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Near critical asymptotics in the frozen Erdős-Rényi

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Abstract. We consider a modification of the standard Erdős-Rényi random graph, where the growth of components containing a unique cycle is slowed down and the apparition of multicyclic components is forbidden. It was proven by Contat and Curien (2023) that the sizes of the connected components of this graph undergo a similar phase transition to that of the classical model. In particular, in the critical window the scaling limit of the sizes of the components is a "frozen" version of Aldous' multiplicative coalescent (Aldous, 1997). Consequently the total number of vertices belonging to a cyclic component also converges after suitable renormalization. The aim of this article is to study the process arising as this scaling limit. More precisely, we describe its long time asymptotics in the critical window.

1. Introduction

1.1. *Definition of the model and motivations.* The *frozen Erdős-Rényi* model, introduced in Contat and Curien (2023), is a discrete-time random graph process. It is a variant of the standard Erdős-Rényi random graph and produces a graph in which every connected component contains at most one cycle.

We first recall the following construction of the Erdős-Rényi process $(\mathrm{ER}(n,m))_{m\in\mathbb{Z}_+}$, for $n \in \mathbb{N}$. Consider independent and identically distributed edges $E_i = \{X_i, Y_i\}, i \in \mathbb{N}$, both endpoints being uniformly distributed over $\{1, ..., n\}$, independently. For $m \in \mathbb{Z}_+$, the graph $\mathrm{ER}(n,m)$ is the (multi)graph with vertex set $\{1, ..., n\}$ and whose edges are $(E_i)_{1\le i\le m}$. The p-frozen Erdős-Rényi process

$$(F_p(n,m) : m \in \mathbb{Z}_+)$$

depends on a parameter $p \in [0,1]$ and is constructed recursively as follows. Consider the same edges $(E_i)_{i\in\mathbb{N}}$ as in the Erdős-Rényi model (this couples both processes). Let also $(U_m)_{m\in\mathbb{N}}$ be a sequence of independent and identically distributed variables with uniform law over $[0,1]$, independent of $(E_i)_{i\in\mathbb{N}}$. For $n \in \mathbb{N}$, the vertex set remains $\{1,\dots,n\}$. Initially, at $m = 0$, $F_p(n,0)$ consists of n isolated vertices. Then, for $m \in \mathbb{N}$

- if E_m connects two trees, the edge E_m is added to $F_p(n,m-1)$ to construct $F_p(n,m)$. If this addition creates a cycle, this produces a *unicycle* component (that is a component with

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exactly one cycle). In the latter case, we declare the vertices of this unicycle component *frozen*.

- if both endpoints of E_m belong to unicycle components, then E_m is discarded and $F_p(n,m) = F_p(n,m-1)$.
- if E_m connects a tree and a unicycle component, then E_m is discarded if $U_m > p$; otherwise it is kept. If E_m is kept, the tree's vertices become part of the unicycle component and are frozen.

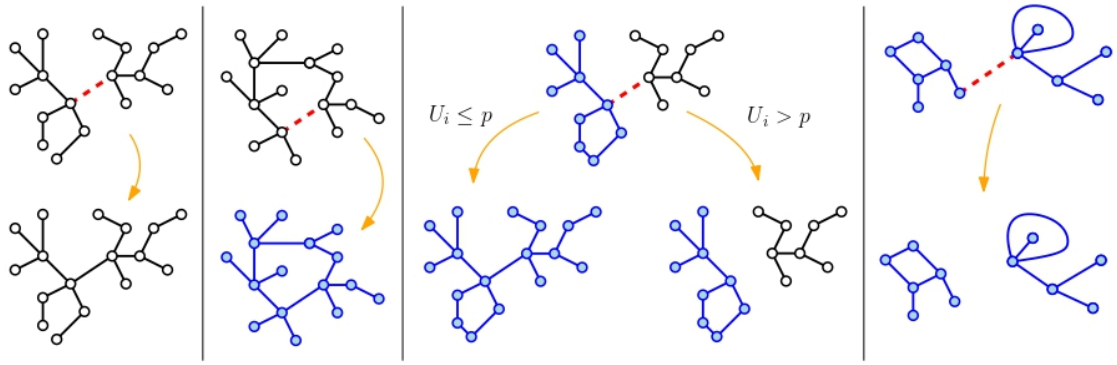

FIGURE 1.1. Transitions of the frozen Erdős-Rényi. The new edge to be examined is in dotted red.

Note that the graph $F_p(n,m)$ has no multicycle component (component with more than one cycle). The *gel* of the graph is $F_p(n,m)$ refers to the set of all frozen vertices, that is the set of vertices belonging to (uni)cycle components. In the sequel, the *forest part* of a graph will refer to its subgraph corresponding to its set of trees.

Remarks 1.1. 1) In the 0-frozen model F_0, the evolution of unicycle components is completely stopped as soon as they become unicycle.
2) In contrast, in the frozen model when $p = 1$, the tree components are always added to the gel when the selected edge connects it with the gel. The only discarded edges are the one which would create a multicycle component. In particular in the natural coupling with the standard Erdős-Rényi graph, the forest part of both graphs coincide, and the set of frozen vertices coincides with vertices involved in cyclic components in the standard model.

The frozen model resonates with several variants of the classical Erdős-Rényi graph, that have been introduced to study how perturbations affect phase transition phenomena. In the standard model it is now well established (Erdős and Rényi, 1959) that the component sizes of the graph $\mathrm{ER}(n,m)$ undergo a phase transition at $m \sim \frac{n}{2}$. Specifically, when $m = \lfloor cn \rfloor$ with $c < \frac{1}{2}$, the largest connected component of $\mathrm{ER}(n,m)$ is of size $O(\log(n))$ while a (unique) *giant component* of size $\Theta(n)$ emerges in the supercritical regime $m = \lfloor cn \rfloor, c > \frac{1}{2}$. In the critical regime $m = \frac{n}{2} + O(n^{2/3})$, the k-th largest components are of sizes $\Theta\left(n^{2/3}\right)$, for any fixed $k \in \mathbb{N}$, (see e.g. Bollobás, 1984, Łuczak, 1990, Łuczak et al., 1994). Moreover the giant component is essentially the only component containing more than one cycle, with the other components being either trees or unicycle components with high probability (see e.g. Bollobás, 2001).

The predominance of trees and unicycle components in the classical model motivates the study of the frozen graph where every component is either a tree or a unicycle. In particular the frozen model clearly obstructs the creation of a giant component, and is therefore in line with several modifications of the Erdős-Rényi model. In this direction, one could refer to Achlioptas et al. (2009); Riordan and Warnke (2012) for the study of "Achlioptas processes". Another example is the "forest fire"

model introduced by Ráth and Tóth (2009) which incorporates the possibility for components to "burn" and which exhibits self-organized criticality. More generally numerous models of "frozen percolation" (Aldous, 2000; Merle and Normand, 2014; Ráth, 2009; Yeo, 2018) (though the term frozen in this context refer to a different mechanism) have been studied to prevent the apparition of the giant component in the Erdős-Rényi graph.

While we were working on the article, a paper (Krapivsky, 2024) with related interests was published in the physics literature. Via a more intuitive approach, Krapisvky studied the supercritical regime $m = \lfloor tn \rfloor, t > \frac{1}{2}$, where the gel becomes macroscopic. He derived the expression of its scaling limit, together with the expression of the scaling limit of the number of trees of size k, $k \geq 1$. Its predictions were confirmed recently in Haas and Viau (2025) where the authors also describe the asymptotic behavior of the first time when every connected component is frozen.

1.2. *Statement of the results.* The frozen Erdős-Rényi model was introduced by Contat and Curien (2023) in the case $p = \frac{1}{2}$, where they constructed a coupling between $F_{1/2}$ and a parking model on Cayley trees. They only focused on the case $p = \frac{1}{2}$ but their results are generalized without additional difficulty for general $p \in [0,1]$ in Viau (2025+). Especially, they exhibited a convergence of the components sizes after suitable renormalization in the so-called *critical regime* $m = \frac{n}{2} + \frac{t}{2}n^{2/3}, t \in \mathbb{R}$. Similar results were already known for the standard model (Aldous, 1997; Broutin and Marckert, 2016). More precisely, for $n \in \mathbb{N}$,$m \in \mathbb{Z}_+$ and $i \in \mathbb{N}$ let us denote by

$$T_p^{(i)}(n,m) \quad \text{and} \quad U_p^{(i)}(n,m) \tag{1.1}$$

the size of the i-th largest tree (respectively the size of the i-th largest unicycle component) in $F_p(n,m)$. In the critical regime Contat and Curien proved that the process

$$n^{-2/3} \cdot \left(\left(\left(T_p^{(i)} \left(n, \left\lfloor \frac{n}{2} + \frac{t}{2} n^{2/3} \right\rfloor \right) \right)_{i \in \mathbb{N}}, \left(U_p^{(i)} \left(n, \left\lfloor \frac{n}{2} + \frac{t}{2} n^{2/3} \right\rfloor \right) \right)_{i \in \mathbb{N}} \right) : t \in \mathbb{R} \right) \tag{1.2}$$

has a limit in distribution in $\ell^2 \times \ell^1$ towards a *frozen multiplicative coalescent* (Contat and Curien, 2023 for $p = \frac{1}{2}$, Viau, 2025+ for $p \in [0,1]$). The limit is a modification of Aldous' multiplicative coalescent (Aldous, 1997) and will be discussed further in Section 5.

Consequently, in the critical window, the process representing the renormalized number of frozen vertices

$$\left(n^{-2/3} \cdot \sum_{i \in \mathbb{N}} U_p^{(i)} \left(n, \left\lfloor \frac{n}{2} + \frac{t}{2} n^{2/3} \right\rfloor \right) : t \in \mathbb{R} \right)$$

converges towards a process X_p. It was shown in Contat and Curien (2023) (see Section 2.2 for some intuition on this result) that X_p is an inhomogeneous pure-jump Feller process whose jump kernel is given by

$$\mathbf{n}_p(t,x,dy) := \frac{1}{2} \frac{\mathrm{d}y}{\sqrt{2\pi y^3}} \cdot (y + 2px) \cdot \frac{p_1(t-x-y)}{p_1(t-x)}, \tag{1.3}$$

where $t \in \mathbb{R}$ is the time parameter, $x \in [0, +\infty)$ the space parameter and $y \in (0, +\infty)$ the size of the jump and

$$p_1(z) = -\frac{\mathrm{e}^{z^3/12}}{2} \left(z \mathrm{Ai} \left(\frac{z^2}{4} \right) + 2\mathrm{Ai}' \left(\frac{z^2}{4} \right) \right), \quad z \in \mathbb{R} \tag{1.4}$$

with Ai the Airy function (see Vallée and Soares, 2010 for a definition). The function p_1 is studied in further detail in Section 2.1. Here again these results are proved in detail for $p = \frac{1}{2}$ in Contat and Curien (2023) and are generalized for $p \in [0,1]$ in Viau (2025+).

The process X_p is the main focus of this article. We are interested in its asymptotic behavior at the frontier of the critical window, that is in the regime $t \to +\infty$ (the so-called *near supercritical*

regime). With the description (1.3) of X_p, Contat and Curien (2023) conjectured that for every $p \in [0,1]$

$$\frac{X_p(t)}{t} \xrightarrow[t\to+\infty]{(\mathbb{P})} 1+p.$$

The aim of this article is to prove this conjecture, and actually to demonstrate a stronger concentration of X_p around the line $(1+p)t$. We distinguish the case $p=0$ from the others as it exhibits a somewhat different behavior compared to $p \in (0,1]$. Our two main results are as follows, starting with the case $p \in (0,1]$ where we establish a very strong concentration around the line $(1+p)t$.

Theorem 1.2. *For every $p \in (0,1]$, the following convergence holds*

$$X_p(t) - (1+p)t \xrightarrow[t\to+\infty]{a.s.} 0.$$

Remark 1.3. As already mentioned, the case $p=1$ is closely related to the Erdős-Rényi model. In this case, Theorem 1.2 informally states that the number of frozen vertices in $F_1(n,m)$ at time $m = \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor$ is approximatively $2tn^{2/3}$ when n is large and t goes to $+\infty$. This has to be related to a result in Łuczak (1990) saying that the size of largest cluster in the standard Erdős-Rényi graph at time $\left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor$ is close to $2tn^{2/3}$ when t goes to $+\infty$. This largest cluster is likely to be formed by the majority of unicycle components in the 1-frozen Erdős-Rényi.

For the 0-frozen model we prove the following.

Theorem 1.4. *For $p=0$, the process $(X_0(t)-t)_{t\geq 0}$ is positive recurrent and converges at an exponential rate to an invariant probability measure μ on $(-\infty,\infty)$.*
Furthermore, μ verifies the following tail inequality: there exists $c>0$ such that for all $t>0$:

$$\mu\left((-\infty,-t]\cup[t,+\infty)\right) \leq \mathrm{e}^{-ct^3}.$$

This theorem has an interesting consequence on the 0-frozen Erdős-Rényi graph: it implies the convergence of its forest part to a stationary law at the frontier of the critical window. In order to state this corollary, let us denote $\left(T_0^{(i)}(t)\right)_{i\in\mathbb{N}}$ the scaling limit of the renormalized tree components sizes in (1.2).

Corollary 1.5. *For all $\epsilon > 0$, the following convergence in distribution holds for the $\ell^{3/2+\epsilon}$ topology*

$$\left(T_0^{(i)}(t)\right)_{i\in\mathbb{N}} \xrightarrow[t\to+\infty]{(d)} L$$

where L is a random variable with explicit distribution.

The distribution of the limit variable L is explicit and involves the stationary law μ. In order to lighten the introduction, we report the description of this law to Section 5 and more precisely to Corollary 5.3. The previous result is an example of "self-organized criticality". This is due to the fact that, when $p=0$, the forest part of the graph has no interaction with frozen components and is thus self-organizing.Consequently, as soon as the forest reaches a critical regime (when its number of edges is closed to the half of its number of vertices), it remains critical. Similar phenomena were observed in Cerf and Forien (2022); Merle and Normand (2014); Ráth (2009); Ráth and Tóth (2009); Yeo (2018) for different mechanisms. In contrast, when $p \in (0,1]$, the forest interacts with the frozen part and is not self-organizing. This interaction with the gel forces the forest to become subcritical (that is its number of edges is "significantly" less than the half of its number of vertices) and foster the freezing of small trees. This makes a self-organized criticality impossible.

1.3. *Organization of the paper.* The rest of the paper is divided into four sections. The first section mainly focuses on providing background on the process X_p and on the function p_1. The second and third are respectively dedicated to the proofs of Theorem 1.2 and Theorem 1.4. The last section explores several applications of these two theorems. Notably, Corollary 5.2 provides new information on Aldous' augmented multiplicative coalescent (Aldous, 1997) and Corollary 5.3 describes the stationary law in the case $p = 0$.

2. Preliminaries

2.1. *Preliminary analysis of p_1 and related quantities.* This section is dedicated to the study of the function p_1 which appears in the jump kernel (1.3) of the process X_p. Let us recall its definition, for $x \in \mathbb{R}$

$$p_1(x) = \frac{-\mathrm{e}^{x^3/12}}{2}\left(x\mathrm{Ai}\left(\frac{x^2}{4}\right) + 2\mathrm{Ai}'\left(\frac{x^2}{4}\right)\right),$$

where Ai is the Airy function. Notably, Ai is solution to the differential equation $y'' = xy$ with the condition $\mathrm{Ai}(x) \to 0$ when $x \to +\infty$ (for further information about the Airy function, see e.g. Vallée and Soares, 2010). It was proven in Banderier et al. (2001) that p_1 is the density of a 3/2-spectrally positive stable Lévy distribution with characteristic function

$$\varphi(t) = \exp\left(-t^{3/2}2^{3/2}\mathrm{e}^{i\pi/4}/3\right), \quad t \in [0, +\infty)$$

($\varphi(t) = \overline{\varphi(-t)}$ for $t < 0$). Banderier et al. (2001) indeed proved that the previous definition of p_1 coincides with the more standard expression

$$p_1(x) = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\frac{2}{3}t^{3/2}}\cos\left(xt + \frac{2}{3}t^{3/2}\right)\mathrm{d}t, \quad x \in \mathbb{R}.$$

The latter expression is classically obtained by applying Fourier inversion to the characteristic function φ. We gather a few useful properties about the function p_1 in the following lemma.

Lemma 2.1.
1) The function p_1 is unimodal: increasing on $(-\infty, x_{max})$ and decreasing on $(x_{max}, +\infty)$ for some $x_{max} \in (-\infty, 0)$.
2) If $x \in [0, +\infty)$ and $y \in [0, -x_{max}]$ the following inequality holds

$$\frac{p_1(x-y)}{p_1(x)} \geq 1. \tag{2.1}$$

3) The function p_1 admits the following expansion at $-\infty$

$$p_1(x) \underset{x\to-\infty}{=} \frac{\mathrm{e}^{-|x|^3/6}}{\sqrt{2\pi}}\left(\sqrt{|x|} + \frac{1}{6|x|^{5/2}} + O\left(|x|^{-11/2}\right)\right), \tag{2.2}$$

and the following equivalent at $+\infty$

$$p_1(x) \underset{x\to+\infty}{\sim} \frac{x^{-5/2}}{\sqrt{2\pi}}. \tag{2.3}$$

The asymptotic expansions (2.2) and (2.3) express the fact that the tails of p_1 are extremely asymetric. Note that even if we only prove that $x_{\max} \in (-\infty, 0)$, numerical simulations give $x_{\max} \approx -0,886...$

Proof: 1) We refer to Zolotarev (1986), Theorem 2.7.6 for the proof of unimodality. Moreover, $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ for every $x \in \mathbb{R}$. Consequently, for all $x \in \mathbb{R}$

$$p_1'(x) = \frac{-\mathrm{e}^{x^3/12}}{2}\left(\left(1 + \frac{x^3}{2}\right)\mathrm{Ai}\left(\frac{x^2}{4}\right) + x^2\mathrm{Ai}'\left(\frac{x^2}{4}\right)\right). \tag{2.4}$$

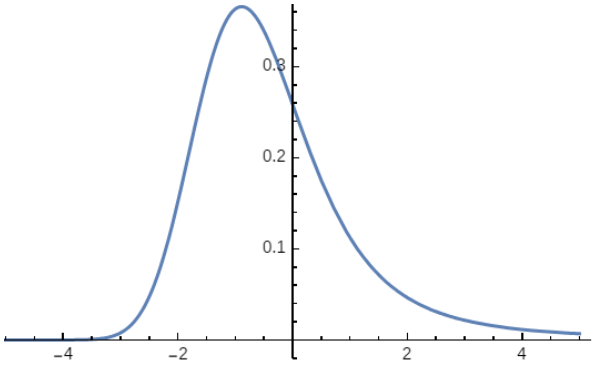

FIGURE 2.2. Plot of the function p_1 on the interval $[-5;5]$.

In particular, $p_1'(0) = \frac{-\mathrm{Ai}(0)}{2} = \frac{-1}{3^{2/3}\Gamma(2/3)} < 0$ (Vallée and Soares, 2010, Equation (2.10)) where Γ denotes the gamma function. By unimodality of p_1 this implies that $x_{\max} \in (-\infty, 0)$.
2) This immediately follows from the first point.
3) Both expansions rely on the following results for Ai and its derivative Ai′ (see e.g. Vallée and Soares, 2010, p.14):

$$\mathrm{Ai}(z) \underset{z\to+\infty}{=} \frac{1}{2\sqrt{\pi}z^{1/4}}\mathrm{e}^{-\frac{2}{3}z^{3/2}}\left(1 - \frac{5}{48z^{3/2}} + O\left(\frac{1}{z^3}\right)\right), \tag{2.5}$$

and

$$\mathrm{Ai}'(z) \underset{z\to+\infty}{=} -\frac{z^{1/4}}{2\sqrt{\pi}}\mathrm{e}^{-\frac{2}{3}z^{3/2}}\left(1 + \frac{7}{48z^{3/2}} + O\left(\frac{1}{z^3}\right)\right). \tag{2.6}$$

The expansions (2.2) and (2.3) then follow using (2.5) and (2.6) with $z = \frac{x^2}{4}$. □

We now turn to a key lemma on the ratio $\frac{p_1(x-y)}{p_1(x)}$ which appears in the expression (1.3) of the jump kernel of X_p.

Lemma 2.2. *Let $y \in (0, +\infty)$ be fixed. The function $f_y : x \mapsto f_y(x) := \frac{p_1(x-y)}{p_1(x)}$ is non-decreasing on $(-\infty, 0)$.*

Proof: First recall that the Airy function is positive and decreasing on $\mathbb{R}_+$, and its derivative is non-decreasing on $\mathbb{R}_+$ (this is implied, e.g., by Equation (2.30) in Vallée and Soares, 2010). Using expression (2.4) of p_1' we get that for every $x \in (-\infty, 0)$

$$f_y'(x) = \frac{\mathrm{e}^{(x-y)^3/12 - x^3/12}}{4p_1(x)^2}\left(\mathrm{Ai}\left(\frac{x^2}{4}\right)\mathrm{Ai}\left(\frac{(x-y)^2}{4}\right)a_y(x) + \mathrm{Ai}'\left(\frac{x^2}{4}\right)\mathrm{Ai}'\left(\frac{(x-y)^2}{4}\right)b_y(x)\right.$$
$$\left. + \mathrm{Ai}\left(\frac{(x-y)^2}{4}\right)\mathrm{Ai}'\left(\frac{x^2}{4}\right)c_y(x) + \mathrm{Ai}\left(\frac{x^2}{4}\right)\mathrm{Ai}'\left(\frac{(x-y)^2}{4}\right)d_y(x)\right)$$

where

$$a_y(x) = x\left(1 + \frac{(x-y)^3}{2}\right) - (x-y)\left(1 + \frac{x^3}{2}\right), \quad b_y(x) = 2y^2 - 4xy,$$
$$c_y(x) = 2 + (x-y)^3 - x^2(x-y) \quad \text{and} \quad d_y(x) = x(x-y)^2 - 2 - x^3.$$

It is easy to see that for $x < 0$ and $y > 0$, we have $a_y(x) > 0$ and $b_y(x) > 0$ so that

$$\mathrm{Ai}\left(\frac{x^2}{4}\right)\mathrm{Ai}\left(\frac{(x-y)^2}{4}\right)a_y(x) + \mathrm{Ai}'\left(\frac{x^2}{4}\right)\mathrm{Ai}'\left(\frac{(x-y)^2}{4}\right)b_y(x) > 0.$$

To show that f'_y is positive on $(-\infty, 0)$, it thus suffices to show that for every $x \in (-\infty, 0)$

$$\mathrm{Ai}\left(\frac{(x-y)^2}{4}\right)\mathrm{Ai}'\left(\frac{x^2}{4}\right)c_y(x) + \mathrm{Ai}\left(\frac{x^2}{4}\right)\mathrm{Ai}'\left(\frac{(x-y)^2}{4}\right)d_y(x) > 0.$$

In order to do so, let us reorganize the terms in the above expression as follows

$$\begin{aligned}
&\mathrm{Ai}\left(\frac{(x-y)^2}{4}\right)\mathrm{Ai}'\left(\frac{x^2}{4}\right)c_y(x) + \mathrm{Ai}\left(\frac{x^2}{4}\right)\mathrm{Ai}'\left(\frac{(x-y)^2}{4}\right)d_y(x)\\
&= (-2x^2y + 3xy^2 - y^3)\mathrm{Ai}\left(\frac{(x-y)^2}{4}\right)\mathrm{Ai}'\left(\frac{x^2}{4}\right)\\
&+ (-2x^2y + xy^2)\mathrm{Ai}\left(\frac{x^2}{4}\right)\mathrm{Ai}'\left(\frac{(x-y)^2}{4}\right)\\
&+ 2\left(\mathrm{Ai}\left(\frac{(x-y)^2}{4}\right)\mathrm{Ai}'\left(\frac{x^2}{4}\right) - \mathrm{Ai}\left(\frac{x^2}{4}\right)\mathrm{Ai}'\left(\frac{(x-y)^2}{4}\right)\right).
\end{aligned}$$

On $\mathbb{R}_+$, Ai is positive and Ai' negative, and we thus easily see that the first two terms are positive (remember that x is negative). Thus, we need to show that the last term of the previous expression is also positive. Since x is negative and y positive, we have $(x-y)^2 \geq x^2$ and it suffices to show that

$$\forall u \geq v \geq 0,\ \mathrm{Ai}(u)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(u) \geq 0.$$

This is trivially true for $u = v \in \mathbb{R}_+$ and we thus introduce, for any constant $c > 0$

$$F_c(v) := \mathrm{Ai}(v+c)\mathrm{Ai}'(v) - \mathrm{Ai}(v)\mathrm{Ai}'(v+c).$$

Since $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ we have for all $c > 0$ and $v \geq 0$

$$\partial_v F_c(v) = -c\mathrm{Ai}(v+c)\mathrm{Ai}(v) \leq 0.$$

Thus, we only have to prove that, for any $c > 0$, $F_c(v)$ is positive for large v where we can use the asymptotics (2.5) and (2.6) to get

$$F_c(v) \underset{v\to+\infty}{\sim} \frac{\mathrm{e}^{-\frac{2}{3}(v+c)^{3/2}}\mathrm{e}^{-\frac{2}{3}v^{3/2}}}{4\pi}\left(\frac{(v+c)^{1/4}}{v^{1/4}} - \frac{v^{1/4}}{(v+c)^{1/4}}\right).$$

As the right-hand side of the previous estimate is positive for every $c > 0$ and $v > 0$, this concludes the proof. □

We end this section with results about integrals involving the p_1 function. These results will help us to control the expectation and variance of X_p in Section 3.1 and to approximate the predictable compensator in Section 3.3 (Lemma 3.9). For $k \in \{1, 2, 3\}$ and $x \geq 0$ we define

$$I_k(x) := \int_0^\infty y^k \cdot \frac{p_1(-x-y)}{p_1(-x)}\frac{\mathrm{d}y}{\sqrt{2\pi y^3}} \tag{2.7}$$

As we shall also consider restrictions to jumps smaller than $-x_{\max} \in (0, +\infty)$, we define

$$I_1^{(x_{\max})}(x) := \int_0^{-x_{\max}} \frac{p_1(-x-y)}{p_1(-x)}\frac{\mathrm{d}y}{\sqrt{2\pi y}}. \tag{2.8}$$

Lemma 2.3. *There exists $x_0 \geq 0$, $C_1 \in (0, +\infty)$, $C_2 \in (0, +\infty)$ and $C_3 \in (0, +\infty)$ such that for all $x \geq x_0$ we have*

$$\left|I_1(x) - \frac{1}{x}\right| \leq \frac{C_1}{x^4}, \quad I_2(x) \leq \frac{C_2}{x^3}\ ,\ I_3(x) \leq \frac{C_3}{x^5} \ \text{ and } \ I_1^{(x_{max})}(x) \geq \frac{1}{2x}.$$

Proof: We start with the upper bound for I_1. From the asymptotic expansion (2.2) of p_1 we conclude that there exists $x_0 \geq 0$ such that for all $x \geq x_0$ and $y \geq 0$ we have

$$\begin{aligned}\frac{p_1(-x-y)}{p_1(-x)} &\leq \mathrm{e}^{-(x+y)^3/6+x^3/6}\frac{\sqrt{x+y}+\frac{1}{2|x+y|^{5/2}}}{\sqrt{x}} \\ &\leq \mathrm{e}^{-x^2y/2}\frac{\sqrt{x+y}+\frac{1}{2x^{5/2}}}{\sqrt{x}} \end{aligned} \tag{2.9}$$

(note that the choice of multiplying $|x+y|^{5/2}$ by 2 in the previous inequality is arbitrary). For any $x \geq x_0$, we get that

$$I_1(x) \leq J_1(x) + J_2(x) \tag{2.10}$$

where

$$J_1(x) = \int_{\mathbb{R}_+} e^{-x^2y/2}\sqrt{1+\frac{y}{x}}\frac{\mathrm{d}y}{\sqrt{2\pi y}} \quad \text{and} \quad J_2(x) = \frac{1}{x^3}\int_{\mathbb{R}_+} e^{-x^2y/2}\frac{\mathrm{d}y}{2\sqrt{2\pi y}}.$$

We perform the change of variables $u = x\sqrt{y}$ in both integrals and get

$$\begin{aligned} J_1(x) &= \frac{1}{x}\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}_+} \mathrm{e}^{-u^2/2}\sqrt{1+\frac{u^2}{x^3}}\mathrm{d}u \\ &\leq \frac{1}{x}\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}_+} \mathrm{e}^{-u^2/2}\left(1+\frac{u^2}{x^3}\right)\mathrm{d}u = \frac{1}{x}+\frac{1}{x^4} \end{aligned}$$

and

$$J_2(x) = \frac{1}{x^4}\int_{\mathbb{R}_+} \mathrm{e}^{-u^2/2}\frac{\mathrm{d}u}{\sqrt{2\pi}} = \frac{1}{2x^4}.$$

Plugging these bounds in (2.10) gives the upper bound for I_1. We now turn to the lower bound for I_1. The proof is similar. We use (2.2) and conclude that there exists $x_0 \geq 0$ such that for all $x \geq x_0$

$$\begin{aligned}\frac{p_1(-x-y)}{p_1(-x)} &\geq \mathrm{e}^{-(x+y)^3/6+x^3/6}\frac{\sqrt{x+y}}{\sqrt{x}+\frac{1}{x^{5/2}}} \\ &\geq \mathrm{e}^{-x^2y/2-xy^2/2-y^3/6}\frac{\sqrt{x}}{\sqrt{x}+\frac{1}{x^{5/2}}} \\ &\geq \mathrm{e}^{-x^2y/2}\left(1-\frac{xy^2}{2}-\frac{y^3}{6}\right)\left(1-x^{-3}\right), \end{aligned} \tag{2.11}$$

where we used that $\mathrm{e}^{-z} \geq 1-z$ for $z \geq 0$ to get the last line. For all $x \geq x_0$ we thus have:

$$I_1(x) \geq \left(1-x^{-3}\right)\int_{\mathbb{R}_+} \mathrm{e}^{-x^2y/2}\left(1-\frac{xy^2}{2}-\frac{y^3}{6}\right)\frac{\mathrm{d}y}{\sqrt{2\pi y}}.$$

Performing again the change of variables $u = x\sqrt{y}$ yields that

$$\begin{aligned} I_1(x) &\geq \sqrt{\frac{2}{\pi}}\frac{1}{x}\left(1-x^{-3}\right)\int_{\mathbb{R}_+} \mathrm{e}^{-u^2/2}\left(1-\frac{u^4}{2x^3}-\frac{u^6}{6x^6}\right)\mathrm{d}u \\ &= \frac{1}{x}\left(1-x^{-3}\right)\left(1-O\left(x^{-3}\right)\right). \end{aligned}$$

This yields the lower bound for the function I_1 provided that x is sufficiently large.

The proof of the result for I_2 is similar. We use (2.9) to get the existence of $x_0 \geq 0$ such that for all $x \geq x_0$ and $y \geq 0$

$$\begin{aligned} I_2(x) &\leq \int_{\mathbb{R}_+} y\mathrm{e}^{-x^2y/2}\left(\sqrt{1+\frac{y}{x}}+\frac{1}{2x^3}\right)\frac{\mathrm{d}y}{2\sqrt{2\pi y}} \\ &\leq \frac{1}{x^3}\int_{\mathbb{R}_+} u^2\mathrm{e}^{-u^2/2}\left(1+\frac{u^2}{x^3}+\frac{1}{2x^3}\right)\frac{\mathrm{d}u}{\sqrt{2\pi}} \end{aligned}$$

where we performed again the change of variables $u = x\sqrt{y}$. The upper bound for I_2 follows from the last inequality. The proof of the bound on I_3 follows the same steps.
The proof of the bound for $I_1^{(x_{\max})}$ is similar. By (2.11) there exists $x_0 \geq 0$ such that for all $x \geq x_0$ and $y \geq 0$

$$\begin{aligned} \frac{p_1(-x-y)}{p_1(-x)} &\geq \frac{3}{4}\mathrm{e}^{-x^2y/2-xy^2/2-y^3/6}\left(1+\frac{y}{x}\right)^{1/2} \\ &\geq \frac{3}{4}\mathrm{e}^{-x^2y/2}\left(1-xy^2/2-y^3/6\right). \end{aligned}$$

Thus for all $x \geq x_0$ we have

$$\begin{aligned} I_1^{(x_{\max})}(x) &\geq \frac{3}{4}\int_0^{-x_{\max}} \mathrm{e}^{-x^2y/2}\left(1-\frac{xy^2}{2}-\frac{y^3}{6}\right)\frac{\mathrm{d}y}{\sqrt{2\pi y}} \\ &\geq \frac{3}{4x}\int_0^{\sqrt{-x_{\max}}\,x} \mathrm{e}^{-u^2/2}\left(1-\frac{u^4}{2x^3}-\frac{u^6}{6x^6}\right)\frac{\sqrt{2}\mathrm{d}u}{\sqrt{\pi}} \\ &\geq \frac{5}{8x}\int_0^{\sqrt{-x_{\max}}\,x} \mathrm{e}^{-u^2/2}\sqrt{\frac{2}{\pi}}\mathrm{d}u \end{aligned}$$

where we take x to be even larger than before, if necessary to make the terms which multiply $\frac{1}{x^3}$ and $\frac{1}{x^6}$ small. Since $\sqrt{-x_{\max}}\,x$ diverges as x goes to infinity and since $\int_0^L \mathrm{e}^{-u^2/2}\sqrt{\frac{2}{\pi}}\mathrm{d}u \to 1$ as $L \to \infty$ we can conclude. □

2.2. *Background on X_p.* We now turn to the process X_p, defined as the scaling limit of the renormalized number of frozen vertices in the critical window. We start with a construction of X_p in terms of a Poisson point process that we widely use through the article.

Proposition 2.4. *There exists a Poisson point process Π over $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+$ with measure intensity $\frac{\mathrm{d}y}{2\sqrt{2\pi y^3}}\cdot\mathrm{d}z\cdot\mathrm{d}s$ such that for all $t \geq 0$*

$$X_p(t) = X_p(0) + \sum_{0\leq S_i\leq t} Y_i\cdot\mathbb{1}_{\left\{Z_i\leq (Y_i+2pX_p(S_i-))\frac{p_1(S_i-X_p(S_i-)-Y_i)}{p_1(S_i-X_p(S_i-))}\right\}}, \tag{2.12}$$

where (Y_i, Z_i, S_i) are the atoms of the process Π.

A proof is given for $p = \frac{1}{2}$ in Contat and Curien (2023), and the proof for general $p \in [0,1]$ can be found in Viau (2025+). In this section we give some intuition for this result and for the expression (1.3) of the jump kernel of X_p as we use both repeatedly in the paper. In that aim, we introduce a few notations. For $n \in \mathbb{N}$, $m \in \mathbb{Z}_+$

- $G_p(n,m)$ is the size of the gel at time m, that is the number of frozen vertices in $F_p(n,m)$
- $D_p(n,m)$ is the number of discarded edges at time m
- $\mathbf{F}_p(n,.)$ is the filtration associated to $(G_p(n,.), D_p(n,.))$
- $V_p(n,m)$ is the number of vertices in the forest at time m
- $E_p(n,m)$ is the number of edges in the forest at time m

Recall also notations (1.1) for the sizes of the tree components and unicycle components. Note that the following relations are valid for any $n \in \mathbb{N}, m \in \mathbb{Z}_+$

$$V_p(n,m) = n - G_p(n,m) \quad \text{and} \quad E_p(n,m) = m - G_p(n,m) - D_p(n,m). \tag{2.13}$$

For $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, we define

$$\Delta G_p(n,m) = G_p(n,m+1) - G_p(n,m)$$

the jump of the gel at time m. In order to lighten the notations we shall denote, for $t \in \mathbb{R}$

$$F_{p,n}(t) := F_p\left(n, \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor \vee 0\right), \tag{2.14}$$

the continuous-time version of the frozen process in the critical window. In this regime we shall also denote by $G_{p,n}(t)$, $D_{p,n}(t)$, $V_{p,n}(t)$, $E_{p,n}(t)$ the continuous-time versions of each process. Finally, for $N \in \mathbb{N}$, $M \in \mathbb{Z}_+$, let us define $\mathcal{W}(N,M)$ the set of unordered unrooted forests with N labeled vertices and M edges (hence $N-M$ components). With these notations, the following lemma holds (proved for $p = \frac{1}{2}$ in Contat and Curien, 2023 and generalized without further difficulty in Viau, 2025+). This lemma was also proved in Haas and Viau (2025) (Proposition 2.1) for a slightly different version of the frozen process.

Lemma 2.5. *1) **Free forest property:** For any $n \in \mathbb{N}, m \in \mathbb{Z}_+$, conditionally on $G_p(n,m)$ and $D_p(n,m)$ the forest part of $F_p(n,m)$ is uniformly distributed over $\mathcal{W}\left(V_p(n,m), E_p(n,m)\right)$.*
2) For every $n \in \mathbb{N}, m \in \mathbb{Z}_+$ and $k \in [\![1; E_p(n,m)+1]\!]$

$$\begin{aligned}&\mathbb{P}\left(\Delta G_p(n,m) = k | \mathbf{F}_p(n,m)\right)\\ &= \binom{V_p(n,m)}{k} \cdot k^{k-2} \cdot \frac{\#\mathcal{W}\left(V_p(n,m)-k, E_p(n,m)-k+1\right)}{\#\mathcal{W}\left(V_p(n,m), E_p(n,m)\right)} \cdot \left(\frac{k^2 + 2pkG_p(n,m)}{n^2}\right)\end{aligned} \tag{2.15}$$

Sketch of proof of Proposition 2.4: • Recall Definition (1.3) of the jump kernel $\mathbf{n}_p$. It should be clear that the jump kernel of a process admitting the representation (2.12) is indeed $\mathbf{n}_p$. Moreover, existence and path-uniqueness of such a process are consequences of classical Lipschitz and linear-growth condition theorems (see e.g. Jacod and Shiryaev, 2003, Theorem 2.32) together with analytic results on the function p_1 that are overlooked here (but are actually similar to the arguments used in the proof of the upcoming Proposition 2.6).
• Recall notation (2.14). In the critical regime $m = \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor$, it were proved in Contat and Curien (2023) that

$$\begin{pmatrix} n^{-2/3} \cdot G_{p,n}(t) \\ n^{-1/3} \cdot D_{p,n}(t) \\ n^{-2/3} \cdot \left(T^{(i)}_{p,n}(t)\right)_{i\in\mathbb{N}} \end{pmatrix}_{t\in\mathbb{R}} \xrightarrow[n\to+\infty]{(d)} \begin{pmatrix} X_p(t) \\ D_p(t) \\ \left(T^{(i)}_p(t)\right)_{i\in\mathbb{N}} \end{pmatrix}_{t\in\mathbb{R}} \tag{2.16}$$

for some processes $\left(D_p, \left(T^{(i)}_p\right)_{i\in\mathbb{N}}\right)$. This suggests to consider, for $m = \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor$

$$\mathbb{P}\left(\Delta G_p(n,m) = yn^{2/3} \,|\, G_p(n,m) = xn^{2/3},\, D_p(n,m) = dn^{1/3}\right)$$

for $x, y, d \in [0, +\infty)$. Using formula (2.15) together with asymptotic enumerations of forests (see e.g. Britikov, 1988) the previous probability is asymptotic to

$$n^{-4/3} \cdot (y + 2px) \cdot \frac{p_1(t-x-y)}{p_1(t-x)} \cdot \frac{1}{\sqrt{2\pi} y^{3/2}}.$$

In particular, the jump kernel of $\left(n^{-2/3}G_{p,n}(t)\right)_{t\in[0,+\infty)}$ converges to $\mathbf{n}_p(t,x,\mathrm{d}y)$. The convergence in distribution of the process $\left(n^{-2/3}G_{p,n}(t)\right)_{t\in[0,+\infty)}$ to a process with jump kernel $\mathbf{n}_p$ then follows

from convergence results for Feller process (see e.g. Jacod and Shiryaev, 2003, Chapter IX, Theorem 3.39) and is not detailed here. □

We finish with a claim on X_p, and refer to Meyn and Tweedie (2009), Chapters 4 and 20 for definitions.

Proposition 2.6. *1) For every $p \in [0,1]$ the process $(X_p(t) - (1+p)t)_{t\geq 0}$ is an aperiodic and irreducible Feller process.*
2) For every $p \in [0,1]$ the process X_p is non-explosive.
3) Moreover, for $p = 0$ the process X_0 only jumps a finite number of times over any compact time interval.

Remark 2.7. The third point is not true for $p \in (0,1]$ where there is an accumulation of small jumps.

Proof: 1) Recall definition (1.3) of $\mathbf{n}_p$. Irreducibility and aperiodicity follow both from the fact that the support of $\mathbf{n}_p$ is the whole space $\mathbb{R}_+$ since its density with respect to Lebesgue measure is everywhere positive.
2) The non-explosivity is a consequence of the fact that for all $x \geq 0$ and $t \in \mathbb{R}$ we have

$$\int_0^t \int_{\mathbb{R}_+} y \cdot \mathbf{n}_p(s, x, \mathrm{d}y) ds < +\infty.$$

In order to prove this we decompose

$$\int_{\mathbb{R}_+} y \cdot \mathbf{n}_p(s, x, \mathrm{d}y) = \int_0^{t-x_{\max}} y \cdot \mathbf{n}_p(s, x, \mathrm{d}y) + \int_{t-x_{\max}}^{+\infty} y \cdot \mathbf{n}_p(s, x, \mathrm{d}y). \tag{2.17}$$

For the first integral, note that for fixed $x \geq 0$, for $s \in [0,t]$ and $y \in [0, t - x_{\max}]$, by compactness, the ratio $\frac{p_1(s-x-y)}{p_1(s-x)}$ is bounded by a constant $C_1 \in (0, +\infty)$ only depending on t and x. Consequently

$$\int_0^{t-x_{\max}} y \cdot \mathbf{n}_p(s, x, \mathrm{d}y) \leq C_1 \int_0^{t-x_{\max}} (y + 2px) \frac{\mathrm{d}y}{2\sqrt{2\pi y}} \leq \tilde{C}_1(t, x)$$

where $\tilde{C}_1(t,x) \in (0, +\infty)$ also depends only on t and x.
For the second integral in (2.17), note that $s - x - y \leq t - y \leq x_{\max}$. Recalling that p_1 is increasing on $(-\infty, x_{\max}]$ we deduce that

$$p_1(s - x - y) \leq p_1(t - y).$$

Moreover, we also have $-x \leq s - x \leq t$. By unimodality of the function p_1 we deduce that $p_1(s-x) \geq \min(p_1(-x), p_1(t))$. Combining the last two inequalities leads to

$$\begin{aligned}\int_{t-x_{\max}}^{+\infty} y \cdot \mathbf{n}_p(s, x, \mathrm{d}y) &\leq \frac{1}{\min(p_1(-x), p_1(t))} \int_{t-x_{\max}}^{+\infty} (y + 2px) p_1(t-y) \frac{\mathrm{d}y}{2\sqrt{2\pi y}} \\ &\leq C_2(t, x)\end{aligned}$$

where $C_2(t,x) \in (0, +\infty)$ is a constant only depending on t and x (the convergence of the last integral follows from the asymptotics (2.2) on p_1).
3) When $p = 0$, the jump kernel rewrites as

$$\mathbf{n}_0(t, x, \mathrm{d}y) = \frac{p_1(t - x - y)}{p_1(t - x)} \frac{\mathrm{d}y}{2\sqrt{2\pi y}}.$$

Following the same steps as above, we show that for any $x \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$

$$\int_0^t \int_{\mathbb{R}_+} \mathbf{n}_0(s, x, \mathrm{d}y) ds < +\infty.$$

This prevents the accumulation of small jumps and prove the third point of the lemma. □

2.3. *Comparison of under/over solutions.* A key point of the proof of Theorem 1.2 is to show that the process X_p (or more precisely its predictable compensator defined in (3.18)) approximately solves a differential equation (Lemma 3.9). The aim of this section is therefore to give a few comparison results of under/over solutions of differential equations (the terminology is defined hereafter). We first recall a basic result when the functions are differentiable and then adapt it to left-differentiable functions. Throughout this section, we consider a differential equation defined as follows. Let $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function and (E) be the differential equation

$$u'(t) = F(t, u(t)) \quad \forall t \in (0, +\infty). \tag{E}$$

Definition 2.8. We say that a differentiable function $u : \mathbb{R}_+ \to \mathbb{R}$
- is a **strict under solution** to (E) if for every $t \in (0, +\infty)$, $u'(t) < F(t, u(t))$.
- is a **strict over solution** to (E) if for every $t \in (0, +\infty)$, $u'(t) > F(t, u(t))$

The following lemma is folklore in differential equation theory.

Lemma 2.9. *Let u_1, u_2, u_3 be respectively under solution, solution and over solution to Equation* (E)*. Let $t_0 \in \mathbb{R}_+$ such that $u_1(t_0) < u_2(t_0) < u_3(t_0)$. Then, for every $t \geq t_0$*

$$u_1(t) < u_2(t) < u_3(t).$$

Note that the differential equation (E) does not need to fill the Cauchy-Lipschitz criteria. As the functions we will work with in Section 3.3 should not be differentiable everywhere we need a slight improvement of this lemma. This motivates the following definitions. In the definition, the notation $\partial_- u(t)$ denotes the left-derivative of u at t.

Definition 2.10. Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that for every $t \in (0, +\infty)$, $\partial_- u(t)$ is finite. We say that
- u is a **strict left under solution** to (E) if for every $t \in (0, +\infty)$, $\partial_- u(t) < F(t, u(t))$.
- u is a **strict left over solution** to (E) if for every $t \in (0, +\infty)$, $\partial_- u(t) > F(t, u(t))$.
- u is a **left solution** to (E) if for every $t \in (0, +\infty)$, $\partial_- u(t) = F(t, u(t))$.

We prove the following analogue of Lemma 2.9.

Lemma 2.11. *Let u_1, u_2, u_3 be respectively strict left under solution, left solution, and strict left over solution to Equation* (E)*. Let also $t_0 \in \mathbb{R}_+$ such that $u_1(t_0) < u_2(t_0) < u_3(t_0)$. Then, for all $t \geq t_0$*

$$u_1(t) < u_2(t) < u_3(t).$$

The proof is adapted from the standard proof of Lemma 2.9.

Proof: We prove the left-hand side inequality and let the right-hand side to the reader. By assumption, $u_1(t_0) < u_2(t_0)$. As u_1 and u_2 are continuous (this is contained in Definition 2.10) there exists $\varepsilon > 0$ such that for every $t \in [t_0, t_0 + \varepsilon]$, $u_1(t) < u_2(t)$. We then introduce:

$$T := \inf\{t \geq t_0 : u_1(t) \geq u_2(t)\} \in [t_0, +\infty]$$

(with the convention $\inf(\emptyset) = +\infty$). The above implies that $T > t_0$. By contradiction, assume that $T < +\infty$. By continuity of both u_1 and u_2 we have $u_1(T) = u_2(T)$. Then for $h < 0$ small,

$$\begin{aligned} u_1(T+h) - u_2(T+h) &= u_1(T) - u_2(T) + h\left(\partial_- u_1(T) - \partial_- u_2(T)\right) + o(h) \\ &= h\left(\partial_- u_1(T) - \partial_- u_2(T)\right) + o(h) \\ &> 0 \end{aligned}$$

since $\partial_- u_1(T) < F(T, u_1(T)) = F(T, u_2(T)) \leq \partial_- u_2(T)$. This contradicts the definition of T as an infimum. □

3. The case $p \in (0,1]$

We start with an overview of the proof of Theorem 1.2. Our strategy is to approximate X_p by its predictable compensator X_p^{pre} defined for every $t \in \mathbb{R}_+$ by

$$X_p^{\mathrm{pre}}(t) = X_p(0) + \int_0^t \int_0^\infty y(y+2pX_p(s)) \frac{p_1(s - X_p(s) - y)}{p_1(s - X_p(s))} \frac{\mathrm{d}y}{2\sqrt{2\pi y^3}} \mathrm{d}s. \tag{3.1}$$

The compensator is easier to work with than the process X_p. We will see in Lemma 3.9 that the couple $(X_p, X_p^{\mathrm{pre}})$ almost surely approximately solves the deterministic integral equation

$$x_p^{\mathrm{pre}}(t) - x_p^{\mathrm{pre}}(t_0) = \int_{t_0}^t \frac{p \cdot x_p(s)}{x_p(s) - s} \mathrm{d}s$$

with final boundary condition $\lim_{t\to+\infty} u(t) - v(t) = m \in \mathbb{R}$. With the results of Section 2.3 we will be able to see that a solution of the latter equation must verify $x_p(t) - (1+p)t \to 0$ as $t \to +\infty$. This reasoning will lead to Theorem 1.2.

Our approach is similar to a "differential equation method" (see e.g. Wormald, 1999) but the non-homogeneity in both time and space as well as the accumulation of small jumps prevents us from applying an existing theorem (to the best of our knowledge). The approximations of X_p and X_p^{pre} rely on asymptotics (2.2) on the p_1 function. In order to use it, we need a preliminary result: for $\epsilon > 0$ small enough, $X_p(t) \geq (1+\epsilon)t$ almost surely for large t (Proposition 3.1). The aim of Section 3.1 is to show this crude lower bound.

3.1. $X_p(t)$ is larger than $(1+\epsilon)t$.

We work on a probability space that we denote by $(\Omega, \mathcal{F}, \mathbb{P})$. The goal of this section is to prove the following.

Proposition 3.1. *For $\epsilon \in (0,p)$ small enough, almost surely, we have $X_p(t) \geq (1+\epsilon)t$ for all t large enough.*

Our strategy is to prove that for $\epsilon \in (0,p)$ small enough (this will be precised during the proof)

$$\sum_{n\geq 0} \mathbb{P}\left(X_p(n) \leq (1+\epsilon)n\right) < +\infty,$$

so that, almost surely, for n large enough, $X_p(n)$ is larger than $(1+\epsilon)n$ and Proposition 3.1 will follow since X is increasing (taking $\epsilon/2$ instead of ϵ for example).

We start with an easy but useful result: if Y follows a Poisson distribution of parameter $\lambda > 0$, then for every $a < 1$

$$\mathbb{P}\left(Y \leq a\lambda\right) \leq \frac{C}{\lambda^2} \tag{3.2}$$

for a constant $C \in (0, +\infty)$ which only depends on a (this result is far from being optimal and exponential bounds can be obtained). This is a mere consequence of Markov's inequality together with the fact that the fourth centered moment of a Poisson distribution of parameter λ is $\lambda(1+3\lambda)$:

$$\mathbb{P}\left(Y \leq a\lambda\right) \leq \mathbb{P}\left((Y-\lambda)^4 \geq (\lambda - a\lambda)^4\right) \leq \frac{\mathbb{E}\left[(Y-\lambda)^4\right]}{(1-a)^4\lambda^4} = \frac{\lambda(1+3\lambda)}{(1-a)^4\lambda^4}.$$

We first prove that X_p tends to infinity as t goes to infinity, which is the "lauching point" of the proof of Proposition 3.1.

Lemma 3.2. *For any $A > 0$, there exists a constant $C(A) \in (0, +\infty)$ and $n_0 \in \mathbb{Z}_+$ such that for every $n \geq n_0$ we have:*

$$\mathbb{P}\left(X_p(n) \leq A\right) \leq \frac{C(A)}{n^2}.$$

Proof: From Lemma 2.1 recall the definition of $x_{\max} \in (-\infty, 0)$ and define

$$B := \frac{-x_{\max}}{2} \int_{-x_{\max}/2}^{-x_{\max}} \frac{\mathrm{d}y}{2\sqrt{2\pi y^3}} > 0 \tag{3.3}$$

Next, by Proposition 2.4, for every $n \geq \max\left(A, \frac{4A}{-x_{\max}B}\right)$ we have

$$\{X_p(n) \leq A\} \subset \left\{ \sum_{A<S_i\leq n} Y_i \cdot \mathbb{1}_{\left\{Z_i \leq (Y_i + 2pX_p(S_i-))\frac{p_1(S_i - X_p(S_i-) - Y_i)}{p_1(S_i - X_p(S_i-))}\right\}} \leq A \right\} \tag{3.4}$$

where (Y_i, Z_i, S_i) are the atoms of a Poisson point process Π with intensity $\frac{\mathrm{d}y}{2\sqrt{2\pi y^3}}\mathrm{d}z\,\mathrm{d}s$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$. Using the inclusion $\{X_p(n) \leq A, A < S_i \leq n\} \subset \{S_i - X_p(S_i-) \geq 0\}$ together with inequality (2.1) we deduce that

$$\left\{X_p(n) \leq A, A < S_i \leq n, Y_i \in \left[-\frac{x_{\max}}{2}, -x_{\max}\right]\right\} \subset \left\{Y_i \cdot \frac{p_1(S_i - X_p(S_i-) - Y_i)}{p_1(S_i - X_p(S_i-)} \geq -\frac{x_{\max}}{2}\right\}.$$

Combined with (3.4) this yields that for every $n \geq \max\left(A, \frac{4A}{-x_{\max}B}\right)$

$$\begin{aligned}\{X_p(n) \leq A\} &\subset \left\{ \sum_{A<S_i\leq n} Y_i \cdot \mathbb{1}_{\left\{Z_i \leq -\frac{x_{\max}}{2}\right\}} \cdot \mathbb{1}_{\left\{Y_i \in [-\frac{x_{\max}}{2}, -x_{\max}]\right\}} \leq A \right\} \\ &\subset \left\{ \Sigma_{A,n} := \sum_{A<S_i\leq n} \mathbb{1}_{\left\{Z_i \leq -\frac{x_{\max}}{2}\right\}} \cdot \mathbb{1}_{\left\{Y_i \in [-\frac{x_{\max}}{2}, -x_{\max}]\right\}} \leq \frac{2A}{-x_{\max}} \leq \frac{Bn}{2} \right\}.\end{aligned}$$

By Definition (3.3) of B and since (Y_i, Z_i, S_i) are the atoms of Π, the random variable $\Sigma_{A,n}$ follows a Poisson distribution of parameter $B(n-A)$. We can therefore conclude with (3.2). □

We now use Lemma 3.2 to prove that X_p is almost surely asymptotically larger than $(1+\epsilon)$ times identity, where $\epsilon \in (0, p)$ is sufficiently small. We define, somewhat arbitrarily

$$c(\epsilon) = \frac{1}{2}\left(1 + \frac{1+\epsilon}{1+p}\right) \quad \text{and} \quad d(\epsilon) = \frac{1}{2}\left(c(\epsilon) + 1\right). \tag{3.5}$$

Remark 3.3. The following inequalities hold

$$\frac{1+\epsilon}{1+p} < c(\epsilon) < d(\epsilon) < 1. \tag{3.6}$$

Let us also introduce

$$T_n := \inf\{s \geq c(\epsilon)n : X_p(s) > s\}. \tag{3.7}$$

(with the convention $\inf\{\emptyset\} = +\infty$) the first time after $c(\epsilon)n$ that $(t, X_p(t))$ enters the region above the diagonal $\{(t,x) : x = t\}$. We intersect $\{X_p(n) \leq (1+\epsilon)n\}$ by including the information on T_n as follows

$$\{X_p(n) \leq (1+\epsilon)n\} = \{X_p(n) \leq (1+\epsilon)n, T_n > d(\epsilon)n\} \cup \{X_p(n) \leq (1+\epsilon)n, T_n \leq d(\epsilon)n\}. \tag{3.8}$$

Lemma 3.4. *There exists a constant $C \in (0, +\infty)$ such that, for all n large enough:*

$$\mathbb{P}(X_p(n) \leq (1+\epsilon)n, T_n > d(\epsilon)n) \leq \frac{C}{n^2}$$

Proof: In order to simplify the notation, we write c and d instead of $c(\epsilon)$ and $d(\epsilon)$. For $n \geq 1$ we define $A_n := \{X_p(n) \leq (1+\epsilon)n, T_n > dn\}$. Let us also define, somewhat arbitrarily

$$A = \frac{1}{p(d-c)\,(-x_{\max})} \cdot \left(\int_{-x_{\max}/2}^{-x_{\max}} \frac{\mathrm{d}y}{2\sqrt{2\pi y^3}} \right)^{-1}. \tag{3.9}$$

As an immediate consequence of Lemma 3.2 there exists $C(A) \in (0, +\infty)$ such that for n large enough

$$\mathbb{P}\left(X_p(cn) < A\right) \leq \frac{C(A)}{n^2}.$$

In order to prove the lemma we can thus focus on

$$\tilde{A}_n := A_n \cap \{X_p(cn) \geq A\}.$$

Note that Definition (3.7) of T_n implies that $\tilde{A}_n \cap \{S_i \in [cn, dn]\} \subset \tilde{A}_n \cap \{X_p(S_i-) \leq S_i\}$. Thus, Equation (2.1) implies that

$$\tilde{A}_n \cap \{S_i \in [cn, dn]\} \cap \left\{ Y_i \in \left[-\frac{x_{\max}}{2}, -x_{\max} \right] \right\} \subset \left\{ \frac{p_1(S_i - X_p(S_i-) - Y_i)}{p_1(S_i - X_p(S_i-))} \geq 1 \right\}.$$

Using Proposition 2.4 we deduce that

$$\begin{aligned} \tilde{A}_n &\subset \left\{ X_p(cn) + \sum_{cn < S_i \leq dn} Y_i \cdot \mathbb{1}_{\left\{ Z_i \leq 2pX_p(S_i-) \frac{p_1(S_i - X_p(S_i-) - Y_i)}{p_1(S_i - X_p(S_i-))} \right\}} \leq dn \right\} \\ &\subset \left\{ \Sigma_{cn,dn} := \sum_{cn < S_i \leq dn} \mathbb{1}_{\left\{ Y_i \in [-\frac{x_{\max}}{2}, -x_{\max}] \right\}} \cdot \mathbb{1}_{\{Z_i \leq 2pA\}} \leq \frac{2}{-x_{\max}} dn \right\}. \end{aligned}$$

Recall Definition (3.9) of the constant A. Since (Y_i, Z_i, S_i) are the atoms of the Poisson point process Π, the variable $\Sigma_{cn,dn}$ follows a Poisson distribution of parameter

$$(d-c)n \cdot 2pA \cdot \int_{-x_{\max}/2}^{-x_{\max}} \frac{\mathrm{d}y}{2\sqrt{2\pi y^3}} = \frac{2}{-x_{\max}} n.$$

As observed in (3.6), we have $d < 1$. Thus $\frac{2}{-x_{\max}} > \frac{2}{-x_{\max}} d$ and we can conclude with (3.2). □

We now turn to the bound of the probability of the last term in (3.8). Note that we have not yet used the "ϵ small enough" hypothesis. This is needed in the following.

Lemma 3.5. *For every $\epsilon \in (0, p)$ small enough, there exists a constant $C > 0$ such that, for all n large enough we have:*

$$\mathbb{P}\left(X_p(n) \leq (1+\epsilon)n, T_n \leq d(\epsilon)n\right) \leq \frac{C}{n^3}.$$

The condition on ϵ relies on the following easy lemma.

Lemma 3.6. *For every $p \in (0, 1]$, the function $f : \epsilon \mapsto \frac{pc(\epsilon)(1-d(\epsilon))}{2(1+\epsilon-d(\epsilon))} - (1+\epsilon-c(\epsilon))$ is strictly positive in a neighborhood of 0.*

Proof: By continuity it suffices to prove that $f(0) > 0$. By definitions (3.5) of c and d we have

$$f(0) = \frac{p(2+p)}{2(2+2p)} - 1 + \frac{2+p}{2+2p} = \frac{p^2}{2(2+2p)} > 0.$$

□

We thus set $\epsilon > 0$ small enough such that

$$\frac{pc(\epsilon)\,(1-d(\epsilon))}{2(1+\epsilon-d(\epsilon))} - (1+\epsilon-c(\epsilon)) =: b > 0, \tag{3.10}$$

and prove that Lemma 3.5 is valid for such choice of ϵ.

Proof of Lemma 3.5: In this proof again we denote c and d instead of $c(\epsilon)$ and $d(\epsilon)$. By Lemma 2.3, there exists $n_0 \geq 0$ and $M \in (0, +\infty)$ such that for all $x \geq n_0 - dn_0$

$$I_1^{(x_{\max})}(x) \geq \frac{1}{2x} \quad \text{and} \quad I_2(x) \leq \frac{M}{x^3}. \tag{3.11}$$

For $n \geq 1$ we define

$$B_n := \{X_p(n) \leq (1+\epsilon)n, T_n \leq dn\}.$$

Recalling Definition (3.7) of T_n and using the fact that X is non-decreasing, we deduce that

$$B_n \cap \{S_i \in (dn, n]\} \subset \{(1+\epsilon)n \geq X_p(S_i-) \geq X_p(T_n) \geq T_n \geq cn\}. \tag{3.12}$$

Using again Proposition 2.4 and that $Y_i + 2pX(S_i-) \geq 2pX_p(S_i-)$ this gives

$$\begin{aligned} B_n &\subset \left\{ X_p(dn) + \sum_{dn<S_i\leq n} Y_i \cdot \mathbb{1}_{\left\{Z_i \leq 2pX_p(S_i-)\frac{p_1(S_i-X_p(S_i-)-Y_i)}{p_1(S_i-X_p(S_i-))}\right\}} \leq (1+\epsilon)n \right\} \\ &\subset \left\{ cn + \sum_{dn<S_i\leq n} Y_i \cdot \mathbb{1}_{\left\{Z_i \leq 2pcn\frac{p_1(S_i-X_p(S_i-)-Y_i)}{p_1(S_i-X_p(S_i-))}\right\}} \leq (1+\epsilon)n \right\}. \end{aligned} \tag{3.13}$$

We want to remove the terms $X_p(S_i-)$ in the indicator function, since they greatly complicate the calculation of expectation and variance of the random variable involved in the previous event. To this end, we prove that the following inclusion holds for sufficiently large n

$$B_n \cap \{S_i \in (dn, n]\} \cap \{Y_i \in [0, -x_{\max}]\} \subset \left\{ \frac{p_1(S_i - X_p(S_i-) - Y_i)}{p_1(S_i - X_p(S_i-))} \geq \frac{p_1\left(dn - (1+\epsilon)n - Y_i\right)}{p_1\left(dn - (1+\epsilon)n\right)} \right\}. \tag{3.14}$$

Indeed if $S_i \in (dn, n]$, the inclusion (3.12) yields that $S_i - X_p(S_i-) \geq dn - (1+\epsilon)n$. Two possibilities may occur:

- if $S_i - X_p(S_i-) \leq 0$, (3.14) is actually valid for every $n \in \mathbb{Z}_+$ by Lemma 2.2.
- if $S_i - X_p(S_i-) > 0$, due to (2.1), for every $Y_i \in [0, -x_{\max}]$, the ratio $\frac{p_1(S_i-X_p(S_i-)-Y_i)}{p_1(S_i-X_p(S_i-))}$ is larger than 1, which yields (3.14) for sufficiently large n.

Combined with (3.13) this gives

$$B_n \subset \left\{ \Sigma_{dn,n} := \sum_{dn<S_i\leq n} Y_i \cdot \mathbb{1}_{\{Y_i\in[0,-x_{\max}]\}} \cdot \mathbb{1}_{\left\{Z_i \leq 2pcn\frac{p_1(dn-(1+\epsilon)n-Y_i)}{p_1(dn-(1+\epsilon)n)}\right\}} \leq (1+\epsilon-c)n \right\} \tag{3.15}$$

We now compute the expectation of $\Sigma_{dn,n}$ using Campbell's formula (see e.g. Last and Penrose, 2018, Proposition 2.7) and get that for all $n \geq n_0$

$$\begin{aligned} \mathbb{E}\left[\Sigma_{dn,n}\right] &= \int_{dn}^{n} \int_{0}^{-x_{\max}} 2pcn \frac{p_1(dn-(1+\epsilon)n-y)}{p_1(dn-(1+\epsilon)n)} \frac{\mathrm{d}y}{2\sqrt{2\pi y}} \mathrm{d}s \\ &= pcn(n-dn) \cdot I_1^{(x_{\max})}\left((1+\epsilon-d)n\right) \\ &\geq \frac{pc(1-d)n}{2(1+\epsilon-d)} \quad \text{from (3.11)}. \end{aligned}$$

Equation (3.10) therefore yields

$$\mathbb{E}\left[\Sigma_{cn,n}\right] - (1+\epsilon-c)n \geq bn. \tag{3.16}$$

Moreover, using again Campbell's formula we have for all $n \geq n_0$

$$\begin{aligned}\mathrm{Var}\left(\Sigma_{dn,n}\right) &= \int_{dn}^{n}\int_{0}^{-x_{\max}} y^2 \cdot 2pcn \frac{p_1(dn-(1+\epsilon)n-y)}{p_1(dn-(1+\epsilon)n)} \frac{\mathrm{d}y}{2\sqrt{2\pi y^3}}\mathrm{d}s. \\ &\leq 2pcn(n-dn)\cdot I_2\left((1+\epsilon-d)n\right) \\ &\leq \frac{C(\epsilon,p)}{n} \qquad \text{from (3.11)} \end{aligned} \tag{3.17}$$

with $C(\epsilon,p) \in (0,+\infty)$. We conclude as follows, taking into account (3.15), (3.16) and (3.17)

$$\mathbb{P}(B_n) \leq \mathbb{P}\left(\Sigma_{dn,n} \leq (1+\epsilon-c)n\right) \leq \mathbb{P}\left(\Sigma_{dn,n} - \mathbb{E}[\Sigma_{dn,n}] \leq -bn\right) \leq \frac{\mathrm{Var}(\Sigma_{dn,n})}{b^2n^2} \leq \frac{A}{n^3}$$

with $A \in (0,+\infty)$. □

Proof of Proposition 3.1: It is a direct consequence of Lemmas 3.4 and 3.5 together with Borel-Cantelli lemma. □

3.2. *Approximation by the predictable compensator.* This section paves the way to the proof of Theorem 1.2. A particular attention is given to the approximation of X_p by its *predictable compensator* defined for $t \geq 0$ by

$$X_p^{\mathrm{pre}}(t) = X_p(0) + \int_0^t \int_{\mathbb{R}_+} y\cdot(y+2pX_p(s))\cdot \frac{p_1(s-X_p(s)-y)}{p_1(s-X_p(s))}\frac{\mathrm{d}y}{2\sqrt{2\pi y^3}}\mathrm{d}s. \tag{3.18}$$

With this definition

$$\left(M(t) := X_p(t) - X_p^{\mathrm{pre}}(t) : t \geq 0\right)$$

is a martingale (see e.g. Jacobsen, 2006, Chapter 4).

Remark 3.7. From Proposition 2.4 one could expect $X_p(s-)$ instead of $X_p(s)$ in the previous expression of the predictable compensator. Nevertheless, as the process X_p is almost surely càdlag and has countably many jumps on any time interval, defining X_p^{pre} with $X_p(s-)$ in the integral would result in a process indistinguishable from the one defined by (3.18).

From the previous section, for $\epsilon > 0$ small enough, almost surely, for t large enough, $X_p(t) \geq (1+\epsilon)t$. This in particular allows us to use the asymptotics (2.2) of p_1 to approximate X_p^{pre}. We first show that the associated martingale M converges almost surely, which justifies the approximation with the compensator. In the sequel we let $\epsilon > 0$ be such that Proposition 3.1 holds.

Lemma 3.8. *The martingale $(M(t) : t \geq 0)$ converges almost surely when $t \to +\infty$, and its limit M_∞ is almost surely finite.*

Proof: To prove the lemma, we compute the quadratic variation $\langle M, M\rangle_t$ of M and show that $\langle M, M\rangle_\infty := \lim_{t\to+\infty}\langle M, M\rangle_t$ is almost surely finite. For $t \geq 0$, we have (Jacobsen, 2006, Chapter 4)

$$\langle M, M\rangle_t = \int_0^t \int_{\mathbb{R}_+} y^2\cdot(y+2pX_p(s))\cdot \frac{p_1(s-X_p(s)-y)}{p_1(s-X_p(s))}\frac{\mathrm{d}y}{2\sqrt{2\pi y^3}}\,\mathrm{d}s.$$

By Proposition 3.1, almost surely there exists $t_0 \geq 0$ such that for all $t \geq t_0$, $X_p(t) \geq (1+\epsilon)t$. We set

$$t_1 := t_0 \vee (x_0/\epsilon) \vee 1,$$

where x_0 has been defined in Lemma 2.3. For every $t \geq t_1$, $X_p(t) - t \geq x_0 \geq 0$, so that, recalling notation (2.7) and Lemma 2.3 we have for $t \geq t_1$

$$\begin{aligned}\langle M, M\rangle_t &= \langle M, M\rangle_{t_1} + \int_{t_1}^{t} I_3\left(X_p(s) - s\right) + 2pX_p(s) \cdot I_2\left(X_p(s) - s\right) \mathrm{d}s \\ &\leq \langle M, M\rangle_{t_1} + \int_{t_1}^{\infty} \frac{C_3}{\left(X_p(s) - s\right)^5} \mathrm{d}s + \int_{t_1}^{\infty} \frac{2p \cdot C_2 X_p(s)}{\left(X_p(s) - s\right)^3} \mathrm{d}s \end{aligned} \tag{3.19}$$

Using Proposition 3.1 again, it is easy to see that the first integral is almost surely finite. For the second one, we write

$$\frac{X_p(s)}{\left(X_p(s) - s\right)^3} = \frac{1}{\left(X_p(s) - s\right)^2} + \frac{s}{\left(X_p(s) - s\right)^3} \underset{s \to +\infty}{=} O\left(s^{-2}\right)$$

to see that the second integral in (3.19) is also almost surely finite. Gathering these information we indeed get that $\langle M, M\rangle_\infty$ is almost surely finite, which leads to the claim of the lemma. □

The previous lemma justifies the approximation of the process X by its predictable compensator X_p^{pre}. The following lemma is key in the proof of Theorem 1.2.

Lemma 3.9. *There exists a deterministic constant $C \in (0, +\infty)$ and a random time $t_0 = t_0(\omega) \geq 0$ verifying $t_0 < +\infty$ almost surely such that for all $t \geq t_1 \geq t_0$ we have:*

$$\int_{t_1}^{t} \frac{pX_p(s)}{X_p(s) - s} \mathrm{d}s - \frac{C}{t_1^2} < X_p^{\mathrm{pre}}(t) - X_p^{\mathrm{pre}}(t_1) < \int_{t_1}^{t} \frac{pX_p(s)}{X_p(s) - s} \mathrm{d}s + \frac{C}{t_1^2}.$$

The following lemma lays the foundations of the proof of Theorem 1.2. It indeed shows that the couple $(X_p, X_p^{\mathrm{pre}})$ almost surely approximately solves the (deterministic) integral equation problem

$$x_p^{\mathrm{pre}}(t) - x_p^{\mathrm{pre}}(t_1) = \int_{t_1}^{t} \frac{p \cdot x_p(s)}{x_p(s) - s} \mathrm{d}s \tag{E}$$

with final boundary condition $\lim_{t \to +\infty} x_p^{\mathrm{pre}}(t) - x_p(t) = m_\infty$ for some $m_\infty \in \mathbb{R}$. We will prove in Section 3.3 (more precisely in Lemma 3.11) that a solution (x, x^{pre}) of (E) must verify $x(t) - (1 + p)t \to 0$ as $t \to \infty$. As the proof of Lemma 3.11 relies on Section 2.3 and especially on Lemma 2.11, a starting point of comparison will be required. For the random couple $(X_p, X_p^{\mathrm{pre}})$ this is the object of the next lemma.

Lemma 3.10. *For every $\delta > 0$, almost surely, there exists t arbitrarily large such that*

$$(1 + p)t - M_\infty - \delta < X_p^{\mathrm{pre}}(t) < (1 + p)t - M_\infty + \delta. \tag{3.20}$$

Combined with Lemma 3.8 this implies in particular that for any $\delta > 0$, almost surely, there exists t arbitrarily large such that

$$(1 + p)t - \delta < X_p(t) < (1 + p)t + \delta.$$

We prove both lemmas, starting with Lemma 3.9.

Proof of Lemma 3.9: By Proposition 3.1, almost surely, there exists $t_0 \in (0, +\infty)$ such that for every $s \geq t_0$, $X_p(s) - s \geq \epsilon s \geq 0$. Recalling Definition (3.18) of X_p^{pre} and using the notations of Lemma 2.3 we deduce that for every $t \geq t_1 \geq t_0$

$$X_p^{\mathrm{pre}}(t) - X_p^{\mathrm{pre}}(t_1) = \int_{t_1}^{t} I_2\left(X_p(s) - s\right) + pX_p(s) I_1\left(X_p(s) - s\right) \mathrm{d}s.$$

Moreover, combining Proposition 3.1 and Lemma 2.3 there almost surely exists $t_0' \geq 0$ such that for every $s \geq t_0'$

$$\left| I_1\left(X_p(s) - s\right) - \frac{1}{X_p(s) - s} \right| \leq \frac{C_1}{\left(X_p(s) - s\right)^4} \quad \text{and} \quad I_2\left(X_p(s) - s\right) \leq \frac{C_2}{\left(X_p(s) - s\right)^3}$$

where $C_1, C_2 \in (0, +\infty)$ are deterministic constants. Thus, for every $t \geq t_1 \geq \max(t_0, t_0')$

$$\begin{aligned} X_p^{\mathrm{pre}}(t) - X_p^{\mathrm{pre}}(t_1) &\leq \int_{t_1}^{t} \frac{C_2}{(X_p(s)-s)^3} + \frac{pX_p(s)}{X_p(s)-s} + \frac{pC_1X_p(s)}{(X_p(s)-s)^4}\mathrm{d}s \\ &< \int_{t_1}^{t} \frac{pX_p(s)}{X_p(s)-s}\mathrm{d}s + \frac{C}{t_1^2} \end{aligned}$$

for a deterministic constant $C \in (0, +\infty)$, since $X_p(s) - s \geq \epsilon s$ for $t \geq t_0$ and where we wrote again $X_p(s) = X_p(s) - s + s$ to deal with $X_p(s)/(X_p(s)-s)^4$. The lower bound is obtained similarly. □

Proof of Lemma 3.10: The proof is a consequence of Lemma 3.9. Fix $\delta > 0$ and define

$$A := \left\{\text{There exists } t \text{ arbitrarily large such that } \; X_p^{\mathrm{pre}}(t) < (1+p)t - M_\infty + \delta.\right\}$$

Since $M(t)$ converges almost surely to a finite limit M_∞ as $t \to +\infty$, it suffices to show that $\mathbb{P}(B) = 0$ where

$$B := A^c \cap \{M(t) \xrightarrow[t\to+\infty]{} M_\infty\} \cap \{M_\infty \text{ is finite}\}.$$

We take $\omega \in B$. The rest of the proof is deterministic and we forget the dependence on ω. By definition of B there exists $t_1 \geq 0$ such that for all $t \geq t_1$

$$X_p^{\mathrm{pre}}(t) \geq (1+p)t - M_\infty + \delta \tag{3.21}$$

Since $\omega \in \{M(t) \xrightarrow[t\to+\infty]{} M_\infty\}$, taking t_1 larger if necessary, we can assume that for all $t \geq t_1$

$$M(t) \geq M_\infty - \frac{\delta}{2}. \tag{3.22}$$

Using Lemma 3.9, Equations (3.21) and (3.22) we get that for every $t \geq t_1$

$$\begin{aligned} (1+p)t - M_\infty + \delta \leq X_p^{\mathrm{pre}}(t) &\leq X_p^{\mathrm{pre}}(t_1) + \frac{C}{t_1^2} + \int_{t_1}^{t} p + \frac{ps}{X_p^{\mathrm{pre}}(s) + M_\infty - \frac{\delta}{2} - s}\mathrm{d}s \\ &\leq X_p^{\mathrm{pre}}(t_1) + \frac{C}{t_1^2} + \int_{t_1}^{t} p + \frac{ps}{ps + \frac{\delta}{2}}\mathrm{d}s \\ &\leq X_p^{\mathrm{pre}}(t_1) + \frac{C}{t_1^2} + (1+p)(t-t_1) - \frac{\delta}{2p}\ln\left(\frac{pt + \frac{\delta}{2}}{pt_1 + \frac{\delta}{2}}\right). \end{aligned}$$

Since M_∞ is finite on B and the logarithmic term tends to infinity as $t \to +\infty$, we get a contradiction, and B is a null event. With the exact same reasoning we get that almost surely

$$X_p^{\mathrm{pre}}(t) > (1+p)t - M_\infty - \delta \text{ for arbitrarily large } t$$

The continuity of X_p^{pre} then implies that both inequalities of (3.20) must be satisfy for arbitrarily large t, which concludes the proof. □

3.3. *Proof of Theorem 1.2.* Theorem 1.2 states the almost sure convergence of $X_p(t) - (1+p)t$ to 0 when t goes to infinity. Equipped with the results of the previous section, the rest of the proof of Theorem 1.2 is actually deterministic. We therefore prove a deterministic result and then apply it to $(X_p, X_p^{\mathrm{pre}})$ with probability 1. Inspired by Proposition 3.1, Lemma 3.8, Lemma 3.9 and Lemma 3.10 we define the following assumptions for a couple of functions $(x_p, x_p^{\mathrm{pre}})$.

(H1). x_p is càdlag and x_p^{pre} is continuous
(H2). There exists $\varepsilon \in (0, p)$ and $t_0 \in [0, +\infty)$ such that for every $t \geq t_0$, $x_p(t) \geq (1+\epsilon)t$.
(H3). $x_p(t) - x_p^{\mathrm{pre}}(t) \to m_\infty$ as $t \to \infty$, for some $m_\infty \in \mathbb{R}$.

(H4). There exists $t_0 \in [0, +\infty)$ such that for every $t \geq t_1 \geq t_0$

$$\left| x_p^{\mathrm{pre}}(t) - x_p^{\mathrm{pre}}(t_1) - \int_{t_1}^{t} \frac{p \cdot x_p(s)}{x_p(s) - s} \mathrm{d}s \right| < \frac{c}{t_1^2}.$$

(H5). For every $\delta > 0$, there exists t arbitrarily large such that

$$\left| x_p^{\mathrm{pre}}(t) - (1+p)t + m_\infty \right| < \delta.$$

We prove the following.

Lemma 3.11. *Let (x_p, x_p^{pre}) be a couple of functions verifying (H1)-(H5). Then*

$$x_p(t) - (1+p)t \to 0 \text{ as } t \to +\infty.$$

This immediately gives the proof of Theorem 1.2.

Proof of Theorem 1.2: By Lemma 3.11 it suffices to show that $(X_p, X_p^{\mathrm{pre}})$ almost surely verifies hypotheses (H1)-(H5). As already mentioned, X_p is almost surely càdlag and X_p^{pre} is almost surely continuous and (H1) is fulfilled. Hypotheses (H2), (H3), (H4) and (H5) are then immediate consequences of respectively Proposition 3.1, Lemma 3.8, Lemma 3.9 and Lemma 3.10. □

To conclude this section and the proof of Theorem 1.2, it thus suffices to prove Lemma 3.11.
• **Set-ups** Fix $\delta > 0$. Gathering hypotheses (H3) and (H5), there exists $t_0 \in [0, +\infty)$ such that

$$\left| x_p^{\mathrm{pre}}(t_0) - (1+p)t_0 + m_\infty \right| < \delta. \tag{3.23}$$

and such that for every $t \geq t_0$

$$\left| x_p(t) - x_p^{\mathrm{pre}}(t) - m_\infty \right| < \delta. \tag{3.24}$$

Moreover, using (H4) and the fact that t_0 can be chosen arbitrarily large in (H4) we can also assume that for every $t \geq t_0$

$$\left| x_p^{\mathrm{pre}}(t) - x_p^{\mathrm{pre}}(t_0) - \int_{t_0}^{t} \frac{p \cdot x_p(s)}{x_p(s) - s} \mathrm{d}s \right| < \delta. \tag{3.25}$$

Let us also define, for every $t \geq t_0$

$$u(t) = \int_{t_0}^{t} \frac{p \cdot x_p(s)}{x_p(s) - s} \mathrm{d}s. \tag{3.26}$$

By assumptions (H1) and (H2), this defines a continuous and left-differentiable function.
Let us finally define, for $D := \{(t, x) \in [t_0, +\infty) \times \mathbb{R}_+ : x \geq (1+\epsilon)t\}$, and for $(t, x) \in D$

$$f_1(t, x) := p + \frac{pt}{x + x_p^{\mathrm{pre}}(t_0) + m_\infty + 2\delta - t} \tag{3.27}$$

and

$$f_2(t, x) := p + \frac{pt}{x + x_p^{\mathrm{pre}}(t_0) + m_\infty - 2\delta - t}. \tag{3.28}$$

With the previous definition of D the functions f_1 and f_2 are well-defined and continuous, provided that t_0 is chosen sufficiently large. Let us then define the associated differential equations

$$(\mathrm{E}_1) : v'(t) = f_1(t, v(t)) \quad \text{and} \quad (\mathrm{E}_2) : v'(t) = f_2(t, v(t)).$$

With these notations, we prove the following, recalling terminology of Section 2.3.

Corollary 3.12. *The function u is a strict left over solution to Equation (E_1) and a strict left under solution to Equation (E_2).*

Proof: Continuity and left-differentiability have already been discussed. Moreover, for every $t > t_0$ we have

$$\partial_- u(t) = \frac{p \cdot x_p(t-)}{x_p(t-) - t} = p + \frac{pt}{x_p^{\mathrm{pre}}(t) + (x_p(t-) - x_p^{\mathrm{pre}}(t)) - t}.$$

Since (3.24) and (3.25) hold we deduce that for every $t > t_0$

$$\partial_- u(t) > p + \frac{pt}{u(t) + x_p^{\mathrm{pre}}(t_0) + m_\infty + 2\delta - t} = f_1(t, u(t))$$

and

$$\partial_- u(t) < p + \frac{pt}{u(t) + x_p^{\mathrm{pre}}(t_0) + M_\infty - 2\delta - t} = f_2(t, u(t)),$$

which yields the desired claim. □

We can turn to the proof of Lemma 3.11. The strategy is the following: we find solutions to Equations (E_1) and (E_2) and then compare them to u using Lemma 2.11 and Lemma 3.12.

Proof of Lemma 3.11: Let us define, for $t \geq t_0$

$$u_1(t) := (1+p)t - x_p^{\mathrm{pre}}(t_0) - m_\infty - 2\delta \quad \text{and} \quad u_2(t) := (1+p)t - x_p^{\mathrm{pre}}(t_0) - m_\infty + 2\delta.$$

The functions u_1 and u_2 are differentiable and it is easy to check that u_1 is solution to Equation (E_1) and u_2 solution to (E_2). For example $f_1(t, u_1(t)) = 1 + p = u_1'(t)$. Moreover, observe that $u(t_0) = 0$ and that

$$u_1(t_0) = (1+p)t_0 - x_p^{\mathrm{pre}}(t_0) - m_\infty - 2\delta \quad \text{and} \quad u_2(t_0) = (1+p)t_0 - x_p^{\mathrm{pre}}(t_0) - m_\infty + 2\delta.$$

Condition (3.23) on t_0 therefore implies that $u_1(t_0) < u(t_0) = 0 < u_2(t_0)$. We thus are exactly in the settings of Lemma 2.11 which yields that for every $t \geq t_0$

$$(1+p)t - x_p^{\mathrm{pre}}(t_0) - m_\infty - 2\delta \leq u(t) \leq (1+p)t - x_p^{\mathrm{pre}}(t_0) - m_\infty + 2\delta.$$

By Definition (3.26) of u and Equation (3.25), this implies that for every $t \geq t_0$

$$(1+p)t - m_\infty - 3\delta \leq x_p^{\mathrm{pre}}(t) \leq (1+p)t - m_\infty + 3\delta.$$

Plugging finally Equation (3.24) this yields that for every $t \geq t_0$

$$(1+p)t - 4\delta \leq x_p(t) \leq (1+p)t + 4\delta.$$

Since δ can be chosen arbitrarily small, this concludes the proof. □

4. The case $p = 0$

The aim of this section is to prove Theorem 1.4 which states the convergence of $(X_0(t) - t)_{t \geq 0}$ to a stationary law as t goes to infinity and gives tail inequalities for the stationary law. We use a different strategy to the one used in the case $p \in (0, 1]$ based on the fact that when $p = 0$ the process $(X_0(t) - t)_{t \geq 0}$ is a homogeneous Markov process. We use a Foster-Lyapunov criterion for Markov processes (Meyn and Tweedie, 2009, Theorem 20.3.2). More precisely, if $\mathcal{A}$ denotes the extended generator of $(X_0(t) - t)_{t \geq 0}$, we aim to find a function $V : \mathbb{R} \to [1, +\infty]$ such that

$$\mathcal{A}V(x) \leq -aV(x) + b\mathbb{1}_C(x) \tag{4.1}$$

where $a > 0$, $b < +\infty$ and C is a petite set. A petite set is a set such that there exists a probability distribution a on $\mathbb{R}_+$ such that for any $x \in C$ and any $B \in \mathcal{B}(\mathbb{R})$

$$K_a(x, B) := \int_0^\infty P_t(x, B) a(\mathrm{d}t) \geq \nu_a(B)$$

where ν_a is a non-trivial measure on $\mathcal{B}(\mathbb{R})$ and where $(P_t)_{t \geq 0}$ denotes the Markov semi-group of Y. Notably, we will see that in our case any compact set is a petite set.

4.1. *Extended generator.* To lighten the notation, let us define $(Y(t))_{t\geq 0} := (X_0(t) - t)_{t\geq 0}$.

Lemma 4.1. *The process $(Y(t))_{t\geq 0}$ is a non-explosive, irreducible, aperiodic, homogeneous Markov process, with extended generator*

$$\mathcal{A}(f)(x) = -f'(x) + \int_{\mathbb{R}_+} (f(y+x) - f(x)) \cdot \frac{p_1(-x-y)}{p_1(-x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}}, \tag{4.2}$$

and whose domain contains functions f of the form $f(x) = \mathrm{e}^{\alpha x^3}\mathbb{1}_{\{x>0\}} + \mathrm{e}^{\beta|x|^3}\mathbb{1}_{\{x\leq 0\}}$ for $0 < \alpha < \frac{1}{6}$ and some $\beta > 0$.

Proof: Non-explosivity, irreducibility and aperiodicity have already been discussed in Proposition 2.6. Let $(L_u)_{u\geq 0}$ be the family of generators associated to the (inhomogeneous) process $(X_0(u))_{u\geq 0}$. We have, for any $\mathcal{C}^2$ function $f : \mathbb{R}_+ \to \mathbb{R}$ and $x \in \mathbb{R}_+$

$$L_u(f)(x) = \int_{\mathbb{R}_+} (f(x+y) - f(x))\, \mathbf{n}_0(u, x, \mathrm{d}y).$$

Then, if $\mathcal{L}$ is the generator of $(t, X_0(t))_{t\geq 0}$, for any regular function $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$

$$\begin{aligned}\mathcal{L}(F)(t,x) &= \partial_t F(t,x) + L_t\left(F(t,.)\right)(x)\\ &= \partial_t F(t,x) + \int_{\mathbb{R}_+} (F(t, x+y) - F(t,x))\, \mathbf{n}_0(t, x, \mathrm{d}y).\end{aligned}$$

Let $f : \mathbb{R} \to \mathbb{R}$ and $F(t,x) := f(x-t)$ for $t, x \in \mathbb{R}_+$. Applying the previous equality to F yields that

$$\mathcal{L}(F)(t,x) = -f'(x-t) + \int_{\mathbb{R}_+} (f(x+y-t) - f(x-t))\, \mathbf{n}_0(t, x, \mathrm{d}y).$$

By evaluating this equation in $(t, X_0(t))$ we therefore have

$$\mathcal{L}(F)\left(t, X_0(t)\right) = -f'\left(Y(t)\right) + \int_{\mathbb{R}_+} (f(y + Y(t)) - f(Y(t))) \cdot \frac{p_1(-Y(t)-y)}{p_1(-Y(t))} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}}.$$

We thus deduce that

$$\mathcal{A}(f)\left(Y(t)\right) = -f'\left(Y(t)\right) + \int_{\mathbb{R}_+} (f(y + Y(t)) - f(Y(t))) \cdot \frac{p_1(-Y(t)-y)}{p_1(-Y(t))} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}},$$

so that if $x \in \mathbb{R}$

$$\mathcal{A}(f)(x) = -f'(x) + \int_{\mathbb{R}_+} (f(y+x) - f(x)) \cdot \frac{p_1(-x-y)}{p_1(-x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}}.$$

Note that we did not talk about domain in this proof, we do not describe it entirely but we need to make sure that the function we will use in (4.1) belongs to $\mathcal{D}(\mathcal{A})$. For this we need

$$M(t) := f\left(Y(t)\right) - f\left(Y(0)\right) - \int_0^t \mathcal{A}(f)\left(Y(s)\right) \mathrm{d}s$$

to be a local martingale. The difficulty is about the integrability condition and this follows from the fact that for all $t \geq 0$ and $x \geq 0$

$$\sup_{0\leq s\leq t} \int_{\mathbb{R}_+} |f(x+y) - f(x)| \mathbf{n}_0(s, x, \mathrm{d}y) < +\infty.$$

We do not detail here why functions of the form $f(x) = \mathrm{e}^{\alpha|x|^3}$ for $0 < \alpha < \frac{1}{6}$ do check this condition, as it would be redundant with what follows, but hopefully it will become clear after the calculations in the next section. □

4.2. *Proof of Theorem 1.4.* This is the technical part of the proof. In a arbitrary way let us define

$$V(x) := \mathrm{e}^{\alpha|x|^3}\mathbb{1}_{\{x\geq 0\}} + \mathrm{e}^{\beta|x|^3}\mathbb{1}_{\{x<0\}} \tag{4.3}$$

with

$$0<\alpha<\frac{1}{6} \quad \text{and} \quad 0<\beta<\min\left(\frac{p_1(0)}{9},\alpha\right).$$

We need a few settings in order to prove Theorem 1.4. Thanks to the asymptotics (2.2) and (2.3) on p_1, there exists $A>0$ be such that for all $x>A$

$$\frac{1}{2}\cdot\frac{x^{-5/2}}{\sqrt{2\pi}} \leq p_1(x) \leq \frac{3}{2}\cdot\frac{x^{-5/2}}{\sqrt{2\pi}} \tag{4.4}$$

and for every $x<-A$

$$\frac{1}{2}\cdot\frac{\sqrt{|x|}\mathrm{e}^{-|x|^3/6}}{\sqrt{2\pi}} \leq p_1(x) \leq \frac{3}{2}\cdot\frac{\sqrt{|x|}\mathrm{e}^{-|x|^3/6}}{\sqrt{2\pi}} \tag{4.5}$$

(the choices of $\frac{1}{2}$ and $\frac{3}{2}$ are arbitrary). We then take $B>\max(A,1)$ (we will see in the proof the required conditions for B) and finally define $\delta := (\beta/\alpha)^{1/3}$ and

$$C := [-B/\delta, B]. \tag{4.6}$$

We prove the following.

Lemma 4.2. *Let $V:\mathbb{R}\to[1,\infty)$ defined in (4.3). There exists $a\in(0,+\infty)$ and $b\in(0,+\infty)$ such that for every $x\in\mathbb{R}$ we have:*

$$\mathcal{A}V(x) \leq -aV(x) + b\mathbb{1}_C(x).$$

With this lemma, Theorem 1.4 follows from Lyapunov criterion for homogeneous Markov processes. Specifically, we refer to Meyn and Tweedie (1993), Theorem 5.1 for the statement and the proof of the criterion. We also derive the tail inequalities for the stationary law.

Proof of Theorem 1.4: From Lemma 4.2 we have

$$\forall x\in\mathbb{R}, \quad \mathcal{A}V(x) \leq -aV(x) + b\mathbb{1}_{[-B/\delta,B]}(x)$$

with $a,b\in(0,+\infty)$. To conclude we need the set $[-B/\delta,B]$ to be petite. It is a consequence that compact sets are petite sets for $(X_0(t)-t)_{t\geq 0}$ (see e.g. Tweedie, 1994, Theorems 5.1 and 7.1). Since, by Proposition 4.1, it is also a non-explosive, irreducible and aperiodic process, we can apply Foster-Lyapunov criterion (Meyn and Tweedie, 1993, Theorem 5.1) and get the first part of Theorem 1.4.

Eventually, the tail inequalities for μ are consequences of inequality (4.1) with

$$V(x) = \mathrm{e}^{\alpha|x|^3}\mathbb{1}_{\{x\geq 0\}} + \mathrm{e}^{\beta|x|^3}\mathbb{1}_{\{x<0\}}, \quad \text{with } 0<\alpha<\frac{1}{6} \text{ and some } \beta>0.$$

We integrate this inequality with respect to the probability measure μ and use that if $f\in\mathcal{D}(\mathcal{A})$ then $\int_{\mathbb{R}}\mathcal{A}f\,\mathrm{d}\mu = 0$. We get

$$0 = \int_{\mathbb{R}} \mathcal{A}V(x)\mu(\mathrm{d}x) \leq \int_{\mathbb{R}} -aV(x) + b\mathbb{1}_{[-B,B]}(x)\mu(\mathrm{d}x).$$

Combining this inequality with the fact that μ is a probability measure, we deduce that for every $0<\alpha<\frac{1}{6}$ and for some $\beta>0$

$$\int_{\mathbb{R}} \left[\mathrm{e}^{\alpha|x|^3}\mathbb{1}_{\{x\geq 0\}} + \mathrm{e}^{\beta|x|^3}\mathbb{1}_{\{x<0\}}\right]\mu(\mathrm{d}x) < +\infty.$$

In particular, both $\int_{\mathbb{R}_+}\mathrm{e}^{\alpha x^3}\mu(\mathrm{d}x)$ and $\int_{\mathbb{R}_-}\mathrm{e}^{\beta|x|^3}\mu(\mathrm{d}x)$ are finite and we finish the proof using Markov's inequality. □

To conclude we still need to prove Lemma 4.2. The proof is quite heavy.

Proof: Recall Definition (4.6) of the set C. We decompose the task whether x is in C or not.
• **Case $x > B$:** For such x the asymptotics (4.5) yield that for all $y \in \mathbb{R}_+$

$$\frac{p_1(-x-y)}{p_1(-x)} \leq 3\frac{\sqrt{x+y}}{\sqrt{x}} \mathrm{e}^{-x^2 y/2 - xy^2/2 - y^3/6}. \tag{4.7}$$

Expression (4.2) of $\mathcal{A}$ together with Definition (4.3) of V then imply that for every $x > B$

$$\begin{aligned}\mathcal{A}V(x) &= -3\alpha x^2 V(x) + \int_{\mathbb{R}_+} \left(\mathrm{e}^{\alpha(x+y)^3} - \mathrm{e}^{\alpha x^3}\right) \cdot \frac{p_1(-x-y)}{p_1(-x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}} \\ &= V(x)\left[-3\alpha x^2 + \int_{\mathbb{R}_+} \left(\mathrm{e}^{\alpha(y^3+3xy^2+3x^2y)} - 1\right) \cdot \frac{p_1(-x-y)}{p_1(-x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}}\right].\end{aligned}$$

Equation (4.7) hence leads to

$$\begin{aligned}\mathcal{A}V(x) &\leq V(x)\left[-3\alpha x^2 + 3\int_{\mathbb{R}_+} \mathrm{e}^{(\alpha-1/6)y^3 + (3\alpha-1/2)xy^2 + (3\alpha-1/2)x^2 y} \frac{\sqrt{x+y}}{\sqrt{x}} \frac{\mathrm{d}y}{2\sqrt{2\pi y}}\right] \\ &\leq V(x)\left[-3\alpha x^2 + \int_{\mathbb{R}_+} \mathrm{e}^{(\alpha-1/6)y^3} \left(1 + \frac{\sqrt{y}}{\sqrt{x}}\right) \frac{\mathrm{d}y}{2\sqrt{2\pi y}}\right],\end{aligned}$$

where we used that $0 < \alpha < 1/6$ so that $3\alpha - 1/2 < 0$. We finally use the following: if $t > 0$

$$\int_{\mathbb{R}_+} \mathrm{e}^{-tu^3}\,\mathrm{d}u = \frac{\Gamma(4/3)}{t^{1/3}} \quad \text{and} \quad \int_{\mathbb{R}_+} \mathrm{e}^{-tu^3} \frac{\mathrm{d}u}{\sqrt{u}} = \frac{2\Gamma(7/6)}{t^{1/6}}$$

where Γ is the Gamma function. Using these with $t = 1/6 - \alpha > 0$, we deduce that for any $x > B$

$$\mathcal{A}V(x) \leq V(x)\left[-3\alpha x^2 + \frac{\Gamma(7/6)}{2\sqrt{2\pi}(1/6-\alpha)^{1/6}} + \frac{\Gamma(4/3)}{2\sqrt{2\pi x}(1/6-\alpha)^{1/3}}\right].$$

Taking B large enough we get that $\mathcal{A}V(x) \leq -\frac{1}{2}V(x)$ for all $x > B$ which concludes in this case.
• **Case $x < -B/\delta$:** This is the trickiest part of the proof. The additional difficulty comes from the fact that the derivative term in the generator expression is positive so that negative contribution needs to be found in the integrals. For $x < -B/\delta$ we have

$$\mathcal{A}V(x) = 3\beta x^2 V(x) + \int_{\mathbb{R}_+} (V(y+x) - V(x)) \cdot \frac{p_1(-x-y)}{p_1(-x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}}.$$

For every $0 \leq y \leq -x$, $V(y+x) - V(x) = \mathrm{e}^{-\beta(x+y)^3} - \mathrm{e}^{-\beta x^3}$ is negative and thus

$$\begin{aligned}\mathcal{A}V(x) &\leq 3\beta x^2 V(x) + \int_{-x}^{+\infty} \left(\mathrm{e}^{\alpha(y+x)^3} - \mathrm{e}^{-\beta x^3}\right) \cdot \frac{p_1(-x-y)}{p_1(-x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}} \\ &\leq V(x)\left[3\beta x^2 + \int_{-x}^{+\infty} \left(\mathrm{e}^{\alpha(y+x)^3 + \beta x^3} - 1\right) \cdot \frac{p_1(-x-y)}{p_1(-x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}}\right].\end{aligned} \tag{4.8}$$

To find a negative contribution in the previous expression, we decompose the integral, whether $-x \leq y \leq -(1+\delta)x$ or not (recall $\delta = (\beta/\alpha)^{1/3}$). In the first case $\mathrm{e}^{\alpha(y+x)^3+\beta x^3} - 1$ is negative and we must therefore lower bound

$$\int_{-x}^{-(1+\delta)x} \left(1 - \mathrm{e}^{\alpha(y+x)^3 + \beta x^3}\right) \cdot \frac{p_1(-x-y)}{p_1(-x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}}.$$

Since $x < -B/\delta < -B$, from (4.4) we get

$$p_1(-x) \leq \frac{3|x|^{-5/2}}{2\sqrt{2\pi}}.$$

Then notice that if $-x \le y \le -x+1$ we have $0 \le x+y \le 1$. Hence, $p_1(-x-y) \ge p_1(0)$ and we deduce that

$$\begin{aligned}\int_{-x}^{-(1+\delta)x} \left(1-\mathrm{e}^{\alpha(y+x)^3+\beta x^3}\right) \cdot \frac{p_1(x-y)}{p_1(x)} \cdot \frac{\mathrm{d}y}{2\sqrt{2\pi y}} &\ge \frac{2}{3}|x|^{5/2} p_1(0) \int_{-x}^{-x+1} \left(1-\mathrm{e}^{\alpha(y+x)^3+\beta x^3}\right) \frac{\mathrm{d}y}{2\sqrt{y}} \\ &\ge \frac{2}{3}|x|^{5/2} p_1(0) \left(1-\mathrm{e}^{\alpha+\beta x^3)}\right) \left(\sqrt{-x+1}-\sqrt{-x}\right) \\ &\ge \frac{2}{3} p_1(0) \left(1-\mathrm{e}^{\alpha+\beta x^3}\right) \frac{x^2}{2} \qquad (4.9)\end{aligned}$$

We finally have to deal with the part of the integral where $y \ge -(1+\delta)x$ in (4.8). Since $x < -B$, (4.4) gives:

$$\frac{|x|^{-5/2}}{2\sqrt{2\pi}} \le p_1(-x).$$

Then $y \ge -(1+\delta)x$ implies $-x-y \le -\delta x \le -B$ and Equation (4.5) gives

$$p_1(-x-y) \le \frac{3}{2} \frac{\sqrt{|x|}\mathrm{e}^{-|x+y|^3/6}}{\sqrt{2\pi}}.$$

Combining the two previous inequalities hence yields that

$$\begin{aligned}\int_{-(1+\delta)x}^{+\infty} \left(\mathrm{e}^{\alpha(y+x)^3+\beta x^3}-1\right) \frac{p_1(-x-y)}{p_1(-x)} \frac{\mathrm{d}y}{2\sqrt{2\pi y}} &\le 3|x|^{5/2} \int_{-(1+\delta)x}^{+\infty} \mathrm{e}^{\alpha(y+x)^3+\beta x^3} \mathrm{e}^{-(y+x)^3/6} \frac{\sqrt{y+x}\mathrm{d}y}{2\sqrt{2\pi y}} \\ &\le \frac{3}{2\sqrt{2\pi}}|x|^{5/2}\mathrm{e}^{\beta x^3} \int_{-(1+\delta)x}^{+\infty} \mathrm{e}^{(y+x)^3(\alpha-1/6)}\,\mathrm{d}y \\ &\le \frac{3}{2\sqrt{2\pi}}|x|^{5/2}\mathrm{e}^{\beta x^3} \int_{-\delta x}^{+\infty} \mathrm{e}^{u^3(\alpha-1/6)}\,\mathrm{d}u \\ &\le \frac{1}{2\sqrt{2\pi}} \frac{|x|^{1/2}}{(1/6-\alpha)} \mathrm{e}^{\beta x^3}\mathrm{e}^{(1/6-\alpha)x^3} \qquad (4.10)\end{aligned}$$

where we used that for any $t>0$ and $A>0$, $\int_A^{+\infty} \mathrm{e}^{-tu^3}\mathrm{d}u \le \frac{\mathrm{e}^{-tA^3}}{3tA^2}$ with $t = 1/6-\alpha$ and $A = -\delta x$. Plugging (4.9) and (4.10) into (4.8) finally yields that for every $x < -B$

$$\mathcal{A}V(x) \le V(x) \left[3\beta x^2 - \frac{1}{3}p_1(0)\left(1-\mathrm{e}^{\alpha+\beta x^3}\right)x^2 + \frac{|x|^{1/2}}{2\sqrt{2\pi}}\mathrm{e}^{\beta x^3}\mathrm{e}^{(1/6-\alpha)x^3}\right].$$

Since $\beta < \min(\alpha, \frac{p_1(0)}{9})$, we easily see that it is possible to choose B such that for all $x < -B$ we have

$$\mathcal{A}V(x) \le -\frac{1}{2}V(x).$$

• **Case** $x \in [-B/\delta, B]$**:** We want to see that $\mathcal{A}V(x) \le b$ for $b \in (0,+\infty)$ independent of x. This follows from the continuity of $x \mapsto \mathcal{A}V(x)$ over the compact set $[-B/\delta, B]$. For $x \in [0,B]$, we have:

$$\mathcal{A}V(x) = -3\alpha x^2 \mathrm{e}^{\alpha x^3} + \int_{\mathbb{R}_+} (\mathrm{e}^{\alpha(x+y)^3} - \mathrm{e}^{\alpha x^3}) \frac{p_1(-x-y)}{p_1(-x)} \frac{\mathrm{d}y}{2\sqrt{2\pi y}},$$

The continuity over $[0,B]$ is a thus consequence of classical results of continuity for parameter-dependent integral (the domination follows from the choice of α). For $x \in [-B/\delta, 0)$ we have

$$\begin{aligned}\mathcal{A}V(x) = 3\beta x^2 \mathrm{e}^{\alpha x^3} &+ \int_0^{-x} (\mathrm{e}^{\beta(x+y)^3} - \mathrm{e}^{\alpha x^3}) \frac{p_1(-x-y)}{p_1(-x)} \frac{\mathrm{d}y}{2\sqrt{2\pi y}} \\ &+ \int_{-x}^{\infty} (\mathrm{e}^{\alpha(x+y)^3} - \mathrm{e}^{\alpha x^3}) \frac{p_1(-x-y)}{p_1(-x)} \frac{\mathrm{d}y}{2\sqrt{2\pi y}}.\end{aligned}$$

This gives us the continuity on $[-B/\delta, 0)$ (from the choice of α and β) and the continuity at 0. □

Remark 4.3. The proof of Theorem 1.4 gives more information on the tail of μ. Indeed we easily see that $\mu([t,+\infty]) \leq Ce^{-\alpha t^3}$ for all $\alpha \in \left(0, \frac{1}{6}\right)$. From the expansion (2.2), 1/6 seems to be optimal. The upper bound $\beta < \frac{p_1(0)}{9} \approx 0,0287$ for the left side of the tail is a bit disappointing, and we could also expect $\beta < 1/6$. Optimizing our method could lead to $\beta < \frac{p_1(0)+p_1(1)}{2}$ but this is still less than 1/6.

5. Applications to multiplicative coalescents

The aim of this section is to give a few applications of our results. Especially, Corollary 5.3 states the convergence of the scaling limit of the forest part of the 0-frozen Erdős-Rényi graph to a stationary law. We also give a consequence on Aldous' multiplicative coalescent in Corollary 5.2. As the results make scaling limit of uniform random forests appear, let us first recall the following facts from Contat and Curien (2023); Martin and Yeo (2018).

5.1. *Scaling limits of uniform random forests.* For $n \in \mathbb{N}, m \in \mathbb{Z}_+$, recall that $\mathcal{W}(n,m)$ denotes the set of unordered unrooted forests with n labeled vertices and m edges. Let $W(n,m)$ be a random forest chosen uniformly over $\mathcal{W}(n,m)$ (beware that there is no obvious coupling of $W(n,m)$ for varying $m \in \mathbb{Z}_+$). The scaling limit of the tree sizes of such forest in the regime $m = \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor, t \in \mathbb{R}$, were studied by Martin and Yeo (2018) and a new description of this scaling limit were proved in Contat and Curien (2023). Inspired by this regime, for $t \in \mathbb{R}$ we denote

$$W_n(t) = W\left(n, \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor\right)$$

and $\mathbb{W}_n(t)$ the sequence of the tree sizes in $W_n(t)$ in decreasing order **renormalized by** $n^{-2/3}$. Martin and Yeo (2018) proved that $\mathbb{W}_n(t)$ converges in distribution to a random variable, whose law is as follows (the following description comes from Contat and Curien, 2023). Let $(\mathbf{L}(s))_{s\geq 0}$ be a stable Lévy process with index 3/2 and only positive jumps, starting from 0 and with Lévy measure $\frac{1}{\sqrt{2\pi}}x^{-5/2}\mathbb{1}_{\{x>0\}}$. For every $s > 0$ the density of $\mathbf{L}(s)$ is $p_s(.)$, where

$$p_s(x) = s^{-2/3} \cdot p_1\left(xs^{-2/3}\right), \quad \forall x \in \mathbb{R}$$

(see e.g. Bertoin, 1996, Chapter VIII or Zolotarev, 1986). For any $t \in \mathbb{R}$, we define the process $\left(\mathbf{L}^{(t)}(s) : 0 \leq s \leq 1\right)$ obtained by conditioning the process $\mathbf{L}$ to be equal to t at time 1 (this is a *Lévy bridge* and is a degenerated conditioning but still can be obtained with the help of $h-$transforms Bertoin, 1996). With these notations, Contat and Curien (2023) proved that for every fixed $t \in \mathbb{R}$ the following convergence holds:

$$\mathbb{W}_n(t) \xrightarrow[n\to+\infty]{(d)} \left(\Delta\mathbf{L}^{(t)}(s) : 0 \leq s \leq 1\right)^{\downarrow} =: \Delta\mathbf{L}^{(t)}, \tag{5.1}$$

where $\Delta\mathbf{L}^{(t)}(s)$ denotes the size of the jump of $\mathbf{L}^{(t)}$ at time s, and where the convergence holds for the $\ell^{3/2+\epsilon}$ topology for any $\epsilon > 0$.

5.2. *Applications to multiplicative coalescents.* As mentioned in the introduction, in the critical window, the sizes of the components of the frozen graph converges after suitable renormalization. More precisely, recalling (1.1) and (1.2) the process

$$n^{-2/3} \cdot \left(\left(\left(T_p^{(i)}\left(n, \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor\right)\right)_{i\in\mathbb{N}}, \left(U_p^{(i)}\left(n, \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor\right)\right)_{i\in\mathbb{N}}\right) : t \in \mathbb{R}\right)$$

converges in $\ell^2 \times \ell^1$ towards a process

$$(\mathcal{FM}_p(t) : t \in \mathbb{R}) = \left(\left(T_p^{(i)}(t) \right)_{i \in \mathbb{N}}, \left(U_p^{(i)}(t) \right)_{i \in \mathbb{N}} : t \in \mathbb{R} \right)$$

called the *frozen multiplicative coalescent*. This process was introduced for $p = \frac{1}{2}$ in Contat and Curien (2023) and is generalized for $p \in [0,1]$ in Viau (2025+).

Just as the frozen graph is a modification of the classical Erdős-Rényi graph, the frozen multiplicative coalescent is a modification of Aldous' standard multiplicative coalescent (Aldous, 1997; Broutin and Marckert, 2016). The latter is a process $(\mathcal{M}(t) : t \in \mathbb{R})$ with values in ℓ^2. Its coordinates can be identified as *particles* and, intuitively, a pair of particles of mass x and y merges to a new particle of mass $x + y$ at rate xy. The idea is to recover the dynamics of the sizes of the connected components of the Erdős-Rényi graph. The heuristic behind the frozen multiplicative coalescent of parameter p $(\mathcal{FM}_p(t) : t \in \mathbb{R})$ is the same. The particles of the frozen multiplicative coalescent $\mathcal{FM}_p(t)$ at time t are of two types: the frozen particles $U_p^{(i)}(t), i \in \mathbb{N}$, whose decreasing masses are in ℓ^1 and the tree particles $T_p^{(i)}(t), i \in \mathbb{N}$, whose decreasing masses are in ℓ^2. Every pair of tree particles of mass x and y merges to a tree particle of mass $x + y$ at rate xy. A tree particle of mass x freezes at rate $\frac{x^2}{2}$ and a tree particle of mass x. Finally a frozen particle of mass y merges to a frozen particle of mass $x + y$ at rate pxy.

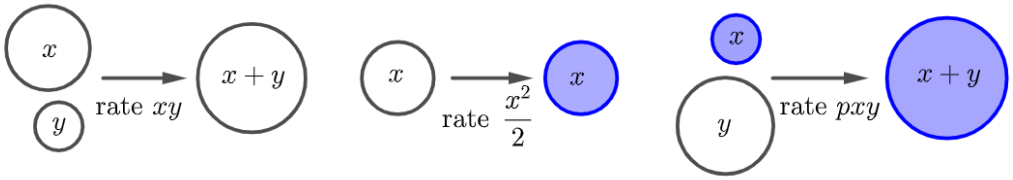

FIGURE 5.3. Dynamics of the frozen multiplicative coalescent of parameter p.

To express the forthcoming results we need to define the notion of *surplus* of a connected component, that is its number of edges minus its number of vertices plus 1. A tree is thus a component with surplus 0, a unicycle component is a component with surplus 1 and a multicyclic component has surplus at least 2. We say that a component *has surplus* if its surplus is positive. As we need it to state Corollary 5.2 note that there exists an *augmented multiplicative coalescent* $(\mathcal{M}(t), \mathcal{S}(t))$ (see Broutin and Marckert, 2016 for a construction) where the process $\mathcal{S}$ is the scaling limit of components' surplus in ER $\left(n, \left\lfloor \frac{n}{2} + \frac{t}{2} n^{2/3} \right\rfloor\right)$. By analogy we will talk about the surplus of the particles of the multiplicative coalescent.

Proposition 5.1. *For every $p \in [0,1]$, for every $t \in \mathbb{R}$, conditionally on the gel mass $X_p(t)$, the law of the sizes of the tree particles $\left(T_p^{(i)}(t)\right)_{i \in \mathbb{N}}$ is the same as the law of $\Delta \mathbf{L}^{(t - X_p(t))}$, i.e. the law of the jumps of the conditioned Lévy process $\mathbf{L}^{(t - X_p(t))}$.*

In particular for $p = 1$, since the only discarded edges are the edges connecting two frozen vertices, the forest parts of $F_1(n,m)$ and $\mathrm{ER}(n,m)$ coincide. In the limit, the vector of the sizes of the tree particles $\left(T_p^{(i)}(t)\right)_{i \in \mathbb{N}}$ is thus equal in distribution to the vector of the sizes of particles without surplus in the augmented multiplicative coalescent. This immediately gives the following corollary.

Corollary 5.2. *For every $t \in \mathbb{R}$, the process of the total mass of the particles with surplus in $\mathcal{M}(t)$ has law $X_1(t)$, and, conditionally on it, the remaining particles are distributed as the jumps of the conditioned Lévy process $\mathbf{L}^{(t - X_1(t))}$.*

We give a short proof of Proposition 5.1, which is based on the free forest property (Lemma 2.5). From Section 2.2 recall the notations $G_p(n,m), D_p(n,m), V_p(n,m)$ and $E_p(n,m)$. Recall also notation (2.14) of the time-continuous versions of the processes in the critical window.

Proof of Proposition 5.1: By Lemma 2.5, conditionally on $G_p(n,m)$ and $D_p(n,m)$, the forest part of $F_p(n,m)$ is uniformly distributed over $\mathcal{W}(V_p(n,m), E_p(n,m))$. In the regime $m = \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor$, relations (2.13) give

$$V_{p,n}(t) = n - G_{p,n}(t) \quad \text{and} \quad E_{p,n}(t) = \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor - G_{p,n}(t) - D_{p,n}(t).$$

Using Skorokhod's representation theorem and convergence (2.16) we have, with a slight abuse of notation

$$\begin{aligned} E_{p,n}(t) &= \left\lfloor \frac{n}{2} + \frac{t}{2}n^{2/3} \right\rfloor - G_{p,n}(t) + O(n^{1/3}) \\ &= \frac{V_{p,n}(t)}{2} + \frac{t - X_p(t)}{2} V_{p,n}(t)^{2/3} + o\left(V_{p,n}(t)^{2/3}\right). \end{aligned}$$

almost surely as n goes to infinity. Combining this with the convergence of the frozen Erdős-Rényi to the frozen multiplicative coalescent and the scaling limit (5.1) of uniform random forests yields the desired proposition. □

We focus on $p = 0$. By Proposition 5.1, conditionally on the size of the gel $X_0(t)$, the particles without surplus are distributed as the jump of the conditioned Lévy process $\mathbf{L}^{(t-X_0(t))}$. Moreover, by Theorem 1.4, $X_0(t) - t$ converges to a stationary law μ as t goes to $+\infty$. We immediately get the following result, already announced in the introduction, together with the following description of the limit.

Corollary 5.3. *For every $\epsilon > 0$, the following convergence in distribution holds for the $\ell^{3/2+\epsilon}$ topology*

$$\left(T_p^{(i)}(t)\right)_{i\in\mathbb{N}} \xrightarrow[t\to+\infty]{(d)} \Delta\mathbf{L}^{-M},$$

where M is a random variable with distribution μ.

As already explained in the introduction, the intuition behind this result is that when $p = 0$ the evolution of frozen components is completely stopped and the only way to increase the frozen mass is to create an intern cycle in a tree. As a consequence only "large" trees become frozen, whereas small trees remain trees. When $p \in (0,1]$ a tree can freeze if it is connected with a unicycle component, so that small trees have more probability to freeze, which prevents the existence of a stationary law. Informally, since $t - X_p(t) \underset{t\to+\infty}{\approx} -pt$, for large t the law of the tree particles in $\left(T_p^{(i)}(t)\right)_{i\in\mathbb{N}}$ is the law of a 3/2-stable Levy process with only positive jumps conditioned to be very negative a time 1 (roughly equal to $-pt$). The jumps of such a process necessarily tends towards 0 as t goes to $+\infty$.

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