Supplement to 'Defining Dispersion: A Fundamental Order for Univariate Discrete Distributions'

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Numbering

Numbers of Sections, Definitions, Theorems etc. always refer to the main article 'Defining Dispersion: A Fundamental Order for Univariate Discrete Distributions'. For equations and figures, single numbers refer to the main article whereas numbers in combination with either the letter 'A' or the letter 'B' refer to this supplement.

Appendix A: Proofs

A.1. Proofs for Section 3

Proof of Proposition 3.2. We define a function $\varphi : \mathcal{D}_0 \to \bigcup_{A\in\mathbb{I}} \mathbb{S}_A$ and show that it is a bijection, which is an even stronger statement than the assertion.

Note that the codomain of φ is a disjoint union of the sets $\mathbb{S}_A, A \in \mathbb{I}$. The value assignment of φ is now defined by cases. Therefore, let $F \in \mathcal{D}_0$, then $\operatorname{supp}(F)$ is order-isomorphic to a subset of $\mathbb{Z}$. Note that since minima and maxima are defined via the order $\leq$, this order-isomorphism preserves minima and maxima.

Case 1: $\min(\operatorname{supp}(F))$ and $\max(\operatorname{supp}(F))$ both exist.
A subset of $\mathbb{Z}$ has a minimum and a maximum, if and only if it is finite. Therefore, $\operatorname{supp}(F)$ is also finite. Let $n = |\operatorname{supp}(F)| \in \mathbb{N}_{\geq 2}$. Define $x_1 = \min(\operatorname{supp}(F))$ and $x_j = \min(\operatorname{supp}(F) \setminus \{x_1, \dots, x_{j-1}\}), j = 2, \dots, n$. Then, we define $\varphi(F) = (x_j, p_j)_{j \in \{1,\dots,n\}} \in \mathbb{S}_{\{1,\dots,n\}}$, where $p_j = \mathbb{P}(X = x_j) > 0, j = 1, \dots, n$. Note that all of the steps of the value assignment in this case are unique, thus ensuring injectivity in this case.

Case 2: $\min(\operatorname{supp}(F))$ exists, but $\max(\operatorname{supp}(F))$ does not.
The only set within $\mathbb{I}$ with existing minimum but non-existing maximum is $\mathbb{N}$. Similarly to Case 1, define $x_1 = \min(\operatorname{supp}(F))$ and $x_j = \min(\operatorname{supp}(F) \setminus \{x_1, \dots, x_{j-1}\}), j = 2, 3, \dots$. Then, the definition $\varphi(F) = (x_j, p_j)_{j \in \mathbb{N}} \in \mathbb{S}_{\mathbb{N}}$ with $p_j = \mathbb{P}(X = x_j) > 0, j = 1, 2, \dots$, is once again unique.

Case 3: $\max(\operatorname{supp}(F))$ exists, but $\min(\operatorname{supp}(F))$ does not.
This is analogous to Case 2 by simply swapping to roles of minima and maxima and replacing $\mathbb{N}$ by $(-\mathbb{N})$.

Case 4: $\min(\operatorname{supp}(F))$ and $\max(\operatorname{supp}(F))$ both do not exist.
The only set within $\mathbb{I}$ with non-existing minimum and non-existing maximum is $\mathbb{Z}$. We now define

$$
\begin{aligned}
x_0 &= F^{-1}(\tfrac{1}{2}) = \inf\{t \in \mathbb{R} : F(t) \geq \tfrac{1}{2}\} \in \operatorname{supp}(F), \\
x_j &= \min\{t \in \operatorname{supp}(F) \setminus \{x_0, \dots, x_{j-1}\} : F(t) \geq \tfrac{1}{2}\}, \quad j = 1, 2, \dots, \\
x_j &= \max\{t \in \operatorname{supp}(F) \setminus \{x_{j+1}, \dots, x_0\} : F(t) < \tfrac{1}{2}\}, \quad j = -1, -2, \dots .
\end{aligned}
$$

Defining $\varphi(F) = (x_j, p_j)_{j \in \mathbb{Z}} \in \mathbb{S}_{\mathbb{Z}}$ with $p_j = \mathbb{P}(X = x_j) > 0, j \in \mathbb{Z}$, now once again leads to a unique value assignment in this case.

It is ensured in every case separately that φ is well-defined and injective. Now let $A \in \mathbb{I}$ and $(x_j, p_j)_{j \in A} \in \mathbb{S}_A$. We define a cdf F by $\mathbb{P}(X = x_j) = p_j > 0, j \in A$. It follows directly that $\operatorname{supp}(F) = \{x_j : j \in A\}$ is order-isomorphic to $A \subseteq \mathbb{Z}$ with $|A| \geq 2$. This implies $F \in \mathcal{D}_0$ and following the above value assignment for φ yields $\varphi(F) = (x_j, p_j)_{j \in A}$. Thus, φ is surjective and therefore a bijection.

Before continuing, we define all single conditions in the definitions of $\leq_{\text{disp}}^{\wedge\text{-disc}}$ and $\leq_{\text{disp}}^{\vee\text{-disc}}$ as separate orders.

Definition A.1. Let $F, G \in \mathcal{D}_0$.

a) G is said to be *at least as discretely dispersed* as F *with respect to the probability mass*, denoted by $F \leq_{\text{D-pm}}^{\text{disc}} G$, if

$$q_b \leq p_a \quad \forall (a, b) \in R(\leftrightharpoons).$$

b) G is said to be *at least as $\wedge$-discretely dispersed* as F *with respect to the support*, denoted by $F \leq^{\wedge\text{-disc}}_{\text{D-supp}} G$, if

$$x_a - x_{a-1} \leq y_b - y_{b-1} \quad \forall (a,b) \in R(\leftrightharpoons_\wedge).$$

c) G is said to be *at least as $\vee$-discretely dispersed* as F *with respect to the support*, denoted by $F \leq^{\vee\text{-disc}}_{\text{D-supp}} G$, if

$$x_a - x_{a-1} \leq y_b - y_{b-1} \quad \forall (a,b) \in R(\leftrightharpoons_\vee).$$

Proof of Theorem 3.5. Since the implication holds in a more general setting, only the equivalence must be proven.

We start out by proving that $F(D_F) \subseteq G(D_G)$ already implies $F \leq^{\text{disc}}_{\text{D-pm}} G$. To this end, let $(a,b) \in R(\leftrightharpoons)$, i.e.

$$(F(x_{a-1}), F(x_a)) \cap (G(y_{b-1}), G(y_b)) \neq \emptyset. \tag{A.1}$$

By assumption, $G(y_b) > F(x_a)$ implies $F(x_a) = G(y_{b-1})$ and $G(y_{b-1}) < F(x_{a-1})$ implies $F(x_{a-1}) = G(y_b)$, so both cases contradict (A.1). This yields

$$(G(y_{b-1}), G(y_b)) \subseteq (F(x_{a-1}), F(x_a)). \tag{A.2}$$

Again by assumption, there exist $b_u \in B, b_\ell \in B \cup \{-\infty\}, b_\ell < b_u$ such that $F(x_a) = G(y_{b_u})$ and $F(x_{a-1}) = G(y_{b_\ell})$. By combining this with (A.2), we obtain $b, b-1 \in \{b_\ell, \ldots, b_u\}$ or, equivalently, $b \in \{b_\ell + 1, \ldots, b_u\}$. It follows

$$p_a = F(x_a) - F(x_{a-1}) = G(y_{b_u}) - G(y_{b_\ell}) = \sum_{j=b_\ell+1}^{b_u} q_j \geq q_b,$$

thus proving $F \leq^{\text{disc}}_{\text{D-pm}} G$.

It remains to be shown that $F \leq_{\text{disp}} G$ is equivalent to $F \leq^{\wedge\text{-disc}}_{\text{D-supp}} G$. Note that $F(D_F) = F(\text{supp}(F)) \setminus \{1\} = F(\text{supp}(F) \setminus \{\max(\text{supp}(F))\})$ holds as well as the analogous identity for G. The following equivalences hold:

$$\begin{aligned}
&F \leq_{\text{disp}} G \Leftrightarrow \lambda^1(F^{-1}(\{r\})) \leq \lambda^1(G^{-1}(\{r\})) \quad \forall r \in F(D_F) \\
&\Leftrightarrow \lambda^1(F^{-1}(\{F(x)\})) \leq \lambda^1(G^{-1}(\{G(y)\})) \\
&\qquad\qquad \forall x \in \text{supp}(F) \setminus \{\max(\text{supp}(F))\}, \\
&\qquad\qquad y \in \text{supp}(G) \setminus \{\max(\text{supp}(G))\} : F(x) = G(y) \\
&\Leftrightarrow \lambda^1(F^{-1}(\{F(x_a)\})) \leq \lambda^1(G^{-1}(\{G(y_b)\})) \ \forall a \in \overline{A}, b \in \overline{B} : F(x_a) = G(y_b) \\
&\Leftrightarrow \lambda^1([x_a, x_{a+1})) \leq \lambda^1([y_b, y_{b+1})) \qquad \forall a \in \overline{A}, b \in \overline{B} : F(x_a) = G(y_b) \\
&\Leftrightarrow x_a - x_{a-1} \leq y_b - y_{b-1} \qquad \forall a \in \underline{A}, b \in \underline{B} : F(x_{a-1}) = G(y_{b-1}),
\end{aligned}$$

where the first equivalence holds because of Proposition 3.15. Comparing the last equivalent characterization with the definition of $F \leq^{\wedge\text{-disc}}_{\text{D-supp}} G$ yields that

only the equivalence of $F(x_{a-1}) = G(y_{b-1})$ and $a \leftrightharpoons_\wedge b$ is left to prove for $a \in \underline{A}, b \in \underline{B}$. To this end, we use Proposition 3.17a) to obtain that, since $a \in \underline{A}$, $a \leftrightharpoons_\wedge b$ is equivalent to

$$\begin{aligned} G(y_{b-1}) \in \mathrm{NN}_F^G(a) &= \{\sup\left(G(D_G) \cap (0, F(x_{a-1})]\right), \inf\left(G(D_G) \cap [F(x_{a-1}), 1)\right)\} \\ &= \{F(x_{a-1})\}. \end{aligned}$$

Proof of Theorem 3.6. Let $H \mathrel{\hat{=}} (C, (z_j, r_j)_{j \in C})$. First, let $(a, c) \in R(\overset{F,H}{\leftrightharpoons})$, i.e.

$$(F(x_{a-1}), F(x_a)) \cap (H(z_{c-1}), H(z_c)) \neq \emptyset.$$

Furthermore, note that $\left(\bigcup_{b \in B} (G(y_{b-1}), G(y_b))\right) \cup G(D_G) = (0, 1)$ holds and the union is disjoint. Since $(F(x_{a-1}), F(x_a)) \cap (H(z_{c-1}), H(z_c))$ is a non-empty open sub-interval of $(0, 1)$, there exists a $b \in B$ such that

$$(F(x_{a-1}), F(x_a)) \cap (G(y_{b-1}), G(y_b)) \cap (H(z_{c-1}), H(z_c)) \neq \emptyset.$$

It follows that $a \overset{F,G}{\leftrightharpoons} b$ as well as $b \overset{G,H}{\leftrightharpoons} c$. By assumption, this yields $r_c \leq q_b \leq p_a$ and since $(a, c) \in R(\overset{F,H}{\leftrightharpoons})$ was arbitrary, the transitivity of $\leq^{\text{disc}}_{\text{D-pm}}$ follows.

Now let $(a, c) \in R(\overset{F,H}{\leftrightharpoons_\vee})$. We prove the existence of a $b \in \underline{B}$ such that $a \overset{F,G}{\leftrightharpoons_\vee} b$ and $b \overset{G,H}{\leftrightharpoons_\vee} c$ via case distinction.

Case 1: $a \leftrightharpoons c$

From the first part of the proof of b), we already know that there exists $b \in B$ such that $a \overset{F,G}{\leftrightharpoons} b$ and $b \overset{G,H}{\leftrightharpoons} c$. Note that $F(x_{a-1}) < G(y_b)$ follows from $a \overset{F,G}{\leftrightharpoons} b$. Assume now $b = \min B$ along with the existence of that minimum. This implies $G(y_{b-1}) = 0$ and therefore $\alpha \overset{F,G}{\leftrightharpoons} b$ for all $\alpha \in A \cap (-\infty, a]$. That latter set is not empty and includes at least the element $a - 1$ since $a > \min A$. We obtain

$$\begin{aligned} G(y_b) = G(y_b) - G(y_{b-1}) &= q_b \\ &\leq p_{a-1} = F(x_{a-1}) - F(x_{a-2}) \leq F(x_{a-1}), \end{aligned}$$

a contradiction, and therefore $b \neq \min B$ or, equivalently, $b \in \underline{B}$. Combining $a \overset{F,G}{\leftrightharpoons} b$ with $(a, b) \in \underline{A} \times \underline{B}$ yields $a \overset{F,G}{\leftrightharpoons_\vee} b$ and in the same way, combining $b \overset{G,H}{\leftrightharpoons} c$ with $(b, c) \in \underline{B} \times \underline{C}$ yields $b \overset{G,H}{\leftrightharpoons_\vee} c$.

Case 2: $a - 1 \leftrightharpoons c - 1$

Analogously to Case 1, there exists a $b \in \underline{B}$ such that $a - 1 \overset{F,G}{\leftrightharpoons} b - 1$ and $b - 1 \overset{G,H}{\leftrightharpoons} c - 1$. It follows directly that $a \overset{F,G}{\leftrightharpoons_\vee} b$ and $b \overset{G,H}{\leftrightharpoons_\vee} c$.

It follows that

$$x_a - x_{a-1} \leq y_b - y_{b-1} \leq z_c - z_{c-1},$$

which yields the transitivity of $\leq^{\vee\text{-disc}}_{\text{D-supp}}$ since $(a, c) \in R(\overset{F,H}{\leftrightharpoons_\vee})$ was arbitrary. Combined with the fact that $\leq^{\text{disc}}_{\text{D-pm}}$ is transitive, the transitivity of $\leq^{\vee\text{-disc}}_{\text{disp}}$ follows.

It remains to give a counterexample for the transitivity of $\leq_{\text{disp}}^{\wedge\text{-disc}}$. Since we already proved the transitivity of $\leq_{\text{D-pm}}^{\text{disc}}$, it needs to be a counterexample for the transitivity of $\leq_{\text{D-supp}}^{\wedge\text{-disc}}$. Let G be defined as F in Example 3.18b) and H be defined as G in that example. Let $F = \text{Bin}(1, \frac{1}{2})$ and

$$\begin{aligned} c_y &= y_2 - y_1 = y_3 - y_2, \\ c_z &= z_3 - z_2 = z_4 - z_3 = z_6 - z_5 = z_7 - z_6, \\ \delta &= z_2 - z_1 = z_5 - z_4 = z_8 - z_7. \end{aligned}$$

Set $c_y = 2$, $c_z = 3$ and $\delta = \frac{1}{2}$. We easily obtain

$$\begin{aligned} &R(\overset{F,G}{\leftrightharpoons_\wedge}) = \{(2,2),(2,3)\},\ R(\overset{G,H}{\leftrightharpoons_\wedge}) = \{(2,3),(2,4),(3,6),(3,7)\}, \\ &R(\overset{F,H}{\leftrightharpoons_\wedge}) = \{(2,5)\}. \end{aligned}$$

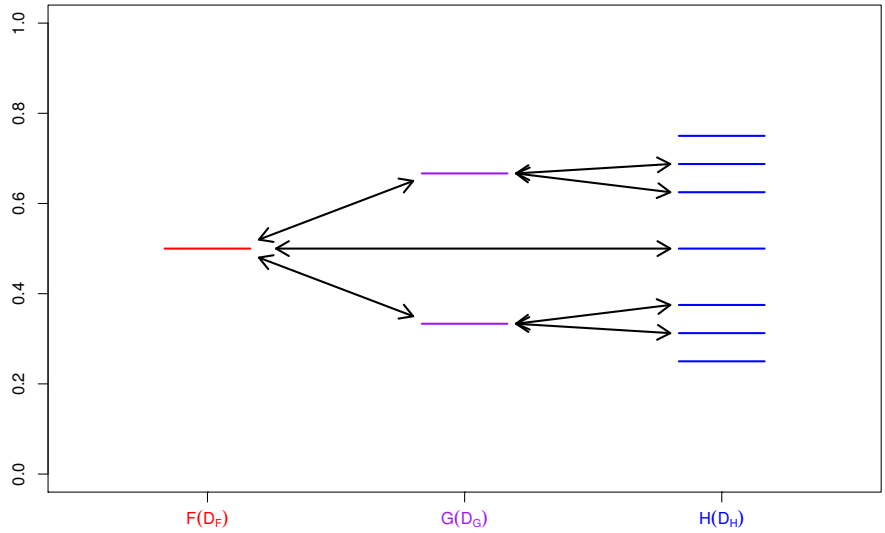

FIG A.1. *Illustration of the counterexample for the transitivity of $\leq_{D\text{-}supp}^{\wedge\text{-}disc}$ and $\leq_{disp}^{\wedge\text{-}disc}$ as given in the proof of Theorem 3.6. The pairs of constant intervals, of which the lengths are to be compared with respect to $\leftrightharpoons_\wedge$, are identified by double-sided arrows. Each constant interval is represented by the value that the corresponding cdf takes on there.*

An illustration of which pairs of constant intervals are compared among the three cdf's is given in Figure A.1. Because of

$$x_2 - x_1 = 1 \leq 2 = c_y = y_2 - y_1 = y_3 - y_2,$$

$F \leq_{\text{D-supp}}^{\wedge\text{-disc}} G$ holds, and since

$$y_2 - y_1 = c_y = 2 \leq 3 = c_z = z_3 - z_2 = z_4 - z_3 \quad \text{and}$$

$$y_3 - y_2 = c_y = 2 \leq 3 = c_z = z_6 - z_5 = z_7 - z_6,$$

$G \leq^{\wedge\text{-disc}}_{\text{D-supp}} H$ holds. However,

$$x_2 - x_1 = 1 > \tfrac{1}{2} = \delta = z_5 - z_4$$

contradicts $F \leq^{\wedge\text{-disc}}_{\text{D-supp}} H$ and thereby contradicts the transitivity of $\leq^{\wedge\text{-disc}}_{\text{D-supp}}$ and of $\leq^{\wedge\text{-disc}}_{\text{disp}}$.

Proof of Proposition 3.7. For $a, b \in A$, we obviously have $a \leftrightharpoons b$, if and only if $a = b$. Consequently, for all $a, b \in \underline{A}$, we have

$$\begin{aligned} a \leftrightharpoons_\wedge b &\iff a = b \text{ and } a - 1 = b - 1 \iff a = b, \\ a \leftrightharpoons_\vee b &\iff a = b \text{ or } a - 1 = b - 1 \iff a = b. \end{aligned}$$

Therefore, $F \leq^{\text{disc}}_{\text{D-pm}} F$ is equivalent to $p_a \leq p_a$ for all $a \in A$; and $F \leq^{\wedge\text{-disc}}_{\text{D-supp}} F$ is equivalent to $F \leq^{\vee\text{-disc}}_{\text{D-supp}} F$, which, in turn, is equivalent to $x_a - x_{a-1} \leq x_a - x_{a-1}$ for all $a \in \underline{A}$. Hence, $F \leq^{\wedge\text{-disc}}_{\text{disp}} F$ and $F \leq^{\vee\text{-disc}}_{\text{disp}} F$ both hold.

Proof of Proposition 3.8. Part a) follows directly from the fact that, under the lattice assumptions, the three statements $F \leq^{\wedge\text{-disc}}_{\text{D-supp}} G$, $F \leq^{\vee\text{-disc}}_{\text{D-supp}} G$, and $c_F \leq c_G$ are equivalent. With that noted, the transitivity of $\leq^{\wedge\text{-disc}}_{\text{D-supp}}$ follows, which also proves part b).

Proof of Theorem 3.9. We start by proving that (i) and (iii) are equivalent.

'(i)⇒(iii)': Let $r \in (0,1) \setminus (F(D_F) \cup G(D_G))$, which is possible since $F(D_F) \cup G(D_G)$ is at most countable and $(0,1)$ is uncountable. Now let $a_r = \min\{a \in A : F(x_a) \geq r\}$ (which exists since $r < 1 = \sup F(\text{supp}(F))$ holds and thus, there exists an $a \in A$ such that $r \in (F(x_{a-1}), F(x_a)))$ and $b_r = \min\{b \in B : G(y_b) \geq r\}$. Then, $a_r \leftrightharpoons b_r$ and, by assumption, $F(x_{a_r}) - F(x_{a_r-1}) = p_{a_r} = q_{b_r} = G(y_{b_r}) - G(y_{b_r-1})$ follows. Rearranging yields

$$F(x_{a_r}) - G(y_{b_r}) = F(x_{a_r-1}) - G(y_{b_r-1}).$$

Assume $F(x_{a_r}) > G(y_{b_r})$. It follows that $a_r \leftrightharpoons b_r + 1$, yielding $F(x_{a_r}) - F(x_{a_r-1}) = p_{a_r} = q_{b_r+1} = G(y_{b_r+1}) - G(y_{b_r})$. Note that $b_r + 1 \in B$ since $G(y_{b_r}) < F(x_{a_r}) \leq 1 = \sup G(\text{supp}(G))$. Because of $G(y_{b_r}) > F(x_{a_r-1})$, it follows that $G(y_{b_r+1}) > F(x_{a_r})$, thus yielding $a_r + 1 \leftrightharpoons b_r + 1$. Now the same line of reasoning applied to a_r and b_r can also be applied to $a_r + 1$ and $b_r + 1$. Inductively, it follows that $p_{a_r} = p_\alpha = q_\beta$ for all $\alpha \in A \cap [a_r, \infty)$ and all $\beta \in B \cap [b_r, \infty)$. We now know that there exist $c_A, c_B \in \mathbb{N}_0$ such that

$$\begin{aligned} F(x_{a_r}) + c_A \cdot p_{a_r} &= F(x_{a_r}) + \sum_{a_r+1}^{\sup A} p_\alpha = 1 = G(y_{b_r}) + \sum_{\beta=b_r+1}^{\sup B} p_\beta \\ &= G(y_{b_r}) + c_B \cdot p_{a_r}, \end{aligned}$$

which, since $0 < F(x_{a_r}) - G(y_{b_r}) < p_{a_r}$ (otherwise $a_r \leftrightharpoons b_r$ would not hold), yields

$$0 = F(x_{a_r}) - G(y_{b_r}) + p_{a_r}(c_A - c_B) \neq 0,$$

a contradiction. By symmetry, the case $G(y_{b_r}) > F(x_{a_r})$ also yields a contradiction, leaving only $F(x_{a_r}) = G(y_{b_r})$. It immediately follows that $a_r \leftrightharpoons b_r$ and $a_r + 1 \leftrightharpoons b_r + 1$, yielding $p_{a_r} = q_{b_r}$ and $p_{a_r+1} = q_{b_r+1}$. This also yields $F(x_{a_r-1}) = G(y_{b_r-1})$ and $F(x_{a_r+1}) = G(y_{b_r+1})$. Inductively, we obtain $p_{a_r+d} = q_{b_r+d}$ and $F(x_{a_r+d}) = G(y_{b_r+d})$ for all $d \in \mathbb{Z}$ such that $a_r + d \in A$ and $b_r + d \in B$. Furthermore, $a_r + d = \min A$ is equivalent to $b_r + d = \min B$ for all $d \in \mathbb{Z}$ and the same is true for the maximums of A and B. It follows that $F(\operatorname{supp}(F)) = G(\operatorname{supp}(G))$ and, since the indexing sets are uniquely determined by the supports, $A = B$ follows along with $a_r = b_r$. The sets of pairs of indices to be compared are given by $R(\leftrightharpoons) = \{(a, a) : a \in A\}$ and $R(\leftrightharpoons_\wedge) = \{(a, a) : a \in \underline{A}\}$.
Now define $\lambda = y_{a_r} - x_{a_r}$. Let $\alpha \in A$ and, without restriction, let $\alpha \geq a_r$. Then,

$$y_\alpha - x_\alpha = y_{a_r} - x_{a_r} + \sum_{j=a_r+1}^{\alpha} ((y_j - y_{j-1}) - (x_j - x_{j-1})) = y_{a_r} - x_{a_r} = \lambda$$

follows from $F =_{\text{disp}}^{\wedge\text{-disc}} G$. Overall, we obtain $(B, (y_j, q_j)_{j\in B}) = (A, (x_j + \lambda, p_j)_{j\in A})$ and since the indexing set and the identifying sequence uniquely identify the corresponding cdf, the assertion follows.

'(iii)⇐(i)': The assumption directly implies $B = A$, $y_j = x_j + \lambda$ and $q_j = p_j$ for all $j \in A$. The latter observation then implies $F(\operatorname{supp}(F)) = G(\operatorname{supp}(G))$ as well as $R(\leftrightharpoons) = \{(a, a) : a \in A\}$ and $R(\leftrightharpoons_\wedge) = \{(a, a) : a \in \underline{A}\}$. While $F =_{\text{D-pm}}^{\text{disc}} G$ is now trivial, $F =_{\text{D-supp}}^{\wedge\text{-disc}} G$ follows from $(x_a+\lambda)-(x_{a-1}+\lambda) = x_a - x_{a-1}$ for all $a \in \underline{A}$, thus concluding the proof.

Since $\leq_{\text{disp}}^{\wedge\text{-disc}}$ is a weakening of $\leq_{\text{disp}}^{\vee\text{-disc}}$, the implication '(ii)⇒(i)' is true generally. For the remaining implication, it is sufficient to show $F \leq_{\text{disp}}^{\wedge\text{-disc}} G \Leftrightarrow F \leq_{\text{disp}}^{\vee\text{-disc}} G$ for all $F, G \in \mathcal{D}_0$ with $F(\operatorname{supp}(F)) = G(\operatorname{supp}(G))$. Assuming $F(\operatorname{supp}(F)) = G(\operatorname{supp}(G))$ directly implies $R(\leftrightharpoons) = \{(a, a) : a \in A\}$, where A is the indexing set of either cdf. This yields

$$R(\leftrightharpoons_\wedge) = \{(a, a) : a \in \underline{A}\} = R(\leftrightharpoons_\vee),$$

thus ensuring the equivalence of $\leq_{\text{disp}}^{\wedge\text{-disc}}$ and $\leq_{\text{disp}}^{\vee\text{-disc}}$ and concluding the proof.

Proof of Proposition 3.10. If $F(D_F) \subseteq G(D_G)$, the order $\leq_{\text{disp}}^{\wedge\text{-disc}}$ is equivalent to $\leq_{\text{disp}}$ and the assertion follows. Otherwise, let $a \in \underline{A}$ such that $F(x_{a-1}) \in F(D_F) \setminus G(D_G) \neq \emptyset$. It follows that

$$\begin{aligned}\operatorname{NN}_F^G(a) \ni \sup(G(D_G) \cap (0, F(x_{a-1})]) &< F(x_{a-1}) \\ &< \inf(G(D_G) \cap [F(x_{a-1}), 1)) \in \operatorname{NN}_F^G(a).\end{aligned}$$

Hence, $|\mathrm{NN}_F^G(a)| = 2$, and we choose $b \in \overline{B}$ such that $\mathrm{NN}_F^G(a) = \{G(y_b), G(y_{b-1})\}$. (Note that $\overline{B} \neq \emptyset$ since $|B| \geq 3$ follows from Lemma A.2.) Particularly, this means that $a \leftrightharpoons_\wedge b+1$ and, therefore, $x_a - x_{a-1} \leq y_{b+1} - y_b$. Due to Lemma A.2 and the beginning of the proof of Proposition 3.17, for all $j \in \underline{A}, j \geq a$, there exists a $k_j \in \underline{B}, k_j \geq k_a = b+1$ such that $G(y_{k_j-1}) = \inf(G(D_G) \cap [F(x_{j-1}), 1)) \in \mathrm{NN}_F^G(j)$. It follows that $j \leftrightharpoons_\wedge k_j$ and, particularly, $j - 1 \leftrightharpoons k_j - 1$ for all $j \in \underline{A}, j \geq a$. Moreover, the k_j's are pairwise distinct. To see this, let $i, j \in \underline{A} \cap [a, \infty)$ with $i > j$; then we have

$$F(x_{j-1}) \leq F(x_{i-2}) = F(x_{i-1}) - p_{i-1} \leq G(y_{k_i-1}) - q_{k_i-1} = G(y_{k_i-2}),$$

which implies $G(y_{k_j-1}) \leq G(_{k_i-1})$ and $k_j \leq k_i - 1 < k_i$. It follows that

$$\sum_{j=a}^{\sup \underline{A}} (x_j - x_{j-1}) \leq \sum_{j=a}^{\sup \underline{A}} (y_{k_j} - y_{k_j-1}) \leq \sum_{k=b+1}^{\sup \underline{B}} (y_k - y_{k-1}). \tag{A.3}$$

Instead of using the index a as a starting point upwards, we can also use it as a starting point downwards. It follows from the structure of the set $\mathrm{NN}_F^G(a)$ that $a \leftrightharpoons_\wedge b$ holds and, subsequently, $x_a - x_{a-1} \leq y_b - y_{b-1}$. By Lemma A.2, we obtain that, for all $j \in \underline{A}, j \leq a$, there exists a $\ell_j \in \underline{B}, \ell_j \leq \ell_a = b$ such that $G(y_{\ell_j-1}) = \sup(G(D_G) \cap (0, F(x_{j-1})]) \in \mathrm{NN}_F^G(j)$. As before, it follows that $j \leftrightharpoons_\wedge \ell_j$ and, particularly, $j \leftrightharpoons \ell_j$ for all $j \in \underline{A}, j \leq a$. To see that the ℓ_j's are also pairwise distinct, let $i, j \in \underline{A} \cap (-\infty, a]$ with $i > j$, yielding

$$F(x_{i-1}) \geq F(x_j) = F(x_{j-1}) + p_j \geq G(y_{\ell_j-1}) + q_{\ell_j} = G(y_{\ell_j})$$

and, therefore, $\ell_i \geq \ell_j + 1 > \ell_j$. It follows that

$$\sum_{j=\inf \underline{A}}^{a} (x_j - x_{j-1}) \leq \sum_{j=\inf \underline{A}}^{a} (y_{\ell_j} - y_{\ell_j-1}) \leq \sum_{\ell=\inf \underline{B}}^{b} (y_\ell - y_{\ell-1}). \tag{A.4}$$

By combining (A.3) and (A.4), we obtain

$$\begin{aligned} \lambda^1(D_F) &= \sum_{j=\inf \underline{A}}^{\sup \underline{A}} (x_j - x_{j-1}) < \sum_{j=\inf \underline{A}}^{\sup \underline{A}} (x_j - x_{j-1}) + (x_a - x_{a-1}) \\ &\leq \sum_{k=\inf \underline{B}}^{\sup \underline{B}} (y_k - y_{k-1}) = \lambda^1(D_G), \end{aligned}$$

if $\lambda^1(D_F) < \infty$. Otherwise, we obtain that D_G also has infinite Lebesgue measure, since then the leftmost sum in (A.3) or (A.4) is already infinite.

Proof of Proposition 3.11. Similarly as in the proof of Proposition 3.10, Lemma A.2 states that for all $a \in \underline{A}$, there exists a $b_a \in \underline{B}$ such that $G(y_{b_a-1}) =$

$\sup(G(D_G) \cap (0, F(x_{a-1})]) \in \mathrm{NN}_F^G(a)$, implying $a \leftrightharpoons_\wedge b_a$. As shown for Proposition 3.10, these b_a's are pairwise distinct. It follows for all $a \in A$ that

$$x_a = \min(\mathrm{supp}(F)) + \sum_{j=\min \underline{A}}^{a} (x_j - x_{j-1}) \leq \min(\mathrm{supp}(G)) + \sum_{j=\min \underline{A}}^{a} (y_{b_j} - y_{b_j-1})$$
$$\leq \min(\mathrm{supp}(G)) + \sum_{k=\min \underline{B}}^{b_a} (y_k - y_{k-1}) = y_{b_a}.$$

Note that for all $(a,b) \in R(\leftrightharpoons)$, $b \geq b_a$ holds because of $G(y_b) > F(x_{a-1}) \geq G(y_{b_a-1})$ and the maximality of b_a. This means that, for all $(a,b) \in R(\leftrightharpoons)$, we obtain $x_a \leq y_{b_a} \leq y_b$. By considering Definition 3.3 and the proof of Proposition 3.13, it is easy to see that this is equivalent to $F \leq_{\mathrm{st}} G$.

Proof of Proposition 3.13. The following chain of equivalences proves the assertion:

$$\begin{aligned}
& g(G^{-1}(p)) \leq f(F^{-1}(p)) && \forall p \in (0,1) \qquad (A.5)\\
\Leftrightarrow\ & g(y_b) \leq f(x_a) && \forall (a,b) \in A \times B \text{ such that } F^{-1}(p) = x_a,\\
& && \quad G^{-1}(p) = y_b \text{ for some } p \in (0,1)\\
\Leftrightarrow\ & g(y_b) \leq f(x_a) && \forall (a,b) \in R(\leftrightharpoons)\\
\Leftrightarrow\ & \mathbb{P}(Y = y_b) \leq \mathbb{P}(X = x_a) && \forall (a,b) \in R(\leftrightharpoons)\\
\Leftrightarrow\ & q_b \leq p_a && \forall (a,b) \in R(\leftrightharpoons).
\end{aligned}$$

Proof of Proposition 3.15. We start by proving the implication from left to right. Due to Proposition 2.5, only the inequality of the Lebesgue measures needs to be shown. This follows immediately as

$$\begin{aligned}
\lambda^1(F^{-1}(\{p\})) &= \lim_{r \searrow p} (F^{-1}(r) - F^{-1}(p))\\
&\leq \lim_{r \searrow p} (G^{-1}(r) - G^{-1}(p)) = \lambda^1(G^{-1}(\{p\})) \qquad (A.6)
\end{aligned}$$

holds for all $p \in F(D_F)$ by assumption since $F(D_F) \subseteq (0,1)$.

For the other implication, let $p, q \in (0,1), p < q$. Since F is discrete, the difference of its quantile function at p and q is equal to the summed lengths of all intervals, on which F is constant at a value between p and q. Thus,

$$F^{-1}(q) - F^{-1}(p) = \sum_{r \in F(D_F) \cap [p,q)} \lambda^1(F^{-1}(\{r\}))$$

and analogously for G. By assumption, we obtain

$$\begin{aligned}
&\big(G^{-1}(q) - G^{-1}(p)\big) - \big(F^{-1}(q) - F^{-1}(p)\big)\\
&\qquad = \sum_{r \in F(D_F) \cap [p,q)} \big(\lambda^1(G^{-1}(\{r\})) - \lambda^1(F^{-1}(\{r\}))\big)
\end{aligned}$$

$$+ \sum_{r\in(G(D_G)\setminus F(D_F))\cap[p,q)} \lambda^1(G^{-1}(\{r\})).$$

Since both of these summands are non-negative, the assertion follows.

As a prerequitite for the proof of Proposition 3.17, we need the following lemma.

Lemma A.2. *Let $F, G \in \mathcal{D}_0$ satisfy $F \leq^{disc}_{D\text{-}pm} G$.*

a) Let $a \in A$. Then, exactly one of the following two statements holds:

(i) $\exists b \in \overline{B} : F(x_{a-1}) < G(y_b) < F(x_a)$,

(ii) $\exists b \in B : F(x_{a-1}) = G(y_{b-1})$ and $F(x_a) = G(y_b)$.

b) For all $a \in \overline{A}$, there exist $b_1, b_2 \in \overline{B}$ such that $G(y_{b_1}) \leq F(x_a) \leq G(y_{b_2})$ holds.

Proof. a) We prove the equivalence $\neg(i) \Leftrightarrow (ii)$. Note that $\neg(i)$ is equivalent to $G(D_G) \cap (F(x_{a-1}), F(x_a)) = \emptyset$.

'$\Leftarrow$': It follows that $(F(x_{a-1}), F(x_a)) \cap G(D_G) = (G(y_{b-1}), G(y_b)) \cap G(D_G) = \emptyset$.

'$\Rightarrow$': We start by proving that there exists a $b_u \in B$ such that $G(y_{b_u}) \geq F(x_a)$. If $F(x_a) < 1$, this follows directly from the fact that $\sup G(\mathrm{supp}(G)) = 1$. If $F(x_a) = 1$, $\sup G(\mathrm{supp}(G)) = 1$ also implies $\max G(\mathrm{supp}(G)) = 1$ since, otherwise, $\lim_{\beta\to\infty} G(y_\beta) - G(y_{\beta-1}) = 0 < 1 - F(x_{a-1}) = F(x_a) - F(x_{a-1})$ along with $\lim_{\beta\to\infty} G(y_\beta) = 1$ would contradict $F \leq^{\mathrm{disc}}_{\mathrm{D\text{-}pm}} G$. Obviously, there also exists a $b_\ell \in B \cup \{\min B - 1\}$ such that $G(y_{b_\ell}) \leq F(x_{a-1})$.

Now, define $b \in B$ by $G(y_b) = \min(G(\mathrm{supp}(G)) \cap [F(x_a), 1]) \leq G(y_{b_u})$, yielding $a \leftrightharpoons b$. By assumption $G(y_{b-1}) \leq F(x_{a-1}) < F(x_a) \leq G(y_b)$ holds and it follows that $q_b \geq p_a$. Equality holds, if and only if $G(y_{b-1}) = F(x_{a-1})$ and $G(y_b) = F(x_a)$, which corresponds to (ii). If equality does not hold, $F \leq^{\mathrm{disc}}_{\mathrm{D\text{-}pm}} G$ is contradicted.

b) Let $a \in \overline{A}$. The assertion follows by applying part a) to both a and $a+1 \in A$, and by considering all four arising cases separately.

□

The following chain of inequalities follows directly from Lemma A.2:

$$\inf G(D_G) \leq \inf F(D_F) \leq \sup F(D_F) \leq \sup G(D_G).$$

Proof of Proposition 3.17. a) First, note that, for all $a \in \underline{A}$, it follows from Lemma A.2b) that $G(D_G) \cap (0, F(x_{a-1})] \neq \emptyset$ and $G(D_G) \cap [F(x_{a-1}), 1) \neq \emptyset$. Thus,

$$\mathrm{NN}^G_F(a) = \{\sup\left(G(D_G) \cap (0, F(x_{a-1})]\right), \inf\left(G(D_G) \cap [F(x_{a-1}), 1)\right)\}.$$

Because of $\sup G(\mathrm{supp}(G)) = 1$ and $F(x_{a-1}) < 1$, there exists a $b \in B$ such that $F(x_{a-1}) \in [G(y_{b-1}), G(y_b))$. It follows that

$$\sup(G(D_G) \cap (0, F(x_{a-1})]) = \max(G(D_G) \cap (0, F(x_{a-1})])$$

and, analogously,

$$\inf(G(D_G) \cap [F(x_{a-1}), 1)) = \min(G(D_G) \cap [F(x_{a-1}), 1)).$$

'$\supseteq$': Let $a \in \underline{A}$. Now $b-1 \in \overline{B}$ (or $b \in \underline{B}$) is defined uniquely by $G(y_{b-1}) = \max(G(D_G) \cap (0, F(x_{a-1})])$. If follows that $G(y_{b-1}) \leq F(x_{a-1}) < G(y_b)$ and thus, $a \leftrightharpoons b$. The case $G(y_{b-1}) = F(x_{a-1})$ is equivalent to

$$\min(G(D_G) \cap [F(x_{a-1}), 1)) = \max(G(D_G) \cap (0, F(x_{a-1})]) = F(x_{a-1}).$$

Then, there exists an $\varepsilon > 0$ such that

$$\begin{aligned}(F(x_{a-2}), F(x_{a-1})) &\ni F(x_{a-1}) - \varepsilon = G(y_{b-1}) - \varepsilon \in (G(y_{b-2}), G(y_{b-1})),\\ (F(x_{a-1}), F(x_a)) &\ni F(x_{a-1}) + \varepsilon = G(y_{b-1}) + \varepsilon \in (G(y_{b-1}), G(y_b)),\end{aligned}$$

thus yielding $a \leftrightharpoons_\wedge b$. The case $G(y_{b-1}) < F(x_{a-1})$ remains to be considered. It immediately follows that $a - 1 \leftrightharpoons b$. This, combined with $a \leftrightharpoons b$, yields

$$\begin{aligned}&G(y_b) - G(y_{b-1}) \leq F(x_{a-1}) - F(x_{a-2}),\\ &G(y_b) - G(y_{b-1}) \leq F(x_a) - F(x_{a-1})\\ &\implies F(x_{a-2}) < G(y_{b-1}) < F(x_{a-1}) < G(y_b) < F(x_a).\end{aligned}$$

It follows that $a - 1 \leftrightharpoons b - 1$ and $a \leftrightharpoons b + 1$, thus, $a \leftrightharpoons_\wedge b$ and $a \leftrightharpoons_\wedge b+1$. Since $G(y_{(b+1)-1}) = G(y_b) = \min(G(D_G) \cap [F(x_{a-1}), 1))$, this concludes the proof of the implication from right to left.

'$\subseteq$': Let $(a, b) \in R(\leftrightharpoons_\wedge) \subseteq \underline{A} \times \underline{B}$.

Case 1: $G(y_{b-1}) > F(x_{a-1})$
Under this assumption, Lemma A.2 states that $b - 1 > \min B$. From $a - 1 \leftrightharpoons b - 1$, we then obtain $G(y_{b-2}) < F(x_{a-1})$. Consequently, $G(y_{b-1}) = \min(G(D_G) \cap [F(x_{a-1}), 1)) \in \mathrm{NN}_F^G(a)$.

Case 2: $G(y_{b-1}) < F(x_{a-1})$
Similarly to Case 1, we have $a \leftrightharpoons b$, which yields $G(y_b) > F(x_{a-1})$ and $G(y_{b-1}) = \max(G(D_G) \cap (0, F(x_{a-1})]) \in \mathrm{NN}_F^G(a)$.

Case 3: $G(y_{b-1}) = F(x_{a-1})$
It immediately follows that

$$\begin{aligned}G(y_{b-1}) &= \min(G(D_G) \cap [F(x_{a-1}), 1))\\ &= \max(G(D_G) \cap (0, F(x_{a-1})]) \in \mathrm{NN}_F^G(a).\end{aligned}$$

b) '$\supseteq$': Let $b \in \underline{B}$. Assume first $F(D_F) \cap [G(y_{b-1}), 1) \neq \emptyset$. Then, with analogous reasoning to part a), there exists an $a \in \underline{A}$ such that

$$F(x_{a-1}) = \inf(F(D_F) \cap [G(y_{b-1}), 1)) = \min(F(D_F) \cap [G(y_{b-1}), 1)).$$

It follows that $F(x_{a-1}) \geq G(y_{b-1}) > F(x_{a-2})$, where it is possible that $a - 1 = \min A$ and therefore, $F(x_{a-2}) = 0$. Nonetheless, we obtain $a - 1 \leftrightharpoons b - 1$ and thus, $a \leftrightharpoons_\vee b$.
Now we assume $F(D_F) \cap (0, G(y_{b-1})] \neq \emptyset$ (which can occur simultaneously to $F(D_F) \cap [G(y_{b-1}), 1) \neq \emptyset$). Again analogously to part a), there exists an $a \in \underline{A}$ such that

$$F(x_{a-1}) = \sup(F(D_F) \cap (0, G(y_{b-1})]) = \max(F(D_F) \cap (0, G(y_{b-1})]).$$

We now infer $F(x_{a-1}) \leq G(y_{b-1}) < F(x_a)$, yielding $a \leftrightharpoons b$ and thereby $a \leftrightharpoons_\vee b$.

'$\subseteq$': Let $(a, b) \in R(\leftrightharpoons_\vee) \subseteq \underline{A} \times \underline{B}$.

Case 1: $F(x_{a-1}) < G(y_{b-1})$
If $a - 1 \leftrightharpoons b - 1$, it follows $G(y_{b-2}) < F(x_{a-1})$, which then yields $a \leftrightharpoons b - 1$. We obtain

$$\begin{aligned} F(x_a) &= F(x_{a-1}) + p_a \geq F(x_{a-1}) + q_{b-1} \\ &> G(y_{b-2}) + q_{b-1} = G(y_{b-1}). \end{aligned}$$

If $a \leftrightharpoons b$, it follows directly that $F(x_a) > G(y_{b-1})$. Since $a \leftrightharpoons_\vee b$ implies $a \leftrightharpoons b$ or $a - 1 \leftrightharpoons b - 1$, the inequality $F(x_a) > G(y_{b-1})$ holds generally. It yields $F(x_{a-1}) = \max(F(D_F) \cap (0, G(y_{b-1})]) \in \mathrm{NN}_G^F(b)$.

Case 2: $F(x_{a-1}) > G(y_{b-1})$
If $a \leftrightharpoons b$, it follows $G(y_b) > F(x_{a-1})$, which then yields $a - 1 \leftrightharpoons b$. We obtain

$$F(x_{a-2}) = F(x_{a-1}) - p_{a-1} \leq F(x_{a-1}) - q_b < G(y_b) - q_b = G(y_{b-1}).$$

If $a - 1 \leftrightharpoons b - 1$, it follows directly that $F(x_{a-2}) < G(y_{b-1})$. Thus, the implication $F(x_{a-1}) = \min(F(D_F) \cap [G(y_{b-1}), 1)) \in \mathrm{NN}_G^F(b)$ generally holds.

Case 3: $F(x_{a-1}) = G(y_{b-1})$
Similarly to part a), it immediately follows that

$$\begin{aligned} F(x_{a-1}) &= \min(F(D_F) \cap [G(y_{b-1}), 1)) \\ &= \max(F(D_F) \cap (0, G(y_{b-1})]) \in \mathrm{NN}_G^F(b). \end{aligned}$$

A.2. Proofs for Section 4

For the proofs in this section, we introduce further notation. By Definition 3.3, every pair $(a, b) \in R(\leftrightharpoons)$ is associated with a non-empty interval subset of the

unit interval. We define

$$r_{(a,b)} = \lambda^1\big((F(x_{a-1}), F(x_a)) \cap (G(y_{b-1}), G(y_b))\big) \tag{A.7}$$

for all $(a,b) \in A \times B$ as the length of that interval. Note that $r_{(a,b)} > 0$ is equivalent to $a \leftrightharpoons b$.

Moreover, there exists a union N of at most countably many atoms in $(0,1)$ such that

$$N \cup \bigcup_{(a,b)\in R(\leftrightharpoons)} (F(x_{a-1}), F(x_a)) \cap (G(y_{b-1}), G(y_b)) = (0,1),$$

with all of the unions on the left hand side being disjoint. Specifically, $N = F(D_F) \cup G(D_G)$. For any two pairs $(a,b), (\alpha,\beta) \in R(\leftrightharpoons)$ with $(a,b) \neq (\alpha,\beta)$, we say that (α,β) is *higher (lower) than* (a,b) if $\gamma > c$ ($\gamma < c$) holds for all $\gamma \in (F(x_{\alpha-1}), F(x_\alpha)) \cap (G(y_{\beta-1}), G(y_\beta))$ and all $c \in (F(x_{a-1}), F(x_a)) \cap (G(y_{b-1}), G(y_b))$. One of these two situations is guaranteed to hold since the interval associated with (a,b) and the interval associated with (α,β) are disjoint.

Proof of Theorem 4.1. Let $(a,b), (\alpha,\beta) \in R(\leftrightharpoons)$. For better cross-reference, we divide the proof into three parts.

Part 1: In this part, we show by contradiction that $|x_\alpha - x_a| > |y_\beta - y_b|$ implies $|\alpha - a| = |\beta - b| + 1$. To this end, we first assume $|\alpha - a| \leq |\beta - b|$. Without restriction, let $\alpha \geq a$. If $\alpha = a$, then $|x_\alpha - x_a| = 0 \leq |y_\beta - y_b|$ follows, contradicting the assumption. Hence, it remains to consider the case $\alpha > a$. Because of $a \leftrightharpoons b$ and $\alpha \leftrightharpoons \beta$, we obtain

$$G(y_{b-1}) < F(x_a) \leq F(x_{\alpha-1}) < G(y_\beta), \tag{A.8}$$

yielding $\beta \geq b$. We either have $G(y_{\beta-1}) \leq F(x_{\alpha-1})$, yielding $G(y_{\beta-1}) = \sup(G(D_G) \cap (0, F(x_{\alpha-1}))) \in \mathrm{NN}_F^G(\alpha)$; in this case we define $k_\alpha = \beta$. Or we have $G(y_{\beta-1}) > F(x_{\alpha-1})$, implying that there exists a $k_\alpha \leq \beta$ such that $G(y_{k_\alpha-1}) = \inf(G(D_G) \cap [F(x_{\alpha-1}), 1)) \in \mathrm{NN}_F^G(\alpha)$. Note that $k_\alpha > b$ holds because of $G(y_{k_\alpha-1}) \geq F(x_{\alpha-1})$ and (A.8). So, considering both cases as well as Proposition 3.17a), there exists a $k_\alpha \in \{b+1, \ldots, \beta\}$ such that $\alpha \leftrightharpoons_\wedge k_\alpha$.

It follows that $\alpha - 1 \leftrightharpoons k_\alpha - 1$ and we can repeat the line of argument from above with $\alpha - 1$ taking the role of α and $k_\alpha - 1$ taking the role of β, as long as $\alpha - 1 > a$. We then obtain a $k_{\alpha-1} \in \{b+1, \ldots, k_\alpha - 1\}$ such that $\alpha - 1 \leftrightharpoons_\wedge k_{\alpha-1}$ and can repeat the line of argument again, starting with $\alpha - 2 \leftrightharpoons k_{\alpha-1} - 1$ as long as $\alpha - 2 > a$. Iteratively, we obtain the following statement:

$\forall j \in \{a+1, \ldots, \alpha\}$ $\exists$ pairwise distinct $k_j \in \{b+1, \ldots, \beta\}$ such that $j \leftrightharpoons_\wedge k_j$.

Since $F \leq^{\wedge\text{-disc}}_{\text{D-supp}} G$ holds, it follows

$$|x_\alpha - x_a| = x_\alpha - x_a = \sum_{j=a+1}^{\alpha} (x_j - x_{j-1})$$

$$\leq \sum_{j=a+1}^{\alpha} (y_{k_j} - y_{k_j-1})$$
$$\leq \sum_{k=b+1}^{\beta} (y_k - y_{k-1}) = y_\beta - y_b = |y_\beta - y_b|,$$

a contradiction to the assumption $|x_\alpha - x_a| > |y_\beta - y_b|$. This closes the case $|\alpha - a| \leq |\beta - b|$.
Second, we assume $|\alpha - a| \geq |\beta - b| + 2$. Again, let $\alpha \geq a$. Because of $\alpha - a = |\alpha - a| \geq |\beta - b| + 2 \geq 2$, we have $\alpha > a$. By combining $a \leftrightharpoons b$ and $\alpha \leftrightharpoons \beta$, we obtain (A.8) and $\beta \geq b$, as before. It follows from (A.8) that, for every $j \in \{a+1, \ldots, \alpha-1\}$, there exists a $k \in \{b, \ldots, \beta\}$ such that $j \leftrightharpoons k$. More specifically, we define $k_j = \min\{k \in \{b, \ldots \beta\} : j \leftrightharpoons k\}$ for every $j \in \{a+1, \ldots, \alpha-1\}$. These k_j are pairwise distinct; otherwise there would exist indices $i, j \in \{a+1, \ldots, \alpha-1\}, i \neq j$ (without restriction $i < j$) such that $k_i = k_j$. This implies $G(y_{k_i-1}) \leq F(x_{i-1})$ because of the minimality of k_i. We obtain

$$G(y_{k_i}) = G(y_{k_i-1}) + q_{k_i} \leq F(x_{i-1}) + p_i = F(x_i) \leq F(x_{j-1}) < G(y_{k_j}),$$

yielding a contradiction, and thereby proving that the mapping $j \mapsto k_j$ is injective. The cardinal number of the domain $\{a+1, \ldots, \alpha-1\}$ of that mapping is $|\alpha - a| - 1$ and by assumption larger than or equal to the cardinal number $|\beta - b| + 1$ of its codomain $\{b, \ldots, \beta\}$. For $|\alpha - a| > |\beta - b| + 2$, this directly contradicts the injectivity of the mapping. For $|\alpha - a| = |\beta - b| + 2$, it follows that the mapping is bijective and we obtain

$$F(x_{\alpha-1}) - F(x_a) = \sum_{j=a+1}^{\alpha-1} p_j \geq \sum_{j=a+1}^{\alpha-1} q_{k_j} = \sum_{k=b}^{\beta} q_k = G(y_\beta) - G(y_{b-1}).$$

This, however, contradicts (A.8). Thus, we have shown the implication

$$|x_\alpha - x_a| > |y_\beta - y_b| \Longrightarrow |\alpha - a| = |\beta - b| + 1 \quad \forall (a,b), (\alpha,\beta) \in R(\leftrightharpoons) \quad \text{(A.9)}$$

by excluding all other possibilities.

Part 2: It becomes apparent in Part 3 of the proof that pairs $(a,b), (\alpha,\beta) \in R(\leftrightharpoons)$ satisfying $|x_\alpha - x_a| \leq |y_\beta - y_b|$ are easy to deal with. Therefore, the critical situation $|x_\alpha - x_a| > |y_\beta - y_b|$ is of particular interest. Considering in the final result (A.9) of Part 1, this implies $|\alpha - a| = |\beta - b| + 1$. It is the purpose of Part 2 to analyze the situation for these kinds of pairs more closely, so let $(a,b), (\alpha,\beta) \in R(\leftrightharpoons)$ with $|\alpha - a| = |\beta - b| + 1$. Furthermore, let $\alpha > a$. As in Part 1, we obtain (A.8) and thereby $\beta \geq b$. It follows from (A.8) and $a \leftrightharpoons b$ that, for every $j \in \{a, \ldots, \alpha-1\}$, there exists a $k \in \{b, \ldots, \beta\}$ such that $j \leftrightharpoons k$. We consider the mapping

$$\varphi_1 : \{a, \ldots, \alpha-1\} \to \{b, \ldots, \beta\}, \quad j \mapsto k_j = \max\{k \in \{b, \ldots \beta\} : j \leftrightharpoons k\}.$$

To see that φ_1 is strictly increasing, let $j, j+1 \in \{a, \dots, \alpha-1\}$. It follows from the maximality of k_{j+1} that $G(y_{k_{j+1}}) \geq F(x_{j+1})$, yielding

$$G(y_{k_{j+1}-1}) = G(y_{k_{j+1}}) - q_{k_{j+1}} \geq F(x_{j+1}) - p_{j+1} = F(x_j) > G(y_{k_j-1})$$

and, thereby, $k_{j+1} > k_j$. Thus, φ_1 is strictly increasing and therefore also injective. Since $\alpha - a - 1 = \beta - b$ means that the domain and the codomain of φ_1 have the same cardinal number, the mapping is bijective. This means that

$$a \leftrightharpoons b,\ a+1 \ \leftrightharpoons\ b+1,\ \dots,\ \alpha-2 \ \leftrightharpoons\ \beta-1,\ \alpha-1 \ \leftrightharpoons\ \beta, \tag{A.10}$$

$$a+1 \leftrightharpoons_\wedge b+1,\ \dots,\ \alpha-2 \leftrightharpoons_\wedge \beta-1,\ \alpha-1 \leftrightharpoons_\wedge \beta. \tag{A.11}$$

Combined with (A.8), this has the following two implications:

$$\begin{aligned} G(y_{b-1}) - F(x_{a-1}) &= \left(G(y_\beta) - \sum_{k=b}^{\beta} q_k\right) - \left(F(x_{\alpha-1}) - \sum_{j=a}^{\alpha-1} p_j\right) \\ &= G(y_\beta) - F(x_{\alpha-1}) + \sum_{j=a}^{\alpha-1} (p_j - q_{k_j}) \geq G(y_\beta) - F(x_{\alpha-1}) > 0, \end{aligned} \tag{A.12}$$

$$\begin{aligned} x_{\alpha-1} - x_a &= \sum_{j=a+1}^{\alpha-1} (x_j - x_{j-1}) \\ &\leq \sum_{j=a+1}^{\alpha-1} (y_{k_j} - y_{k_j-1}) = \sum_{k=b+1}^{\beta} (y_k - y_{k-1}) = y_\beta - y_b. \end{aligned} \tag{A.13}$$

It follows from (A.8), $a \leftrightharpoons b$ and $\alpha \leftrightharpoons \beta$ that, for every $j \in \{a, \dots, \alpha\}$, there exists a $k \in \{b-1, \dots, \beta\}$ such that $j \leftrightharpoons k$. Note that $b - 1 \in B$ because of (A.12), which implies $G(y_{b-1}) > F(x_{a-1}) \geq 0$. We now consider the mapping

$$\varphi_2 : \{a, \dots, \alpha\} \to \{b-1, \dots, \beta\}, \quad j \mapsto k_j = \min\{k \in \{b-1, \dots \beta\} : j \leftrightharpoons k\}.$$

To see that φ_2 is strictly increasing, let $j, j+1 \in \{a, \dots, \alpha\}$. It follows from the minimality of k_j that $G(y_{k_j-1}) \leq F(x_{j-1})$, yielding

$$G(y_{k_j}) = G(y_{k_j-1}) + q_{k_j} \leq F(x_{j-1}) + p_j = F(x_j) = F(x_{(j+1)-1}) < G(y_{k_{j+1}})$$

and, thereby, $k_j < k_{j+1}$. Thus, φ_2 is strictly increasing and therefore also injective. Since $\alpha - a = \beta - b + 1$ means that the domain and the codomain of φ_2 have the same cardinal number, the mapping is bijective. This means that

$$a \leftrightharpoons b-1,\ a+1 \ \leftrightharpoons\ b,\ \dots,\ \alpha-1 \ \leftrightharpoons\ \beta-1,\ \alpha \ \leftrightharpoons\ \beta, \tag{A.14}$$

$$a+1 \leftrightharpoons_\wedge b,\ \ldots,\ \alpha-1 \leftrightharpoons_\wedge \beta-1,\ \alpha \leftrightharpoons_\wedge \beta. \tag{A.15}$$

Combined with (A.8), this has the following two implications:

$$\begin{aligned} F(x_{\alpha-1}) - G(y_{\beta-1}) &= \left(F(x_a) + \sum_{j=a+1}^{\alpha-1} p_j\right) - \left(G(y_{b-1}) + \sum_{k=b}^{\beta-1} q_k\right) \\ &= F(x_a) - G(y_{b-1}) + \sum_{j=a+1}^{\alpha-1} (p_j - q_{k_j}) \geq F(x_a) - G(y_{b-1}) > 0, \end{aligned} \tag{A.16}$$

and

$$\begin{aligned} x_\alpha - x_a &= \sum_{j=a+1}^{\alpha} (x_j - x_{j-1}) \leq \sum_{j=a+1}^{\alpha} (y_{k_j} - y_{k_j-1}) \\ &= \sum_{k=b}^{\beta} (y_k - y_{k-1}) = y_\beta - y_{b-1}. \end{aligned} \tag{A.17}$$

Part 3: Recall the definition of $r_{(a,b)}$ in (A.7) to see that the following holds true for all $t \in \mathbb{R}$:

$$\begin{aligned} H_{|X-X'|}(t) &= \mathbb{P}(|X - X'| \leq t) \\ &= \sum_{x,x' \in \operatorname{supp}(F)} \mathbb{P}(X = x)\mathbb{P}(X' = x')\mathbb{1}\{|x - x'| \leq t\} \\ &= \sum_{a,\alpha \in A} p_a p_\alpha \mathbb{1}\{|x_a - x_\alpha| \leq t\} \\ &= \sum_{a,\alpha \in A} \left(\sum_{b \in B: a \leftrightharpoons b} r_{(a,b)}\right)\left(\sum_{\beta \in B: \alpha \leftrightharpoons \beta} r_{(\alpha,\beta)}\right) \mathbb{1}\{|x_a - x_\alpha| \leq t\} \\ &= \sum_{(a,b),(\alpha,\beta) \in R(\leftrightharpoons)} r_{(a,b)} r_{(\alpha,\beta)} \mathbb{1}\{|x_a - x_\alpha| \leq t\}. \end{aligned} \tag{A.18}$$

Analogously, we obtain

$$H_{|Y-Y'|}(t) = \mathbb{P}(|Y - Y'| \leq t) = \sum_{(a,b),(\alpha,\beta) \in R(\leftrightharpoons)} r_{(a,b)} r_{(\alpha,\beta)} \mathbb{1}\{|y_b - y_\beta| \leq t\} \tag{A.19}$$

for all $t \in \mathbb{R}$. The claim of the theorem is equivalent to

$$\begin{aligned} 0 &\leq H_{|X-X'|}(t) - H_{|Y-Y'|}(t) \\ &= \sum_{(a,b),(\alpha,\beta) \in R(\leftrightharpoons)} r_{(a,b)} r_{(\alpha,\beta)} \big(\mathbb{1}\{|x_\alpha - x_a| \leq t\} - \mathbb{1}\{|y_\beta - y_b| \leq t\}\big) \end{aligned} \tag{A.20}$$

for all $t \in \mathbb{R}$. By Part 1, we have for all $(a,b),(\alpha,\beta) \in R(\leftrightharpoons)$ that

$$\begin{aligned}|\alpha - a| \neq |\beta - b| + 1 &\Longrightarrow |x_\alpha - x_a| \leq |y_\beta - y_b| \\ &\iff \big(|y_\beta - y_b| \leq t \Longrightarrow |x_\alpha - x_a| \leq t\big) \quad \forall t \in \mathbb{R} \\ &\iff \mathbb{1}\{|x_\alpha - x_a| \leq t\} - \mathbb{1}\{|y_\beta - y_b| \leq t\} \geq 0 \quad \forall t \in \mathbb{R}.\end{aligned}$$

The sums in (A.18) and (A.19) can each be split into three separate sums, one with $\alpha > a$, one with $\alpha = a$ and one with $\alpha < a$. Since all the summands in (A.18) and (A.19) are symmetric in (a,b) and (α,β), the sum with $\alpha > a$ is equal to the sum with $\alpha < a$. Because $\alpha = a$ and $|\alpha - a| = |\beta - b| + 1$ are not possible simultaneously, we obtain for all $t \in \mathbb{R}$ that

$$\begin{aligned}&H_{|X-X'|}(t) - H_{|Y-Y'|}(t) \\ &\quad = 2 \sum_{\substack{(a,b),(\alpha,\beta)\in R(\leftrightharpoons): \\ \alpha > a}} r_{(a,b)} r_{(\alpha,\beta)} \big(\mathbb{1}\{|x_\alpha - x_a| \leq t\} - \mathbb{1}\{|y_\beta - y_b| \leq t\}\big) \\ &\qquad + \sum_{\substack{(a,b),(\alpha,\beta)\in R(\leftrightharpoons): \\ \alpha = a}} r_{(a,b)} r_{(\alpha,\beta)} \big(\mathbb{1}\{|x_\alpha - x_a| \leq t\} - \mathbb{1}\{|y_\beta - y_b| \leq t\}\big) \\ &\quad \geq 2 \sum_{\substack{(a,b),(\alpha,\beta)\in R(\leftrightharpoons): \\ |\alpha - a| = |\beta - b| + 1, \\ \alpha > a}} \Big[r_{(a,b)} r_{(\alpha,\beta)} \big(\mathbb{1}\{x_\alpha - x_a \leq t\} - \mathbb{1}\{y_\beta - y_b \leq t\}\big) \\ &\qquad + r_{(a,b-1)} r_{(\alpha-1,\beta)} \big(\mathbb{1}\{x_{\alpha-1} - x_a \leq t\} - \mathbb{1}\{y_\beta - y_{b-1} \leq t\}\big) \Big].\end{aligned} \tag{A.21}$$

For the validity of the inequality we use that, under the assumptions $(a,b),(\alpha,\beta) \in R(\leftrightharpoons)$, $|\alpha - a| = |\beta - b| + 1$ and $\alpha > a$, it follows that $(a,b-1),(\alpha-1,\beta) \in R(\leftrightharpoons)$ according to (A.10) and (A.14). Note that $|(\alpha - 1) - a| \neq |\beta - (b-1)| + 1$ holds true in all summands of the last sum, so no summand is used twice in that sum. For the differences of the indicator functions, it holds that

$$\begin{aligned}&\mathbb{1}\{x_\alpha - x_a \leq t\} - \mathbb{1}\{y_\beta - y_b \leq t\} = -1 \iff t \in [y_\beta - y_b, x_\alpha - x_a) \text{ and} \\ &\mathbb{1}\{x_{\alpha-1} - x_a \leq t\} - \mathbb{1}\{y_\beta - y_{b-1} \leq t\} = 1 \iff t \in [x_{\alpha-1} - x_a, y_\beta - y_{b-1})\end{aligned}$$

for all $(a,b),(\alpha,\beta) \in R(\leftrightharpoons)$ with $|\alpha - a| = |\beta - b| + 1$ and $\alpha > a$. Because of (A.13) and (A.17), $[y_\beta - y_b, x_\alpha - x_a) \subseteq [x_{\alpha-1} - x_a, y_\beta - y_{b-1})$ follows, yielding

$$\begin{aligned}&\mathbb{1}\{x_\alpha - x_a \leq t\} - \mathbb{1}\{y_\beta - y_b \leq t\} = -1 \\ \Longrightarrow &\mathbb{1}\{x_{\alpha-1} - x_a \leq t\} - \mathbb{1}\{y_\beta - y_{b-1} \leq t\} = 1.\end{aligned}$$

This means that in order to prove that (A.21) is larger than or equal to zero and thereby complete the proof, it remains to show that $r_{(a,b)} r_{(\alpha,\beta)} \leq$

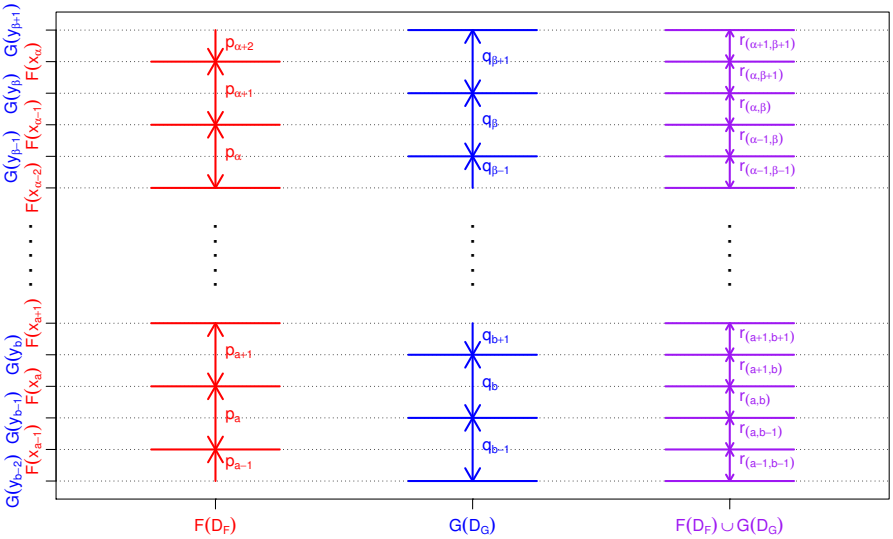

FIG A.2. *Exemplary visualization of crucial situation in the proof of Theorem 4.1. The horizontal lines represent the elements of the sets given on the x-axis. The double-sided arrows and the variable names next to them represent the distances between those horizontal lines.*

$r_{(a,b-1)}r_{(\alpha-1,\beta)}$ holds for all $(a,b),(\alpha,\beta) \in R(\leftrightharpoons)$ with $|\alpha-a| = |\beta-b|+1$ and $\alpha > a$. In that setting, by combining (A.12) and (A.16), we obtain

$$\begin{aligned} r_{(a,b)}r_{(\alpha,\beta)} &= \big(F(x_a) - G(y_{b-1})\big)\big(G(y_\beta) - F(y_{\alpha-1})\big) \\ &\leq \big(F(x_{\alpha-1}) - G(y_{\beta-1})\big)\big(G(y_{b-1}) - F(x_{a-1})\big) = r_{(\alpha-1,\beta)}r_{(a,b-1)}. \end{aligned}$$

The starting point of the proof of Theorem 4.1 is the representation of the cdf of $|X-X'|$ given in (A.18) as a sum of products $r_{(a,b)}r_{(\alpha,\beta)}$ of probability masses on two given pairs in $R(\leftrightharpoons)$. An analogous sum can be used to represent the cdf of $|Y-Y'|$ with the sole difference that each summand is multiplied with a different indicator than for $|X-X'|$. The implication (A.9), which is shown in Part 1 of the proof, states the indicators for a given summand are one for $|X-X'|$ and zero for $|Y-Y'|$ if and only if $|\alpha-a| = |\beta-b|+1$, which is a crucial restriction. In Part 3 of the proof, it is shown that, for every two pairs of indices $(a,b),(\alpha,\beta)$ for which this occurs, there exists another set of indices $(a,b-1),(\alpha-1,\beta)$ for which the indicators take the opposite values. Additionally, it is shown that $r_{(a,b-1)}r_{(\alpha-1,\beta)}$, the coefficient for the latter set, is always larger than $r_{(a,b)}r_{(\alpha,\beta)}$, the coefficient for the former set. So for every negative summand in the difference of the two cdf's, there exists a corresponding positive summand with equal or larger absolute value. The situation is illustrated in Figure A.2. The fact that the values of $F(D_F)$ and $G(D_G)$ in question are alternating, follows from $|\alpha-a| = |\beta-b|+1$ and is derived in Part 2 of the proof. The plot also hints at the fact that, for reasons of symmetry, $(a+1,b),(\alpha,\beta+1)$ would have been

a viable alternative of $(a, b-1), (\alpha-1, \beta)$ in its role as compensatory set of indices.

Proof of Lemma 4.2. Let $X \sim F$, $Y \sim G$ and, as assumed $\mathbb{E}[X] = \mathbb{E}[Y]$ with $F \neq G$.

a) For all $a \in A$, define the set $B_a = \{b \in \underline{B} : F(x_{a-1}) \leq G(y_{b-1}) \leq F(x_a)\}$. Note that, for all $a \in \underline{A}$, $B_a \neq \emptyset$ holds. Otherwise, there would exist a $b \in \underline{B}$ such that $G(y_{b-1}) < F(x_{a-1}) < F(x_a) < G(y_b)$, yielding $q_b > p_a$ in spite of $a \leftrightharpoons b$ and thus contradicting $F \leq^{\wedge\text{-disc}}_{\text{disp}} G$. With similar reasoning, $B_{\min A} \neq \emptyset$ follows (provided that the minimum exists). We now prove part a) by contradiction and therefore assume that $x_a \leq y_{b-1}$ or $y_b < x_a$ holds for all $b \in B_a$ and all $a \in A$. This is contradicted by case distinction.

Case 1: $x_a \leq y_{b-1}\ \forall (a,b) \in \bigcup_{a \in A} (\{a\} \times B_a)$
Let $a \in \underline{A}$. Obviously, $a \leftrightharpoons b$ holds for all $b \in B_a$, except if $G(y_{b-1}) = F(x_a)$. More precisely, if there exists a $b \in B$ such that $F(x_{a-1}) = G(y_{b-1})$, then $\{b \in B : a \leftrightharpoons b\} \subseteq B_a$. Otherwise, we have $\{b \in B : a \leftrightharpoons b\} \subseteq B_a \cup \{\min B_a - 1\}$. Overall, $x_a \leq y_b$ follows for all $(a,b) \in R(\leftrightharpoons) \cap (\underline{A} \times B)$. (This is because, for all $a \in \underline{A}$, we have $x_a \leq y_{b-1} < y_b$ for $b \in B_a$ and $x_a \leq y_{\min B_a - 1}$ by assumption.)
Now, let $a = \min A$ and assume that this minimum exists. Similarly as before, we have

$$\{b \in B : a \leftrightharpoons b\} \subseteq B_a \cup \{\min B_a - 1\} = B_a \cup \{\min B\}$$

and we can infer $x_a \leq y_b$ for all $(a,b) \in R(\leftrightharpoons) \cap (\{\min A\} \times B)$. Combined with the results for $a \in \underline{A}$, it follows that $x_a \leq y_b$ holds for all $(a,b) \in R(\leftrightharpoons)$. Furthermore, at least one of these inequalities is strict since equality for all these pairs would imply $F = G$. It follows

$$0 = \mathbb{E}[Y] - \mathbb{E}[X] = \sum_{(a,b) \in R(\leftrightharpoons)} r_{(a,b)}(y_b - x_a) > 0,$$

a contradiction.

Case 2: $y_b < x_a\ \forall (a,b) \in \bigcup_{a \in A} (\{a\} \times B_a)$
Let $a \in A$. Analogously to Case 1, it can be shown that $\{b \in B : a \leftrightharpoons b\} \subseteq B_a \cup \{\min B_a - 1\}$. If $b \in B_a$, $y_b < x_a$ holds by assumption; if $b = \min B_a - 1$, $y_b < y_{\min B_a} < x_a$ holds. Overall, we have $y_b < x_a$ for all $(a,b) \in R(\leftrightharpoons)$, yielding

$$0 = \mathbb{E}[X] - \mathbb{E}[Y] = \sum_{(a,b) \in R(\leftrightharpoons)} r_{(a,b)}(x_a - y_b) > 0, \tag{A.22}$$

a contradiction.

The remaining cases all consist of $x_a \leq y_{b-1}$ holding for some pairs $(a,b) \in \bigcup_{a \in A} (\{a\} \times B_a)$ and $y_b < x_a$ holding for others. For that, we order all

these pairs from low to high, or, in other words, primarily by $a \in A$ and secondarily by $b \in B_a$, i.e.

$$\ldots, (a-1, \min B_{a-1}), \ldots, (a-1, \max B_{a-1}), \\ (a, \min B_a), \ldots, (a, \max B_a), \ldots.$$

This gives us three possible kinds of successive pairs. The first one is $(a,b),(a,b+1)$, where $a \in A$ and $b, b+1 \in B_a$ (denoted by (P1)). Both the second and the third kind are of the form $(a, \max B_a), (a+1, \min B_{a+1})$ for an $a \in \overline{A}$. If $G(y_{\max B_a - 1}) \in F(D_F)$, then $\max B_a = \min B_{a+1}$ holds by definition of B_a. This gives the second kind of successive pairs, which is of the form $(a,b),(a+1,b)$, where $a \in \overline{A}$ and $G(y_{b-1}) = F(x_a)$ (denoted by (P2)). (Note that $G(y_{\max B_a - 1}) = F(x_{a-1})$ is not possible because $G(y_{\max B_a}) > F(x_a)$ then contradicts $q_{\max B_a} \leq p_a$ and therefore $a \leftharpoons \max B_a$, and $G(y_{\max B_a}) \leq F(x_a)$ then contradicts the maximality of $\max B_a$.) If $G(y_{\max B_a - 1}) \notin F(D_F)$, then $\max B_a = \min B_{a+1} - 1$ holds. This gives us the third kind of successive pairs, which is of the form $(a,b),(a+1,b+1)$, where $a \in \overline{A}$ and $b = \max B_a$ (denoted by (P3)).
For each of these kinds of successive pairs, $x_a \leq y_{b-1}$ can hold for the former and $y_b < x_a$ can hold for the latter or vice versa. Overall, this gives us six cases that remain to be considered.

Case 3: (P1) with $x_a \leq y_{b-1}$ and $y_{b+1} < x_a$
This directly gives $y_{b+1} < x_a \leq y_{b-1}$, a contradiction.

Case 4: (P1) with $y_b < x_a$ and $x_a \leq y_b$
These two statements directly contradict each other.

Case 5: (P2) with $x_a \leq y_{b-1}$ and $y_b < x_{a+1}$
Since $G(y_{b-1}) = F(x_a)$ holds for any successive pair of index pairs of the second kind, it follows $a+1 \leftharpoons_\wedge b$. Then,

$$x_{a+1} = x_a + (x_{a+1} - x_a) \leq y_{b-1} + (x_{a+1} - x_a) \leq y_{b-1} + (y_b - y_{b-1}) = y_b, \tag{A.23}$$

which contradicts the assumption $y_b < x_{a+1}$.

Case 6: (P2) with $y_b < x_a$ and $x_{a+1} \leq y_{b-1}$
This directly gives $x_{a+1} \leq y_{b-1} < y_b < x_a$, a contradiction.

Case 7: (P3) with $x_a \leq y_{b-1}$ and $y_{b+1} < x_{a+1}$
Since $G(y_{b-1}) < F(x_a) < G(y_b)$ holds for any successive pair of pairs of the third kind, it follows from Proposition 3.17a) that $a+1 \leftharpoons_\wedge b$. Then, (A.23) again holds, contradicting the assumption $y_b < y_{b+1} < x_{a+1}$.

Case 8: (P3) with $y_b < x_a$ and $x_{a+1} \leq y_b$
This directly gives $x_{a+1} \leq y_b < x_a$, a contradiction.

Now, we have considered all relevant cases and have thereby proved part a).

b) We prove this part by showing that (i) follows, if (ii) does not hold. This is done in several steps.

Step 1: If (ii) does not hold, part a) yields that there exists a pair $(a,b) \in A \times \overline{B}$ such that $F(x_a) = G(x_b)$ and $y_b < x_a$ or there exists a pair $(a,b) \in \overline{A} \times \overline{B}$ such that $F(x_a) = G(x_b)$ and $x_{a+1} \leq y_{b+1}$.

Step 2: In the first case from Step 1, it directly follows from $b \in \overline{B}$ and $F(x_a) = G(y_b)$ that $a \in \overline{A}$. Then, either $x_{a+1} \leq y_{b+1}$ holds (in which case we are finished) or $y_{b+1} < x_{a+1}$. Hence, assume $y_{b+1} < x_{a+1}$. We proceed by showing $b+1 \in \overline{B}$. We prove this by contradiction and therefore assume $G(y_{b+1}) = 1$, from which $1 = F(x_{a+1}) \geq G(y_{b+1})$ directly follows because of $a+1 \leftrightharpoons b+1$. Thus, $p_{a+1} = q_{b+1}$ holds, which yields $R(\leftrightharpoons) \cap (\{a+1\} \times B) = \{(a+1, b+1)\}$. We now seek to show $y_\beta < x_\alpha$ for all $(\alpha, \beta) \in R(\leftrightharpoons)$, which then implies $\mathbb{E}[Y] < \mathbb{E}[X]$ in the same way as in (A.22) and thereby concludes the proof of $b+1 \in \overline{B}$ by contradiction. Obviously, $y_\beta < x_a$ is true for all $\beta \in B$ such that $a \leftrightharpoons \beta$ because of $y_b < x_a$ and since all such β's are no larger than b. From Proposition 3.17a), it follows that there exists a $\beta_0 \leq b-1$ such that $a \leftrightharpoons_\wedge \beta_0 + 1$ with β_0 being the largest element of B, for which $a-1 \leftrightharpoons \beta_0$ holds. (This can be achieved by defining $\beta_0 = \min \mathrm{NN}_F^G(a) - 1$.) From this,

$$\begin{aligned} y_{\tilde{\beta}_0} < y_{\beta_0} = y_{\beta_0+1} - (y_{\beta_0+1} - y_{\beta_0}) &\leq y_{\beta_0+1} - (x_a - x_{a-1}) \\ &\leq y_b - (x_a - x_{a-1}) < x_a - (x_a - x_{a-1}) = x_{a-1} \end{aligned}$$

follows for all $\tilde{\beta}_0 \in B$ such that $a-1 \leftrightharpoons \tilde{\beta}_0$. This can be recursively continued for $a-2, a-3, \ldots$ as long as these indices are in A. Overall, this proves that $y_\beta < x_\alpha$ holds for all $(\alpha, \beta) \in R(\leftrightharpoons)$.

Step 3: Still assuming $y_{b+1} < x_{a+1}$, it follows from Step 2 that $b+1 \in \overline{B}$. Now, if $F(x_{a+1}) = 1$, along with $y_\beta < x_{a+1}$ for all $\beta > b+1$, $\mathbb{E}[Y] < \mathbb{E}[X]$ follows similarly as before, leading to a contradiction. Assuming $F(x_{a+1}) = 1$ along with the existence of a $\beta > b+1$ such that $y_{\beta-1} < x_{a+1} \leq y_\beta$ yields that $(a+1, \beta)$ satisfies condition (ii), which also poses a contradiction. Hence, $F(x_{a+1}) < 1$ and $a+1 \in \overline{A}$.

Step 4: Since (ii) is assumed to not hold, $y_{\beta+1} < x_{a+1}$ follows for all $\beta \in \overline{B}$ with $F(x_a) < G(y_\beta) < F(x_{a+1})$. Hence, there exists a $\beta_1 \in \overline{B}$ with $F(x_{a+1}) \leq G(y_{\beta_1}) < F(x_{a+2})$ and $y_{\beta_1} < x_{a+1} < x_{a+2}$. Now we can use our earlier procedure recursively in the following sense. If $G(y_{\beta_1}) > F(x_{a+1})$, $y_{\beta_1+1} < x_{a+2}$ follows since (ii) does not hold. This gives us the setting from the beginning of Step 4 with (a, β) replaced by $(a+1, \beta_1)$. Here, $a+2 \in \overline{A}$ is guaranteed since otherwise, $\mathbb{E}[Y] < \mathbb{E}[X]$ would follow. This can be repeated recursively as long as the strict inequality analogous to $G(y_{\beta_1}) > F(x_{a+1})$ is satisfied. The recursion has to stop at some point (in the sense that equality holds) since otherwise, $\mathbb{E}[Y] < \mathbb{E}[X]$ would follow.

If $G(y_{\beta_1}) = F(x_{a+1})$ holds (or an analogous inequality later in the recursion), we can reuse the proof from Step 2 onwards, replacing the pair (a,b) by $(a+1, \beta_1)$. With this new recursion, we end up at

some point with a pair that satisfies condition (i) since, otherwise, the contradiction $\mathbb{E}[Y] < \mathbb{E}[X]$ would follow again.

Step 5: It remains to consider the second case in Step 1, so that there exists a pair $(a, b) \in \overline{A} \times \overline{B}$ such that $x_{a+1} \leq y_{b+1}$. Here, we can use the exact same procedure as in Steps 2–4, only going through the cdf's F and G in the opposite direction. The fact that the inequality $x_{a+1} \leq y_{b+1}$ is not a strict inequality does not change anything, because if equality held every time, we would end up with $F = G$, a contradiction. (In Steps 2–4, the contradiction $\mathbb{E}[Y] < \mathbb{E}[X]$ always follows from $y_\beta < x_\alpha$ holding for all $(\alpha, \beta) \in R(\leftrightharpoons)$. Hence, if the inequality is not strict, $\mathbb{E}[X] = \mathbb{E}[Y]$ can only occur if $y_\beta = x_\alpha$ holds for all $(\alpha, \beta) \in R(\leftrightharpoons)$. However, in that case, we already have $F = G$.)

Proof of Theorem 4.3. Assume without restriction that $\mathbb{E}[X] = \mathbb{E}[Y]$ and $F \neq G$. We proceed by showing $F \leq_{\text{dil}} G$ in cases (i) and (ii) of Lemma 4.2b), adopting the notation from there. According to the assumption $\mathbb{E}[X] = \mathbb{E}[Y]$, the characterization of the dilation order given in Proposition 2.4a) and the identity

$$\mathbb{E}[(X-t)_+] = \mathbb{E}[X-t] + \mathbb{E}[(X-t)_-] = \mathbb{E}[X] - t + \mathbb{E}[(X-t)_-]$$

for all $t \in \mathbb{R}$, $F \leq_{\text{dil}} G$ is equivalent to

$$\mathbb{E}[(X-t)_+] \leq \mathbb{E}[(Y-t)_+] \quad \forall t \geq t_0 \quad \text{and} \quad \mathbb{E}[(X-t)_-] \leq \mathbb{E}[(Y-t)_-] \quad \forall t \leq t_0 \tag{A.24}$$

for any $t_0 \in \mathbb{R}$.

(i) Let (a, b) be the pair, the existence of which is guaranteed by (i) in Lemma 4.2b) and choose $t_0 = \frac{x_a + x_{a+1}}{2}$. The first inequality in (A.24) is now equivalent to

$$\sum_{(\alpha,\beta) \in R(\leftrightharpoons)} r_{(\alpha,\beta)}((y_\beta - t)_+ - (x_\alpha - t)_+) \geq 0. \tag{A.25}$$

for all $t \geq t_0$. The summand corresponding to any pair $(\alpha, \beta) \in R(\leftrightharpoons)$ with $\alpha \leq a$ is zero because $x_\alpha < t_0$ and $y_\beta < t_0$ hold in that case. So, let $\alpha \geq a+1$, also yielding $\beta \geq b+1$. For all $k > a+1$, there exists exactly one $\ell_k > b+1$ with $G(y_{\ell_k - 1}) \leq F(x_{k-1})$ and $k \leftrightharpoons_\wedge \ell_k$, according to Proposition 3.17a). ($\ell_k > b+1$ holds because $a+1$ is the only element of A, for which $a+1 \leftrightharpoons_\wedge b+1$ holds, since $F(x_a) = G(x_b)$.) Furthermore, $a+1 < k-1 < k$ implies $b+1 < \ell_{k-1} < \ell_k$ because of $G(y_{\ell_{k-1}-1}) \leq F(x_{k-2}) < G(y_{\ell_k - 1})$ (see Lemma A.2a)); thus, the ℓ_k's are pairwise distinct. Hence, for all $\alpha > a+1$, there exists a $\beta_0 > b+1$ with $G(y_{\beta_0 - 1}) \leq F(x_{\alpha-1})$ and

$$\begin{aligned} x_\alpha - x_{a+1} = \sum_{k=a+2}^{\alpha} (x_k - x_{k-1}) &\leq \sum_{k=a+2}^{\alpha} (y_{\ell_k} - y_{\ell_k - 1}) \\ &\leq \sum_{\ell=b+2}^{\beta_0} (y_\ell - y_{\ell-1}) = y_{\beta_0} - y_{b+1}. \end{aligned}$$

Because of $G(y_{\beta_0-1}) \leq F(x_{\alpha-1})$, $\beta \geq \beta_0$ holds for all $\beta \in B$ such that $\alpha \leftrightharpoons \beta$. Overall, all such β satisfy

$$\begin{aligned} y_\beta \geq y_{\beta_0} &= (y_{\beta_0} - y_{b+1}) + y_{b+1} \geq (x_\alpha - x_{a+1}) + y_{b+1} \\ &\geq (x_\alpha - x_{a+1}) + x_{a+1} = x_\alpha, \end{aligned}$$

yielding $(y_\beta - t)_+ \geq (x_\alpha - t)_+$ for all $t \geq t_0$. Hence, (A.25) is true. The second inequality in (A.24) is shown completely analogously by extending the inequality $y_b < x_a$ down the cdf's F and G instead of extending the inequality $x_{a+1} \leq y_{b+1}$ up the cdf's.

(ii) Let $(a,b),(\alpha,\beta) \in A \times \underline{B}$ satisfy condition (ii) with $\alpha - 1 > a$, which yields $\beta - 1 > b$. This implies $(a, b-1), (\alpha, \beta-1) \in R(\leftrightharpoons)$ as well as $y_{b-1} < x_a \leq y_b < y_{\beta-1} < x_\alpha \leq y_\beta$, which yields $x_\alpha - x_a > y_{\beta-1} - y_b$. According to (A.9) in Part 1 of the proof of Theorem 4.1, $\alpha - a = (\beta-1) - b + 1 = \beta - b$ follows. Part 2 of the same proof then implies

$$\begin{aligned} R(\leftrightharpoons_\wedge) \supseteq \{&(a+1,b),(a+1,b+1),(a+2,b+1),(a+2,b+2),\dots \\ &\dots,(\alpha-2,\beta-2),(\alpha-1,\beta-2),(\alpha-1,\beta-1),(\alpha,\beta-1)\} \end{aligned}$$

(see (A.10), (A.11), (A.14) and (A.15)). Obviously, $\alpha \leftrightharpoons \beta$ also holds. Thus, for all $k \in \{1,\dots,\alpha-a\} = \{1,\dots,\beta-b\}$, we have $a+k \leftrightharpoons_\wedge b+k-1$ and $a+k \leftrightharpoons_\wedge b+k$. Let $k \in \{1,\dots,\alpha-a-1\}$. Now $G(y_{b+k-1}) \in (F(x_{a+k-1}), F(x_{a+k}))$ follows from $a+k \leftrightharpoons b+k-1$ and $a+k \leftrightharpoons b+k$. Furthermore,

$$\begin{aligned} x_{a+k} &= x_a + \sum_{j=1}^{k} (x_{a+j} - x_{a+j-1}) \leq y_b + \sum_{j=1}^{k} (x_{a+j} - x_{a+j-1}) \\ &\leq y_b + \sum_{j=1}^{k} (y_{b+j} - y_{b+j-1}) = y_{b+k} \end{aligned}$$

and

$$\begin{aligned} x_{a+k} &= x_\alpha - \sum_{j=k+1}^{\alpha-a} (x_{a+j} - x_{a+j-1}) > y_{\beta-1} - \sum_{j=k+1}^{\alpha-a} (x_{a+j} - x_{a+j-1}) \\ &\geq y_{\beta-1} - \sum_{j=k+1}^{\beta-b} (y_{b+j-1} - y_{b+j-2}) = y_{b+k-1}. \end{aligned}$$

Overall, the pair $(a+k, b+k)$ satisfies condition (ii) for all $k \in \{1,\dots,\alpha-a-1\}$.

The entirety of the proof for case (ii) so far states that, if there are multiple pairs of indices satisfying condition (ii), they are of the form $\dots,(a,b),(a+1,b+1),\dots$. Assume now that this chain of pairs has a lower end, so that there exists a pair $(a,b) \in \underline{A} \times \underline{B}$ that satisfies condition (ii), but the pair

$(a-1, b-1)$ does not. Additionally, let $(\alpha, \beta) \in R(\leftrightharpoons)$ be a lower pair than (a, b) in the sense that the interval measured by $r_{(\alpha,\beta)}$ is lower on the unit interval than the interval measured by $r_{(a,b)}$. We seek to prove $y_\beta < x_\alpha$ for all such pairs. If $x_a - x_\alpha \leq y_{b-1} - y_\beta$ holds, then $y_\beta < x_\alpha$ directly follows because of $y_{b-1} < x_a$. According to (A.9) in Part 1 of the proof of Theorem 4.1, $a - \alpha = b - \beta = (b-1) - \beta + 1$ is a necessary condition for $x_a - x_\alpha > y_{b-1} - y_\beta$. In that case, Part 2 of the same proof implies that the values of F and G alternate in the sense of $F(x_{\alpha-1}) < G(y_{\beta-1}) < F(x_\alpha)$. If $y_\beta < x_\alpha$ already holds, we are done; if $y_\beta \geq x_\alpha$ holds along with $x_\alpha > y_{\beta-1}$, the pair (α, β) satisfies condition (ii), posing a contradiction. Therefore, assume $x_\alpha \leq y_{\beta-1}$. Again invoking (A.9), this is only possible if $a - \alpha = (b-1) - (\beta-1) + 1 = b - \beta + 1$, which contradicts $a - \alpha = b - \beta$. Overall, it follows that $y_\beta < x_\alpha$ holds for all pairs $(\alpha, \beta) \in R(\leftrightharpoons)$ lower than the chain of pairs satisfying condition (ii). Analogously, it can be shown that $x_\alpha \leq y_\beta$ holds for all pairs $(\alpha, \beta) \in R(\leftrightharpoons)$ higher than the chain of pairs satisfying condition (ii).

Due to the proven structure of this chain, we can find a unique indexing set $C \in \{\{0, \dots, n\} : n \in \mathbb{N}_0\} \cup \{\mathbb{N}_0, -\mathbb{N}_0, \mathbb{Z}\}$ and an initial pair $(a, b) \in A \times \underline{B}$ such that $((a+c, b+c))_{c \in C}$ is the sequence of all pairs satisfying condition (ii). We now define the following two mappings:

$$\tilde{m}_{pm} : C \to (0,1), \quad c \mapsto \frac{G(y_{b+c-1}) - F(x_{a+c-1})}{F(x_{a+c}) - F(x_{a+c-1})},$$
$$\tilde{m}_{supp} : C \to (0,1], \quad c \mapsto \frac{x_{a+c} - y_{b+c-1}}{y_{b+c} - y_{b+c-1}}$$

Both mappings can be extended to $\mathbb{Z}$ by assigning the value 1 to both, if the argument is smaller than the minimum of C, and assigning the value 0 to both, if the argument is larger than the maximum of C. We now show that both mappings are decreasing on C. For that, let $c, c+1 \in C$ and define $d_F = \min\{F(x_{a+c}) - F(x_{a+c-1}), F(x_{a+c+1}) - F(x_{a+c})\}$ as well as $d_y = \min\{y_{b+c} - y_{b+c-1}, y_{b+c+1} - y_{b+c}\}$. Note that, because of

$$F(x_{a+c-1}) < G(y_{b+c-1}) < F(x_{a+c}) < G(y_{b+c}) < F(x_{a+c+1}),$$

$a+c \leftrightharpoons b+c, a+c+1 \leftrightharpoons b+c$ follows as well as $a+c+1 \leftrightharpoons_\wedge b+c, a+c+1 \leftrightharpoons_\wedge b+c+1$. Then,

$$\begin{aligned}
\tilde{m}_{pm}(c+1) - \tilde{m}_{pm}(c) =& \frac{G(y_{b+c}) - F(x_{a+c})}{F(x_{a+c+1}) - F(x_{a+c})} \\
&+ \frac{F(x_{a+c}) - G(y_{b+c-1})}{F(x_{a+c}) - F(x_{a+c-1})} - 1 \\
\leq& \frac{G(y_{b+c}) - G(y_{b+c-1})}{d_F} - 1 \leq 0, \qquad \text{(A.26)} \\
\tilde{m}_{supp}(c+1) - \tilde{m}_{supp}(c) =& \frac{x_{a+c+1} - y_{b+c}}{y_{b+c+1} - y_{b+c}} + \frac{y_{b+c} - x_{a+c}}{y_{b+c} - y_{b+c-1}} - 1
\end{aligned}$$

$$\leq \frac{x_{a+c+1} - x_{a+c}}{d_y} - 1 \leq 0.$$

It follows that the mappings

$$m_{pm} : C \to (0, \infty), \quad c \mapsto \left(\tilde{m}_{pm}(c)^{-1} - 1\right)^{-1} = \frac{G(y_{b+c-1}) - F(x_{a+c-1})}{F(x_{a+c}) - G(y_{b+c-1})},$$
$$m_{supp} : C \to (0, \infty], \quad c \mapsto \left(\tilde{m}_{supp}(c)^{-1} - 1\right)^{-1} = \frac{x_{a+c} - y_{b+c-1}}{y_{b+c} - x_{a+c}}$$

are also decreasing. Now, define the mapping

$$m : C \to (0, \infty], \quad c \mapsto m_{pm}(c) \cdot m_{supp}(c) = \frac{(G(y_{b+c-1}) - F(x_{a+c-1}))(x_{a+c} - y_{b+c-1})}{(F(x_{a+c}) - G(y_{b+c-1}))(y_{b+c} - x_{a+c})}.$$

m can also be extended to the domain $\mathbb{Z}$ by assigning $m(c) = \infty$ if $c < \min C$ and $m(c) = 0$ if $c > \max C$. Furthermore, m inherits from m_{pm} and m_{supp} that it is decreasing. Define $c_0 = \min\{c \in \mathbb{Z} : m(c) \leq 1\}$. Since m is monotone, the existence of this minimum follows, if we can show that the range of m is neither a subset of $[0, 1]$ nor a subset of $(1, \infty]$. To prove this, first assume that $m(c) \leq 1$ for all $c \in \mathbb{Z}$. It follows $\inf C = -\infty$, which implies $C \in \{-\mathbb{N}_0, \mathbb{Z}\}$. If $C = -\mathbb{N}_0$, $y_\beta \geq x_\alpha$ holds for all pairs $(\alpha, \beta) \in R(\leftrightharpoons)$ higher than (a, b). Hence, for $C \in \{-\mathbb{N}_0, \mathbb{Z}\}$,

$$\begin{aligned}\mathbb{E}[Y] - \mathbb{E}[X] &\geq \sum_{c=\inf C}^{\sup C} r_{(a+c,b+c)}(y_{b+c} - x_{b+c}) + r_{(a+c,b+c-1)}(y_{b+c-1} - x_{a+c}) \\ &= \sum_{c=\inf C}^{\sup C} (F(x_{a+c}) - G(y_{b+c-1}))(y_{b+c} - x_{a+c}) \qquad \text{(A.27)} \\ &\qquad - (G(y_{b+c-1}) - F(x_{a+c-1}))(x_{a+c} - y_{b+c-1}).\end{aligned}$$

The non-negativity of the c-th summand in (A.27) is equivalent to $m(c) \leq 1$, which was assumed for all $c \in C$. The assumption $\mathbb{E}[X] = \mathbb{E}[Y]$ now implies $m(c) = 1$ for all $c \in C$. Because of $\inf C = -\infty$, the sequence $(F(x_{a+c}) - F(x_{a+c-1}))_{c \in C}$ converges to zero as $c \to -\infty$. This means that there exists a $c \in C$ such that $F(x_{a+c-1}) - F(x_{a+c-2}) < F(x_{a+c}) - F(x_{a+c-1})$, meaning that the first inequality in (A.26) is strict. This contradicts that $\tilde{m}_{pm}$ is constant on C and therefore also that $m(c) = 1$ holds for all $c \in C$. Overall, $m(c) \leq 1$ cannot hold for all $c \in \mathbb{Z}$. Since assuming $m(c) > 1$ for all $c \in \mathbb{Z}$ analogously leads to a contradiction, we have proved that the minimum in the definition of c_0 exists.

Define $(a_0, b_0) = (a + c_0, b + c_0)$. We now prove $F \leq_{\text{dil}} G$ by showing (A.24), which is equivalent to

$$\sum_{(\alpha,\beta) \in R(\leftrightharpoons)} r_{(\alpha,\beta)}((y_\beta - t)_+ - (x_\alpha - t)_+) \geq 0 \quad \forall t \geq t_0 \quad \text{and} \qquad \text{(A.28)}$$

$$\sum_{(\alpha,\beta)\in R(\leftrightharpoons)} r_{(\alpha,\beta)}((y_\beta - t)_- - (x_\alpha - t)_-) \geq 0 \quad \forall t \leq t_0. \tag{A.29}$$

We proceed via case distinction.

Case 1: $c_0 - 1, c_0 \in C$

In this case, we choose $t_0 = y_{b_0-1}$. It holds that

$$\begin{aligned} &y_{b_0-2} < x_{a_0-1} \leq y_{b_0-1} < x_{a_0} \leq y_{b_0}, \\ &F(x_{a_0-2}) < G(y_{b_0-2}) < F(x_{a_0-1}) < G(y_{b_0-1}) < F(x_{a_0}). \end{aligned}$$

Hence, $(a_0, b_0 - 1)$ is the lowest pair in $R(\leftrightharpoons)$ such that the corresponding summand in (A.28) is not zero. The next pairs in $R(\leftrightharpoons)$ are then $(a_0, b_0), (a_0 + 1, b_0), (a_0 + 1, b_0 + 1), \ldots$ as long as the pairs of the form $(a_0 + k, b_0 + k), k \in \mathbb{N}$, still satisfy condition (ii). If $\max C$ exists, $y_\beta \geq x_\alpha$ holds for any pair $(x_\alpha, y_\beta) \in R(\leftrightharpoons)$ higher than $(a + \max C, b + \max C)$ (as was shown earlier). Therefore,

$$\begin{aligned} 0 \leq \sum_{c=c_0}^{\max C} \Big[& r_{(a+c,b+c)}((y_{b+c} - t)_+ - (x_{a+c} - t)_+) \qquad (A.30) \\ & + r_{(a+c,b+c-1)}((y_{b+c-1} - t)_+ - (x_{a+c} - t)_+) \Big] \\ = \sum_{c=c_0}^{\max C} \Big[& (F(x_{a+c}) - G(y_{b+c-1}))\,((y_{b+c} - t)_+ - (x_{a+c} - t)_+) \qquad (A.31) \\ & - (G(y_{b+c-1}) - F(x_{a+c-1}))\,((x_{a+c} - t)_+ - (y_{b+c-1} - t)_+) \Big] \end{aligned}$$

for all $t \geq t_0$ is sufficient to show (A.28). We now consider each summand in (A.31) separately. First, in the case $t < y_{b+c-1}$, the summand is equal to

$$\begin{aligned} &(F(x_{a+c}) - G(y_{b+c-1}))\,(y_{b+c} - x_{a+c}) \\ &- (G(y_{b+c-1}) - F(x_{a+c-1}))\,(x_{a+c} - y_{b+c-1})\,. \end{aligned} \tag{A.32}$$

The non-negativity of this is equivalent to $m(c) \leq 1$, which is true since $c \geq c_0$. In the case $y_{b+c-1} \leq t \leq x_{a+c}$, the summand is equal to

$$\begin{aligned} &(F(x_{a+c}) - G(y_{b+c-1}))\,(y_{b+c} - x_{a+c}) \\ &- (G(y_{b+c-1}) - F(x_{a+c-1}))\,(x_{a+c} - t)\,, \end{aligned}$$

which is no smaller than (A.32) and therefore also non-negative. In the case $x_{a+c} \leq t \leq y_{b+c}$, the summand is equal to $(F(x_{a+c}) - G(y_{b+c-1}))$ $(y_{b+c} - t)$, which is non-negative because both factors are. Finally, in the case $t > y_{b+c}$, the summand is zero. Overall, inequality (A.30) is

satisfied, leaving (A.29) to be shown. For that, proceeding similarly to before, it is sufficient to show

$$0 \leq \sum_{c=\min C}^{c_0-1} \Big[\left(F(x_{a+c}) - G(y_{b+c-1})\right) \left((y_{b+c} - t)_- - (x_{a+c} - t)_-\right) - \left(G(y_{b+c-1}) - F(x_{a+c-1})\right) \left((x_{a+c} - t)_- - (y_{b+c-1} - t)_-\right) \Big] \tag{A.33}$$

for all $t \leq t_0$. We again only consider the c-th summand, beginning with the case $t > y_{b+c}$, in which it is equal to negative (A.32). The non-negativity of that term is equivalent to $m(c) \geq 1$, which is true since $c < c_0$. The non-negativity of the c-th summand in the remaining cases follows in a similar fashion. Hence, both (A.28) and (A.29) are satisfied, yielding $F \leq_{\text{dil}} G$.

Case 2: $c_0 - 1 = \max C$
In this case, $(a_0 - 1, b_0 - 1) \in A \times \underline{B}$ obviously holds and we choose $t_0 = x_{a_0-1}$. Because of $y_{b_0-2} < x_{a_0-1} \leq y_{b_0-1}$, only $(a_0 - 1, b_0 - 1)$ and pairs $(\alpha, \beta) \in R(\leftrightharpoons)$ higher than $(a_0 - 1, b_0 - 1)$ have non-zero summands in (A.28). However, since $x_\alpha \leq y_\beta$ and, consequently, $(x_\alpha - t)_+ \leq (y_\beta - t)_+$ holds for all those pairs, (A.28) is true.
Because of $y_{b_0-2} < x_{a_0-1} \leq y_{b_0-1}$, the highest pair in $R(\leftrightharpoons)$ with a non-zero summand in (A.29) is $(a_0 - 1, b_0 - 2)$. As before, all pairs $(\alpha, \beta) \in R(\leftrightharpoons)$ below the chain of pairs in $R(\leftrightharpoons)$ satisfying condition (ii) can be disregarded as they satisfy $y_\beta < x_\alpha$. It follows that (A.33) is a sufficient condition for (A.29) and it can be shown analogously to Case 1 as the chain indexed by C has the same properties from $c_0 - 1$ downwards. (In fact, condition (A.33) can even be weakened since it includes the non-positive summand associated with the pair $(a_0 - 1, b_0 - 1)$, which is not relevant here.)

Case 3: $c_0 = \min C$
For this case, we can proceed analogously to Case 2 after choosing $t_0 = x_{a_0}$ and by flipping the procedure upside-down.

By using the stop-loss transform characterization of the dilation order, the proof of Theorem 4.3 is relatively simple, if G is larger than (or equal to) F for $t \leq t_0$ and G is smaller than F for $t \geq t_0$. For all situations depicted in Figure 7 except for the lower right panel, t_0 can be chosen in such a way that this holds. In particular, this is always possible in case (i), making the proof in this case a lot less difficult.

The only situation left to be considered is the one shown in the lower right panel of Figure 7, where F and G intersect multiple times. The structure of those intersections can be further narrowed down before then being divided in three cases in the proof. The line of reasoning in Case 1 is illustrated in Figure A.3. The main focus lies on the ratio between the two areas between each pair of vertical lines. If the left area is divided by the right area, the resulting ratio

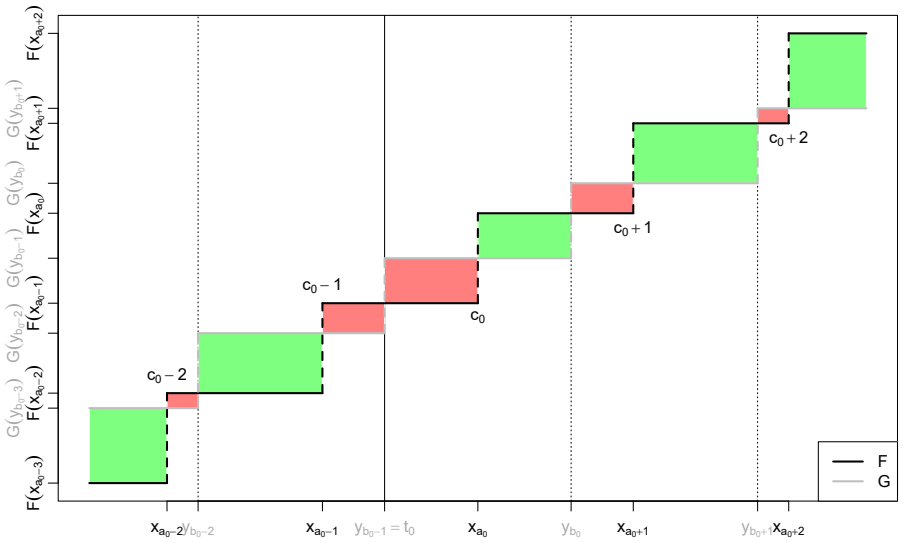

FIG A.3. *Exemplary situation in the proof of Theorem 4.3, part (ii), Case 1. (F and G are assumed to be standardized w.r.t. the mean.)*

decreases as the corresponding index $(c_0 - 2, c_0 - 1, c_0, \ldots)$ increases. (In the proof, the ratio for an index c is denoted by $m(c)$.) Now, t_0 is chosen such that this ratio is smaller than 1 on the left side of t_0 and larger than or equal to 1 on the right side of t_0. In Figure A.3, the smaller area of each pair is filled in red and the larger one is filled in green. For all $t \geq t_0$, the sum of the green areas on the right side of t minus the sum of the red areas on the right side of t is required to be non-negative. For $t = t_0$, this is obviously true since the ratio between the green and the red areas is no smaller than 1 for each index. If t is increased from t_0 towards ∞, the red areas are always reduced before the corresponding green areas are. The situation on the left side of t_0, so for $t \leq t_0$, is analogous.

This begs the question why Cases 2 and 3 in part (ii) of the proof of Theorem 4.3 need to be considered separately and cannot be included in the line of reasoning illustrated by Figure A.3. This is illustrated in Figure A.4, where the left panel is exemplary for Case 2 (as the chain of intersections of F and G ends at index c_0, which is chosen as before) and the right panel is exemplary for Case 3 (as the chain of intersections of F and G begins at index c_0). In Case 3, if $t_0 = y_{b_0-1}$ is chosen as before, it is possible for F to exhibit more steep jumps on the right side of t_0, thus expanding the corresponding red area to be larger than the corresponding green area. This is possible because the chain of intersections of F and G ends at index c_0 and does not extend any further below that index. The problem can be solved by choosing $t_0 = x_{a_0}$ instead. On the left side of t_0, F is then smaller than G; on the right side of t_0, there is an additional green area before the first full pair of red and green areas, once again ensuring the

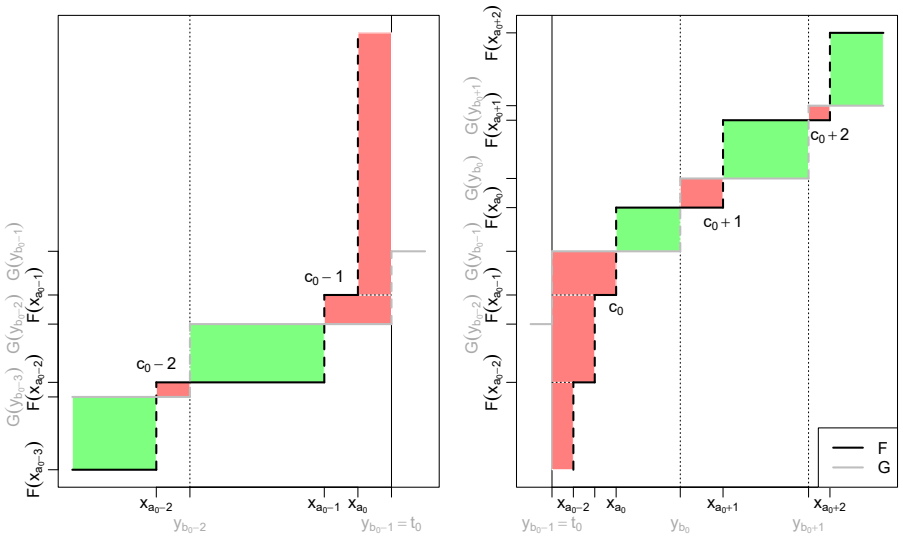

FIG A.4. *Exemplary situation in the proof of Theorem 4.3, part (ii), Cases 2 and 3, if* $t_0 = y_{b_0-1}$ *was chosen instead of* $t_0 = x_{a_0-1}$ *and* $t_0 = x_{a_0}$*. (*F *and* G *are assumed to be standardized w.r.t. the mean.)*

non-negativity required in (A.31). The line of reasoning in Case 2 is analogous, just going down instead of up.

Proof of Theorem 4.7. We prove this by constructing a counterexample for arbitrary values of α and β. First, define $n = \left\lfloor \frac{\beta-\alpha}{\min(\alpha,1-\beta)} \right\rfloor + 1 \in \mathbb{N}$ and choose $\delta \in \left(\frac{\beta-\alpha}{n}, \min(\alpha, 1-\beta) \right) \subseteq (0,1)$. The latter interval is not empty since

$$n > \frac{\beta-\alpha}{\min(\alpha,1-\beta)} \Leftrightarrow \min(\alpha,1-\beta) > \frac{\beta-\alpha}{n}$$

holds. Note that, by definition, $0 \leq (n-1)\delta < \beta - \alpha < n\delta < 1$ is true. The last inequality specifically follows from

$$\begin{aligned} n\delta &< \left(\left\lfloor \frac{\beta-\alpha}{\min(\alpha,1-\beta)} \right\rfloor + 1 \right) \cdot \min(\alpha,1-\beta) \\ &< (\beta-\alpha) + \min(\alpha,1-\beta) = 1 - \max(\alpha,1-\beta) < 1. \end{aligned}$$

Further, let $\varepsilon_p = \frac{(\beta-\alpha)-(n-1)\delta}{2} > 0$, $\varepsilon_q = \frac{n\delta-(\beta-\alpha)}{2} > 0$. Now we define $F, G \in \mathcal{D}_0$ by $F \hat{=} (A, (x_j, p_j)_{j\in A})$ and $G \hat{=} (B, (y_j, q_j)_{j\in B})$, where $A = \{1, \ldots, n+1\}$ with $x_a = a$ for all $a \in A$ and $B = \{1, \ldots, n+2\}$ with $y_b = b$ for all $b \in B$. Furthermore, let

$$p_1 = \alpha + \varepsilon_p, \quad p_j = \delta \ \forall j \in \{2, \ldots, n\}, \qquad p_{n+1} = (1-\beta) + \varepsilon_p,$$

$$q_1 = \alpha - \varepsilon_q, \quad q_j = \delta \ \forall j \in \{2, \dots, n+1\}, \quad q_{n+2} = (1-\beta) - \varepsilon_q.$$

For all $j \in \{1, \dots, n\}$, it holds that

$$\begin{aligned} G(y_j) &= \alpha - \varepsilon_q + (j-1)\delta = \frac{\alpha+\beta}{2} + \delta\left(j - 1 - \frac{n}{2}\right) \\ &< \frac{\alpha+\beta}{2} + \delta\left(j - \frac{n}{2} - \frac{1}{2}\right) = \alpha + \varepsilon_p + (j-1)\delta = F(x_j) \\ &< \frac{\alpha+\beta}{2} + \delta\left(j - \frac{n}{2}\right) = \alpha + \varepsilon_p + j\delta = G(y_{j+1}), \end{aligned}$$

yielding $R(\leftrightharpoons) = \bigcup_{j=1}^{n+1}\{(j,j),(j,j+1)\}$. Hence, $F \leq_{\text{D-pm}}^{\text{disc}} G$. Since F and G are both lattice distributions with the same defining distance, it follows that $F \leq_{\text{disp}}^{\wedge\text{-disc}} G$. However,

$$\text{IQR}(\alpha,\beta)(F) = (n+1) - 1 = n > n - 1 = (n+1) - 2 = \text{IQR}(\alpha,\beta)(G).$$

Proof of Theorem 4.8. Without restriction, let $F^{-1}(\frac{1}{2}) = G^{-1}(\frac{1}{2}) = 0$. Furthermore, let $a_0 \in A$ and $b_0 \in B$ be the unique indices that satisfy $x_{a_0} = F^{-1}(\frac{1}{2}) = 0 = G^{-1}(\frac{1}{2}) = y_{b_0}$. Because of $F(x_{a_0-1}) < \frac{1}{2} \leq G(y_{b_0})$ and $G(y_{b_0-1}) < \frac{1}{2} \leq F(x_{a_0})$, $a_0 \leftrightharpoons b_0$ holds. Hence, $F(x_{a_0}) - F(x_{a_0-1}) = p_{a_0} \geq q_{b_0} = G(y_{b_0}) - G(y_{b_0-1})$ and it follows that $G(y_{b_0}) \leq F(x_{a_0})$ or $G(y_{b_0-1}) \geq F(x_{a_0-1})$.

We begin by assuming $G(y_{b_0}) \leq F(x_{a_0})$ and show that, consequently, $y_\beta \geq x_\alpha$ holds for all pairs $(\alpha,\beta) \in R(\leftrightharpoons)$ higher than (a_0,b_0). Choosing $G(y_{b_k-1}) = \min \text{NN}_F^G(a_0+k)$ for all $k \in \mathbb{N}$ such that $a_0 + k \in A$ gives pairwise distinct indices $b_k \in \underline{B}$ such that $a_0 + k \leftrightharpoons_\wedge b_k$. This is guaranteed by Lemma A.2. Note that, for all $k \in \mathbb{N}$, b_k is also the smallest index in B such that $a_0 + k \leftrightharpoons b_k$. Note also that $b_0 < b_1$. Hence, for all pairs $(\alpha,\beta) \in R(\leftrightharpoons)$ higher than (or equal to) (a_0,b_0), there exists a $k \in \mathbb{N}_0$ such that

$$\begin{aligned} x_\alpha = x_{a_0+k} &= x_{a_0} + \sum_{j=1}^{k}(x_{a_0+j} - x_{a_0+j-1}) = \sum_{j=1}^{k}(x_{a_0+j} - x_{a_0+j-1}) \\ &\leq \sum_{j=1}^{k}(y_{b_j} - y_{b_{j-1}}) = y_{b_0} + \sum_{j=1}^{k}(y_{b_j} - y_{b_{j-1}}) \leq y_{b_k} \leq y_\beta. \end{aligned}$$

Analogously, the assumption $G(y_{b_0-1}) \geq F(x_{a_0-1})$ yields $y_\beta \leq x_\alpha$ holds for all pairs $(\alpha,\beta) \in R(\leftrightharpoons)$ lower than (a_0,b_0). With this in mind, we prove the assertion via case distinction.

Case 1: $F(x_{a_0-1}) \leq G(y_{b_0-1}) < G(y_{b_0}) \leq F(x_{a_0})$
It holds that

$$\begin{aligned} \text{MDMAD}(G) - \text{MDMAD}(F) &= \sum_{(a,b)\in R(\leftrightharpoons)} r_{(a,b)}(|y_b - y_{b_0}| - |x_a - x_{a_0}|) \\ &= \sum_{(a,b)\in R(\leftrightharpoons)} r_{(a,b)}(|y_b| - |x_a|) \end{aligned}$$

$$= \sum_{\substack{(a,b)\in R(\leftrightharpoons)\\ \text{lower than}(a_0,b_0)}} r_{(a,b)}(x_a - y_b) + \sum_{\substack{(a,b)\in R(\leftrightharpoons)\\ \text{higher than}(a_0,b_0)}} r_{(a,b)}(y_b - x_a). \quad \text{(A.34)}$$

Since $x_a \geq y_b$ holds for all pairs $(a,b) \in R(\leftrightharpoons)$ lower than (a_0,b_0) and $y_b \geq x_a$ holds for all pairs $(a,b) \in R(\leftrightharpoons)$ higher than (a_0,b_0), (A.34) is non-negative. Thus, the proof in this case is completed.

Case 2: $F(x_{a_0-1}) < G(y_{b_0-1}) < F(x_{a_0}) < G(y_{b_0})$

We start out by considering pairs $(a,b) \in R(\leftrightharpoons)$ that are higher than (a_0,b_0). According to (A.9), $x_a > y_b$ is only possible for any such pair, if $a - a_0 = b - b_0 + 1$ holds. Hence, let $(a,b) \in R(\leftrightharpoons)$ be higher than (a_0,b_0) and satisfy $a - a_0 = b - b_0 + 1$. Part 2 of the proof of Theorem 4.1 states that, in this case, the values of F and G alternate for index pairs between (a_0,b_0) and (a,b). In that area, the pairs in $R(\leftrightharpoons)$ are of the form

$$(a_0,b_0),(a_0+1,b_0),(a_0+1,b_0+1),(a_0+2,b_0+1),\ldots \\ \ldots,(a-2,b-1),(a-1,b-1),(a-1,b),(a,b). \quad \text{(A.35)}$$

Hence,

$$\begin{aligned} &\sum_{\substack{(a,b)\in R(\leftrightharpoons)\\ \text{higher than}(a_0,b_0)}} r_{(a,b)}(y_b - x_a) \\ \geq\ &- \sum_{\substack{c\in\mathbb{N}_0:\\ (a_0+1+c,b_0+c)\in R(\leftrightharpoons)}} r_{(a_0+1+c,b_0+c)}(x_{a_0+1+c} - y_{b_0+c}) \\ =\ &- \sum_{\substack{c\in\mathbb{N}_0:\\ F(x_{a_0+c})<G(y_{b_0+c})}} (G(y_{b_0+c}) - F(x_{a_0+c}))(x_{a_0+1+c} - y_{b_0+c}). \quad \text{(A.36)} \end{aligned}$$

Note that, due to the structure of $R(\leftrightharpoons)$ given in (A.35), $F(x_{a_0+c}) < G(y_{b_0+c})$ is equivalent to $(a_0+1+c, b_0+c) \in R(\leftrightharpoons)$ for any $c \in \mathbb{N}_0$. Since $(a_0+1+c) - (a_0+1) < (b_0+c) - b_0 + 1$ holds for all $c \in \mathbb{N}_0$ such that $(a_0+1+c, b_0+c) \in R(\leftrightharpoons)$, (A.9) implies $x_{a_0+1+c} - y_{b_0+c} \leq x_{a_0+1} - y_{b_0}$. Combined with (A.36), this yields

$$\begin{aligned} &\sum_{\substack{(a,b)\in R(\leftrightharpoons)\\ \text{higher than}(a_0,b_0)}} r_{(a,b)}(y_b - x_a) \geq -(x_{a_0+1} - y_{b_0}) \\ &\cdot \sum_{\substack{c\in\mathbb{N}_0:\\ F(x_{a_0+c})<G(y_{b_0+c})}} (G(y_{b_0+c}) - F(x_{a_0+c})). \quad \text{(A.37)} \end{aligned}$$

For any $c \in \mathbb{N}_0$ with $F(x_{a_0+c}) < G(y_{b_0+c})$ and $F(x_{a_0+c+1}) < G(y_{b_0+c+1})$,

it can be shown analogously to (A.26) that

$$\frac{G(y_{b_0+c+1}) - F(x_{a_0+c+1})}{F(x_{a_0+c+2}) - F(x_{a_0+c+1})} \leq \frac{G(y_{b_0+c}) - F(x_{a_0+c})}{F(x_{a_0+c+1}) - F(x_{a_0+c})}.$$

By applying this inequality recursively, we obtain

$$\frac{G(y_{b_0}) - F(x_{a_0})}{F(x_{a_0+1}) - F(x_{a_0})} \geq \frac{G(y_{b_0+c}) - F(x_{a_0+c})}{F(x_{a_0+c+1}) - F(x_{a_0+c})}$$

for all $c \in \mathbb{N}_0$ with $F(x_{a_0+c}) < G(y_{b_0+c})$. With this, we can continue inequality (A.37) by writing

$$\begin{aligned}
&\sum_{\substack{(a,b)\in R(\leftrightharpoons)\\ \text{higher than}(a_0,b_0)}} r_{(a,b)}(y_b - x_a) \\
&\geq -(x_{a_0+1} - y_{b_0}) \cdot \sum_{\substack{c\in\mathbb{N}_0:\\ F(x_{a_0+c})<G(y_{b_0+c})}} (G(y_{b_0+c}) - F(x_{a_0+c})) \\
&= -(x_{a_0+1} - y_{b_0}) \cdot \sum_{\substack{c\in\mathbb{N}_0:\\ F(x_{a_0+c})<G(y_{b_0+c})}} \left(\frac{G(y_{b_0+c}) - F(x_{a_0+c})}{F(x_{a_0+c+1}) - F(x_{a_0+c})} \cdot \right. \\
&\qquad\qquad \left. (F(x_{a_0+c+1}) - F(x_{a_0+c})) \right) \\
&\geq -(x_{a_0+1} - y_{b_0}) \cdot \frac{G(y_{b_0}) - F(x_{a_0})}{F(x_{a_0+1}) - F(x_{a_0})} \\
&\qquad\qquad \cdot \sum_{\substack{c\geq 0\\ F(x_{a_0+c})<G(y_{b_0+c})}} (F(x_{a_0+c+1}) - F(x_{a_0+c})) \qquad \text{(A.38)} \\
&\geq -(x_{a_0+1} - y_{b_0}) \cdot \frac{G(y_{b_0}) - F(x_{a_0})}{F(x_{a_0+1}) - F(x_{a_0})} \cdot (1 - F(x_{a_0})) \\
&\geq -\frac{x_{a_0+1} - y_{b_0}}{2} \cdot \frac{G(y_{b_0}) - F(x_{a_0})}{F(x_{a_0+1}) - F(x_{a_0})}. \qquad \text{(A.39)}
\end{aligned}$$

Now we consider the pairs $(a, b) \in R(\leftrightharpoons)$ lower than (a_0, b_0). According to (A.9), $x_a - y_b < x_{a_0+1} - y_{b_0}$ is only possible for this kind of pair if $a_0 - a = b_0 - b$ holds. In this case, the values of F and G between these pairs of indices once again alternate so that the corresponding elements of $R(\leftrightharpoons)$ are of the form

$$\begin{aligned}
&(a_0, b_0), (a_0, b_0 - 1), (a_0 - 1, b_0 - 1), (a_0 - 1, b_0 - 2), \dots \\
&\qquad \dots, (a + 2, b + 1), (a + 1, b + 1), (a + 1, b), (a, b).
\end{aligned}$$

It follows

$$\sum_{\substack{(a,b)\in R(\leftrightharpoons)\\ \text{lower than}(a_0,b_0)}} r_{(a,b)}(x_a - y_b)$$

$$\geq (x_{a_0+1} - y_{b_0}) \cdot \left(F(x_{a_0}) - \sum_{\substack{c \in \mathbb{N}_0: \\ (a_0-c, b_0-c) \in R(\leftrightharpoons)}} r_{(a_0-c, b_0-c)} \right)$$

$$= (x_{a_0+1} - y_{b_0}) \cdot \left(F(x_{a_0}) - \sum_{\substack{c \in \mathbb{N}_0: \\ F(x_{a_0-c}) > G(y_{b_0-c-1})}} (F(x_{a_0-c}) - G(y_{b_0-c-1})) \right). \tag{A.40}$$

Note for the inequality that $x_a - y_b < 0 = x_{a_0} - y_{b_0}$ would imply $a_0 - a = b_0 - b + 1$, which contradicts $a_0 - a = b_0 - b$. For any $c \in \mathbb{N}_0$ with $F(x_{a_0-c}) > G(y_{b_0-c-1})$ and $F(x_{a_0-c-1}) > G(y_{b_0-c-2})$, one can now again show similarly to (A.26) that

$$\frac{F(x_{a_0-c}) - G(y_{b_0-c-1})}{F(x_{a_0-c}) - F(x_{a_0-c-1})} \geq \frac{F(x_{a_0-c-1}) - G(y_{b_0-c-2})}{F(x_{a_0-c-1}) - F(x_{a_0-c-2})}$$

holds. Inductively, it follows that

$$\frac{F(x_{a_0}) - G(y_{b_0-1})}{F(x_{a_0}) - F(x_{a_0-1})} \geq \frac{F(x_{a_0-c}) - G(y_{b_0-c-1})}{F(x_{a_0-c}) - F(x_{a_0-c-1})}$$

is true for all $c \in \mathbb{N}_0$ with $F(x_{a_0-c}) > G(y_{b_0-c-1})$. Continuing from (A.40), we obtain

$$\sum_{\substack{(a,b) \in R(\leftrightharpoons) \\ \text{lower than}(a_0,b_0)}} r_{(a,b)}(x_a - y_b)$$

$$\geq (x_{a_0+1} - y_{b_0}) \cdot \left(F(x_{a_0}) - \sum_{\substack{c \in \mathbb{N}_0: \\ F(x_{a_0-c}) > G(y_{b_0-c-1})}} \left(\frac{F(x_{a_0-c}) - G(y_{b_0-c-1})}{F(x_{a_0-c}) - F(x_{a_0-c-1})} \cdot (F(x_{a_0-c}) - F(x_{a_0-c-1})) \right) \right)$$

$$\geq (x_{a_0+1} - y_{b_0}) \cdot \left(F(x_{a_0}) - \frac{F(x_{a_0}) - G(y_{b_0-1})}{F(x_{a_0}) - F(x_{a_0-1})} \cdot \sum_{\substack{c \in \mathbb{N}_0: \\ F(x_{a_0-c}) > G(y_{b_0-c-1})}} (F(x_{a_0-c}) - F(x_{a_0-c-1})) \right)$$

$$\geq (x_{a_0+1} - y_{b_0}) \cdot \left(F(x_{a_0}) - \frac{F(x_{a_0}) - G(y_{b_0-1})}{F(x_{a_0}) - F(x_{a_0-1})} \cdot F(x_{a_0}) \right)$$

$$\geq \frac{x_{a_0+1} - y_{b_0}}{2} \cdot \left(1 - \frac{F(x_{a_0}) - G(y_{b_0-1})}{F(x_{a_0}) - F(x_{a_0-1})} \right). \tag{A.41}$$

Now, by combining (A.34), which is also valid in Case 2, with (A.39) and (A.41), it follows that

$$\begin{aligned}
&\mathrm{MDMAD}(G) - \mathrm{MDMAD}(F) \\
&\geq \frac{x_{a_0+1} - y_{b_0}}{2} \cdot \left(1 - \frac{G(y_{b_0}) - F(x_{a_0})}{F(x_{a_0+1}) - F(x_{a_0})} - \frac{F(x_{a_0}) - G(y_{b_0-1})}{F(x_{a_0}) - F(x_{a_0-1})}\right) \\
&\geq \frac{x_{a_0+1} - y_{b_0}}{2} \cdot \left(1 - \frac{q_{b_0}}{\min\{p_{a_0}, p_{a_0+1}\}}\right) \geq 0,
\end{aligned}$$

where the last inequality holds since $(a_0, b_0), (a_0 + 1, b_0) \in R(\leftrightharpoons)$ is true by assumption.

Case 3: $G(y_{b_0-1}) < F(x_{a_0-1}) < G(y_{b_0}) < F(x_{a_0})$
This case is completely analogous to Case 2 for reasons of symmetry.

A.3. Proofs for Section 5

Proof of Theorem 5.3. Let $\lambda = 1 - \pi_G \in (0,1)$ and $\varrho = \frac{\log(1-\pi_F)}{\log(1-\pi_G)} > 1$, then $\pi_F = 1 - \lambda^\varrho$ and $\pi_G = 1 - \lambda$. Note that $F \leq^{\wedge\text{-disc}}_{\text{D-supp}} G$ holds because both are lattice distributions with defining distance 1. We start by finding an equivalent condition for $(a,b) \in R(\leftrightharpoons)$, where $a, b \in \mathbb{N} = A = B$. The statement $(a,b) \in R(\leftrightharpoons)$ is equivalent to $F(a-1) < G(b)$ and $G(b-1) < F(a)$ holding simultaneously. For these two inequalities, the following equivalences hold:

$$\begin{aligned}
F(a-1) < G(b) &\Leftrightarrow 1 - \lambda^{\varrho(a-1)} < 1 - \lambda^b \Leftrightarrow \lambda^{\varrho(a-1)} > \lambda^b \\
&\Leftrightarrow \varrho(a-1) < b \Leftrightarrow a < \tfrac{b}{\varrho} + 1, \\
G(b-1) < F(a) &\Leftrightarrow 1 - \lambda^{b-1} < 1 - \lambda^{\varrho a} \Leftrightarrow \lambda^{b-1} > \lambda^{\varrho a} \\
&\Leftrightarrow b - 1 < \varrho a \Leftrightarrow a > \tfrac{b-1}{\varrho}.
\end{aligned}$$

Overall, $(a,b) \in R(\leftrightharpoons)$ is equivalent to $a \in (\frac{b-1}{\varrho}, \frac{b}{\varrho} + 1)$. Because a is required to be a natural number and $(\frac{b}{\varrho} + 1) - \frac{b-1}{\varrho} = 1 + \frac{1}{\varrho} \in (1,2)$, there are either one or two possible values for a for a fixed $b \in \mathbb{N}$. Particularly, it follows that

$$a \in \left\{ \left\lfloor \frac{b}{\varrho} \right\rfloor + 1, \left\lceil \frac{b-1}{\varrho} \right\rceil \right\}.$$

The two values are different, if there exists an $n \in \mathbb{N}$ such that $\frac{b-1}{\varrho} \leq n \leq \frac{b}{\varrho}$. We consider the two elements separately. To this end, note that

$$\begin{aligned}
\lambda^{b-1} - \lambda^b = \mathbb{P}(Y = b) \leq \mathbb{P}(X = a) = \lambda^{\varrho(a-1)} - \lambda^{\varrho a} \\
\forall (a,b) \in \bigcup_{b \in \mathbb{N}} \left\{ \left(\left\lfloor \frac{b}{\varrho} \right\rfloor + 1, b \right), \left(\left\lceil \frac{b-1}{\varrho} \right\rceil, b \right) \right\}
\end{aligned}$$

is equivalent to $F \leq^{\text{disc}}_{\text{D-pm}} G$, and therefore to $F \leq^{\wedge\text{-disc}}_{\text{disp}} G$.

Case 1: Let $a = \left\lceil \frac{b-1}{\varrho} \right\rceil$.
It holds that

$$\begin{aligned}\lambda^{\varrho(a-1)} - \lambda^{\varrho a} &= \lambda^{\varrho(a-1)}(1-\lambda^{\varrho}) > \lambda^{\varrho \cdot \frac{b-1}{\varrho}}(1-\lambda^{\varrho}) \\ &= \lambda^{b-1} - \lambda^{b-1+\varrho} > \lambda^{b-1} - \lambda^{b},\end{aligned}$$

where the last inequality is true because $\varrho > 1$ implies $\lambda^{b-1+\varrho} < \lambda^{b}$.

Case 2: Let $a = \left\lfloor \frac{b}{\varrho} \right\rfloor + 1$.
It holds that

$$\lambda^{\varrho(a-1)} - \lambda^{\varrho a} = \lambda^{\varrho(a-1)}(1-\lambda^{\varrho}) > \lambda^{\varrho \cdot \frac{b}{\varrho}}(1-\lambda^{\varrho}) = \lambda^{b}(1-\lambda^{\varrho}).$$

Thus, it is sufficient to show

$$\lambda^{b-1}(\lambda - \lambda^{\varrho+1}) \geq \lambda^{b-1} - \lambda^{b}$$

or, equivalently,

$$2\lambda \geq \lambda^{\varrho+1} + 1.$$

This, in turn, is equivalent to

$$\begin{aligned}\lambda(2-\lambda^{\varrho}) \geq 1 &\Leftrightarrow \lambda^{\varrho} \leq 2 - \tfrac{1}{\lambda} \\ &\Leftrightarrow \varrho \geq \frac{\log(2-\frac{1}{\lambda})}{\log(\lambda)} = \frac{\log(2\lambda-1)}{\log(\lambda)} - 1,\end{aligned}$$

which is true by assumption.

Appendix B: Behaviour of further discrete distributions in the new framework

B.1. Binomial distribution

The binomial distribution has two parameters to be varied, namely the sample size n and the success probability π. However, if we consider two distributions $F = \mathrm{Bin}(n, \pi_F)$ and $G = \mathrm{Bin}(n, \pi_G)$ with $n \in \mathbb{N}_{\geq 2}$ and $\pi_F, \pi_G \in (0,1), \pi_F \neq \pi_G$, Proposition 3.10 states that neither $F \leq_{\mathrm{disp}}^{\wedge\text{-disc}} G$ nor $G \leq_{\mathrm{disp}}^{\wedge\text{-disc}} F$ holds. That is because of $D_F = D_G = [0, n)$, which yields $\lambda^1(D_F) = \lambda^1(D_G) = n < \infty$. Heuristically, the binomial distribution seems to be most dispersed when it is symmetric. Its dispersion declines, if the success probability becomes markedly high or low as then, the probability mass is concentrated heavily on one side. This observation is reflected in the left panel of Figure B.1, which depicts the behaviour of the dispersion measures from Chapter 4 for fixed n and varying π.

Furthermore, the plot shows that the dispersion measures display differing degrees of smoothness as a function of the success probability. While IQR is the only measure that is not continuous, MAD, MDMAD and IER also exhibit

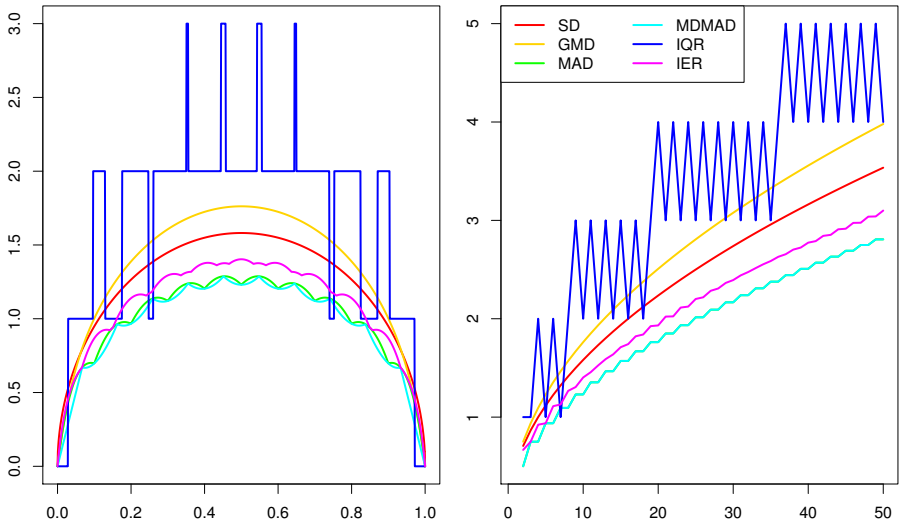

FIG B.1. *Plot of $\tau(X)$ for six different dispersion measures τ and $X \sim \mathrm{Bin}(n, \pi)$. Left panel: $n = 10$ fixed, $\tau(X)$ as a function of π. Right panel: $\pi = \frac{1}{2}$ fixed, $\tau(X)$ as a function of n.*

some lack of smoothness. Solely the graphs of SD and GMD look like they could stem from an infinitely often differentiable function.

For binomial distributions with fixed success probability π and varying sample size n, we restrict ourselves to the symmetric case $\pi = \frac{1}{2}$. If we consider two distributions $F = \mathrm{Bin}(m, \frac{1}{2})$ and $G = \mathrm{Bin}(n, \frac{1}{2})$ with $m, n \in \mathbb{N}_{\geq 2}$ and $m < n$, we can once again invoke Proposition 3.10 to obtain $G \not\leq_{\mathrm{disp}}^{\wedge\text{-disc}} F$. The remaining question is: if at all, under which conditions concerning m and n does $F \leq_{\mathrm{disp}}^{\wedge\text{-disc}} G$ hold? Because of the non-explicit structure of the cdf of the binomial distribution, no theoretical result answering this question could be proved. Instead, we have to rely solely on numerical computations.

The results are depicted in Figure B.2. They generally support the graphical impression that the (symmetric) binomial distribution becomes more dispersed as its sample size increases. However, the difference between the two sample sizes m and n needs to be quite large for $F \leq_{\mathrm{disp}}^{\wedge\text{-disc}} G$ to hold. For $n < 5m$, $F \leq_{\mathrm{disp}}^{\wedge\text{-disc}} G$ only holds very sporadically. For $5m \leq n \leq 10m$, $F \leq_{\mathrm{disp}}^{\wedge\text{-disc}} G$ holds in some cases, depending on the compatibility of the two distributions. However, as n approaches $10m$, the share of positive results seems to increase. Finally, for $n > 10m$, $F \leq_{\mathrm{disp}}^{\wedge\text{-disc}} G$ always holds with very few exceptions if $n \approx 10m$. It is notable that the borders between the red and the mixed area as well as between the mixed and the green area both seem to be approximately linear. According to further numerical evaluations for larger sample sizes, the factors of 5 and 10 seem to grow a bit further to approximately 8.5 and 12 at $m = 400$.

The behaviour of the dispersion measures when applied to symmetric binomial distributions is similar to our previous observations. All graphs of the

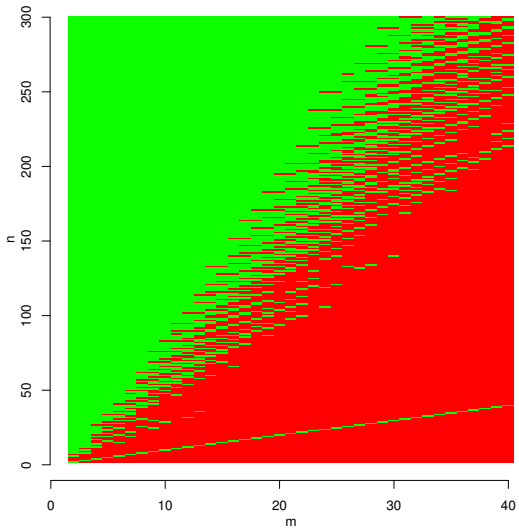

FIG B.2. *Plot of the numerical results concerning* $F \leq_{disp}^{\wedge\text{-}disc} G$ *for selected parameter values for* $F = \mathrm{Bin}(m, \frac{1}{2})$ *and* $G = \mathrm{Bin}(n, \frac{1}{2})$. *In green areas,* $F \leq_{disp}^{\wedge\text{-}disc} G$ *holds; in red areas,* $F \not\leq_{disp}^{\wedge\text{-}disc} G$ *holds. Note the different scales of the two axes.*

corresponding plot in the right panel of Figure B.1 are increasing, once again with the exception of IQR. Their slopes slightly decrease as n is increasing. Their smoothness properties coincide with our observations from the left panel of Figure B.1, where π varies instead of n.

B.2. Poisson distribution

The last discrete distribution considered in this section is the Poisson distribution. For that, we consider two distributions $F = \mathrm{Pois}(\lambda_F)$ and $G = \mathrm{Pois}(\lambda_G)$ with $\lambda_F, \lambda_G > 0$. Similarly to the geometric distribution, it is easy to show that $F \leq_{\mathrm{st}} G$ and even $F <_{st} G$ holds, if $\lambda_F < \lambda_G$. By Proposition 3.11, $G \not\leq_{\mathrm{disp}}^{\wedge\text{-}\mathrm{disc}} F$ follows in that case. Whether $F \leq_{\mathrm{disp}}^{\wedge\text{-}\mathrm{disc}} G$ holds can once again only be analyzed numerically since the cdf of the Poisson distribution also does not have an explicit form.

The results for selected values of λ_F and λ_G are depicted in the left panel of Figure B.3. As for the binomial distribution, $F \leq_{\mathrm{disp}}^{\wedge\text{-}\mathrm{disc}} G$ only seems to hold, if λ_G is sufficiently large compared to λ_F. However, the differing factor between λ_F and λ_G at the border between the red and the green area decreases for increasing λ_F. For $\lambda_F = 0.05$, that factor is equal to 600, and it is subsequently reduced: to 70 for $\lambda_F = 1$, to 10 for $\lambda_F = 10$, and to 5 for $\lambda_F = 100$. It is unclear whether this reduction is representative of the actual interaction between the Poisson distribution and the order $\leq_{\mathrm{disp}}^{\wedge\text{-}\mathrm{disc}}$ or it is a numerical phenomenon. The

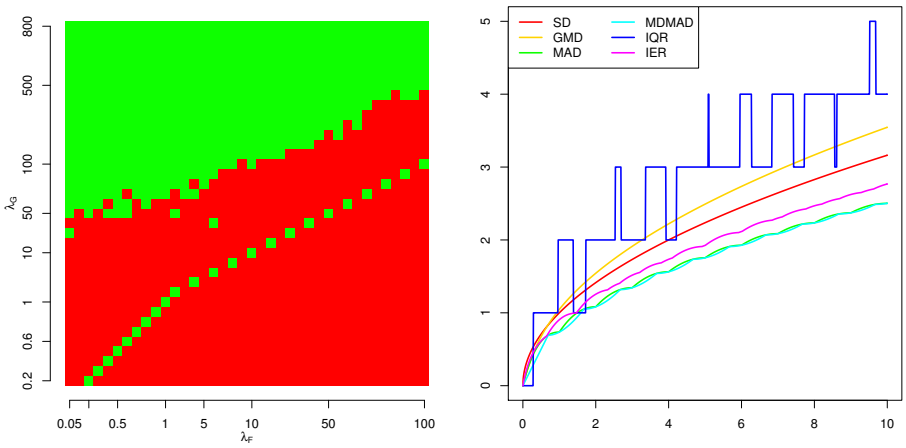

FIG B.3. *Left panel: Plot of the numerical results concerning* $F \leq_{disp}^{\wedge\text{-}disc} G$ *(green: yes; red: no) for selected parameter values for* $F = \text{Pois}(\lambda_F)$ *and* $G = \text{Pois}(\lambda_G)$*. Note that the scale on both axes is not linear and only contains exemplary values.*
Right panel: Plot of $\tau(X)$ *for six different dispersion measures* τ *and* $X \sim \text{Pois}(\lambda)$*, as a function of* λ*.*

latter explanation is supported by the fact that, with increasing parameter λ, the amount of probability mass within jumps too small to register numerically also increases. Therefore, more relevant jumps cannot be compared properly.

The behaviour of the dispersion measures plotted in the right panel of Figure B.3 is similar to the previous distribution families. The declining slope of the graphs is indicative of smaller differences in dispersion for higher parameter values and therefore suggests that our observations about the left panel are indeed due to numerical inaccuracies.