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# Large deviations for a spatial average of stochastic heat and wave equations

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**Abstract.** We consider the one-dimensional stochastic heat and wave equations driven by Gaussian noises with constant initial conditions. We study the spatial average of the solutions on an interval of length $R$ and show that the family of laws of the spatial average satisfies the large deviation principle as $R$ goes to infinity. We also present the large deviation principle in the space of continuous functions. We prove these results using the tools of Malliavin calculus to evaluate the covariance of nonlinear functionals of the solution.

## 1. Introduction

This paper considers the following form of one-dimensional stochastic heat and wave equations

$$\mathcal{L}u(t,x) = \sigma(u(t,x))\dot{W}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}, \quad \mathcal{L} = \partial_t - \frac{1}{2}\partial_x^2 \text{ or } \mathcal{L} = \partial_t^2 - \partial_x^2, \tag{1.1}$$

with constant initial conditions. Here $T > 0$ is fixed, $\sigma\colon \mathbb{R} \to \mathbb{R}$, and $\dot{W}(t,x)$ is the formal notation of a centered Gaussian noise. We analyze a random field solution $\{u(t,x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$ to (1.1) (see Section 2.1 for details), and the goal of this paper is to show the Large Deviation Principle (LDP for short) for the spatial average of the solution process defined by

$$\frac{1}{R}F_R(t) \coloneqq \frac{1}{R}\int_0^R (u(t,x) - \mathbb{E}[u(t,x)])dx, \quad t \in [0,T].$$

In recent years, the asymptotic behavior of spatial averages of solutions to stochastic PDEs has been actively studied, and various limit theorems have been established in different settings using tools from Malliavin calculus. For example, the works Huang et al. (2020a,b); Delgado-Vences et al. (2020); Nualart and Zheng (2022) treat the equation (1.1) and establish quantitative central limit theorems and its functional version for the spatial averages of the solution using Malliavin-Stein's method. Following these results, Chen et al. (2021) and Nualart and Zheng (2020) investigate in

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detail the spatial ergodicity of the solutions to stochastic heat and wave equations and show that the law of large numbers

$$\frac{1}{R}F_R(t) \xrightarrow{R\to\infty} 0, \qquad \text{a.s. and in } L^p(\Omega),\ p\in[1,\infty) \tag{1.2}$$

holds under some mild conditions on noises. Furthermore, the law of the iterated logarithm for $F_R(t)$, when $\mathcal{L}$ is the heat operator, is also shown in Li and Zhang (2023).

The common underlying idea in all of these works is that the spatial average of the solution can be approximately viewed as a sum of weakly dependent random variables, depending on the spatial correlation of the driving noise. With this and (1.2) in mind, it is then natural to study the associated LDP for the law of spatial average $R^{-1}F_R(t)$. However, it remains unknown whether the LDP holds, except in the trivial case where $\sigma$ is a constant function, in which the law of $F_R(t)$ follows a Gaussian distribution. For general $\sigma$, a large deviation upper bound can be obtained if $\sigma$ is a bounded Lipschitz function, using a similar argument as in Mueller (1991, Lemma 2.1). However, this argument cannot provide a large deviation lower bound and, therefore, cannot establish the LDP. This paper aims to address this issue and to present that the LDP holds for the family of the laws of $\{R^{-1}F_R(t)\}_{R>0}$ under stronger conditions on $\sigma$.

1.1. *Settings and main results.* Throughout the paper, we assume that a centered Gaussian noise $\dot{W}(t,x)$ in (1.1) has the following covariance structure

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)\Gamma(x-y),$$

where $\delta_0$ is the Dirac delta function and $\Gamma$ is a nonnegative, nonnegative definite tempered measure.

To establish the LDP results, we need to impose certain conditions on $\sigma$ in (1.1) and on $\Gamma$.

**Assumption 1.1.** We work under the following assumptions:

(1) $\sigma\colon \mathbb{R}\to\mathbb{R}$ belongs to $C_{\mathrm{b}}^1(\mathbb{R})$, the space of bounded $C^1$ functions with bounded derivatives.
(2) $\Gamma$ satisfies one of the following two cases:
  (I) $\Gamma(dx) = \delta_0(dx)$, *i.e.* $\dot{W}$ is the space-time white noise.
  (II) $\Gamma(dx) = \Gamma(x)dx$ is a nonnegative, nonnegative definite measure, and for some $\rho>0$ and $\eta>1$,
$$\Gamma(x) = O(\exp(-\rho|x|^\eta)), \quad \text{as } |x|\to\infty.$$

*Remark* 1.2. (1) For technical reasons, we assume that $\sigma$ is bounded so that the quadratic variations associated with the solution $u$ can be controlled almost surely. See Section 4 for details.

(2) It is known that the LDP holds for stationary sequences of random variables under sufficiently strong mixing-type conditions (see *e.g.* Bryc and Dembo, 1996; Dembo and Zeitouni, 2010, Section 6.4). Given this, and recalling that the spatial integral of the solution can heuristically be interpreted as a sum of weakly dependent random variables, it is natural to impose an appropriate mixing-type condition in order to establish the LDP for the spatial average. The assumption on the spatial correlation $\Gamma$ of the noise is introduced to ensure such a mixing-type condition (cf. Proposition 3.4).

(3) In case (II), $\Gamma(x)dx$ becomes a tempered measure, and consequently, $\Gamma(x)\in L^1(\mathbb{R})$. A function $\Gamma$ satisfying (II) can be easily constructed using convolution. Specifically, for a locally integrable nonnegative function $f$ with $f(x) = O(\exp(-\rho 2^\eta|x|^\eta))$, the function $\Gamma(x) = (f(-\cdot) * f(\cdot))(x)$ satisfies assumption (II) (*cf.* Lemma 3.5).

For any $k\in\mathbb{N}$ and $(t_1,\dots,t_k) =: \mathbb{T}_k \in [0,T]^k$, let $\mu_R^{\mathbb{T}_k}$ denote the law on $\mathbb{R}^k$ of

$$\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k) := \frac{1}{R}(F_R(t_1),\dots,F_R(t_k)) = \left(\frac{1}{R}F_R(t_1),\dots,\frac{1}{R}F_R(t_k)\right).$$

The first main result is the LDP for the finite-dimensional distributions of the spatial average.

**Theorem 1.3.** *Suppose that Assumption 1.1 is satisfied. Then, for any $k \in \mathbb{N}$ and $(t_1, \dots, t_k) =: \mathbb{T}_k \in [0,T]^k$, the family of measures $\{\mu_R^{\mathbb{T}_k}\}_{R>0}$ satisfies the LDP with speed $R$ and the convex good rate function*

$$I^{\mathbb{T}_k}(x) = \sup_{\lambda \in \mathbb{R}^k} \{\lambda \cdot x - \Lambda^{\mathbb{T}_k}(\lambda)\}, \qquad \text{where} \quad \Lambda^{\mathbb{T}_k}(\lambda) := \lim_{R\to\infty} \frac{1}{R} \log \mathbb{E}[e^{\lambda \cdot \mathbb{F}_R(\mathbb{T}_k)}].$$

*In particular, the limit $\Lambda^{\mathbb{T}_k}(\lambda)$ exists for any $\lambda \in \mathbb{R}^k$.*

As we will see in Lemma 5.1, for each $R > 0$, $\{R^{-1}F_R(t)\}_{t\in[0,T]}$ admits a continuous modification. Let $\mu_R$ denote the law of this continuous modification. By applying Theorem 1.3, we can extend the LDP to the space of continuous functions $C([0,T])$, which is our second main result.

**Theorem 1.4.** *Suppose that Assumption 1.1 is satisfied. The family of measures $\{\mu_R\}_{R>0}$ satisfies the LDP in $C([0,T])$ endowed with the uniform convergence topology, with speed $R$ and the convex good rate function*

$$I(f) = \sup_{k\in\mathbb{N}} \sup_{0\le t_1<\cdots<t_k\le T} \left\{ I^{\mathbb{T}_k}\Big((f(t_1), \dots, f(t_k))\Big)\right\}, \quad f \in C([0,T]).$$

*Remark* 1.5. A similar LDP also holds for the laws of a spatial average $(2R)^{-1}\int_{-R}^{R}(u(t,x) - \mathbb{E}[u(t,x)])dx$ because, in our setting, a solution process $\{u(t,x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$ to (1.1) satisfies the property (S). See Section 2.1 and Lemma 2.3 for more details.

*Remark* 1.6. Since the law of large numbers and central limit theorems have also been established for the spatial average of stochastic heat and wave equations in two or more spatial dimensions (see *e.g.* Chen et al., 2021; Nualart and Zheng, 2020; Huang et al., 2020b; Nualart and Zheng, 2022; Ebina, 2024, 2025), we can expect that the LDP should hold in the higher-dimensional case under the same assumptions. Unfortunately, the approach used to prove Theorems 1.3 and 1.4 does not appear to apply easily to the higher-dimensional case. To establish the LDP results, we rely on an approximate subadditivity argument to ensure the existence of certain limits in Section 4. However, some of the estimates provided there become too crude to guarantee the existence of such limits, essentially due to the absence of a natural ordering in higher dimensions.

After introducing some notation and conventions below, the structure of the paper is as follows. In Section 2, we briefly present the formulation of (1.1) and recall some preliminary results on the large deviation theory and the Malliavin calculus used in the paper. Section 3 presents a key estimate that is crucial for proving the main results. Then, we prove Theorem 1.3 and Theorem 1.4 in Section 4 and Section 5, respectively.

**Notation.** We will use the following standard notations throughout the paper.

- For $x, y \in \mathbb{R}^k$, let $|x|$ and $x \cdot y$ denote the usual Euclidean norm and inner product on $\mathbb{R}^k$, respectively.
- For $a, b \in \mathbb{R}$, we use $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.
- For a Lipschitz function $f$, we write $\mathrm{Lip}(f)$ for its Lipschitz constant.
- For some $l$ quantities $Q_1, \dots Q_l$ $(l \in \mathbb{N})$, we denote by $C_{Q_1,\dots,Q_l}$ a constant depending on $Q_1, \dots Q_l$ that may vary from line to line. These constants are not necessarily the same in different places and are used to simplify expressions.

In this paper, we adopt the following notational convention (with slight abuse of notation) to treat both cases of $\Gamma$ in Assumption 1.1 in a unified manner.

**Convention.** Throughout the paper, we set

$$\|\delta_0\|_{L^1(\mathbb{R})} := 1 \quad \text{and} \quad \int_{\mathbb{R}^2} f(x)g(y)\delta_0(x-y)dxdy := \int_{\mathbb{R}} f(x)g(x)dx.$$

When $\Gamma = \delta_0$, we interpret $\delta_0$ as $\delta_0(x) = O(\exp(-\rho|x|^\eta))$ with $\eta = \infty$, in analogy with case (II) of Assumption 1.1. For example, expressions like $2 \wedge \eta$ that appear in later sections should be understood as $2 \wedge \infty = 2$ when $\Gamma = \delta_0$. On the other hand, for constants that are stated to depend on $\eta$ but are not explicitly written, we assume that they are independent of $\eta$ when $\Gamma = \delta_0$. For example, a constant like $C_{T,\eta,\rho}$ that appears in later sections should be understood as not depending on $\eta$ when $\Gamma = \delta_0$.

## 2. Preliminaries

2.1. *Stochastic heat and wave equations.* In this section, we recall the formulation of (1.1) using the random field approach. Let us define the separable real Hilbert space $\mathcal{H}$ by completing the space $\mathcal{S}(\mathbb{R})$ of real-valued rapidly decreasing functions on $\mathbb{R}$ with the real inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} \varphi(x)\psi(y)\Gamma(x-y)dxdy, \qquad \varphi, \psi \in \mathcal{S}(\mathbb{R}).$$

We set $\mathcal{H}_T = L^2([0,T];\mathcal{H})$. When $\Gamma = \delta_0$, then $\mathcal{H} = L^2(\mathbb{R})$ and $\mathcal{H}_T = L^2([0,T];L^2(\mathbb{R})) \simeq L^2([0,T]\times\mathbb{R})$. Note that $\mathbf{1}_{[0,t]}\mathbf{1}_A \in \mathcal{H}_T$ for all $t \geq 0$ and $A \in \mathcal{B}_{\mathrm{b}}(\mathbb{R})$, where $\mathcal{B}_{\mathrm{b}}(\mathbb{R})$ is the collection of all bounded Borel sets of $\mathbb{R}$.

Let $W = \{W(h) \mid h \in \mathcal{H}_T\}$ be a centered Gaussian process with the covariance function

$$\mathbb{E}[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}_T},$$

defined on a complete probability space $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}$ is assumed to be generated by $W$. We write $\mathscr{F}_t^0$ for the $\sigma$-algebra generated by $\{W(\mathbf{1}_{[0,s]}\mathbf{1}_A) \mid 0 \leq s \leq t,\ A \in \mathcal{B}_{\mathrm{b}}(\mathbb{R})\}$ and the $P$-null sets, and we set $\mathscr{F}_t = \cap_{s>t}\mathscr{F}_s^0$ for $t \in [0,T)$ and $\mathscr{F}_T = \mathscr{F}_T^0$. Then, we can construct a worthy martingale measure $\{W(\mathbf{1}_{[0,t]}\mathbf{1}_A) \mid t \in [0,T], A \in \mathcal{B}_{\mathrm{b}}(\mathbb{R})\}$ with respect to $\{\mathscr{F}_t\}_{t\in[0,T]}$. With this measure, we can apply the theory of stochastic integration developed in Walsh (1986) to formulate the equation (1.1). For further details, we refer the reader to Walsh (1986); Dalang and Quer-Sardanyons (2011).

We consider (1.1) with the constant initial conditions

$$u(0,x) = c_H \ \text{ for } \mathcal{L} = \partial_t - \frac{1}{2}\partial_x^2 \quad \text{and} \quad u(0,x) = c_{W1},\ \partial_t u(0,x) = c_{W2} \ \text{ for } \mathcal{L} = \partial_t^2 - \partial_x^2.$$

A solution to (1.1) is defined as a predictable process $\{u(t,x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$ satisfying

$$u(t,x) = \mathcal{I}(t) + \int_0^t \int_{\mathbb{R}} G(t-s, x-y)\sigma(u(s,y))W(ds,dy) \quad \text{a.s.}, \tag{2.1}$$

for all $(t,x) \in [0,T]\times\mathbb{R}$. Here, $\mathcal{I}(t)$ is the contribution from initial conditions and $G$ is the associated fundamental solution of the operator $\mathcal{L}$ in (1.1). In particular, when $\mathcal{L} = \partial_t - \frac{1}{2}\partial_x^2$,

$$G(t,x) = (2\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{2t}}, \quad \mathcal{I}(t) = G(t) * c_H = c_H, \tag{2.2}$$

and when $\mathcal{L} = \partial_t^2 - \partial_x^2$,

$$G(t,x) = \frac{1}{2}\mathbf{1}_{\{y\in\mathbb{R}:|y|<t\}}(x), \quad \mathcal{I}(t) = (G(t) * c_{W2}) + \partial_t(G(t) * c_{W1}) = tc_{W2} + c_{W1}.$$

A solution $u$ to (1.1) is said to be unique if, for any other solution $\{v(t,x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$, we have $P(u(t,x) = v(t,x)) = 1$ for every $(t,x) \in [0,T]\times\mathbb{R}$. The following is a standard result (see *e.g.* Dalang, 1999; Dalang and Quer-Sardanyons, 2011).

**Proposition 2.1.** *Let $\sigma\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. The equation (1.1) has a unique solution $\{u(t,x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$, and the following holds.*

(i) *For all $p \geq 1$, $\sup_{(t,x)\in[0,T]\times\mathbb{R}}\|u(t,x)\|_{L^p(\Omega)} < \infty$ and $(t,x) \mapsto u(t,x)$ is $L^p(\Omega)$-continuous.*

(ii) *For each $t \geq 0$, the process $\{u(t,x)\}_{x\in\mathbb{R}}$ is strictly stationary.*

Moreover, it is known (*cf.* Dalang, 1999, Lemma 18) that the solution $u$ to (1.1) satisfies the property (S). For the definition of property (S), see Dalang (1999, Definition 5.1). One of the consequences of property (S) is that for any $n \in \mathbb{N}$ and any $(t_1, x_1), \dots, (t_n, x_n) \in [0, T] \times \mathbb{R}$,

$$(u(t_1, x_1), \dots, u(t_n, x_n)) \overset{d}{=} (u(t_1, x_1 + a), \dots, u(t_n, x_n + a)) \tag{2.3}$$

holds for each $a \in \mathbb{R}$. Here, $\overset{d}{=}$ denotes equality in distribution between the left-hand side and the right-hand side.

*Remark* 2.2. Strictly speaking, Dalang (1999, Lemma 18) shows that a sequence of approximations $u_n$ converging in $L^2(\Omega)$ to a solution $u$ of the SPDE considered there satisfies property (S) for each $n$. However, it is easy to see that $u$ itself also satisfies property (S), as a consequence of the $L^2(\Omega)$-convergence. It is also worth noting that although a zero initial condition is considered in Dalang (1999), if the contribution from the initial data is deterministic and independent of the spatial variable, then property (S) can still be shown to hold by an argument similar to that in Ebina (2025, Lemma 4.2).

For simplification, let us introduce the following notation: For $a, b \in \mathbb{R}$, $t \in [0, T]$, and $\mathbb{T}_k = (t_1, \dots, t_k) \in [0, T]^k$,

$$F_b^a(t) \coloneqq \int_a^b (u(t, x) - \mathbb{E}[u(t, x)])dx, \quad \mathbb{F}_b^a(\mathbb{T}_k) \coloneqq (F_b^a(t_1), \dots, F_b^a(t_k)), \quad \text{and} \quad \mathbb{F}_b(\mathbb{T}_k) \coloneqq \mathbb{F}_b^0(\mathbb{T}_k).$$

Using property (S) and the $L^2(\Omega)$-continuity of the solution $u$, we obtain the following invariance for its spatial average.

**Lemma 2.3.** *Let $\{u(t, x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$ be the solution to (1.1). For every $k \in \mathbb{N}$, $\mathbb{T}_k = (t_1, \dots, t_k) \in [0, T]^k$, and $a, b \in \mathbb{R}$ with $a < b$, we have*

$$\mathbb{F}_b^a(\mathbb{T}_k) \overset{d}{=} \mathbb{F}_{b+c}^{a+c}(\mathbb{T}_k)$$

*for any $c \in \mathbb{R}$. In particular, the distribution of $\mathbb{F}_{b+c}^{a+c}(\mathbb{T}_k)$ does not depend on $c$.*

*Proof*: For each $i = 1, \dots, k$, by Proposition 2.1, $x \mapsto u(t_i, x)$ is uniformly $L^2(\Omega)$-continuous on compact sets of $\mathbb{R}$. Thus, for any $\varepsilon > 0$, there exists $\delta_{i,\varepsilon} > 0$ such that for any $x, x' \in \mathbb{R}$ with $|x - x'| < \delta_{i,\varepsilon}$,

$$\|u(t_i, x) - u(t_i, x')\|_{L^2(\Omega)} < \varepsilon.$$

For each $i$ and $\varepsilon$, let $a = r_0^{i,\varepsilon} < r_1^{i,\varepsilon} < \cdots < r_{n_{i,\varepsilon}}^{i,\varepsilon} = b$ be a partition of $[a, b]$ such that

$$\max_{0\le j\le n_{i,\varepsilon}-1} |r_j^{i,\varepsilon} - r_{j+1}^{i,\varepsilon}| < \delta_{i,\varepsilon},$$

and take $y_j^{i,\varepsilon} \in [r_j^{i,\varepsilon}, r_{j+1}^{i,\varepsilon}]$ arbitrarily for each $j$. Then, we can easily check that

$$\left( \sum_{j=0}^{n_{1,\varepsilon}-1} (u(t_1, y_j^{1,\varepsilon}) - \mathbb{E}[u(t_1, y_j^{1,\varepsilon})]) \int_{r_j^{1,\varepsilon}}^{r_{j+1}^{1,\varepsilon}} dx, \dots, \sum_{j=0}^{n_{k,\varepsilon}-1} (u(t_k, y_j^{k,\varepsilon}) - \mathbb{E}[u(t_k, y_j^{k,\varepsilon})]) \int_{r_j^{k,\varepsilon}}^{r_{j+1}^{k,\varepsilon}} dx \right)$$

converges in $L^2(\Omega;\mathbb{R}^k)$ to $\mathbb{F}_b^a(\mathbb{T}_k)$ as $\varepsilon \to 0$. Now observe from (2.3) that for any $c \in \mathbb{R}$, we have

$$\begin{aligned}
&\left( \sum_{j=0}^{n_{1,\varepsilon}-1} (u(t_1, y_j^{1,\varepsilon}) - \mathbb{E}[u(t_1, y_j^{1,\varepsilon})]) \int_{r_j^{1,\varepsilon}}^{r_{j+1}^{1,\varepsilon}} dx, \dots, \sum_{j=0}^{n_{k,\varepsilon}-1} (u(t_k, y_j^{k,\varepsilon}) - \mathbb{E}[u(t_k, y_j^{k,\varepsilon})]) \int_{r_j^{k,\varepsilon}}^{r_{j+1}^{k,\varepsilon}} dx \right) \\
&\overset{d}{=} \left( \sum_{j=0}^{n_{1,\varepsilon}-1} (u(t_1, y_j^{1,\varepsilon} - c) - \mathbb{E}[u(t_1, y_j^{1,\varepsilon} - c)]) \int_{r_j^{1,\varepsilon}}^{r_{j+1}^{1,\varepsilon}} dx, \dots \right. \\
&\qquad \left. \dots, \sum_{j=0}^{n_{k,\varepsilon}-1} (u(t_k, y_j^{k,\varepsilon} - c) - \mathbb{E}[u(t_k, y_j^{k,\varepsilon} - c)]) \int_{r_j^{k,\varepsilon}}^{r_{j+1}^{k,\varepsilon}} dx \right).
\end{aligned}$$

Since the right-hand side converges in $L^2(\Omega;\mathbb{R}^k)$ to $\mathbb{F}_{b+c}^{a+c}(\mathbb{T}_k)$ as $\varepsilon \to 0$, we can conclude that $F_b^a(\mathbb{T}_k) \overset{d}{=} F_{b+c}^{a+c}(\mathbb{T}_k)$, and the lemma follows. □

2.2. *Malliavin calculus.* Here we introduce some elements of the Malliavin calculus based on the Gaussian process $W = \{W(h) \mid h \in \mathcal{H}_T\}$ defined in Section 2.1. For much more information, the reader is referred to Nualart (2006).

We say that $F\colon \Omega \to \mathbb{R}$ is a smooth cylindrical random variable with respect to $W$ if $F$ is of the form $F = f(W(h_1), \dots, W(h_k))$ for some $k \in \mathbb{N}$, $h_1, \dots, h_k \in \mathcal{H}_T$, and a smooth function $f\colon \mathbb{R}^k \to \mathbb{R}$ such that $f$ and all orders of its partial derivatives are at most of the polynomial growth.

We will write $D$ for the Malliavin derivative operator. If $F = f(W(h_1), \dots, W(h_k))$ is a smooth cylindrical random variable, then the Malliavin derivative $DF$ is the $\mathcal{H}_T$-valued random variable given by

$$DF = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_k)) h_i.$$

The operator $D$ is a closed operator from $L^p(\Omega)$ into $L^p(\Omega;\mathcal{H}_T)$ for any $p \in [1,\infty)$, and the domain of $D$ in $L^p(\Omega)$ is the Sobolev space $\mathbb{D}^{1,p}$ defined as the closure of the set of all smooth cylindrical random variables in $L^p(\Omega)$ under the norm

$$\|F\|_{1,p} := (\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathcal{H}_T}^p])^{\frac{1}{p}}.$$

For any $F \in \mathbb{D}^{1,p}$ and a Lipschitz function $\phi\colon \mathbb{R} \to \mathbb{R}$, $\phi(F)$ belongs to $\mathbb{D}^{1,p}$ and we have the following chain rule: there exists a real-valued random variable $\Phi_F$ with $|\Phi_F| \le \mathrm{Lip}(\phi)$ almost surely such that

$$D\phi(F) = \Phi_F DF.$$

If $\phi$ is a continuously differentiable function with bounded derivative, we have $\Phi_F = \phi'(F)$.

Let $F, G \in \mathbb{D}^{1,2}$. If the derivatives $DF$ and $DG$ have good enough versions $D_{t,x}F$, $D_{t,x}G$ which are measurable functions on $[0,T] \times \mathbb{R} \times \Omega$, then we can derive from the Clark-Ocone formula that

$$|\mathrm{Cov}(F,G)| \le \int_0^T \int_{\mathbb{R}^2} \|D_{s,y}F\|_{L^2(\Omega)} \|D_{s,z}G\|_{L^2(\Omega)} \Gamma(y-z)\,dydzds. \tag{2.4}$$

We will use this to obtain the covariance estimates for the spatial averages of the solution in Section 3.

2.3. *Tools from large deviation theory.* This section collects some basic notions and tools from the large deviation theory. See Dembo and Zeitouni (2010) for more details.

Let $\mathcal{X}$ be a topological space and $\mathcal{B}(\mathcal{X})$ be its Borel $\sigma$-algebra. A function $I\colon \mathcal{X} \to [0,\infty]$ is called a rate function if $I$ is lower semicontinuous. We say that a family of probability measures $\{\nu_R\}_{R>0}$

on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ satisfies the LDP with speed $R$ and a rate function $I$ if, for all $A \in \mathcal{B}(\mathcal{X})$,

$$-\inf_{x\in A^o} I(x) \le \liminf_{R\to\infty} \frac{1}{R}\log \nu_R(A) \le \limsup_{R\to\infty} \frac{1}{R}\log \nu_R(A) \le -\inf_{x\in \bar{A}} I(x),$$

where $A^o$ and $\bar{A}$ denote the interior and the closure of $A$, respectively. A rate function $I$ is said to be a good rate function if, for all $r \in [0,\infty)$, $I^{-1}([0,r])$ is compact. A family $\{\nu_R\}_{R>0}$ is called exponentially tight if for every $\alpha < \infty$, there exists a compact set $K_\alpha \subset \mathcal{X}$ such that

$$\limsup_{R\to\infty} \frac{1}{R}\log \nu_R(K_\alpha^c) < -\alpha.$$

Let us recall the definition of a well-separating class given in Dembo and Zeitouni (2010, Definition 4.4.7). A class $\mathcal{G}$ of continuous real-valued functions on $\mathcal{X}$ is called well-separating if it satisfies the following:

(i) $\mathcal{G}$ contains the all constant functions.
(ii) If $g_1, g_2 \in \mathcal{G}$, then $g_1 \wedge g_2 \in \mathcal{G}$, where $g_1 \wedge g_2(x) := \min\{g_1(x), g_2(x)\}$.
(iii) For any $x, y \in \mathcal{X}$ with $x \neq y$ and $a, b \in \mathbb{R}$, there exists $g \in \mathcal{G}$ such that $g(x) = a$ and $g(y) = b$.

For instance, when $\mathcal{X}$ is a locally convex Hausdorff topological vector space, the class of all continuous, bounded above, concave functions on $\mathcal{X}$ is well-separating, as shown in Dembo and Zeitouni (2010, Lemma 4.4.8).

In Section 4, we establish the LDP for $\{\mu_R^{\mathbb{T}_k}\}_{R>0}$ by using the following form of inverse Varadhan's lemma. Let $C_{\mathrm{b}}(\mathbb{R}^k)$ be the space of continuous bounded functions on $\mathbb{R}^k$.

**Proposition 2.4** (Dembo and Zeitouni, 2010, Theorems 4.4.2 and 4.4.10)**.** *Let $\{\nu_R\}_{R>0}$ be a family of probability measures on $\mathbb{R}^k$. Let $\mathcal{G}(\mathbb{R}^k)$ be a well-separating class on $\mathbb{R}^k$. If $\{\nu_R\}_{R>0}$ is exponentially tight and the limit*

$$\Lambda_g := \lim_{R\to\infty} \frac{1}{R}\log\left(\int_{\mathbb{R}^k} e^{Rg(x)}\nu_R(dx)\right) \tag{2.5}$$

*exists for all $g \in \mathcal{G}(\mathbb{R}^k)$, then $\Lambda_f$ exists for each $f \in C_{\mathrm{b}}(\mathbb{R}^k)$, and $\{\nu_R\}_{R>0}$ satisfies the LDP with speed $R$ and the good rate function*

$$I(x) = \sup_{f\in C_{\mathrm{b}}(\mathbb{R}^k)} \{f(x) - \Lambda_f\}.$$

In this paper, we will use the specific well-separating class $\mathcal{G}_{\mathrm{BLC}}(\mathbb{R}^k)$ on $\mathbb{R}^k$.

**Lemma 2.5** (*cf.* Dembo and Zeitouni, 2010, Lemma 4.4.8)**.** *The class $\mathcal{G}_{\mathrm{BLC}}(\mathbb{R}^k)$ of all bounded above, Lipschitz continuous, concave functions on $\mathbb{R}^k$ is well-separating.*

*Proof*: The same argument as in Dembo and Zeitouni (2010, Lemma 4.4.8) works since the pointwise minimum of two Lipschitz continuous functions is again a Lipschitz continuous function. □

If the rate function $I$ in Proposition 2.4 is convex, we can show under a mild assumption that $I$ is the Fenchel-Legendre transform of

$$\Lambda(\lambda) := \lim_{R\to\infty} \frac{1}{R}\log\left(\int_{\mathbb{R}^k} e^{R\lambda\cdot x}\nu_R(dx)\right). \tag{2.6}$$

**Proposition 2.6** (Dembo and Zeitouni, 2010, Theorem 4.5.10)**.** *Let $\{\nu_R\}_{R>0}$ be a family of probability measures on $\mathbb{R}^k$ such that*

$$\limsup_{R\to\infty} \frac{1}{R}\log\left(\int_{\mathbb{R}^k} e^{R\lambda\cdot x}\nu_R(dx)\right) < \infty, \quad \text{for any } \lambda \in \mathbb{R}^k.$$

*If $\{\nu_R\}_{R>0}$ satisfies the LDP with speed $R$ and a good rate function $I$, then for each $\lambda \in \mathbb{R}^k$, the limit (2.6) exists, is finite, and satisfies $\Lambda(\lambda) = \sup_{x\in\mathbb{R}^k}\{\lambda\cdot x - I(x)\}$. Moreover, if the rate function $I$ is convex, then it is the Fenchel-Legendre transform of $\Lambda$. That is, for all $x \in \mathbb{R}^k$,*

$$I(x) = \sup_{\lambda\in\mathbb{R}^k}\{\lambda\cdot x - \Lambda(\lambda)\}.$$

In Section 4.2, we will use the following result shown in Hammersley (1962) to ensure the existence of a limit (2.5).

**Lemma 2.7** (Approximate subadditivity, Hammersley, 1962, Theorem 2)**.** *Let $\zeta\colon (0,\infty)\to\mathbb{R}$ be a non-decreasing function on $[\xi,\infty)$ for some $\xi > 0$ with $\int_\xi^\infty \frac{\zeta(x)}{x^2}dx < \infty$. If $f\colon (0,\infty)\to\mathbb{R}$ satisfies*

$$f(x+y) \le f(x) + f(y) + \zeta(x+y), \quad x, y \ge c_f,$$

*for some $c_f > 0$ which may depend on $f$, then the limit $\bar{f} = \lim_{x\to\infty}\frac{f(x)}{x} \in [-\infty,\infty)$ exists.*

We also record the large deviations result for projective limits by Dawson and Gärtner (1987) in a particular form that can be applied to prove Theorem 1.4. For the general result and more details, we refer to Dembo and Zeitouni (2010, Section 4.6).

Let

$$\mathcal{J} := \bigcup_{k\in\mathbb{N}}\{(t_1,\dots,t_k)\in\mathbb{R}^k \mid 0\le t_1<\cdots<t_k\le T\}.$$

That is, for each $j\in\mathcal{J}$, there exists $k\in\mathbb{N}$ such that $j=(t_i,\dots,t_k)$ with $0\le t_1<\cdots<t_k\le T$, and we write $|j|$ for this $k\in\mathbb{N}$ for simplicity. We define a partial order $\preceq$ on $\mathcal{J}$ as follows: For $i=(s_1,\dots,s_{|i|})\in\mathcal{J}$ and $j=(t_1,\dots,t_{|j|})\in\mathcal{J}$, $i\preceq j$ if and only if $\{s_1,\dots,s_{|i|}\}\subset\{t_1,\dots,t_{|j|}\}$. When $i\preceq j$, there exists an injection $a\colon\{1,\dots,|i|\}\to\{1,\dots,|j|\}$ and we define the projection $p_{ij}$ by

$$p_{ij}\colon\mathbb{R}^{|j|}\to\mathbb{R}^{|i|},\quad (x_1,\dots,x_{|j|})\mapsto(x_{a(1)},\dots,x_{a(|i|)}).$$

Then, $(\mathcal{J},\preceq)$ is a partially ordered right-filtering set and $((\mathbb{R}^{|j|})_{j\in\mathcal{J}},(p_{ij})_{i\preceq j\in\mathcal{J}})$ is a projective system. The projective limit $\varprojlim\mathbb{R}^{|j|}$ of the system is given by

$$\varprojlim\mathbb{R}^{|j|} := \left\{(x_j)_{j\in\mathcal{J}}\in\prod_{j\in\mathcal{J}}\mathbb{R}^{|j|}\;\middle|\; x_i = p_{ij}(x_j)\quad\text{for every } i,j\in\mathcal{J}\text{ with } i\preceq j\right\},$$

and the topology on $\varprojlim\mathbb{R}^{|j|}$ is the topology induced by the product topology of $\prod_{j\in\mathcal{J}}\mathbb{R}^{|j|}$. For each $j\in\mathcal{J}$, we write $p_j\colon\varprojlim\mathbb{R}^{|j|}\to\mathbb{R}^{|j|}$ for the canonical projection.

Let $\mathbb{R}^{[0,T]}$ be the product space and let $\widetilde{p}_j\colon\mathbb{R}^{[0,T]}\to\mathbb{R}^{|j|}$ be the map given by

$$\widetilde{p}_j(f) = (f(t_1),\dots,f(t_{|j|})),\quad\text{for each } j=(t_1,\dots,t_{|j|})\in\mathcal{J}.$$

Then, $\Phi\colon\mathbb{R}^{[0,T]}\to\varprojlim\mathbb{R}^{|j|}$ defined by $\Phi(f)=(\widetilde{p}_j(f))_{j\in\mathcal{J}}$ gives a homeomorphism, and we can identify $\varprojlim\mathbb{R}^{|j|}$ with $\mathbb{R}^{[0,T]}$ and $p_j$ with $\widetilde{p}_j$. Moreover, a Borel measure $\nu$ on $\mathbb{R}^{[0,T]}$ can naturally be regarded as a Borel measure on $\varprojlim\mathbb{R}^{|j|}$. This identification allows us to use the results of large deviations for Borel probability measures on the projective limit $\varprojlim\mathbb{R}^{|j|}$ as results for Borel probability measures on $\mathbb{R}^{[0,T]}$.

**Proposition 2.8.** *Let $\{\nu_R\}_{R>0}$ be a family of probability measures on $(\mathbb{R}^{[0,T]},\mathcal{B}(\mathbb{R}^{[0,T]}))$. If, for every $j\in\mathcal{J}$, the family of Borel probability measures $\{\nu_R\circ\widetilde{p}_j^{-1}\}_{R>0}$ on $\mathbb{R}^{|j|}$ satisfies the LDP with the good rate function $I_j$, then $\{\nu_R\}_{R>0}$ satisfies the LDP with the good rate function*

$$I(f) = \sup_{j\in\mathcal{J}}\{I_j(\widetilde{p}_j(f))\},\qquad f\in\mathbb{R}^{[0,T]}.$$

## 3. Covariance estimates

For the rest of the paper, we proceed under Assumption 1.1. We also recall that the notational convention introduced at the end of Section 1 applies throughout the paper. Unless otherwise specified, we treat the cases of the stochastic heat and wave equations simultaneously in what follows.

We use the Malliavin calculus tool to bound covariance functions. If $\sigma$ is a continuously differentiable Lipschitz function, it is known that the solution $u(t,x)$ to (1.1) belongs to $\mathbb{D}^{1,p}$ for any $(t,x) \in [0,T] \times \mathbb{R}$ and $p \in [1,\infty)$. Moreover, we can take a version $D_{s,y}u(t,x)$ of the derivative $Du(t,x)$ such that $D_{s,y}u(t,x)$ is jointly measurable in the variables $(t,x,s,y,\omega)$ and satisfies

$$D_{s,y}u(t,x) = G(t-s,x-y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}} G(t-r,x-z)\sigma'(u(r,z))D_{s,y}u(r,z)W(dr,dz). \quad (3.1)$$

We have the following moment estimate for $D_{s,y}u(t,x)$.

**Lemma 3.1.** *Let $u(t,x)$ be the solution to (1.1). For any $(t,x) \in [0,T]\times\mathbb{R}$ and $p \in [1,\infty)$, we have*

$$\|D_{s,y}u(t,x)\|_{L^p(\Omega)} \le C_{p,T,\sigma}G(t-s,x-y), \qquad \text{for almost every } (s,y) \in [0,T]\times\mathbb{R}.$$

For the proof of Lemma 3.1, see Huang et al. (2020a, Lemma A.1) for the stochastic heat equation case and see Delgado-Vences et al. (2020, Lemma 2.2) and Nualart and Zheng (2022, Theorem 1.3) for the stochastic wave equation case.

*Remark* 3.2. To be precise, in Huang et al. (2020a); Delgado-Vences et al. (2020); Nualart and Zheng (2022), the estimate in Lemma 3.1 is derived only under the setting where $\mathcal{I}(t) = 1$. In our case, $\mathcal{I}(t)$ may depend on time, but as long as it is deterministic, its Malliavin derivative vanishes, and we still have (3.1), just as in the case $\mathcal{I}(t) = 1$. Therefore, Lemma 3.1 can be proved by the same argument as in the cited works, where the case $\mathcal{I}(t) = 1$ is considered.

Recall from Section 2.1 the following notation: For $a, b \ge 0$, $t \in [0,T]$, and $\mathbb{T}_k = (t_1,\dots,t_k) \in [0,T]^k$,

$$F_b^a(t) := \int_a^b (u(t,x) - \mathbb{E}[u(t,x)])dx, \quad \mathbb{F}_b^a(\mathbb{T}_k) := (F_b^a(t_1),\dots,F_b^a(t_k)), \quad \text{and} \quad \mathbb{F}_b(\mathbb{T}_k) := \mathbb{F}_b^0(\mathbb{T}_k).$$

**Lemma 3.3.** *For any $a, b \in \mathbb{R}$, $t \in [0,T]$, and $p \in [1,\infty)$, we have*

$$\|D_{s,y}F_b^a(t)\|_{L^p(\Omega)} \le C_{p,T,\sigma}(\mathbf{1}_{[a,b]} * G(t-s))(y), \qquad \text{for almost every } (s,y) \in [0,T]\times\mathbb{R}.$$

*Proof*: $F_b^a(t)$ can be viewed as a Bochner integral valued in $L^p(\Omega)$, and, from the closedness of the differential operator $D$, it follows that $DF_b^a(t) = \int_a^b Du(t,x)dx$. It is easily seen that $\int_a^b D_{s,y}u(t,x)dx$ is a version of $DF_b^a(t)$ (*i.e.* $DF_b^a(t) = \int_a^b D_{\cdot,\star}u(t,x)dx$ in $L^p(\Omega;\mathcal{H}_T)$), and Lemma 3.3 follows from Lemma 3.1. □

By combining (2.4) and Lemma 3.3, we obtain the following covariance estimate.

**Proposition 3.4.** *Let $L, R > 0$, $k \in \mathbb{N}$, and $\varphi$, $\psi$ be Lipschitz functions on $\mathbb{R}^k$. For any $\mathbb{T}_k = (t_1,\dots,t_k) \in [0,T]^k$ and for sufficiently large $\Theta > 0$, we have*

$$\left|\mathrm{Cov}\left[\varphi(\mathbb{F}_L(\mathbb{T}_k)), \psi(\mathbb{F}_{L+\Theta+R}^{L+\Theta}(\mathbb{T}_k))\right]\right| \le C_T \mathrm{Lip}(\varphi)\mathrm{Lip}(\psi)k^2LR\exp\{-C_{T,\eta,\rho}\Theta^{2\wedge\eta}\}.$$

*Proof*: We first consider the stochastic heat equation case (*i.e.* $u(t,x)$ satisfies (2.1) with (2.2)). By the chain rule, $\varphi(\mathbb{F}_L(\mathbb{T}_k))$ and $\psi(\mathbb{F}_{L+\Theta+R}^{L+\Theta}(\mathbb{T}_k))$ belong to $\mathbb{D}^{1,2}$ and

$$D\varphi(\mathbb{F}_L(\mathbb{T}_k)) = \sum_{i=1}^k \Phi_i DF_L(t_i), \qquad D\psi(\mathbb{F}_{L+\Theta+R}^{L+\Theta}(\mathbb{T}_k)) = \sum_{i=1}^k \Psi_i DF_{L+\Theta+R}^{L+\Theta}(t_i),$$

where $\Phi_i, \Psi_i$ are random variables bounded by $\mathrm{Lip}(\varphi), \mathrm{Lip}(\psi)$, respectively. Applying (2.4) and Lemma 3.3, we get

$$\begin{aligned}
&\Big|\mathrm{Cov}\Big[\varphi(\mathbb{F}_L(\mathbb{T}_k)), \psi(\mathbb{F}^{L+\Theta}_{L+\Theta+R}(\mathbb{T}_k))\Big]\Big| \\
&\le C_{T,\sigma}\mathrm{Lip}(\varphi)\mathrm{Lip}(\psi) \\
&\quad\times \sum_{i,j=1}^{k} \int_0^{t_i\wedge t_j} \int_{\mathbb{R}^2} \big(\mathbf{1}_{[0,L]} * G(t_i - s)\big)(y)\big(\mathbf{1}_{[L+\Theta,L+\Theta+R]} * G(t_j - s)\big)(z)\Gamma(y-z)dydzds.
\end{aligned}$$

By a simple computation,

$$\begin{aligned}
&\int_0^{t_i\wedge t_j} \int_{\mathbb{R}^2} \big(\mathbf{1}_{[0,L]} * G(t_i - s)\big)(y)\big(\mathbf{1}_{[L+\Theta,L+\Theta+R]} * G(t_j - s)\big)(z)\Gamma(y-z)dydzds \\
&= \int_0^{t_i\wedge t_j} ds \int_0^{L} dw \int_{L+\Theta}^{L+\Theta+R} d\widetilde{w}(G(t_i - s) * G(t_j - s) * \Gamma)(w - \widetilde{w}) \\
&\le C_T \int_0^{L} dw \int_{L+\Theta}^{L+\Theta+R} d\widetilde{w}(\widetilde{G} * \Gamma)(w - \widetilde{w}),
\end{aligned}$$

where $\widetilde{G}(x) = e^{-\frac{|x|^2}{2T}}$. Since $(\widetilde{G}*\Gamma)(x) = O\big(\exp\big\{-C_{T,\eta,\rho}|x|^{2\wedge\eta}\big\}\big)$ as $|x| \to \infty$ by Lemma 3.5 below, the desired estimate follows. The same estimate also holds for the stochastic wave equation case since $\frac{1}{2}\mathbf{1}_{\{|y|<t\}}(x) \le C_T \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x|^2}{2t}}$ for $t \in (0,T]$ and $x \in \mathbb{R}$. This completes the proof. □

**Lemma 3.5.** *Let $f_i\colon \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ be locally integrable functions such that for some constants $K_i > 1$, $C_i > 0$, $\alpha_i > 0$, and $\beta_i > 0$,*

$$|f_i(x)| \le C_i \exp\Big(-\alpha_i|x|^{\beta_i}\Big), \qquad |x| > K_i, \quad i = 1, 2.$$

*Then $f_1, f_2 \in L^1(\mathbb{R})$ and for any $x$ with $|x| > 2(K_1 \vee K_2)$, we have*

$$|(f_1 * f_2)(x)| \le 2(C_1\|f_2\|_{L^1(\mathbb{R})} \vee C_2\|f_1\|_{L^1(\mathbb{R})}) \exp\Big(-\Big(\frac{\alpha_1}{2^{\beta_1}} \wedge \frac{\alpha_2}{2^{\beta_2}}\Big)|x|^{\beta_1\wedge\beta_2}\Big).$$

*Proof*: Obviously, $f_1, f_2 \in L^1(\mathbb{R})$. If $|x| > 2(K_1 \vee K_2)$ and $|y| \le \frac{|x|}{2}$, then $|x-y| \ge |x| - |y| \ge \frac{|x|}{2} > (K_1 \vee K_2)$, and we have

$$\begin{aligned}
&|(f_1 * f_2)(x)| \\
&\le \int_{\{|y|\le\frac{|x|}{2}\}} |f_1(x-y)||f_2(y)|dy + \int_{\{|y|>\frac{|x|}{2}\}} |f_1(x-y)||f_2(y)|dy \\
&\le C_1 \int_{\{|y|\le\frac{|x|}{2}\}} \exp(-\alpha_1|x-y|^{\beta_1})|f_2(y)|dy + C_2 \int_{\{|y|>\frac{|x|}{2}\}} |f_1(x-y)| \exp(-\alpha_2|y|^{\beta_2})dy \\
&\le C_1 \int_{\{|y|\le\frac{|x|}{2}\}} |f_2(y)|dy \exp\Big(-\frac{\alpha_1}{2^{\beta_1}}|x|^{\beta_1}\Big) + C_2 \int_{\{|y|>\frac{|x|}{2}\}} |f_1(x-y)|dy \exp\Big(-\frac{\alpha_2}{2^{\beta_2}}|x|^{\beta_2}\Big) \\
&\le C_1\|f_2\|_{L^1(\mathbb{R})} \exp\Big(-\frac{\alpha_1}{2^{\beta_1}}|x|^{\beta_1}\Big) + C_2\|f_1\|_{L^1(\mathbb{R})} \exp\Big(-\frac{\alpha_2}{2^{\beta_2}}|x|^{\beta_2}\Big).
\end{aligned}$$

The desired inequality now follows from a straightforward estimate. □

## 4. LDP for the finite-dimensional distributions

We prove Theorem 1.3 in this section. The flow of the proof is essentially the same as in Dembo and Zeitouni (2010, Section 6.4), but in our setting, the boundedness of random variables assumed in Dembo and Zeitouni (2010) is not satisfied. Instead, we proceed by using tail estimates of spatial integrals.

Section 4.1 and Section 4.2 are devoted to proving the exponential tightness of $\{\mu_R^{\mathbb{T}_k}\}_{R>0}$ and the existence of the limit $\lim_{R\to\infty} R^{-1}\log\mathbb{E}[e^{Rg(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k))}]$ for each $g\in\mathcal{G}_{\mathrm{BLC}}(\mathbb{R}^k)$, respectively. It follows that $\{\mu_R^{\mathbb{T}_k}\}_{R>0}$ satisfies the LDP with speed $R$ and the good rate function $I^{\mathbb{T}_k}$, thanks to Proposition 2.4. We then apply Proposition 2.6 to show that $I^{\mathbb{T}_k}$ is convex and can be expressed as the Fenchel-Legendre transform in Section 4.3.

4.1. *Exponential tightness.*

**Proposition 4.1.** *For every $k\in\mathbb{N}$ and $\mathbb{T}_k=(t_1,\dots,t_k)\in[0,T]^k$, the family of probability measures $\{\mu_R^{\mathbb{T}_k}\}_{R>0}$ is exponentially tight.*

*Proof*: If we define

$$F_{R,t}(r)=\int_0^r\int_{\mathbb{R}}(G(t-s)*\mathbf{1}_{[0,R]})(y)\sigma(u(s,y))W(ds,dy),$$

then $\{F_{R,t}(r)\}_{r\in[0,T]}$ is a square-integrable continuous martingale such that $F_{R,t}(t)=F_R(t)$ and

$$\begin{aligned}\langle F_{R,t}\rangle_t&=\int_0^t\int_{\mathbb{R}}(G(t-s)*\mathbf{1}_{[0,R]})(y)(G(t-s)*\mathbf{1}_{[0,R]})(z)\sigma(u(s,y))\sigma(u(s,z))\Gamma(y-z)dydzds\\&\le C_T\|\sigma\|_\infty^2\|\Gamma\|_{L^1(\mathbb{R})}R=:C_{T,\Gamma,\sigma}R\end{aligned}$$

Let $\alpha>0$. The exponential inequality for martingales (*cf.* Nualart, 2006, Section A.2) yields

$$\begin{aligned}&\mu_R^{\mathbb{T}_k}\left(\left(\left[-\sqrt{2\alpha C_{T,\Gamma,\sigma}},\sqrt{2\alpha C_{T,\Gamma,\sigma}}\right]^k\right)^c\right)\\&\le\sum_{i=1}^k P\left(\langle F_{R,t_i}\rangle_{t_i}\le C_{T,\Gamma,\sigma}R,\ |F_{R,t_i}(t_i)|>\sqrt{2\alpha C_{T,\Gamma,\sigma}}R\right)\\&\le 2ke^{-\alpha R},\end{aligned}$$

which implies the exponential tightness. □

4.2. *Existence of the limit.* In this section, we show that the limit

$$\Lambda_g^{\mathbb{T}_k}:=\lim_{R\to\infty}\frac{1}{R}\log\mathbb{E}\left[\exp\left\{Rg\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k)\right)\right\}\right]$$

exists for any $k\in\mathbb{N}$, $\mathbb{T}_k=(t_1,\dots,t_k)\in[0,T]^k$, and $g\in\mathcal{G}_{\mathrm{BLC}}(\mathbb{R}^k)$, by using approximate subadditivity lemma (Lemma 2.7). To this end, we first observe that it suffices to prove the existence of the limit $\Lambda_g^{\mathbb{T}_k}$ for any $k\in\mathbb{N}$, $\mathbb{T}_k\in[0,T]^k$, and $g\in\mathcal{G}_{\mathrm{BLC}}(\mathbb{R}^k)$ satisfying $g(x)<0$ for all $x\in\mathbb{R}^k$. Indeed, if this is established, then for any $h\in\mathcal{G}_{\mathrm{BLC}}(\mathbb{R}^k)$ satisfying $h(x)<M_h$ for all $x\in\mathbb{R}^k$ for some constant $M_h>0$, the function $h-M_h$ belongs to $\mathcal{G}_{\mathrm{BLC}}(\mathbb{R}^k)$ and the limit

$$\Lambda_h^{\mathbb{T}_k}=M_h+\Lambda_{h-M_h}^{\mathbb{T}_k}$$

also exists.

In what follows, we fix $\mathbb{T}_k\in[0,T]^k$ and a function $g\in\mathcal{G}_{\mathrm{BLC}}(\mathbb{R}^k)$ with $g(x)<0$ for all $x\in\mathbb{R}^k$, and prove the existence of the limit $\Lambda_g^{\mathbb{T}_k}$. Let us write

$$H_R^{\mathbb{T}_k}(g):=\mathbb{E}\left[\exp\left\{Rg\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k)\right)\right\}\right].$$

**Lemma 4.2.** *For sufficiently large $R>0$,*

$$H_R^{\mathbb{T}_k}(g)\ge\frac{1}{2}e^{-m_gR},\qquad\text{where } m_g:=-\min_{x\in[-1,1]^k}g(x)\in(0,\infty).$$

*Proof*: By a simple estimate,

$$H_R^{\mathbb{T}_k}(g) \geq e^{-m_g R} P\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k) \in [-1,1]^k\right) \geq e^{-m_g R}\left(1 - \sum_{i=1}^k P(|F_R(t_i)| > R)\right).$$

Since $\lim_{R\to\infty} P(|F_R(t_i)| > R) = 0$ by the same argument as in the proof of Proposition 4.1, we have

$$1 - \sum_{i=1}^k P(|F_R(t_i)| > R) > \frac{1}{2}, \qquad \text{for sufficiently large } R,$$

and the lemma follows. □

To show that $R \mapsto -\log H_R^{\mathbb{T}_k}(g)$ is approximately subadditive, we begin by evaluating $H_{L+R}^{\mathbb{T}_k}(g)$.

**Lemma 4.3.** *Let $\Theta, L, R > 0$ and $p \in (0,1)$. It holds*

$$H_{L+R}^{\mathbb{T}_k}(g) \geq \frac{\mathbb{E}\Big[\exp\Big\{Lg\big(\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k)\big) + Rg\Big(\frac{1}{R}\mathbb{F}_{L+\Theta+R}^{L+\Theta}(\mathbb{T}_k)\Big)\Big\}\Big]^{\frac{1}{p}}}{\mathbb{E}\Big[\exp\Big\{\frac{p}{1-p}\cdot \mathrm{Lip}(g)\Big|\mathbb{F}_{L+\Theta}^{L}(\mathbb{T}_k) - \mathbb{F}_{L+R+\Theta}^{L+R}(\mathbb{T}_k)\Big|\Big\}\Big]^{\frac{1-p}{p}}}. \tag{4.1}$$

*Proof*: By the Lipschitz continuity of $g$,

$$\begin{aligned}&\left|g\left(\frac{1}{L+R}\mathbb{F}_{L+R}(\mathbb{T}_k)\right) - g\left(\frac{L}{L+R}\cdot\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k) + \frac{R}{L+R}\cdot\frac{1}{R}\mathbb{F}_{L+\Theta+R}^{L+\Theta}(\mathbb{T}_k)\right)\right| \\ &\leq \frac{\mathrm{Lip}(g)}{L+R}\left|\mathbb{F}_{L+\Theta}^{L}(\mathbb{T}_k) - \mathbb{F}_{L+R+\Theta}^{L+R}(\mathbb{T}_k)\right|.\end{aligned}$$

Hence, the concavity of $g$ yields

$$\begin{aligned}&H_{L+R}^{\mathbb{T}_k}(g) \\ &\geq \mathbb{E}\left[\exp\left\{Lg\left(\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k)\right) + Rg\left(\frac{1}{R}\mathbb{F}_{L+\Theta+R}^{L+\Theta}(\mathbb{T}_k)\right)\right\}\exp\left\{-\mathrm{Lip}(g)\left|\mathbb{F}_{L+\Theta}^{L}(\mathbb{T}_k) - \mathbb{F}_{L+R+\Theta}^{L+R}(\mathbb{T}_k)\right|\right\}\right].\end{aligned} \tag{4.2}$$

Applying a reverse Hölder inequality (*cf.* Adams and Fournier, 2003, THEOREM 2.12) to the right-hand side of (4.2), we obtain

$$H_{L+R}^{\mathbb{T}_k}(g) \geq \frac{\mathbb{E}\Big[\exp\Big\{Lg\big(\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k)\big) + Rg\Big(\frac{1}{R}\mathbb{F}_{L+\Theta+R}^{L+\Theta}(\mathbb{T}_k)\Big)\Big\}\Big]^{\frac{1}{p}}}{\mathbb{E}\Big[\exp\Big\{\frac{p}{1-p}\cdot \mathrm{Lip}(g)\Big|\mathbb{F}_{L+\Theta}^{L}(\mathbb{T}_k) - \mathbb{F}_{L+R+\Theta}^{L+R}(\mathbb{T}_k)\Big|\Big\}\Big]^{\frac{1-p}{p}}},$$

where we use the fact that $g(x) < 0$ for all $x \in \mathbb{R}^k$. □

From now on, we fix $\alpha \in (\frac{1}{2\wedge\eta}, 1)$ and $\beta > 0$ satisfying $\alpha + \beta < 1$. Choosing

$$\Theta = (L+R)^\alpha \quad \text{and} \quad p = \frac{(L+R)^\beta}{(L+R)^\beta + 1} \in (0,1)$$

in Lemma 4.3, we further estimate from below the right-hand side of (4.1), *i.e.*

$$\frac{\mathbb{E}\Big[\exp\Big\{Lg\big(\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k)\big) + Rg\Big(\frac{1}{R}\mathbb{F}_{L+(L+R)^\alpha+R}^{L+(L+R)^\alpha}(\mathbb{T}_k)\Big)\Big\}\Big]^{1+\frac{1}{(L+R)^\beta}}}{\mathbb{E}\Big[\exp\Big\{\mathrm{Lip}(g)(L+R)^\beta\Big|\mathbb{F}_{L+(L+R)^\alpha}^{L}(\mathbb{T}_k) - \mathbb{F}_{L+R+(L+R)^\alpha}^{L+R}(\mathbb{T}_k)\Big|\Big\}\Big]^{\frac{1}{(L+R)^\beta}}}. \tag{4.3}$$

We first bound from above the denominator of (4.3).

**Lemma 4.4.** *Under the same setting as in Lemma 4.3,*

$$\mathbb{E}\Big[\exp\Big\{\mathrm{Lip}(g)(L+R)^{\beta}\Big|\mathbb{F}^{L}_{L+(L+R)^{\alpha}}(\mathbb{T}_k)-\mathbb{F}^{L+R}_{L+R+(L+R)^{\alpha}}(\mathbb{T}_k)\Big|\Big\}\Big]^{\frac{1}{(L+R)^{\beta}}}$$
$$\le\Big(C_{\mathrm{Lip}(g),T,\Gamma,\sigma}(L+R)^{\frac{\alpha}{2}+\beta}\Big)^{\frac{1}{(L+R)^{\beta}}}\exp\Big\{C_{\mathrm{Lip}(g),T,\Gamma,\sigma}(L+R)^{\alpha+\beta}\Big\}.$$

*Proof*: Since $\mathbb{E}[e^{cX}]=c\int_{\mathbb{R}}e^{cr}P(X\ge r)dr$ holds for any random variable $X$ and for any constant $c>0$, we get

$$\mathbb{E}\Big[\exp\Big\{\mathrm{Lip}(g)(L+R)^{\beta}\Big|\mathbb{F}^{L}_{L+(L+R)^{\alpha}}(\mathbb{T}_k)-\mathbb{F}^{L+R}_{L+R+(L+R)^{\alpha}}(\mathbb{T}_k)\Big|\Big\}\Big]$$
$$\le\mathrm{Lip}(g)(L+R)^{\beta}\sum_{i=1}^{k}\int_{\mathbb{R}}e^{\mathrm{Lip}(g)(L+R)^{\beta}r}P\bigg(\Big|F^{L}_{L+(L+R)^{\alpha}}(t_i)-F^{L+R}_{L+R+(L+R)^{\alpha}}(t_i)\Big|\ge\frac{r}{\sqrt{k}}\bigg)dr,\quad(4.4)$$

and we need to evaluate the probability inside the integral. If we let

$$M_t(r)$$
$$:=\int_0^{r\wedge t}\int_{\mathbb{R}}\big(\mathbf{1}_{[L,L+(L+R)^{\alpha}]}*G(t-s)(y)-\mathbf{1}_{[L+R,L+R+(L+R)^{\alpha}]}*G(t-s)(y)\big)\sigma(u(s,y))W(ds,dy),$$

then $\{M_t(r)\}_{r\in[0,T]}$ is a square-integrable continuous martingale such that $M_t(t)=F^{L}_{L+(L+R)^{\alpha}}(t)-F^{L+R}_{L+R+(L+R)^{\alpha}}(t)$ and

$$\langle M_t(\cdot)\rangle_t\le C_T\|\sigma\|_{\infty}^2\|\Gamma\|_{L^1(\mathbb{R})}(L+R)^{\alpha}=:C_{T,\Gamma,\sigma}(L+R)^{\alpha}$$

as in the proof of Proposition 4.1. Applying the exponential inequality for martingales, we obtain

$$P\bigg(\Big|F^{L}_{L+(L+R)^{\alpha}}(t)-F^{L+R}_{L+R+(L+R)^{\alpha}}(t)\Big|\ge r\bigg)\le 2e^{-\frac{r^2}{2C_{T,\Gamma,\sigma}(L+R)^{\alpha}}}.\quad(4.5)$$

Hence, combining (4.4) and (4.5), we have

$$\mathbb{E}\Big[\exp\Big\{\mathrm{Lip}(g)(L+R)^{\beta}\Big|\mathbb{F}^{L}_{L+(L+R)^{\alpha}}(\mathbb{T}_k)-\mathbb{F}^{L+R}_{L+R+(L+R)^{\alpha}}(\mathbb{T}_k)\Big|\Big\}\Big]$$
$$\le\sqrt{8\pi C_{T,\Gamma,\sigma}}k^{\frac{3}{2}}\mathrm{Lip}(g)(L+R)^{\frac{\alpha}{2}+\beta}\exp\bigg\{\frac{k\mathrm{Lip}(g)^2C_{T,\Gamma,\sigma}}{2}(L+R)^{\alpha+2\beta}\bigg\},$$

and the lemma follows. □

Next, we bound from below the numerator of (4.3) using the covariance estimate in Proposition 3.4.

**Lemma 4.5.** *For sufficiently large $L$ and $R$,*

$$\mathbb{E}\bigg[\exp\bigg\{Lg\Big(\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k)\Big)+Rg\Big(\frac{1}{R}\mathbb{F}^{L+(L+R)^{\alpha}}_{L+(L+R)^{\alpha}+R}(\mathbb{T}_k)\Big)\bigg\}\bigg]^{1+\frac{1}{(L+R)^{\beta}}}$$
$$\ge\frac{1}{16}H_L^{\mathbb{T}_k}(g)H_R^{\mathbb{T}_k}(g)e^{-m_g(L+R)^{1-\beta}}$$

*Proof*: By Proposition 3.4, Lemmas 4.2 and 2.3, we have for sufficiently large $L$ and $R$,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{Lg\left(\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k)\right)+Rg\left(\frac{1}{R}\mathbb{F}^{L+(L+R)^\alpha}_{L+(L+R)^\alpha+R}(\mathbb{T}_k)\right)\right\}\right]\\
&\geq \mathbb{E}\left[\exp\left\{Lg\left(\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k)\right)\right\}\right]\mathbb{E}\left[\exp\left\{Rg\left(\frac{1}{R}\mathbb{F}^{L+(L+R)^\alpha}_{L+(L+R)^\alpha+R}(\mathbb{T}_k)\right)\right\}\right]\\
&\quad - C_{\mathrm{Lip}(g),T}k^2LRe^{-C_{T,\eta,\rho}(L+R)^{(2\wedge\eta)\alpha}}\\
&\geq H_L^{\mathbb{T}_k}(g)H_R^{\mathbb{T}_k}(g)\left(1-4C_{\mathrm{Lip}(g),T}k^2(L+R)^2e^{-C_{T,\eta,\rho}(L+R)^{(2\wedge\eta)\alpha}+m_g(L+R)}\right).
\end{aligned}$$

Because $(2\wedge\eta)\alpha>1$ and $\lim_{x\to\infty}x^2e^{-C_{T,\eta,\rho}x^{(2\wedge\eta)\alpha}+m_gx}=0$, we can assume

$$\frac{1}{2}<1-4C_{\mathrm{Lip}(g),T}k^2(L+R)^2e^{-C_{T,\eta,\rho}(L+R)^{(2\wedge\eta)\alpha}+m(L+R)}<1$$

by taking $L$ and $R$ large enough. Thus, we can apply Lemma 4.2 again to obtain

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{Lg\left(\frac{1}{L}\mathbb{F}_L(\mathbb{T}_k)\right)+Rg\left(\frac{1}{R}\mathbb{F}^{L+(L+R)^\alpha}_{L+(L+R)^\alpha+R}(\mathbb{T}_k)\right)\right\}\right]^{1+\frac{1}{(L+R)^\beta}}\\
&\geq\left(\frac{1}{2}\right)^{1+\frac{1}{(L+R)^\beta}}H_L^{\mathbb{T}_k}(g)H_R^{\mathbb{T}_k}(g)\left(\frac{1}{4}e^{-m_g(L+R)}\right)^{\frac{1}{(L+R)^\beta}}\\
&\geq\frac{1}{16}H_L^{\mathbb{T}_k}(g)H_R^{\mathbb{T}_k}(g)e^{-m_g(L+R)^{1-\beta}},
\end{aligned}$$

which is the desired estimate. □

Now, we combine all the computations to show the approximate subadditivity of $R\mapsto-\log H_R^{\mathbb{T}_k}(g)$.

**Proposition 4.6.** *There exists $M>0$ such that for any $L,R>M$,*

$$-\log H_{L+R}^{\mathbb{T}_k}(g)\leq-\log H_L^{\mathbb{T}_k}(g)-\log H_R^{\mathbb{T}_k}(g)+(m_g+1)(L+R)^{1-\beta}+C_{\mathrm{Lip}(g),T,\Gamma,\sigma}(L+R)^{\alpha+\beta}.$$

*Consequently, the limit $\Lambda_g^{\mathbb{T}_k}=\lim_{R\to\infty}\frac{1}{R}\log H_R^{\mathbb{T}_k}(g)$ exists.*

*Proof*: Take $L$ and $R$ large enough to apply Lemma 4.5. By Lemmas 4.3, 4.4, and 4.5, we have

$$\begin{aligned}
&\frac{H_{L+R}^{\mathbb{T}_k}(g)}{H_L^{\mathbb{T}_k}(g)H_R^{\mathbb{T}_k}(g)}\\
&\geq\frac{1}{16\left(C_{\mathrm{Lip}(g),T,\Gamma,\sigma}(L+R)^{\frac{\alpha}{2}+\beta}\right)^{\frac{1}{(L+R)^\beta}}}\exp\left\{-m_g(L+R)^{1-\beta}-C_{\mathrm{Lip}(g),T,\Gamma,\sigma}(L+R)^{\alpha+\beta}\right\}
\end{aligned}$$

It follows by letting $L+R$ large enough that

$$-\log H_{L+R}^{\mathbb{T}_k}(g)\leq-\log H_L^{\mathbb{T}_k}(g)-\log H_R^{\mathbb{T}_k}(g)+(m_g+1)(L+R)^{1-\beta}+C_{\mathrm{Lip}(g),T,\Gamma,\sigma}(L+R)^{\alpha+\beta}.$$

Since $1-\beta<1$ and $\alpha+\beta<1$, the existence of the limit follows from Lemma 2.7. □

4.3. *Convexity of the rate function.* By Propositions 2.4, 4.1, and 4.6, we have already shown that $\{\mu_R^{\mathbb{T}_k}\}_{R>0}$ satisfies the LDP with speed $R$ and the good rate function

$$I^{\mathbb{T}_k}(x)=\sup_{f\in C_{\mathrm{b}}(\mathbb{R}^k)}\{f(x)-\Lambda_f^{\mathbb{T}_k}\},\tag{4.6}$$

where

$$\Lambda_f^{\mathbb{T}_k}=\lim_{R\to\infty}\frac{1}{R}\log\mathbb{E}\left[\exp\left\{Rf\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k)\right)\right\}\right],\quad f\in C_{\mathrm{b}}(\mathbb{R}^k).$$

In this section, we show the convexity of the rate function $I^{\mathbb{T}_k}$ in Proposition 4.8 below. It then follows from Proposition 2.6 that $I^{\mathbb{T}_k}$ is the Fenchel-Legendre transform of

$$\Lambda^{\mathbb{T}_k}(\lambda) = \lim_{R\to\infty} \frac{1}{R} \log \mathbb{E}[e^{\lambda\cdot \mathbb{F}_R(\mathbb{T}_k)}], \quad \lambda \in \mathbb{R}^k, \tag{4.7}$$

thereby completing the proof of Theorem 1.3. Hereafter, we use the notations

$$\mathbb{B}_k(x,r) := \{y \in \mathbb{R}^k : |y-x| < r\} \quad \text{and} \quad \mathbb{B}_k(r) := \mathbb{B}_k(0,r), \quad x \in \mathbb{R}^k, r > 0.$$

**Lemma 4.7.** *If $x_1, x_2 \in \mathbb{R}^k$ and $\delta > 0$ satisfy*

$$\liminf_{R\to\infty} \frac{1}{R} \log \mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_i, \frac{\delta}{4}\right)\right) > -\infty, \quad i = 1,2,$$

*then there exists $R_{x_1,x_2,\delta} > 0$ such that for any $R \geq R_{x_1,x_2,\delta}$,*

$$\mu_{2R}^{\mathbb{T}_k}\left(\mathbb{B}_k\left(\frac{x_1+x_2}{2}, \delta\right)\right) \geq \frac{1}{2}\mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_1, \frac{\delta}{4}\right)\right)\mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_2, \frac{\delta}{4}\right)\right).$$

*Proof*: Let $\Theta > 0$. By a simple estimate,

$$\begin{aligned}
&\mu_{2R}^{\mathbb{T}_k}\left(\mathbb{B}_k\left(\frac{x_1+x_2}{2}, \delta\right)\right) = P\left(\left|\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k) - x_1 + \frac{1}{R}\mathbb{F}_{2R}^R(\mathbb{T}_k) - x_2\right| < 2\delta\right) \\
&\geq P\left(\left|\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k) - x_1\right| < \frac{\delta}{2}, \left|\frac{1}{R}\mathbb{F}_{2R+\Theta}^{R+\Theta}(\mathbb{T}_k) - x_2\right| < \frac{\delta}{2}, \left|\frac{1}{R}\mathbb{F}_{2R}^R(\mathbb{T}_k) - \frac{1}{R}\mathbb{F}_{2R+\Theta}^{R+\Theta}(\mathbb{T}_k)\right| < \delta\right) \\
&= \mathbb{E}\left[\mathbf{1}_{\mathbb{B}_k(x_1,\frac{\delta}{2})}\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k)\right)\mathbf{1}_{\mathbb{B}_k(x_2,\frac{\delta}{2})}\left(\frac{1}{R}\mathbb{F}_{2R+\Theta}^{R+\Theta}(\mathbb{T}_k)\right)\mathbf{1}_{\mathbb{B}_k(\delta)}\left(\frac{1}{R}\mathbb{F}_{R+\Theta}^R(\mathbb{T}_k) - \frac{1}{R}\mathbb{F}_{2R+\Theta}^{2R}(\mathbb{T}_k)\right)\right].
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\mathbb{E}\left[\mathbf{1}_{\mathbb{B}_k(x_1,\frac{\delta}{2})}\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k)\right)\mathbf{1}_{\mathbb{B}_k(x_2,\frac{\delta}{2})}\left(\frac{1}{R}\mathbb{F}_{2R+\Theta}^{R+\Theta}(\mathbb{T}_k)\right)\mathbf{1}_{\mathbb{B}_k(\delta)}\left(\frac{1}{R}\mathbb{F}_{R+\Theta}^R(\mathbb{T}_k) - \frac{1}{R}\mathbb{F}_{2R+\Theta}^{2R}(\mathbb{T}_k)\right)\right]\right. \\
&\quad \left. - \mathbb{E}\left[\mathbf{1}_{\mathbb{B}_k(x_1,\frac{\delta}{2})}\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k)\right)\mathbf{1}_{\mathbb{B}_k(x_2,\frac{\delta}{2})}\left(\frac{1}{R}\mathbb{F}_{2R+\Theta}^{R+\Theta}(\mathbb{T}_k)\right)\right]\right| \\
&\leq P\left(\left|\mathbb{F}_{R+\Theta}^R(\mathbb{T}_k) - \mathbb{F}_{2R+\Theta}^{2R}(\mathbb{T}_k)\right| \geq R\delta\right) \\
&\leq 2ke^{-\frac{\delta^2R^2}{2k\Theta C_{T,\Gamma,\sigma}}}
\end{aligned}$$

by a similar argument as in the proof of Lemma 4.4 (see (4.5)), we have

$$\begin{aligned}
&\mu_{2R}^{\mathbb{T}_k}\left(\mathbb{B}_k\left(\frac{x_1+x_2}{2}, \delta\right)\right) \\
&\geq \mathbb{E}\left[\mathbf{1}_{\mathbb{B}_k(x_1,\frac{\delta}{2})}\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k)\right)\mathbf{1}_{\mathbb{B}_k(x_2,\frac{\delta}{2})}\left(\frac{1}{R}\mathbb{F}_{2R+\Theta}^{R+\Theta}(\mathbb{T}_k)\right)\right] - 2ke^{-\frac{\delta^2R^2}{2k\Theta C_{T,\Gamma,\sigma}}}.
\end{aligned} \tag{4.8}$$

Let $f_i^\delta : \mathbb{R}^k \to [0,1]$, $(i = 1,2)$ be Lipschitz functions such that $\mathbf{1}_{\mathbb{B}_k(x_i,\frac{\delta}{4})} \leq f_i^\delta \leq \mathbf{1}_{\mathbb{B}_k(x_i,\frac{\delta}{2})}$. As before, we deduce from Proposition 3.4 that

$$\left|\mathrm{Cov}\left(f_1^\delta\left(\frac{1}{R}\mathbb{F}_R(\mathbb{T}_k)\right), f_2^\delta\left(\frac{1}{R}\mathbb{F}_{2R+\Theta}^{R+\Theta}(\mathbb{T}_k)\right)\right)\right| \leq C_T \mathrm{Lip}(f_1^\delta)\mathrm{Lip}(f_2^\delta)k^2 \exp\{-C_{T,\eta,\rho}\Theta^{2\wedge\eta}\}.$$

This together with (4.8) and Lemma 2.3 yields

$$\mu_{2R}^{\mathbb{T}_k}\left(\mathbb{B}_k\left(\frac{x_1+x_2}{2}, \delta\right)\right) \geq \mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_1, \frac{\delta}{4}\right)\right)\mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_2, \frac{\delta}{4}\right)\right) - \mathbf{A}(\Theta, R), \tag{4.9}$$

where $\mathbf{A}(\Theta,R) \coloneqq C_T \mathrm{Lip}(f_1^\delta)\mathrm{Lip}(f_2^\delta)k^2 \exp\{-C_{T,\eta,\rho}\Theta^{2\wedge\eta}\} + 2ke^{-\frac{\delta^2 R^2}{2k\Theta C_{T,\Gamma,\sigma}}}$. Now observe that by the assumption of the lemma, there exists $M>0$ and $R^{(1)}_{x_1,x_2,\delta}>0$ such that for any $R \geq R^{(1)}_{x_1,x_2,\delta}$,

$$\mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_1,\frac{\delta}{4}\right)\right)\mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_2,\frac{\delta}{4}\right)\right) \geq e^{-MR}. \tag{4.10}$$

Hence, combining (4.9) and (4.10), we obtain for any $R \geq R^{(1)}_{x_1,x_2,\delta}$,

$$\mu_{2R}^{\mathbb{T}_k}\left(\mathbb{B}_k\left(\frac{x_1+x_2}{2},\delta\right)\right) \geq \mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_1,\frac{\delta}{4}\right)\right)\mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_2,\frac{\delta}{4}\right)\right)\{1-\mathbf{A}(\Theta,R)e^{MR}\}.$$

Set $\Theta = R^\gamma$ for some $\gamma \in \left(\frac{1}{2\wedge\eta},1\right)$. Then we can choose $R^{(2)}_{x_1,x_2,\delta}>0$ such that for any $R \geq R^{(2)}_{x_1,x_2,\delta}$,

$$1-\mathbf{A}(\Theta,R)e^{MR} \geq \frac{1}{2}.$$

Therefore, letting $R_{x_1,x_2,\delta} \coloneqq R^{(1)}_{x_1,x_2,\delta} \vee R^{(2)}_{x_1,x_2,\delta}$ completes the proof. □

**Proposition 4.8.** *For each $\lambda \in \mathbb{R}^k$, the limit $\Lambda^{\mathbb{T}_k}(\lambda)$ in (4.7) exists and is finite. Moreover, the rate function $I^{\mathbb{T}_k}$ in (4.6) is convex. Consequently, $I^{\mathbb{T}_k}$ is the Fenchel-Legendre transform of $\Lambda^{\mathbb{T}_k}$, i.e.*

$$I^{\mathbb{T}_k}(x) = \sup_{\lambda\in\mathbb{R}^k}\{\lambda\cdot x - \Lambda^{\mathbb{T}_k}(\lambda)\}.$$

*Proof*: Recall that $\{F_{R,t}(r)\}_{r\in[0,T]}$ defined in the proof of Proposition 4.1 is a square-integrable continuous martingale such that $F_{R,t}(t) = F_R(t)$ and $\langle F_{R,t}\rangle_t \leq C_{T,\Gamma,\sigma}R$. A simple computation shows that for any $\lambda \in \mathbb{R}^k$,

$$\begin{aligned}
\limsup_{R\to\infty}\frac{1}{R}\log\mathbb{E}[e^{\lambda\cdot\mathbb{F}_R(\mathbb{T}_k)}] &\leq \limsup_{R\to\infty}\frac{1}{R}\log\left(\prod_{i=1}^k \mathbb{E}\left[e^{k\lambda_i F_{R,t_i}(t_i)}\right]\right)\\
&\leq \sum_{i=1}^k \limsup_{R\to\infty}\frac{1}{R}\log\left(\mathbb{E}\left[e^{k\lambda_i F_{R,t_i}(t_i)-\frac{k^2\lambda_i^2}{2}\langle F_{R,t_i}\rangle_{t_i}}\right]e^{\frac{k^2\lambda_i^2 C_{T,\Gamma,\sigma}}{2}R}\right)\\
&\leq \frac{k^2|\lambda|^2 C_{T,\Gamma,\sigma}}{2} < \infty,
\end{aligned} \tag{4.11}$$

where the first inequality follows from Hölder's inequality, and the third inequality holds because the process $\{e^{k\lambda_i F_{R,t_i}(r)-\frac{k^2\lambda_i^2}{2}\langle F_{R,t_i}\rangle_r}\}_{r\in[0,T]}$ is a supermartingale and we have

$$\mathbb{E}\left[e^{k\lambda_i F_{R,t_i}(t_i)-\frac{k^2\lambda_i^2}{2}\langle F_{R,t_i}\rangle_{t_i}}\right] \leq \mathbb{E}\left[e^{k\lambda_i F_{R,t_i}(0)-\frac{k^2\lambda_i^2}{2}\langle F_{R,t_i}\rangle_0}\right] = 1.$$

Now we show the convexity of $I^{\mathbb{T}_k}$ in (4.6). By Dembo and Zeitouni (2010, Theorem 4.1.18), we have

$$-I^{\mathbb{T}_k}(x) = \inf_{\delta>0,\ y\in\mathbb{B}_k(x,\delta)}\liminf_{R\to\infty}\frac{1}{R}\log\mu_R^{\mathbb{T}_k}(\mathbb{B}_k(y,\delta)) = \inf_{\delta>0,\ y\in\mathbb{B}_k(x,\delta)}\limsup_{R\to\infty}\frac{1}{R}\log\mu_R^{\mathbb{T}_k}(\mathbb{B}_k(y,\delta)).$$

Let us take $x_1,x_2\in\mathbb{R}$ such that $I^{\mathbb{T}_k}(x_i)<\infty,\ (i=1,2)$. Then, for this $x_1,x_2$, it holds

$$\liminf_{R\to\infty}\frac{1}{R}\log\mu_R^{\mathbb{T}_k}(\mathbb{B}_k(x_i,\delta)) > -\infty \quad \text{for any } \delta>0 \quad (i=1,2). \tag{4.12}$$

Because $y \in \mathbb{B}_k(\frac{x_1+x_2}{2}, \delta)$ implies $\mathbb{B}_k(\frac{x_1+x_2}{2}, \delta') \subset \mathbb{B}_k(y, \delta)$ for all $\delta' > 0$ small enough, we see that

$$\begin{aligned} -I^{\mathbb{T}_k}\left(\frac{x_1+x_2}{2}\right) &= \inf_{\delta>0, y\in\mathbb{B}_k\left(\frac{x_1+x_2}{2},\delta\right)} \limsup_{R\to\infty} \frac{1}{R} \log \mu_R^{\mathbb{T}_k}(\mathbb{B}_k(y,\delta)) \\ &= \inf_{\delta'>0} \limsup_{R\to\infty} \frac{1}{R} \log \mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(\frac{x_1+x_2}{2}, \delta'\right)\right) \\ &\geq \inf_{\delta>0} \liminf_{R\to\infty} \frac{1}{2R} \log \mu_{2R}^{\mathbb{T}_k}\left(\mathbb{B}_k\left(\frac{x_1+x_2}{2}, \delta\right)\right). \end{aligned} \tag{4.13}$$

By (4.12), we can apply Lemma 4.7 to obtain

$$\begin{aligned} &\liminf_{R\to\infty} \frac{1}{2R} \log \mu_{2R}^{\mathbb{T}_k}\left(\mathbb{B}_k\left(\frac{x_1+x_2}{2}, \delta\right)\right) \\ &\geq \frac{1}{2} \liminf_{R\to\infty} \frac{1}{R} \log \mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_1, \frac{\delta}{4}\right)\right) + \frac{1}{2} \liminf_{R\to\infty} \frac{1}{R} \log \mu_R^{\mathbb{T}_k}\left(\mathbb{B}_k\left(x_2, \frac{\delta}{4}\right)\right). \end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14), we obtain

$$-I^{\mathbb{T}_k}\left(\frac{x_1+x_2}{2}\right) \geq -\frac{1}{2} I^{\mathbb{T}_k}(x_1) - \frac{1}{2} I^{\mathbb{T}_k}(x_2). \tag{4.15}$$

Since (4.15) trivially holds for $x_i$ such that $I^{\mathbb{T}_k}(x_i) = \infty$, we have (4.15) for all $x_1, x_2 \in \mathbb{R}^k$. This is sufficient to show the convexity of $I^{\mathbb{T}_k}$, thanks to its lower semicontinuity. Finally, by Proposition 2.6, (4.11) and the convexity of $I^{\mathbb{T}_k}$ imply that $I^{\mathbb{T}_k}$ is the Fenchel-Legendre transform of $\Lambda^{\mathbb{T}_k}$, and this completes the proof. □

## 5. Sample path LDP

In this section, we prove Theorem 1.4. We make use of the following estimate.

**Lemma 5.1.** *For any $0 \leq s < t \leq T$, $R > 0$, and $r \geq 0$,*

$$P\left(\left|\frac{1}{R}F_R(t) - \frac{1}{R}F_R(s)\right| \geq r\right) \leq 2\exp\left(-\frac{Rr^2}{2C_{T,\Gamma,\sigma}|t-s|}\right). \tag{5.1}$$

*In particular, for any $p \in \mathbb{N}$, we have*

$$\mathbb{E}\left[\left|\frac{1}{R}F_R(t) - \frac{1}{R}F_R(s)\right|^{2p}\right] \leq 2^{p+1}(p!)C_{T,\Gamma,\sigma}^p R^{-p}|t-s|^p.$$

*Proof*: Let

$$\begin{aligned} &F_{R,s,t}(q) \\ &= \int_0^q \int_{\mathbb{R}} \left\{\mathbf{1}_{[0,t)}(r)(G(t-r) * \mathbf{1}_{[0,R]})(z) - \mathbf{1}_{[0,s)}(r)(G(s-r) * \mathbf{1}_{[0,R]})(z)\right\}\sigma(u(r,z))W(dr,dz). \end{aligned}$$

We have $F_{R,s,t}(T) = F_R(t) - F_R(s)$ and

$$\begin{aligned} &\langle F_{R,s,t}\rangle_T \\ &= \int_0^T \int_{\mathbb{R}^2} \left\{\mathbf{1}_{[0,t)}(r)(G(t-r) * \mathbf{1}_{[0,R]})(z) - \mathbf{1}_{[0,s)}(r)(G(s-r) * \mathbf{1}_{[0,R]})(z)\right\}\sigma(u(r,z))\Gamma(z-\widetilde{z}) \\ &\qquad \times \left\{\mathbf{1}_{[0,t)}(r)(G(t-r) * \mathbf{1}_{[0,R]})(\widetilde{z}) - \mathbf{1}_{[0,s)}(r)(G(s-r) * \mathbf{1}_{[0,R]})(\widetilde{z})\right\}\sigma(u(r,\widetilde{z}))dzd\widetilde{z}dr \\ &\leq \|\sigma\|_\infty^2 \|\Gamma\|_{L^1(\mathbb{R})} \int_0^T \int_{\mathbb{R}} \left|\mathbf{1}_{[0,t)}(r)\mathcal{F}G(t-r)(\xi) - \mathbf{1}_{[0,s)}(r)\mathcal{F}G(s-r)(\xi)\right|^2 \left|\mathcal{F}\mathbf{1}_{[0,R]}(\xi)\right|^2 d\xi dr, \end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. Here, in the last inequality, we used the boundedness of $\sigma$, Young's convolution inequality, and Plancherel's theorem, in that order. It is easy to check that

$$\int_0^T \left|\mathbf{1}_{[0,t)}(r)\mathcal{F}G(t-r)(\xi) - \mathbf{1}_{[0,s)}(r)\mathcal{F}G(s-r)(\xi)\right|^2 dr \le C_T|t-s|$$

holds in both the heat and wave cases. See, *e.g.*, the proofs of Proposition 4.1 in Huang et al. (2020a) and Delgado-Vences et al. (2020). Therefore, we obtain $\langle F_{R,s,t}\rangle_T \le C_{T,\Gamma,\sigma}|t-s|R$, and the exponential inequality for martingales now gives (5.1). □

Lemma 5.1 implies that the process $\{R^{-1}F_R(t)\}_{t\in[0,T]}$ has a continuous modification. We write $\mu_R$ for the law on $(C([0,T]), \mathcal{B}(C([0,T])))$ of a continuous modification of this process.

Let $\mathcal{O}_{\mathrm{p}}$ (resp. $\mathcal{O}_{\mathrm{u}}$) denote the topology of pointwise (resp. uniform) convergence on $\mathbb{R}^{[0,T]}$. Let $\widetilde{\mathcal{O}_{\mathrm{p}}}$ (resp. $\widetilde{\mathcal{O}_{\mathrm{u}}}$) be the subspace topology on $C([0,T])$ induced by $\mathcal{O}_{\mathrm{p}}$ (resp. $\mathcal{O}_{\mathrm{u}}$). We write $\mathcal{B}(\mathcal{O})$ for the Borel $\sigma$-algebra of a topology $\mathcal{O}$. Note that $\mu_R$ on $(C([0,T]), \mathcal{B}(\widetilde{\mathcal{O}_{\mathrm{u}}}))$ can naturally be regarded as a measure on $(\mathbb{R}^{[0,T]}, \mathcal{B}(\mathcal{O}_{\mathrm{p}}))$ since $\mathcal{O}_{\mathrm{p}} \subset \mathcal{O}_{\mathrm{u}}$ and $\mathcal{B}(\widetilde{\mathcal{O}_{\mathrm{u}}}) = \mathcal{B}(\mathcal{O}_{\mathrm{u}}) \cap C([0,T])$.

The following proposition is a direct consequence of Theorem 1.3 and Proposition 2.8.

**Proposition 5.2.** *The family of probability measures $\{\mu_R\}_{R>0}$ on $(\mathbb{R}^{[0,T]}, \mathcal{B}(\mathcal{O}_p))$ satisfies the LDP with speed $R$ and the convex good rate function*

$$I(f) = \sup_{k\in\mathbb{N}} \sup_{0\le t_1<\cdots<t_k\le T} \left\{ I^{\mathbb{T}_k}\Big((f(t_1),\dots,f(t_k))\Big)\right\}, \qquad f \in \mathbb{R}^{[0,T]},$$

*where $I^{\mathbb{T}_k}$ is the rate function specified in Theorem 1.3.*

In light of Proposition 5.2, the proof of Theorem 1.4 is completed once we establish the exponential tightness of $\{\mu_R\}_{R>0}$ with respect to the finer topology $\mathcal{O}_{\mathrm{u}}$. To this end, we use the following estimate, which is a consequence of the embedding result between Hölder and Orlicz spaces given in Schied (1997, Section 7).

**Lemma 5.3** (Schied, 1997, Corollary 7.1)**.** *Let $n \in \mathbb{N}$ and $\{\xi(t)\}_{t\in[0,T]^n}$ be a real-valued continuous process. Suppose that there exists $p, q, r \in (0,\infty)$ such that*

$$\sup_{s,t\in[0,T]^n, s\ne t} \mathbb{E}\left[\exp\left(\frac{p}{|t-s|^q}|\xi(t)-\xi(s)|\right)\right] \le r.$$

*Then, for any $q' \in (0,q)$ and $M \ge 0$,*

$$P\left(\sup_{s,t\in[0,T]^n, s\ne t} \frac{|\xi(t)-\xi(s)|}{|t-s|^{q'}} \ge M\right) \le (1+r)\exp\left(-\frac{Mp}{c_{n,q,q',T}}\right),$$

*where the constant $c_{n,q,q',T}$ depends only on $n$, $q$, $q'$, and $T$, and can be taken, for instance, as*

$$c_{n,q,q',T} = \frac{T^{q-q'}(1+Q!)2^{q'+1+\frac{n}{Q}}}{1-2^{-(q-q')+\frac{n}{Q}}}, \qquad \text{with} \quad Q := \left\lfloor \frac{n}{q-q'} \right\rfloor + 1.$$

*Remark* 5.4. Although Schied (1997, Section 7) considers continuous processes indexed by $[0,1]^n$, we note for completeness that the same argument applies to the case $[0,T]^n$ without any essential modification.

**Proposition 5.5.** *The family of probability measures $\{\mu_R\}_{R>0}$ on $(\mathbb{R}^{[0,T]}, \mathcal{B}(\mathcal{O}_{\mathrm{u}}))$ is exponentially tight. Consequently, $\{\mu_R\}_{R>0}$ on $C([0,T], \mathcal{B}(\widetilde{\mathcal{O}_{\mathrm{u}}}))$ satisfies the same LDP as in Proposition 5.2.*

*Proof*: For $M > 0$ and $a > 0$, set

$$H_M^a := \left\{ f \in C([0,T]) \;\middle|\; f(0) = 0, \sup_{s,t\in[0,T], s\ne t} \frac{|f(t)-f(s)|}{|t-s|^a} \le M \right\}.$$

By the Ascoli-Arzelà theorem, $H_M^a$ is a compact subset of $(C([0,T]), \widetilde{\mathcal{O}_{\mathrm{u}}})$, and hence $\iota(H_M^a)$ is compact in $(\mathbb{R}^{[0,T]}, \mathcal{O}_{\mathrm{u}})$, where $\iota\colon (C([0,T], \widetilde{\mathcal{O}_{\mathrm{u}}})) \to (\mathbb{R}^{[0,T]}, \mathcal{O}_{\mathrm{u}})$ is the inclusion map.

Let $0 \le s < t \le T$ and $A > 0$. Using $\mathbb{E}[e^{cX}] = c\int_{\mathbb{R}} e^{cr} P(X \ge r)dr$, which holds for any random variable $X$ and $c > 0$, and applying (5.1), we obtain

$$\mathbb{E}[e^{\frac{A}{R}|F_R(t)-F_R(s)|}] \le \sqrt{\frac{8\pi C_{T,\Gamma,\sigma}|t-s|A^2}{R}} \exp\left(\frac{C_{T,\Gamma,\sigma}|t-s|A^2}{2R}\right).$$

Thus, taking $A = R|t-s|^{-\frac{1}{2}}$, we have

$$\sup_{s,t\in[0,T],s\neq t} \mathbb{E}\left[\exp\left(\frac{R}{|t-s|^{\frac{1}{2}}}\left|\frac{1}{R}F_R(t) - \frac{1}{R}F_R(s)\right|\right)\right] \le \sqrt{8\pi C_{T,\Gamma,\sigma}R}\exp\left(\frac{C_{T,\Gamma,\sigma}R}{2}\right).$$

This, together with Lemma 5.3, implies that for any $\delta < \frac{1}{2}$,

$$\begin{aligned} &P\left(\sup_{s,t\in[0,T],s\neq t} \frac{1}{|t-s|^{\delta}}\left|\frac{1}{R}F_R(t) - \frac{1}{R}F_R(s)\right| > M\right) \\ &\le \left(1 + \sqrt{8\pi C_{T,\Gamma,\sigma}R}\exp\left(\frac{C_{T,\Gamma,\sigma}R}{2}\right)\right)\exp\left(-\frac{MR}{C_{T,\delta}}\right). \end{aligned} \tag{5.2}$$

Since the left-hand side of above equals $\mu_R((\iota(H_M^{\delta}))^c)$, it follows from (5.2) that

$$\limsup_{R\to\infty} \frac{1}{R}\log \mu_R((\iota(H_M^{\delta}))^c) \le \left(-\frac{M}{C_{T,\delta}}\right) \vee \left(\frac{C_{T,\Gamma,\sigma}}{2} - \frac{M}{C_{T,\delta}}\right) = \frac{C_{T,\Gamma,\sigma}}{2} - \frac{M}{C_{T,\delta}}.$$

Therefore, we conclude that the family $\{\mu_R\}_{R>0}$ is exponentially tight on $(\mathbb{R}^{[0,T]}, \mathcal{B}(\mathcal{O}_{\mathrm{u}}))$. Combining this exponential tightness and Proposition 5.2, we deduce from Dembo and Zeitouni (2010, Corollary 4.2.6) that $\{\mu_R\}_{R>0}$ on $(\mathbb{R}^{[0,T]}, \mathcal{B}(\mathcal{O}_{\mathrm{u}}))$ satisfies the same LDP as in Proposition 5.2. Therefore, the same LDP also holds in $(C([0,T]), \mathcal{B}(\widetilde{\mathcal{O}_{\mathrm{u}}}))$ since $C([0,T])$ is a closed subset of $(\mathbb{R}^{[0,T]}, \mathcal{O}_{\mathrm{u}})$ (*cf.* Dembo and Zeitouni, 2010, Lemma 4.1.5). □

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