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Stochastic Kimura Equation

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Abstract. In this work we study the one-dimensional stochastic Kimura equation $\partial_t u(z,t) = z\partial_z^2 u(z,t) + u(z,t)\dot{W}(z,t)$ for $z>0$ and $t \geq 0$, equipped with constant initial data and the Dirichlet boundary condition at 0, with $\dot{W}$ being a Gaussian space-time noise. This equation can be seen as a degenerate analog of the parabolic Anderson model. We combine the Wiener chaos theory from the Malliavin calculus, the Duhamel perturbation technique from PDEs, and the kernel analysis of (deterministic) degenerate diffusion equations to develop a solution theory for the stochastic Kimura equation. We establish results on existence, uniqueness, moments, and continuity for the solution $u(z,t)$. In particular, we investigate how the stochastic potential and the degeneracy in the diffusion operator jointly affect the properties of $u(z,t)$ near the boundary. We also derive explicit estimates on the comparison under the L^2-norm between $u(z,t)$ and its deterministic counterpart for (z,t) from a proper range.

1. Introduction

In this work we study a type of one-dimensional stochastic degenerate diffusion equation, which we call *stochastic Kimura equations*, that are one-dimensional diffusion equations on the non-negative half real line with diffusion coefficients degenerating linearly at the boundary 0 and subject to Gaussian noise potentials. This work is a marriage between the study of degenerate diffusion, which is a classical topic with a long-celebrated history, and the techniques from stochastic partial differential equations (SPDEs), which is a fast developing field with many recent breakthroughs. By combining analytic methods and stochastic analysis, we derive both qualitative (e.g., existence and uniqueness) and quantitative (e.g., moments and regularity) results on solutions to stochastic Kimura equations. We put a particular emphasis on the short-term behavior of the solution near the boundary, investigating the joint effect of the degenerate diffusion operator and the space-time noise potential. To the best of our knowledge, these aspects of stochastic degenerate diffusion equations have not been previously studied. While in this work, for the purpose of demonstration, we only

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treat models that combine typical degenerate diffusion equations with typical Gaussian noise, we expect similar techniques and part of the results to be applicable to more general degenerate diffusion equations under reasonable stochastic noise.

1.1. *Kimura Diffusion and Parabolic Anderson Model.* In this subsection we give a coarse review of the two fields, degenerate diffusion equations and SPDEs, at the intersection of which our work lies. Given that both fields have rich and vast literature, our reviews will be limited as we will only mention the results that are most relevant to the problem concerned in this work. In particular, the two pillars based on which our work is carried out are the *Kimura diffusion*, a classical example of one-dimensional degenerate diffusion, and the *parabolic Anderson model*, a well studied SPDE model. We hereby give a brief introduction to both models, but leave the technical details for the next section.

1.1.1. Kimura Diffusion. The Kimura diffusion is an important mathematical tool in population genetics as a model of the propagation of a certain genotype among a population (Crow and Kimura, 1970). In the classical setting, it is described by the stochastic differential equation (SDE) $Z_t = \sqrt{2Z_t}dB_t + b\left(Z_t\right)dt$ with $Z_0 = z > 0$, b being a drift coefficient and 0 being an absorbing boundary. Correspondingly, the *Kimura equation* is the partial differential equation (PDE)

$$\partial_t u\left(z,t\right) = z\partial_z^2 u\left(z,t\right) + b\left(z\right)\partial_z u\left(z,t\right) \tag{1.1}$$

for $z > 0, t \geq 0$, equipped with initial data u_{int} and the Dirichlet boundary condition at 0. Characterized by the degeneracy in diffusion, this pair of equations has been of great interest to analysts and probabilists. Properties of the solutions (e.g., existence, uniqueness, regularity, estimation, etc.) have been extensively studied for the Kimura diffusion, as well as for its certain generalizations, in various contexts (e.g., Ethier (1976); Fleming and Viot (1979); Cerrai and Clément (2004); Epstein and Mazzeo (2013); Athreya et al. (2002)). Both analytic and probabilistic tool-kits have been developed to treat degenerate diffusion operators, replacing classical techniques that are restricted to strict elliptic/parabolic operators.

The specific Kimura diffusion that will be examined later under a stochastic potential is the model with $b \equiv 0$, but the methods developed in this study are extendable to cases with general b. Let $q_0\left(z,w,t\right)$ denote the fundamental solution to (1.1) with $b \equiv 0$, explicitly determined in terms of the Bessel function (Epstein and Mazzeo, 2010; Chen and Stroock, 2010; Chen and Weih-Wadman, 2020):

$$q_0\left(z,w,t\right) := \sqrt{\frac{z}{w}}\frac{e^{-\frac{z+w}{t}}}{t}I_1\left(\frac{2\sqrt{zw}}{t}\right) = \frac{ze^{-\frac{z+w}{t}}}{t^2}\sum_{n=0}^{\infty}\frac{z^n w^n}{t^{2n}n!(n+1)!} \tag{1.2}$$

for every $z,w,t > 0$, where I_1 is the modified Bessel function of the first kind and admits a series representation (see (6.3) in the Appendix). That is, if the Kimura equation is equipped with the initial data $u_{\text{int}}\left(z\right)$, then the solution is given by

$$u_0\left(z,t\right) = \int_0^\infty u_{\text{int}}\left(w\right)q_0\left(z,w,t\right)\mathrm{d}w \tag{1.3}$$

for every $z,t > 0$. We will refer to $u_0\left(z,t\right)$ as the *deterministic solution*, which, as we will see later, serves as the starting point in the construction of the solution to the stochastic Kimura equation.

In latter sections a perturbation of the Kimura equation will also be considered:

$$\partial_t u\left(z,t\right) = z\partial_z^2 u\left(z,t\right) + u\left(z,t\right)V\left(z,t\right)$$

where V is a potential function, whose fundamental solution will be denoted by $q_0^V\left(z,w,t\right)$. When V is bounded and time-independent, a sharp regularity theory on $q_0^V\left(z,w,t\right)$ is developed in Chen and Stroock (2010) via the Duhamel perturbation method, which produces accurate estimates on the comparison between $q_0^V(z,w,t)$ and $q_0(z,w,t)$ near the boundary for small time. Details of these results will be presented in §2.1.

1.1.2. Parabolic Anderson Model. SPDEs have become a novel and popular tool in modeling phenomena observed in scientific experiments in random environments. Following the seminal papers Dalang (1999); Hu et al. (2015), many classical deterministic PDEs have been re-visited when the domain is subject to certain random noise, and a fast-growing set of theories is being developed for solutions to such SPDEs (e.g., Motyl (2013); Balan and Song (2017); Hausenblas and Razafimandimby (2020); Hairer and Rosati (2024); Berger et al. (2023)). Most notably, the parabolic Anderson model (PAM) is the classical heat equation perturbed by a Gaussian random potential. The one-dimensional model can be written as

$$\partial_t u(z,t) = \frac{1}{2}\partial_z^2 u(z,t) + u(z,t)\dot{W}(z,t) \text{ for } z\in\mathbb{R} \text{ and } t>0, \tag{1.4}$$

where $\dot{W}(z,t)$ denotes a Gaussian space-time noise on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Formally speaking, $\{\dot{W}(z,t) \mid z\in\mathbb{R}, t>0\}$ can be viewed as a centered Gaussian field with covariance

$$\mathbb{E}\left[\dot{W}(z_1,t_1)\dot{W}(z_2,t_2)\right] = f(z_1-z_2)\gamma(t_1-t_2) \tag{1.5}$$

for $z_1,z_2\in\mathbb{R}$ and $t_1,t_2>0$, where $f:\mathbb{R}\to\mathbb{R}_+$ and $\gamma:\mathbb{R}\to\mathbb{R}_+$ are two non-negative definite, symmetric, locally integrable kernels. To solve (1.4) with initial (deterministic) data $u_{\text{int}}(z)$, we seek a jointly measurable random field $u=\{u(z,t)\mid z\in\mathbb{R}, t\geq 0\}\subseteq L^2(\Omega)$, to which we refer as a *(mild) solution* to (1.4), such that for every $z\in\mathbb{R}$ and $t>0$,

$$u(z,t) = \int_{\mathbb{R}} g_t(z,w)\,u_{\text{int}}(w)\,\mathrm{d}w + \int_0^t\int_{\mathbb{R}} g_{t-\tau}(z,w)\,u(w,\tau)\,W(\mathrm{d}w,\mathrm{d}\tau) \text{ a.s.,}$$

where $g_t(z,w) = e^{-|z-w|^2/2}/\sqrt{2\pi t}$ is the standard heat kernel, and the stochastic integral with respect to "$W(\mathrm{d}w,\mathrm{d}\tau)$" is defined as a generalized Itô integral, following the theory of Malliavin calculus. In §2.2, we review all the necessary elements from the Malliavin calculus, including the construction and the properties of such stochastic integrals.

Common approaches toward the study of PAM combine the Malliavin calculus with the classical heat kernel analysis, yielding abundant results on the solution (e.g., Carmona and Molchanov (1995); Dalang (1999); Balan and Tudor (2008); Hu et al. (2015); Balan and Chen (2018); Balan et al. (2019)). One alternative approach is the theory of rough paths developed in Hairer (2014); Hairer and Labbé (2015). The PAM has also been studied in other settings, e.g., on $\mathbb{Z}^d$ (Gärtner and Molchanov, 1998; Gärtner and König, 2005), on the hypercube (Avena et al., 2020), and on Heisenberg groups (Baudoin et al., 2023), as well as subject to Lévy random noise (Berger et al., 2023). For nonlinear degenerate parabolic SPDEs, the wellposedness of the Cauchy problem has been studied in the semilinear and quasilinear cases (Hofmanová, 2013; Debussche et al., 2016).

1.2. *Stochastic Kimura Equation and Main Results.* Inspired by the works reviewed in the previous subsection, we decide to bring the two models, the Kimura diffusion and the PAM, together and investigate the following stochastic Kimura equation:

$$\partial_t u(z,t) = z\partial_z^2 u(z,t) + u(z,t)\dot{W}(z,t) \text{ for } z,t>0, \tag{1.6}$$

where $\dot{W}$ is the same as in (1.5). Similarly as in the PAM case, a mild solution to (1.6) is a random field $u=\{u(z,t)\mid z>0, t\geq 0\}\subseteq L^2(\Omega)$ such that u has a jointly measurable modification, and for every $z>0$ and $t\geq 0$,

$$u(z,t) = u_0(z,t) + \int_0^t\int_0^\infty q_0(z,w,t-\tau)\,u(w,\tau)\,W(\mathrm{d}w,\mathrm{d}\tau) \text{ a.s.,} \tag{1.7}$$

where $u_0(z,t)$ is the deterministic solution as defined in (1.3).

The purpose of studying (1.6) has multiple folds. First, we aim to develop a solution theory for stochastic degenerate diffusion equations, which would yield applications in modeling diffusion processes in random mediums with absorbing boundaries, e.g., genotype propagation with random hibernation/revival. The model of the stochastic Kimura equation is a good starting point since the deterministic equation is well studied with abundant results on the fundamental solution $q_0(z,w,t)$. Second, we want to extend the existing literature on the PAM to the settings of degenerate diffusions, and investigate how the current methods in constructing the solution to the PAM, as well as the properties of the solution, may change in the presence of degeneracy in the diffusion. Furthermore, we are interested in studying the interaction between the degenerate diffusion operator and the stochastic potential, in order to analyze their "joint effect" on the solution; to this end, we will pay particular attention to the behavior of the solution near the boundary 0, and compare the solution to the deterministic solution in a proper sense.

In §3.1 we prove that (1.6) admits a mild solution $\{u(z,t) \mid z>0, t\geq 0\}$ that satisfies (1.7) and this solution is unique up to modification (Theorem 3.4 and Theorem 3.8). We adopt a strategy that is similar to the one invoked in Hu et al. (2015); Balan et al. (2019), which is to combine the Wiener chaos expansion from the Malliavin calculus and the Duhamel method from PDEs. However, the techniques we apply are substantially different from those utilized in the PAM case, because (1) although $q_0(z,w,t)$ is explicitly determined, unlike $g_t(z,w)$, $q_0(z,w,t)$ does not have a closed-form expression, nor certain desirable features such as symmetry, and (2) since our model is restricted on the non-negative half real line, convenient technical elements based on Fourier analysis are not applicable in our case. For each aspect of the problem that we investigate, we establish our results in two steps: first we solve (1.6) when $\dot{W}$ is the space-time white noise, i.e., when both f and γ in (1.5) are Dirac delta δ_0, and then extend the construction of the solution to the colored noise setting with f and γ satisfying proper conditions.

Taking a close look at the behaviors of $u(z,t)$, we observe that the impact of the stochastic potential is most significant near the boundary, which makes the behavior of $u(z,t)$ for z near the boundary fundamentally different from that of $u_0(z,t)$, the solution to the deterministic Kimura equation. To explore the role of the stochastic potential near the boundary, in §4 we also introduce a degenerate coefficient to $\dot{W}$ and consider the variation of the stochastic Kimura equation with a degenerate noise. That is, for some $\beta>0$, we study

$$\partial_t u(z,t) = z\partial_z^2 u(z,t) + u(z,t)\,\hat{z}^{\beta}\dot{W}(z,t), \tag{1.8}$$

where $\hat{z} := \min\{z,1\}$. Similarly, (1.8) admits a unique mild solution $\{u(z,t) \mid z>0, t\geq 0\}$. It turns out that so long as $\beta>0$, the degeneracy in the stochastic potential will "tame" $u(z,t)$ enough so that it remains comparable to $u_0(z,t)$ under the L^2-norm for some finite time t and for z all the way to the boundary.

To be specific, we first prove in Theorem 3.10 that at every fixed $z>0$,

$$\lim_{t\searrow 0}\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}-1\right)^2\right]=0,$$

which applies to both (1.6) and (1.8). We further establish in Theorem 4.4 and Theorem 4.6 that, in the setting of (1.8), there exists $T\in(0,1)$ such that

$$\sup_{z>0,\, t\in[0,T]}\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right]<\infty.$$

These results indicate that $u(z,t)$ stays comparable to $u_0(z,t)$, in the L^2-sense, for t sufficiently small and z arbitrarily close to the boundary. Since $u_0(z,t)$ is deterministic and explicit, we obtain the accurate short-term near-boundary asymptotics of the L^2-norm of $u(z,t)$. Similar properties have been established in Chen and Stroock (2010); Chen and Weih-Wadman (2020) for the deterministic equation but with a bounded and time-independent potential.

In §4.2 and §4.3 we examine the higher moments and the regularity of $u(z,t)$, particularly its continuity in z at the boundary. The methods adopted in this part also requires the presence of the degeneracy in the stochastic potential. Again, in the setting of (1.8) with any $\beta > 0$, we obtain results on L^p-integrability and moment bounds for $u(z,t)$ for all $p \geq 2$, $z > 0$ and $t \geq 0$ (Theorem 4.8 and Theorem 4.9), which enables us to prove a Hölder continuity result on $u(z,t)$ in z all the way to the boundary (Theorem 4.14 and Theorem 4.16).

The aforementioned results are part of our study of the stochastic Kimura equation, as the first step in an attempt to advance our knowledge on general stochastic degenerate diffusion equations. There are other aspects of the work that are still ongoing and will not be addressed here. For example, our methods can be extended to study the following two variations of (1.6) without substantial difficulty: (1) with an order α of degeneracy in the diffusion operator for α from some proper range, i.e.,

$$\partial_t u(z,t) = z^\alpha \partial_z^2 u(z,t) + u(z,t)\dot{W}(z,t),$$

and (2) with a drift coefficient b satisfying some proper conditions, i.e.,

$$\partial_t u(z,t) = z\partial_z^2 u(z,t) + b(z)\partial_z u(z,t) + u(z,t)\dot{W}(z,t).$$

In addition, the framework developed in this paper also allows us to study long-term asymptotics of the solution and to investigate intermittency in the stochastic degenerate diffusion equation setting.

1.3. *Contents and Notations.* In §2 we review some technical results from the study of the deterministic Kimura equation and the necessary elements from the Malliavin calculus. In §3 we set up the model of the stochastic Kimura equation, establish existence and uniqueness of the solution, and discuss the ratio estimate away from the boundary $z = 0$. In §4, we proceed to the model with degenerate noise and prove ratio bounds, moment estimates, and continuity of the solution. §5 outlines some related problems and future directions. Useful formulas and estimates utilized in direct computations are deposited in §6.

We write $\mathbb{R}_+$ to mean the interval $(0,\infty)$. For $a, b \in \mathbb{R}$, we write $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$. For $z > 0$, we denote $\hat{z} := z \wedge 1$. We denote by $\mathcal{D}\left(\mathbb{R}_+^2\right)$ the set of all smooth, compactly supported functions defined on $\mathbb{R}_+^2$. We denote by $B(\mathbb{R}_+)$ the space of all bounded measurable functions defined on $\mathbb{R}_+$, equipped with the uniform norm

$$\|f\|_\infty := \sup\{|f(x)| : x > 0\}.$$

For $\alpha \in \mathbb{C}$, we denote by J_α and I_α the Bessel function and the modified Bessel function of the first kind of order α, respectively. For $q \geq p \geq 0$, $a_1, \cdots, a_p$, $b_1, \cdots, b_q \in \mathbb{R}$, and $x \in \mathbb{C}$, we denote by

$${}_pF_q\left[\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| \, x\right]$$

the generalized hypergeometric function. The definitions of these special functions are reviewed in §6.

2. Background

2.1. *The Kimura Equation.* In this section, we discuss the *deterministic* Kimura equation. We will rely on the results on the fundamental solution to the deterministic Kimura equation to investigate the stochastic analog (1.6). Recall that we are interested in the Kimura equation (1.1) with $b \equiv 0$. In this section, following Chen and Weih-Wadman (2020), we present an introduction to the slightly more general case of a constant drift $b \equiv \nu < 1$. That is, we explore the equation

$$\partial_t u(z,t) = z\partial_z^2 u(z,t) + \nu\partial_z u(z,t), \tag{2.1}$$

for $z, t > 0$ equipped with an absorbing boundary condition at 0, which has the fundamental solution

$$q_\nu(z,w,t) := \frac{z^{\frac{1-\nu}{2}} w^{\frac{\nu-1}{2}}}{t} e^{-\frac{z+w}{t}} I_{1-\nu}\left(\frac{2\sqrt{zw}}{t}\right) = \frac{z^{1-\nu} e^{-\frac{z+w}{t}}}{t^{2-\nu}} \sum_{n=0}^{\infty} \frac{z^n w^n}{t^{2n} n! \Gamma(n+2-\nu)}. \tag{2.2}$$

In particular, for $\nu = 0$, we recover (1.2). Furthermore, when $\nu = 0$, the following technical results on $q_0(z,w,t)$ will be useful later.

Lemma 2.1. *For every $z, w, t > 0$, we have that*

$$q_0(z,w,t) \le \frac{z}{t^2} e^{-\frac{(\sqrt{z}-\sqrt{w})^2}{t}}. \tag{2.3}$$

Further, when $zw \ge t^2$, it also holds that

$$q_0(z,w,t) \le C \frac{z^{\frac{1}{4}} w^{-\frac{3}{4}}}{\sqrt{t}} e^{-\frac{(\sqrt{z}-\sqrt{w})^2}{t}} \tag{2.4}$$

for some universal constant $C > 0$, which should be viewed as a refinement of (2.3).

Proof: We first consider the series representation of $q_0(z,w,t)$ in (1.2) and bound the series as

$$\sum_{n=0}^{\infty} \frac{z^n w^n}{t^{2n} n!(n+1)!} \le \sum_{n=0}^{\infty} \frac{2^{2n}(\sqrt{zw}/t)^{2n}}{(2n)!} \le e^{2\frac{\sqrt{zw}}{t}},$$

from which the first upper bound (2.3) follows immediately.

Now let us assume that $zw \ge t^2$. By the integral representation of I_1 in (6.4), $q_0(z,w,t)$ can be rewritten as

$$q_0(z,w,t) = \frac{2ze^{-\frac{z+w}{t}}}{\pi t^2} \int_0^\pi e^{2\frac{\sqrt{zw}}{t}\cos\theta} \sin^2\theta \mathrm{d}\theta = \frac{2ze^{-\frac{(\sqrt{z}-\sqrt{w})^2}{t}}}{\pi t^2} \int_0^\pi e^{-2\frac{\sqrt{zw}}{t}(1-\cos\theta)} \sin^2\theta \mathrm{d}\theta.$$

We note that $\sin^2\theta \le \theta^2 \wedge 1$ and $1-\cos\theta \ge \frac{1}{2}\mathbb{1}_{[\frac{\pi}{3},\pi]}(\theta) + \frac{1}{4}\theta^2 \mathbb{1}_{[0,\frac{\pi}{3})}(\theta)$, and hence the integral above can be bounded as

$$\int_0^\pi e^{-2\frac{\sqrt{zw}}{t}(1-\cos\theta)} \sin^2\theta \mathrm{d}\theta \le \int_0^{\pi/3} e^{-\frac{\sqrt{zw}}{2t}\theta^2} \theta^2 \mathrm{d}\theta + \frac{2\pi}{3} e^{-\frac{\sqrt{zw}}{t}} \le C\left(\frac{\sqrt{zw}}{t}\right)^{-\frac{3}{2}}$$

for some universal constant $C > 0$, which yields (2.4). $\square$

Back to the general case with $\nu < 1$, a comparison is established in Chen and Weih-Wadman (2020) between $q_\nu(z,w,t)$ and the solution to the following variation of (2.1) subject to a (deterministic) potential $V \in B(\mathbb{R}_+)$:

$$\partial_t u^V(z,t) = z\partial_z^2 u^V(z,t) + \nu \partial_z u^V(z,t) + u^V(z,t) V(z).$$

Namely, if $q_\nu^V(z,w,t)$ denotes the fundamental solution to the equation above, then

$$\sup_{z,w>0} \left| \frac{q_\nu^V(z,w,t)}{q_\nu(z,w,t)} - 1 \right| \le e^{t\|V\|_\infty} - 1, \tag{2.5}$$

which allows one to establish a sharp regularity theory on $q_\nu^V(z,w,t)$ for small times t and z, w arbitrarily close to the boundary.

The main strategy in obtaining $q_\nu^V(z,w,t)$ is to follow the Duhamel perturbation method and express $q_\nu^V(z,w,t)$ as a series of functions defined in terms of $q_\nu(z,w,t)$. In §3.1, we apply a similar method to (1.6); therefore, as a primer we give a brief review of the Duhamel perturbation method arguments. We observe that for $z, w, t > 0$, $q_\nu^V(z,w,t)$ satisfies the integral equation

$$q_\nu^V(z,w,t) = q_\nu(z,w,t) + \int_0^t \int_0^\infty q_\nu(z,x,t-\tau) q_\nu^V(x,w,\tau) V(x) \mathrm{d}x \mathrm{d}\tau. \tag{2.6}$$

Since $q_\nu^V(z,w,t)$ appears recursively in the integrand, by formally unfolding (2.6) once, we obtain

$$\begin{aligned}
q_\nu^V(z,w,t) &= q_\nu(z,w,t) + \int_0^t\int_0^\infty q_\nu(z,x,t-\tau)\Big(q_\nu(x,w,\tau) \\
&\quad + \int_0^\tau\int_0^\infty q_\nu(x,x',\tau-\tau')\, q_\nu(x',w,\tau')\, V(x')\, \mathrm{d}x'\mathrm{d}\tau'\Big) V(x)\,\mathrm{d}x\mathrm{d}\tau \\
&= q_\nu(z,w,t) + \int_0^t\int_0^\infty q_\nu(z,x,t-\tau)\, q_\nu(x,w,\tau)\, V(x)\,\mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t\int_0^\tau\int_{\mathbb{R}_+^2} q_\nu(z,x,t-\tau)\, q_\nu(x,x',\tau-\tau')\, q_\nu^V(x',w,\tau')\, V(x')\, V(x)\,\mathrm{d}x'\mathrm{d}x\mathrm{d}\tau'\mathrm{d}\tau.
\end{aligned}$$

Continuing this procedure, if we set $q_{\nu,0}(z,w,t) := q_\nu(z,w,t)$ and define inductively for $n \geq 1$,

$$q_{\nu,n}(z,w,t) := \int_0^t\int_0^\infty q_\nu(z,z_n,t-t_n)\, q_{\nu,n-1}(z_n,w,t_n)\, V(z_n)\,\mathrm{d}z_n\mathrm{d}t_n,$$

then after unfolding n times, we obtain

$$q_\nu^V(z,w,t) = \sum_{k=0}^n q_{\nu,k}(z,w,t) + \int_{\Delta_{n+1}t}\int_{\mathbb{R}_+^{n+1}} q_\nu^V(z_0,w,t_0)\prod_{i=1}^{n+1} q_\nu(z_i,z_{i-1},t_i-t_{i-1})\, V(z_i)\,\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{t},$$

where $\Delta_{n+1}t := \Big\{(t_0,\ldots,t_n)\in(0,t)^{n+1} \mid 0<t_0<\cdots<t_n<t\Big\}$, $\mathbf{z}=(z_0,\ldots,z_n)$, $\mathbf{t}=(t_0,\ldots,t_n)$, and we put $z_{n+1}=z$ and $t_{n+1}=t$. Iterating this process at infinitum, it can be shown that

$$q_\nu^V(z,w,t) := \sum_{n=0}^\infty q_{\nu,n}(z,w,t)$$

converges absolutely and satisfies (2.6). Using the fact that $q_\nu(z,w,t)$ satisfies the Kolmogorov-Chapman equations, by a simple inductive argument, we have

$$q_{\nu,n}(z,w,t) \leq \frac{t^n\|V\|_\infty^n}{n!} q_\nu(z,w,t) \text{ for every } n\geq 1,$$

from which (2.5) directly follows.

Intuitively, when t is small, the potential V has not yet generated enough "perturbation", so $q_\nu^V(z,w,t)$ is expected to be close to $q_\nu(z,w,t)$. Equation (2.5) not only gives a precise interpretation of this fact, since it implies that the difference

$$\left|\frac{q_\nu^V(z,w,t)}{q_\nu(z,w,t)} - 1\right| = O(t)$$

for small t, but the estimate is also uniform in the spatial variables $z,w>0$ which is considerably more accurate than the standard heat kernel estimate.

2.2. *The Malliavin Calculus.* In this section, we review some prerequisite materials regarding the Malliavin calculus which is essential for the study of stochastic Kimura equations. For further details regarding Skorohod integrals and chaos decomposition, we refer the reader to Dalang et al. (2008); Nualart (2006); Markus (2013); Hairer (2021).

Let f and γ be as in (1.5), and consider the inner product

$$\langle\varphi,\psi\rangle_H := \int_{\mathbb{R}_+^4} f(x-y)\,\gamma(r-s)\,\varphi(x,r)\,\psi(y,s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s$$

for $\varphi, \psi \in \mathcal{D}\left(\mathbb{R}_+^2\right)$. We define the Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ as the completion of $\mathcal{D}\left(\mathbb{R}_+^2\right)$ with respect to $\langle\cdot,\cdot\rangle_H$. It can be shown that there exists an isonormal Gaussian process $W = \{W(h) \mid h \in H\}$ on an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}\left[W(h_1)\right] \equiv 0 \text{ and } \mathbb{E}\left[W(h_1)\,W(h_2)\right] = \langle h_1, h_2\rangle_H$$

for every $h_1, h_2 \in H$. When viewed as a map between Hilbert spaces, $W : H \to W(H) =: \mathcal{H} \subseteq L^2(\Omega)$, is an isometry. This framework is convenient because it allows us to study the noise process W through well-behaved deterministic functions.

As it turns out, if $\mathcal{F}$ is the σ-algebra generated by W, then $L^2(\Omega)$ admits a decomposition into countably many closed images of W under a special class of polynomials. Let $n \geq 0$ and H_n denote the $n-$th Hermite polynomial, which is defined as

$$\mathrm{H}_n(x) := (-1)^n e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-\frac{x^2}{2}}\right).$$

For example, the first few Hermite polynomials are

$$\mathrm{H}_0 \equiv 1, \qquad \mathrm{H}_1(x) = x, \qquad \mathrm{H}_2(x) = x^2 - 1, \qquad \mathrm{H}_3(x) = x^3 - 3x.$$

Then, the *$n-$th Wiener chaos space* $\mathcal{H}_n$ is defined as

$$\mathcal{H}_n := \overline{\operatorname{span}\left\{\mathrm{H}_n\left(W(h)\right) \mid h \in H, \|h\|_H = 1\right\}},$$

where the closure is taken in $L^2(\Omega)$. Then,

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

i.e., for any $X \in L^2(\Omega)$, for each $n \geq 0$ there exists a unique $X_n \in \mathcal{H}_n$ such that

$$X = \sum_{n=0}^{\infty} X_n, \tag{2.7}$$

where the series converges in $L^2(\Omega)$. We remark that since $\mathrm{H}_0 \equiv 1$, $\mathcal{H}_0 \cong \mathbb{R}$ is a space of constants and it can be shown that $X_0 = \mathbb{E}[X]$. By the hypercontractivity property of Gaussian fields (see, e.g., §7 of Hairer (2021)), for every $n \geq 1$ and $p \geq 2$, $\mathcal{H}_n \subseteq L^p(\Omega)$. In fact, if $Y \in \mathcal{H}_n$, then

$$\left(\mathbb{E}\left[|Y|^p\right]\right)^{\frac{1}{p}} \leq (p-1)^{\frac{n}{2}} \left(\mathbb{E}\left[Y^2\right]\right)^{\frac{1}{2}}. \tag{2.8}$$

Below, we will demonstrate representations for these projections X_n in terms of stochastic integrals in the sense of Skorohod, whose moments are related to the inner product $\langle\cdot,\cdot\rangle_H$. To this end, we first define a suitable notion of derivative. Let $C_{\mathrm{p}}^{\infty}\left(\mathbb{R}^N\right)$ denote the space of smooth functions $F : \mathbb{R}^N \to \mathbb{R}$ which (along with their partial derivatives) have at most polynomial growth. We say a random variable $X \in L^2(\Omega)$ is *smooth* if there exists $F \in C_{\mathrm{p}}^{\infty}\left(\mathbb{R}^N\right)$ and $h_1, \ldots, h_N \in H$ such that $X = F\left(W(h_1), \ldots, W(h_N)\right)$. Moreover, we denote by $\mathcal{P}$ the set of all such smooth random variables. It is clear that $\mathcal{P}$ is a dense subspace of $L^2(\Omega)$.

Consequently, for some $X = F\left(W(h_1), \ldots, W(h_N)\right) \in \mathcal{P}$, the *Malliavin derivative* is an unbounded, closed linear operator $D : \mathcal{P} \to L^2(\Omega; H)$ given by

$$DX := \sum_{j=1}^{N} \partial_j F\left(W(h_1), \ldots, W(h_N)\right) h_j, \tag{2.9}$$

where for $z, t > 0$ we formally write[1]

$$D_{z,t}X := \sum_{j=1}^{N} \partial_j F\left(W\left(h_1\right), \ldots, W\left(h_N\right)\right) h_j\left(z, t\right),$$

and without loss of generality, we may assume that $h_1, \ldots, h_N$ are orthonormal under $\langle \cdot, \cdot \rangle_H$.

As with classical Sobolev spaces, we define the inner product

$$\langle X, Y \rangle_{1,2} := \mathbb{E}\left[XY\right] + \mathbb{E}\left[\langle DX, DY \rangle_H\right]$$

for $X, Y \in \mathcal{P}$, and let $\mathbb{D}^{1,2}$ denote the closure of $\mathcal{P}$ with respect to $\langle \cdot, \cdot \rangle_{1,2}$. Note that $\mathcal{P} \subseteq \mathbb{D}^{1,2} \subseteq L^2\left(\Omega\right)$, and since $\mathcal{P}$ is dense in $L^2\left(\Omega\right)$, $\mathcal{P}$ is dense in $\mathbb{D}^{1,2}$ and hence D can be extended to $\mathbb{D}^{1,2}$.

Viewing $D : \mathbb{D}^{1,2} \to L^2\left(\Omega; H\right)$, we are ready to introduce the notion of stochastic integral relevant to (1.6). Let the *divergence operator* δ be the adjoint operator of D. Namely, the domain $\mathrm{Dom}\left(\delta\right)$ of δ is the set of random variables $u \in L^2\left(\Omega; H\right)$ for which there exists $C_u > 0$ such that

$$\left|\mathbb{E}\left[\langle DX, u \rangle_H\right]\right| \leq C_u \|X\|_{1,2}$$

for all $X \in \mathbb{D}^{1,2}$; in this case, $\delta\left(u\right)$ is the unique element of $L^2\left(\Omega\right)$ such that

$$\mathbb{E}\left[\delta\left(u\right) X\right] = \mathbb{E}\left[\langle DX, u \rangle_H\right]$$

for all $X \in \mathbb{D}^{1,2}$. As the adjoint of an unbounded and densely defined operator, δ is in turn also closed. Furthermore, we use the notation

$$\delta\left(u\right) := \int_{\mathbb{R}_+^2} u\left(z, t\right) W\left(\mathrm{d}z, \mathrm{d}t\right),$$

where $W\left(\mathrm{d}z, \mathrm{d}t\right) = \dot{W}\left(z, t\right) \mathrm{d}z\mathrm{d}t$ (formally, $\dot{W}(z, t) = \partial_t \partial_z W\left(z, t\right)$). This nomenclature is motivated by the fact that the divergence operator possesses many desirable properties of a typical integral and coincides with the Itô stochastic integral in certain cases. For this reason, we also refer to δ as the *Skorohod integral*, and if $u \in \mathrm{Dom}\left(\delta\right)$, we say that u is *Skorohod integrable*.

Continuing along these lines, we define the multiple Skorohod integral to arrive at representations for the random variables X_n in (2.7). For $n \geq 1$, let $H^{\otimes n}$ denote the $n-$th tensor product of H, with the corresponding inner product and norm denoted by $\langle \cdot, \cdot \rangle_{H^{\otimes n}}$ and $\| \cdot \|_{H^{\otimes n}}$, respectively. Then for $f_n \in H^{\otimes n}$, recursively applying the Skorohod integral yields

$$\mathcal{I}_n\left(f_n\right) := \int_{\mathbb{R}_+^{2n}} f_n\left(z_1, t_1, \ldots, z_n, t_n\right) W\left(\mathrm{d}z_1, \mathrm{d}t_1\right) \cdots W\left(\mathrm{d}z_n, \mathrm{d}t_n\right).$$

Given $f_n \in H^{\otimes n}$, we denote by $\tilde{f}_n$ the symmetrization of f_n, given by

$$\tilde{f}_n\left(z_1, t_1, \ldots, z_n, t_n\right) := \frac{1}{n!} \sum_{\sigma \in S_n} f\left(z_{\sigma(1)}, t_{\sigma(1)}, \ldots, z_{\sigma(n)}, t_{\sigma(n)}\right),$$

where S_n is the set of all permutations on n elements. Then $\mathcal{I}_n\left(f_n\right) = \mathcal{I}_n(\tilde{f}_n)$, and for $g_m \in H^{\otimes m}$,

$$\mathbb{E}\left[\mathcal{I}_n\left(f_n\right) \mathcal{I}_m\left(g_m\right)\right] = \begin{cases} n! \langle \tilde{f}_n, \tilde{g}_m \rangle_{H^{\otimes n}} & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

For notational simplicity, from now on we will write the above $n! \langle \tilde{f}_n, \tilde{g}_n \rangle_{H^{\otimes n}}$ as $\langle f_n, g_n \rangle_n$ and similarly $n! \|\tilde{f}_n\|_{H^{\otimes n}}^2$ as $\|f_n\|_n^2$.

[1]Depending on the regularity governed by $\langle \cdot, \cdot \rangle_H$, it is possible that the elements of H (and $H^{\otimes n}$ below) are generalized functions, so "$h_j(z, t)$" may only be a formal representation of h_j.

It holds that, for every $n \geq 1$, there exists a unique $f_n : \mathbb{R}_+^{2n} \to \mathbb{R}$ symmetric such that $f_n \in H^{\otimes n}$ and $X_n = \mathcal{I}_n(f_n)$. In light of this, (2.7) becomes

$$X = \sum_{n=0}^{\infty} \mathcal{I}_n(f_n) \quad \text{with} \quad \mathcal{I}_0(f_0) = \mathbb{E}[X]. \tag{2.11}$$

We refer to this expansion as the *Wiener chaos expansion* of X. Further, $X \in \mathbb{D}^{1,2}$ if and only if $\sum_{n=1}^{\infty} n\|f_n\|_n^2 < \infty$, in which case $DX \in L^2(\Omega; H)$ and formally for $z, t > 0$,

$$D_{z,t}X = \sum_{n=1}^{\infty} n\mathcal{I}_{n-1}(f_n(\cdot, z, t)).$$

On the other hand, if $u \in L^2(\Omega; H)$, then there exists a unique mapping $(z, t) \in \mathbb{R}_+^2 \mapsto f_n(\cdot; z, t) \in H^{\otimes n}$ such that $f_n(\cdot; z, t)$ is symmetric in its first n pairs of variables and

$$u(z, t) = \sum_{n=0}^{\infty} \mathcal{I}_n(f_n(\cdot; z, t)) \quad \text{with} \quad \mathcal{I}_0(f_0(z, t)) = \mathbb{E}[u(z, t)]. \tag{2.12}$$

Based on the discussions above, we have the following criterion for Skorohod integrability.

Proposition 2.2. (Proposition 1.3.7 of Nualart (2006)) *Let $u \in L^2(\Omega; H)$ with the expansion (2.12). As a function $\mathbb{R}_+^{2(n+1)} \to \mathbb{R}$, $f_n \in H^{\otimes(n+1)}$. Then $u \in Dom(\delta)$ if and only if the series*

$$\delta(u) = \sum_{n=0}^{\infty} \mathcal{I}_{n+1}(f_n) \tag{2.13}$$

converges in $L^2(\Omega)$, which is equivalent to $\sum_{n=0}^{\infty} \|f_n\|_{n+1}^2 < \infty$.

To check convergence of the series (2.13) in $L^2(\Omega)$, we often make use of the property (2.10) of the multiple Skorohod integral.

Remark 2.3. We close this section with a remark on general-order Malliavin derivatives. It is possible to iterate the operator in (2.9) and define the *k-th order Malliavin derivative* operator $D^k : \mathcal{P} \to L^2(\Omega; H^{\otimes k})$ for $k \geq 1$. Similarly, if $\mathbb{D}^{k,2}$ is the closure of $\mathcal{P}$ under the inner product

$$\langle X, Y \rangle_{k,2} := \mathbb{E}[XY] + \sum_{j=1}^{k} \mathbb{E}\left[\langle D^j X, D^j Y \rangle_{H^{\otimes j}}\right]$$

for all $X, Y \in \mathcal{P}$, then $\mathbb{D}^{k,2}$ is a subspace of $L^2(\Omega)$ and D^k can be extended to $\mathbb{D}^{k,2}$. Moreover, for $X \in L^2(\Omega)$ in the form of Wiener chaos expansion (2.11), $X \in \mathbb{D}^{k,2}$ if and only if $\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\|f_n\|_n^2 < \infty$, and in this case $D_*^k X = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\mathcal{I}_{n-k}(f_n(\cdot; *)) \in H^{\otimes k}$.

3. Stochastic Kimura Equation with Non-Degenerate Noise

In this section, we combine the methods from Hu et al. (2015); Balan and Song (2017); Balan et al. (2019) and the results from Chen and Stroock (2010); Chen and Weih-Wadman (2020) to study the existence and the uniqueness of the solution $u(z, t)$ to the stochastic Kimura equation (1.6) (§3.1). We will also examine the relation between $u(z, t)$ and $u_0(z, t)$, where we recall that $u_0(z, t)$ is the deterministic solution defined in (1.3) (§3.2). In this work, we restrict our study to the constant initial value problem, i.e., setting $u_{\text{int}} \equiv 1$. However, our methods and results can be extended to bounded initial data problems, i.e., $u_{\text{int}} \in B(\mathbb{R}_+)$, in a straightforward way.

3.1. *Existence and Uniqueness of Solution.* The main subject under investigation in this work is the following SPDE equipped with constant initial data and the Dirichlet boundary condition.

$$\begin{cases} \partial_t u(z,t) = z\partial_z^2 u(z,t) + u(z,t)\dot{W}(z,t) & \text{for } z,t>0, \\ u(z,0) = 1 & \text{for } z>0, \\ u(0,t) = 0 & \text{for } t\geq 0. \end{cases} \tag{3.1}$$

In this section, we prove that (3.1) admits a mild solution which is unique up to a modification. First, we make precise the meaning of a "mild solution". We adopt the definition from the case of the one-dimensional PAM (see, e.g., Definition 2.2 in Balan and Chen (2018)) and adapt it to our setting.

Definition 3.1. A square-integrable random field $u = \{u(z,t) \mid z>0, t\geq 0\}$ is a *mild solution* to (3.1) if
(a) u is jointly measurable (or has one such modification) and

$$\sup_{z>0, t\in[0,T]} \mathbb{E}\left[u^2(z,t)\right] < \infty \text{ for every } T\geq 0;$$

(b) For every $z,t>0$, the process $\left\{\mathbb{1}_{(0,t)}(\tau)\, q_0(z,w,t-\tau)\, u(w,\tau) \mid w,\tau>0\right\}$ is Skorohod integrable and

$$u(z,t) = u_0(z,t) + \int_0^t \int_0^\infty q_0(z,w,t-\tau)\, u(w,\tau)\, W(\mathrm{d}w,\mathrm{d}\tau). \tag{3.2}$$

It follows automatically from **(a)** and **(b)** that $u(0,t)=0$ for every $t\geq 0$ and $u(z,0)=1$ for every $z>0$.

Suppose u is a mild solution to (3.1), then for every $z,t>0$, $u(z,t)$ admits the following Wiener chaos expansion

$$u(z,t) = \sum_{n=0}^\infty \mathcal{I}_n(f_n(\cdot;z,t)), \tag{3.3}$$

for some $f_n(\cdot;z,t) \in H^{\otimes n}$. Consequently, we obtain

$$\mathbb{E}\left[u^2(z,t)\right] = \sum_{n=0}^\infty \mathbb{E}\left[\mathcal{I}_n^2(f_n(\cdot;z,t))\right] = \sum_{n=0}^\infty \|f_n(\cdot;z,t)\|_n^2,$$

where we recall that $\|f_n(\cdot;z,t)\|_n^2 = n!\|\tilde{f}_n(\cdot;z,t)\|_{H^{\otimes n}}^2$ with $\tilde{f}_n(\cdot;z,t)$ being the symmetrization of $f_n(\cdot;z,t)$ in the first n pairs of variables. So, we turn our attention to finding an expression for $f_n(\cdot;z,t)$ and computing its norm. Meanwhile, since u satisfies the integral equation (3.2), by following a similar argument as in §2.1 to unfold (3.2) recursively in Skorohod integrals, we see that if we set $f_0'(z,t) = u_0(z,t)$ and inductively define for $n\geq 1$,

$$\begin{aligned} f_n'(z_1,t_1,\ldots,z_n,t_n;z,t) = \mathbb{1}_{(0,t)}(t_n)\, q_0(z,z_n,t-t_n) \\ \cdot f_{n-1}'(z_1,t_1,\ldots,z_{n-1},t_{n-1};z_n,t_n), \end{aligned} \tag{3.4}$$

then, by condition **(b)** of Definition 3.1, formally we have

$$u(z,t) = \sum_{n=0}^\infty \mathcal{I}_n\left(f_n'(\cdot;z,t)\right).$$

In other words, for every $z,t>0$ and $n\geq 0$, $f_n(\cdot;z,t) = f_n'(\cdot;z,t)$, and hence we have obtained a way to construct these functions. Following (3.4), we obtain explicit expressions:

$$\tilde{f}_n(z_1,t_1,\ldots,z_n,t_n;z,t) = \frac{1}{n!} q_0(z,z_{\sigma(n)},t-t_{\sigma(n)}) \cdot \prod_{i=1}^{n-1} q_0(z_{\sigma(i+1)},z_{\sigma(i)},t_{\sigma(i+1)}-t_{\sigma(i)})$$

where σ is the permutation of the indices $\{1, 2, \dots, n\}$ such that $\{(z_{\sigma(1)}, t_{\sigma(1)}), \dots, (z_{\sigma(n)}, t_{\sigma(n)})\}$ is the reordering of $\{(z_1, t_1), \dots, (z_n, t_n)\}$ with $0 < t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(n)} < t$. Thus,

$$\begin{aligned}
&\|f_n(\cdot; z, t)\|_n^2 \\
&= \frac{1}{n!} \int_{[0,t]^{2n}} \int_{\mathbb{R}_+^{2n}} q_0(z, x_{\sigma(n)}, t - r_{\sigma(n)}) \cdot \prod_{i=1}^{n-1} q_0\left(x_{\sigma(i+1)}, x_{\sigma(i)}, r_{\sigma(i+1)} - r_{\sigma(i)}\right) \\
&\qquad \cdot q_0(z, y_{\varsigma(n)}, t - s_{\varsigma(n)}) \cdot \prod_{i=1}^{n-1} q_0\left(y_{\varsigma(i+1)}, y_{\varsigma(i)}, s_{\varsigma(i+1)} - s_{\varsigma(i)}\right) \\
&\qquad \cdot u_0(x_{\sigma(1)}, r_{\sigma(1)}) u_0(y_{\varsigma(1)}, s_{\varsigma(1)}) \cdot \prod_{i=1}^{n} f(x_i - y_i)\, \gamma(r_i - s_i)\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s},
\end{aligned} \tag{3.5}$$

where σ and ς are the permutations of the indices $\{1, 2, \dots, n\}$, such that $0 < r_{\sigma(1)} < r_{\sigma(2)} < \cdots < r_{\sigma(n)} < t$ and $0 < s_{\varsigma(1)} < s_{\varsigma(2)} < \cdots < s_{\varsigma(n)} < t$.

Of course, we still need to prove rigorously that (3.3) with the derived expression in (3.4) for f_n is in fact the unique mild solution to (3.1). We build towards the proof by first studying the case of the space-time white noise via a treatment similar to Hu et al. (2015); Balan and Song (2017).

3.1.1. *Under Space-Time White Noise.* We are interested in (3.1) when the centered Gaussian noise $\dot{W} := \{\dot{W}(z, t) \mid z, t > 0\}$ is the *space-time white noise*, i.e., the covariance is given by

$$\mathbb{E}\left[\dot{W}(z_1, t_1)\, \dot{W}(z_2, t_2)\right] = \delta_0(z_1 - z_2)\, \delta_0(t_1 - t_2)$$

for $z_1, z_2, t_1, t_2 > 0$, where δ_0 denotes the Dirac delta. In other words, $\langle \cdot, \cdot \rangle_H$ coincides with the standard $L^2\left(\mathbb{R}_+^2\right)$ inner product.

Let f_n be defined by (3.4), we first aim to prove that for every $n \ge 0$ and $z, t > 0$, we have $f_n(\cdot; z, t) \in H^{\otimes n}$. Recall that, in the constant initial data case,

$$f_0(z, t) = u_0(z, t) = \int_0^\infty q_0(z, w, t)\, \mathrm{d}w,$$

and we have an explicit formula for this integral.

Lemma 3.2. *For every $z, t > 0$,*

$$\int_0^\infty q_0(z, w, t)\, \mathrm{d}w = 1 - e^{-\frac{z}{t}}. \tag{3.6}$$

Proof: It follows directly from (1.2) that for every $z, t > 0$,

$$\begin{aligned}
\int_0^\infty q_0(z, w, t)\, \mathrm{d}w &= \frac{z e^{-\frac{z}{t}}}{t^2} \sum_{n=0}^{\infty} \frac{z^n}{t^{2n} n! (n+1)!} \int_0^\infty e^{-\frac{w}{t}} w^n \mathrm{d}w \\
&= e^{-\frac{z}{t}} \sum_{n=0}^{\infty} \frac{z^{n+1}}{t^{n+1}(n+1)!} = 1 - e^{-\frac{z}{t}}.
\end{aligned}$$

□

For $n \ge 1$, in order to show that $f_n(\cdot; z, t) \in H^{\otimes n}$, it suffices to verify the finiteness of $\|f_n(\cdot; z, t)\|_n^2$. In this case the general expression (3.5) can be reduced to the following recursive relation:

$$\|f_n(\cdot; z, t)\|_n^2 = \int_0^t \int_0^\infty q_0^2(z, w, t - \tau)\, \|f_{n-1}(\cdot; w, \tau)\|_{n-1}^2 \mathrm{d}w \mathrm{d}\tau. \tag{3.7}$$

By (3.6), for $n=1$ we observe

$$\| f_1(\cdot;z,t)\|_1^2 = \int_0^t\int_0^\infty q_0^2(z,w,t-\tau)\,u_0^2(w,\tau)\,\mathrm{d}w\mathrm{d}\tau \le \int_0^t\int_0^\infty q_0^2(z,w,\tau)\,\mathrm{d}w\mathrm{d}\tau,$$

where the last integral again can be computed directly.

Lemma 3.3. *For $x>0$, define $U(x) := e^{-x}I_0(x) - \frac{1}{2}e^{-2x}$. Then, for every $z,t>0$,*

$$\int_0^t\int_0^\infty q_0^2(z,w,\tau)\,\mathrm{d}w\mathrm{d}\tau = U\left(\frac{z}{t}\right) \le \frac{1}{2}. \tag{3.8}$$

Proof: Again, by (1.2) and (6.3), we have that

$$\begin{aligned}\int_0^\infty q_0^2(z,w,\tau)\,\mathrm{d}w &= \int_0^\infty \frac{zw^{-1}}{\tau^2}e^{-\frac{2(z+w)}{\tau}} I_1^2\left(2\frac{\sqrt{zw}}{\tau}\right)\mathrm{d}w \\ &= \frac{ze^{-\frac{2z}{\tau}}}{\tau^2 i^2}\int_0^\infty w^{-1}e^{-\frac{2w}{\tau}} J_1^2\left(\frac{2i\sqrt{zw}}{\tau}\right)\mathrm{d}w.\end{aligned}$$

Next, we apply (6.7) to obtain

$$i^{-2}\int_0^\infty w^{-1}e^{-\frac{2w}{\tau}}J_1^2\left(\frac{2i\sqrt{zw}}{\tau}\right)\mathrm{d}w = e^{\frac{z}{\tau}}\left(I_0\left(\frac{z}{\tau}\right) - I_1\left(\frac{z}{\tau}\right)\right) - 1,$$

and hence we get

$$\int_0^t\int_0^\infty q_0^2(z,w,\tau)\,\mathrm{d}w\mathrm{d}\tau = \int_0^t \frac{ze^{-\frac{2z}{\tau}}}{\tau^2}\left[e^{\frac{z}{\tau}}\left(I_0\left(\frac{z}{\tau}\right) - I_1\left(\frac{z}{\tau}\right)\right) - 1\right]\mathrm{d}\tau. \tag{3.9}$$

Using the fact that $I_0' = I_1$, we have

$$z\int_0^t \frac{e^{-\frac{z}{\tau}}}{\tau^2}I_1\left(\frac{z}{\tau}\right)\mathrm{d}\tau = -\int_0^t e^{-\frac{z}{\tau}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[I_0\left(\frac{z}{\tau}\right)\right]\mathrm{d}\tau = e^{-\frac{z}{t}}I_0\left(\frac{z}{t}\right) + z\int_0^t \frac{e^{-\frac{z}{\tau}}}{\tau^2}I_0\left(\frac{z}{\tau}\right)\mathrm{d}\tau,$$

and substituting this last expression into (3.9), (3.8) directly follows. Finally, $\|U\|_\infty = \frac{1}{2}$ follows from straightforward calculations. □

We now have a bound on $\| f_1(\cdot;z,t)\|_1^2$ that is uniform in $z,t>0$, which leads to a straightforward induction argument. We claim that for every $z,t>0$ and $n\ge 1$,

$$\| f_n(\cdot;z,t)\|_n^2 \le 2^{-n+1}U\left(\frac{z}{t}\right) \le 2^{-n}. \tag{3.10}$$

The case $n=1$ is confirmed by Lemma 3.3. Next, assuming (3.10) holds for n for some $n\ge 1$, then by (3.7), for every $z,t>0$,

$$\| f_{n+1}(\cdot;z,t)\|_{n+1}^2 \le 2^{-n}\int_0^t\int_0^\infty q_0^2(z,w,t-\tau)\,\mathrm{d}w\mathrm{d}\tau = 2^{-n}U\left(\frac{z}{t}\right) \le 2^{-n-1}.$$

It directly follows that, for every $n\ge 1$ and $z,t>0$, $f_n(\cdot;z,t)\in H^{\otimes n}$, and hence

$$u_n(z,t) := \mathcal{I}_n(f_n(\cdot;z,t)) \in \mathcal{H}_n. \tag{3.11}$$

We are now equipped to present the main result of this subsection.

Theorem 3.4. *Let $\dot{W}$ be the space-time white noise. Then, $u = \{u(z,t) \mid z>0, t\ge 0\}$ as in (3.3) is well-defined as a convergent series in $L^2(\Omega)$. Moreover, u is the unique mild solution to (3.1), and for every $z,t>0$,*

$$\mathbb{E}\left[u^2(z,t)\right] \le \left(1-e^{-\frac{z}{t}}\right)^2 + 2U\left(\frac{z}{t}\right) \le 2. \tag{3.12}$$

Proof: The convergence of the series in (3.3) to $u(z,t)$ in $L^2(\Omega)$ and (3.12) follow by the summability of the bounds (3.10) over $n \geq 0$. To see that u has a jointly measurable modification, we first show that $(z,t) \mapsto u(z,t)$ is continuous in $L^2(\Omega)$. To this end, we have that, for every $z_1, z_2 > 0$, $t_1, t_2 \geq 0$ and $n \geq 1$,

$$\begin{aligned}
&\mathbb{E}\left[\left|u_n(z_1,t_1) - u_n(z_2,t_2)\right|^2\right] \\
&= \int_0^\infty \int_0^\infty \left|\mathbb{1}_{(0,t_1)}(\tau) q_0(z_1,w,t_1-\tau) - \mathbb{1}_{(0,t_2)}(\tau) q_0(z_2,w,t_2-\tau)\right|^2 \mathbb{E}\left[u_{n-1}^2(w,\tau)\right] \mathrm{d}w\mathrm{d}\tau, \\
&\leq 2^{-n+1} \int_0^\infty \int_0^\infty \left|\mathbb{1}_{(0,t_1)}(\tau) q_0(z_1,w,t_1-\tau) - \mathbb{1}_{(0,t_2)}(\tau) q_0(z_2,w,t_2-\tau)\right|^2 \mathrm{d}w\mathrm{d}\tau.
\end{aligned}$$

Therefore, to show the $L^2(\Omega)$-continuity of u in (z,t), it is sufficient to show that

$$\int_0^\infty \int_0^\infty \left|\mathbb{1}_{(0,t_1)}(\tau) q_0(z_1,w,t_1-\tau) - \mathbb{1}_{(0,t_2)}(\tau) q_0(z_2,w,t_2-\tau)\right|^2 \mathrm{d}w\mathrm{d}\tau \to 0$$

as $|(z_1,t_1) - (z_2,t_2)| \to 0$. We may assume $t_1 < t_2$ without loss of generality, in which case the integral above becomes

$$\int_0^{t_1} \int_0^\infty \left|q_0(z_1,w,t_1-\tau) - q_0(z_2,w,t_2-\tau)\right|^2 \mathrm{d}w\mathrm{d}\tau + \int_{t_1}^{t_2} \int_0^\infty q_0^2(z_2,w,t_2-\tau)\,\mathrm{d}w\mathrm{d}\tau.$$

The first term above vanishes as $|(z_1,t_1) - (z_2,t_2)| \to 0$ by the dominated convergence theorem and the continuity of q_0, and the second term vanishes as $t_2 - t_1 \to 0$ by the integrability of $\int_0^\infty q_0^2(z_2,w,t_2-\tau)\mathrm{d}w$ in $\tau \in (0,t_2)$. Therefore, we have proven that $(z,t) \mapsto u(z,t)$ is continuous in $L^2(\Omega)$, and hence by Theorem 30 in Chapter IV of Dellacherie and Meyer (2008), we know that u possesses a jointly measurable modification, which will again be denoted by u. We have verified the condition **(a)** in Definition 3.1.

Now we turn to the condition **(b)**. For every $z,t > 0$, let

$$u^{z,t}(w,\tau) := \mathbb{1}_{(0,t)}(\tau)\, q_0(z,w,t-\tau)\, u(w,\tau) \text{ for } w,\tau > 0. \tag{3.13}$$

The measurability of u guarantees the measurability of $(w,\tau) \mapsto u^{z,t}(w,\tau)$. Moreover, by (3.12), for every $w,\tau > 0$, $\mathbb{E}[(u^{z,t}(w,\tau))^2] < \infty$ and hence $u^{z,t}(w,\tau)$ admits the Wiener chaos expansion

$$u^{z,t}(w,\tau) = \sum_{n=0}^\infty \mathcal{I}_n\left(g_n^{z,t}(\cdot;w,\tau)\right),$$

where

$$g_n^{z,t}(\cdot;w,\tau) = \mathbb{1}_{(0,t)}(\tau)\, q_0(z,w,t-\tau)\, f_n(\cdot;w,\tau) = f_{n+1}(\cdot,w,\tau;z,t).$$

Clearly, $g_n^{z,t} \in H^{\otimes(n+1)}$ and hence

$$\delta\left(u^{z,t}\right) = \sum_{n=0}^\infty \mathcal{I}_{n+1}\left(g_n^{z,t}\right) = \sum_{n=0}^\infty \mathcal{I}_{n+1}\left(f_{n+1}(\cdot;z,t)\right) = \sum_{n=1}^\infty u_n(z,t)$$

which converges in $L^2(\Omega)$. Therefore, $u^{z,t}$ is indeed Skorohod integrable and by Proposition 2.2,

$$\int_0^t \int_0^\infty q_0(z,w,t-\tau)\, u(w,\tau)\, W(\mathrm{d}w,\mathrm{d}\tau) = \delta\left(u^{z,t}\right) = u(z,t) - u_0(z,t).$$

Thus, u also satisfies the condition **(b)** and hence is a mild solution to (3.1).

Next, we prove that u is the unique mild solution to (3.1). Let $v = \{v(z,t) \mid z > 0, t \geq 0\}$ be another solution to (3.1) with the Wiener chaos expansion

$$v(z,t) = \sum_{n=0}^\infty \mathcal{I}_n\left(k_n(\cdot;z,t)\right)$$

for some symmetric non-negative functions $k_n(\cdot;z,t)\in H^{\otimes n}$ for every $z,t>0$ and $n\geq 0$. Clearly, $k_0(z,t)=u_0(z,t)=f_0(z,t)$. For every $w,\tau>0$, let $v^{z,t}(w,\tau)$ be defined similarly to (3.13). Then, by definition of a solution, the process $v^{z,t}=\left\{v^{z,t}(w,\tau)\mid w,\tau>0\right\}$ is Skorohod integrable and admits the Wiener chaos expansion

$$v^{z,t}(w,\tau)=\sum_{n=0}^{\infty}\mathcal{I}_n\left(h_n^{z,t}(\cdot;w,\tau)\right)$$

where $h_n^{z,t}(\cdot;w,\tau)=\mathbb{1}_{(0,t)}(\tau)\,q_0(z,w,t-\tau)\,k_n(\cdot;w,\tau)$. By Proposition 2.2,

$$\sum_{n=0}^{\infty}\mathcal{I}_{n+1}\left(k_{n+1}(\cdot;z,t)\right)=v(z,t)-\mathbb{E}\left[v(z,t)\right]=\delta\left(v^{z,t}\right)=\sum_{n=0}^{\infty}\mathcal{I}_{n+1}\left(\widetilde{h_n^{z,t}}\right),$$

from which it follows that $k_{n+1}(\cdot;z,t)=\widetilde{h_n^{z,t}}$ for every $n\geq 0$ by the uniqueness of the Wiener chaos expansion with symmetric kernels.

We now express $k_{n+1}(\cdot;z,t)$ recursively. For $n=0$, we obtain

$$k_1(z_1,t_1;z,t)=\widetilde{h_0^{z,t}}(z_1,t_1)=\mathbb{1}_{(0,t)}(t_1)\,q_0(z,z_1,t-t_1)\,f_0(z_1,t_1)=f_1(z_1,t_1;z,t).$$

For $n=1$, we have $k_2(z_1,t_1,z_2,t_2;z,t)=\widetilde{h_1^{z,t}}(z_1,t_1,z_2,t_2)$ which further equals

$$\begin{aligned}
&\frac{1}{2}\left[h_1^{z,t}(z_1,t_1,z_2,t_2)+h_1^{z,t}(z_2,t_2,z_1,t_1)\right]\\
=&\frac{1}{2}\left[\mathbb{1}_{(0,t)}(t_2)\,q_0(z,z_2,t-t_2)\,k_1(z_1,t_1,z_2,t_2)+\mathbb{1}_{(0,t)}(t_1)\,q_0(z,z_1,t-t_1)\,k_1(z_2,t_2,z_1,t_1)\right]\\
=&\frac{1}{2}\left[\mathbb{1}_{(0,t)}(t_2)\,q_0(z,z_2,t-t_2)\,f_1(z_1,t_1,z_2,t_2)+\mathbb{1}_{(0,t)}(t_1)\,q_0(z,z_1,t-t_1)\,f_1(z_2,t_2,z_1,t_1)\right]\\
=&\frac{1}{2}\left[f_2(z_1,t_1,z_2,t_2;z,t)+f_2(z_2,t_2,z_1,t_1;z,t)\right].
\end{aligned}$$

That is, $k_2(\cdot;z,t)=\tilde{f}_2(\cdot;z,t)$. Following an induction argument, we arrive at the conclusion that $k_n(\cdot;z,t)=\tilde{f}_n(\cdot;z,t)$ for every $n\geq 0$. Thus,

$$v(z,t)=\sum_{n=0}^{\infty}\mathcal{I}_n\left(k_n(\cdot;z,t)\right)=\sum_{n=0}^{\infty}\mathcal{I}_n\left(\tilde{f}_n(\cdot;z,t)\right)=\sum_{n=0}^{\infty}\mathcal{I}_n\left(f_n(\cdot;z,t)\right)=u(z,t).$$

We have completed the proof of the uniqueness of the solution. □

3.1.2. *Under Colored Noise.* We continue to the case of colored noise. We refer to the centered Gaussian process $\dot{W}=\{\dot{W}(z,t)\mid z,t>0\}$ as the $(f,\gamma)-$*colored noise* if its covariance is formally given by

$$\mathbb{E}\left[\dot{W}(z_1,t_1)\,\dot{W}(z_2,t_2)\right]=f(z_1-z_2)\,\gamma(t_1-t_2)$$

for $z_1,z_2,t_1,t_2>0$, where $f,\gamma:\mathbb{R}\to\mathbb{R}_+$ are non-negative definite symmetric kernels. As in the previous section, we seek to show that the process u defined in (3.3) is the unique mild solution to (3.1). To this end, we will prove that the series in (3.3) converges in $L^2(\Omega)$ for every $z,t>0$ by, again, proving $\|f_n(\cdot;z,t)\|_n^2$ forms a convergent series in n. However, in the colored noise case, the general formula (3.5) of $\|f_n(\cdot;z,t)\|_n^2$ does not yield a simple recursive relation as (3.7) for the white noise model. In addition, we also miss some of the technical advantages compared with the white noise case. For example, after applying Lemma 3.2, for $n=1$ we arrive at

$$\|f_1(\cdot;z,t)\|_1^2\leq\int_{[0,t]^2}\gamma(r-s)\int_{\mathbb{R}_+^2}f(x-y)\,q_0(z,x,t-r)\,q_0(z,y,t-s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s$$

which cannot be directly computed as in Lemma 3.3. Instead, we seek to extract γ and f from the integrals by imposing additional assumptions.

For $\varepsilon, t > 0$, we introduce the following notations

$$R_\varepsilon := \left\{(x,y) \in \mathbb{R}_+^2 \mid |x-y| < \varepsilon\right\}, \tag{3.14}$$

$$F_\varepsilon := \int_{-\varepsilon}^{\varepsilon} f(x)\,\mathrm{d}x = 2\int_0^{\varepsilon} f(x)\,\mathrm{d}x, \tag{3.15}$$

$$\Gamma_t := \int_{-t}^{t} \gamma(\tau)\,\mathrm{d}\tau = 2\int_0^{t} \gamma(\tau)\,\mathrm{d}\tau. \tag{3.16}$$

Assumptions on (f,γ) that we adopt throughout our study are:

- **(i)** For every $\varepsilon > 0$, f is bounded on $\mathbb{R}\setminus[-\varepsilon,\varepsilon]$. Without loss of generality, we may assume that $f(x) \le f(\varepsilon)$ for every $x \in \mathbb{R}\setminus[-\varepsilon,\varepsilon]$.
- **(ii)** For every $\varepsilon, t > 0$, $F_\varepsilon < \infty$, $\Gamma_t < \infty$ and $F_\varepsilon \to 0$ as $\varepsilon \to 0$.

We proceed to make our strategy concrete.

Lemma 3.5. *Let $z, t > 0$ and U be as in Lemma 3.3. Then, for every $\varepsilon > 0$ we have*

$$\begin{aligned}\int_{[0,t]^2\times\mathbb{R}_+^2} &\gamma(r-s)\,f(x-y)q_0(z,x,t-r)\,q_0(z,y,t-s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s \\ &\le \Gamma_t\left(F_\varepsilon U\left(\frac{z}{t}\right) + tf(\varepsilon)\right).\end{aligned} \tag{3.17}$$

Proof: Let $\varepsilon > 0$, and define

$$\mathcal{L}_{1,1} := \int_{[0,t]^2\times R_\varepsilon} \gamma(r-s)\,f(x-y)\,q_0(z,x,t-r)\,q_0(z,y,t-s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s,$$

$$\mathcal{L}_{1,2} := \int_{[0,t]^2\times R_\varepsilon^{\complement}} \gamma(r-s)\,f(x-y)\,q_0(z,x,t-r)\,q_0(z,y,t-s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s.$$

We will treat $\mathcal{L}_{1,1}$ and $\mathcal{L}_{1,2}$ separately. By (6.2) applied twice (once for the spatial variables, once for the temporal variables), we get

$$\begin{aligned}\mathcal{L}_{1,1} &\le \int_{[0,t]^2} \gamma(r-s)\,F_\varepsilon\left(\int_0^\infty q_0^2(z,x,t-r)\,\mathrm{d}x\cdot\int_0^\infty q_0^2(z,y,t-s)\,\mathrm{d}y\right)^{\frac{1}{2}}\mathrm{d}r\mathrm{d}s \\ &\le \Gamma_t F_\varepsilon\left(\int_0^t\int_0^\infty q_0^2(z,x,t-r)\,\mathrm{d}x\mathrm{d}r\cdot\int_0^t\int_0^\infty q_0^2(z,y,t-s)\,\mathrm{d}y\mathrm{d}s\right)^{\frac{1}{2}}.\end{aligned}$$

Notice that these last integrals are precisely the quantities studied in the white noise case. Therefore, Lemma 3.3 immediately implies that

$$\mathcal{L}_{1,1} \le \Gamma_t F_\varepsilon U\left(\frac{z}{t}\right). \tag{3.18}$$

On the other hand, by our assumptions on f, we have

$$\begin{aligned}\mathcal{L}_{1,2} &\le f(\varepsilon)\int_{[0,t]^2}\gamma(r-s)\int_{\mathbb{R}_+^2} q_0(z,x,t-r)\,q_0(z,y,t-s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s \\ &= f(\varepsilon)\int_{[0,t]^2}\gamma(r-s)\left(1-e^{-\frac{z}{t-r}}\right)\left(1-e^{-\frac{z}{t-s}}\right)\mathrm{d}r\mathrm{d}s \\ &\le f(\varepsilon)\int_{[0,t]^2}\gamma(r-s)\,\mathrm{d}r\mathrm{d}s \le f(\varepsilon)\,\Gamma_t t,\end{aligned}$$

where the last inequality is due to (6.2) again. Combining the respective bounds on $\mathcal{L}_{1,1}$ and $\mathcal{L}_{1,2}$ above, we have proven the desired inequality. □

As in the white noise case, we aim to derive a bound on $\|f_n(\cdot;z,t)\|_n^2$ for every $z,t>0$. Moreover, upon replacing the function U by its uniform norm $\frac{1}{2}$, the bound in (3.17) is increasing with respect to the time variables. Hence, it is tempting to replicate (3.10) and obtain a geometric bound on $\|f_n(\cdot;z,t)\|_n^2$ by induction. However, this approach may not work in the general colored noise setting. Even when the noise is white in time, i.e., $\gamma=\delta_0$, we observe that (3.5) becomes

$$\|f_n(\cdot;z,t)\|_n^2=\int_0^t\int_{\mathbb{R}_+^2} f(x-y)\,q_0(z,x,t-r)\,q_0(z,y,t-r)\\ \cdot\langle f_{n-1}(\cdot;x,r),f_{n-1}(\cdot;y,r)\rangle_{n-1}\,\mathrm{d}x\mathrm{d}y\mathrm{d}r,$$

and from here a straightforward induction argument leads to

$$\|f_n(\cdot;z,t)\|_n^2\le\left(\frac{F_\varepsilon}{2}+f(\varepsilon)\,t\right)^n, \tag{3.19}$$

which is not guaranteed to form a convergent series for general f and $z,t>0$. We will further illustrate this point with the example below.

Example 3.6. Suppose $\gamma=\delta_0$ and $f(x)=|x|^{-1/2}$ for $x>0$. Then, by (3.19),

$$\sum_{n=0}^{\infty}\|f_n(\cdot;z,t)\|_n^2\le\sum_{n=0}^{\infty}\left(2\sqrt{\varepsilon}+\frac{t}{\sqrt{\varepsilon}}\right)^n,$$

whose right hand side converges if and only if $2\sqrt{\varepsilon}+\frac{t}{\sqrt{\varepsilon}}<1$. Yet when t is sufficiently large, no choice of ε exists for which this inequality holds.

In light of the example above, we are required to adopt a different approach to bound $\|f_n(\cdot;z,t)\|_n^2$ in the case with general $(f,\gamma)-$colored noise. Let us examine more carefully the case $n=2$. Note that, for $z,t>0$, by (3.5), $\|f_2(\cdot;z,t)\|_2^2$ is bounded from above by

$$\frac{1}{2}\int_{[0,t]^2}\int_{\mathbb{R}_+^2} q_0(z,x_{\sigma(2)},t-r_{\sigma(2)})q_0(x_{\sigma(2)},x_{\sigma(1)},r_{\sigma(2)}-r_{\sigma(1)})\\ \cdot\int_{[0,t]^2}\int_{\mathbb{R}_+^2} q_0(z,y_{\varsigma(2)},t-s_{\varsigma(2)})q_0(y_{\varsigma(2)},y_{\varsigma(1)},s_{\varsigma(2)}-s_{\varsigma(1)})\\ \cdot f(x_1-y_1)f(x_2-y_2)\gamma(r_1-s_1)\gamma(r_2-s_2)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}, \tag{3.20}$$

where we recall that σ and ς are the permutations of $\{1,2\}$ such that $0<r_{\sigma(1)}<r_{\sigma(2)}<t$ and, respectively, $0<s_{\varsigma(1)}<s_{\varsigma(2)}<t$.

Given $\varepsilon>0$, we split the integral in (3.20) into four parts, according to the range of $|x_1-y_1|$ and $|x_2-y_2|$ with respect to ε.

Case 1: Set $\mathbf{R}_1:=\{(\mathbf{x},\mathbf{y})\in\mathbb{R}_+^4 \mid |x_1-y_1|\le\varepsilon,|x_2-y_2|\le\varepsilon\}$ and denote by $\mathcal{L}_{2,1}$ the integral in (3.20) with $(\mathbf{x},\mathbf{y})$ over $\mathbf{R}_1$ instead of $\mathbb{R}_+^4$. By repeatedly applying (6.2) to $\gamma(r_i-s_i)$ and $f(x_i-y_i)$,

$i = 1, 2$, we obtain that

$$\begin{aligned}\mathcal{L}_{2,1} &\leq \frac{1}{2}\Gamma_t^2 F_\varepsilon^2 \left(\int_{[0,t]^2}\int_{\mathbb{R}_+^2} q_0^2(z, x_{\sigma(2)}, t - r_{\sigma(2)}) q_0^2(x_{\sigma(2)}, x_{\sigma(1)}, r_{\sigma(2)} - r_{\sigma(1)}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{r}\right)^{\frac{1}{2}} \\ &\qquad \cdot \left(\int_{[0,t]^2}\int_{\mathbb{R}_+^2} q_0^2(z, y_{\varsigma(2)}, t - s_{\varsigma(2)}) q_0^2(y_{\varsigma(2)}, y_{\varsigma(1)}, s_{\varsigma(2)} - s_{\varsigma(1)}) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{s}\right)^{\frac{1}{2}} \\ &\leq \Gamma_t^2 F_\varepsilon^2 \left(\int_{0<r_1<r_2<t}\int_{\mathbb{R}_+^2} q_0^2(z, x_2, t - r_2) q_0^2(x_2, x_1, r_2 - r_1) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{r}\right)^{\frac{1}{2}} \\ &\qquad \cdot \left(\int_{0<s_1<s_2<t}\int_{\mathbb{R}_+^2} q_0^2(z, y_2, t - s_2) q_0^2(y_2, y_1, s_2 - s_1) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{s}\right)^{\frac{1}{2}},\end{aligned}$$

where the second inequality is simply due to the symmetry of both integrands with respect to the diagonal of $[0, t]^2$. The two integrals on the right hand side above are exactly the kind of integrals we treated in the white noise case. Recall that if for $n \geq 1$ and $z, t > 0$,

$$g_n(z, t) := \int \cdots \int_{\Delta_n t \times \mathbb{R}_+^n} q_0^2(z, x_n, t - r_n) \prod_{i=1}^{n-1} q_0^2(x_{i+1}, x_i, r_{i+1} - r_i) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{r}, \tag{3.21}$$

where $\Delta_n t := \{0 < r_1 < r_2 < \cdots < r_n < t\}$, then $g_n(z, t) \leq 2^{-n+1} U(\frac{z}{t}) \leq 2^{-n}$, and hence

$$\mathcal{L}_{2,1} \leq \frac{1}{2}\Gamma_t^2 F_\varepsilon^2 U\left(\frac{z}{t}\right).$$

Case 2: Set $\mathbf{R}_2 := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^4 \mid |x_1 - y_1| > \varepsilon, |x_2 - y_2| \leq \varepsilon\}$ and denote by $\mathcal{L}_{2,2}$ the integral in (3.20) with $(\mathbf{x}, \mathbf{y})$ over $\mathbf{R}_2$ instead of $\mathbb{R}_+^4$. To treat $\mathcal{L}_{2,2}$, we will first integrate with respect to x_1, y_1 and then apply the inequality (6.2) to $\gamma(r_1 - s_1)$, $\gamma(r_2 - s_2)$ and $f(x_2 - y_2)$. We begin with the following immediate estimate:

$$\begin{aligned}\mathcal{L}_{2,2} \leq \frac{1}{2} f(\varepsilon) \int_{[0,t]^4}\int_{R_\varepsilon} &\left(\int_{\mathbb{R}_+} q_0(z, x_{\sigma(2)}, t - r_{\sigma(2)}) q_0(x_{\sigma(2)}, x_{\sigma(1)}, r_{\sigma(2)} - r_{\sigma(1)}) \mathrm{d}x_1\right) \\ &\cdot \left(\int_{\mathbb{R}_+} q_0(z, y_{\varsigma(2)}, t - s_{\varsigma(2)}) q_0(y_{\varsigma(2)}, y_{\varsigma(1)}, s_{\varsigma(2)} - s_{\varsigma(1)}) \mathrm{d}y_1\right) \\ &\cdot f(x_2 - y_2) \mathrm{d}x_2 \mathrm{d}y_2 \cdot \gamma(r_1 - s_1)\gamma(r_2 - s_2) \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}.\end{aligned}$$

Depending on whether $\sigma(1) = 1$ or $\sigma(2) = 1$, the integral in x_1 produces either

$$\int_{\mathbb{R}_+} q_0(z, x_2, t - r_2) q_0(x_2, x_1, r_2 - r_1) \mathrm{d}x_1 \leq q_0(z, x_2, t - r_2)$$

or

$$\int_{\mathbb{R}_+} q_0(z, x_1, t - r_1) q_0(x_1, x_2, r_1 - r_2) \mathrm{d}x_1 = q_0(z, x_2, t - r_2),$$

the latter of which is due to the semigroup property of the kernel q_0. Thus, performing the integral in x_1 within $\mathcal{L}_{2,2}$ will not only preserve the pattern of the integrand but also eliminate the dependence on r_1 from the product of q_0's, while the ordering of the remaining temporal variables will not be disrupted. The integral in y_1 possesses similar properties. Therefore, we get a further estimate as

$$\begin{aligned}\mathcal{L}_{2,2} \leq \frac{1}{2} f(\varepsilon) \int_{[0,t]^4}\int_{R_\varepsilon} &q_0(z, x_2, t - r_2) q_0(z, y_2, t - s_2) f(x_2 - y_2) \mathrm{d}x_2 \mathrm{d}y_2 \\ &\cdot \gamma(r_1 - s_1)\gamma(r_2 - s_2) \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}.\end{aligned}$$

We then switch to the modified Young' inequality (6.2) and apply it repeatedly to $f(x_2-y_2)$, $\gamma(r_1-s_1)$ and $\gamma(r_2-s_2)$. Finally, we arrive at

$$\mathcal{L}_{2,2} \leq \frac{1}{2} f(\varepsilon) F_\varepsilon \Gamma_t^2 t U\left(\frac{z}{t}\right).$$

Case 3: Set $\mathbf{R}_3 := \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}_+^4 \mid |x_1-y_1| \leq \varepsilon, |x_2-y_2| > \varepsilon\}$ and denote by $\mathcal{L}_{2,3}$ the integral in (3.20) with $(\mathbf{x},\mathbf{y})$ over $\mathbf{R}_3$ instead of $\mathbb{R}_+^4$. It is clear that the treatment of $\mathcal{L}_{2,3}$ is exactly the same as that of $\mathcal{L}_{2,2}$, only with the subscripts "1" and "2" swapped everywhere. We get the same estimate:

$$\mathcal{L}_{2,3} \leq \frac{1}{2} f(\varepsilon) F_\varepsilon \Gamma_t^2 t U\left(\frac{z}{t}\right).$$

Case 4: We now consider $\mathbf{R}_4 := \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}_+^4 \mid |x_1-y_1| > \varepsilon, |x_2-y_2| > \varepsilon\}$ and let $\mathcal{L}_{2,4}$ be integral in (3.20) with $(\mathbf{x},\mathbf{y})$ over $\mathbf{R}_4$ instead of $\mathbb{R}_+^4$. In this scenario, $f(x_i-y_i) \leq f(\varepsilon)$ for both $i=1,2$, so we can directly integrate in (x_i,y_i) and apply (6.2) to $\gamma(r_i-s_i)$, $i=1,2$. That is,

$$\mathcal{L}_{2,4} \leq \frac{1}{2} f^2(\varepsilon) \int_{[0,t]^2} \int_{[0,t]^2} \gamma(r_1-s_1)\gamma(r_2-s_2) \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s} \leq \frac{1}{2} f^2(\varepsilon)\Gamma_t^2 t^2.$$

Finally, we combine all the cases above and conclude that

$$\|f_2(\cdot; z,t)\|_2^2 \leq \frac{1}{2}\Gamma_t^2\left(F_\varepsilon^2 U\left(\frac{z}{t}\right) + 2f(\varepsilon)F_\varepsilon t U\left(\frac{z}{t}\right) + f^2(\varepsilon)t^2\right).$$

We now have a bound on $\|f_2(\cdot; z,t)\|_2^2$ that is uniform in z but increasing in t and dependent on F_ε and $f(\varepsilon)$ throughout. In order to tackle $\|f_n(\cdot; z,t)\|_n^2$ for larger values of n, the key observation is that the approach above can be expanded to produce bounds that maintain the same general pattern.

Lemma 3.7. *Let $z,t > 0$ and $n \geq 1$. Then, for every $\varepsilon > 0$, we have*

$$\|f_n(\cdot; z,t)\|_n^2 \leq \frac{(\Gamma_t f(\varepsilon)t)^n}{n!} + 2U\left(\frac{z}{t}\right)\left(\frac{\Gamma_t F_\varepsilon}{2}\right)^n \cdot \sum_{\ell=0}^{n-1} \frac{1}{\ell!}\left(\frac{2f(\varepsilon)t}{F_\varepsilon}\right)^\ell. \tag{3.22}$$

Proof: Following a similar argument as above for the case of $n=2$, we consider the integral (3.5) concerned for $\|f_n(\cdot; z,t)\|_n^2$ over a partition of $(\mathbf{x},\mathbf{y}) \in \mathbb{R}_+^{2n}$ according to the range of $|x_i-y_i|$ with respect to ε for each $i=1,\dots,n$. Assume that there are ℓ pairs of (x_i,y_i) such that $|x_i-y_i| > \varepsilon$ and the rest pairs satisfy the opposite inequality. There are $\binom{n}{\ell}$ number of such configurations, and due to the symmetry in spatial coordinates, we only need to consider the case when $|x_i-y_i| > \varepsilon$ for $i=1,\dots,\ell$, and $|x_j-y_j| \leq \varepsilon$ for $j=\ell+1,\dots,n$. Let us denote by $\mathcal{L}$ the integral in (3.5) where $(\mathbf{x},\mathbf{y})$ is restricted to such a region.

Let us first look at the special case when $\ell = n$, i.e., $|x_i-y_i| > \varepsilon$ for all $i=1,\dots,n$. In this case we can directly integrate with respect to all the spatial variables, which leads to

$$\mathcal{L} \leq \frac{1}{n!} f^n(\varepsilon)\Gamma_t^n t^n.$$

Now we turn to the case when $\ell \leq n-1$. Following the same procedure as above, we will first integrate with respect to (x_i,y_i), $i=1,\dots,\ell$, within $\mathcal{L}$. This step will produce an integral whose integrand is still in the form of two products of q_0 functions in $(\mathbf{x},\mathbf{r})$ and $(\mathbf{y},\mathbf{s})$, respectively, both of which contain the factor $q_0(z,\cdot,t-*)$ and depend on $(x_j,r_j),(y_j,s_j)$ for $j=\ell+1,\dots,n$ through the permutation that orders the temporal variables in the ascending order. Concretely, we have the following intermediate estimate:

$$\mathcal{L} \leq \frac{f^\ell(\varepsilon)}{n!} \int_{[0,t]^{2\ell}} \prod_{i=1}^{\ell} \gamma(r_i - s_i) \cdot \int_{[0,t]^{2(n-\ell)}} \int_{R_\varepsilon^{2(n-\ell)}} \prod_{j=\ell+1}^{n} f(x_j - y_j)\gamma(r_j - s_j)$$
$$\cdot \prod_{j=\ell+1}^{n} q_0(x_{\sigma'(j+1)}, x_{\sigma'(j)}, r_{\sigma'(j+1)} - r_{\sigma'(j)}) q_0(y_{\varsigma'(j+1)}, y_{\varsigma'(j)}, s_{\varsigma'(j+1)} - s_{\varsigma'(j)}) \mathrm{d}\mathbf{x}^\ell \mathrm{d}\mathbf{y}^\ell \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s},$$

where σ' and ς' are the permutations of $\{\ell+1, \dots, n\}$ such that $0 < r_{\sigma'(\ell+1)} < r_{\sigma'(\ell+2)} < \cdots < r_{\sigma'(n)} < t$ and $0 < s_{\varsigma'(\ell+1)} < s_{\varsigma'(\ell+2)} < \cdots < s_{\varsigma'(n)} < t$, and $\mathbf{x}^\ell := (x_{\ell+1}, \dots, x_n)$ and $\mathbf{y}^\ell := (y_{\ell+1}, \dots, y_n)$. We also identify $\sigma'(n+1) = \varsigma'(n+1) = n+1$ and $(x_{n+1}, r_{n+1}) = (y_{n+1}, s_{n+1}) = (z, t)$.

Note that the product of q_0's above does not depend on (r_i, s_i) for $i = 1, \dots, \ell$. Thus, we can apply (6.2) to $\prod_{i=1}^{\ell} \gamma(r_i - s_i)$ and update the estimate above to:

$$\mathcal{L} \leq \frac{f^\ell(\varepsilon)}{n!} \Gamma_t^\ell t^\ell \int_{[0,t]^{2(n-\ell)}} \int_{R_\varepsilon^{2(n-\ell)}} \prod_{j=\ell+1}^{n} f(x_j - y_j)\gamma(r_j - s_j)$$
$$\cdot \prod_{j=\ell+1}^{n} q_0(x_{\sigma'(j+1)}, x_{\sigma'(j)}, r_{\sigma'(j+1)} - r_{\sigma'(j)}) q_0(y_{\varsigma'(j+1)}, y_{\varsigma'(j)}, s_{\varsigma'(j+1)} - s_{\varsigma'(j)}) \mathrm{d}\mathbf{x}^\ell \mathrm{d}\mathbf{y}^\ell \mathrm{d}\mathbf{r}^\ell \mathrm{d}\mathbf{s}^\ell,$$

where $\mathbf{r}^\ell = (r_{\ell+1}, \dots, r_n)$ and $\mathbf{s}^\ell = (s_{\ell+1}, \dots, s_n)$. This integral over $[0,t]^{2(n-\ell)} \times R_\varepsilon^{2(n-\ell)}$ is the exact analog of the integral $\mathcal{L}_{2,1}$ we encountered in the case of $n = 2$, only in $n - \ell$ groups of variables. Therefore, similarly as in the treatment of $\mathcal{L}_{2,1}$, by applying (6.2), this time, to $\prod_{j=\ell+1}^{n} f(x_j - y_j)\gamma(r_j - s_j)$, we arrive at

$$\mathcal{L} \leq \frac{f^\ell(\varepsilon)}{n!} \Gamma_t^\ell t^\ell F_\varepsilon^{n-\ell} \left(\int_{[0,t]^{n-\ell}} \int_{\mathbb{R}_+^{n-\ell}} \prod_{j=\ell+1}^{n} q_0^2(x_{\sigma'(j+1)}, x_{\sigma'(j)}, r_{\sigma'(j+1)} - r_{\sigma'(j)}) \mathrm{d}\mathbf{x}^\ell \mathrm{d}\mathbf{r}^\ell \right)^{\frac{1}{2}}$$
$$\cdot \left(\int_{[0,t]^{n-\ell}} \int_{\mathbb{R}_+^{n-\ell}} \prod_{j=\ell+1}^{n} q_0^2(y_{\varsigma'(j+1)}, y_{\varsigma'(j)}, s_{\varsigma'(j+1)} - s_{\varsigma'(j)}) \mathrm{d}\mathbf{y}^\ell \mathrm{d}\mathbf{s}^\ell \right)^{\frac{1}{2}}$$
$$\leq \frac{f^\ell(\varepsilon)}{n!} \Gamma_t^\ell t^\ell F_\varepsilon^{n-\ell} (n-\ell)! \left(\int_{\{0<r_{\ell+1}<\ldots<r_n<t\}} \int_{\mathbb{R}_+^{n-\ell}} \prod_{j=\ell+1}^{n} q_0^2(x_{j+1}, x_j, r_{j+1} - r_j) \mathrm{d}\mathbf{x}^\ell \mathrm{d}\mathbf{r}^\ell \right)^{\frac{1}{2}}$$
$$\cdot \left(\int_{\{0<s_{\ell+1}<\ldots<s_n<t\}} \int_{\mathbb{R}_+^{n-\ell}} \prod_{j=\ell+1}^{n} q_0^2(y_{j+1}, y_j, s_{j+1} - s_j) \mathrm{d}\mathbf{y}^\ell \mathrm{d}\mathbf{s}^\ell \right)^{\frac{1}{2}}$$
$$\leq \frac{(n-\ell)!}{n!} f^\ell(\varepsilon) \Gamma_t^\ell t^\ell F_\varepsilon^{n-\ell} 2^{-(n-\ell-1)} U\left(\frac{z}{t}\right),$$

where the second inequality is due to the invariance of both integrands under permutations of the variables, and the last inequality follows from the estimate on integrals of the form (3.21) with a "length" $n - \ell$.

Finally, (3.22) follows from multiplying $\mathcal{L}$ by $\binom{n}{\ell}$ and summing over $\ell = 0, \dots, n$. ☐

We are now ready to present the results on the solution to (3.1) under the colored noise.

Theorem 3.8. *Let $\dot{W}$ be the $(f, \gamma)-$colored noise with (f, γ) satisfying Assumptions **(i)** and **(ii)**. Then, $u = \{u(z,t) \mid z > 0, t \geq 0\}$ as in (3.3) is well-defined as a convergent series in $L^2(\Omega)$. Moreover, u is the unique mild solution to (3.1), and for every $z, t > 0$ and $\varepsilon > 0$ sufficiently small such* *that* $\Gamma_t F_\varepsilon < 2$,

$$\mathbb{E}\left[u^2(z,t)\right] \le u_0^2(z,t) + \left(e^{\Gamma_t f(\varepsilon)t} - 1\right) + 2U\left(\frac{z}{t}\right)\exp\left(\frac{2f(\varepsilon)t}{F_\varepsilon}\right)\frac{\Gamma_t F_\varepsilon}{2-\Gamma_t F_\varepsilon}. \tag{3.23}$$

Proof: We only need to verify the convergence of the series in (3.3) to $u(z,t)$ in $L^2(\Omega)$. The rest of the statement can be proved similarly as in Theorem 3.4 in the white noise case. Given $z, t, \varepsilon > 0$, (3.22) implies that

$$\|f_n(\cdot; z,t)\|_n^2 \le \frac{(\Gamma_t f(\varepsilon)t)^n}{n!} + 2U\left(\frac{z}{t}\right)\left(\frac{\Gamma_t F_\varepsilon}{2}\right)^n \exp\left(\frac{2f(\varepsilon)t}{F_\varepsilon}\right).$$

The first term above clearly forms a convergent series for every $z, t, \varepsilon > 0$. Further, according to Assumption **(ii)** on f, given $z, t > 0$, we can always choose $\varepsilon > 0$ sufficiently small such that $\Gamma_t F_\varepsilon < 2$, which will guarantee the summability of the second term above. With such choices of z, t, ε, the desired estimate (3.23) follows from computing the series involving the right hand side above over $n \ge 0$, while identifying $\|f_0(\cdot; z,t)\|_0^2$ as $u_0^2(z,t)$. $\square$

We close this section with a remark on the Malliavin differentiability of $u(z,t)$, although the rest of the work does not concern the Malliavin derivatives.

Remark 3.9. Let $u = \{u(z,t) \mid z > 0, t \ge 0\}$ be the solution to (3.1) constructed above, either under the space-time white noise or under the $(f,\gamma)-$colored noise with (f,γ) satisfying Assumptions **(i)** and **(ii)**. Recall that $\mathbb{D}^{k,2}$, $k \ge 1$, is the subspace of $L^2(\Omega)$ on which the $k-$th order Malliavin derivative operator is defined. We observe that, according to either (3.10) or (3.22), for every $z, t > 0$ and every $k \ge 1$,

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\|f_n(\cdot; z,t)\|_n^2 < \infty,$$

which is sufficient for us to conclude that $u(z,t) \in \mathbb{D}^{k,2}$. In other words, thanks to the geometric bounds (3.10) and (3.22), $u(z,t)$, as an L^2-convergent series at every (z,t), is in fact Malliavin differentiable to all orders.

3.2. *Comparison between $u(z,t)$ and $u_0(z,t)$*. With existence and uniqueness of the solution established, we turn to the behavior of the solution near the boundary of the domain. In the deterministic setting, as illustrated in Chen and Weih-Wadman (2020), to examine the near-boundary properties of the solution to the Kimura equation perturbed by a potential, it is insightful to compare it with the solution to the original (unperturbed) equation. In the stochastic setting, since the solution u is a random field, instead of a point-wise comparison, we examine the ratio between $u(z,t)$ and the deterministic solution $u_0(z,t)$ under the L^2-norm.

Recall that for every $z, t > 0$ and $n \ge 1$, $u_n(z,t) = \mathcal{I}_n(f_n(\cdot; z,t))$ with f_n as defined in (3.4). We will study

$$\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right] = \sum_{n=0}^{\infty}\mathbb{E}\left[\left(\frac{u_n(z,t)}{u_0(z,t)}\right)^2\right]. \tag{3.24}$$

Intuitively speaking, similarly as in the deterministic case, as long as z is away from 0, one expects that $u(z,t)$ is "comparable" to $u_0(z,t)$ at least for sufficiently small t since the perturbation produced by the stochastic potential only becomes more effective as time grows. We will turn this intuition into a rigorous statement in the next theorem.

Theorem 3.10. *Consider the stochastic Kimura equation* (3.1) *with $\dot{W}$ being either the space-time white noise or the $(f,\gamma)-$colored noise with (f,γ) satisfying Assumptions **(i)** and **(ii)**. Let*

$u=\{u(z,t)\mid z>0,t\geq 0\}$ *be the unique mild solution to* (3.1). *Then, for every* $z>0$,

$$\lim_{t\searrow 0}\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}-1\right)^2\right]=0 \tag{3.25}$$

and the convergence is uniform in z *bounded away from* 0.

Proof: For $z>0$ fixed, as $t\searrow 0$, $\frac{z}{t}\nearrow\infty$. Thus, by (3.10),

$$\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}-1\right)^2\right]=\sum_{n=1}^{\infty}\mathbb{E}\left[\left(\frac{u_n(z,t)}{u_0(z,t)}\right)^2\right]\leq\frac{1}{(1-e^{-1})^2}\sum_{n=1}^{\infty}\mathbb{E}\left[u_n^2(z,t)\right].$$

In the case when $\dot{W}$ is the space-time white noise, by (3.10), the right hand above is bounded by a constant multiple of $U\left(\frac{z}{t}\right)$. When $\dot{W}$ is the colored noise with (f,γ) satisfying Assumptions (i) and (ii), by (3.23), for some $\varepsilon>0$, the above is bounded by a constant multiple of

$$\left(e^{\Gamma_t f(\varepsilon)t}-1\right)+2U\left(\frac{z}{t}\right)\exp\left(\frac{2f(\varepsilon)t}{F_\varepsilon}\right)\frac{\Gamma_t F_\varepsilon}{2-\Gamma_t F_\varepsilon}.$$

Recall that $U(x)=e^{-x}I_0(x)-\frac{1}{2}e^{-2x}$, and it is easy to check that $e^{-x}I_0(x)\searrow 0$ as $x\nearrow\infty$, from where it follows that $\lim_{t\searrow 0}\sup_{z>c}U\left(\frac{z}{t}\right)=0$ for all $c>0$. Therefore, in either the white noise or the colored noise case, (3.25) and the uniform convergence statement hold. □

We want to point out that the convergence established in Theorem 3.10 is weaker than results expected in general in the case with deterministic potential. In fact, in the deterministic setting, when the potential is (locally) bounded, we would expect the quantity concerned in (3.25) to converge to 0, as $t\searrow 0$, uniformly in spatial variables all the way to the boundary, just like the result (2.5) reviewed in §2.1. In other words, in the deterministic setting, $u(z,t)$ is uniformly "close" to $u_0(z,t)$ in a neighborhood of the boundary for small time. However, in the stochastic setting, $u(z,t)$ and $u_0(z,t)$ do not enjoy such a property in general, and we will demonstrate in the example below that $u(z,t)$ can be intrinsically different from $u_0(z,t)$ as $z\searrow 0$, no matter how small t is.

Example 3.11. Let $u=\{u(z,t)\mid z>0,t\geq 0\}$ be the unique mild solution to (3.1) with $\dot{W}$ being the space-time white noise. By the fact that $I_1(x)\geq x/2$ for every $x>0$, we have

$$\begin{aligned}\mathbb{E}\left[u_1^2(z,t)\right]&=\int_0^t\int_0^\infty q_0^2(z,w,t-\tau)\left(1-e^{-\frac{w}{\tau}}\right)^2\mathrm{d}w\mathrm{d}\tau\\&\geq z^2\int_0^t\frac{e^{-\frac{2z}{t-\tau}}}{(t-\tau)^4}\int_0^\infty e^{-\frac{2w}{t-\tau}}\left(1-e^{-\frac{w}{\tau}}\right)^2\mathrm{d}w\mathrm{d}\tau\\&=\frac{z^2}{2t}\int_0^t\frac{e^{-\frac{2z}{t-\tau}}}{(t-\tau)(t+\tau)}\mathrm{d}\tau\\&\geq\frac{z^2}{4t^2}\int_0^t\frac{e^{-\frac{2z}{t-\tau}}}{t-\tau}\mathrm{d}\tau=\frac{z^2}{4t^2}\int_{2z/t}^\infty\frac{e^{-y}}{y}\mathrm{d}y.\end{aligned}$$

Therefore, for any $t>0$ fixed, we have

$$\lim_{z\searrow 0}\mathbb{E}\left[\left(\frac{u_1(z,t)}{u_0(z,t)}\right)^2\right]\geq\lim_{z\searrow 0}\frac{\frac{z^2}{4t^2}\int_{2z/t}^\infty\frac{e^{-y}}{y}\mathrm{d}y}{\left(1-e^{-z/t}\right)^2}=\infty,$$

which implies that the ratio in (3.24) also becomes unbounded as $z\searrow 0$, no matter how small t is.

The computation above suggests that the perturbation due to the stochastic noise is most significant near the boundary, and this is not surprising. Heuristically speaking, due to the degeneracy at the boundary, the "effect" of diffusion is weak in the neighborhood of 0; as a consequence, for small values of z, as soon as $t > 0$, the stochastic noise, as a potential perturbation, drives the stochastic (perturbed) solution $u(z,t)$ to deviate from the deterministic (non-perturbed) solution $u_0(z,t)$ in a substantial way. Therefore, for the stochastic Kimura equation in the current form (3.1), we do not expect $\frac{u(z,t)}{u_0(z,t)}$ can be controlled uniformly in z all the way to the boundary 0. In order to re-produce a ratio bound, for the stochastic Kimura equation, analogous to (2.5) in the deterministic setting, we need to "tame" the noise near the boundary in a proper way.

4. Stochastic Kimura Equation with Degenerate Noise

As mentioned above, the stochastic noise in (3.1) is too strong near the boundary for $u(z,t)$ and $u_0(z,t)$ to be comparable for z in a neighborhood of 0, even when t is small. To remedy this issue, we introduce a degenerate coefficient of order $\beta > 0$ to $\dot{W}$ at the boundary and consider the following SPDE, again, equipped with constant initial data and the Dirichlet boundary condition:

$$\begin{cases} \partial_t u\left(z,t\right) & = z\partial_z^2 u\left(z,t\right) + \hat{z}^{\beta} u\left(z,t\right)\dot{W}\left(z,t\right) & \text{for } z,t>0, \\ u\left(z,0\right) & = 1 & \text{for } z>0, \\ u\left(0,t\right) & = 0 & \text{for } t\geq 0, \end{cases} \tag{4.1}$$

where $\hat{z} := z \wedge 1$. It can be shown similarly as in §3.1 that (4.1) admits a unique mild solution $u = \{u\left(z,t\right) \mid z > 0, t \geq 0\}$, and for every (z,t), $u\left(z,t\right)$ also possesses the series representation

$$u\left(z,t\right) = \sum_{n=0}^{\infty} u_n\left(z,t\right)$$

where the convergence is in $L^2\left(\Omega\right)$. Similarly as in the previous chapter, for every $n \geq 1$, the L^2-norm of $u_n\left(z,t\right)$ can be explicitly computed by

$$\begin{aligned} &\mathbb{E}\left[u_n^2\left(z,t\right)\right] \\ &\quad = \frac{1}{n!}\int_{[0,t]^{2n}}\int_{\mathbb{R}_+^{2n}} q_0(z,x_{\sigma(n)},t-r_{\sigma(n)})\cdot\prod_{i=1}^{n-1} q_0\left(x_{\sigma(i+1)},x_{\sigma(i)},r_{\sigma(i+1)}-r_{\sigma(i)}\right) \\ &\qquad\qquad \cdot q_0(z,y_{\varsigma(n)},t-s_{\varsigma(n)})\cdot\prod_{i=1}^{n-1} q_0\left(y_{\varsigma(i+1)},y_{\varsigma(i)},s_{\varsigma(i+1)}-s_{\varsigma(i)}\right) \\ &\qquad\qquad \cdot u_0(x_{\sigma(1)},r_{\sigma(1)})u_0(y_{\varsigma(1)},s_{\varsigma(1)})\cdot\prod_{i=1}^{n} f\left(x_i-y_i\right)\hat{x}_i{}^{\beta}\hat{y}_i{}^{\beta}\gamma\left(r_i-s_i\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s}, \end{aligned} \tag{4.2}$$

where σ and ς are the permutations of $\{1,2,\ldots,n\}$ such that $0 < r_{\sigma(1)} < r_{\sigma(2)} < \cdots < r_{\sigma(n)} < t$ and $0 < s_{\varsigma(1)} < s_{\varsigma(2)} < \cdots < s_{\varsigma(n)} < t$.

In this section, under the setting of (4.1), we will derive an accurate L^2-estimate on $u\left(z,t\right)$ for z near the boundary 0 and for t small through the comparison with $u_0(z,t)$ (§4.1). We further establish higher moment bounds for $u\left(z,t\right)$ (§4.2) and the continuity of $u\left(z,t\right)$ in z all the way to the boundary (§4.3).

4.1. *Comparison between $u\left(z,t\right)$ and $u_0\left(z,t\right)$, revisited.* Let us re-examine the ratio between $u(z,t)$ and $u_0(z,t)$ in the degenerate noise setting. We will prove that so long as $\beta > 0$, $u\left(z,t\right)$ and $u_0\left(z,t\right)$ are "comparable" in a proper sense. As before, we begin our analysis in the case of the space-time white noise and then extend it to the case of colored noise.

4.1.1. *Under Space-Time White Noise.* Our strategy for the white noise case is similar to that in the proof of the convergence of (3.3) in $L^2(\Omega)$. Namely, we use (3.7) to recursively compute a bound with certain repetitive pattern. We first make use of the following technical lemma for $n = 1$.

Lemma 4.1. *Given $z, \beta > 0$, $t \in (0,1)$, and $\alpha \in (0, \beta \wedge \frac{1}{2}]$, we define*

$$K(z,t;2\beta-\alpha)(\tau) := \int_0^\infty \hat{w}^{2\beta-\alpha} q_0^2(z,w,t-\tau) \frac{u_0^2(w,\tau)}{u_0^2(z,t)} \mathrm{d}w \text{ for } 0<\tau<t. \tag{4.3}$$

Then, there exists $C_{\alpha,\beta} > 0$, such that for every $0<\tau<t$,

$$K(z,t;2\beta-\alpha)(\tau) \le C_{\alpha,\beta}(t-\tau)^{\alpha-1}\Pi(z,t), \tag{4.4}$$

where $\Pi(z,t) := 1$ if $0<z<t$ and $\Pi(z,t) := z^{-\alpha}$ if $0<t\le z$.

Proof: Let z, β, t, α be as above and fixed. Throughout, the value of the constant C may change from line to line, but it remains solely dependent on α, β.

When $0<t\le z$, we observe that $u_0(z,t) \ge 1-e^{-1}$ and hence for every $\tau \in (0,t)$,

$$\begin{aligned} K(z,t;2\beta-\alpha)(\tau) &\le \frac{1}{(1-e^{-1})^2}\int_0^\infty \hat{w}^{2\beta-\alpha} q_0^2(z,w,t-\tau)u_0^2(w,\tau)\mathrm{d}w \\ &\le C\int_0^\infty q_0^2(z,w,t-\tau)\mathrm{d}w \le Cz^{-\alpha}(t-\tau)^{\alpha-1}. \end{aligned}$$

The last inequality above is exactly (4.17), which is derived from an estimate on the concerned generalized hypergeometric function. We will delay the proof of this inequality until Lemma 4.7.

Now we focus on the case when $0<z<t$ and hence $u_0(z,t) \ge \frac{z}{2t}$. We further split $K(z,t;2\beta-\alpha)(\tau)$ into two terms:

$$K_1 := \int_0^{\frac{(t-\tau)^2}{z}} \hat{w}^{2\beta-\alpha} q_0^2(z,w,t-\tau)\frac{u_0^2(w,\tau)}{u_0^2(z,t)}\mathrm{d}w$$

and

$$K_2 := \int_{\frac{(t-\tau)^2}{z}}^\infty \hat{w}^{2\beta-\alpha} q_0^2(z,w,t-\tau)\frac{u_0^2(w,\tau)}{u_0^2(z,t)}\mathrm{d}w.$$

We will treat K_1 and K_2 in further separate cases.

Case $\tau \le t/2$. In this case, we apply (2.3) to bound K_1 as

$$\begin{aligned} K_1 &\le \frac{\frac{z^2}{(t-\tau)^4}\int_0^{\frac{(t-\tau)^2}{z}} \hat{w}^{2\beta-\alpha} e^{-\frac{2(\sqrt{z}-\sqrt{w})^2}{t-\tau}}\mathrm{d}w}{\frac{z^2}{4t^2}} \\ \left(\text{by } t-\tau \ge \frac{t}{2}\right) &\le \frac{C}{(t-\tau)^2}\int_0^{\frac{(t-\tau)^2}{z}} \hat{w}^{2\beta-\alpha} e^{-\frac{2(\sqrt{z}-\sqrt{w})^2}{t-\tau}}\mathrm{d}w \\ &= \frac{C}{(t-\tau)^{\frac{3}{2}}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_0^{\frac{t-\tau}{\sqrt{z}}} \hat{u}^{4\beta-2\alpha} u e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right). \end{aligned}$$

By (6.1), if $z \le t-\tau$, then

$$\begin{aligned} K_1 &\le \frac{C}{(t-\tau)^{\frac{3}{2}}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_0^{\frac{t-\tau}{\sqrt{z}}} u^{4\beta-2\alpha+1} e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right) \\ &\le C(t-\tau)^{2\beta-\alpha-1} \le C(t-\tau)^{\alpha-1}, \end{aligned}$$

and if $z > t-\tau$, then we proceed as

$$\begin{aligned}
K_1 &\le \frac{C}{\sqrt{t-\tau}}\frac{1}{t-\tau}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_0^{\frac{t-\tau}{\sqrt{z}}}\hat{u}^{4\beta-2\alpha}ue^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right)\\
\left(\text{by } \frac{1}{t-\tau}\le\frac{1}{u\sqrt{z}}\right) &\le \frac{C}{\sqrt{z}\sqrt{t-\tau}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_0^{\frac{t-\tau}{\sqrt{z}}}\hat{u}^{4\beta-2\alpha}e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right)\\
&\le \frac{C\hat{z}^{2\beta-\alpha}}{t-\tau}\left(\frac{t-\tau}{z}\right)^{\frac12}\le\frac{C\hat{z}^{2\beta-\alpha}}{t-\tau}\left(\frac{t-\tau}{z}\right)^{\alpha}\\
&\le C\hat{z}^{2\beta-2\alpha}(t-\tau)^{\alpha-1}\le C(t-\tau)^{\alpha-1},
\end{aligned}$$

where in both inequalities we make use of the assumptions that $0<\alpha\le\beta\wedge\frac12$ and $0<z,t-\tau<1$.

For K_2, we apply (2.4) to get

$$\begin{aligned}
K_2 &\le \frac{\frac{\sqrt{z}}{t-\tau}\int_{\frac{(t-\tau)^2}{z}}^{\infty}\hat{w}^{2\beta-\alpha}w^{-\frac32}e^{-\frac{2(\sqrt{z}-\sqrt{w})^2}{t-\tau}}\mathrm{d}w}{\frac{z^2}{4t^2}}\\
\left(\text{by } t-\tau\ge\frac{t}{2}\right) &\le Cz^{-\frac32}(t-\tau)\int_{\frac{(t-\tau)^2}{z}}^{\infty}\hat{w}^{2\beta-\alpha}w^{-\frac32}e^{-\frac{2(\sqrt{z}-\sqrt{w})^2}{t-\tau}}\mathrm{d}w\\
&= Cz^{-\frac32}(t-\tau)^{\frac32}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_{\frac{t-\tau}{\sqrt{z}}}^{\infty}\hat{u}^{4\beta-2\alpha}u^{-2}e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right)\\
\left(\text{by } u^{-2}\le\frac{z}{(t-\tau)^2}\right) &\le \frac{C}{\sqrt{z}\sqrt{t-\tau}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_{\frac{t-\tau}{\sqrt{z}}}^{\infty}\hat{u}^{4\beta-2\alpha}e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right).
\end{aligned}$$

If $z>t-\tau$, then similarly as above, again by (6.1), we have that $K_2\le C(t-\tau)^{\alpha-1}$; if $z\le t-\tau$, we further derive

$$\begin{aligned}
K_2 &\le \frac{C}{\sqrt{z}(t-\tau)^{\frac32}}(t-\tau)\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_{\frac{t-\tau}{\sqrt{z}}}^{\infty}\hat{u}^{4\beta-2\alpha}e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right)\\
(\text{by } t-\tau\le u\sqrt{z}) &\le \frac{C}{(t-\tau)^{\frac32}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_{\frac{t-\tau}{\sqrt{z}}}^{\infty}u^{4\beta-2\alpha+1}e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right)\\
&\le C(t-\tau)^{2\beta-\alpha-1}\le C(t-\tau)^{\alpha-1}.
\end{aligned}$$

Case $\tau>t/2$. The steps to estimate K_1 and K_2 are exactly the same as in the previous case, but here we will make use of the fact that $u_0(w,\tau)\le\frac{w}{\tau}$ for every $w,\tau>0$ throughout. We will omit the detailed arguments but only write down the result of each step. We start with K_1:

$$\begin{aligned}
K_1 &\le \frac{\frac{z^2}{(t-\tau)^4}\int_0^{\frac{(t-\tau)^2}{z}}\hat{w}^{2\beta-\alpha}e^{-\frac{2(\sqrt{z}-\sqrt{w})^2}{t-\tau}}\frac{w^2}{\tau^2}\mathrm{d}w}{\frac{z^2}{4t^2}}\\
&\le \frac{C}{(t-\tau)^{\frac72}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_0^{\frac{t-\tau}{\sqrt{z}}}\hat{u}^{4\beta-2\alpha}u^5e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right).
\end{aligned}$$

If $z \le t-\tau$, then $K_1 \le C(t-\tau)^{\alpha-1}$; if $z > t-\tau$, then

$$\begin{aligned} K_1 &\le \frac{C}{\sqrt{t-\tau}}\left(\frac{(t-\tau)^{-3}}{\sqrt{\pi(t-\tau)}}\int_0^{\frac{t-\tau}{\sqrt{z}}} \hat{u}^{4\beta-2\alpha}u^5 e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right) \\ &\le \frac{Cz^{-\frac{3}{2}}}{\sqrt{t-\tau}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_0^{\frac{t-\tau}{\sqrt{z}}} \hat{u}^{4\beta-2\alpha}u^2 e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right) \\ &\le \frac{C}{\sqrt{t-\tau}}\hat{z}^{2\beta-\alpha}z^{-\frac{1}{2}} \le C(t-\tau)^{\alpha-1}. \end{aligned}$$

Next, for K_2, we have

$$\begin{aligned} K_2 &\le \frac{\dfrac{\sqrt{z}}{t-\tau}\displaystyle\int_{\frac{(t-\tau)^2}{z}}^{\infty} \hat{w}^{2\beta-\alpha}\frac{w^{\frac{1}{2}}}{\tau^2}e^{-\frac{2(\sqrt{z}-\sqrt{w})^2}{t-\tau}}\mathrm{d}w}{\dfrac{z^2}{4t^2}} \\ &\le \frac{Cz^{-\frac{3}{2}}}{\sqrt{t-\tau}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_{\frac{t-\tau}{\sqrt{z}}}^{\infty} \hat{u}^{4\beta-2\alpha}u^2 e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right). \end{aligned}$$

If $z > t-\tau$, then $K_2 \le C(t-\tau)^{\alpha-1}$; if $z \le t-\tau$, then

$$\begin{aligned} K_2 &\le \frac{Cz^{-\frac{3}{2}}}{(t-\tau)^{\frac{7}{2}}}\left(\frac{(t-\tau)^3}{\sqrt{\pi(t-\tau)}}\int_{\frac{t-\tau}{\sqrt{z}}}^{\infty} \hat{u}^{4\beta-2\alpha}u^2 e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right) \\ &\le \frac{C}{(t-\tau)^{\frac{7}{2}}}\left(\frac{1}{\sqrt{\pi(t-\tau)}}\int_{\frac{t-\tau}{\sqrt{z}}}^{\infty} u^{4\beta-2\alpha+5} e^{-\frac{2(\sqrt{z}-u)^2}{t-\tau}}\mathrm{d}u\right) \le C(t-\tau)^{\alpha-1}. \end{aligned}$$

Combining all these case, we have shown that $K(z,t;2\beta-\alpha)(\tau) = K_1 + K_2 \le C(t-\tau)^{\alpha-1}$. ☐

Regarding the estimates derived above, we make the following observations which will be useful later.

Remark 4.2. For every $z, t, \beta > 0$ and $0 < \alpha \le \beta \wedge \frac{1}{2}$, if we define

$$\tilde{K}\left(z,t;\beta-\frac{\alpha}{2}\right)(\tau) := \int_0^{\infty} \hat{w}^{\beta-\frac{\alpha}{2}} q_0(z,w,t-\tau)\frac{u_0(w,\tau)}{u_0(z,t)}\mathrm{d}w \text{ for } 0 < \tau < t, \tag{4.5}$$

then following the proof of Lemma 3.5 step-by-step with minor changes, we obtain

$$\tilde{K}\left(z,t;\beta-\frac{\alpha}{2}\right)(\tau) \le \tilde{C}_{\alpha,\beta} \text{ for every } 0 < \tau < t, \tag{4.6}$$

where, again, the constant $\tilde{C}_{\alpha,\beta}$ only depends on β and α. This estimate on $\tilde{K}\left(z,t;\beta-\frac{\alpha}{2}\right)(\tau)$ will be useful in the next section in treating the ratio $u(z,t)/u_0(z,t)$ under the colored noise.

Since $\hat{w}^{2\beta} \le \hat{w}^{2\beta-\alpha}$ for every $w > 0$, By integrating the right hand side of (4.4) in τ in $(0,t)$ and modifying the constant $C_{\alpha,\beta}$ if necessary, we immediately obtain the following corollary.

Corollary 4.3. *Let β and α be the same as in Lemma 4.1 and fixed. Let u_1 be as in (3.11). Then, there exists constant $C_{\alpha,\beta} > 0$ such that for every $z, t > 0$,*

$$\mathbb{E}\left[\left(\frac{u_1(z,t)}{u_0(z,t)}\right)^2\right] \le \int_0^t K(z,t;2\beta-\alpha)(\tau)\,\mathrm{d}\tau \le C_{\alpha,\beta}t^{\alpha}\Pi(z,t). \tag{4.7}$$

By iterating the estimate (4.7), we can prove the main result for this section which states that the solution to the stochastic Kimura equation (4.1) remains comparable to the deterministic solution under L^2-norm uniformly in all $z > 0$ and all t sufficiently small.

Theorem 4.4. *Let $\beta > 0$. Assume that $\dot{W}$ is the space-time white noise and*

$$u = \{u(z,t) \mid z > 0, t \geq 0\}$$

is the unique mild solution to (4.1). Then, there exists $T = T_\beta \in (0,1)$ such that

$$\sup_{z>0, t\in[0,T]} \mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right] < \infty. \tag{4.8}$$

Proof: First, we observe that by (3.12),

$$\sup_{0<t\leq z} \mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right] < \frac{2}{(1-e^{-1})^2}. \tag{4.9}$$

Therefore, we only need to focus on the case when $0 < z < t$. In light of (3.24), we proceed to show that for $z, \beta > 0$, $t \in (0,1)$ and $\alpha \in (0, \beta \wedge \frac{1}{2}]$, for every $n \geq 1$,

$$\mathbb{E}\left[\left(\frac{u_n(z,t)}{u_0(z,t)}\right)^2\right] \leq C^n_{\alpha,\beta} t^{n\alpha} \Pi(z,t). \tag{4.10}$$

The base case $n = 1$ is given by Corollary 4.3. Next, assume that (4.10) holds for $n-1$ for some $n \geq 2$. By (4.10), we have

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{u_n(z,t)}{u_0(z,t)}\right)^2\right] &= \frac{\int_0^t \int_0^\infty \hat{w}^{2\beta} q_0^2(z,w,t-\tau)\, \mathbb{E}\left[u^2_{n-1}(w,\tau)\right] \mathrm{d}w\mathrm{d}\tau}{u_0^2(z,t)} \\
&= \frac{\int_0^t \left(\int_0^\tau + \int_\tau^\infty\right) \hat{w}^{2\beta} q_0^2(z,w,t-\tau)\, \mathbb{E}\left[u^2_{n-1}(w,\tau)\right] \mathrm{d}w\mathrm{d}\tau}{u_0^2(z,t)} \\
&\leq C^{n-1}_{\alpha,\beta} \int_0^t \tau^{(n-1)\alpha} \int_0^\tau \hat{w}^{2\beta} q_0^2(z,w,t-\tau) \frac{u_0^2(w,\tau)}{u_0^2(z,t)} \mathrm{d}w\mathrm{d}\tau \\
&+ C^{n-1}_{\alpha,\beta} \int_0^t \tau^{(n-1)\alpha} \int_\tau^\infty \hat{w}^{2\beta-\alpha} q_0^2(z,w,t-\tau) \frac{u_0^2(w,\tau)}{u_0^2(z,t)} \mathrm{d}w\mathrm{d}\tau \\
&\leq C^{n-1}_{\alpha,\beta} t^{(n-1)\alpha} \int_0^t K(z,t;2\beta-\alpha)(\tau)\, \mathrm{d}\tau \leq C^n_{\alpha,\beta} t^{n\alpha} \Pi(z,t).
\end{aligned}$$

Hence, (4.10) holds for $n \geq 1$. Therefore, whenever $C_{\alpha,\beta} t^\alpha < 1$ and $0 < z < t$, by substituting (4.10) into (3.24),

$$\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right] \leq \sum_{n=0}^\infty C^n_{\alpha,\beta} t^{n\alpha} = \frac{1}{1 - C_{\alpha,\beta} t^\alpha}. \tag{4.11}$$

Set $\alpha = \beta \wedge \frac{1}{2}$ and choose $T > 0$ such that $C_{\alpha,\beta} T^\alpha < 1$. Combining (4.9) and (4.11) it follows that for every $t \in [0,T]$ and $0 < z < t$,

$$\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right] \leq \frac{1}{1 - C_{\alpha,\beta} t^\alpha} \leq \frac{1}{1 - C_{\alpha,\beta} T^\alpha},$$

which leads to (4.8). □

4.1.2. *Under Colored Noise.* As before, we seek to bootstrap the results from the case of space-time white noise. Fortunately, the same techniques we developed in §3.1.2 are still applicable to us in this part. Once again, we build to the main result by first treating the base case $n = 1$.

Lemma 4.5. *Let $\beta, \varepsilon > 0$ and $0 < \alpha \leq \beta \wedge \frac{1}{2}$ and assume that (f, γ) satisfies Assumptions **(i)** and **(ii)**. Given $z > 0$, $t \in (0,1)$, we have that*

$$\mathbb{E}\left[\left(\frac{u_1(z,t)}{u_0(z,t)}\right)^2\right] \leq \int_{[0,t]^2} \gamma(r-s) \int_{\mathbb{R}_+^2} \hat{x}^{\beta-\frac{\alpha}{2}} \hat{y}^{\beta-\frac{\alpha}{2}} f(x-y) \cdot q_0(z,x,t-r)\, q_0(z,y,t-s) \frac{u_0(x,r)\, u_0(y,s)}{u_0^2(z,t)} \mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}s \leq \Gamma_t (F_\varepsilon + f(\varepsilon))\, C_{\alpha,\beta} t^\alpha \tilde{\Pi}(z,t), \tag{4.12}$$

where F_ε and Γ_t are as defined in (3.15) and (3.16), for every $z, t > 0$, $\tilde{\Pi}(z,t) := \Pi(z,t) \vee 1$, and the constant $C_{\alpha,\beta}$ may be greater than in Corollary 4.3.

Proof: The first inequality is due to the trivial fact that $\hat{x}^\beta \hat{y}^\beta \leq \hat{x}^{\beta-\frac{\alpha}{2}} \hat{y}^{\beta-\frac{\alpha}{2}}$. Although here we will not follow the same recursive steps as in the space-time white noise case, we do need to use the estimates established in the previous case, and hence we still need to invoke the parameter α in the estimates. To prove the second inequality above, let R_ε be as defined in (3.14), and denote by V_1 the integral (4.12) with (x,y) over R_ε instead of $\mathbb{R}_+^2$ and by V_2 the same integral with (x,y) over $R_\varepsilon^\complement$ instead of $\mathbb{R}_+^2$. We will again treat the two integrals separately. First, taking K to be as in (4.3) and applying (6.2) and Lemma 4.1, we see

$$V_1 \leq \Gamma_t F_\varepsilon \int_0^t K(z,t;2\beta-\alpha)(\tau)\mathrm{d}\tau \leq \Gamma_t F_\varepsilon C_{\alpha,\beta} t^\alpha \Pi(z,t). \tag{4.13}$$

For the second term, recall that for $\tilde{K}$ as defined in (4.5), by (6.2) and (4.6), we have

$$V_2 \leq \Gamma_t f(\varepsilon) \int_0^t \tilde{K}^2\left(z,t;\beta-\frac{\alpha}{2}\right)(\tau)\mathrm{d}\tau \leq \Gamma_t f(\varepsilon) \tilde{C}_{\alpha,\beta} t. \tag{4.14}$$

Combining (4.13) and (4.14) leads to (4.12). □

Before we move on to treat $\mathbb{E}\left[\left(\frac{u_n(z,t)}{u_0(z,t)}\right)^2\right]$ for general $n \geq 1$, we first recall Example 3.6, which motivated the need for a different L^2-bound in case one could not guarantee the existence of ε such that the concerned series in §3.1.2 is convergent for all time t. However, the mere issue of convergence will not occur here by virtue of only studying the solution u for small values of t. Given $\varepsilon > 0$ from a proper range, whenever needed, we may assume that t is sufficiently small to ensure the convergence of a certain series.

Theorem 4.6. *Let $\beta, \varepsilon > 0$. Assume that $\dot{W}$ is the $(f,\gamma)-$colored noise with (f,γ) satisfying Assumptions **(i)** and **(ii)** and $u = \{u(z,t) \mid z > 0, t \geq 0\}$ is the unique mild solution to (4.1). Then there exists $T = T_{\beta,\varepsilon} \in (0,1)$ such that*

$$\sup_{z>0, t\in[0,T]} \mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right] < \infty. \tag{4.15}$$

Proof: Choose and fix $\beta > 0$, $0 < \alpha \leq \beta \wedge \frac{1}{2}$ and $\varepsilon > 0$ sufficiently small such that $F_\varepsilon \Gamma_1 < 2$. Again, by (3.23), whenever $0 < t \leq z$,

$$\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right] \leq \left(1-e^{-1}\right)^2 \left[1 + \left(e^{\Gamma_t f(\varepsilon) t} - 1\right) + 2U\left(\frac{z}{t}\right) \exp\left(\frac{2f(\varepsilon)t}{F_\varepsilon}\right) \frac{\Gamma_t F_\varepsilon}{2 - \Gamma_t F_\varepsilon}\right]$$

which is bounded in all $t \in (0,1)$ and $z \geq t$. Therefore, we only need to treat the case when $0 < z < t < 1$.

We will follow a similar procedure as in the proof of Lemma 3.7 to establish a bound on $\mathbb{E}\left[\left(\frac{u_n(z,t)}{u_0(z,t)}\right)^2\right]$ in the colored noise setting. Again, we approach the integral formula of $\mathbb{E}\left[u_n^2(z,t)\right]$ in (4.2) by considering the specific region where $|x_i - y_i| > \varepsilon$ for $i = 1, \ldots, \ell$ and $|x_j - y_j| \le \varepsilon$ for $j = \ell+1, \ldots, n$ for some $\ell = 0, 1, \ldots, n$, and denote by $\mathcal{L}$ the integral in (4.2) with $(\mathbf{x}, \mathbf{y})$ restricted in this region. First, we treat the case $\ell = n$. By replacing $\hat{x}_i^\beta \hat{y}_i^\beta$ with its upper bound 1 and integrating with respect to (x_i, y_i) for all $i = 1, \ldots, n$, through the semigroup property of the kernel q_0 in conjunction with (6.2), (4.5) and (4.6), we obtain that

$$\frac{\mathcal{L}}{u_0^2(z,t)} \le \frac{1}{n!} f^n(\varepsilon) \Gamma_t^n \int_{[0,t]^n} \tilde{K}^2\Big(z, t; \beta - \frac{\alpha}{2}\Big)(\min(r_1, \ldots, r_n)) \mathrm{d}\mathbf{r} \le \frac{1}{n!} f^n(\varepsilon) \Gamma_t^n t^n \tilde{C}_{\alpha,\beta},$$

where we write $r_{\sigma(1)} = \min(r_1, \ldots, r_n)$. Next, for the remaining scenarios of ℓ, we argue in the same way as in Lemma 3.7 and invoke the estimate in (4.10) to see that

$$\frac{\mathcal{L}}{u_0^2(z,t)} \le \frac{1}{n!} \Gamma_t^n f^\ell(\varepsilon) F_\varepsilon^{n-\ell} t^\ell (n-\ell)! C_{\alpha,\beta}^{n-\ell} t^{(n-\ell)\alpha} \Pi(z,t) \,.$$

Note that $\Pi(z,t) = 1$ when $0 < z < t$, and without loss generality, we may assume $\tilde{C}_{\alpha,\beta} \ge 1$. Summarizing all of the above, we arrive at

$$\begin{aligned} \mathbb{E}\left[\left(\frac{u_n(z,t)}{u_0(z,t)}\right)^2\right] &\le \tilde{C}_{\alpha,\beta} \left(C_{\alpha,\beta} \Gamma_t F_\varepsilon t^\alpha\right)^n \sum_{\ell=0}^n \frac{1}{\ell!} \left(\frac{t^{1-\alpha} f(\varepsilon)}{C_{\alpha,\beta} F_\varepsilon}\right)^\ell \\ &\le \tilde{C}_{\alpha,\beta} \left(C_{\alpha,\beta} \Gamma_t F_\varepsilon t^\alpha\right)^n \exp\left(\frac{t^{1-\alpha} f(\varepsilon)}{C_{\alpha,\beta} F_\varepsilon}\right). \end{aligned} \tag{4.16}$$

With $\varepsilon > 0$ fixed and $\alpha = \beta \wedge \frac{1}{2}$, we can always choose $T > 0$ such that $C_{\alpha,\beta} \Gamma_T F_\varepsilon T^\alpha < 1$. Then, whenever $t \in [0, T]$ and $0 < z < t$,

$$\mathbb{E}\left[\left(\frac{u(z,t)}{u_0(z,t)}\right)^2\right] \le \tilde{C}_{\alpha,\beta} \exp\left(\frac{t^{1-\alpha} f(\varepsilon)}{C_{\alpha,\beta} F_\varepsilon}\right) \sum_{n=0}^\infty \left(C_{\alpha,\beta} \Gamma_t F_\varepsilon t^\alpha\right)^n \le \frac{\tilde{C}_{\alpha,\beta} \exp\left(\frac{T^{1-\alpha} f(\varepsilon)}{C_{\alpha,\beta} F_\varepsilon}\right)}{1 - C_{\alpha,\beta} \Gamma_T F_\varepsilon T^\alpha},$$

from which (4.15) follows. □

4.2. *Higher Moments of Solution.* So far, our discussions on the solution to the Kimura equations, either (3.1) or (4.1), under either the space-time white noise or the colored noise, have been focusing on L^2-theory, and all the bounds and estimates we have derived are based on the L^2-norm of the solution. Naturally, one would inquire about properties of the solution involving L^p-norms with $p > 2$, but unlike for the space $L^2(\Omega)$, there is no isometry or expansion property in $L^p(\Omega)$ that allows us to directly treat higher moments of the solution. Nevertheless, for a solution u to either (3.1) or (4.1) with the Wiener chaos expansion given by (3.3), (2.8) allows us to return to the treatment of the L^2-norm at the cost of an additional geometric term depending on p.

Unfortunately, for every $z, t > 0$, $n \ge 1$, while the methods developed in §3.1 give $u_n(z,t) \in L^p(\Omega)$ for every all $p \ge 2$, they do not produce sufficiently sharp bounds to guarantee the convergence of the series in (3.3) to $u(z,t)$ in $L^p(\Omega)$ for $p \ge 2$. For example, in the case when u is the mild solution to (3.1) and $\dot{W}$ is the non-degenerate space-time white noise, directly applying the bound (3.10) with (2.8) and (3.3) gives

$$\sum_{n=0}^\infty \mathbb{E}\left[|u_n(z,t)|^p\right]^{\frac{1}{p}} \le \sum_{n=0}^\infty \left(\frac{p-1}{2}\right)^{\frac{n}{2}},$$

the right-hand side of which fails to converge for $p \ge 3$.

As it turns out, the technical difficulty described above can be overcome in the scheme of (4.1). In other words, at the presence of a degenerate diffusion operator, modifying the stochastic noise term so that it degenerates at the same location as the diffusion operator will help "tame" the behaviors of the stochastic solution. We will prove in this subsection that the solution to (4.1) has finite L^p-norms for all $p \geq 2$. In fact, in the following subsection, we will also prove the continuity of the solution all the way to the boundary with the help of the degenerate stochastic noise in (4.1). Throughout this subsection, we choose and fix $\beta > 0$ and $\alpha \in (0, 2\beta \wedge \frac{1}{2})$. Note that in this and the following subsection, the required range for α is relaxed compared with in the previous subsection.

4.2.1. *Under Space-Time White Noise.* Let $u = \{u(z,t) \mid z > 0, t \geq 0\}$ be the unique mild solution to (4.1) with $\dot{W}$ being the space-time white noise, and u_n be as in (3.11) for $n \geq 1$. We begin by re-examining our bound for $\mathbb{E}\left[u_1^2(z,t)\right]$. That is, recall that by Lemma 3.3, we have

$$\mathbb{E}\left[u_1^2(z,t)\right] \leq U\left(\frac{z}{t}\right) \leq \frac{1}{2}.$$

While previously we have mostly adopted the uniform bound of U for convenience, notice that in fact $U(x) \to 0$ as $x \to \infty$. Thus, it is possible to obtain a tighter estimate of U for large argument values. We make this notion precise in the next lemma.

Lemma 4.7. *There exists a constant $C > 0$ such that for every $z, t > 0$ and $0 < \tau < t$,*

$$\int_0^\infty q_0^2(z,w,t-\tau)\,dw \leq Cz^{-\alpha}(t-\tau)^{\alpha-1}. \tag{4.17}$$

Let U be the function defined in Lemma 3.3. Then for every $z, t > 0$,

$$U\left(\frac{z}{t}\right) = \int_0^t \int_0^\infty q_0^2(z,w,t-\tau)\,\mathrm{d}w\mathrm{d}\tau \leq \frac{C}{\alpha} z^{-\alpha} t^{\alpha}. \tag{4.18}$$

Proof: As in the proof of Lemma 3.3, we have

$$\int_0^\infty q_0^2(z,w,t-\tau)\,dw = \frac{2z}{(t-\tau)^2} e^{-\frac{2z}{t-\tau}} \int_0^\infty u^{-1} e^{-\frac{(t-\tau)u^2}{2z}} I_1^2(u)\,du.$$

In fact, the integral in the right-hand side above is a representation of the confluent hypergeometric function ${}_2F_2$ with parameters $a_1 = 3/2, a_2 = 1$ and $b_1 = 2, b_2 = 3$ (see §6). The tail of said function is well-studied; by (6.8) and (6.5), there exists a constant $C > 0$ depending only on a_1, a_2, b_1, b_2 and α, such that

$$\begin{aligned}\int_0^\infty q_0^2(z,w,t-\tau)\,dw &= \frac{2z}{(t-\tau)^2} e^{-\frac{2z}{t-\tau}} \frac{z}{4(t-\tau)} {}_2F_2\left[\begin{array}{c|c} 3/2, 1 & \frac{2z}{t-\tau} \\ 2, 3 & \end{array}\right] \\ &\leq C\frac{z^2}{(t-\tau)^3}\left(\frac{z}{t-\tau}\right)^{-2-\alpha} = Cz^{-\alpha}(t-\tau)^{\alpha-1},\end{aligned}$$

which confirms (4.17). (4.18) follows from integrating (4.17) in $\tau \in (0,t)$. □

With a little more effort, we can refine the estimate for $\mathbb{E}\left[u_n^2(z,t)\right]$ for every $z, t > 0$ and $n \geq 1$ by using Lemma 4.7 together with the recursive expression (4.2), which will enable us to show the convergence of the series $u_n(z,t)$ in $L^p(\Omega)$ for all $p \geq 2$.

Theorem 4.8. *Let $C > 0$ be the same as in Lemma 4.7. Then, for every $n \geq 1$, $p \geq 2$, and $z, t > 0$, we have*

$$\left(\mathbb{E}\left[|u_n(z,t)|^p\right]\right)^{\frac{1}{p}} \leq \frac{\left[(p-1)Ct^\alpha \Gamma(\alpha)\right]^{\frac{n}{2}}}{z^{\frac{\alpha}{2}}\sqrt{\Gamma(n\alpha+1)}}. \tag{4.19}$$

Moreover, for every $z, t > 0$ and $p \geq 2$, we have $u(z,t) \in L^p(\Omega)$ and the series in (3.3) converges to $u(z,t)$ in $L^p(\Omega)$.

Proof: We first revisit the case $p=2$. For $n\geq 1$, we will show that for $z,t>0$,

$$\mathbb{E}\left[u_n^2(z,t)\right]\leq \frac{C^n(\Gamma(\alpha))^n}{\Gamma(n\alpha+1)}z^{-\alpha}t^{n\alpha}. \tag{4.20}$$

We proceed by induction on n. The base $n=1$ follows from Lemma 4.7 upon writing $\frac{1}{\alpha}$ as $\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}$. Now assume that (4.20) holds for $n-1$ with $n\geq 2$. By (4.2) together with (4.17),

$$\begin{aligned}
\mathbb{E}\left[u_n^2(z,t)\right] &= \int_0^t\int_0^\infty \hat{w}^{2\beta}q_0^2(z,w,t-\tau)\,\mathbb{E}\left[u_{n-1}^2(w,\tau)\right]dwd\tau\\
&\leq \frac{C^{n-1}(\Gamma(\alpha))^{n-1}}{\Gamma((n-1)\alpha+1)}\int_0^t \tau^{(n-1)\alpha}\int_0^\infty \hat{w}^{2\beta}q_0^2(z,w,t-\tau)\,w^{-\alpha}dwd\tau\\
&\leq \frac{C^{n}(\Gamma(\alpha))^{n-1}}{\Gamma((n-1)\alpha+1)}z^{-\alpha}\int_0^t \tau^{(n-1)\alpha}(t-\tau)^{\alpha-1}d\tau\\
&\leq \frac{C^{n}(\Gamma(\alpha))^{n-1}}{\Gamma((n-1)\alpha+1)}z^{-\alpha}t^{n\alpha}\frac{\Gamma(\alpha)\Gamma((n-1)\alpha+1)}{\Gamma(n\alpha+1)}\\
&= \frac{C^n(\Gamma(\alpha))^n}{\Gamma(n\alpha+1)}z^{-\alpha}t^{n\alpha}.
\end{aligned}$$

Therefore, for every $n\geq 1$, (4.20) holds and hence (4.19) follows.

Finally, for every $z,t>0$ and $p\geq 2$,

$$\left(\mathbb{E}\left[|u(z,t)|^p\right]\right)^{\frac{1}{p}}\leq \sum_{n=0}^\infty\left(\mathbb{E}\left[|u_n(z,t)|^p\right]\right)^{\frac{1}{p}}\leq u_0(z,t)+\sum_{n=1}^\infty\frac{\left[(p-1)Ct^\alpha\Gamma(\alpha)\right]^{\frac{n}{2}}}{z^{\frac{\alpha}{2}}\sqrt{\Gamma(n\alpha+1)}}.$$

It follows from a simple convergence test that the series on the right hand side above converges for every $z,t>0$ and $p\geq 2$. We have completed the proof of the theorem. □

4.2.2. *Under Colored Noise.* With §4.2.1 and §3.1.2, we almost have all the necessary tools to prove a version of Theorem 4.8 corresponding to the colored noise case.

Let us temporarily return to (3.1) the original stochastic Kimura equation without degeneracy in the noise. Assume that $u=\{u(z,t)\mid z>0,t\geq 0\}$ is the unique mild solution to (3.1) with $\dot{W}$ being the $(f,\gamma)-$colored noise with (f,γ) satisfying Assumptions **(i)** and **(ii)**. For $z,t,\varepsilon>0$, let Γ_t and F_ε be the same as (3.16) and (3.15), and notice that by (2.8) and (3.22), we have

$$\left(\mathbb{E}\left[|u(z,t)|^p\right]\right)^{\frac{1}{p}}\leq u_0(z,t)+\sum_{n=1}^\infty\left(\frac{(p-1)\Gamma_tF_\varepsilon}{2}\right)^{\frac{n}{2}}\exp\left(\frac{2f(\varepsilon)t}{F_\varepsilon}\right) \tag{4.21}$$

Then, given $z,t>0$, for every $p\geq 2$, there exists $\varepsilon>0$ sufficiently small such that $(p-1)\Gamma_tF_\varepsilon<2$, in which case the series on the right hand side above converges. Therefore, with the help of the properties of (f,γ) in the colored noise setting, we do achieve L^p-integrability of the solution, without forcing the noise to degenerate at the boundary.

Now switching to the scheme of (4.1), with the degeneracy in the noise term, we have developed (4.20) which is a refined version of the bound on the L^2-norm of $u_n(z,t)$. Therefore, by incorporating (4.20), we expect to also obtain a refined version of (4.21). In particular, we will show that it is possible to achieve L^p-convergence of the series of $u_n(z,t)$ for all choices of $z,t,\varepsilon>0$.

Theorem 4.9. *For every $n\geq 1$, $p\geq 2$, and $z,t,\varepsilon>0$,*

$$\left(\mathbb{E}\left[|u_n(z,t)|^p\right]\right)^{\frac{1}{p}}\leq \hat{z}^{-\frac{\alpha}{2}}\left((p-1)C\Gamma(\alpha)\Gamma_tF_\varepsilon t^\alpha\right)^{\frac{n}{2}}\left(\sum_{\ell=0}^{n-1}\frac{\left(f(\varepsilon)t^{1-\alpha}\right)^\ell\left(C\Gamma(\alpha)F_\varepsilon\right)^{-\ell}}{\ell!\Gamma((n-\ell)\alpha+1)}\right)^{\frac{1}{2}}.$$

Furthermore, for every $z, t > 0$ we have $u(z,t) \in L^p(\Omega)$ and the series in (3.3) converges to $u(z,t)$ in $L^p(\Omega)$.

Proof: Let us first review the bound on $\mathbb{E}\left[u_n^2(z,t)\right]$ developed in §3.1.2. In particular, the estimate (3.22) yields

$$\mathbb{E}\left[u_n^2(z,t)\right] \leq \frac{(\Gamma_t f(\varepsilon)t)^n}{n!} + \left(\frac{\Gamma_t F_\varepsilon}{2}\right)^n \cdot \sum_{\ell=0}^{n-1} \frac{1}{\ell!}\left(\frac{2f(\varepsilon)t}{F_\varepsilon}\right)^\ell.$$

Recall that the first term on the right hand side above was derived by focusing on the relevant integral over the "off-diagonal" region $(R_\varepsilon^\complement)^n$, i.e., where the concerned spatial variables $\mathbf{x}$ and $\mathbf{y}$ are at least ε apart in all coordinates. We do not need to make any changes to this component of the estimate. The remaining terms on the right hand side above were based on the L^2-estimate (3.10) from the space-time white noise case. Here we will use (4.20) to replace (3.10) and re-derive these estimates; specifically, following the steps in the proof of Lemma 3.7, the updated upper bound for the affected terms above becomes

$$z^{-\alpha}\Gamma_t^n \sum_{\ell=0}^{n-1} \frac{1}{\ell!} f^\ell(\varepsilon) t^\ell F_\varepsilon^{n-\ell} \frac{C^{n-\ell}\left(\Gamma(\alpha)\right)^{n-\ell}}{\Gamma\left((n-\ell)\alpha+1\right)} t^{(n-\ell)\alpha}.$$

Putting the two terms together, we have a refined estimate

$$\mathbb{E}\left[u_n^2(z,t)\right] \leq \hat{z}^{-\alpha}\left(C\Gamma(\alpha)\Gamma_t F_\varepsilon t^\alpha\right)^n \sum_{\ell=0}^{n} \frac{\left(f(\varepsilon)t^{1-\alpha}\right)^\ell \left(C\Gamma(\alpha)F_\varepsilon\right)^{-\ell}}{\ell!\Gamma((n-\ell)\alpha+1)}.$$

The desired estimate on $\left(\mathbb{E}\left[|u_n(z,t)|^p\right]\right)^{\frac{1}{p}}$, for all $p \geq 2$, is an immediate consequence of (2.8). Again, by a straightforward convergence test, the series on the right hand side above converges for all $z, t, \varepsilon > 0$ and $p \geq 2$. We have proven the convergence statement in the theorem. $\square$

Remark 4.10. In light of Theorem 4.9, we see that in the scheme where the stochastic noise becomes degenerate at the boundary, we no longer need the assumption that $F_\varepsilon \to 0$ as $\varepsilon \to 0$ for the existence of the solution $u(z,t)$ to (4.1), as the L^2-integrability of $u(z,t)$ and the L^2-convergence of the series of $u_n(z,t)$ to $u(z,t)$ can be achieved for all choices of $z, t, \varepsilon > 0$.

4.3. *Hölder Continuity of Solution (up to Boundary).* In the previous subsection, for the solution to (4.1) the stochastic Kimura equation with degenerate noise, we derived bounds for L^p-norms for $p \geq 2$ based on refining the L^2-norm estimates. In this subsection, we continue to study the continuity of the solution. Let $u = \{u(z,t) \mid z > 0, t \geq 0\}$ be the unique mild solution to (4.1) with $\beta > 0$. Assume that $\dot{W}$ is either the space-time white noise or the $(f,\gamma)-$colored noise with (f,γ) satisfying Assumptions **(i)** and **(ii)**. Since u satisfies (3.2), with $z > 0$ fixed, the continuity $t \mapsto u(z,t)$ follows from the L^2-boundedness of $\{u(w,\tau) \mid w > 0, 0 < \tau < t\}$. Therefore, we will focus on the continuity of $z \mapsto u(z,t)$. Our strategy is to apply Kolmogorov's continuity theorem (see, e.g., Theorem 4.3.2 in Stroock (2011)). To this end, we need to examine $\mathbb{E}[(u(z_1,t) - u(z_2,t))^2]$ for $z_1, z_2, t > 0$, and hope to bound it in terms of a positive power of $|z_1 - z_2|$.

Remark 4.11. In the deterministic setting, the boundary regularity of the solution is often an important aspect in the study of degenerate diffusions. Similarly, for the stochastic Kimura equation, we are particularly interested in investigating the Hölder continuity of the solution in the spatial variable near the boundary 0, which will help us understand the behaviors of the solution near the boundary under the joint effect of the degeneracy in the diffusion operator and the stochastic noise. With this purpose in mind, we need to develop bounds for $\mathbb{E}[(u(z_1,t) - u(z_2,t))^2]$ that are "robust" for z_1 or z_2 all the way to 0, which requires techniques that are more involved than treating the case when z_1 and z_2 are bounded away from 0. We expect that the solution possesses better Hölder continuity in the interior of the domain than at the boundary, but deriving sharp results on the interior regularity is not a goal for this work.

Similarly to the derivation of (3.4), $u(z_1,t)-u(z_2,t)$ admits the following Wiener chaos expansion:

$$u(z_1,t)-u(z_2,t)=\sum_{n=0}^{\infty}\mathcal{I}_n\left(f_n(\cdot;z_1,z_2,t)\right), \tag{4.22}$$

where we slightly abuse the notation by denoting $f_0(\cdot;z_1,z_2,t):=u_0(z_1,t)-u_0(z_2,t)$ and

$$f_n(\cdot;z_1,z_2,t):=f_n(\cdot;z_1,t)-f_n(\cdot;z_2,t)$$

for $f_n(\cdot;z_1,t)$ and $f_n(\cdot;z_2,t)$ as in (3.4). Hence, setting

$$d_0(z_1,z_2,t)(w):=q_0(z_1,w,t)-q_0(z_2,w,t),$$

we have that for every $n\geq 1$,

$$\begin{aligned}
&\mathbb{E}\left[\left(u_n(z_1,t)-u_n(z_2,t)\right)^2\right]\\
&=\frac{1}{n!}\int_{[0,t]^{2n}\times\mathbb{R}_+^{2n}}d_0(z_1,z_2,t-r_{\sigma(n)})(x_{\sigma(n)})\cdot\prod_{i=1}^{n-1}q_0\left(x_{\sigma(i+1)},x_{\sigma(i)},r_{\sigma(i+1)}-r_{\sigma(i)}\right)\\
&\qquad\cdot d_0(z_1,z_2,t-s_{\varsigma(n)})(y_{\varsigma(n)})\cdot\prod_{i=1}^{n-1}q_0\left(y_{\varsigma(i+1)},y_{\varsigma(i)},s_{\varsigma(i+1)}-s_{\varsigma(i)}\right)\\
&\qquad\cdot u_0(x_{\sigma(1)},r_{\sigma(1)})u_0(y_{\varsigma(1)},s_{\varsigma(1)})\cdot\prod_{i=1}^{n}f(x_i-y_i)\,\hat{x}_i{}^{\beta}\hat{y}_i{}^{\beta}\gamma(r_i-s_i)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{s},
\end{aligned} \tag{4.23}$$

where σ and ς are the permutations of $\{1,2,\ldots,n\}$ such that $0<r_{\sigma(1)}<r_{\sigma(2)}<\cdots<r_{\sigma(n)}<t$ and $0<s_{\varsigma(1)}<s_{\varsigma(2)}<\cdots<s_{\varsigma(n)}<t$.

Thus we can reuse the refined bounds developed in §4.2. Again, throughout this subsection, we fix $\beta>0$ and $0<\alpha<2\beta\wedge\frac{1}{2}$.

4.3.1. *Under Space-Time White Noise.* As usual, we first examine the case of when $\dot{W}$ is the degenerate space-time white noise. In this case, for $z_1,z_2,t>0$ and $n\geq 1$, (4.23) becomes

$$\mathbb{E}\left[\left(u_n(z_1,t)-u_n(z_2,t)\right)^2\right]=\int_0^t\int_0^{\infty}\hat{w}^{2\beta}d_0^2(z_1,z_2,t-\tau)(w)\,\mathbb{E}\left[u_{n-1}^2(w,\tau)\right]\mathrm{d}w\mathrm{d}\tau,$$

which, according to (4.20), is bounded from above by

$$\frac{\left(C\Gamma(\alpha)t^{\alpha}\right)^{n-1}}{\Gamma\left((n-1)\alpha+1\right)}\int_0^t\int_0^{\infty}\hat{w}^{2\beta}w^{-\alpha}d_0^2(z_1,z_2,t-\tau)(w)\,\mathrm{d}w\mathrm{d}\tau.$$

So, no recursive argument is needed here. Instead, we only need to focus on estimating

$$Q_t(z_1,z_2):=\int_0^t\int_0^{\infty}\hat{w}^{2\beta}w^{-\alpha}d_0^2(z_1,z_2,t-\tau)(w)\,\mathrm{d}w\mathrm{d}\tau. \tag{4.24}$$

To this end, we first present some technical results on the regularity of the function $z\mapsto q_0(z,w,t-\tau)$ for arbitrary $w,t-\tau>0$.

Lemma 4.12. *There exists a constant $M>0$ such that for every $z_1,z_2,t-\tau>0$,*

$$\sup_{w>0}\left|d_0(z_1,z_2,t-\tau)(w)\right|\leq M\frac{|z_1-z_2|^{\frac{1}{2}}}{(t-\tau)^{\frac{3}{2}}}. \tag{4.25}$$

Proof: By direct computation, we have that for every $z, w, t-\tau > 0$,

$$\partial_z q_0(z,w,t-\tau) = \frac{1}{t-\tau}\left(q_1(z,w,t-\tau) - q_0(z,w,t-\tau)\right),$$

where q_1 is given by (2.2) with $\nu = 1$. Hence,

$$\begin{aligned}|d_0(z_1,z_2,t-\tau)(w)| &= \left|\int_{z_1}^{z_2} \partial_z q_0(z,w,t-\tau)\,\mathrm{d}z\right| \le \frac{1}{t-\tau}\int_{z_1}^{z_2} |q_1(z,w,t-\tau) - q_0(z,w,t-\tau)|\,\mathrm{d}z \\ &\le \frac{|z_1-z_2|^{\frac{1}{2}}}{t-\tau}\left[\int_0^\infty \left(q_1^2(z,w,t-\tau) + q_0^2(z,w,t-\tau)\right)\mathrm{d}z\right]^{\frac{1}{2}}.\end{aligned}$$

Again, by (2.2) and (6.8), we have

$$\int_0^\infty q_1^2(z,w,t-\tau)\,\mathrm{d}z = \frac{e^{-\frac{2w}{t-\tau}}}{2(t-\tau)} \cdot {}_2F_2\left[\begin{matrix}1/2,1\\1,1\end{matrix}\,\middle|\,\frac{2w}{t-\tau}\right],$$

$$\int_0^\infty q_0^2(z,w,t-\tau)\,\mathrm{d}z = \frac{e^{-\frac{2w}{t-\tau}}}{4(t-\tau)} \cdot {}_2F_2\left[\begin{matrix}3/2,3\\2,3\end{matrix}\,\middle|\,\frac{2w}{t-\tau}\right].$$

(4.25) follows from these last two computations combined with (6.5). □

Therefore, we observe that it is possible to extrapolate the Hölder continuity of the stochastic solution from the Hölder continuity of $z \mapsto q_0(z,w,t-\tau)$. It thus remains to treat the aftermath of substituting (4.25) into (4.24), aiming at obtaining a bound independent of z_1 and z_2.

Lemma 4.13. *For $z_1, z_2, t > 0$ and any $0 < \lambda < (4\beta - 2\alpha) \wedge \frac{1}{2}$, there exists a constant $M_{\alpha,\beta,\lambda} > 0$ such that*

$$Q_t(z_1,z_2) \le M_{\alpha,\beta,\lambda} t^\eta |z_1 - z_2|^{\frac{\lambda}{2}},$$

where $\eta = \left((2\beta-\alpha)\wedge\frac{1}{4}\right) - \frac{\lambda}{2}$.

Proof: By Lemma 4.12, for any $\lambda \in (0,2)$, there exists a constant $M_\lambda > 0$ such that

$$\begin{aligned}Q_t(z_1,z_2) &\le M_\lambda |z_1-z_2|^{\frac{\lambda}{2}} \int_0^t \frac{1}{(t-\tau)^{\frac{3\lambda}{2}}}\int_0^\infty \hat{w}^{2\beta-\alpha}\,|d_0(z_1,z_2,t-\tau)(w)|^{2-\lambda}\,\mathrm{d}w\mathrm{d}\tau, \\ &\le M_\lambda |z_1-z_2|^{\frac{\lambda}{2}} \\ &\quad \cdot \int_0^t \frac{1}{(t-\tau)^{\frac{3\lambda}{2}}}\int_0^\infty \hat{w}^{2\beta-\alpha}\left[q_0^{2-\lambda}(z_1,w,t-\tau) + q_0^{2-\lambda}(z_2,w,t-\tau)\right]\mathrm{d}w\mathrm{d}\tau.\end{aligned}$$

It remains to treat, for any $z, t > 0$, the following integral:

$$\int_0^t \frac{1}{(t-\tau)^{\frac{3\lambda}{2}}}\int_0^\infty \hat{w}^{2\beta-\alpha} q_0^{2-\lambda}(z,w,t-\tau)\,\mathrm{d}w\mathrm{d}\tau.$$

Note that $\hat{w}^{2\beta-\alpha} \le \hat{w}^{(2\beta-\alpha)\wedge\frac{1}{4}} \le w^{(2\beta-\alpha)\wedge\frac{1}{4}}$, and hence we can upper bound the integral above by

$$\begin{aligned}&\int_0^t \frac{1}{(t-\tau)^{\frac{3\lambda}{2}}}\int_0^\infty w^{(2\beta-\alpha)\wedge\frac{1}{4}} q_0^{2-\lambda}(z,w,t-\tau)\,\mathrm{d}w\mathrm{d}\tau \\ &= \int_0^t \frac{1}{(t-\tau)^{\frac{3\lambda}{2}}}\int_0^\infty w^{(2\beta-\alpha)\wedge\frac{1}{4}} q_0^{1-\lambda}(z,w,t-\tau)\cdot q_0(z,w,t-\tau)\,\mathrm{d}w\mathrm{d}\tau \\ &\le \int_0^t \frac{1}{(t-\tau)^{\frac{3\lambda}{2}}}\left[\int_0^\infty w^{\frac{(2\beta-\alpha)\wedge\frac{1}{4}}{1-\lambda}} q_0^2(z,w,t-\tau)\,\mathrm{d}w\right]^{1-\lambda}\mathrm{d}\tau,\end{aligned}$$

where the last line is an application of Jensen's inequality. Set $\xi := \frac{(2\beta-\alpha)\wedge\frac{1}{4}}{1-\lambda}$. Then, by (6.8) and (6.5), the integral inside the power above can be bounded as

$$\int_0^\infty q_0^2(z,w,t-\tau)\, w^\xi dw = \frac{\Gamma(1+\xi)}{2^{3+\xi}}(t-\tau)^{\xi-1}\left(\frac{2z}{t-\tau}\right)^2 e^{-\frac{2z}{t-\tau}}\,{}_2F_2\left[\begin{array}{c} 3/2,\xi+1 \\ 2,3 \end{array}\,\middle|\, \frac{2z}{t-\tau}\right]$$
$$\leq M_{\alpha,\beta,\lambda}(t-\tau)^{\xi-1},$$

provided that $\frac{3}{2}+\xi+1-2-3\leq -2$, i.e., $(2\beta-\alpha)\wedge\frac{1}{4}\leq\frac{1}{2}(1-\lambda)$. Meanwhile, when the above inequality holds, we have that

$$Q_t(z_1,z_2)\leq M_{\alpha,\beta,\lambda}|z_1-z_2|^{\frac{\lambda}{2}}\int_0^t (t-\tau)^{(\xi-1)(1-\lambda)-\frac{3}{2}\lambda}\,\mathrm{d}\tau = M_{\alpha\beta,\lambda}t^{(\xi-1)(1-\lambda)-\frac{3}{2}\lambda+1}|z_1-z_2|^{\frac{\lambda}{2}},$$

provided that $(\xi-1)(1-\lambda)-\frac{3}{2}\lambda>-1$, i.e., $(2\beta-\alpha)\wedge\frac{1}{4}>\frac{\lambda}{2}$. Combining the constraints above, we get the proper range of λ:

$$0<\lambda<\begin{cases} 4\beta-2\alpha & \text{if } 0<2\beta-\alpha<\frac{1}{4}, \\ \frac{1}{2} & \text{if } 2\beta-\alpha\geq\frac{1}{4}. \end{cases}$$

$\square$

We are ready to present the main result of this section.

Theorem 4.14. *Let $u=\{u(z,t)\mid z>0,t\geq 0\}$ be the unique mild solution to (4.1) with $\dot{W}$ being the space-time white noise. Then, for fixed $t>0$ and any $\theta\in\left(0,\beta\wedge\frac{1}{8}\right)$, there exists an almost surely $\theta-$Hölder continuous modification of the process $z\in[0,\infty)\mapsto u(z,t)\in\mathbb{R}$. In particular, almost surely $z\mapsto u(z,t)$ is continuous all the way to the boundary 0.*

Proof: For any $p\geq 2$ and $n\geq 1$, by (2.8) and Lemma 4.13,

$$\left(\mathbb{E}\left[|u_n(z_1,t)-u_n(z_2,t)|^p\right]\right)^{\frac{1}{p}}\leq\sqrt{M_{\alpha,\beta,\lambda}}\,t^{\frac{\eta}{2}}|z_1-z_2|^{\frac{\lambda}{4}}\frac{\left((p-1)C\Gamma(\alpha)t^\alpha\right)^{\frac{n-1}{2}}}{\sqrt{\Gamma((n-1)\alpha+1)}},$$

which, combined with (4.22), implies that

$$\left(\mathbb{E}\left[|u(z_1,t)-u(z_2,t)|^p\right]\right)^{\frac{1}{p}}\leq|u_0(z_1,t)-u_0(z_2,t)|$$
$$+\sqrt{M_{\alpha,\beta,\lambda}}\,t^{\frac{\eta}{2}}|z_1-z_2|^{\frac{\lambda}{4}}\sum_{n=0}^\infty\frac{\left((p-1)C\Gamma(\alpha)t^\alpha\right)^{\frac{n}{2}}}{\sqrt{\Gamma(n\alpha+1)}}$$
$$\leq M_{\alpha,\beta,\lambda,p,t}|z_1-z_2|^{\frac{\lambda}{4}}$$

for some constant $M_{\alpha,\beta,\lambda,p,t}>0$. Therefore, applying Kolmogorov's continuity theorem, we conclude that, when p is sufficiently large, for every $t>0$, $z\mapsto u(z,t)$ is almost surely θ-Hölder continuous for any $\theta\in(0,\frac{\lambda}{4}-\frac{1}{p})$.

Since p can be arbitrarily large and λ can be arbitrarily close to $(4\beta-2\alpha)\wedge\frac{1}{2}$, θ can be arbitrarily close to $\left(\beta-\frac{\alpha}{2}\right)\wedge\frac{1}{8}$. On the other hand, α can be arbitrarily close to 0. Therefore, we conclude that for every $t>0$, $z\mapsto u(z,t)$ is almost surely θ-Hölder continuous on $[0,\infty)$ for any $\theta\in\left(0,\beta\wedge\frac{1}{8}\right)$. $\square$

The result in Theorem 4.14 suggests that, when β is small, increasing the order of degeneracy in the stochastic potential does improve the regularity of the solution at the boundary, but this effect is limited. In other words, the continuity of the solution at the boundary is subject to, first, how "tamed" the noise is at the boundary, and then the stochasticity nature of the solution itself.

4.3.2. *Under Colored Noise.* We turn to the case when $\dot{W}$ is the $(f,\gamma)-$colored noise with (f,γ) satisfying Assumptions **(i)** and **(ii)**. As before, we seek to apply the usual approach. Let $z_1, z_2, t, \varepsilon > 0$. It is sufficient to study the integral in (4.23) over the region where $|x_i - y_i| > \varepsilon$ for $i = 1, \ldots, \ell$ and $|x_j - y_j| \le \varepsilon$ for $j = \ell+1, \ldots, n$ for some $\ell = 0, 1, \ldots, n$. It turns out that in this step, the integrals arising from the case of $\ell \le n-1$ are easier to adapt, because we simply replace the white-noise L^2-bound involving $q_0^2(z, \cdot, t-*)$ by that which corresponds to $d_0^2(z_1, z_2, t-*)(\cdot)$. That is, by Lemma 4.13, we know that one contributor to the bound of $\mathbb{E}\left[\left(u_n\left(z_1,t\right) - u_n\left(z_2,t\right)\right)^2\right]$ is

$$M_{\alpha,\beta,\lambda} t^{\eta} |z_1 - z_2|^{\frac{\lambda}{2}} \Gamma_t^n \sum_{\ell=0}^{n-1} \frac{1}{\ell!} f^{\ell}(\varepsilon) t^{\ell} F_{\varepsilon}^{n-\ell} \frac{\left(C\Gamma\left(\alpha\right) t^{\alpha}\right)^{n-\ell-1}}{\Gamma\left((n-\ell-1)\alpha+1\right)},$$

where $\lambda, \eta, M_{\alpha,\beta,\lambda}$ are as in Lemma 4.13. As for the region with $\ell = n$, we need to treat the following quantity:

$$\frac{1}{n!} f^n(\varepsilon) \Gamma_t^n \int_{[0,t]^n} \left(\int_0^{\infty} \hat{w}^{\beta} d_0\left(z_1, z_2, t - \min(r_1, \ldots, r_n)\right)(w)\, \mathrm{d}w \right)^2 \mathrm{d}\mathbf{r},$$

and extract from this quantity a positive power of $|z_1 - z_2|$. This is done in the following lemma.

Lemma 4.15. *Let λ be as in Lemma 4.13. There exists a constant $M_\lambda > 0$ such that for every $z_1, z_2, t > 0$,*

$$\int_{[0,t]^n} \left(\int_0^{\infty} \hat{w}^{\beta} d_0\left(z_1, z_2, t - \min(r_1, \ldots, r_n)\right)(w)\, \mathrm{d}w \right)^2 \mathrm{d}\mathbf{r} \le M_\lambda t^{n-\frac{\lambda}{2}} |z_1 - z_2|^{\frac{\lambda}{2}}.$$

Proof: For simplicity, we write $m(\mathbf{r}) := \min(r_1, \ldots, r_n)$. It is clear that

$$\left| \int_0^{\infty} \hat{w}^{\beta} d_0\left(z_1, z_2, t - m(\mathbf{r})\right)(w)\, \mathrm{d}w \right| \le 2.$$

Therefore, the concerned integral is bounded from above by

$$\begin{aligned}
&2^{2-\frac{\lambda}{2}} \int_{[0,t]^n} \left| \int_0^{\infty} \hat{w}^{\beta} d_0\left(z_1, z_2, t - m(\mathbf{r})\right)(w)\, \mathrm{d}w \right|^{\frac{\lambda}{2}} \mathrm{d}\mathbf{r}, \\
\le& 2^{2-\frac{\lambda}{2}} \int_{[0,t]^n} \frac{1}{\left(t - m(\mathbf{r})\right)^{\frac{\lambda}{2}}} \left[\int_{z_1}^{z_2} \left(\int_0^{\infty} q_1\left(z, w, t - m(\mathbf{r})\right) \mathrm{d}w + \int_0^{\infty} q_0\left(z, w, t - m(\mathbf{r})\right) \mathrm{d}w \right) \mathrm{d}z \right]^{\frac{\lambda}{2}} \mathrm{d}\mathbf{r} \\
\le& 4\left|z_1 - z_2\right|^{\frac{\lambda}{2}} \int_{[0,t]^n} \frac{1}{\left(t - m(\mathbf{r})\right)^{\frac{\lambda}{2}}} \mathrm{d}\mathbf{r} = M_\lambda \left|z_1 - z_2\right|^{\frac{\lambda}{2}} t^{n-\frac{\lambda}{2}},
\end{aligned}$$

where we used the fact that for every $z, \tau > 0$, $\int_0^{\infty} q_1\left(z, w, \tau\right) \mathrm{d}w = 1$. □

Combining the result of the lemma and the analysis above, we see that

$$\mathbb{E}\left[\left(u_n\left(z_1,t\right) - u_n\left(z_2,t\right)\right)^2\right] \le M_{\alpha,\beta,\lambda,t} |z_1 - z_2|^{\frac{\lambda}{2}} \left(C\Gamma(\alpha)\Gamma_t F_{\varepsilon} t^{\alpha}\right)^n \sum_{\ell=0}^{n} \frac{\left(f(\varepsilon) t^{1-\alpha}\right)^{\ell} \left(C\Gamma(\alpha) F_{\varepsilon}\right)^{-\ell}}{\ell! \Gamma((n-\ell-1)\alpha+1)},$$

where $M_{\alpha,\beta,\lambda,t} > 0$ is a uniform constant in $z_1, z_2 > 0$.

We now state the counterpart of Theorem 4.14 in the colored noise setting.

Theorem 4.16. *Let $u = \{u\left(z,t\right) \mid z > 0, t \ge 0\}$ be the unique mild solution to (4.1) with $\dot{W}$ being the $(f,\gamma)-$colored noise with (f,γ) satisfying Assumptions **(i)** and **(ii)**. Then, for fixed $t, \varepsilon > 0$ and any $\theta \in \left(0, \beta \wedge \frac{1}{8}\right)$, there exists an almost surely $\theta-$Hölder continuous modification of the process $z \in [0,\infty) \mapsto u\left(z,t\right) \in \mathbb{R}$. In particular, $z \mapsto u\left(z,t\right)$ is continuous all the way to the boundary 0.*

Proof: Given $\alpha, \beta, \varepsilon$ as described at the beginning of the section, let η, λ be as in Lemma 4.13. For any $z_1, z_2, t > 0$ and $n \geq 1$, following the estimate on $\mathbb{E}\left[\left(u_n\left(z_1, t\right) - u_n\left(z_2, t\right)\right)^2\right]$ developed above, we have that

$$\left(\mathbb{E}\left[\left|u_n\left(z_1, t\right) - u_n\left(z_2, t\right)\right|^p\right]\right)^{\frac{1}{p}} \leq M^{\frac{1}{2}}_{\alpha,\beta,\lambda,t}|z_1 - z_2|^{\frac{\lambda}{4}} \left((p-1)C\Gamma(\alpha)\Gamma_t F_\varepsilon t^\alpha\right)^{\frac{n}{2}} \cdot \left(\sum_{\ell=0}^{n} \frac{\left(f(\varepsilon)t^{1-\alpha}\right)^\ell \left(C\Gamma(\alpha)F_\varepsilon\right)^{-\ell}}{\ell!\Gamma((n-\ell-1)\alpha+1)}\right)^{\frac{1}{2}},$$

which, again, forms a convergent series in $n \geq 0$. The rest follows exactly the same steps as in the proof of Theorem 4.14. □

5. Conclusion and Further Questions

Beyond the results demonstrated above for existence, uniqueness, asymptotics, moment bounds, and continuity of the solution to (3.1) and (4.1), we are also interested in exploring other well studied properties of SPDEs, such as long-term asymptotics and intermittency (e.g., Carmona and Molchanov (1995); Gärtner et al. (2007); Foondun and Khoshnevisan (2009); Berger et al. (2023)).

Moreover, while we have demonstrated results in the special case of a drift-less ($\nu = 0$) stochastic Kimura equation, the deterministic problem (2.1) studied in Chen and Weih-Wadman (2020) possesses a more general drift, i.e., $\partial_t u = z\partial_z^2 u + b\left(z\right)\partial_z u$. In the particular case when the drift is constant $b\left(z\right) \equiv \nu < 1$, the fundamental solution $q_\nu(z, w, t)$ to the deterministic Kimura equation can also be expressed explicitly in terms of Bessel functions. At present, we have no cause to believe that the introduction of such a drift term would have a qualitative impact on the results established in this study. We can follow exactly the same steps as explained in §3 to construct and study the solution to the stochastic Kimura equation $\partial_t = z\partial_z^2 + b\left(z\right)\partial_z + \dot{W}u$. Under some mild conditions on $b\left(z\right)$, we expect the same techniques employed in the above to remain applicable, albeit more involved to apply. We believe the same qualitative results will hold but the explicit expressions on estimates and bounds will be hard to recover for a general drift.

Similarly, we should also be able to conduct this line of study on the stochastic degenerate diffusion equation $\partial_t u = z^\rho \partial_z^2 u + \dot{W}u$ with a more general order of degeneracy $\rho \in (0, 2)$, as the fundamental solution to the deterministic problem is well understood. It would be interesting to compare the properties of the stochastic solutions under different values of ρ, i.e., under different strengths of degeneracy.

In addition, in the deterministic setting, the Wright-Fisher equation $\partial_t u = z\left(1-z\right)\partial_z^2 u$, the two-sided-boundary analog of the Kimura equation, can be transformed into a one-sided-boundary model through a localization method (Chen and Stroock, 2010; Chen and Weih-Wadman, 2022). It would be interesting to investigate and to adapt the localization method in the stochastic setting, which would allow us to develop a solution theory for the stochastic Wright-Fisher equation.

Finally, it is not clear whether the estimates presented in this work, particularly the integral bounds derived in §3.1 and §3.2, are sharp. This is in part due to the fact that the spatial integrals over $\mathbb{R}_+$ are mostly treated with global estimates on Bessel functions or hypergeometric functions, which may not be tight. Employing more accurate estimates on these special functions may yield sharper bounds overall. In the same vein, it remains to be examined whether the requirement on the degeneracy of the noise at the boundary is necessary for obtaining the estimates and the continuity results in §4.2 and §4.3. In light of improving sharpness overall in the results, it may be possible to recover the bounds on the higher moments and the Hölder continuity of the solution under non-degenerate Gaussian noise. In addition, it is yet to be explored whether modifying the temporal kernel γ of the noise (instead of introducing the degeneracy to the spatial kernel f) would similarly achieve the "taming" effect on the solution.

6. Appendix

In this section, we collect some inequalities and identities that are used in §3. We omit all the proofs.

Integral Estimate of Gaussian Distribution: Given the Gaussian distribution $N\left(m, \sigma^2\right)$, for every $\kappa > -1$, there exists a constant $C_\kappa > 0$ such that

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} |x|^\kappa e^{-\frac{(x-m)^2}{2\sigma^2}} \mathrm{d}x \leq C_\kappa \left(m^\kappa + \sigma^\kappa\right). \tag{6.1}$$

Young's Convolution Inequality (Theorem 3.9.4 in Bogachev (2007)): Consider some sets $A, B, C \subseteq \mathbb{R}$ and locally integrable functions $f, g, h : \mathbb{R} \to \mathbb{R}$. Denote $D = \{(x, y) \in B \times C \mid x - y \in A\}$. Then, for all $p, q, r \geq 1$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$,

$$\left| \int_D f(x-y)\, g(x)\, h(y)\, \mathrm{d}x\mathrm{d}y \right| \leq \left(\int_A |f|^r\right)^{\frac{1}{r}} \left(\int_B |g|^p\right)^{\frac{1}{p}} \left(\int_C |h|^q\right)^{\frac{1}{q}}. \tag{6.2}$$

Next, we present various special functions. For more details, see Watson (1995).

Bessel Functions. Let $\alpha \in \mathbb{C}$, the *Bessel function of the first kind* J_α have various representations, for example,

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

The *modified Bessel function of the first kind* I_α is given by

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}, \tag{6.3}$$

or equivalently, in the integral representation as

$$I_\alpha(x) = \frac{(x/2)^\alpha}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi e^{x\cos\theta} \sin^{2\alpha}\theta \mathrm{d}\theta. \tag{6.4}$$

When α takes integer values, the functions I_n, $n \in \mathbb{N}$, satisfy the recurrence relation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-n} I_n(x)\right) = x^{-n} I_{n+1}(x).$$

In particular for $n = 0$, $I_0'(x) = I_1(x)$.

Hypergeometric Functions. For each $n \geq 0$ and $a \in \mathbb{R}$, we define the *Pochhammer symbol* as

$$(a)_n := \begin{cases} 1 & \text{if } n = 0. \\ a(a+1)(a+2)\cdots(a+n-1) & \text{if } n \geq 1. \end{cases}$$

Given $q \geq p \geq 0$, and $a_1, \ldots, a_p, b_1, \ldots, b_q \in \mathbb{R}$, if none of $b_1, \ldots, b_q$ are non-positive integers, then for $x \in \mathbb{C}$, the *generalized Hypergeometric function* ${}_pF_q$ is defined as

$${}_pF_q\left[\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}.$$

Here are two technical results involving these functions:

${}_1F_1$ and Incomplete Gamma Function (https://dlmf.nist.gov/13.6): Let $n \in \mathbb{N}$ and $x \in \mathbb{C}$, then

$${}_1F_1\left[\begin{array}{c} 1 \\ n+1 \end{array} \middle| x\right] = ne^x x^{-n}\left(\Gamma(n) - \Gamma(n, a)\right).$$

Asymptotic Expansion of ${}_2F_2$ (Volkmer and Wood (2014)): Let $x \in \mathbb{R}_+$. Suppose that neither c nor d are non-positive integers, then ${}_2F_2$ has the following asymptotic expansion:

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)} {}_2F_2 \left[\begin{array}{c} a, b \\ c, d \end{array} \middle| x \right] \sim e^x x^\nu \sum_{k=0}^{\infty} c_k x^{-k}$$

as $x \to \infty$, where $\nu = a + b - c - d$. If $\nu < 0$, then for every $0 \leq k \leq -\nu$,

$$\sup_{z>0} e^{-x} x^k {}_2F_2 \left[\begin{array}{c} a, b \\ c, d \end{array} \middle| x \right] < \infty. \tag{6.5}$$

We are interested in the connections between Bessel functions and Hypergeometric functions in the form of certain integral quantities. These are given in the following results from §13.3 of Watson (1995).

Given $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{R}_+$,

$$\int_0^\infty J_1(at) \exp\left(-bt^2\right) \mathrm{d}t = \frac{a}{4b} {}_1F_1 \left[\begin{array}{c} 1 \\ 2 \end{array} \middle| -\frac{a^2}{4b} \right] = \frac{1}{a} \left(1 - e^{-\frac{a^2}{4b}} \right), \tag{6.6}$$

and

$$\int_0^\infty J_1^2(at) \exp\left(-bt^2\right) t^{-1} \mathrm{d}t = \frac{1}{2} \left(1 - e^{-\frac{a^2}{2b}} I_0\left(-\frac{a^2}{2b} \right) + e^{-\frac{a^2}{2b}} I_1\left(-\frac{a^2}{2b} \right) \right). \tag{6.7}$$

More generally, if $\lambda + \nu > 0$, then

$$\int_0^\infty J_\nu^2(at) \exp\left(-bt^2\right) t^{2\lambda - 1} \mathrm{d}t = \frac{a^{2\nu}\Gamma(\nu + \lambda)}{2^{1+2\nu} b^{\lambda+\nu} \Gamma^2(\nu + 1)} {}_2F_2 \left[\begin{array}{cc} \nu + \frac{1}{2} & \nu + \lambda \\ \nu + 1 & 2\nu + 1 \end{array} \middle| -\frac{a^2}{b} \right]. \tag{6.8}$$

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