any). Since $\xi'_j(0) = 0$, we have $k \neq 0$. Let us define $\ell = k - 1$ if $k > 0$ and $\ell = k + 1$ if $k < 0$, so that $D(k, \ell) = 1$.

We now distinguish between two cases. First, let us assume that $\xi'_j(k) = 1$. Then, item (iii) tells us that in the block B_k the configuration ξ_j is a copy of η. We consider γ a τ-legal toppling sequence for (ξ_j, h_j) which topples the sites of B_k (and only these sites) until all sites of B_k are stable (with probability 1 such a sequence exists, let us assume that this is the case).

By definition of p as the probability of a good stabilization, with probability p no particles jump out of B_k while performing this sequence of topplings γ. In this case, we let τ'_{j+1} be the array obtained from τ'_j by replacing the instruction number $h'_j(k) + 1$ at site k with a sleep instruction, and we define $\alpha_{j+1} = \gamma$ and $\alpha'_{j+1} = (k)$, so that by construction items (i) and (ii) hold for $j+1$. We then have $\xi'_{j+1}(k) = \mathfrak{s}$ and ξ_{j+1} is stable in the block B_k, with nothing changed out of this block compared to ξ_j, which shows that item (iii) still holds at rank $j+1$. Besides, the minimality of $|k|$ ensures that items (iv) and (v) remain satisfied at rank $j+1$. Property (vi) holds for $j+1$ because α'_{j+1} uses precisely the instruction that we replaced. Lastly, item (vii) is inherited because we performed topplings only in the block B_k, with $k \neq 0$.

Assume now that a bad stabilization happens: a particle jumps out of B_k while applying γ. We then let τ'_{j+1} be the array obtained from τ'_j by replacing the instruction number $h'_j(k) + 1$ at site k with a jump instruction towards the origin. Then, we force these $\|\eta\|$ particles which were in the block B_k in the configuration ξ_j to walk, with topplings which are τ-acceptable for $\Phi^\tau_\gamma(\xi_j, h_j)$, until these particles form a copy of η in the block B_ℓ (recall that B_ℓ is the neighbour block of B_k in the direction of the origin). We do this by performing only topplings on the sites of the blocks B_m with $m \geq \ell$ if $k > 0$ or on the sites of the blocks B_m with $m \leq \ell$ if $k < 0$ (to see this, label the particles, assign to each particle a destination site in B_ℓ and move each particle one by one until it reaches its destination, which eventually happens with probability 1, and doing so no particle walks closer to the origin than its destination, because the jumps are only to the nearest neighbours). We call δ the obtained acceptable sequence, and we then let $\alpha_{j+1} = (\gamma, \delta)$ and $\alpha'_{j+1} = (k)$. Here also, items (i) and (ii) hold by construction. Item (iv), together with item (iii), ensures that none of these blocks on which we perform topplings can contain sleeping particles, except maybe the block B_ℓ. This entails that items (iii) and (iv) still hold at rank $j+1$. As in the previous case, property (v) holds for $j+1$ by minimality of $|k|$ and property (vi) because α'_{j+1} uses the replaced instruction. The property (vii) is also inherited because if $\ell \neq 0$ then we did not topple the block B_0, whereas if $\ell = 0$ then $\xi'_{j+1}(0) \neq 0$.

We now deal with the case $\xi'_j(k) \geq 2$. Then, item (iii) tells us that $\xi_j \geq \bar{\eta}$ in B_k and that the configuration $\xi_j - \bar{\eta}$ contains at least $\|\eta\|$ particles. We move $\|\eta\|$ of these particles with topplings read from τ and acceptable for (ξ_j, h_j) until these particles form a copy of η in the block B_ℓ, performing topplings only on the sites of the blocks B_m with $m \geq \ell$ if $k > 0$ or $m \leq \ell$ if $k < 0$. It is important that we move these extra particles, leaving aside $\|\eta\|$ other particles which already form a copy of η in the block B_k, so that the configuration will still contain a copy of η in this block B_k at rank $j+1$. We then let α_{j+1} be the acceptable sequence of topplings performed. Here we do not need to modify the array τ'_j because, since $\xi'_j(k) \geq 2$, we can move one particle from k to ℓ with legal topplings read from τ'_j, by simply reading instructions until we find a jump instruction (and this jump instruction necessarily points to ℓ, by definition of the directed jump distribution D). Hence, we let $\tau'_{j+1} = \tau'_j$ and we let α'_{j+1} be the sequence consisting of m_0 occurrences of k, where $m_0 = \inf\{m \geq 1 : (\tau'_j)_{k,\, h'_j(k)+m} \neq \mathfrak{s}\}$ (which is almost surely finite, let us assume that it is the case). As before, items (i) to (vii) are inherited at rank $j+1$.

We constructed α_{j+1}, α'_{j+1} and τ'_{j+1} in the different cases, and there now remains to check that item (viii) remains true at rank $j+1$. To show this, consider the σ-field

$$\mathcal{F}_j \;=\; \sigma\big(\xi',\, h_j,\, ((\tau_j)_{x,i})_{x\in\mathbb{Z},\, i\leq h_j(x)},\, h'_j,\, ((\tau'_j)_{x,i})_{x\in\mathbb{Z},\, i\leq h'_j(x)}\big)\,,$$

which contains the information of the initial configuration and of the instructions used by β_j and β'_j. Then, ξ_j and ξ'_j are $\mathcal{F}_j$-measurable, so the event A that ξ'_j is unstable, with $\xi'_j(0) = 0$ and that the site k that we consider in the construction at step j is such that $\xi'_j(k) = 1$, is $\mathcal{F}_j$-measurable. Recall that we replace an instruction in τ'_{j+1} compared to τ'_j only on this event A, and that the instruction that we write is a sleep instruction if and only if we get a good stabilization. Yet, conditioned on $\mathcal{F}_j$, on this event A the conditional probability to obtain a good stabilization is constant and equal to p, because it depends only on instructions of τ that were not used by β_j, and thus are not revealed yet. Hence, when we rewrite an instruction, then the instruction that we write is independent of $\mathcal{F}_j$, as was the instruction that was present at this position in τ'_j, and moreover these two instructions have the same distribution, since $p = \lambda'/(1+\lambda')$ (the instruction that we write is a sleep instruction with probability p, while $\lambda'/(1+\lambda')$ is the probability that a given instruction in τ'_j is a sleep instruction). Hence, τ'_j and τ'_{j+1} have the same conditional distribution knowing $\mathcal{F}_j$. A fortiori, since $\sigma(\xi') \subset \mathcal{F}_j$, we deduce that τ'_j and τ'_{j+1} also have the same conditional distribution knowing only ξ'. Yet, item (viii) at rank j ensures that τ'_j is independent of ξ' and has distribution $\mathcal{P}_{\lambda',D}$: therefore, it remains true at rank $j+1$.

Thus, we constructed τ'_j, α_j and α'_j for every $j \in \mathbb{N}$, which satisfy the eight above properties. Note that property (vi) entails that for every fixed $k \in \mathbb{Z}$ and $i \geq 1$, in the sequence $(\tau'_j)_{j\in\mathbb{N}}$ the instruction at position (k,i) changes at most once, so that the sequence $((\tau'_j)_{k,i})_{j\in\mathbb{N}}$ is stationary. Hence, the sequence of arrays $(\tau'_j)_{j\in\mathbb{N}}$ weakly converges to a limit that we denote τ'. The property (viii) passes to the limit: this array τ' is also independent of ξ' and has distribution $\mathcal{P}_{\lambda',D}$.

There now remains to show that the implication (4.1) holds. From now on and until the end of the proof, we assume that $m^{\tau'}_{\xi'}(0) = 0$. Since by definition $\xi'(0) \neq \mathfrak{s}$, this implies that $\xi'(0) = 0$, and that the site 0 remains empty after applying any τ'-legal sequence of topplings. Note that item (vi) ensures that for every $j \in \mathbb{N}$ the instructions of τ'_j that are used by β'_j are not modified in subsequent steps, and are therefore identical in τ'. Therefore, the sequence β'_j is not only τ'_j-legal but also τ'-legal for $(\xi',\, 0)$. By what we just said, this implies that $\xi'_j(0) = 0$ for every $j \in \mathbb{N}$.

This implies that for every $k \in \mathbb{Z}\setminus\{0\}$, the sequence β'_j performs at most $|k|-1$ jumps at the site k. Indeed, if β'_j performs at least $|k|$ jumps at a site $k \in \mathbb{Z}\setminus\{0\}$ then the resulting configuration ξ'_j contains at least $|k|$ particles strictly between the origin and k and then one of these particles will eventually reach the origin, due to the priority rule of item (v), which contradicts our finding that $\xi'_j(0) = 0$ for every $j \in \mathbb{N}$.

Let now $k \geq 1$, $V' = \{-k,\, \ldots,\, k\}$ and $V = \cup_{\ell\in V'} B_\ell$ and let us show that for every $x \in B_0$, $m^{\tau}_{\xi,V}(x) = 0$. For every $j \in \mathbb{N}$, let u_j be the total number of jumps performed by the sequence α'_j on the sites of V'. The above paragraph ensures that this (non-decreasing) sequence $(u_j)_{j\in\mathbb{N}}$ is bounded, and therefore constant from a certain rank j_0, which means that after step j_0 we do not perform any more jumps on V'. Yet, for every $j \geq j_0$ such that ξ'_j is still unstable in V', the priority rule of item (v) forces us to topple a site of V', but this toppling can only be a sleep because $j \geq j_0$, so that one particle of V' falls asleep. Hence, at the latest at step $j = j_0 + 2k$, the configuration ξ'_j must be stable in V'.

By virtue of the connection given by item (iii) between ξ_j and ξ'_j, this implies that the configuration ξ_j is stable in V. Thus, β_j is a τ-acceptable sequence which stabilizes ξ in V. By Lemma 3.2, this implies that $m^\tau_{V,\xi} \leq m_{\beta_j} = h_j$.

Yet, since $\xi'_j(0) = 0$ the property (vii) entails that $h_j(x) = 0$ for every $x \in B_0$, whence $m^\tau_{V,\xi}(x) = 0$ for every $x \in B_0$. Since this holds for every $k \geq 1$, we deduce that for every $x \in B_0$ we have $m^\tau_\xi(x) = 0$, which concludes the proof of the implication (4.1). □

4.3. *Positive probability that the block of the origin is never visited.* Using the coupling given by Proposition 4.1 and combining it with Theorem 2.1, we obtain:

Corollary 4.2. *Let $\lambda > 0$, let P be a translation-invariant nearest-neighbour jump distribution, let $n \geq 1$, let $\eta : V_n \to \mathbb{N}$ with $\eta \neq 0$ and let $p = \mathbb{P}_\eta(M_n = 0)$. Then for every i.i.d. initial distribution ν with mean density of particles $q < p$ we have*

$$\nu_\eta \otimes \mathcal{P}_{\lambda,P}\big(\forall x \in B_0\,,\ m^\tau_\xi(x) = 0\big) \ > \ 0\,.$$

Proof: Proposition 4.1 shows that

$$\nu_\eta \otimes \mathcal{P}_{\lambda,P}\big(\forall x \in B_0\,,\ m^\tau_\xi(x) = 0\big) \ \geq \ \nu \otimes \mathcal{P}_{\lambda',D}\big(m^\tau_\xi(0) = 0\big)\,, \tag{4.2}$$

with $\lambda' = p/(1-p)$ and D a directed jump distribution, as defined in the statement of Proposition 4.1. Yet, since $q < p = \lambda'/(1+\lambda')$, Theorem 2.1 entails that the right-hand side of (4.2) is strictly positive, whence the result. □

4.4. *Making the initial configuration translation-invariant.* Now, we would like to deduce that for ν and ν_η as above, the mean density of particles in this initial distribution ν_η is at most ζ_c, using Theorem 1.1. But there remains to deal with a small issue: this distribution is not translation-invariant. To obtain an initial distribution that is invariant by translation, we simply apply a translation by a random offset. Thus, we take ξ distributed according to ν_η and Y a uniform variable in $B_0 = V_n$, independent of ξ, and we define the configuration ξ^{inv} by writing, for every $x \in \mathbb{Z}$,

$$\xi^{\mathrm{inv}}(x) \ = \ \xi(x+Y)\,.$$

We call ν^{inv}_η the distribution of ξ^{inv}, and we now check that this distribution is translation-ergodic. First, by construction it is translation-invariant. There remains to see that every translation-invariant event has probability 0 or 1. Let ξ' be a random configuration with distribution ν and let $\xi = \varphi_\eta(\xi')$, so that ξ has distribution ν_η, and let ξ^{inv} be defined as above, with Y independent of ξ. Then, if A is a Borel set of $(\mathbb{N} \cup \{\mathfrak{s}\})^{\mathbb{Z}}$ which is invariant by translation, we can write

$$\begin{aligned}
\nu^{\mathrm{inv}}_\eta(A) \ &= \ \mathbb{P}(\xi^{\mathrm{inv}} \in A) \ = \ \sum_{y \in B_0} \mathbb{P}\big(\{Y = y\} \cap \{(x \mapsto \xi(x+y)) \in A\}\big) \ = \ \sum_{y \in B_0} \mathbb{P}(Y = y)\,\mathbb{P}(\xi \in A) \\
&= \ \mathbb{P}(\xi \in A) \ = \ \mathbb{P}\big(\xi' \in (\varphi_\eta)^{-1}(A)\big) \ = \ \nu\big((\varphi_\eta)^{-1}(A)\big)
\end{aligned}$$

where, in the third equality we used the independence of Y and ξ and the fact that A is invariant by translation. Yet, the distribution ν is translation-ergodic because it is i.i.d., and the event $(\varphi_\eta)^{-1}(A)$ is invariant by translation because A itself is invariant by translation, so we have $\nu_q\big((\varphi_\eta)^{-1}(A)\big) \in \{0,1\}$, which allows us to deduce that $\nu^{\mathrm{inv}}_\eta(A) \in \{0,1\}$. Hence, the distribution ν^{inv}_η is translation-ergodic.

Lastly, note that the density of particles in this random initial configuration is given by

$$\mathbb{E}\big[\xi^{\mathrm{inv}}(0)\big] \ = \ \mathbb{E}\big[\xi(Y)\big] \ = \ \mathbb{E}\big[\xi'(0)\eta(Y)\big] \ = \ q \times \mathbb{E}\big[\eta(Y)\big] \ = \ q \times \frac{\|\eta\|}{n}\,. \tag{4.3}$$

4.5. *Conclusion.* We now put the pieces together to obtain Theorem 1.5.

Proof of Theorem 1.5: Let λ, P, ζ, n and η be as in the statement, let us write $p = \mathbb{P}_\eta(M_n = 0)$ and consider $q \in [0,\, p)$.

Let ν be an i.i.d. probability distribution on $\mathbb{N}^{\mathbb{Z}}$ with mean density q. With the initial distribution ν_η^{inv} defined in Section 4.4, the relation (3.3) between fixation and the odometer allows us to write

$$\begin{aligned}\mathbb{P}_{\lambda,P}^{\nu_\eta^{\mathrm{inv}}}\big(\text{the system fixates}\big) \;&=\; \nu_\eta^{\mathrm{inv}} \otimes \mathcal{P}_{\lambda,P}\big(m_\xi^\tau(0) < \infty\big) \;\geq\; \nu_\eta^{\mathrm{inv}} \otimes \mathcal{P}_{\lambda,P}\big(m_\xi^\tau(0) = 0\big)\\ &=\; \frac{1}{n}\sum_{y\in B_0} \nu_\eta \otimes \mathcal{P}_{\lambda,P}\big(m_\xi^\tau(y) = 0\big) \;\geq\; \nu_\eta \otimes \mathcal{P}_{\lambda,P}\big(\forall x \in B_0\,,\; m_\xi^\tau(x) = 0\big)\,,\end{aligned}$$

which is strictly positive by Corollary 4.2. Thus, the model with initial distribution ν_η^{inv} fixates with positive probability.

Since ν_η^{inv} is translation-ergodic, Theorem 1.1 allows us to deduce that this initial distribution is not supercritical, that is to say, $\nu_\eta^{\mathrm{inv}}\big[\xi(0)\big] \leq \zeta_c$. Given (4.3) and recalling that $\|\eta\| \geq \zeta n$, we obtain that $q\zeta \leq \zeta_c$. This being true for every $q < p$, we eventually deduce that $p\zeta \leq \zeta_c$, which concludes the proof of Theorem 1.5. □

5. Proof of Theorem 1.4: a fraction jumps out of the segment

The aim of this section is to deduce Theorem 1.4 from Theorem 1.5.

5.1. *Preliminary: a no man's land around a segment.* The following Lemma tells us that adding empty intervals around the segment V_n where particles are not allowed to sleep and stabilizing the configuration in V_n and in these intervals does not increase the number of particles which exit during stabilization, at least in distribution.

Lemma 5.1. *Let $\lambda > 0$ and let P be a nearest-neighbour jump distribution on $\mathbb{Z}$. Let $n \geq 1$, let $\eta : V_n \to \mathbb{N}$ be a fixed deterministic initial configuration on V_n with only active particles, and let $a,\, b \in \mathbb{Z}$ be such that $W = \{a,\, \ldots,\, b\} \supset V_n$. Starting from the initial configuration η and performing legal topplings in V_n and acceptable topplings in $W \setminus V_n$ until the resulting configuration is stable in V_n and empty in $W \setminus V_n$, we denote by M_n^W the number of particles which jump out of W. Then M_n stochastically dominates M_n^W.*

This result can seem counter-intuitive, since forcing particles to not only exit V_n but also to cross these empty intervals could lead some of them to come back inside V_n and wake up sleeping particles, causing more of them to jump out. Indeed, for a fixed realization of the array of toppling instructions, it is not true in general that $M_n^W \leq M_n$, but the lemma indicates that this inequality turns out to hold in distribution.

Proof: Let λ, P, n, η, a, b, W be as in the statement. Let us first consider the simpler case of a no man's land being added only on one side of V_n, say the left side. That is to say, we assume that $b = \max V_n$.

Let τ be a random array of independent instructions, with no sleep instructions in $W \setminus V_n$, that is to say, τ is obtained from the usual array (as considered in the site-wise construction presented in Section 3.1) by removing the sleep instructions on the sites of $W \setminus V_n$. Then M_n^W is the number of particles which jump out of W during the stabilization of W with topplings which are legal for τ.

We then consider the following toppling strategy, which is represented in Figure 5.4:

- **Step 1:** Topple the leftmost active particle in V_n.
- **Step 2:** If a particle just left V_n by the left exit, force it to walk with acceptable topplings until it leaves W.

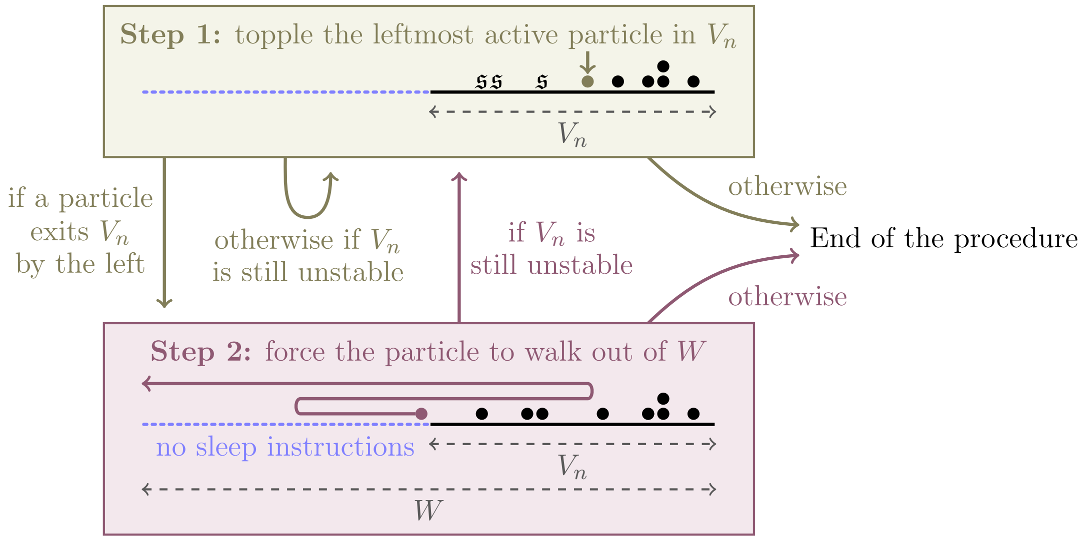


FIGURE 5.4. Strategy to prove Lemma 5.1 in the case $\max W = \max V_n$. The key point is that we always topple the leftmost active particle in V_n, so that whenever a particle jumps out of V_n from the left, all the other particles are active, allowing us to force this particle to walk out of W with no effect on the other particles.

- Repeat these steps 1 and 2 as long as there remains active particles in V_n.

Let N be the number of particles which jump out of W during this procedure. If the procedure terminates (which happens almost surely), it yields an acceptable sequence of topplings α which stabilizes W. Thus, by the monotonicity property given by Lemma 3.2, the odometer of this sequence satisfies $m_\alpha \geq m^\tau_{W,\eta}$. Yet, for any given realization of the array of instructions, the number of particles which jump out of W when applying a toppling procedure is an increasing function of the odometer of this procedure. Therefore, we have $N \geq M_n^W$.

We now show that N has the same distribution as M_n.

Note that, after each time that step 1 is performed, then all the sleeping particles are located on the left of the leftmost active particle. That is to say, for any two sites $x, y \in V_n$ with $x < y$, it cannot be that x contains an active particle while y contains a sleeping one. Indeed, if this was not true, then consider the first instant when two such sites $x < y$ exist. Then necessarily the last instruction used must have been a sleep instruction at y, which contradicts the rule that we always topple the leftmost active site, since x was active.

As a consequence, each time that step 2 is triggered then all the particles in V_n must be active. Indeed, when a particle leaves V_n by the left exit during step 1, we know that we just toppled the leftmost site of V_n, since we consider a nearest-neighbour jump distribution. Following the above observation, this implies that all the particles in V_n are active.

Thus, the configuration inside V_n is left unchanged after performing step 2, the only change in V_n being that step 2 has used some toppling instructions, but the remaining instructions remain i.i.d. with the same distribution. Let τ' be the field of instructions obtained from τ by removing the instructions used during all the occurrences of step 2. Denote by $M_n(\tau')$ the number of particles which jump out of V_n during the stabilization of V_n using the instructions in τ', ignoring particles once they jump out of V_n. Then, we have $N = M_n(\tau')$. Yet, this field τ' has the same distribution as τ, whence the equality in distribution $M_n(\tau') \stackrel{d}{=} M_n(\tau) = M_n$. To sum up, we have $M_n^W \leq N = M_n(\tau') \stackrel{d}{=} M_n$, which shows that M_n stochastically dominates M_n^W, concluding the proof in the case $b = \max V_n$.

We now turn to the general case of $W = \{a, \ldots, b\} \supset V_n$. Writing $U = \{a, \ldots, \max V_n\}$, the above proof shows that M_n stochastically dominates M_n^U. Then, we repeat a similar strategy to show that M_n^U dominates M_n^W. More precisely, considering an array τ with no sleeps out of V_n, we now adopt the following strategy:

- **Step 1:** Topple the rightmost active particle in U.
- **Step 2:** If a particle just left U by the right exit, force it to walk with acceptable topplings until it leaves W.
- Repeat these steps 1 and 2 as long as there remains active particles in U.

Then, the same arguments as before show that the number of particles which exit W with this procedure is at least M_n^W and is distributed as M_n^U, which concludes the proof. □

5.2. *If few particles jump out, then no one leaves a slightly larger segment.* We now prove the following lower bound on the cost to stabilize in the no man's lands all the particles which jump out of V_n:

Lemma 5.2. *In dimension $d = 1$, for every $\lambda > 0$ and every nearest-neighbour jump distribution P, for every $n \geq 1$ and every deterministic initial configuration $\eta : V_n \to \mathbb{N}$, for any $k, \ell \in \mathbb{N}$, we have*

$$\mathbb{P}_\eta\big(M_{n+4\ell} = 0\big) \;\geq\; \mathbb{P}_\eta\big(M_n \leq k\big) \times \mathbb{P}\big(G_1 + \cdots + G_k \leq \ell\big)\,,$$

where $(G_j)_{j\geq 1}$ are i.i.d. Geometric variables with parameter $\lambda/(1+\lambda)$.

Note that, when we write $\mathbb{P}_\eta\big(M_{n+4\ell} = 0\big)$, we implicitly extend the configuration $\eta : V_n \to \mathbb{N}$ to the configuration on $V_{n+4\ell}$ which coincides with η on V_n and has no particles on $V_{n+4\ell} \setminus V_n$.

Proof: Let λ, P, n, η, k, ℓ and $(G_j)_{j\geq 1}$ be as in the statement. To stabilize η in $V_{n+4\ell}$, we proceed in two steps, as explained in the sketch of the proof in Section 2.

First, we perform legal topplings in V_n and acceptable topplings in $V_{n+2\ell} \setminus V_n$ until all the sites of V_n are stable and all the sites of $V_{n+2\ell} \setminus V_n$ are empty. Let us denote by M'_n the number of particles which jump out of $V_{n+2\ell}$ during this step. It follows from Lemma 5.1 that M_n stochastically dominates M'_n (note that M'_n corresponds to $M_n^{V_{m+2\ell}}$ in the notation of Lemma 5.1). Thus, we have

$$\mathbb{P}_\eta\big(M'_n \leq k\big) \;\geq\; \mathbb{P}_\eta\big(M_n \leq k\big)\,. \tag{5.1}$$

Then, in the second stage, we try to stabilize these M'_n particles inside $V_{n+4\ell} \setminus V_n$, using the trapping procedure introduced in Rolla and Sidoravicius (2012) (in the proof of their Theorem 2). This procedure shows that, if $M'_n \leq k$, then with probability at least $\mathbb{P}\big(G_1 + \cdots + G_k \leq \ell\big)$, this second stage succeeds, yielding an acceptable toppling sequence which stabilizes these M'_n particles inside $V_{n+4\ell} \setminus V_n$ with none of these particles jumping out of $V_{n+4\ell} \setminus V_n$.

Thus, we deduce that

$$\mathbb{P}_\eta\big(M_{n+4\ell} = 0\big) \;\geq\; \mathbb{P}_\eta\big(M'_n \leq k\big) \times \mathbb{P}\big(G_1 + \cdots + G_k \leq \ell\big)$$

which, combined with (5.1), concludes the proof of the Lemma. □

5.3. *Concluding proof of Theorem 1.4.* We now put the pieces together to obtain the claimed bound.

Proof of Theorem 1.4: Let $\lambda > 0$, let P be a nearest-neighbour jump distribution on $\mathbb{Z}$, let $\zeta > \zeta_c$ and consider ε, α and β such that

$$0 \;\leq\; \frac{(1+\lambda)\varepsilon}{\lambda} \;<\; \alpha \;<\; \beta \;<\; \frac{\zeta - \zeta_c}{4\zeta_c}\,. \tag{5.2}$$

Let $(G_j)_{j\geq 1}$ be i.i.d. Geometric variables with parameter $\lambda/(1+\lambda)$. Since $(1+\lambda)\varepsilon/\lambda < \alpha$, the weak law of large numbers ensures that

$$\lim_{n\to\infty} \mathbb{P}\big(G_1 + \cdots + G_{\lfloor \varepsilon n \rfloor} \leq \alpha n\big) \;=\; 1\,.$$

Thus, we can take $n_0 \geq 1$ such that, for every $n \geq n_0$,

$$\mathbb{P}\big(G_1 + \cdots + G_{\lfloor \varepsilon n \rfloor} \,\leq\, \alpha n\big) \,\geq\, \frac{1+4\alpha}{1+4\beta}\,.$$

Now, let $n \geq n_0$ and let $\eta : V_n \to \mathbb{N}$ be a fixed deterministic initial configuration such that $\|\eta\| \geq \zeta n$. Applying Lemma 5.2 with $k = \lfloor \varepsilon n \rfloor$ and $\ell = \lfloor \alpha n \rfloor$, we get

$$\mathbb{P}_\eta\big(M_{n+4\ell} = 0\big) \,\geq\, \mathbb{P}_\eta\big(M_n \leq \varepsilon n\big) \times \mathbb{P}\big(G_1 + \cdots + G_{\lfloor \varepsilon n \rfloor} \,\leq\, \alpha n\big) \,\geq\, \mathbb{P}_\eta\big(M_n \leq \varepsilon n\big) \times \frac{1+4\alpha}{1+4\beta}\,. \quad (5.3)$$

Then, note that

$$\frac{\|\eta\|}{n+4\ell} \,\geq\, \frac{\zeta}{1+4\alpha} \,>\, \zeta_c\,,$$

by virtue of (5.2). Thus, applying Theorem 1.5 to η, seen as a configuration on $V_{n+4\ell}$, we have

$$\mathbb{P}_\eta\big(M_{n+4\ell} = 0\big) \,\leq\, \frac{(1+4\alpha)\zeta_c}{\zeta}\,. \quad (5.4)$$

Combining (5.3) and (5.4), we get

$$\mathbb{P}_\eta\big(M_n \leq \varepsilon n\big) \,\leq\, \frac{1+4\beta}{1+4\alpha} \times \frac{(1+4\alpha)\zeta_c}{\zeta} \,=\, \frac{(1+4\beta)\zeta_c}{\zeta}\,.$$

Taking the supremum over all configurations $\eta : V_n \to \mathbb{N}$ with $\|\eta\| \geq \zeta n$, we obtain that

$$\forall \beta \in \left(\frac{(1+\lambda)\varepsilon}{\lambda}, \frac{\zeta - \zeta_c}{4\zeta_c}\right) \quad \exists\, n_0 \geq 1 \quad \forall n \geq n_0 \qquad \sup_{\substack{\eta : V_n \to \mathbb{N}:\\ \|\eta\| \geq \zeta n}} \mathbb{P}_\eta\big(M_n \leq \varepsilon n\big) \,\leq\, \frac{(1+4\beta)\zeta_c}{\zeta}\,,$$

which is precisely the claim of Theorem 1.4. □

6. Proof of Theorem 1.3

We now prove the equivalence claimed in Theorem 1.3. Let $\lambda > 0$, let P be a nearest-neighbour translation-invariant jump distribution on $\mathbb{Z}$ and let η_0 be an i.i.d. initial distribution with mean ζ and all particles initially active.

6.1. *Direct implication.* The direct implication is an easy consequence of Theorem 1.4. It follows from the Central Limit Theorem that $\mathbb{P}\big(\|\eta_0\|_{V_n} \geq \zeta n\big) \to 1/2$ when $n \to \infty$. Thus, choosing whatever $\varepsilon > 0$ in the range indicated by Theorem 1.4, we have

$$\begin{aligned}\liminf_{n\to\infty} \frac{\mathbb{E}\big[M_n\big]}{n} \,&\geq\, \liminf_{n\to\infty} \mathbb{P}\big(\|\eta_0\|_{V_n} \geq \zeta n\big) \inf_{\substack{\eta : V_n \to \mathbb{N}:\\ \|\eta\| \geq \zeta n}} \mathbb{P}_\eta\big(M_n > \varepsilon n\big)\,\varepsilon \\ &\geq\, \frac{1}{2} \times \left[1 - \frac{\zeta_c}{\zeta}\left(1 + \frac{4(1+\lambda)\varepsilon}{\lambda}\right)\right] \times \varepsilon \,>\, 0\,.\end{aligned}$$

6.2. *Reciprocal: proof of Proposition 1.6.* The reciprocal implication of Theorem 1.3 follows from Theorem 1.2 and Proposition 1.6, which deals with the particular case of $\zeta = \zeta_c$, and which we now prove.

Proof of Proposition 1.6: Let d, λ, P and η_0 be as in the statement. By monotonicity (see for example Lemma 2.5 of Rolla, 2020), we can assume without loss of generality that all the particles are active in the configuration η_0. Let $\varepsilon \in (0, \zeta_c)$. Then, let us consider another i.i.d. initial distribution η_0' with mean $\zeta_c - \varepsilon$, which is coupled with η_0 in such a way that $\eta_0'(x) \leq \eta_0(x)$ for every $x \in \mathbb{Z}^d$. This can be done for example by taking $\eta_0'(x) = Y_x \eta_0(x)$, where $(Y_x)_{x\in\mathbb{Z}^d}$ are i.i.d. Bernoulli variables with parameter $(\zeta_c - \varepsilon)/\zeta_c$, independent of everything else.

Then, for every $n \geq 1$, denoting by M_n and M'_n the numbers of particles which jump out of the box V_n starting respectively with η_0 and with η'_0, we claim that

$$\mathbb{E}\big[M_n\big] \;\leq\; \varepsilon\,|V_n| + \mathbb{E}\big[M'_n\big]\,. \tag{6.1}$$

Indeed, starting from the configuration η_0, we may first apply acceptable topplings to the configuration $\eta_0 - \eta'_0$, until all particles exit, leaving us with only the configuration η'_0 remaining inside V_n. During this first stage, the average number of particles which jump out of the box is equal to $\mathbb{E}\big[\|\eta_0\| - \|\eta'_0\|\big] = \varepsilon\,|V_n|$. Then, we stabilize in V_n with legal topplings, which gives a number of particles jumping out of the box which is distributed as M'_n. Since we performed acceptable topplings during the first stage, we obtain an upper bound on M_n, whence (6.1).

Then, by the contrapositive of Theorem 1.2, combined with Theorem 1.1, we know that

$$\lim_{n\to\infty} \frac{\mathbb{E}\big[M'_n\big]}{|V_n|} \;=\; 0\,.$$

Combining this with (6.1), we deduce that

$$\limsup_{n\to\infty} \frac{\mathbb{E}\big[M_n\big]}{|V_n|} \;\leq\; \varepsilon \,+\, \lim_{n\to\infty} \frac{\mathbb{E}\big[M'_n\big]}{|V_n|} \;=\; \varepsilon\,.$$

This being true for every $\varepsilon \in (0,\, \zeta_c)$, the proof of Proposition 1.6 is complete. □

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