ALEA, Lat. Am. J. Probab. Math. Stat. **22**, 1067–1083 (2025)
DOI: 10.30757/ALEA.v22-42

On a non-linear birth process with multiple arrivals

Kuldeep Kumar Kataria and Mostafizar Khandakar

Department of Mathematics, Indian Institute of Technology Bhilai, Durg-491002, India.
E-mail address: kuldeepk@iitbhilai.ac.in

Department of Science and Mathematics, Indian Institute of Information Technology Guwahati, Bongora-781015, India.
E-mail address: mostafizar@iiitg.ac.in

Abstract. In this paper, we introduce a generalized birth process (GBP) which performs jumps of size $1, 2, \ldots, k$ whose rates depend on the state of the process. We derive a non-exploding condition for it. The system of differential equations that governs its state probabilities is obtained. In this governing system of differential equations, we replace the first order derivative with Caputo fractional derivative to obtain a fractional variant of the GBP, namely, the generalized fractional birth process (GFBP). The Laplace transform of the state probabilities of this fractional variant is obtained whose inversion yields its one-dimensional distribution. It is shown that the GFBP is equal in distribution to a time-changed version of the GBP. A limiting case of the GFBP is considered in which jump of any size $j \geq 1$ is possible. Also, we discuss a state dependent version of it.

1. Introduction

The pure birth process is a counting process in which the jump of unit size occurs in an infinitesimal interval of length h with probability $\lambda_n h + o(h)$, that is, the jump intensity $\lambda_n > 0$ depends on the state n of the process. Orsingher and Polito (2010) introduced and studied a fractional version of the pure birth process, namely, the fractional pure birth process (FPBP). It is denoted by $\{\mathcal{N}^\alpha(t)\}_{t\geq 0}$, $0 < \alpha \leq 1$ whose state probabilities $q^\alpha(n,t) = \Pr\{\mathcal{N}^\alpha(t) = n\}$, $n \geq 1$ solve the following system of fractional differential equations:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} q^\alpha(n,t) = -\lambda_n q^\alpha(n,t) + \lambda_{n-1} q^\alpha(n-1,t),$$

with the initial conditions

$$q^\alpha(n,0) = \begin{cases} 1, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Received by the editors January 19th, 2025; accepted August 15th, 2025.
2010 *Mathematics Subject Classification.* 60G22, 26A33, 60J80.
Key words and phrases. Birth process, state dependent process, fractional Poisson process.

Here, $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ is the Caputo fractional derivative defined as (see Kilbas et al. (2006))

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} f'(s)\,\mathrm{d}s, & 0 < \alpha < 1, \\ f'(t), & \alpha = 1, \end{cases}$$

where $f'(t)$ is the first order derivative of $f(t)$.

Its Laplace transform is given by (see Kilbas et al. (2006), Eq. (5.3.3))

$$\mathcal{L}\left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t); s\right) = s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0), \quad s > 0, \tag{1.1}$$

where $\tilde{f}(s)$ is the Laplace transform of $f(t)$. For $\alpha = 1$, the FPBP reduces to the pure birth process (see Feller (1968)).

The Poisson process is another important pure birth type process mainly used in modeling the count data. Its jump intensity is a constant $\lambda > 0$, that is, it does not depend on the state of the process. It's a Lévy as well as a renewal process with exponentially distributed waiting times. It has certain limitations in modeling the phenomena with long memory due to its light-tailed distributed waiting times. To overcome these limitations, many researchers have introduced and studied various fractional generalizatons of the Poisson process in the past two decades. These fractional variants are generally characterized by their heavy-tailed distributed waiting times. Some commonly used fractional versions of the Poisson process are the time fractional Poisson process (TFPP) (see Beghin and Orsingher (2009)), the space fractional Poisson process (SFPP) (see Orsingher and Polito (2012)) and their generalization, *viz.*, the space-time fractional Poisson process (STFPP) (see Orsingher and Polito (2012)) *etc.*

Di Crescenzo et al. (2016) introduced and studied a generalization of the Poisson process, namely, the generalized counting process (GCP) $\{M(t)\}_{t\geq 0}$ which performs k kinds of jumps of amplitude $1, 2, \ldots, k$ with positive rates $\lambda_1, \lambda_2, \ldots, \lambda_k$, respectively. For $h > 0$ small enough, its transition probabilities are given by (see Di Crescenzo et al. (2016), Section 2)

$$\Pr\{M(t+h) = n | M(t) = m\} = \begin{cases} 1 - (\lambda_1 + \lambda_2 + \cdots + \lambda_k)h + o(h), & n = m, \\ \lambda_j h + o(h), & n = m + j,\ j = 1, 2, \ldots, k, \\ o(h), & n > m + k, \end{cases}$$

where $o(h) \to 0$ as $h \to 0$. It is important to note that the jump intensities in GCP do not depend on its state. For $k = 1$, the GCP reduces to the Poisson process. Moreover, they considered a fractional variant of the GCP, namely, the generalized fractional counting process (GFCP) by taking the Caputo fractional derivative in the system of differential equations that governs the state probabilities of the GCP. A limiting case of the GFCP, namely, the convoluted fractional Poisson process (CFPP) is obtained by letting $k \to \infty$ and suitably choosing the intensity parameters (see Kataria and Khandakar (2021)). For $k = 1$, the GFCP reduces to the TFPP. For more properties and special cases of the GFCP, we refer the reader to Kataria and Khandakar (2022).

In this paper, we introduce a generalized birth process (GBP) $\{\mathcal{X}(t)\}_{t\geq 0}$ which performs jumps of size $1, 2, \ldots, k$ with positive rates $\lambda_{(n)_1}, \lambda_{(n)_2}, \ldots, \lambda_{(n)_k}$, respectively. Here, $n \geq n_0$ denotes the state of the process at any given time $t \geq 0$ such that the initial state n_0 is a fixed non-negative integer. We obtain a condition of no explosion for the GBP. We consider a fractional version of it, namely, the generalized fractional birth process (GFBP) by taking Caputo fractional derivative in the system of differential equations thats governs the state probabilities of GBP. We denote it by $\{\mathcal{X}^\alpha(t)\}_{t\geq 0}$, $0 < \alpha \leq 1$ and its state probabilities $p^\alpha(n,t) = \Pr\{\mathcal{X}^\alpha(t) = n\}$ satisfy the following

system of fractional differential equations:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}p^\alpha(n,t) = -\sum_{i=1}^{k}\lambda_{(n)_i}p^\alpha(n,t) + \sum_{i=1}^{\min\{n-n_0,k\}}\lambda_{(n-i)_i}p^\alpha(n-i,t), \quad n \geq n_0,$$

with the initial conditions

$$p^\alpha(n,0) = \begin{cases} 1, & n = n_0, \\ 0, & n > n_0. \end{cases}$$

The GFBP reduces to FPBP for $n_0 = k = 1$ and distinct $\lambda_{(n)_1}$'s for all $n \geq 1$. It is shown that the TFPP, SFPP, STFPP, GFCP and CFPP are its other particular cases. We obtain the Laplace transform of its state probabilities whose inversion yields its one-dimensional distribution. We establish the following time-changed relationship between the GBP and GFBP:

$$\mathcal{X}^\alpha(t) \stackrel{d}{=} \mathcal{X}(T_{2\alpha}(t)),$$

where $\stackrel{d}{=}$ denotes the equality in distribution and $\{T_{2\alpha}(t)\}_{t>0}$ is a random process independent of the GBP whose distribution solves (2.14). A limiting case of the GFBP is discussed which has jumps of any size $j \geq 1$. We obtain the state probabilities and a time-changed relation for this special case. Later, we discuss a state dependent version of the GFBP.

The involvement of the Caputo fractional derivative in the system of differential equations that govern the state probabilities of GFBP induces a global memory in that system. Note that the TFPP, GFCP and CFPP exhibit the long-range dependence property. It is known that the processes with long-range dependence property has applications in finance (see Ding et al. (1993)), data traffic modeling (see Karagiannis et al. (2004)), hydrology (see Doukhan et al. (2003), p. 461–472), *etc.* Also, the TFPP has applications in economics, in optics to describe light propagation through non-homogeneous media, and in the analysis of the transport of charged carriers. Moreover, the FPBP is useful in modeling explosively developing epidemics, accelerated cosmic showers and rapidly growing populations (see Orsingher and Polito (2010)). Furthermore, the SFPP can be utilized to model catastrophic events such as floods, earthquakes *etc.* Since the GFBP generalizes the TFPP, GFCP, SFPP and FBP, it might have potential applications in those areas.

The paper is organized as follows: In Section 2, we introduce the GBP and a fractional variant of it, namely, the GFBP. We derive a sufficient condition for no explosion in GBP. Then, we obtain the Laplace transform of state probabilities of GFBP whose inversion yields its one-dimensional distribution. A time-changed result connecting GBP and GFBP is established. Also, a limiting case of the GFBP is discussed. In Section 3, we consider a state dependent version of the GFBP and discuss its state probabilities. Some concluding remarks are given in Section 4.

2. Generalized fractional birth process

In this section, we introduce a generalized birth process (GBP) which performs jumps of size $1, 2, \ldots, k$ with positive rates $\lambda_{(n)_1}, \lambda_{(n)_2}, \ldots, \lambda_{(n)_k}$, respectively. Here, n denotes the state of the process at a given time $t \geq 0$. We denote the GBP by $\{\mathcal{X}(t)\}_{t\geq 0}$ and assume that it has n_0 progenitors at time $t = 0$, that is, $\Pr\{\mathcal{X}(0) = n_0\} = 1$, where $n_0 \geq 0$ is a fixed integer. The probability of its jumps in an infinitesimal interval of length h is given by

$$\Pr\{\mathcal{X}(t+h) = n + i | \mathcal{X}(t) = n\} = \begin{cases} 1 - \sum_{i=1}^{k}\lambda_{(n)_i}h + o(h), & i = 0, \\ \lambda_{(n)_i}h + o(h), & i = 1, 2, \ldots, k, \\ o(h), & i > k. \end{cases}$$

Note that the rates of jumps of GBP depend on its state, and its transition probabilities are function of states and jump sizes. Thus, the GBP is a stationary process.

Remark 2.1. In the scenario of no progenitors, that is, $n_0 = 0$ the usage of the term 'birth process' in GBP may seem unjustified. However, all the calculations and analysis that we shall do for the GBP hold for $n_0 = 0$. In this case, it may be viewed as an immigration model as the inclusion of positive rate at state 0 describes the effect of immigration in the population.

Let us denote $p(n,t) = \Pr\{\mathcal{X}(t) = n\}$, $n \geq n_0$. With these assumptions, we have

$$p(n,t+h) = p(n,t)\left(1 - \sum_{i=1}^{k} \lambda_{(n)_i} h\right) + \sum_{i=1}^{\min\{n-n_0,k\}} \lambda_{(n-i)_i} p(n-i,t)h + o(h)$$

which reduces to the following form as $h \to 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}p(n,t) = -\sum_{i=1}^{k} \lambda_{(n)_i} p(n,t) + \sum_{i=1}^{\min\{n-n_0,k\}} \lambda_{(n-i)_i} p(n-i,t), \quad n \geq n_0, \tag{2.1}$$

with the initial conditions

$$p(n,0) = \begin{cases} 1, & n = n_0, \\ 0, & n > n_0. \end{cases}$$

A fractional version of the GBP, namely, the generalized fractional birth process (GFBP) can be obtained by taking Caputo fractional derivative in the system of differential equations given in (2.1). We denote it by $\{\mathcal{X}^\alpha(t)\}_{t\geq 0}$, $0 < \alpha \leq 1$, and its state probabilities $p^\alpha(n,t) = \Pr\{\mathcal{X}^\alpha(t) = n\}$ satisfy the following system of fractional differential equations:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}p^\alpha(n,t) = -\sum_{i=1}^{k} \lambda_{(n)_i} p^\alpha(n,t) + \sum_{i=1}^{\min\{n-n_0,k\}} \lambda_{(n-i)_i} p^\alpha(n-i,t), \quad n \geq n_0, \tag{2.2}$$

with the initial conditions

$$p^\alpha(n,0) = \begin{cases} 1, & n = n_0, \\ 0, & n > n_0. \end{cases}$$

Equivalently,

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}p^\alpha(n,t) = -\sum_{i=1}^{k} \lambda_{(n)_i} p^\alpha(n,t) + \sum_{i=1}^{k} \lambda_{(n-i)_i} p^\alpha(n-i,t), \quad n \geq n_0, \tag{2.3}$$

as $p^\alpha(n-i,t) = 0$ for all $i > n$. For $\alpha = 1$, we get an equivalent expression for the GBP.

Remark 2.2. The TFPP is obtained as a particular case of the GFBP for $n_0 = 0$, $k = 1$ and $\lambda_{(n)_1} = \lambda$ for all $n \geq 0$. The GFBP reduces to the FPBP when $n_0 = k = 1$ and $\lambda_{(n)_1}$'s are distinct for all $n \geq 1$. Also, for $n_0 = 0$, $\lambda_{(n)_i} = \lambda_i$ for all $n \geq 0$ and $1 \leq i \leq k$, the GFBP reduces to the GFCP. In case of infinite jumps, that is, letting $k \to \infty$ and taking $\lambda_{(n)_i} = \lambda_i = \beta_{i-1} - \beta_i$, $i \geq 1$, $n \geq 0$, the GFCP further reduces to the CFPP, where $\{\beta_i\}_{i\in\mathbb{Z}}$ be a sequence of intensity parameters such that $\beta_i = 0$ for all $i < 0$ and $\beta_i > \beta_{i+1} > 0$ for all $i \geq 0$ with $\lim_{i\to\infty} \beta_{i+1}/\beta_i < 1$. However, if we choose $\lambda_{(n)_i} = \lambda_i = (-1)^{i+1}\lambda^\beta \beta(\beta-1)\cdots(\beta-i+1)/i!$, $i \geq 1$, $n \geq 0$ for some $0 < \beta \leq 1$ and letting $k \to \infty$, we get the STFPP which reduces to the SFPP for $\alpha = 1$.

The next result gives the sufficient condition for no explosion in GBP.

Theorem 2.3. *If* $\sum_{m=n_0}^{\infty} \left(\sum_{i=1}^{k}\sum_{j=1}^{i} \lambda^2_{(m-j+1)_i}\right)^{-1/2} = \infty$ *then* $\sum_{m=n_0}^{\infty} p(m,t) = 1$ *for all* $t \geq 0$.

Proof: Let $S(n_0+n,t)=\sum_{m=n_0}^{n_0+n}p(m,t)$, $n\geq 0$. On taking $\alpha=1$ in the system of differential equations (2.3) and adding its first $n+1$ equations, we get

$$\sum_{m=n_0}^{n_0+n}\frac{\mathrm{d}}{\mathrm{d}t}p(m,t)=-\sum_{m=n_0}^{n_0+n}\sum_{i=1}^{k}\lambda_{(m)_i}p(m,t)+\sum_{m=n_0}^{n_0+n}\sum_{i=1}^{k}\lambda_{(m-i)_i}p(m-i,t)$$

which reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}S(n_0+n,t)=-\sum_{j=1}^{k}\sum_{i=j}^{k}\lambda_{(n_0+n-j+1)_i}p(n_0+n-j+1,t),\quad n\geq 0.$$

Equivalently,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(m,t)&=-\sum_{j=1}^{k}\sum_{i=j}^{k}\lambda_{(m-j+1)_i}p(m-j+1,t)\\&=-\sum_{i=1}^{k}\sum_{j=1}^{i}\lambda_{(m-j+1)_i}p(m-j+1,t),\quad m\geq n_0.\end{aligned}\tag{2.4}$$

Note that $S(m,0)=1$ for all $m\geq n_0$. On integrating (2.4), we get

$$1-S(m,t)=\sum_{i=1}^{k}\sum_{j=1}^{i}\lambda_{(m-j+1)_i}\int_0^t p(m-j+1,s)\mathrm{d}s.\tag{2.5}$$

Let $\eta(t)=\lim_{m\to\infty}(1-S(m,t))$. Using the monotonicity of $S(m,t)$ and applying the Cauchy-Schwarz inequality in (2.5) successively, we get

$$\begin{aligned}0\leq\eta(t)&\leq\sum_{i=1}^{k}\left(\sum_{j=1}^{i}\lambda_{(m-j+1)_i}^2\right)^{1/2}\left(\sum_{j=1}^{i}\left(\int_0^t p(m-j+1,s)\mathrm{d}s\right)^2\right)^{1/2}\\&\leq\left(\sum_{i=1}^{k}\sum_{j=1}^{i}\lambda_{(m-j+1)_i}^2\right)^{1/2}\left(\sum_{i=1}^{k}\sum_{j=1}^{i}\left(\int_0^t p(m-j+1,s)\mathrm{d}s\right)^2\right)^{1/2}\\&\leq\left(\sum_{i=1}^{k}\sum_{j=1}^{i}\lambda_{(m-j+1)_i}^2\right)^{1/2}\sum_{i=1}^{k}\sum_{j=1}^{i}\int_0^t p(m-j+1,s)\mathrm{d}s.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{m=n_0}^{n_0+n}\left(\sum_{i=1}^{k}\sum_{j=1}^{i}\lambda_{(m-j+1)_i}^2\right)^{-1/2}\eta(t)&\leq\sum_{i=1}^{k}\sum_{j=1}^{i}\int_0^t\sum_{m=n_0}^{n_0+n}p(m-j+1,s)\mathrm{d}s\\&\leq k^2\int_0^t S(n_0+n,s)\mathrm{d}s.\end{aligned}$$

If $\sum_{m=n_0}^{\infty}\left(\sum_{i=1}^{k}\sum_{j=1}^{i}\lambda_{(m-j+1)_i}^2\right)^{-1/2}=\infty$ then $\eta(t)=0$ for all t. This implies $S(n_0+n,t)\to 1$ as $n\to\infty$, that is, $\sum_{m=n_0}^{\infty}p(m,t)=1$. □

Remark 2.4. On taking $n_0=k=1$ and $\lambda_{(n)_1}$'s distinct for all $n\geq 1$ in Theorem 2.3, we get the no explosion condition, that is, $\sum_{n=1}^{\infty}\lambda_{(n)_1}^{-1}=\infty$ for the pure birth process (see Feller (1968), p. 452).

Let $\tilde{p}^{\alpha}(n,s)=\int_0^{\infty} e^{-st}p^{\alpha}(n,t)\mathrm{d}t$, $s>0$ denote the Laplace transform of the state probabilities of GFBP. In Proposition 2.6, we obtain the explicit expressions for $\tilde{p}^{\alpha}(n,s)$, $n\geq n_0$.

First, we set some notations.

Let $I_{n\times n}$ denote an identity matrix, $O_{n\times m}$ be a zero matrix and $\bar{e}_n^j$ be a n-tuple unit row vector with unity at jth place and 0 elsewhere. For $m>n$, we define a matrix Q of order $n\times m$ as $Q_{n\times m}=\begin{bmatrix} I_{n\times n} & O_{n\times(m-n)}\end{bmatrix}$. For any positive integer k, let $\Theta_n^k\subset\mathbb{R}^n$ be defined as follows: $\Theta_1^k=\{(1)\}$, $k\geq 1$ and

$$\Theta_n^k=\begin{cases}\bigcup\limits_{i=1}^{k}\{\Theta_{n-i}^k Q_{(n-i)\times n}+i\bar{e}_n^{n-i+1}\}, & 1\leq k\leq n-1,\\ \bigcup\limits_{i=1}^{n-1}\{\Theta_{n-i}^n Q_{(n-i)\times n}+i\bar{e}_n^{n-i+1}\}\cup\{n\bar{e}_n^1\}, & k\geq n,\end{cases}\tag{2.6}$$

where $\Theta_{n-i}^k Q_{(n-i)\times n}\coloneqq\{\bar{x}Q_{(n-i)\times n}:\bar{x}=(x_1,x_2,\ldots,x_{n-i})\in\Theta_{n-i}^k\}$.

Using (2.6), we explicitly write the set Θ_n^k for $n=2,3,4$ as follows:

$$\Theta_2^k=\begin{cases}\{(1,1)\}, & k=1,\\ \{(1,1),(2,0)\}, & k\geq 2,\end{cases}$$

$$\Theta_3^k=\begin{cases}\{(1,1,1)\}, & k=1,\\ \{(1,1,1),(1,2,0),(2,0,1)\}, & k=2,\\ \{(1,1,1),(1,2,0),(2,0,1),(3,0,0)\}, & k\geq 3,\end{cases}$$

$$\Theta_4^k=\begin{cases}\{(1,1,1,1)\}, & k=1,\\ \{(1,1,1,1),(1,2,0,1),(1,1,2,0),(2,0,1,1),(2,0,2,0)\}, & k=2,\\ \{(1,1,1,1),(1,2,0,1),(1,1,2,0),\\ \quad(1,3,0,0),(2,0,1,1),(2,0,2,0),(3,0,0,1)\}, & k=3,\\ \{(1,1,1,1),(1,2,0,1),(1,1,2,0),(1,3,0,0),\\ \quad(2,0,1,1),(2,0,2,0),(3,0,0,1),(4,0,0,0)\}, & k\geq 4.\end{cases}$$

For $n\geq 1$, let $\Lambda_n=\mathbb{N}_0^n\setminus\{1\leq j\leq n-1:x_{j+1}=0\}$ where $(x_1,x_2,\ldots,x_n)\in\Theta_n^k$ and $\mathbb{N}_0^n=\{0,1,\ldots,n\}$. Let $n^*=|\Lambda_n|$ and

$$(j)_l=\begin{cases}0, & l=1,\\ \inf\{(j)_{l-1}<j\leq n^*:x_{j+1}\neq 0\}, & 1<l<n^*,\\ n, & l=n^*.\end{cases}\tag{2.7}$$

Remark 2.5. Note that the set Θ_n^k contains all possible ways the process can reach state n by taking jumps of size 1 to k. Each element in the set shows a specific combination of jump sizes that add up to n. These sets help in writing the expressions for the state probabilities of different states and their Laplace transforms. They are built step-by-step using a clear pattern to include all possible jump paths.

Proposition 2.6. *For $n=n_0$, the Laplace transform of the state probability of GFBP is given by*

$$\tilde{p}^{\alpha}(n_0,s)=\frac{s^{\alpha-1}}{s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0)_i}},\quad s>0,\tag{2.8}$$

and for $n = n_0 + m$, $m \geq 1$, *it is given by*

$$\tilde{p}^{\alpha}(n_0+m,s) = \begin{cases} \displaystyle\sum_{\Theta_m^k} \frac{s^{\alpha-1}\prod_{j=0}^{m-1}\lambda_{(n_0+j)_{x_{j+1}}}}{\displaystyle\prod_{j\in\Lambda_m}\left(s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0+j)_i}\right)}, & 1\leq k\leq m-1, \\ \displaystyle\sum_{\Theta_m^m} \frac{s^{\alpha-1}\prod_{j=0}^{m-1}\lambda_{(n_0+j)_{x_{j+1}}}}{\displaystyle\prod_{j\in\Lambda_m}\left(s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0+j)_i}\right)}, & k\geq m, \end{cases} \tag{2.9}$$

for $s > 0$, *where* $(x_1, x_2, \ldots, x_m) \in \Theta_m^k$ *and* $\lambda_{(n)_0}$ *is interpreted as unity for all* $n \geq n_0$.

Proof: On substituting $n = n_0$ in (2.3), we get

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}p^{\alpha}(n_0,t) = -\sum_{i=1}^{k}\lambda_{(n_0)_i}p^{\alpha}(n_0,t), \tag{2.10}$$

where we have used the fact that $p^{\alpha}(m,t) = 0$ for all $m < n_0$. On taking the Laplace transform in (2.10) and using (1.1), we obtain

$$\tilde{p}^{\alpha}(n_0,s) = \frac{s^{\alpha-1}}{s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0)_i}}.$$

On taking $n = n_0 + 1$ in (2.3), we get

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}p^{\alpha}(n_0+1,t) = -\sum_{i=1}^{k}\lambda_{(n_0+1)_i}p^{\alpha}(n_0+1,t) + \lambda_{(n_0)_1}p^{\alpha}(n_0,t),$$

which on taking Laplace transform gives

$$\begin{aligned}\tilde{p}^{\alpha}(n_0+1,s) &= \frac{\lambda_{(n_0)_1}\tilde{p}^{\alpha}(n_0,s)}{s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0+1)_i}} \\ &= \frac{\lambda_{(n_0)_1}s^{\alpha-1}}{\left(s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0)_i}\right)\left(s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0+1)_i}\right)}.\end{aligned}$$

For $n = n_0 + 2$ in (2.3), we get

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}p^{\alpha}(n_0+2,t) = -\sum_{i=1}^{k}\lambda_{(n_0+2)_i}p^{\alpha}(n_0+2,t) + \sum_{i=1}^{k}\lambda_{(n_0+2-i)_i}p^{\alpha}(n_0+2-i,t),$$

which gives

$$\tilde{p}^{\alpha}(n_0+2,s) = \frac{\sum_{i=1}^{k}\lambda_{(n_0+2-i)_i}\tilde{p}^{\alpha}(n_0+2-i,s)}{s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0+2)_i}}.$$

Thus,

$$\tilde{p}^{\alpha}(n_0+2,s) = \begin{cases} \dfrac{\lambda_{(n_0)_1}\lambda_{(n_0+1)_1}s^{\alpha-1}}{\displaystyle\prod_{j=0}^{2}\left(s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0+j)_i}\right)}, & k=1, \\ \dfrac{\lambda_{(n_0)_1}\lambda_{(n_0+1)_1}s^{\alpha-1}}{\displaystyle\prod_{j=0}^{2}\left(s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0+j)_i}\right)} + \dfrac{\lambda_{(n_0)_2}s^{\alpha-1}}{\displaystyle\prod_{\substack{j=0\\ j\neq 1}}^{2}\left(s^{\alpha}+\sum_{i=1}^{k}\lambda_{(n_0+j)_i}\right)}, & k\geq 2. \end{cases}$$

Similarly, we get the following expressions in the case of $n = n_0 + 3$:

$$\tilde{p}^{\alpha}(n_0+3,s) = \frac{\sum_{i=1}^{k} \lambda_{(n_0+3-i)_i} \tilde{p}^{\alpha}(n_0+3-i,s)}{s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+3)_i}}.$$

For $k = 1$, we get

$$\tilde{p}^{\alpha}(n_0+3,s) = \frac{\lambda_{(n_0)_1}\lambda_{(n_0+1)_1}\lambda_{(n_0+2)_1} s^{\alpha-1}}{\prod_{j=0}^{3}\left(s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)}.$$

For $k = 2$, we get

$$\tilde{p}^{\alpha}(n_0+3,s) = \frac{\lambda_{(n_0)_1}\lambda_{(n_0+1)_1}\lambda_{(n_0+2)_1} s^{\alpha-1}}{\prod_{j=0}^{3}\left(s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)} + \frac{\lambda_{(n_0)_2}\lambda_{(n_0+2)_1} s^{\alpha-1}}{\prod_{\substack{j=0 \\ j\neq 1}}^{3}\left(s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)} + \frac{\lambda_{(n_0)_1}\lambda_{(n_0+1)_2} s^{\alpha-1}}{\prod_{\substack{j=0 \\ j\neq 2}}^{3}\left(s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)}.$$

For $k \geq 3$, we get

$$\tilde{p}^{\alpha}(n_0+3,s) = \frac{\lambda_{(n_0)_1}\lambda_{(n_0+1)_1}\lambda_{(n_0+2)_1} s^{\alpha-1}}{\prod_{j=0}^{3}\left(s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)} + \frac{\lambda_{(n_0)_2}\lambda_{(n_0+2)_1} s^{\alpha-1}}{\prod_{\substack{j=0 \\ j\neq 1}}^{3}\left(s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)} + \frac{\lambda_{(n_0)_1}\lambda_{(n_0+1)_2} s^{\alpha-1}}{\prod_{\substack{j=0 \\ j\neq 2}}^{3}\left(s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)} + \frac{\lambda_{(n_0)_3} s^{\alpha-1}}{\prod_{\substack{j=0 \\ j\neq 1,2}}^{3}\left(s^{\alpha} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)}.$$

Thus,

$$\tilde{p}^{\alpha}(n_0+3,s) = \begin{cases} \displaystyle\sum_{\Theta_3^k} \frac{s^{\alpha-1}\prod_{j=0}^{2}\lambda_{(n_0+j)_{x_{j+1}}}}{\displaystyle\prod_{j\in\Lambda_3}\left(s^{\alpha} + \sum_{i=1}^{k}\lambda_{(n_0+j)_i}\right)}, & 1 \leq k \leq 2, \\ \displaystyle\sum_{\Theta_3^3} \frac{s^{\alpha-1}\prod_{j=0}^{2}\lambda_{(n_0+j)_{x_{j+1}}}}{\displaystyle\prod_{j\in\Lambda_3}\left(s^{\alpha} + \sum_{i=1}^{k}\lambda_{(n_0+j)_i}\right)}, & k \geq 3. \end{cases}$$

By iterating this procedure, we get the required result. □

The next result follows by taking $\nu_n = \alpha$ for all $n \geq 1$ in Eq. (9) and Eq. (39) of Kataria and Vellaisamy (2019).

Proposition 2.7. *Let λ_j, $j = 1, 2, \ldots, n$ be any positive real numbers. Then,*

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha-1}}{\prod_{j=1}^{n}(s^{\alpha}+\lambda_j)};t\right) = \frac{(-1)^{n-1}}{\prod_{j=2}^{n}\lambda_j}\sum_{i=n-1}^{\infty}\frac{(-t^{\alpha})^i}{\Gamma(i\alpha+1)}\sum_{\Omega_n^i}\prod_{j=1}^{n}\lambda_j^{y_j}, \ s>0, \ t\geq 0,$$

where $\Omega_n^i = \{(y_1, y_2, \ldots, y_n) : \sum_{j=1}^{n} y_j = i, \ y_1 \in \mathbb{N}_0, \ y_j \in \mathbb{N}_0 \setminus \{0\}, \ 2 \leq j \leq n\}$.

In case λ_j's are distinct, the above result has a simplified form. The following result is proved using method of induction by Alipour et al. (2015). Here, we give an alternate proof.

Lemma 2.8. *Let*

$$p(x) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x+\lambda_j}{\lambda_j - \lambda_i} - 1, \quad n > 1,$$

where $x \in \mathbb{R}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct real numbers. Then, $p(x) \equiv 0$.

Proof: Note that $p(-\lambda_j) = 0$ for $j = 1, 2, \ldots, n$, that is, $p(x)$ is a polynomial of degree $n-1$ with n distinct roots. This implies that it is a zero polynomial. This proves the result. □

The following result will be used (see Kilbas et al. (2006)):

$$\int_0^\infty e^{-st} E_{\alpha,1}(-\omega t^\alpha)\, \mathrm{d}t = \frac{s^{\alpha-1}}{s^\alpha + \omega}, \tag{2.11}$$

where $E_{\alpha,1}(\cdot)$ is the Mittag-Leffler function defined as

$$E_{\alpha,1}(x) := \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j\alpha+1)}, \quad x \in \mathbb{R}, \quad \alpha > 0.$$

Proposition 2.9. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct positive real numbers. Then,*

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha-1}}{\prod_{j=1}^{n}(s^\alpha+\lambda_j)}; t\right) = \sum_{i=1}^{n} \frac{E_{\alpha,1}(-\lambda_i t^\alpha)}{\prod\limits_{j=1, j\neq i}^{n} (\lambda_j - \lambda_i)}, \ s > 0, \ t \geq 0, \ n > 1.$$

Proof: Using Lemma 2.8, we get

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s^{\alpha-1}}{\prod_{j=1}^{n}(s^\alpha+\lambda_j)}; t\right) &= \mathcal{L}^{-1}\left(\sum_{i=1}^{n} \frac{s^{\alpha-1}}{s^\alpha+\lambda_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{(\lambda_j - \lambda_i)}; t\right) \\ &= \sum_{i=1}^{n} \frac{E_{\alpha,1}(-\lambda_i t^\alpha)}{\prod\limits_{j=1, j\neq i}^{n} (\lambda_j - \lambda_i)},\end{aligned}$$

where the last step follows from (2.11). □

Theorem 2.10. *For $n = n_0$, the state probability of GFBP is given by*

$$p^\alpha(n_0, t) = E_{\alpha,1}\left(-\sum_{i=1}^{k} \lambda_{(n_0)_i} t^\alpha\right), \ t \geq 0,$$

and for $n = n_0 + m$, $m \geq 1$, it is given by

$$p^\alpha(n_0+m, t) = \begin{cases} \displaystyle\sum_{\Theta_m^k} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \frac{(-1)^{m^*-1}}{\prod\limits_{l=2}^{m^*} \mu_l} \sum_{i=m^*-1}^{\infty} \frac{(-t^\alpha)^i}{\Gamma(i\alpha+1)} \sum_{\Omega_{m^*}^i} \prod_{l=1}^{m^*} \mu_l^{y_l}, & 1 \leq k \leq m-1, \\ \displaystyle\sum_{\Theta_m^m} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \frac{(-1)^{m^*-1}}{\prod\limits_{l=2}^{m^*} \mu_l} \sum_{i=m^*-1}^{\infty} \frac{(-t^\alpha)^i}{\Gamma(i\alpha+1)} \sum_{\Omega_{m^*}^i} \prod_{l=1}^{m^*} \mu_l^{y_l}, & k \geq m, \end{cases} \tag{2.12}$$

for all $t \geq 0$, *where* $(x_1, \ldots, x_m) \in \Theta_m^k$, $(y_1, \ldots, y_{m^*}) \in \Omega_{m^*}^i$, $m^* = |\Lambda_m|$, $\mu_l = \sum_{i=1}^k \lambda_{(n_0+(j)_l)_i}$, $(j)_l$*'s are given by (2.7).*

If μ_l, $l = 1, 2, \ldots, m^*$ *are distinct then*

$$p^\alpha(n_0+m,t) = \begin{cases} \displaystyle\sum_{\Theta_m^k} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \sum_{i=1}^{m^*} \frac{E_{\alpha,1}(-\mu_i t^\alpha)}{\displaystyle\prod_{l=1,l\neq i}^{m^*} (\mu_l - \mu_i)}, & 1 \leq k \leq m-1, \\ \displaystyle\sum_{\Theta_m^m} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \sum_{i=1}^{m^*} \frac{E_{\alpha,1}(-\mu_i t^\alpha)}{\displaystyle\prod_{l=1,l\neq i}^{m^*} (\mu_l - \mu_i)}, & k \geq m, \end{cases} \tag{2.13}$$

for all $t \geq 0$.

Proof: For the case $n = n_0$, the result follows by taking inverse Laplace transform in (2.8) and then by using (2.11). For $n > n_0$, the Laplace transform (2.9) of GFBP can be rewritten as

$$\tilde{p}^\alpha(n_0+m,s) = \begin{cases} \displaystyle\sum_{\Theta_m^k} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \frac{s^{\alpha-1}}{\displaystyle\prod_{l=1}^{m^*} (s^\alpha + \mu_l)}, & 1 \leq k \leq m-1, \\ \displaystyle\sum_{\Theta_m^m} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \frac{s^{\alpha-1}}{\displaystyle\prod_{l=1}^{m^*} (s^\alpha + \mu_l)}, & k \geq m. \end{cases}$$

The result follows on taking inverse Laplace transform in the above equation and using Proposition 2.7 and Proposition 2.9, respectively. □

Remark 2.11. Let W_1^α denote the first waiting time of GFBP. Its distribution is given by

$$\Pr\{W_1^\alpha > t\} = \Pr\{\mathcal{X}^\alpha(t) = n_0\} = E_{\alpha,1}\left(-\sum_{i=1}^k \lambda_{(n_0)_i} t^\alpha\right).$$

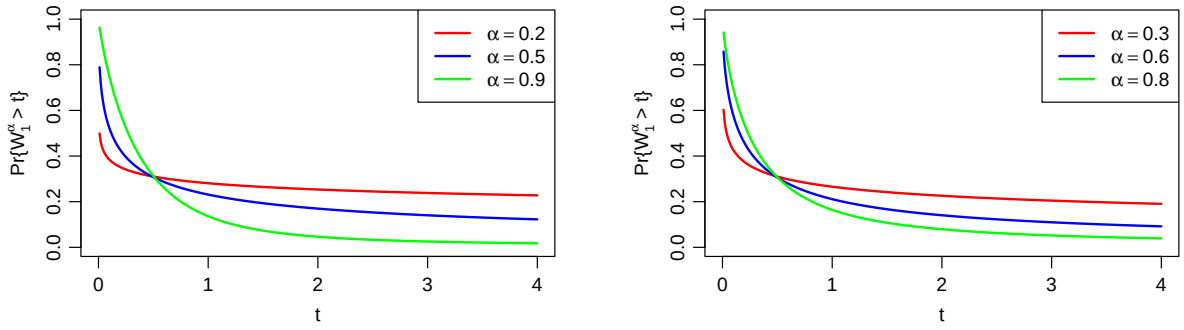

FIGURE 2.1. Plots of $\Pr\{W_1^\alpha > t\}$, $0 \leq t \leq 4$ for different values of α with $\lambda_{(n_0)_i} = \frac{1}{2^i}$, $1 \leq i \leq 3$ and $k = 3$.

Remark 2.12. Recall that the TFPP is obtained as a particular case of the GFBP when $n_0 = 0$, $k = 1$ and $\lambda_{(n)_1} = \lambda$ for all $n \geq 0$. In this case, we have $\Theta_m^1 = \{(1, 1, \ldots, 1)\}$, $m^* = |\mathbb{N}_0^m| = m + 1$, $|\Omega_{m+1}^i| = \frac{i!}{m!(i-m)!}$ and $\mu_l = \lambda_{((j)_l)_1} = \lambda$ as $(j)_l = l - 1$, for all $1 \leq l \leq m + 1$. On substituting these values in (2.12), we get

$$\begin{aligned} p^\alpha(m,t) &= \sum_{\Theta_m^1} \prod_{j=0}^{m-1} \lambda_{(j)_1} \frac{(-1)^m}{\prod_{l=2}^{m+1} \mu_l} \sum_{i=m}^{\infty} \frac{(-t^\alpha)^i}{\Gamma(i\alpha+1)} \sum_{\Omega_{m+1}^i} \prod_{l=1}^{m+1} \mu_l^{y_l} \\ &= (-1)^m \sum_{i=m}^{\infty} \frac{(-\lambda t^\alpha)^i}{\Gamma(i\alpha+1)} \sum_{\Omega_{m+1}^i} 1 \\ &= (-1)^m \sum_{i=m}^{\infty} \frac{(-\lambda t^\alpha)^i}{\Gamma(i\alpha+1)} \frac{i!}{m!(i-m)!} \\ &= \frac{(\lambda t^\alpha)^m}{m!} \sum_{i=0}^{\infty} \frac{(i+m)!}{i!} \frac{(-\lambda t^\alpha)^i}{\Gamma((i+m)\alpha+1)}, \end{aligned}$$

which agrees with the probability mass function (pmf) of TFPP (see Beghin and Orsingher (2009), Eq. (2.10)).

Remark 2.13. Recall that the FPBP is obtained as a particular case of the GFBP when $n_0 = k = 1$ and $\lambda_{(n)_1}$'s are distinct. For notational convenience, we write $\lambda_{(n)_1} = \lambda_n$ for all $n \geq 1$. In this case, we have $\Theta_m^1 = \{(1, 1, \ldots, 1)\}$, $m^* = |\mathbb{N}_0^m| = m + 1$ and $\mu_l = \lambda_{(1+(j)_l)_1} = \lambda_l$ as $(j)_l = l - 1$, for all $1 \leq l \leq m + 1$. On substituting these values in (2.13), we get

$$\begin{aligned} p^\alpha(m+1,t) &= \sum_{\Theta_m^1} \prod_{j=0}^{m-1} \lambda_{j+1} \sum_{i=1}^{m+1} \frac{E_{\alpha,1}(-\lambda_i t^\alpha)}{\prod_{l=1, l\neq i}^{m+1} (\lambda_l - \lambda_i)} \\ &= \prod_{j=1}^{m} \lambda_j \sum_{i=1}^{m+1} \frac{E_{\alpha,1}(-\lambda_i t^\alpha)}{\prod_{l=1, l\neq i}^{m+1} (\lambda_l - \lambda_i)}, \end{aligned}$$

which agrees with the pmf of FPBP (see Orsingher and Polito (2010), Eq. (2.3)).

Let $f_{T_{2\alpha}}(x,t)$ be the folded solution of the following fractional diffusion equation (see Orsingher and Beghin (2004)):

$$\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}t^{2\alpha}} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{2.14}$$

with $u(x, 0) = \delta(x)$ for $0 < \alpha \leq 1$ and $\frac{\partial}{\partial t} u(x, 0) = 0$ for $1/2 < \alpha \leq 1$.

Its Laplace transform is given by (see Orsingher and Polito (2010), Eq. (2.29))

$$\int_0^\infty e^{-st} f_{T_{2\alpha}}(x,t)\mathrm{d}t = s^{\alpha-1} e^{-xs^\alpha}, \quad x > 0. \tag{2.15}$$

Let $\{T_{2\alpha}(t)\}_{t>0}$ denote a random process whose one-dimensional distribution is given by $f_{T_{2\alpha}}(x,t)$.

Theorem 2.14. *Let μ_l's be as defined in Theorem 2.10 and $\{T_{2\alpha}(t)\}_{t>0}$, $0 < \alpha \leq 1$, be a process whose distribution solves (2.14). If μ_l's are distinct then the following holds for GFBP:*

$$\mathcal{X}^\alpha(t) \stackrel{d}{=} \mathcal{X}(T_{2\alpha}(t)), \tag{2.16}$$

where $\{T_{2\alpha}(t)\}_{t>0}$ is independent of the GBP $\{\mathcal{X}(t)\}_{t>0}$.

Proof: Let $\mu_0 = \sum_{i=1}^{k} \lambda_{(n_0)_i}$. The Laplace transform of the probability generating function of GFBP can be written as

$$\begin{aligned}
\tilde{G}^{\alpha}(u,s) &= \int_0^{\infty} e^{-st} \sum_{n=n_0}^{\infty} u^n p^{\alpha}(n,t)\mathrm{d}t \\
&= \int_0^{\infty} e^{-st} \left(u^{n_0} p^{\alpha}(n_0,t) + \sum_{n=n_0+1}^{n_0+k} u^n p^{\alpha}(n,t) + \sum_{n=n_0+k+1}^{\infty} u^n p^{\alpha}(n,t) \right) \mathrm{d}t \\
&= u^{n_0} \int_0^{\infty} e^{-st} \left(p^{\alpha}(n_0,t) + \sum_{m=1}^{k} u^m p^{\alpha}(n_0+m,t) + \sum_{m=k+1}^{\infty} u^m p^{\alpha}(n_0+m,t) \right) \mathrm{d}t \\
&= u^{n_0} \Bigg(\frac{s^{\alpha-1}}{s^{\alpha}+\mu_0} + \sum_{m=1}^{k} u^m \sum_{\Theta_m^m} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \sum_{i=1}^{m^*} \frac{s^{\alpha-1}}{\prod_{l=1,l\neq i}^{m^*} (\mu_l-\mu_i)(s^{\alpha}+\mu_i)} \\
&\quad + \sum_{m=k+1}^{\infty} u^m \sum_{\Theta_m^k} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \sum_{i=1}^{m^*} \frac{s^{\alpha-1}}{\prod_{l=1,l\neq i}^{m^*} (\mu_l-\mu_i)(s^{\alpha}+\mu_i)} \Bigg), \quad \text{(using (2.11) and (2.13))} \\
&= s^{\alpha-1} \int_0^{\infty} e^{-\xi s^{\alpha}} \Bigg(u^{n_0} e^{-\xi\mu_0} + \sum_{m=1}^{k} u^{m+n_0} \sum_{\Theta_m^m} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \sum_{i=1}^{m^*} \frac{e^{-\xi\mu_i}}{\prod_{l=1,l\neq i}^{m^*} (\mu_l-\mu_i)} \\
&\quad + \sum_{m=k+1}^{\infty} u^{m+n_0} \sum_{\Theta_m^k} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \sum_{i=1}^{m^*} \frac{e^{-\xi\mu_i}}{\prod_{l=1,l\neq i}^{m^*} (\mu_l-\mu_i)} \Bigg) \mathrm{d}\xi \\
&= s^{\alpha-1} \int_0^{\infty} e^{-\xi s^{\alpha}} \left(u^{n_0} p(n_0,\xi) + \sum_{m=1}^{k} u^{m+n_0} p(n_0+m,\xi) + \sum_{m=k+1}^{\infty} u^{m+n_0} p(n_0+m,\xi) \right) \mathrm{d}\xi \\
&= \int_0^{\infty} s^{\alpha-1} e^{-\xi s^{\alpha}} G(u,\xi)\mathrm{d}\xi \\
&= \int_0^{\infty} G(u,\xi) \int_0^{\infty} e^{-st} f_{T_{2\alpha}}(\xi,t)\mathrm{d}t\mathrm{d}\xi, \quad \text{(using (2.15))} \\
&= \int_0^{\infty} e^{-st} \left(\int_0^{\infty} G(u,\xi) f_{T_{2\alpha}}(\xi,t)\mathrm{d}\xi \right) \mathrm{d}t.
\end{aligned}$$

By uniqueness of Laplace transform, we get

$$G^{\alpha}(u,t) = \int_0^{\infty} G(u,\xi) f_{T_{2\alpha}}(\xi,t)\mathrm{d}\xi.$$

This completes the proof. □

Remark 2.15. Let $\{H_\alpha(t)\}_{t>0}$, $0 < \alpha < 1$, be an inverse stable subordinator. As the density functions of $H_\alpha(t)$ and $T_{2\alpha}(t)$ coincides (see Meerschaert et al. (2011)), it follows that

$$\mathcal{X}^{\alpha}(t) \stackrel{d}{=} \mathcal{X}(H_\alpha(t)), \quad t > 0,$$

where the inverse stable subordinator is independent of the GBP.

A subordinator $\{D_f(t)\}_{t\geq 0}$ associated with a Bernstein function f is a one-dimensional Lévy process whose sample paths are non-decreasing (see Applebaum (2009), Section 1.3.2). Its Laplace

transform is given by:

$$\mathbb{E}\left(e^{-sD_f(t)}\right) = e^{-tf(s)},\ s > 0.$$

Its first passage time is called the inverse subordinator. It is defined as

$$Y_f(t) := \inf\{x \geq 0 : D_f(x) > t\},\ t \geq 0.$$

We consider the following time-changed process:

$$\mathcal{X}^f(t) := \mathcal{X}(Y_f(t)),\ t \geq 0,$$

where the inverse subordinator is independent of the GBP. It can be shown that the state probabilities $p^f(n,t) = \Pr\{\mathcal{X}^f(t) = n\}$ of $\{\mathcal{X}^f(t)\}_{t\geq 0}$ governs the following system of differential equations:

$$ {}^f\mathcal{D}_t p^f(n,t) = -\sum_{i=1}^{k} \lambda_{(n)_i} p^f(n,t) + \sum_{i=1}^{\min\{n-n_0,k\}} \lambda_{(n-i)_i} p^f(n-i,t),\quad n \geq n_0, \tag{2.17}$$

with the initial conditions

$$p^f(n,0) = \begin{cases} 1, & n = n_0, \\ 0, & n > n_0. \end{cases}$$

Here, ${}^f\mathcal{D}_t$ is the generalized Caputo derivative associated with the Bernštein function f. For the definition of ${}^f\mathcal{D}_t$, we refer the reader to Kochubei (2011) and Toaldo (2015). For $f(s) = s^\alpha$, $\alpha \in (0,1)$, the derivative ${}^f\mathcal{D}_t$ reduces to the Caputo fractional derivative, thus the system of fractional differential equations given in (2.17) reduces to that of (2.2).

Remark 2.16. For $\alpha = 1/2$, the process $\{T_{2\alpha}(t)\}_{t>0}$ becomes a reflecting Brownian motion $\{|B(t)|\}_{t>0}$ (see Beghin and Orsingher (2009)) as the diffusion equation (2.14) reduces to the heat equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \delta(x). \end{cases}$$

So, $\mathcal{X}^{1/2}(t)$ coincides with GBP at a Brownian time, that is, $\mathcal{X}^{1/2}(t) \stackrel{d}{=} \mathcal{X}(|B(t)|)$, $t > 0$.

Remark 2.17. Using (2.16), we get the following relationship between the pmfs of GFBP and GBP:

$$p^\alpha(m,t) = \int_0^\infty p(m,\xi) f_{T_{2\alpha}}(\xi,t)\mathrm{d}\xi.$$

It implies that $\sum_{m=n_0}^\infty p^\alpha(m,t) = 1$ if and only if $\sum_{m=n_0}^\infty p(m,t) = 1$. From Theorem 2.3, it follows that $\sum_{m=n_0}^\infty \left(\sum_{i=1}^k \sum_{j=1}^i \lambda^2_{(m-j+1)_i}\right)^{-1/2} = \infty$ is the non-exploding condition for GFBP.

2.1. *A limiting case of GFBP.* If we consider a process $\{\mathcal{X}_1^\alpha(t)\}_{t\geq 0}$ which has the jumps of any size $j \geq 1$ then from the system (2.2) of GFBP, we get

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} p_1^\alpha(n,t) = -\sum_{i=1}^{\infty} \lambda_{(n)_i} p_1^\alpha(n,t) + \sum_{i=1}^{n} \lambda_{(n-i)_i} p_1^\alpha(n-i,t),\quad n \geq n_0,$$

where $p_1^\alpha(n,t) = \Pr\{\mathcal{X}_1^\alpha(t) = n\}$, $0 < \alpha \leq 1$. Here, we assume that $\sum_{i=1}^\infty \lambda_{(n)_i} < \infty$ for all $n \geq n_0$.

From Theorem 2.10, we get its state probabilities as $p_1^\alpha(n_0,t) = E_{\alpha,1}\left(-\sum_{i=1}^\infty \lambda_{(n_0)_i} t^\alpha\right)$, and

$$p_1^\alpha(n_0+m,t) = \sum_{\Theta_m^m} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \frac{(-1)^{m^*-1}}{\prod_{l=2}^{m^*} \mu_l} \sum_{i=m^*-1}^{\infty} \frac{(-t^\alpha)^i}{\Gamma(i\alpha+1)} \sum_{\Omega_{m^*}^i} \prod_{l=1}^{m^*} \mu_l^{y_l},\quad m \geq 1,$$

where $(x_1, x_2, \ldots, x_m) \in \Theta_m^m$, $(y_1, y_2, \ldots, y_{m^*}) \in \Omega_{m^*}^i$, $m^* = |\Lambda_m|$, $\mu_l = \sum_{i=1}^{\infty} \lambda_{(n_0+(j)_l)_i}$, $(j)_l$'s are given by (2.7). If μ_l, $l = 1, 2, \ldots, m^*$ are distinct then

$$p_1^{\alpha}(n_0 + m, t) = \sum_{\Theta_m^m} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \sum_{i=1}^{m^*} \frac{E_{\alpha,1}(-\mu_i t^{\alpha})}{\prod_{l=1, l\neq i}^{m^*} (\mu_l - \mu_i)}.$$

Let $\{\mathcal{X}_1(t)\}_{t\geq 0}$ denote the non-fractional version of $\{\mathcal{X}_1^{\alpha}(t)\}_{t>0}$. The following relationship holds:

$$\mathcal{X}_1^{\alpha}(t) \stackrel{d}{=} \mathcal{X}_1(T_{2\alpha}(t)),$$

where $\{T_{2\alpha}(t)\}_{t>0}$ is independent of $\{\mathcal{X}_1(t)\}_{t>0}$. Its proof follows along the similar lines to that of Theorem 2.14.

3. A state dependent version of the GFBP

Garra et al. (2015) introduced and studied the state dependent versions of TFPP and FPBP. Here, we introduce a state dependent version of the GFBP. We denote it by $\{\mathcal{Q}(t)\}_{t\geq 0}$. It is defined as the stochastic process whose state probabilities $q(n,t) = \Pr\{\mathcal{Q}(t) = n\}$ satisfy the following system of fractional differential equations:

$$\frac{\mathrm{d}^{\alpha_n}}{\mathrm{d}t^{\alpha_n}} q(n,t) = -\sum_{i=1}^{k} \lambda_{(n)_i} q(n,t) + \sum_{i=1}^{\min\{n-n_0,k\}} \lambda_{(n-i)_i} q(n-i,t), \quad 0 < \alpha_n \leq 1, \ n \geq n_0, \tag{3.1}$$

with the initial conditions

$$q(n,0) = \begin{cases} 1, & n = n_0, \\ 0, & n > n_0. \end{cases}$$

Note that the System (3.1) is obtained by taking variable order fractional derivatives in (2.2). The Caputo fractional derivative of state probability is the convolution of rate of change in state probability and a suitable weight function, that is,

$$\frac{\mathrm{d}^{\alpha_n}}{\mathrm{d}t^{\alpha_n}} q(n,t) = \frac{\mathrm{d}}{\mathrm{d}t} q(n,t) * \frac{t^{-\alpha_n}}{\Gamma(1-\alpha_n)}.$$

Here, the power of the weight function depends on the actual state of the process at time $t \geq 0$. Thus, the number of events that have occurred till time t modifies the order of fractional derivative.

Next, we give the Laplace transform of the state probabilities of state dependent GFBP. Its proof follows similar lines to that of Proposition 2.6. Let Θ_n^k, Λ_n *etc.* be as defined in Section 2.

Proposition 3.1. *For $n = n_0$, the Laplace transform of the state probability of $\{\mathcal{Q}(t)\}_{t\geq 0}$ is given by*

$$\tilde{q}(n_0, s) = \frac{s^{\alpha_{n_0}-1}}{s^{\alpha_{n_0}} + \sum_{i=1}^{k} \lambda_{(n_0)_i}}$$

and for $n = n_0 + m$, $m \geq 1$, it is given by

$$\tilde{q}(n_0+m, s) = \begin{cases} \displaystyle\sum_{\Theta_m^k} \frac{s^{\alpha_{n_0}-1} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}}}{\displaystyle\prod_{j\in\Lambda_m} \left(s^{\alpha_{n_0+j}} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)}, & 1 \leq k \leq m-1, \\ \displaystyle\sum_{\Theta_m^m} \frac{s^{\alpha_{n_0}-1} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}}}{\displaystyle\prod_{j\in\Lambda_m} \left(s^{\alpha_{n_0+j}} + \sum_{i=1}^{k} \lambda_{(n_0+j)_i}\right)}, & k \geq m, \end{cases}$$

where $(x_1, x_2, \ldots, x_m) \in \Theta_m^k$.

The following result holds (see Kataria and Vellaisamy (2019), Eq. (9) and Eq. (39)):

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha_1-1}}{\prod_{j=1}^{n}(s^{\alpha_j}+\lambda_j)};t\right)=\frac{(-1)^{n-1}}{\prod_{j=2}^{n}\lambda_j}\sum_{i=n-1}^{\infty}(-1)^i\sum_{\Omega_n^i}\frac{t^{\sum_{j=1}^{n}y_j\alpha_j}\prod_{j=1}^{n}\lambda_j^{y_j}}{\Gamma(1+\sum_{j=1}^{n}y_j\alpha_j)},\quad n\geq 1, \tag{3.2}$$

where λ_j's are positive real numbers and $\Omega_n^i=\{(y_1,y_2,\ldots,y_n):\sum_{j=1}^{n}y_j=i,\ y_1\in\mathbb{N}_0,\ y_j\in\mathbb{N}_0\setminus\{0\},\ 2\leq j\leq n\}$.

The state probabilities $q(n,t)$ for all $n\geq n_0$ can be obtained by taking inverse Laplace transform in Proposition 3.1 and by using (3.2).

Theorem 3.2. *For $n=n_0$, the state probability of $\{\mathcal{Q}(t)\}_{t\geq 0}$ is given by*

$$q(n_0,t)=E_{\alpha_{n_0},1}\left(-\sum_{i=1}^{k}\lambda_{(n_0)_i}t^{\alpha_{n_0}}\right),$$

and for $n=n_0+m$, $m\geq 1$, it is given by

$$q(n_0+m,t)=\begin{cases}\displaystyle\sum_{\Theta_m^k}\prod_{j=0}^{m-1}\lambda_{(n_0+j)_{x_{j+1}}}\mathcal{L}^{-1}\left(\frac{s^{\alpha_{n_0}-1}}{\prod_{l=1}^{m^*}(s^{\alpha_{n_0+(j)_l}}+\mu_l)};t\right), & 1\leq k\leq m-1,\\ \displaystyle\sum_{\Theta_m^m}\prod_{j=0}^{m-1}\lambda_{(n_0+j)_{x_{j+1}}}\mathcal{L}^{-1}\left(\frac{s^{\alpha_{n_0}-1}}{\prod_{l=1}^{m^*}(s^{\alpha_{n_0+(j)_l}}+\mu_l)};t\right), & k\geq m,\end{cases}$$

where $m^*=|\Lambda_m|$, $\mu_l=\sum_{i=1}^{k}\lambda_{(n_0+(j)_l)_i}$, $(j)_l$'s are given by (2.7) and

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha_{n_0}-1}}{\prod_{l=1}^{m^*}(s^{\alpha_{n_0+(j)_l}}+\mu_l)};t\right)=\frac{(-1)^{m^*-1}}{\prod_{l=2}^{m^*}\mu_l}\sum_{i=m^*-1}^{\infty}(-1)^i\sum_{\Omega_{m^*}^i}\frac{t^{\sum_{l=1}^{m^*}y_l\alpha_{n_0+(j)_l}}}{\Gamma\left(1+\sum_{l=1}^{m^*}y_l\alpha_{n_0+(j)_l}\right)}\prod_{l=1}^{m^*}\mu_l^{y_l}.$$

Remark 3.3. Let W_1 denote the first waiting time of state dependent GFBP. Its distribution is given by

$$\Pr\{W_1>t\}=\Pr\{\mathcal{Q}(t)=n_0\}=E_{\alpha_{n_0},1}\left(-\sum_{i=1}^{k}\lambda_{(n_0)_i}t^{\alpha_{n_0}}\right).$$

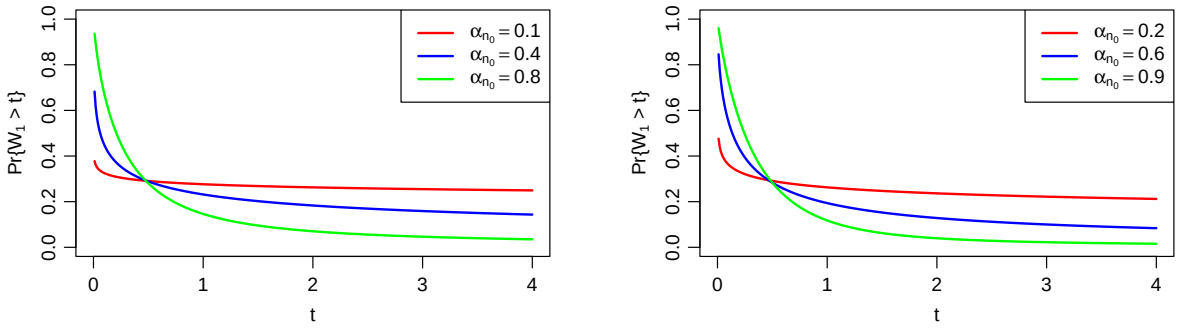

FIGURE 3.2. Plots of $\Pr\{W_1>t\}$, $0\leq t\leq 4$ for different values of α_{n_0} with $\lambda_{(n_0)_i}=e^{-i}$, $1\leq i\leq 5$ and $k=5$.

If we consider a process $\{\mathcal{Q}_1(t)\}_{t\geq 0}$ which has the jumps of any size $j \geq 1$ then from the system (3.1) of $\{\mathcal{Q}(t)\}_{t\geq 0}$, we get

$$\frac{\mathrm{d}^{\alpha_n}}{\mathrm{d}t^{\alpha_n}} q_1(n,t) = -\sum_{i=1}^{\infty} \lambda_{(n)_i} q_1(n,t) + \sum_{i=1}^{n} \lambda_{(n-i)_i} q_1(n-i,t), \quad 0 < \alpha_n \leq 1, \ n \geq n_0,$$

where $q_1(n,t) = \Pr\{\mathcal{Q}_1(t) = n\}$. Here, we assume that $\sum_{i=1}^{\infty} \lambda_{(n)_i} < \infty$ for all $n \geq n_0$.

Its state probabilities are given by $q_1(n_0,t) = E_{\alpha_{n_0},1}\left(-\sum_{i=1}^{\infty} \lambda_{(n_0)_i} t^{\alpha_{n_0}}\right)$ and

$$q_1(n_0+m,t) = \sum_{\Theta_m^m} \prod_{j=0}^{m-1} \lambda_{(n_0+j)_{x_{j+1}}} \mathcal{L}^{-1}\left(\frac{s^{\alpha_{n_0}-1}}{\prod_{l=1}^{m^*}(s^{\alpha_{n_0+(j)_l}} + \mu_l)}; t\right), \quad m \geq 1,$$

where $\mu_l = \sum_{i=1}^{\infty} \lambda_{(n_0+(j)_l)_i}$.

4. Concluding remarks

We introduced and studied a non-linear birth process, namely, the GBP which allows the jumps of sizes $1, 2, \ldots, k$. Its transition rates depends on the current state of the process. This model extends the classical pure birth process, and we derived a sufficient condition for no explosion in the GBP. Also, we considered its fractional variant, namely, the GFBP for which we obtained a closed form expression for the one-dimensional distribution. Also, we established a subordination relationship between the GBP and GFBP. A limiting case of the GFBP is discussed where jumps of any size $j \geq 1$ are permitted. Further, we explored a state-dependent version of the GFBP and obtained its state probabilities.

It is shown that various recently introduced time-changed counting processes, for example, the STFPP, TFPP, SFPP, FPBP, GFCP, CFPP, and *etc.* are special cases of the GFBP. These processes have potential applications in different fields that include finance, data traffic modeling, hydrology, economics, optics, transport of charged particles, modeling rapidly spreading epidemics, accelerated cosmic ray showers, population dynamics and catastrophic events like floods, earthquakes, *etc.* So, the GFBP may also find potential applications across these fields.

5. Acknowledgements

The authors thank the anonymous referee for providing comments and suggestions which led to improvements.

References

Alipour, M., Beghin, L., and Rostamy, D. Generalized fractional nonlinear birth processes. *Methodol. Comput. Appl. Probab.*, **17** (3), 525–540 (2015). MR3377847.

Applebaum, D. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press (2009). ISBN 9780511809781. DOI: 10.1017/CBO9780511809781.

Beghin, L. and Orsingher, E. Fractional Poisson processes and related planar random motions. *Electron. J. Probab.*, **14** (61), 1790–1827 (2009). MR2535014.

Di Crescenzo, A., Martinucci, B., and Meoli, A. A fractional counting process and its connection with the Poisson process. *ALEA Lat. Am. J. Probab. Math. Stat.*, **13** (1), 291–307 (2016). MR3479478.

Ding, Z., Granger, C. W. J., and Engle, R. F. A long memory property of stock market returns and a new model. *J. Empir. Finance*, **1** (1), 83–106 (1993). DOI: 10.1016/0927-5398(93)90006-D.

Doukhan, P., Oppenheim, G., and Taqqu, M. S., editors. *Theory and applications of long-range dependence*. Birkhäuser, Boston (2003). ISBN 978-0-8176-4168-9. ISBN 978-0-8176-4168-9.

Feller, W. *An introduction to probability theory and its applications. Vol. I.* John Wiley & Sons, New York, third edition (1968). ISBN 978-0471257080. MR0228020.

Garra, R., Orsingher, E., and Polito, F. State-dependent fractional point processes. *J. Appl. Probab.*, **52** (1), 18–36 (2015). MR3336844.

Karagiannis, T., Molle, M., and Faloutsos, M. Long-range dependence ten years of internet traffic modeling. *IEEE Internet Comput.*, **8** (5), 57–64 (2004). DOI: 10.1109/MIC.2004.46.

Kataria, K. K. and Khandakar, M. Convoluted fractional Poisson process. *ALEA Lat. Am. J. Probab. Math. Stat.*, **18** (2), 1241–1265 (2021). MR4282188.

Kataria, K. K. and Khandakar, M. Generalized fractional counting process. *J. Theoret. Probab.*, **35** (4), 2784–2805 (2022). MR4509086.

Kataria, K. K. and Vellaisamy, P. On distributions of certain state-dependent fractional point processes. *J. Theoret. Probab.*, **32** (3), 1554–1580 (2019). MR3979680.

Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. *Theory and applications of fractional differential equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam (2006). ISBN 978-0-444-51832-3; 0-444-51832-0. MR2218073.

Kochubei, A. N. General fractional calculus, evolution equations, and renewal processes. *Integral Equations Operator Theory*, **71** (4), 583–600 (2011). MR2854867.

Meerschaert, M. M., Nane, E., and Vellaisamy, P. The fractional Poisson process and the inverse stable subordinator. *Electron. J. Probab.*, **16** (59), 1600–1620 (2011). MR2835248.

Orsingher, E. and Beghin, L. Time-fractional telegraph equations and telegraph processes with Brownian time. *Probab. Theory Related Fields*, **128** (1), 141–160 (2004). MR2027298.

Orsingher, E. and Polito, F. Fractional pure birth processes. *Bernoulli*, **16** (3), 858–881 (2010). MR2730651.

Orsingher, E. and Polito, F. The space-fractional Poisson process. *Statist. Probab. Lett.*, **82** (4), 852–858 (2012). MR2899530.

Toaldo, B. Convolution-type derivatives, hitting-times of subordinators and time-changed C_0-semigroups. *Potential Anal.*, **42** (1), 115–140 (2015). MR3297989.