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Long Time Behavior of General Markov Additive Processes

Celal Umut Yaran* and Mine Çağlar

İstanbul Technical University, Department of Mathematics Engineering, Sarıyer, 34469, İstanbul, Turkey
*Corresponding Author
E-mail address: cyaran13@ku.edu.tr

Koç University, Department of Mathematics, Sarıyer, 34450, İstanbul, Turkey
E-mail address: mcaglar@ku.edu.tr

Abstract. We study general Markov additive processes when the state space of the modulator is a Polish space. Under some regularity assumptions, our main result is the characterization of the long-time behavior of the ordinate in terms of the associated ladder time process and the excursion measure. An important application of Markov additive processes is the Lamperti-Kiu transform, which gives a correspondence between $\mathbb{R}^d\backslash\{0\}$-valued self-similar Markov processes and $S^{d-1}\times\mathbb{R}$-valued Markov additive processes. The asymptotic behavior of the radial distance from the origin of a self-similar Markov process can be characterized by the long-time behavior of the ordinate of the corresponding Markov additive process. We show the applicability of our assumptions on some well-known self-similar Markov processes.

1. Introduction

In 1970s, Markov additive processes (MAP) were introduced in Çinlar (1972), essentially as a real-valued stochastic process $(\xi_t)_{t\geq 0}$ such that the increments of ξ are modulated by a Markov process $(\Theta_t)_{t\geq 0}$. Formally, the process (ξ,Θ) is called a MAP if given $\{(\xi_s,\Theta_s), s\leq t\}$ for any $t\geq 0$, the process $(\xi_{s+t}-\xi_t,\Theta_{s+t})_{s\geq 0}$ has the same law as $(\xi_s,\Theta_s)_{s\geq 0}$ under $\mathbb{P}_{0,v}$ with $v=\Theta_t$, where $\mathbb{P}_{x,\theta}(\xi_0=x,\Theta_0=\theta)=1$. The following studies on general MAPs can be found first in Çinlar (1975, 1976); Kaspi (1982, 1983), and in Ney and Nummelin (1987a,b). Although MAPs have been introduced in such generality, most of the previous work has focused on the case the *modulator* Θ is a Markov chain with a countable state space (e.g. Alsmeyer and Buckmann, 2018; Asmussen and Albrecher, 2010; Çinlar, 1977).

A key application of MAPs is the representation of real-valued self-similar Markov processes, which can be seen as a generalization of the relation between positive self-similar Markov processes and potentially killed Lévy processes given in Lamperti (1972). A real-valued Feller process $(Z_t)_{t\geq 0}$ is called a real self-similar Markov process with index $\alpha>0$ if the process $(Z_t)_{t\geq 0}$ is equal in

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distribution to $(cZ_{c^{-\alpha}t})_{t\geq 0}$ for every $c > 0$, with $P_z(Z_0 = z) = 1$. There exists a one-to-one correspondence between real self-similar Markov processes and $\mathbb{R}\times\{\pm 1\}$-valued MAPs (Chaumont et al., 2013; Kuznetsov et al., 2014). Here, the modulator Θ controls the sign of Z, and e^{ξ} specifies how far away Z is from the origin at time $t \geq 0$. A natural question whether a similar relation can be found for $\mathbb{R}^d\backslash\{0\}$-valued self-similar Markov processes has been resolved by recognizing the radial distance from the origin of the process as a real-valued stochastic process modulated by the angular part of the process. There exists a one-to-one correspondence between $\mathbb{R}\times S^{d-1}$-valued MAPs and $\mathbb{R}^d\backslash\{0\}$-valued self-similar Markov processes killed at the origin Kiu (1980). This relation, known as Lamperti-Kiu transform, has brought back attention to MAPs when the state space of the modulator is a Polish space (Alili et al., 2017; Kyprianou, 2018; Kyprianou and Pardo, 2022; Kyprianou et al., 2020, 2024).

Let S be a Polish space and (ξ,Θ) be an $\mathbb{R}\times S$-valued MAP with an invariant probability measure π for the modulator Θ. We are interested in the long-time behavior of the *ordinate* ξ. It has recently been shown in Kyprianou et al. (2020) that ξ_t exhibits exactly one of the three behaviors, namely, drifts to $+\infty$, oscillates, or drifts to $-\infty$ as $t\to\infty$, if the MAP satisfies a Harris-type condition and Θ has an invariant distribution. The proof follows from the strong law of large numbers for MAPs Kyprianou et al. (2020, Prop.2.15) as inspired by a similar approach for Lévy processes. In fact, a MAP can be considered as a natural extension of a Lévy process. Therefore, further ideas from the fluctuation theory of Lévy processes can also be applicable to general MAPs (Bertoin, 1996). In this paper, we use the distributions of last time at supremum $\bar{g}_\infty$ and infimum $\underline{g}_\infty$ arising from the corresponding ladder processes in the characterization of the long-time behavior of ξ.

In Proposition 3.1, we show under some regularity assumptions for (ξ,Θ) that ξ_t drifts to $-\infty$, that is, $\lim_{t\to\infty}\xi_t = -\infty$ $\mathbb{P}_{0,\pi}$-a.s., if

$$\lim_{\lambda\to 0^+}\mathbb{E}_{0,\pi}[\exp(-\lambda\bar{g}_\infty)] = 1 \quad \text{and} \quad \lim_{\lambda\to 0^+}\mathbb{E}_{0,\pi}[\exp(-\lambda\underline{g}_\infty)] = 0,$$

as well as the other cases which imply that either the process ξ_t oscillates, that is, $\limsup_{t\to\infty}\xi_t = -\liminf_{t\to\infty}\xi_t = \infty$ or ξ_t drifts to $+\infty$, that is, $\lim_{t\to\infty}\xi_t = \infty$. Indeed, for a real-valued process, $\lim_{\lambda\to 0^+}\mathbb{E}[\exp(-\lambda\bar{g}_\infty)] = 1$ implies that $\mathbb{P}(\bar{g}_\infty < \infty) = 1$ and $\lim_{\lambda\to 0^+}\mathbb{E}[\exp(-\lambda\underline{g}_\infty)] = 0$ implies that $\mathbb{P}(\underline{g}_\infty < \infty) = 0$. When these limits hold, the asymptotic behavior of a Lévy process X is found as $\lim_{t\to\infty}X_t = -\infty$ in Bertoin (1996). Along these lines, we determine the long-time behavior of ξ depending on whether $\bar{g}_\infty$ and $\underline{g}_\infty$ are finite or infinite using the strong Markov property and the invariant probability measure.

The main result of this paper is the characterization of the long-time behavior of the ordinate ξ in terms of the associated ladder time process and the excursion measure as given in Theorem 3.3. The distribution of the last time at supremum at an exponentially distributed random time e_q with parameter $q > 0$, which is independent of the MAP, is given in Kyprianou et al. (2020, Prop.2.3) as

$$\mathbb{E}_{0,\pi}[\exp(-\lambda\bar{g}_{e_q})] = \int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)\exp(-\lambda r - qr)[q\ell^+(v) + \eta_v^+(\mathbf{1}_{e_q<\zeta^+})]$$

where V_π^+ is the potential measure associated with the corresponding ascending ladder process $(\bar{L}_t^{-1},\xi_t^+,\Theta_t^+)$, η_v^+ denotes the excursion measure at the supremum, ζ^+ denotes the lifetime of an excursion, and ℓ^+ is a function related to the local time at the supremum $\bar{L}_t$. Under a certain duality assumption called the reversibility property, which implies that $\underline{g}_t$ and $t-\bar{g}_t$ are equal in distribution (Alili et al., 2017), we first obtain the distribution of $\underline{g}_{e_q}$ in terms of the ascending ladder process. Then, based on this result and Proposition 3.1, we show that ξ_t drifts to $-\infty$ $\mathbb{P}_{0,\pi}$-a.s. if

$$\int_{S\times\mathbb{R}^+} U_\pi^+(dv,dz)\eta_v^+(\zeta=\infty) = 1$$

where

$$U_\pi^+(dv,dz)=\mathbb{E}_{0,\pi}[\int_0^{\bar{L}_\infty}\mathbf{1}_{\{\Theta_s^+\in dv,\xi_s^+\in dz\}}ds].$$

The conditions which determine either ξ drifts to $+\infty$ or it oscillates are also examined in Theorem 3.3. Furthermore, we extend this result by replacing the reversibility property with its weaker version, called the weak reversibility property (Kyprianou et al., 2020). In Theorem 3.5, we examine the long-time behavior of the ordinate under the assumption of weak reversibility.

We study two important examples of well-known self-similar Markov processes to show that the corresponding MAPs via Lamperti-Kiu transform satisfy the assumptions of Theorem 3.3. The first example is the free 2-dimensional Bessel process $X=(X^{(1)},X^{(2)})$ whose components $X^{(i)}$ are independent Bessel processes. The second example is the radial Dunkl process X^W in $\mathbb{R}^2\backslash\{0\}$. The long-time behavior of the ordinate of the MAP determines the asymptotic behavior of the radial distance from the origin of the corresponding $\mathbb{R}^d\backslash\{0\}$-valued self-similar Markov process. Although the latter asymptotic behavior for both examples is known, we still prove some new properties of these processes using the Lamperti-Kiu transform. For instance, the modulators are Harris recurrent for both examples. We take $d=2$ for simplicity. However, we believe that these results can be generalized to higher dimensions. Moreover, the assumptions of Theorem 3.3 could be used for other classes of self-similar Markov processes with unknown asymptotic behavior. Further possible examples can be found in Alili et al. (2019).

The paper is organized as follows. The fluctuation identities and duality properties of general MAPs are reviewed in Section 2. Our results on the long-time behavior of the ordinate of MAPs are presented in Section 3. In Section 4, we discuss some examples of self-similar Markov processes and their Lamperti-Kiu transforms in detail.

2. Preliminaries

Let $(\xi_t,\Theta_t)_{t\geq 0}$ be a possibly killed Markov process on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}_{x,\theta})$ with $\mathbb{P}_{x,\theta}(\xi_0=x,\Theta_0=\theta)=1$ and state space $(\mathbb{R}\times S,\mathcal{B}(\mathbb{R}\times S))$ with an extra isolated state ∂, where S is a Polish space, $\mathcal{B}(\mathbb{R},S)$ denotes the Borel σ-algebra on $\mathbb{R}\times S$, and $(\mathcal{F}_t)_{t\geq 0}$ is the minimal augmented admissible filtration. The process $(\xi_t,\Theta_t)_{t\geq 0}$ is called a MAP on $\mathbb{R}\times S$, if

$$\mathbb{P}_{0,\theta}((\xi_{t+s}-\xi_t)\in\Gamma\,,\,\Theta_{t+s}\in A\,|\,\mathcal{F}_t)=\mathbb{P}_{0,v}(\xi_s\in\Gamma\,,\,\Theta_s\in A)$$

for all $\theta\in S$, $\Gamma\in\mathcal{B}(\mathbb{R})$, $A\in\mathcal{B}(S)$, and $t\geq 0$, where $v=\Theta_t$, and denoted by $((\xi,\Theta),\mathbb{P})=((\xi_t,\Theta_t)_{t\geq 0},\mathbb{P}_{0,\theta})$. Let $\zeta:=\inf\{t>0:(\xi_t,\Theta_t)=\partial\}$ denote the lifetime of the MAP. We call ξ as the ordinate and Θ as the modulator of the MAP. By the defining property, a MAP is translation invariant in ξ, that is, (ξ_t,Θ_t) under $\mathbb{P}_{x,\theta}$ has the same law as (ξ_t+x,Θ_t) under $\mathbb{P}_{0,\theta}$ for all $x\in\mathbb{R}$ and $\theta\in S$. A MAP has strong Markov property, that is, for a stopping time T, we have

$$\mathbb{P}_{0,\theta}((\xi_{T+t}-\xi_T)\in\Gamma\,,\,\Theta_{T+t}\in A\,|\,\mathcal{F}_T)=\mathbb{P}_{0,v}(\xi_t\in\Gamma\,,\,\Theta_t\in A)\tag{2.1}$$

for all $\theta\in S$, $\Gamma\in\mathcal{B}(\mathbb{R})$, $A\in\mathcal{B}(S)$, and $t>0$, where $v=\Theta_T$ (Çinlar, 1972, Thm.3.2).

Throughout this section, we assume that Θ is a Hunt process, ξ is quasi-left continuous on $[0,\zeta]$ where ζ is the lifetime of the process and the MAP (ξ,Θ) is upwards regular, that is,

$$\mathbb{P}_{0,\theta}(\tau_0^+=0)=1$$

$\forall\theta\in S$ where $\tau_0^+:=\inf\{t>0:\xi_t>0\}$. Let $\bar{\xi}_t:=\sup_{s\leq t}\xi_s$ denote the running supremum process for the ordinate ξ and $U_t:=\bar{\xi}_t-\xi_t$ be the corresponding reflected process. Denote the zeros of the reflected process by $\bar{M}:=\{t\geq 0:U_t=0\}$ and let $\bar{M}^{cl}$ be the closure of $\bar{M}$ in $\mathbb{R}^+$. The complement

of $\bar{M}^{cl}$ consists of open random intervals. For an interval with endpoints g and d, define

$$(\epsilon_s^{(g)}, v_s^{(g)}) := \begin{cases} (U_{g+s}, \Theta_{g+s}) & \text{if } 0 \leq s \leq d-g \\ (U_d, \Theta_d) & \text{if } d-g \leq s. \end{cases}$$

Notice that $(\epsilon_s^{(g)}, v_s^{(g)})$ corresponds to an excursion from the supremum. Let ζ^+ denote the lifetime of the excursion and η_θ^+ be the excursion measure defined in Kyprianou et al. (2020, §2.2). Let $\bar{L}$ be the local time at the supremum associated to the ordinate ξ. Then, by Kaspi (1983), there exists a measurable function $\ell^+ : S \to \mathbb{R}^+$ such that

$$\int_0^t \mathbf{1}_{\{s\in\bar{M}\}} ds = \int_0^t \mathbf{1}_{\{s\in\bar{M}^{cl}\}} ds = \int_0^t \ell^+(\Theta_s) d\bar{L}_s \qquad \mathbb{P}_{0,\theta}\text{-a.s.}$$

Let $\bar{L}_t^{-1} := \inf\{s : L_s > t\}$ be the inverse local time process and define $\xi_t^+ = \xi_{\bar{L}_t^{-1}}$ and $\Theta_t^+ = \Theta_{\bar{L}_t^{-1}}$ for all t such that $\bar{L}_t^{-1} < \infty$, and otherwise assign both ξ_t^+ and Θ_t^+ to the cemetery state ∂. Then, $(\bar{L}_t^{-1}, \xi_t^+, \Theta_t^+)_{t\geq 0}$ defines a MAP called *ascending ladder process*, where both $\bar{L}_t$ and ξ_t^+ are ordinates (Kyprianou et al., 2020). Let $\bar{g}_t = \sup\{s \leq t : s \in \bar{M}^{cl}\}$ and $\bar{\Theta}_t = \Theta_{\bar{g}_t}\mathbf{1}_{\{\bar{\xi}_t=\xi_{\bar{g}_t}\}} + \Theta_{\bar{g}_t^-}\mathbf{1}_{\{\bar{\xi}_t>\xi_{\bar{g}_t}\}}$ and recall that e_q is an exponentially distributed random time with parameter $q > 0$, which is independent of (ξ, Θ). Then, by Kyprianou et al. (2020, Prop.2.3), for bounded measurable functions $F, G : [0,\infty) \times \mathbb{R} \times S \to \mathbb{R}$ and for $\theta \in S$, we have

$$\mathbb{E}_{0,\theta}[G(\bar{g}_{e_q}, \bar{\xi}_{e_q}, \bar{\Theta}_{e_q}) F(e_q - \bar{g}_{e_q}, \bar{\xi}_{e_q} - \xi_{e_q}, \Theta_{e_q})] =$$
$$\int_{\mathbb{R}^+\times\mathbb{R}^+\times S} e^{-qr} G(r,z,v)[q\ell^+(v)F(0,0,v) + \eta_v^+(F(e_q, \epsilon_{e_q}, v_{e_q})\mathbf{1}_{\{e_q<\zeta^+\}}]V_\theta^+(dr,dv,dz) \quad (2.2)$$

where

$$V_\theta^+(dr,dv,dz) := \mathbb{E}_{0,\theta}[\int_0^{\bar{L}_\infty} \mathbf{1}_{\{\bar{L}_s^{-1}\in dr, \Theta_s^+\in dv, \xi_s^+\in dz\}} ds].$$

Remark 2.1. We say that the MAP $((\xi,\Theta),\mathbb{P})$ is downwards regular if

$$\mathbb{P}_{0,\theta}(\tau_0^- = 0) = 1$$

$\forall\theta \in S$ where $\tau_0^- := \inf\{t > 0 : \xi_t < 0\}$. Now, suppose that the MAP is downwards regular. Then, we can construct the *descending ladder process* $(\underline{L}_t^{-1}, \xi_t^-, \Theta_t^-)$ by considering the ascending ladder process corresponding to the MAP $((\xi_t,\Theta_t)_{t\geq 0}, \hat{\mathbb{P}}_{-x,\theta})$ obtained from $((\xi_t,\Theta_t)_{t\geq 0}, \mathbb{P}_{x,\theta})$ by replacing ξ by its negative $-\xi$. In that case, the running infimum $\underline{\xi}_t := \inf_{s\leq t}\xi_s$ under $\mathbb{P}_{0,\theta}$ corresponds to the running supremum $\bar{\xi}_t := \sup_{s\leq t}(\xi_s)$ under $\hat{\mathbb{P}}_{0,\theta}$. Similarly, the last time at the infimum $\underline{g}_t$, the lifetime of the excursion from the infimum ζ^-, the excursion measure at the the infimum η^- and the local time at the infimum $\underline{L}_t$ with density ℓ^- under $\mathbb{P}_{x,\theta}$ correspond to $\bar{g}_t$, ζ^+, η^+, $\bar{L}_t$ and ℓ^+ under $\hat{\mathbb{P}}_{-x,\theta}$, respectively. Using these notations, the excursion from the infimum version of Equation (2.2) can be stated with potential measure associated with $(\underline{L}_t^{-1}, \xi_t^-, \Theta_t^-)$ defined as

$$\begin{aligned} V_\theta^-(dr,dv,dz) :=& \mathbb{E}_{0,\theta}[\int_0^{\underline{L}_\infty} \mathbf{1}_{\{\underline{L}_s^{-1}\in dr, \Theta_s^-\in dv, \xi_s^-\in dz\}} ds] \\ =& \hat{\mathbb{E}}_{0,\theta}[\int_0^{\bar{L}_\infty} \mathbf{1}_{\{\bar{L}_s^{-1}\in dr, \Theta_s^+\in dv, \xi_s^+\in dz\}} ds] \end{aligned}$$

(Kyprianou et al., 2024, §2.5).

Let E be a Polish space. A pair of E-valued Markov processes with respective transition probabilities $(P_t)_{t\geq 0}$ and $(\widetilde{P}_t)_{t\geq 0}$ are set to be *weak duality* with respect to some σ-finite measure $m(dx)$

if for all positive measurable functions f and g

$$\int_E g(x)P_t f(x)m(dx) = \int_E f(x)\widetilde{P}_t g(x)m(dx).$$

The details of weak duality of Markov processes can be found in Chung and Walsh (2005, Ch.13). Consider the MAPs $((\xi,\Theta),\mathbb{P})$ and $((\xi,\Theta),\hat{\mathbb{P}})$. The problem is to determine whether these two MAPs are in weak duality with respect to the measure $m(ds)dy$ where dy is the Lebesgue measure and $m(ds)$ is some σ-finite measure on $\mathcal{B}(S)$. For weak duality, Alili et al. (2017, Lem.3.1) provides a practical criterion. Namely, the MAPs $((\xi,\Theta),\mathbb{P})$ and $((\xi,\Theta),\hat{\mathbb{P}})$ are in weak duality with respect to measure $m(ds)dy$ if and only if

$$\mathbb{P}_{0,v}(\xi_t \in dx, \Theta_t \in dw)m(dv) = \mathbb{P}_{0,w}(\xi_t \in dx, \Theta_t \in dv)m(dw), \tag{2.3}$$

which is called the *reversibility property* of $((\xi,\Theta),\mathbb{P})$. We have a more general version of this property from Kyprianou et al. (2020). Let $((\xi,\Theta),\tilde{\mathbb{P}})$ be another MAP and this time denote by $((\xi,\Theta),\hat{\mathbb{P}})$ the process obtained from $((\xi,\Theta),\tilde{\mathbb{P}})$ by replacing ξ by its negative. Assume that the MAPs $((\xi,\Theta),\mathbb{P})$ and $((\xi,\Theta),\tilde{\mathbb{P}})$ satisfy the relation

$$\mathbb{P}_{0,v}(\xi_t \in dx, \Theta_t \in dw)m(dv) = \tilde{\mathbb{P}}_{0,w}(\xi_t \in dx, \Theta_t \in dv)m(dw) \tag{2.4}$$

where $m(ds)$ is some σ-finite measure on S. Then, Equation (2.4) is called the *weak reversibility property* of MAPs $((\xi,\Theta),\mathbb{P})$ and $((\xi,\Theta),\tilde{\mathbb{P}})$. Notice that the reversibility property is a particular case of Equation (2.4) if $\tilde{\mathbb{P}} = \mathbb{P}$. Under the weak reversibility property, we have the following duality properties borrowed from Kyprianou et al. (2020, Lem.3.8,Prop.3.9).

Lemma 2.2. *Consider a MAP $((\xi,\Theta),\mathbb{P})$ such that there exists a MAP $((\xi,\Theta),\tilde{\mathbb{P}})$ with weak reversibility property. Let $\hat{\mathbb{P}}_{0,\theta}$ be the law of $(-\xi,\Theta)$ under $\tilde{\mathbb{P}}_{0,\theta}$. Then, for every $t>0$, we have*

(1) for $x \in \mathbb{R}$, the process $(\xi_{(t-s)-}-\xi_t, \Theta_{(t-s)-})_{0\le s\le t}$ under $\mathbb{P}_{x,m}$ has the same law as $(\xi_s,\Theta_s)_{0\le s\le t}$ under $\hat{\mathbb{P}}_{0,m}$.

(2) $(\Theta_0, t-\bar{g}_t, \Theta_t, \bar{\xi}_t-\xi_t, \bar{g}_t, \bar{\Theta}_t, \bar{\xi}_t)$ under $\hat{\mathbb{P}}_{0,m}$ is equal in distribution to $(\Theta_t, \bar{g}_t, \Theta_0, \bar{\xi}_t, t-\bar{g}_t, \bar{\Theta}_t, \bar{\xi}_t-\xi_t)$ under $\mathbb{P}_{0,m}$.

3. Long Time Behavior

It has recently been shown in Kyprianou et al. (2020) that ξ_t exhibits exactly one of the three behaviors, namely, drifts to $+\infty$, oscillates, or drifts to $-\infty$ as $t\to\infty$ almost surely if the MAP satisfies a Harris-type condition and Θ has an invariant distribution. A regenerative structure defined by the Harris-type condition (Ney and Nummelin, 1987a) is closely related to the ergodic properties of the process. Therefore, the strong law of large numbers is proved first as inspired by a similar approach for Lévy processes, and is given by $\lim_{t\to\infty}\xi_t/t = \mathbb{E}_{0,\pi}[\xi_1]$, $\mathbb{P}_{0,\theta}$-a.s. for every $\theta\in\mathcal{S}$. The trichotomy follows whether $\mathbb{E}[\xi_1]$ is positive, negative, or equal to zero (Kyprianou et al., 2020).

Another idea for determining the long-time behavior of the ordinate is the use of fluctuation theory, also borrowed from Lévy processes (Bertoin, 1996). The distributions of the last time at supremum $\bar{g}_\infty$ and the infimum $\underline{g}_\infty$ arising from the corresponding ladder processes can characterize the long-time behavior of ξ. We follow this direction without assuming Harris-type recurrence, but only the existence of an invariant distribution π. Therefore, the characterization is a $\mathbb{P}_{0,\pi}$-a.s. result rather than holding for any starting $\theta\in\mathcal{S}$. We make the following assumptions in this section.

Assumption 1: Θ is a Hunt process with invariant probability measure π and ξ is quasi-left continuous.

Assumption 2: The MAP $((\xi,\Theta),\mathbb{P})$ is both upwards and downwards regular.

Assumption 3: $\mathbb{E}_{0,\pi}[\sup_{s\in[0,1]}|\xi_s|]$ is finite.

The following proposition gives a criterion for the long-time behavior of MAPs in terms of the distributions of $\bar{g}_t$ and $\underline{g}_t$. In Kyprianou et al. (2020, Prop.2.16), this behavior is determined in terms of $\mathbb{E}_{0,\pi}[\xi_t]$ with some extra assumptions using the strong law of large numbers. Therefore, we achieve a different type of characterization thanks to the alternative approach resulting from the generalization of Bertoin (1996, Ch.6.3).

Proposition 3.1. *Consider a MAP $((\xi,\Theta),\mathbb{P})$, such that Assumptions 1,2, and 3 are satisfied. Then, we have*

(i) ξ_t drifts to $-\infty$ $\mathbb{P}_{0,\pi}$-a.s., if and only if $\bar{g}_\infty < \infty$ and $\underline{g}_\infty = \infty$ $\mathbb{P}_{0,\pi}$-a.s.
(ii) ξ_t drifts to $+\infty$ $\mathbb{P}_{0,\pi}$-a.s., if and only if $\bar{g}_\infty = \infty$ and $\underline{g}_\infty < \infty$ $\mathbb{P}_{0,\pi}$-a.s.
(iii) ξ_t oscillates $\mathbb{P}_{0,\pi}$-a.s., if and only if $\bar{g}_\infty = \infty$ and $\underline{g}_\infty = \infty$ $\mathbb{P}_{0,\pi}$-a.s.

Proof: The necessity part is clear, thus we proceed by showing the sufficiency part.
(i) Since the MAP is downwards regular by Assumption 2, and $\underline{g}_\infty = \infty$ $\mathbb{P}_{0,\pi}$-a.s., it follows that, ξ visits $(-\infty,-x]$ $\forall x>0$ and the passage time $T_{(-\infty,-x]} := \inf\{t>0 \,|\, \xi_t \in (-\infty,-x]\}$ is $\mathbb{P}_{0,\pi}$-a.s. finite. Assumption 3 implies that $\mathbb{E}_{0,\pi}[\sup_{s\in[0,t]}|\xi_s|] < \infty$ for all $t>0$ by Kyprianou et al. (2020, Lem.2.13). Hence, $\bar{\xi}_\infty < \infty$ $\mathbb{P}_{0,\pi}$-a.s. since $\bar{g}_\infty < \infty$ $\mathbb{P}_{0,\pi}$-a.s. It means that, we can pick x_ϵ large enough such that

$$\mathbb{P}_{0,\pi}(\xi_t > x/2 \text{ for some } t\geq 0) \leq \epsilon \quad \forall x \geq x_\epsilon.$$

Denote $T := T_{(-\infty,-x]}$. Then, for $x \geq x_\epsilon$, we have

$$\begin{aligned}
&\mathbb{P}_{0,\pi}(\xi_t > -x/2 \text{ for some } t\geq T) \\
&= \mathbb{P}_{0,\pi}(\xi_{t+T} > -x/2 \text{ for some } t\geq 0) \\
&\leq \mathbb{P}_{0,\pi}(\xi_{t+T(-\infty,-x]} - \xi_{T(-\infty,-x]} > x/2 \text{ for some } t\geq 0) && (3.1)\\
&= \int_S \mathbb{P}_{0,\theta}(\xi_{t+T} - \xi_T > x/2 \,\text{for some } t\geq 0)\pi(d\theta) && (3.2)\\
&= \int_S\int_S \mathbb{P}_{0,\theta}(\xi_{t+T} - \xi_T > x/2 \,\text{for some } t\geq 0 \,|\, \Theta_T = \phi)\mathbb{P}_{0,\theta}(\Theta_T \in d\phi)\pi(d\theta) \\
&= \int_S\int_S \mathbb{P}_{0,\phi}(\xi_t > x/2 \text{ for some } t\geq 0)\, \mathbb{P}_{0,\theta}(\Theta_T \in d\phi)\pi(d\theta) && (3.3)\\
&= \int_S \mathbb{P}_{0,\phi}(\xi_t > x/2 \text{ for some } t\geq 0)\, \mathbb{P}_{0,\pi}(\Theta_T \in d\phi) \\
&= \int_S \mathbb{P}_{0,\phi}(\xi_t > x/2 \text{ for some } t\geq 0)\pi(d\phi) && (3.4)\\
&= \mathbb{P}_{0,\pi}(\xi_t > x/2 \text{ for some } t\geq 0) \quad < \quad \epsilon && (3.5)
\end{aligned}$$

where we use $\xi_{T(-\infty,-x]} \leq -x$ $\mathbb{P}_{0,\pi}$-a.s. in (3.1), definition of $\mathbb{P}_{0,\pi}$ in (3.2) and (3.5), strong Markov property (2.1) in (3.3) and the fact that π is an invariant measure in (3.4). Hence, ξ drifts to $-\infty$ $\mathbb{P}_{0,\pi}$-a.s.
(ii) Apply the previous part of the proof to the process $((\xi,\Theta),\hat{\mathbb{P}})$ using the upwards regularity of $((\xi,\Theta),\mathbb{P})$, where $\hat{\mathbb{P}}$ is defined by replacing ξ by its negative.
(iii) Since the MAP is both upwards and downwards regular by Assumption 2, ξ visits $(-\infty,-x]$ and $[x,\infty)$ $\forall x>0$ and both $T_{(-\infty,-x]}$ and $T_{[x,\infty)}$ are finite $\mathbb{P}_{0,\pi}$-a.s. Hence, ξ oscillates. □

Assumption 4: The reversibility property (2.3) holds.

Consider a MAP $((\xi,\Theta),\mathbb{P})$ such that also Assumption 4 is satisfied. By taking $\tilde{\mathbb{P}} = \mathbb{P}$ in the second part of Lemma 2.2, thanks to the reversibility property (2.3), we can deduce that $\underline{g}_t$ and $t-\bar{g}_t$ have the same distribution under $\mathbb{P}_{0,\pi}$ as they are both equal in distribution to $\bar{g}_t$ under $\hat{\mathbb{P}}_{0,\pi}$

where $\hat{\mathbb{P}}_{0,\pi}$ is the law of $(-\xi,\Theta)$ under $\mathbb{P}_{0,\pi}$. Hence, using Proposition 3.1, we obtain the following identities

$$\mathbb{E}_{0,\pi}[e^{-\lambda \bar{g}_{e_q}}] = \int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-\lambda r-qr}[q\ell^+(v)+\eta_v^+(\mathbf{1}_{\{e_q<\zeta^+\}})] \tag{3.6}$$

$$\mathbb{E}_{0,\pi}[e^{-\lambda \underline{g}_{e_q}}] = \int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-qr}[q\ell^+(v)+\eta_v^+(e^{-\lambda e_q}\mathbf{1}_{\{e_q<\zeta^+\}})] \tag{3.7}$$

where

$$V_\pi^+(dr,dv,dz)=\mathbb{E}_{0,\pi}[\int_0^{\bar{L}_\infty}\mathbf{1}_{\{\bar{L}_s^{-1}\in dr\,,\,\Theta_s^+\in dv\,,\,\xi_s^+\in dz\}}ds].$$

We have assumed that MAP is also downwards regular, so we can apply (2.2) for the process $((\xi,\Theta),\hat{\mathbb{P}})$. We obtain the following analogs of (3.6) and (3.7) as

$$\hat{\mathbb{E}}_{0,\pi}[e^{-\lambda \bar{g}_{e_q}}] = \int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^-(dr,dv,dz)e^{-\lambda r-qr}[q\ell^-(v)+\eta_v^-(\mathbf{1}_{\{e_q<\zeta^-\}})] \tag{3.8}$$

$$\hat{\mathbb{E}}_{0,\pi}[e^{-\lambda \underline{g}_{e_q}}] = \int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^-(dr,dv,dz)e^{-qr}[q\ell^-(v)+\eta_v^-(e^{-\lambda e_q}\mathbf{1}_{\{e_q<\zeta^-\}})]$$

where

$$V_\pi^-(dr,dv,dz)=\mathbb{E}_{0,\pi}[\int_0^{\underline{L}_\infty}\mathbf{1}_{\{\underline{L}_s^{-1}\in dr\,,\,\Theta_s^-\in dv\,,\,\xi_s^-\in dz\}}ds].$$

Remark 3.2. If we only use identities (3.6) and (3.7), we will not be able to distinguish the cases where ξ_t drifts to $+\infty$ or oscillates. Therefore, we have also obtained these identities in terms of the descending ladder process $(\underline{L}_t^{-1},\xi_t^-,\Theta_t^-)$.

The following is our main theorem.

Theorem 3.3. *Let $(\xi,\Theta),\mathbb{P})$ be a MAP such that Assumptions 1,2,3, and 4 are satisfied. Then, we have*

(i) ξ_t drifts to $-\infty$ $\mathbb{P}_{0,\pi}$-a.s., if and only if

$$\int_S U_\pi^+(dv,[0,\infty))\eta_v^+(\zeta^+=+\infty)=1 \tag{3.9}$$

(ii) ξ_t drifts to $+\infty$ $\mathbb{P}_{0,\pi}$-a.s., if and only if

$$\int_S U_\pi^-(dv,(-\infty,0])\eta_v^-(\zeta^-=+\infty)=1 \tag{3.10}$$

(iii) ξ_t oscillates $\mathbb{P}_{0,\pi}$-a.s., if and only if

$$\int_S U_\pi^+(dv,[0,\infty))\eta_v^+(\zeta^+=+\infty)=0 \quad \text{and} \quad \int_S U_\pi^-(dv,(-\infty,0])\eta_v^-(\zeta^-=+\infty)=0 \tag{3.11}$$

where

$$U_\theta^+(dv,dz)=\mathbb{E}_{0,\theta}[\int_0^{\bar{L}_\infty}\boldsymbol{1}_{\{\Theta_s^+\in dv,\xi_s^+\in dz\}}ds] \quad \text{and} \quad U_\theta^-(dv,dz)=\mathbb{E}_{0,\theta}[\int_0^{\underline{L}_\infty}\boldsymbol{1}_{\{\Theta_s^-\in dv\,,\,\xi_s^-\in dz\}}ds].$$

Proof: **(i)** We can rewrite (3.6) as

$$\mathbb{E}_{0,\pi}[e^{-\lambda \bar{g}_{e_q}}]=\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-\lambda r-qr}[q\ell^+(v)+\eta_v^+(1-e^{-q\zeta^+})].$$

For any $0<q<\lambda/2$, the integrand is bounded from above by

$$e^{-\lambda r}[\frac{\lambda}{2}\ell^+(v)+\eta_v^+(1-e^{-\lambda\zeta^+/2})].$$

Also, we have

$$\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-\lambda r}[\frac{\lambda}{2}\ell^+(v)+\eta_v^+(1-e^{-\lambda\zeta^+/2})]=\mathbb{E}_{0,\pi}[e^{-\frac{\lambda}{2}\bar{g}_{e_{\lambda/2}}}]<+\infty.$$

Then, by applying the dominated convergence theorem on both sides of (3.6) as $q\to 0^+$, we get

$$\mathbb{E}_{0,\pi}[e^{-\lambda\bar{g}_\infty}]=\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-\lambda r}\eta_v^+(\zeta^+=+\infty).$$

By letting $\lambda\to 0^+$, we obtain

$$\lim_{\lambda\to 0^+}\mathbb{E}_{0,\pi}[e^{-\lambda\bar{g}_\infty}]=\int_S U_\pi^+(dv,[0,\infty))\eta_v^+(\zeta^+=+\infty). \tag{3.12}$$

Note that if ξ_t drifts to $-\infty$ $\mathbb{P}_{0,\pi}$-a.s., then the left-hand side of (3.12) is equal to 1 by the first part of Proposition 3.1. Hence (3.9) holds.

Now, assume that (3.9) holds. Then, (3.12) implies that $\lim_{\lambda\to 0^+}\mathbb{E}_{0,\pi}[e^{-\lambda\bar{g}_\infty}]=1$. Hence, we have shown that $\bar{g}_\infty<\infty$ $\mathbb{P}_{0,\pi}$-a.s. By considering (3.7), we have

$$\mathbb{E}_{0,\pi}[e^{-\lambda\underline{g}_{e_q}}]=\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-qr}[q\ell^+(v)+\frac{q}{q+\lambda}\eta_v^+(1-e^{-(\lambda+q)\zeta^+})]. \tag{3.13}$$

Since $\bar{g}_\infty$ is $\mathbb{P}_{0,\pi}$-a.s. finite, for each $0<\epsilon<1$, there exists $t_\epsilon>0$ such that

$$\mathbb{P}_{0,\pi}(\bar{g}_\infty>t_\epsilon)<\epsilon$$

which implies that

$$\mathbb{P}_{0,\pi}(\bar{L}_\infty>t_\epsilon)<\epsilon.$$

We choose $(t_\epsilon)_{\epsilon\geq 0}$ such that t_ϵ increases to infinity as ϵ goes to zero. Note that, for each $\epsilon>0$, we can rewrite the right-hand side of (3.13) as

$$\begin{aligned}&\int_{[0,t_\epsilon]\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-qr}[q\ell^+(v)+\frac{q}{q+\lambda}\eta_v^+(1-e^{-(\lambda+q)\zeta^+})]\\&\qquad+\int_{(t_\epsilon,\infty)\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-qr}[q\ell^+(v)+\frac{q}{q+\lambda}\eta_v^+(1-e^{-(\lambda+q)\zeta^+})].\end{aligned} \tag{3.14}$$

The second integral in (3.14) can be rewritten as

$$\mathbb{E}_{0,\pi}[\int_{\bar{L}_{t_\epsilon}}^{\bar{L}_\infty} ds\, e^{-q\bar{L}_s^{-1}}[q\ell^+(\Theta_s^+)+\frac{q}{q+\lambda}\eta_{\Theta_s^+}^+(1-e^{-(\lambda+q)\zeta^+})]].$$

Note that as ϵ goes to 0, t_ϵ increases to infinity. It follows that $\bar{L}_{t_\epsilon}$ increases to $\bar{L}_\infty$. Furthermore, the integrand is positive, so the integral decreases monotonically as ϵ goes to 0. Hence, by the monotone convergence theorem, we have

$$\begin{aligned}&\lim_{\epsilon\to 0}\mathbb{E}_{0,\pi}[\int_{\bar{L}_{t_\epsilon}}^{\bar{L}_\infty} ds\, e^{-q\bar{L}_s^{-1}}[q\ell^+(\Theta_s^+)+\frac{q}{q+\lambda}\eta_{\Theta_s^+}^+(1-e^{-(\lambda+q)\zeta^+})]]=\\&\qquad\mathbb{E}_{0,\pi}[\lim_{\epsilon\to 0}\int_{\bar{L}_{t_\epsilon}}^{\bar{L}_\infty} ds\, e^{-q\bar{L}_s^{-1}}[q\ell^+(\Theta_s^+)+\frac{q}{q+\lambda}\eta_{\Theta_s^+}^+(1-e^{-(\lambda+q)\zeta^+})]]=0.\end{aligned}$$

It means that we can fix a sufficiently small $\epsilon>0$ such that the second integral in (3.14) is bounded. For fixed λ and $\epsilon>0$, there exists an integer $n\in\mathbb{N}$ such that $\frac{-\lambda t_\epsilon+\sqrt{\lambda^2t_\epsilon^2+4\lambda t_\epsilon}}{2t_\epsilon}>\frac{\lambda}{n}$. Then, for any $0<q<\lambda/n$, the integrand in the first integral in (3.14) is bounded from above by

$$e^{-\frac{\lambda}{n}r}[\frac{\lambda}{n}\ell^+(v)+\frac{1}{n+1}\eta_v^+(1-e^{-\frac{n+1}{n}\lambda\zeta^+})].$$

Therefore, the first integral in (3.14) is bounded from above by

$$\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-\frac{\lambda}{n}r}[\frac{\lambda}{n}\ell^+(v)+\frac{1}{n+1}\eta_v^+(1-e^{-\frac{n+1}{n}\lambda\zeta^+})]=\mathbb{E}_{0,\pi}[e^{-(\lambda/n)\underline{g}_{e_{\lambda/n}}}]<\infty.$$

Then, applying the dominated convergence theorem as $q\to 0^+$ on both sides of (3.7) yields

$$\mathbb{E}_{0,\pi}[e^{-\lambda\underline{g}_\infty}]=0 \qquad \forall\lambda>0$$

which implies that $\underline{g}_\infty=\infty$ $\mathbb{P}_{0,\pi}$-a.s. Hence, by Proposition 3.1, we conclude that ξ_t drifts to $-\infty$ $\mathbb{P}_{0,\pi}$-a.s.

(ii) Assume (3.10) holds. Then, by the proof of part (i), ξ drifts to $-\infty$ under $\hat{\mathbb{P}}_{0,\pi}$. Hence, ξ drifts to $+\infty$ $\mathbb{P}_{0,\pi}$-a.s.

Now, suppose that ξ drifts to $+\infty$ $\mathbb{P}_{0,\pi}$-a.s. Then, ξ drifts to $-\infty$ under $\hat{\mathbb{P}}_{0,\pi}$-a.s, and by Proposition 3.1, we have $\hat{\mathbb{P}}_{0,\pi}(\bar{g}_\infty<\infty)=1$ which implies (3.10) using an analogue of (3.12) obtained using (3.8).

(iii) Assume (3.11) holds. First equality implies that $\bar{g}_\infty=\infty$ $\mathbb{P}_{0,\pi}$-a.s. by (3.12). Similarly, for the process $((\xi,\Theta),\hat{\mathbb{P}})$, second equality implies that $\bar{g}_\infty=\infty$ $\hat{\mathbb{P}}_{0,\pi}$-a.s. which implies that $\underline{g}_\infty=\infty$ $\mathbb{P}_{0,\pi}$-a.s. Then, by part (iii) of Proposition 3.1, ξ oscillates. The necessity part follows similar as part (i) and (ii). □

Remark 3.4. The identites (3.9) and (3.10) are exclusive. Indeed, assume that (3.9) holds. By the proof of first part, it implies that $\mathbb{E}_{0,\pi}[e^{-\lambda\underline{g}_\infty}]=0$, which is equivalent to say that $\hat{\mathbb{E}}_{0,\pi}[e^{-\lambda\bar{g}_\infty}]=0$. Hence, by (3.8), we have $\int_S U_\pi^-(dv,(-\infty,0])\eta_v^-(\zeta^-=+\infty)=0$.

For Theorem 3.3, one of our assumptions is that the reversibility property holds for the MAP. Now, we replace this property with the weak reversibility property.

Assumption 4* Assume that there exists another MAP $((\xi,\Theta),\tilde{\mathbb{P}})$ such that it satisfies the weak reversibility property (2.4) with $((\xi,\Theta),\mathbb{P})$.

Assume that there exists MAPs $((\xi,\Theta),\mathbb{P})$ and $((\xi,\Theta),\tilde{\mathbb{P}})$ such that Assumptions $1,2,3$, and 4^* hold. By the first part of Lemma 2.2, it can be deduced that

$$\mathbb{E}_{0,\pi}[e^{-\lambda\underline{g}_{e_q}}]=\widetilde{\mathbb{E}}_{0,\pi}[e^{-\lambda(e_q-\bar{g}_{e_q})}]. \tag{3.15}$$

Indeed, let $\hat{\mathbb{P}}_{0,\pi}$ denotes the law of $(-\xi,\Theta)$ under $\tilde{\mathbb{P}}_{0,\pi}$. Then, the second part of Lemma 2.2 implies that $\underline{g}_t$ under $\mathbb{P}_{0,\theta}$ is equal in distribution to $t-\underline{g}_t$ under $\hat{\mathbb{P}}_{0,\theta}$ which is equal in distribution to $t-\bar{g}_t$ under $\tilde{\mathbb{P}}_{0,\theta}$. Then, by (2.2) and (3.15), we have the following identities for the process $((\xi,\Theta),\mathbb{P})$

$$\mathbb{E}_{0,\pi}[e^{-\lambda\bar{g}_{e_q}}]=\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^+(dr,dv,dz)e^{-\lambda r-qr}[q\ell^+(v)+\eta_v^+(\mathbf{1}_{\{e_q<\zeta^+\}})]$$

$$\mathbb{E}_{0,\pi}[e^{-\lambda\underline{g}_{e_q}}]=\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} \widetilde{V}_\pi^+(dr,dv,dz)e^{-qr}[q\widetilde{\ell}^+(v)+\widetilde{\eta}_v^+(e^{-\lambda e_q}\mathbf{1}_{\{e_q<\tilde{\zeta}^+\}})]$$

where

$$\widetilde{V}_\pi^+(dr,dv,dz)=\widetilde{\mathbb{E}}_{0,\pi}[\int_0^{\bar{L}_\infty}\mathbf{1}_{\{\bar{L}_s^{-1}\in dr\,,\,\Theta_s^+\in dv\,,\,\xi_s^+\in dz\}}ds].$$

Similarly, for the process $((\xi,\Theta),\check{\mathbb{P}})$ where $\check{\mathbb{P}}_{0,\theta}$ is the law of $(-\xi,\Theta)$ under $\mathbb{P}_{0,\theta}$, we have the following identities as

$$\check{\mathbb{E}}_{0,\pi}[e^{-\lambda\bar{g}_{e_q}}]=\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} V_\pi^-(dr,dv,dz)e^{-\lambda r-qr}[q\ell^-(v)+\eta_v^-(\mathbf{1}_{\{e_q<\zeta^-\}})]$$

$$\check{\mathbb{E}}_{0,\pi}[e^{-\lambda\underline{g}_{e_q}}]=\int_{\mathbb{R}^+\times S\times\mathbb{R}^+} \widetilde{V}_\pi^-(dr,dv,dz)e^{-qr}[q\tilde{\ell}^-(v)+\tilde{\eta}_v^-(e^{-\lambda e_q}\mathbf{1}_{\{e_q<\tilde{\zeta}^-\}})]$$

where

$$\widetilde{V}_{\pi}^{-}(dr,dv,dz)=\widetilde{\mathbb{E}}_{0,\pi}[\int_{0}^{\underline{L}_{\infty}}\mathbf{1}_{\{\underline{L}_{s}^{-1}\in dr\,,\Theta_{s}^{-}\in dv\,,\xi_{s}^{-}\in dz\}}ds].$$

The following theorem is a generalized version of Theorem 3.3 since Assumption 4* is a weaker condition than Assumption 4.

Theorem 3.5. *Let $((\xi,\Theta),\mathbb{P})$ be a MAP such that Assumptions 1,2,3, and 4* satisfied. Then,*

(i) ξ_t drifts to $-\infty$ $\mathbb{P}_{0,\pi}$-a.s., if and only if

$$\int_{S}U_{\pi}^{+}(dv,[0,\infty))\eta_{v}^{+}(\zeta^{+}=+\infty)=1\quad\text{and}\quad\int_{S}\widetilde{U}_{\pi}^{+}(dv,[0,\infty))\widetilde{\eta}_{v}^{+}(\tilde{\zeta}^{+}=+\infty)=1$$

(ii) ξ_t drifts to $+\infty$ $\mathbb{P}_{0,\pi}$-a.s., if and only if

$$\int_{S}U_{\pi}^{-}(dv,(-\infty,0])\eta_{v}^{-}(\zeta^{-}=+\infty)=1\quad\text{and}\quad\int_{S}\widetilde{U}_{\pi}^{-}(dv,(-\infty,0])\widetilde{\eta}_{v}^{-}(\tilde{\zeta}^{-}=+\infty)=1$$

(iii) ξ_t oscillates $\mathbb{P}_{0,\pi}$-a.s., if and only if

$$\int_{S}U_{\pi}^{+}(dv,[0,\infty))\eta_{v}^{+}(\zeta^{+}=+\infty)=0\quad\text{and}\quad\int_{S}U_{\pi}^{-}(dv,(-\infty,0])\eta_{v}^{-}(\zeta^{-}=+\infty)=0$$

Proof: The necessity parts follow from Proposition 3.1 in view of the following proof of the sufficiency parts.

(i) As shown in (3.12) in the proof of the first part of Theorem 3.3, the hypothesis

$$\int_{S}U_{\pi}^{+}(dv,[0,\infty))\eta_{v}^{+}(\zeta^{+}=+\infty)=1$$

implies that

$$\lim_{\lambda\to0^{+}}\mathbb{E}_{0,\pi}[e^{-\lambda\bar{g}_{\infty}}]=\int_{S}U_{\pi}^{+}(dv,[0,\infty))\eta_{v}^{+}(\zeta^{+}=+\infty)=1.$$

It implies that $\bar{g}_{\infty}$ is $\mathbb{P}_{0,\pi}$-a.s. finite. Similar arguments can be applied to the MAP $((\xi,\Theta),\tilde{\mathbb{P}})$ to get that

$$\lim_{\lambda\to0^{+}}\tilde{\mathbb{E}}_{0,\pi}[e^{-\lambda\bar{g}_{\infty}}]=\int_{S}\widetilde{U}_{\pi}^{+}(dv,[0,\infty))\widetilde{\eta}_{v}^{+}(\tilde{\zeta}^{+}=+\infty)=1$$

Hence, $\bar{g}_{\infty}$ is finite $\tilde{\mathbb{P}}_{0,\pi}$-a.s. Then, as in the proof of the first part of Theorem 3.3, we can conclude that

$$\mathbb{E}_{0,\pi}[e^{-\lambda\underline{g}_{\infty}}]=\lim_{q\to0}\tilde{\mathbb{E}}_{0,\pi}[e^{-\lambda(e_{q}-\bar{g}_{e_{q}})}]=0$$

Hence, $\underline{g}_{\infty}$ is infinite $\mathbb{P}_{0,\pi}$-a.s. By Proposition 3.1, ξ_t drifts to $-\infty$ $\mathbb{P}_{0,\pi}$-a.s.

(ii) By the proof of the first part, we have that ξ_t drifts to $-\infty$ under $\check{\mathbb{P}}_{0,\pi}$; hence, ξ_t drifts to $+\infty$ $\mathbb{P}_{0,\pi}$-a.s.

(iii) Same as part *(iii)* of Theorem 3.3. □

4. Examples

Let $\mathcal{H}$ be a locally compact subset of $\mathbb{R}^{d}\backslash\{0\}$ where $d\geq1$. Consider an $\mathcal{H}$-valued cádlág Markov process $(X,P)=((X_{t})_{t\geq0},(P_{x})_{x\in\mathcal{H}})$ killed at the origin with $P_{x}(X_{0}=x)=1$. X is called an *$\mathcal{H}$-valued self-similar Markov process* (ssMp for short) with index $\alpha>0$ if for all $c,t>0$ and $x\in\mathcal{H}$, we have the following scaling property

$$(X_{t},P_{x})=(cX_{c^{-\alpha}t},P_{c^{-1}x})$$

The scaling property implies that $\mathcal{H}=c\mathcal{H}$ for every $c>0$; hence, $\mathcal{H}$ is necessarily a cone of $\mathbb{R}^{d}\backslash\{0\}$ in the form of

$$\mathcal{H}=\Phi(\mathbb{R}\times\mathcal{S})$$

where S is a locally compact subset of unit sphere S^{d-1} in $\mathbb{R}^d$ and Φ is a homeomorphism from $\mathbb{R}\times S^{d-1}$ to $\mathbb{R}\backslash\{0\}$ defined by $\Phi(y,\theta)=\theta e^y$.

One way to examine $\mathcal{H}$-valued self-similar Markov processes is to use skew-product type decomposition of the process. Consequently, the radial distance of the process becomes a real-valued stochastic process modulated by the angular part of the process. This approach forms the basis of Lamperti-Kiu transform of the self-similar Markov processes. The origin of Lamperti-Kiu transform is due to Lamperti (1972) for the case $d=1$, and a higher dimensional version can be found in Kiu (1980). We give the generalized version of this transform in Theorem 4.1 below, restating Alili et al. (2017, Thm.2.3) for the sake of completeness, to reveal the correspondence between $\mathbb{R}\times S$-valued MAPs, where S is a locally compact subset of the unit sphere S^{d-1}, and self-similar Markov processes in terms of the homeomorphism Φ and the skew-product decomposition.

Theorem 4.1. *(Alili et al., 2017, Thm.2.3) Let $\alpha>0$ and $((\xi,\Theta),\mathbb{P})$ be a MAP in $S\times\mathbb{R}$ with an extra isolated state ∂ and lifetime ζ. Define a process X by*

$$X_t=\begin{cases}\Theta_{\tau_t}e^{\xi_{\tau_t}} & \text{if } t<\int_0^\zeta e^{\alpha\xi_s}ds\\ 0 & \text{if } t\geq\int_0^\zeta e^{\alpha\xi_s}ds\end{cases}$$

where the time change is given by

$$\tau_t:=\inf\{s:\int_0^s e^{\alpha\xi_u}du>t\}$$

for $t<\int_0^\zeta e^{\alpha\xi_s}ds$. Define $P_x=\mathbb{P}_{x/||x||,\log||x||}$ for $x\in\mathcal{H}$ and $P_0=\mathbb{P}_\partial$. Then, the process (X,P_x) is an $\mathcal{H}$-valued ssMp with index $\alpha>0$ and lifetime $\int_0^\zeta e^{\alpha\xi_s}ds$.
Conversely, let (X,P_x) be an $\mathcal{H}$-valued self-similar Markov process with index $\alpha>0$ and lifetime ζ'. Define the process (ξ,Θ) by

$$\begin{cases}\xi_t=\log||X_{A_t}|| \quad \text{and} \quad \Theta_t=\frac{X_{A_t}}{||X_{A_t}||} & \text{if } t<\int_0^{\zeta'}\frac{ds}{||X_s||^\alpha}\\ (\xi_t,\Theta_t)=\partial & \text{if } t\geq\int_0^{\zeta'}\frac{ds}{||X_s||^\alpha}\end{cases}$$

where the time change is given by

$$A_t:=\inf\{s:\int_0^s\frac{du}{||X_u||^\alpha}>t\}$$

for $t<\int_0^{\zeta'}e^{\alpha\xi_s}ds$. Define $\mathbb{P}_{y,\theta}=P_{\theta e^y}$ for $\theta\in S$ and $y\in\mathbb{R}$, and $\mathbb{P}_\partial=P_0$. Then, the process $((\xi,\Theta),\mathbb{P}_{y,\theta})$ is a MAP in $S\times\mathbb{R}$ with lifetime $\int_0^{\zeta'}\frac{ds}{||X_s||^\alpha}$.

Note that whether the lifetime ζ' of the ssMp X is finite or not, either $\int_0^{\zeta'}\frac{ds}{||X_s||^\alpha}=\infty$ and the corresponding MAP (ξ,Θ) has infinite lifetime, or $\int_0^{\zeta'}\frac{ds}{||X_s||^\alpha}<\infty$ and the corresponding MAP (ξ,Θ) has finite lifetime (Alili et al., 2017).

An example of a well-known class of self-similar Markov processes (X,P) and the corresponding MAP $((\xi,\Theta),\mathbb{P})$ via Theorem 4.1 is given as follows. We say that $((\xi,\Theta),\mathbb{P})$ and (X,P) have *skew-product property* if the transition probabilities of (ξ,Θ) are of the form

$$\begin{aligned}\mathbb{P}_{y,\theta}(\xi_t\in dx,\Theta_t\in d\omega)&=e^{-\lambda t}\,\mathbb{P}_y^{\xi'}(\xi'_t\in dx)\mathbb{P}_\theta^{\Theta'}(\Theta'_t\in d\omega)\\ \mathbb{P}_{y,\theta}((\xi_t,\Theta_t)=\partial)&=1-e^{-\lambda t}\end{aligned}\tag{4.1}$$

where $\lambda\geq 0$ is some constant, $(\xi',\mathbb{P}_y^{\xi'})$ is a non-killed real Lévy process, and $(\Theta',\mathbb{P}_\theta^{\Theta'})$ is a Markov process on S with infinite lifetime. One of the important properties of MAPs with skew-product property is that they have the reversibility property (2.3) due to Alili et al. (2017, Prop.3.2).

In this section, we will give two important examples of a ssMp with skew-product property and show that their corresponding MAPs satisfy the assumptions of our main result Theorem 3.3.

4.1. *Free 2-dimensional Bessel Processes.* Let $X = ((X_t^{(1)}, X_t^{(2)}), t \geq 0)$ be a free 2-dimensional Bessel processes where $X^{(i)}$'s are independent Bessel processes of dimension 3. Note that X is an $\mathcal{H}$-valued ssMp with index 2 where $\mathcal{H} = [0, \infty)^2$. Consider the MAP $((\xi, \Theta), \mathbb{P})$ that corresponds to the free Bessel process (X, P_x) via Theorem 4.1. Note that the state space of the modulator $\mathcal{H}_0$ is the restriction of the unit circle to the first quadrant, that is, the set $\mathcal{H}_0 := \{(x, y) : x, y \geq 0\,,\, x^2 + y^2 = 1\}$. As a particular case of Alili et al. (2017, Remark 4.6), there exists a 2-dimensional Brownian motion $W = (W^{(1)}, W^{(2)})$ such that

$$\xi_t = \int_0^t \Theta_s^{(1)} dW_s^{(1)} + \int_0^t \Theta_s^{(2)} dW_s^{(2)} + 2t$$

and $\Theta_t = (\Theta_t^{(1)}, \Theta_t^{(2)})$ satisfies the following SDE system

$$d\Theta_t = b(\Theta_t)dt + \sigma(\Theta_t)dW_t \tag{4.2}$$

with

$$b(x, y) = \begin{bmatrix} 1/x - 5x/2 \\ 1/y - 5y/2 \end{bmatrix} \quad \sigma(x, y) = \begin{bmatrix} 1 - x^2 & -xy \\ -xy & 1 - y^2 \end{bmatrix} = \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix} \tag{4.3}$$

for $(x, y) \in \mathcal{H}_0$.

Lemma 4.2. *The diffusion process $(\Theta, \mathbb{P}_\theta)$ with SDE representation (4.2) is a T-process in the sense of Meyn and Tweedie (1993a).*

Proof: According to Stramer and Tweedie (1997, Prop.2.2), it is sufficient to show that Θ is irreducible with respect to Lebesgue measure, and it is a Feller process. We can deduce from Revuz and Yor (1999, Ch.XI) that the free Bessel process (X, P_x) is Feller. Therefore, the corresponding MAP (ξ, Θ), and Θ are also Feller by Kyprianou (2018).

Consider $B := \{(x, y) : x \in (a, b), y = \sqrt{1 - x^2}\}$ for some arbitrary $0 \leq a < b \leq 1$. We need to show that

$$\int_0^\infty \mathbb{P}_\theta(\Theta_s \in B)ds > 0 \tag{4.4}$$

for each $\theta = (\theta_1, \theta_2) \in \mathcal{H}_0$. Consider the time change $\tau_t := \inf\{u : \int_0^u e^{\alpha\xi_v} dv > t\}$. Then, (4.4) is implied by

$$\int_0^\infty \mathbb{P}_\theta(\Theta_{\tau_t} \in B)dt > 0. \tag{4.5}$$

Indeed, assume that (4.5) holds. Then, $\exists t_2 > t_1 \geq 0$ such that

$$\int_{t_1}^{t_2} \mathbb{P}_\theta(\Theta_{\tau_t} \in B)dt > 0.$$

Since τ_t is continuous and strictly increasing, then the image of the interval $[t_1, t_2]$ under the time change is another interval $[s_1, s_2]$ for $s_2 > s_1 \geq 0$ where $s_1 := \tau_{t_1}$, $s_2 := \tau_{t_2}$. Then, we have

$$0 < \int_{t_1}^{t_2} \mathbb{P}_\theta(\Theta_{\tau_t} \in B)dt = \int_\Omega \mathbb{P}(d\omega) \int_{t_1}^{t_2} \mathbf{1}_{\{\Theta_{\tau_t} \in B\}}(\omega)dt = \int_\Omega \mathbb{P}(d\omega) \int_{s_1}^{s_2} \mathbf{1}_{\{\Theta_s \in B\}}(\omega) e^{-\alpha\xi_s} ds.$$

Since $e^{-\alpha\xi_s}$ is strictly positive, we get

$$\int_{s_1}^{s_2} \mathbb{P}_\theta(\Theta_s \in B)ds > 0$$

which implies (4.4).

Note that a Bessel process of dimension at least 2 never hits 0 (Rogers and Williams, 2000, §4.35). By Lamperti-Kiu transform, we have

$$\{\Theta_{\tau_t} \in B\} = \{X_t = (X_t^{(1)}, X_t^{(2)}) \in C\}$$

where

$$C = \{(\alpha s, \alpha\sqrt{1-s^2}) \,|\, \alpha \in \mathbb{R}\backslash\{0\}\,, s \in (a,b)\}. \tag{4.6}$$

Then, (4.5) is equivalent to

$$\int_0^\infty P_q(X_t \in C)dt > 0 \tag{4.7}$$

which implies (4.4), where $q = (q_1, q_1) := (c\theta_1, c\theta_2)$ for some $c > 0$.

To show (4.7), we introduce a transformation to polar coordinates given by $R_t = ||X_t||$, and $\Phi_t = \arctan(X_t^{(2)}/X_t^{(1)})$. Then, using the fact that $X^{(1)}$ and $X^{(2)}$ are independent Bessel processes of dimension 3, and the probability transition functions of each $X^{(i)}, i = 1, 2$, from Çinlar (2011, Ch.8), we find that

$$\int_0^\infty P_q(X_t \in C)dt = \int_0^\infty dt \int_I d\varphi \int_{(0,\infty)} dr\, r\, \frac{r\cos\varphi}{q_1}\Big[\frac{e^{-(r\cos\varphi - q_1)^2/2t}}{\sqrt{2\pi t}} - \frac{e^{-(r\cos\varphi + q_1)^2/2t}}{\sqrt{2\pi t}}\Big]$$
$$\cdot \frac{r\sin\varphi}{q_2}\Big[\frac{e^{-(r\sin\varphi - q_2)^2/2t}}{\sqrt{2\pi t}} - \frac{e^{-(r\sin\varphi + q_2)^2/2t}}{\sqrt{2\pi t}}\Big]$$

for the case $q_1, q_2 \neq 0$ where $I := (\arccos a, \arccos b)$. Note that q_1 and q_2 cannot be simultaneously zero since we take $||q|| > 0$. In the case that one of them is 0, say $q_1 = 0$, we have

$$\int_0^\infty P_q(X_t \in C)dt = \int_0^\infty dt \int_I d\varphi \int_{(0,\infty)} dr\, r\, \frac{2r^2\cos^2\varphi e^{-r^2\cos^2\varphi/2t}}{\sqrt{2\pi t^3}}$$
$$\cdot \frac{r\sin\varphi}{q_2}\Big[\frac{e^{-(r\sin\varphi - q_2)^2/2t}}{\sqrt{2\pi t}} - \frac{e^{-(r\sin\varphi + q_2)^2/2t}}{\sqrt{2\pi t}}\Big]$$

Since the integrand in both cases is strictly positive, (4.7) holds. It follows that Θ is irreducible with respect to Lebesgue measure. □

Proposition 4.3. *The MAP $((\xi, \Theta), \mathbb{P})$ corresponding to free 2-dimensional Bessel process $X = ((X_t^{(1)}, X_t^{(2)}), t \geq 0)$ where each $X^{(i)}$ is a Bessel process of dimension 3 satisfies the assumptions of Theorem 3.3.*

Proof: According to Alili et al. (2017), (X, P_x) satisfies the skew-product property. Hence, we first show that $\mathbb{P}_{0,\theta}(\zeta = \infty) = 1$ for all $\theta \in S$ to deduce that $\lambda = 0$ in the expression (4.1) so that ξ and Θ are independent in the corresponding MAP $((\xi, \Theta), \mathbb{P}_{0,\theta})$. By Theorem 4.1 and the fact that (X, P_x) has infinite lifetime, (ξ, Θ) has infinite lifetime if the following holds

$$\int_0^\infty \frac{ds}{||X_s||^2} = \infty.$$

According to Revuz and Yor (1999, §11.1), we have the following law of the iterated logarithm

$$\limsup_{t\to\infty} \frac{||X_t||}{\sqrt{2t\ln\ln t}} = 1. \tag{4.8}$$

Hence, for each $\epsilon > 0$, $\exists t' > 0$ such that $\forall t > t'$, we have

$$||X_t||^2 < 2t\ln\ln t + \epsilon.$$

Choose $\epsilon < e$. Then,

$$\int_0^\infty \frac{ds}{||X_s||^2} \geq \int_e^\infty \frac{ds}{||X_s||^2}$$
$$\geq \int_e^\infty \frac{ds}{2s\ln\ln s + \epsilon}$$

$$\geq \int_e^{\infty} \frac{ds}{2s \ln \ln s + s}$$
$$= \int_1^{\infty} \frac{du}{2 \ln u + 1}$$
$$\geq \int_{(1,\infty)} \frac{du}{2u - 1} = \infty$$

where we use the change of variable $\ln s = u$, and the upper bound $\ln x \leq x - 1$.

By the proof of Lemma 4.2, the MAP (ξ, Θ) is a Feller process. Hence, ξ is quasi-left continuous, and Θ is a Hunt process (Blumenthal and Getoor, 1967). Moreover, Θ has infinite lifetime. We will prove that Θ is positive Harris recurrent to show the existence of an invariant probability measure π. Let $\mathcal{L}$ denote the infinitesimal generator of Θ. Actually, using the SDE representation (4.2), we can write the explicit form of $\mathcal{L}$ such that

$$\mathcal{L} = \sum_{i=1}^{2} b_i(x, y)\partial_i + \frac{1}{2} \sum_{i,j=1}^{2} a_{ij}(x, y)\partial_i \partial_j$$

where $a = \sigma\sigma^T$ and, b and σ are given in (4.3). It is straightforward to show that $a = \sigma$ in our case. According to Meyn and Tweedie (1993b, Thm.4.2), if there exist $c, d > 0$, a positive function $V : \mathcal{H}_0 \to \mathbb{R}^+$, and a petite set $C \subseteq \mathcal{H}_0$ with V bounded on C such that

$$\mathcal{L}V(x, y) \leq -c + d\mathbf{1}_C(x, y) \tag{4.9}$$

then, Θ is positive Harris recurrent. We have shown that Θ is a T-process in Lemma 4.2; hence, by Meyn and Tweedie (1993b, Thm.4.4), we can take C as any compact set. Let $V(x, y) = x^3 + y^3$. Then, we have

$$\mathcal{L}V(x, y) = \frac{1}{2}(x + y)(-21 + 54xy)$$

Note that $\mathcal{L}V(x, y)$ and $V(x, y)$ are both bounded on $\mathcal{H}_0$, so (4.9) is straightforward by taking $C = \mathcal{H}_0$. Hence, Θ has an invariant probability measure π, so Assumption 1 holds for (ξ, θ).

By the regularity properties of the Wiener processes, we can deduce that the process X is sphere-exterior and sphere-interior regular. In other words, if $\tau_r^+ := \inf\{t > 0 \,|\, ||X_t|| > r\}$ and $\tau_r^- := \inf\{t > 0 \,|\, ||X_t|| < r\}$, then, we have $P_x(\tau_1^+ = 0) = P_x(\tau_1^- = 0) = 1$ for each $x \in \mathcal{H}_0$. Hence, by Lamperti-Kiu transform, the MAP (ξ, Θ) is both upwards and downwards regular, so Assumption 2 is satisfied.

Note that since $X^{(1)}$ and $X^{(2)}$ are independent Bessel processes of dimension 3, $||X||$ is a Bessel process of dimension 6, and Assumption 3 follows from the following maximal inequality for Bessel process

$$\mathbb{E}[\sup_{t \in (0,\tau)} ||X_t||] \leq \gamma \sqrt{\mathbb{E}[X_\tau]}$$

where γ is a constant depending on the dimension of the Bessel process and τ is a stopping time (Dubins et al., 1993). Indeed, we use the definition $\xi_t = \log ||X_{A_t}||$ from Theorem 4.1 and the fact that A_t is a continuous time change to deduce Assumption 3. Finally, the MAP has the reversibility property by Alili et al. (2017, Ex.B), so Assumption 4 holds. □

4.2. *Radial Dunkl Processes.* We recall some definitions and properties about Dunkl processes from Chybiryakov (2008); Alili et al. (2017); Demni (2009). For $\alpha \in \mathbb{R}^n \backslash \{0\}$, let $H_\alpha := \{\beta \in \mathbb{R}^n | \alpha \cdot \beta = 0\}$ and σ_α denotes the reflection with respect to the hyperplane H_α. A finite set $R \subset \mathbb{R}^n \backslash \{0\}$ is called a *root system* if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ $\forall \alpha \in R$ and $\sigma_\alpha(R) = R$ $\forall \alpha \in R$. The subgroup $W \subset O(\mathbb{R}^n)$, which is generated by the reflections σ_α for $\alpha \in R$, is called the Weyl reflection group associated with R, where $O(\mathbb{R}^n)$ is the orthogonal group of $\mathbb{R}^n$. Let R^+ be a positive subsystem of R and k be a non-negative function on R such that $k \circ w = k$ $\forall w \in W$. Then, a Dunkl processes X^k is

a cádlág Markov process with generator $L_k := \frac{1}{2}\Delta_k$, where $\Delta_k = \sum_{i=1}^d T_i^2$ is the Dunkl Laplacian and $T_i f(x) := \partial_i f(x) + \sum_{\alpha\in R^+} k(\alpha)\alpha^i \frac{f(x)-f(\sigma_\alpha x)}{\alpha\cdot x}$.

Let $\mathcal{C}$ be a connected component of $\mathbb{R}^n \backslash \cup_{\alpha\in R} H_\alpha$, and $\mathbb{R}^n/W$ be the space of W-orbits in $\mathbb{R}^n$. Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}^n/W$ and $\pi_2 : \mathbb{R}^n/W \to \bar{\mathcal{C}}$ be the canonical projections. Then, $X^W := \pi_2 \circ \pi_1(X^k)$ is a Feller self-similar process with index 2 called as a *radial Dunkl process*. Let W_t be a n-dimensional Wiener process. Then, X^W is the unique strong solution of the following SDE

$$dX_t^W = dW_t + \sum_{\alpha\in R_+} \frac{k(\alpha)}{< \alpha, X_t^W >}\alpha\, dt \tag{4.10}$$

(Demni, 2009; Chybiryakov, 2006). Furthermore, if $k(\alpha) \geq 1/2$ for any $\alpha \in R$, then, $P_y(X_t \in \bar{\mathcal{C}}\backslash\mathcal{C}) = 0$ for all $y \in \bar{\mathcal{C}}$ and $t > 0$.

Throughout this section, we assume $k(\alpha) = k$ for some $k \geq 1/2$ for all $\alpha \in R_+$. We consider some important examples of radial Dunkl processes. For $n = 1$, we can choose $R := \{\pm 1\}$ as a root system. Let $R_+ = \{+1\}$ and $\mathcal{C} := (0, \infty)$. Then, the corresponding radial Dunkl process is the Bessel process with dimension k (Revuz and Yor, 1999). In this case, (4.10) takes the form

$$dX_t^W = dW_t + \frac{k}{X_t^W}dt$$

Let $n > 1$ and $(e_i)_{1\leq i\leq n}$ be the standard basis of $\mathbb{R}^n$. The root system $R := \{\pm(e_i - e_j) \,|\, i, j \in \{1, 2, \ldots, n\}, i \neq j\}$ is called A_{n-1} *type root system*. Take $R_+ = \{e_i - e_j \,|\, i < j\}$ and $\mathcal{C}^A := \{x \in \mathbb{R}^d \,|\, x_1 > x_2 > \ldots > x_n\}$. The resulting radial Dunkl process X^W is Dyson's Brownian motion with parameter k (Chybiryakov, 2008).

The following three different types of root systems are related to Wischart processes (Demni, 2010; Humphreys, 1990).

- The root system $R := \{\pm e_i \,|\, i \geq 1\} \cup \{\pm(e_i \pm e_j) \,|\, i < j\}$ is called B_n *type root system*. Take $R_+ = \{e_i \,|\, i \geq 1\} \cup \{(e_i \pm e_j) \,|\, i < j\}$ and $\mathcal{C}^B := \{x \in \mathbb{R}^d \,|\, x_1 > x_2 > \ldots > x_n > 0\}$.
- The root system $R := \{\pm 2e_i \,|\, i \geq 1\} \cup \{\pm(e_i \pm e_j) \,|\, i < j\}$ is called C_n *type root system*. Take $R_+ = \{2e_i \,|\, i \geq 1\} \cup \{(e_i \pm e_j) \,|\, i < j\}$ and $\mathcal{C}^C := \{x \in \mathbb{R}^d \,|\, x_1 > x_2 > \ldots > x_n > 0\}$.
- The root system $R := \{\pm(e_i \pm e_j) \,|\, i < j\}$ is called D_n *type root system*. Take $R_+ = \{(e_i \pm e_j) \,|\, i < j\}$ and $\mathcal{C}^D := \{x \in \mathbb{R}^d \,|\, x_1 > x_2 > \ldots > |x_n| > 0\}$.

The following proposition gives the characterization of the MAP $((\xi, \Theta), \mathbb{P})$ corresponding to a radial Dunkl process using the SDE representation (4.10).

Proposition 4.4. *Let (ξ, Θ) be the MAP corresponding to the radial Dunkl process X^W via the Lamperti-Kiu transform. Then, there exists an n dimensional Wiener process $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ such that ξ satisfies the SDE*

$$d\xi_t = \sum_{i=1}^n \Theta_t^{(i)} dW_t^{(i)} + \Big(\sum_{\alpha\in R_+} k(\alpha) + \frac{1}{2}(n-2)\Big)dt$$

and $\Theta = (\Theta^{(1)}, \ldots, \Theta^{(n)})$ satisfies the SDE system

$$d\Theta_t^{(i)} = dW_t^{(i)} - \Theta^{(i)} \sum_{j=1}^n \Theta_t^{(j)} dW_t^{(j)} + \Big(\sum_{\alpha\in R_+} \frac{k(\alpha)}{< \alpha, \Theta_t >}\alpha^i - \Theta^{(i)} \sum_{\alpha\in R_+} k(\alpha) - \frac{\Theta^{(i)}(n-1)}{2}\Big)dt$$

In particular, for $n = 2$, and $k(\alpha) = k$ for all $\alpha \in R_+$, we have

$$d\xi_t = \Theta_t^{(1)} dW_t^{(1)} + \Theta_t^{(2)} dW_t^{(2)} + k|R_+|dt$$

$$d\Theta_t^{(i)} = (1 - (\Theta_t^{(i)})^2) dW_t^{(1)} - \Theta_t^{(1)}\Theta_t^{(2)} dW_t^{(2)} + \Big(\sum_{\alpha=(\alpha^1,\alpha^2)\in R_+} \frac{k\alpha^i}{\alpha^1\Theta_t^{(1)} + \alpha^2\Theta_t^{(2)}} - \Theta_t^{(i)}(k|R_+| - \frac{1}{2})\Big)dt$$

where $i = 1, 2$ and $|R_+|$ is the number of elements in the set R_+.

Proof: Note that, by Lamperti-Kiu transform

$$\xi_t = \log ||X^W||_{A_t} \quad , \quad \Theta_t^{(i)} = \frac{X^i_{A_t}}{||X^W_{A_t}||} \tag{4.11}$$

where $X^W = (X^{(1)}, X^{(2)}, \ldots, X^{(n)})$ and $A_t = \inf\{s : \int_0^s \frac{du}{||X_u^W||^2} > t\}$. By (4.10), we have

$$dX_t^{(i)} = dW_t^{(i)} + \sum_{\alpha \in R_+} \frac{k(\alpha)}{\alpha^1 X_t^{(1)} + \ldots + \alpha^n X_t^{(n)}} \alpha^i \, dt \tag{4.12}$$

for $\alpha = (\alpha^1, \ldots, \alpha^n) \in R_+$ and $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ is an n-dimensional Wiener process.

Consider the time change A_t. The right inverse of A_t is $\bar{A}_t = \int_0^t \frac{1}{||X_s^W||^2 ds}$. Hence, by (Øksendal, 2003, Thm.8.5.7), we have $\sqrt{\frac{d}{dt}(\bar{A}_t)} dW_{A_t} = \frac{1}{||X_t^W||} dW_{A_t} = dW_t$. Then, for the function $f(x_1, \ldots, x_n) = \log \sqrt{x_1^2 + \ldots x_n^2}$, using Itô's lemma, we obtain

$$\begin{aligned}
d\xi_t &= ||X_t^W|| \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_t^{(1)}, \ldots, X_t^{(n)}) dW_t^{(i)} \\
&\quad + ||X_t^W||^2 [\sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_t^{(1)}, \ldots, X_t^{(n)}) \sum_{\alpha \in R_+} \frac{k(\alpha)}{< \alpha, X_t >} \alpha^i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f(X_t^{(1)}, \ldots, X_t^{(n)})] dt \\
&= \sum_{i=1}^n \frac{X_t^{(i)}}{||X_t^W||} dW_t^{(i)} + [\sum_{i=1}^n X_t^{(i)} \sum_{\alpha \in R_+} \frac{k(\alpha)}{< \alpha, X_t >} \alpha^i + \frac{1}{2} \sum_{i=1}^n \frac{||X_t^W||^2 - (X_t^{(i)})^2}{||X_t^W||^2}] dt \\
&= \sum_{i=1}^n \Theta_t^{(i)} dW_t^{(i)} + [\sum_{\alpha \in R_+} \frac{k(\alpha)}{< \alpha, X_t >} \sum_{i=1}^n X_t^{(i)} \alpha^i + \frac{1}{2}(n-2)] dt \\
&= \sum_{i=1}^n \Theta_t^{(i)} dW_t^{(i)} + [\sum_{\alpha \in R_+} k(\alpha) + \frac{1}{2}(n-2)] dt
\end{aligned}$$

For a fixed $i \in \{1, \ldots, n\}$,and $g^{(i)}(x_1, \ldots, x_n) = \frac{x_i}{\sqrt{x_1^2 + \ldots, x_n^2}}$, we obtain

$$\begin{aligned}
d\Theta_t^{(i)} &= ||X_t^W|| \sum_{i=1}^n \frac{\partial}{\partial x_i} g^{(i)}(X_t^{(1)}, \ldots, X_t^{(n)}) dW_t^{(i)} \\
&\quad + ||X_t^W||^2 [\sum_{i=1}^n \frac{\partial}{\partial x_i} g^{(i)}(X_t^{(1)}, \ldots, X_t^{(n)}) \sum_{\alpha \in R_+} \frac{k(\alpha)}{< \alpha, X_t >} \alpha^i + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} g^{(i)}(X_t^{(1)}, \ldots, X_t^{(n)})] dt \\
&= \frac{(||X_t^W||^2 - (X_t^{(i)})^2}{||X_t^W||^2} dW_t^{(i)} + \frac{-X_t^{(i)}}{||X_t^W||} \sum_{j \neq i} \frac{X^{(j)}}{||X_t^W||} dW_t^{(j)} \\
&\quad + [\frac{||X_t^W||^2 - (X_t^{(i)})^2}{||X_t^W||} \sum_{\alpha \in R_+} \frac{k(\alpha)}{< \alpha, X_t >} \alpha^i + \frac{-X_t^{(i)}}{||X_t^W||} \sum_{j \neq i} X_t^{(j)} \sum_{\alpha \in R_+} \frac{k(\alpha)}{< \alpha, X_t >} \alpha^j \\
&\quad + \frac{1}{2}(\frac{-3X_t^{(i)}[||X_t^W||^2 - (X_t^{(i)})^2]}{||X_t^W||^3} + \frac{-X_t^{(i)}}{||X_t^W||} \sum_{j \neq i} \frac{||X_t^W||^2 - 3(X_t^{(j)})^2}{||X_t^W||^2})] dt \\
&= (1 - (\Theta_t^{(i)})^2) dW_t^{(i)} + -\Theta_t^{(i)} \sum_{j \neq i} \Theta^{(j)} dW_t^{(j)}
\end{aligned}$$

$$+ [||X_t^W|| \sum_{\alpha\in R_+} \frac{k(\alpha)}{<\alpha, X_t>}\alpha^i - \frac{(X_t^{(i)})^2}{||X_t^W||}\sum_{i=1}^n X_t^{(i)} \sum_{\alpha\in R_+}\frac{k(\alpha)}{<\alpha, X_t>}\alpha^i$$
$$+ \frac{-X_t^{(i)}}{2||X_t^W||}(3 - 3\frac{(X_t^{(i)})^2}{||X_t^W||^2} + \sum_{j\neq i}(1 - 3\frac{(X_t^{(j)})^2}{||X_t^W||^2}))]dt$$
$$= dW_t^{(i)} - \Theta^{(i)}\sum_{j=1}^n \Theta_t^{(j)} dW_t^{(j)} + [\sum_{\alpha\in R_+}\frac{k(\alpha)}{<\alpha, \Theta_t>}\alpha^i - \Theta^{(i)}\sum_{\alpha\in R_+} k(\alpha) - \Theta^{(i)}\frac{(n-1)}{2}]dt. \quad \square$$

We consider the case $d = 2$ for simplicity. Let (ξ, Θ) be the corresponding MAP of the radial Dunkl process $X^W = (X^{(1)}, X^{(2)})^W$ in $\mathbb{R}^2\backslash\{0\}$. We will show that (ξ, Θ) satisfies the assumptions of Theorem 3.3 for the cases in which the root system R is of type A_1, B_2, C_2, or D_2.

Lemma 4.5. *The diffusion process $(\Theta, \mathbb{P}_\theta)$ is a T-process in the sense of Meyn and Tweedie (1993a).*

Proof: The process X^W is Feller by Chybiryakov (2008). Hence, according to the proof of Lemma 4.2, it is sufficient to show that

$$\int_0^\infty P_q(X_t \in C)dt > 0 \tag{4.13}$$

where q and C are defined as in (4.6) and (4.7) for appropriate a, b in each case of the root system R. Using the explicit form of the transition density function of X_t^W from Chybiryakov (2008, Eq.6), the change of variables $R_t = ||X_t^W||$, $\Phi_t = \arctan(X^{(2)}, X^{(1)})$, and the fact that $P_y(X_t \in \bar{\mathcal{C}}\backslash\mathcal{C}) = 0$, we get

$$\int_0^\infty P_q(X_t \in C)dt = \int_0^\infty dt \int_I d\varphi \int_{(0,\infty)} dr\, \frac{r}{c_k t^{\gamma+1}} \exp(-\frac{c^2 + r^2}{2t})$$
$$D_k^W(\frac{q}{\sqrt{t}}, \frac{(r\cos\varphi, r\sin\varphi)}{\sqrt{t}})\omega_k((r\cos\varphi, r\sin\varphi))$$

where $I = (\arccos a, \arccos b)$, $D_k(u, v) > 0$ is the Dunkl kernel, $\gamma = \sum_{\alpha\in R_+} k > 0$, $\omega_k(y) = \prod_{\alpha\in R_+} |\alpha \cdot y|^{2k}$ and $c_k = \int_{\mathbb{R}^2} e^{-|x|^2/2}\omega_k(x)dx$. The fact that $P_y(X_t \in \bar{\mathcal{C}}\backslash\mathcal{C}) = 0$ implies that $P_q(X_t \in \cup_{\alpha\in R_+} H_\alpha) = 0$, so $\omega_k(y) > 0$ and $c_k > 0$ a.s. Hence, the integrand is strictly positive, and (4.13) holds, which in turn implies that Θ is irreducible. $\square$

Proposition 4.6. *The MAPs $((\xi, \Theta), \mathbb{P})$ corresponding to radial Dunkl processes (X, P_x) in $\mathbb{R}^2\backslash\{0\}$ of types A_1, B_2, C_2, or D_2 satisfy the assumptions of Theorem 3.3.*

Proof: According to Alili et al. (2017), (X, P_x) satisfies the skew-product property. Hence, we first show that $\mathbb{P}_{0,\theta}(\zeta = \infty) = 1$ for all $\theta \in S$ to deduce that ξ and Θ are independent as in the proof of Proposition 4.3. Note that the property $P_y(X_t \in \bar{\mathcal{C}}\backslash\mathcal{C}) = 0$ for all $y \in \bar{\mathcal{C}}$ implies that the lifetime of (X, P_x) is infinite a.s. According to this fact and Theorem 4.1, the MAP (ξ, Θ) has an infinite lifetime if

$$\int_0^\infty \frac{ds}{||X_s||^2} = \infty. \tag{4.14}$$

According to Alili et al. (2017, Ex.C), $||X_s||$ is a Bessel process of dimension $2 + 2\gamma$ where $\gamma > 0$ is defined in the proof of Lemma 4.5. Hence, by the law of the iterated logarithm (4.8), Equation (4.14) holds.

We have shown that the MAP (ξ, Θ) is a Feller process in Lemma 4.5. Hence, the ordinate ξ is quasi-left continuous, and the modulator Θ is a Hunt process. Moreover, Θ has infinite lifetime. We will prove that Θ is positive Harris recurrent to show the existence of an invariant probability

measure π. Let $\mathcal{L}$ denote the infinitesimal generator of Θ. Using the particular case $n = 2$ of the SDE representation in Proposition 4.4, we have $\mathcal{L} = \sum_{i=1}^{2} b_i(x,y)\partial_i + \frac{1}{2}\sum_{i=1}^{2} a_{ij}\partial_i\partial_j$ where

$$b(x,y) = \begin{bmatrix} \mathcal{A}_1 - x(k|R_+| - 1/2) \\ \mathcal{A}_2 - y(k|R_+| - 1/2) \end{bmatrix} \quad \sigma(x,y) = \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}$$

for $\mathcal{A}_i := \sum_{\alpha \in R^+} \frac{k\alpha^i}{\alpha^1 x + \alpha^2 y}$ with $i = 1, 2$, and $a := \sigma\sigma^T = \sigma$. Denote the corresponding state space of Θ by $\mathcal{C}_0$ for given $\mathcal{C}$. According to Meyn and Tweedie (1993b, Thm.4.2), if there exist $c, d > 0$, a positive function $V : \mathcal{C}_0 \to \mathbb{R}^+$, and a petite set $C \subset \mathcal{C}_0$ with V bounded on C such that

$$\mathcal{L}V(x,y) \leq -c + d\mathbf{1}_C(x,y) \tag{4.15}$$

then, Θ is positive Harris recurrent. By Lemma 4.5, Θ is a T-process; hence, by Meyn and Tweedie (1993b, Thm.4.4), we can take C as any compact set. In each case, we take C as the state space of the modulator and find appropriate $V(x,y)$ for each mentioned type of finite root system such that both $\mathcal{L}V(x,y)$ and $V(x,y)$ are bounded on C. Then, (4.15) will follow immediately.

Type A_1: In this case, $R_+ = \{(1,-1)\}$. We have $\mathcal{A}_1 = \frac{k}{x-y}$ and $\mathcal{A}_2 = \frac{-k}{x-y}$. Let $V(x,y) = x^3 + y^3$. Then, we obtain

$$\mathcal{L}V(x,y) = \frac{3}{2}(x+y)[(4k+2)xy - 1] + (x+y)(12xy - 3)$$

$\mathcal{L}V(x,y)$ and $V(x,y)$ are both finite on $\mathcal{C}_0^A$.

Type B_2: In this case, $R_+ = \{(1,-1),(1,1),(1,0),(0,1)\}$. Then, $\mathcal{A}_1 = \frac{k}{x-y} + \frac{k}{x+y} + \frac{k}{x}$ and $\mathcal{A}_2 = \frac{-k}{x-y} + \frac{k}{x+y} + \frac{k}{y}$. Take $V(x,y) = x^3 + y^3$. Then, we get

$$\mathcal{L}V(x,y) = \frac{6k}{x+y}(8x^2y^2 - 1) + \frac{-3}{2}(x+y)(x-y)^2 + (x+y)(12xy - 3)$$

In this case, $\mathcal{C}_0^B = \{(\cos\alpha, \sin\alpha) : \alpha \in (\pi/4, \pi/2)\}$; hence, the term $x + y$ is always greater than 1. Therefore, $\mathcal{L}V(x,y)$ and $V(x,y)$ are both finite on $\mathcal{C}_0^B$.

Type C_2: In this case, $R_+ = \{(1,-1),(1,1),(2,0),(0,2)\}$. Then, we have $\mathcal{A}_1 = \frac{k}{x-y} + \frac{k}{x+y} + \frac{2k}{x}$ and $\mathcal{A}_2 = \frac{-k}{x-y} + \frac{k}{x+y} + \frac{2k}{y}$. Take $V(x,y) = x^3 + y^3$. Then, we have

$$\mathcal{L}V(x,y) = \frac{12k}{x+y}(6x^2y^2 - 1) - \frac{3}{2}(x-y)^2(x+y) + (x+y)(12xy - 3)$$

In this case, we have $\mathcal{C}_0^C = \mathcal{C}_0^B$; hence, the term $x + y$ is greater than 1. Therefore, $\mathcal{L}V(x,y)$ and $V(x,y)$ are both finite on $\mathcal{C}_0^C$.

Type D_2: In this case, $R_+ = \{(1,-1),(1,1)\}$. We have $\mathcal{A}_1 = \frac{k}{x-y} + \frac{k}{x+y}$ and $\mathcal{A}_2 = \frac{-k}{x-y}$. Take $V(x,y) = (x^3 + y^3)^2$. In this case, we have

$$\mathcal{L}V(x,y) = 48kx^2y^2(1-xy) + 3(4x^2y^2 - 1)(1-xy) + 6(x+y)^2(1-xy)(4xy-1) + (x+y)(12xy-3)$$

It can be verified that $\mathcal{L}V(x,y)$ and $V(x,y)$ are both finite on $\mathcal{C}_0^D$.

Hence, Assumption 1 holds for MAPs (ξ, Θ) corresponding to respective root systems. $||X_s||$ is a Bessel process of dimension $2 + \gamma \geq 3$, and ξ and θ are independent. Then, Assumption 2 follows from the regularity properties of Bessel processes from Rogers and Williams (2000, §5.48). Assumption 3 follows exactly as in the proof of Proposition 4.3. Finally, the MAP has reversibility property by Alili et al. (2017, Ex.C), so Assumption 4 holds. □

Remark 4.7. We have established that Θ is positive Harris recurrent for both examples in our pursuit of verifying the assumptions of Theorem 3.3. Therefore, the angular part $X_t/||X_t||$ of each self-similar Markov process is also positive Harris recurrent since the time change τ_t in Theorem 4.1 is continuous and Assumption 3 holds.

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