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Branching Interval Partition Diffusions

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Abstract. We introduce and study branching interval partition diffusions in their natural generality. We let interval widths evolve independently according to a general real-valued diffusion subject only to conditions that ensure finite lifetimes of intervals and allow the continuous generation of new intervals. The latter is governed by the Pitman-Yor excursion measure of the real-valued diffusion and an associated spectrally positive Lévy process to order both these excursions and their start times. This generalises previous work by Forman, Pal, Rizzolo and Winkel on the self-similar case and gives rise to a new class of general Markovian homogeneous Crump-Mode-Jagers-type branching processes with characteristics varying diffusively during their lifetimes.

1. Introduction

Continuous-time branching processes with finite birth rates and independent identically distributed (iid) lifetimes of individuals are known as Crump–Mode–Jagers (CMJ) processes, as studied in Jagers (1975, Chapter 6). For these processes individuals are born from single parents, and during their lives they can in turn give birth to more individuals until they die after some iid positive random times after their births; those with constant birth rates are known as homogeneous and those for which all births produce exactly one offspring are known as binary. Specifically, CMJ processes are the population size processes and so take values in $\mathbb{N}$. In the homogeneous binary cases individuals produce single offspring after iid exponential random times and this repeats until their deaths (or never occurs if they die before their first reproduction event). We call the difference between birth and death the *lifetime* of the individual and the interval of time between the birth and death the *life* of the individual. To produce Markovian homogeneous binary CMJ processes we require either the lifetimes to be infinite or for the lifetimes to be exponentially distributed.

We study homogeneous binary branching processes with infinite birth rates where individuals are marked with non-negative characteristics that continuously vary during their lives, and we construct continuous Markovian interval-partition-valued processes by the method shown in Forman et al.

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(2020a). Interval partitions are ordered collections of intervals with summable lengths; we define these below following Aldous (1985, Section 17) and Pitman (2006, Chapter 4).

Definition 1.1 (Interval partitions)**.** For $M > 0$, an *interval partition of length* M is a set β of intervals of $[0, M]$ that are open and disjoint, and which cover $[0, M]$ up to a Lebesgue-null set. We write $\|\beta\|$ for the length M. An *interval partition* is an interval partition of length M for some M. We define $\mathcal{I}_H$ to be the space of interval partitions.

Interval partitions can be constructed by concatenating ordered collections of intervals, e.g. the concatenation of intervals $(0, b)$ and $(0, c)$ would be given by $\{(0, b), (b, b+c)\}$ where we have shifted the interval $(0, c)$ by b. More generally we can concatenate interval partitions β and γ by shifting the all intervals of γ by $\|\beta\|$ and we denote the resulting interval partition by $\beta \star \gamma$ in this case.

In our model individuals do not live forever and the characteristics take the value 0 at the births and deaths and are positive during the rest of the lives. We construct interval partitions in Definition 1.2 which are concatenations of intervals with lengths given by these characteristics. The characteristics themselves, as functions of the ages of the individuals they mark, are continuous non-negative excursions away from 0 of length given by the lifetime of the individual they mark. For this reason we refer to a characteristic at a point in time as the *level of health*, or just as the *health*, for the individual it marks at that point in time, and we refer to the continuous excursion of marks over an individual's life as its *health excursion*.

Lambert (2010) and Lambert and Uribe Bravo (2018) study spectrally positive Lévy processes (SPLPs) that are the contour functions of classes of splitting trees and are known as jumping chronological contour processes (JCCPs). For any $x \geq 0$ the jumps across level x represent the individuals alive at that point in time. We construct interval partitions from the health excursions and the SPLP by the method from Forman et al. (2020a). Definition 1.2 formally describes this construction and Figure 1.1 uses terminology from Forman et al. (2020a) which we explain in Section 1.1. The result is that at every point in time an interval partition is produced with a one-to-one correspondence of intervals and individuals alive at that point in time, and each interval has a length given by its corresponding individual's health at that point in time. These interval partitions therefore, as functions of the time for the branching process, form an interval-partition-valued process. This paper continues the investigation into these processes. We use the same terminology as in Forman et al. (2020a): we refer to stochastic processes which take values in $\mathcal{I}_H$ as *interval partition evolutions* or more simply as *IP-evolutions*. We refer to IP-evolutions that are continuous and have the Markov property with respect to some filtered probability space as *interval partition diffusions* or *IP-diffusions*. The construction can be considered to give rise to branching processes as new intervals are generated between existing intervals and we discuss this further after Theorem 1.6. We generalise the class of IP-diffusions that have this branching property beyond the self-similar cases that have already been established in Forman et al. (2020a). We use a complete and separable metric on interval partitions d'_H, from Forman et al. (2020b), which we state in Definition 1.5.

When we consider the health excursions rather than just the lifetimes of the individuals, the birth rate is a measure on the space of continuous non-negative excursions away from 0, and it is this rate that we are referring to when we say the birth rate is constant. We set the birth rate to be the Pitman-Yor excursion measure of a non-negative diffusion, from Pitman and Yor (1982), and we explore the range of possible non-negative diffusions for which this is applicable. The Pitman-Yor excursion measure is more general than the standard Itô excursion measure as it includes cases where 0 is an exit boundary of the diffusion whereas the Itô excursion measure only includes cases where 0 is a regular boundary point of the diffusion (we give the relevant details in Section 1.2). Under certain conditions for summability of the health levels, that we specify in Theorem 1.4, this leads to IP-evolutions where the interval lengths all evolve independently with the same continuous Markovian dynamics given by the specified non-negative diffusion. It is these dynamics on the interval lengths which make possible the continuity and Markov property for the IP-evolutions.

This is also why we choose excursions away from 0, as this preserves continuity at the points of birth and death of individuals; when an individual is born a new interval grows continuously out of 0 in the IP-evolution, with the parent's interval on the left, and it dies when this health level returns to 0.

Excursion measures of diffusions are not finite but σ-finite, and this is why the processes that we study have an infinite birth rate. However, most contribute small health levels and have small lifetimes in the sense that, for any positive threshold $\varepsilon > 0$ there is a finite rate of births with lifetimes or maximum health levels greater than ε. The continuity and the Markov property of the IP-evolutions do not follow immediately from the continuity and Markov property of the one-dimensional diffusions, indeed, it is not even the case that we can produce an IP-evolution from the Pitman-Yor excursion measure of any diffusion. This is because we may not be able to construct the SPLP "scaffolding" or, even if we can, the health levels may not be summable. In Section 1.2, Theorem 1.3 gives conditions for the construction of the SPLP, Theorem 1.4 gives conditions for the summability of health levels, and Theorems 1.6 and 1.7 give conditions for the existence of continuous Markovian IP-evolutions.

1.1. *Construction of the SPLP "scaffolding" and the "skewer process".* Let Λ be a Lévy measure on $(0,\infty)$ for a recurrent Lévy process, i.e. a measure such that $\int_{x=0}^{\infty}(x^2\wedge x)\Lambda(dx)$ is finite, and let $\mathbf{J}$ be a Poisson random measure (PRM) with intensity $\mathrm{Leb}\otimes\Lambda$ on $[0,\infty)\times[0,\infty)$ (where "Leb" is the Lebesgue measure). If $\int_{x=0}^{\infty}x\Lambda(dx)$ is finite then we can construct a recurrent SPLP of bounded variation

$$\mathbf{X}_t := -t\int_{x=0}^{\infty}x\Lambda(dx) + \sum_{\substack{(s,j)\text{ atom of }\mathbf{J}:\\ 0<s<t}} j.$$

Now suppose that Λ instead satisfies the condition $\int_{x=0}^{\infty}(x^2\wedge x)\Lambda(dx) < \infty = \int_{x=0}^{\infty}x\Lambda(dx)$. In this case, for any $\varepsilon > 0$, we can construct a recurrent SPLP of bounded variation using the measure $\Lambda_\varepsilon(dx) := \mathbb{1}_{x>\varepsilon}\Lambda(dx)$ as we did above and take the limit $\varepsilon\downarrow 0$ is to obtain a recurrent SPLP of unbounded variation, see (1.1). For our applications the Lévy measure Λ we will use is of the form $\nu(\zeta\in\cdot)$ where ν is the Pitman-Yor excursion measure for our specified diffusion (a measure on $\mathcal{E}$, the space of continuous positive excursions) and ζ is a function, $\zeta:\mathcal{E}\to[0,\infty)$, which gives the length of excursions which we call the lifetimes.

More formally, continuous positive excursions are functions $f:\mathbb{R}\to\mathbb{R}$ for which $f(y)=0$ for all $y\le 0$, and there exists some $x>0$ such that $f(y)=0$ for all $y>x$, and finally for $y\in(0,x)$ that $f(y)>0$ and f is continuous on $[0,x]$; the lifetime ζ of f is then $\zeta(f)=x$. We use the lifetimes as the jump sizes for the SPLP. Suppose $\int_{x=0}^{\infty}(x^2\wedge x)\nu(\zeta\in dx)<\infty$, and let $\mathbf{N}'$ be a PRM with intensity $\mathrm{Leb}\otimes\nu$ on $[0,\infty)\times\mathcal{E}$. We define the recurrent SPLP $\mathbf{X}=(\mathbf{X}_t,t\ge 0)$ by the almost sure limits

$$\mathbf{X}_t := \lim_{\varepsilon\downarrow 0}\left(-t\int_{x=\varepsilon}^{\infty}x\nu(\zeta\in dx) + \sum_{\substack{(s,f)\text{ atom of }\mathbf{N}'\\ 0<s<t}}\zeta(f)\mathbb{1}_{\zeta(f)>\varepsilon}\right). \tag{1.1}$$

The PRM $\mathbf{N}'$ takes values in a certain class of σ-finite point measures on $[0,\infty)\times\mathcal{E}$ which we denote by $\mathcal{N}$, and the SPLP $\mathbf{X}$ is a càdlàg process and we denote the set of càdlàg paths by $\mathcal{D}$. Specifically, the class $\mathcal{N}$ consists of all the point measures N for which the limit in (1.1) exists for all $t>0$ when replacing $\mathbf{N}'$ by N. The Lévy measure for $\mathbf{X}$ is $\nu(\zeta\in\cdot)$ which we refer to as the lifetime under the Pitman-Yor excursion measure. More generally, for a suitable point measure $N\in\mathcal{N}$, let $\xi(N)$ be the càdlàg function, $(X_t,t\ge 0)$, obtained when replacing $\mathbf{N}'$ by N in (1.1).

We call $\mathbf{X}$ the scaffolding and $\mathbf{N}'$ the point measure of spindles. The reason for these names can be seen in the simplified version (N,X) in Figure 1.1. In the SPLP, there is a one-to-one correspondence between individuals and jumps. Moreover, the jump heights are equal to the individuals' lifetimes, and levels over which each jump crosses are the times for which the corresponding individual is

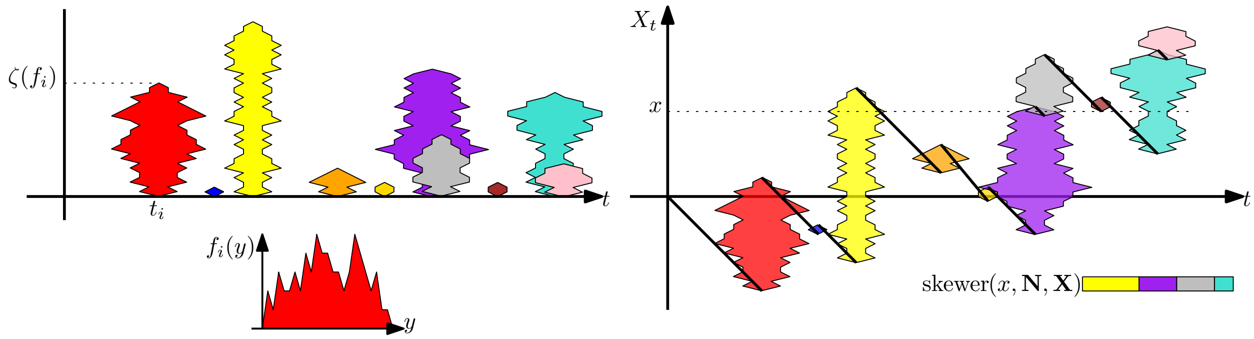

FIGURE 1.1. Left: A PRM of health excursions represented with time vertically moving up page and reflected to produce spindle-like shapes, with the excursion of a single spindle underneath. Right: the scaffolding process (black) constructed from the PRM, along with the interval partition generated by the skewer process at a fixed level x.

alive in our model. Therefore, in the figure, the health excursion can be placed vertically in each jump. The excursions themselves look like spindles as we have width equal to health and they are symmetric for better visual representation on the diagram. We next define the skewer function.

Definition 1.2 (Skewer function)**.** Let J be a countable set and let $N' \coloneqq \sum_{i\in J}\delta_{(r_i,f_i)} \in \mathcal{N}$. Let $I \subseteq J$ be a subset and let $N \coloneqq \sum_{i\in I}\delta_{(r_i,f_i)}$. In this setting we write $N \le N'$ to denote the fact that N' has all the atoms of N. Let $X = \xi(N')$. We define the *aggregate health process* $((M^y_{N,X})(t), t \in [0,\infty))$ by

$$M^y_{N,X}(t) \coloneqq \sum_{\substack{(r,f) \text{ atom of } N:\\ 0\le r<t}} f(y - X_{r-}).$$

We also define the *total health process* at $t \in [0,\infty)$ as $(M^y_{N,X}(t), y \in \mathbb{R})$. When $M^y_{N,X}(\infty) < \infty$ we define the skewer for (N, X) by

$$\text{skewer}(y, N, X) \coloneqq \{(M^y_{N,X}(t-), M^y_{N,X}(t)) : t \in [0,\infty), M^y_{N,X}(t-) < M^y_{N,X}(t)\}.$$

Note that in this case skewer(y, N, X) is an interval partition of length $M^y_{N,X}(\infty)$. We define skewer(y, N, X) to be $\emptyset$, the empty interval partition, if $M^y_{N,X}(\infty)$ is infinite. For random measures $\mathbf{N} \le \mathbf{N}'$ and scaffolding $\mathbf{X} = \xi(\mathbf{N}')$ we define the scaffolding-and-spindles construction of $(\mathbf{N}, \mathbf{X})$ to be the IP-evolution given by $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X}) \coloneqq (\text{skewer}(y, \mathbf{N}, \mathbf{X}), y \ge 0)$. Unless stated otherwise, we take $\mathbf{N}'$ to be a PRM and we take

$$\mathbf{N} = \sum_{\substack{(r,f) \text{ atom of } \mathbf{N}':\\ 0\le r\le T}} \delta_{(r,f)}$$

where T is a.s. positive finite random variable and then in this case we say $\mathbf{N}$ is *stopped at* T, we refer to $\mathbf{N}$ as a *stopped PRM* and we refer to $(\mathbf{N}, \mathbf{X})$ as a *stopped scaffolding-and-spindles pair.*

1.2. *Main results.* We define one-dimensional diffusions with the framework given by Revuz and Yor (1991, Chapter VII.3). In our setting we consider a Markov process $\mathbf{Y}$ whose state space E is of the form $E = [0, h]$ where h is finite or of the form $E = [0, h)$ where h is either finite or infinity. We assume that the paths of $\mathbf{Y}$ are continuous on E and that $\mathbf{Y}$ has the strong Markov property. We further assume that our diffusion is regular on $(0, h)$, which means that for all points x, $y \in (0, h)$ the probability of a diffusion reaching point y having started from point x is positive, i.e. $\mathbb{P}_x(T_y < \infty) > 0$ where T_y is the hitting time of the point y by the diffusion. With these assumptions, there exists a strictly increasing function $s : E \to [0, \infty)$ such that for any a, w, x in E with $0 \le a < x < w \le h$ we have

$$\mathbb{P}_x(T_w < T_a) = \frac{s(x) - s(a)}{s(w) - s(a)}, \tag{1.2}$$

where the function s is called a scale function of $\mathbf{Y}$. The scale function is unique up to a normalising constant and we take $s(0) = 0$. We also define for $v \in (a, w)$

$$G_{(a,w)}(x, v) = \frac{s(x) - s(a)}{s(w) - s(a)} \left(s(w) - s(v)\right) \mathbb{1}_{x<v<w} + \frac{s(w) - s(x)}{s(w) - s(a)} \left(s(v) - s(a)\right) \mathbb{1}_{a<v<x} \tag{1.3}$$

which we call the Green's function for the diffusion on (a, w). Then there exists a unique measure M on $(0, h)$, which is known as the speed measure, such that for any $0 \le a < x < w \le h$ we have

$$\mathbb{E}_x(T_a \wedge T_w) = \int_{v=a}^{w} G_{(a,w)}(x, v) M(dv). \tag{1.4}$$

Certain diffusions can be expressed as solutions to stochastic differential equations as follows. Specifically, there exist locally bounded Borel-measurable functions $\mu : (0, h) \to \mathbb{R}$, $\sigma : (0, h) \to (0, \infty)$ and a Brownian motion $\mathbf{B}$ such that under $\mathbb{P}_x$, $\mathbf{Y}$ satisfies the stochastic differential equation

$$d\mathbf{Y}_t = \mu(\mathbf{Y}_t)dt + \sigma(\mathbf{Y}_t)d\mathbf{B}_t; \qquad \mathbf{Y}_0 = x, \tag{1.5}$$

for all $x \in (0, h)$. More details on diffusions of this type can be found in e.g. Revuz and Yor (1991, Chapter VII.2).

We now state assumptions on the diffusions that we consider in this paper.

Assumption A. A diffusion $\mathbf{Y}$ on state space E satisfies *Assumption A* if:

(1) We have that $\mathbb{P}_x(T_0 < \infty) > 0$ for some $x \in (0, h)$ which is equivalent to $\int_{y=0}^{x} s(y)M(dy) < \infty$ (equivalently for all $x \in (0, h)$ in both cases).
(2) For all $0 < y < x < h$ we have that $\mathbb{E}_x(T_y)$ is finite. This is satisfied if and only if $M(E\backslash[0, x])$ is finite for all $x \in (0, h)$.
(3) There exist locally bounded Borel-measurable functions $\mu : (0, h) \to \mathbb{R}$, $\sigma : (0, h) \to (0, \infty)$ and a Brownian motion $\mathbf{B}$ such that $\mathbf{Y}$ satisfies the stochastic differential equation in (1.5).
(4) The functions μ and σ defined in $A3$ are continuous on $(0, b]$ for some $b \in E\backslash\{0\}$.

We label these conditions $A1$, $A2$, $A3$, $A4$.

We use the representation of stochastic differential equations later in Section 4.2 but for now we simply note that for a process Y that under Assumption A we have

$$\mu(x) = \lim_{z \downarrow 0} \frac{1}{z}\mathbb{E}_x(Y_z - x); \qquad \sigma^2(x) := (\sigma(x))^2 = \lim_{z \downarrow 0} \frac{1}{z}\mathbb{E}_x((Y_z - x)^2) \qquad \text{for } x \in (0, b)$$

where b is the constant in $A4$. The functions μ and σ^2 are referred to as the infinitesimal drift and the infinitesimal variance of the diffusion. Heuristically, for $x \in (0, h)$, the infinitesimal drift $\mu(x)$ describes the average direction of a diffusion at position x will move in and the infinitesimal variance $\sigma^2(x)$ describes the roughness of the path of a diffusion position x.

The first condition $A1$ is equivalent to 0 being an attainable boundary point for the diffusion, i.e. a regular or exit boundary point according to the Feller boundary point classification, see Karlin and Taylor (1981, Table 15.6.1). The boundaries of the diffusion are the points 0 and h. In words, a regular boundary point is one where you can restart the diffusion from the boundary once the diffusion first hits the boundary, i.e. $\mathbb{P}_0(T_x < \infty) > 0$; a typical example is the boundary point 0 for a Brownian motion in $[0, \infty)$ that is reflected at 0. An exit boundary point is where this behaviour does not occur, and instead, you have that the diffusion starting in the interior of the state space can reach the point 0 with positive probability, but the diffusion cannot reenter the interior after hitting 0. We note that the boundary point h is regular exactly when $s(h) := \lim_{x \uparrow h} s(x)$ is finite equivalently when the state space E is equal to $[0,h]$ and not $[0,h)$. In later sections this case sometimes needs special consideration, and we may identify it by any of the equivalent conditions just stated depending on which is most relevant to the section. Furthermore, as some of the measures that we use in this paper are not necessarily atom-free we take care when we define the limits of integration. We write $\int_{w=a}^{b}$ for an integral over (a,b) and we write $\int_{w\in(a,b]}$ to integrate over $(a,b]$ (and then similarly to integrate over $[a,b)$ or $[a,b]$).

We now state our first results which use the hypotheses in Assumption A.

Theorem 1.3 (Lévy measure). *Consider a diffusion that satisfies $A1$, $A2$ of Assumption A. Then it has a Pitman-Yor excursion measure ν and we have that*

$$\int_{x=0}^{\infty} x\nu(\zeta\in dx)<\infty \tag{1.6}$$

if and only if

$$M(E)<\infty, \tag{1.7}$$

which is equivalent to 0 being a regular boundary point for the diffusion. In this setting, the lifetime under ν is a Lévy measure for a recurrent SPLP of bounded variation.

On the other hand, if we have that: the diffusion satisfies all of Assumption A; the condition in (1.7) does not hold (i.e. 0 is an exit boundary of the diffusion); and we have

$$\int_{v=0}^{b}\int_{y=0}^{v} s(y)M(dy)M(dv)<\infty \text{ where } b\in(0,h) \text{ is as in } A4 \tag{1.8}$$

then we have

$$\int_{x=0}^{\infty}(x^2\wedge x)\nu(\zeta\in dx)<\infty=\int_{x=0}^{\infty} x\nu(\zeta\in dx). \tag{1.9}$$

In this setting, the lifetime under ν is a Lévy measure for a recurrent SPLP of unbounded variation.

Theorem 1.3 is proven in Section 3.1. We believe that, under $A1$, $A2$, the condition in (1.8) is necessary as well as sufficient for the lifetime under ν to be a Lévy measure for a recurrent process (i.e. we believe that $A3$ and $A4$ are not necessary), but proving this is beyond the scope of this paper. The Lévy measure condition of Theorem 1.3 means that we can construct a scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$. Later, in Proposition 3.12, we define and construct $(\mathbf{N}_\beta,\mathbf{X}_\beta)$, for $\beta\in\mathcal{I}_H$. This is a scaffolding-and-spindles pair for which we prove that $\overline{\text{skewer}}(\mathbf{N}_\beta,\mathbf{X}_\beta)$ an IP-evolution such that $\text{skewer}(0,\mathbf{N}_\beta,\mathbf{X}_\beta)=\beta$ a.s.

Throughout the rest of the paper we consider diffusions along with spatial transformations $g:E\to g(E)$ that are strictly increasing and continuous with $g(0)=0$. We define

$$g(h)=\lim_{x\uparrow h} g(x)\le\infty$$

and therefore $g(E)=[0,g(h))$ or $g(E)=[0,g(h)]$. Let $\mathbf{Y}=(\mathbf{Y}_t,t\ge0)$ be a diffusion in state space E and let $\mathbf{Z}=(\mathbf{Z}_t,t\ge0)$ be the process defined by $\mathbf{Z}_t=g(\mathbf{Y}_t)$ for all $t\ge0$. Then $\mathbf{Z}$ is a diffusion and has scale function $s_{\mathbf{Z}}(x)=s_{\mathbf{Y}}(g^{-1}(x))$ where $s_{\mathbf{Y}}$ is the scale function of $\mathbf{Y}$ and speed measure $M_{\mathbf{Z}}$ which is the image of the speed measure for $\mathbf{Y}$, $M_{\mathbf{Y}}$, under g, see e.g. Revuz and Yor (1991, Exercise VII.3.18). Therefore we can note that $\mathbf{Y}$ satisfies $A1$ (respectively $A2$) if and only if $\mathbf{Z}$ satisfies $A1$ (respectively $A2$). Furthermore we see that under $A1$, $\mathbf{Y}$ and $\mathbf{Z}$ have Pitman-Yor excursion measures $\nu_{\mathbf{Y}}$ and $\nu_{\mathbf{Z}}$, see Pitman and Yor (1982, Section 3), and as the spatial transformation g does not affect the lifetime ζ of an excursion we have $\nu_{\mathbf{Z}}(\zeta\in\cdot)=\nu_{\mathbf{Y}}(\zeta\in\cdot)$.

We refer to $\mathbf{Y}$ as the untransformed diffusion throughout the paper to avoid confusion between $\mathbf{Y}$ and $\mathbf{Z}$. The motivation of the g transform is that we can consider different diffusions that give rise to the same scaffolding. This technique of using spatial transformations on untransformed diffusions to construct IP-diffusions was first used in Forman et al. (2020a), and we give the details of their setting in Example 1.8.

Theorem 1.4 (Aggregate health summability)**.** *In the setting described above, consider a diffusion* $\mathbf{Z}$ *constructed from an untransformed diffusion* $\mathbf{Y}$ *along with a spatial transformation* g. *Suppose that* $\nu = \nu_{\mathbf{Z}}$ *(equivalently* $\nu = \nu_{\mathbf{Y}}$*) satisfies* (1.6) *or* (1.9) *and let* $(\mathbf{N}, \mathbf{X})$ *be scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb* $\otimes\, \nu_{\mathbf{Z}}$ *and stopped at an a.s. finite random variable* T.

If $\nu = \nu_{\mathbf{Z}}$ *satisfies* (1.6)*, then the aggregate health process* $M^0_{\mathbf{N},\mathbf{X}}(T)$ *is finite a.s. If* $\nu = \nu_{\mathbf{Z}}$ *satisfies* (1.9)*, then* $M^0_{\mathbf{N},\mathbf{X}}(T)$ *is finite a.s. if and only if*

$$\int_{y=0}^{b} g(y) M_{\mathbf{Y}}(dy) < \infty \text{ for some (equivalently all) } b \in (0, h), \tag{1.10}$$

or equivalently that $\int_{z=0}^{b} z M_{\mathbf{Z}}(dz)$ *is finite for some (or all)* $b \in (0, g(h))$. *Furthermore in this case we have that* $M^0_{\mathbf{N},\mathbf{X}}(T)$ *is infinite a.s. when* (1.10) *does not hold.*

Theorem 1.4 is proven in Section 3.2.

Our next results involve continuity of IP-evolutions and so we now give the definition of the complete and separable metric space of interval partitions $(\mathcal{I}_H, d'_H)$ that we use from Forman et al. (2020b).

Definition 1.5 (Interval partition metric space $(\mathcal{I}_H, d'_H)$)**.** Recall $\mathcal{I}_H$, the set of interval partitions, from Definition 1.1. To define d'_H, we first define the notion of a correspondence between two interval partitions, and the distortion of a correspondence. We use the standard notation of $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

(1) A *correspondence* between two interval partitions β and γ is a tuple $(\beta, \gamma, (U_i, V_i)_{i\in[N]})$ involving two collections of intervals $U_1, \ldots, U_N$ from β and $V_1, \ldots, V_N$ from γ, for some $N \in \mathbb{N}$, such that each is ordered with respect to the left-to-right ordering of their interval partitions.

(2) The *distortion* of a correspondence $(\beta, \gamma, (U_i, V_i)_{i\in[N]})$ is defined to be

$$\operatorname{dis}(\beta, \gamma, (U_i, V_i)_{i\in[N]}) := \max \left\{ \begin{matrix} \sum_{i=1}^{N} |\mathrm{Leb}(U_i) - \mathrm{Leb}(V_i)| + \|\beta\| - \sum_{i=1}^{N} \mathrm{Leb}(U_i), \\ \sum_{i=1}^{N} |\mathrm{Leb}(U_i) - \mathrm{Leb}(V_i)| + \|\gamma\| - \sum_{i=1}^{N} \mathrm{Leb}(V_j) \end{matrix} \right\}.$$

(3) The distance $d'_H(\beta, \gamma)$ between interval partitions β and γ is defined to be the infimum of distortions over the set of all possible correspondences:

$$d'_H(\beta, \gamma) := \inf_{N \geq 0, (U_i, V_i)_{i\in[N]}} \operatorname{dis}(\beta, \gamma, (U_i, V_i)_{i\in[N]}).$$

We use this metric in our main results below. Our main results below in Theorems 1.6 and 1.7 show the existence of IP-diffusions. Now we give a set of assumptions for one-dimensional diffusions in Assumption B.

Assumption B. A diffusion $\mathbf{Z}$ satisfies *Assumption B* if it is of the form $\mathbf{Z} = g(\mathbf{Y})$, where $\mathbf{Y}$ is a diffusion that satisfies $A2$, $A3$ of Assumption A with state space $E \subseteq [0, \infty)$ of the form $E = [0, h]$ or $E = [0, h)$ for some $h \leq \infty$ and furthermore is such that:

(1) the Pitman-Yor excursion measure $\nu_{\mathbf{Y}}$ satisfies (1.9) where $\nu = \nu_{\mathbf{Y}}$ (equivalently $\nu = \nu_{\mathbf{Z}}$);
(2) $\sigma^2_{\mathbf{Y}}(x) = 4x$;

there exists $\varepsilon_0 \in (0, h)$ such that

(3) for some $0 < \alpha^- \leq \alpha^+ < 1$ we have

$$-2\alpha^+ \leq \mu_{\mathbf{Y}}(x) \leq -2\alpha^- \text{ for all } x \in (0, \varepsilon_0]$$

(where the subscript $\mathbf{Y}$ in μ and σ^2 denotes that they are the infinitesimal drift and variance for $\mathbf{Y}$), and $g : E \to g(E)$ is a strictly increasing continuous function with $g(0) = 0$, $g(E) \subseteq [0, \infty)$ and

(4) there exists $q_0 > \alpha^+$ such that the transformation g is q-Hölder continuous on $[0, \varepsilon_0)$ for all $q < q_0$, i.e. for all $q < q_0$ there exists $C_q \geq 0$ such that for all x, $y \in [0, \varepsilon_0)$ we have $|g(x) - g(y)| \leq C_q |x - y|^q$.

Finally, we say $\mathbf{Z} = g(\mathbf{Y})$ satisfies Strong Assumption B if as well as the above we have that

(5) the transformation g satisfies $0 < \liminf_{x \to 0} \frac{g(x)}{x} \leq \infty$ or $0 \leq \limsup_{x \to 0} \frac{g(x)}{x} < \infty$.

We believe that $B1$ follows from $B2$ and $B3$ but we could not show this. We do show however in Lemma 4.1 that $A1$ follows from Assumption B and in Proposition 4.2 we show that if the untransformed diffusion additionally satisfies $A4$ then we have that $B2$ and $B3$ implies $B1$. We also show in Proposition 4.2 that under Assumption B and that the diffusion $\mathbf{Z}$ and the untransformed diffusion $\mathbf{Y}$ both satisfy $A1$. We give intuition behind the nature of hypothesis $B5$ with an example of a transformation g that does not satisfy the hypothesis in Example 4.29. The hypothesis $B1$ ensures the scaffolding process is a SPLP of unbounded variation. The reason behind our fixed choice of $\sigma^2_{\mathbf{Y}}$ in $B2$ is that we compare our untransformed diffusions to squared Bessel processes with infinitesimal variance $4x$ and infinitesimal drift $-2\alpha^-$ or $-2\alpha^+$. We see in Example 1.8 that squared Bessel processes of this type featured previously as the untransformed diffusions for the IP-diffusions in Forman et al. (2020a). This comparison with squared Bessel processes is used in a Kolmogorov–Čentsov continuity argument for the IP-evolution in Section 4.4 along with $B3$ and $B4$. The condition $B2$ should not be seen as restrictive as we can apply suitable transformations g to consider diffusions with different infinitesimal variances $\sigma^2_{\mathbf{Z}}$ and we discuss this in Section 1.3.

The hypothesis $B5$ is used in Section 4.6 to show the continuity of several related functions on interval partition space. These continuity results lead to a proof of the Strong Markov property based on a continuity in the initial state proof, adapting the arguments used to show the Strong Markov property for self-similar IP-diffusions in Forman et al. (2020a, Section 6.4).

Theorem 1.6 (Branching IP-diffusions when 0 is an exit point)**.** *Consider a diffusion $\mathbf{Z}$ that satisfies Assumption B and let $\nu_{\mathbf{Z}}$ be its Pitman-Yor excursion measure. Let $(\mathbf{N}, \mathbf{X})$ be a scaffolding-and-spindles pair where $\mathbf{N}$ has intensity $\mathrm{Leb} \otimes \nu_{\mathbf{Z}}$ stopped at an a.s. finite valued random variable T. Then there exists a modification $(\gamma^y, y \geq 0)$ of $\overline{\mathit{skewer}}(\mathbf{N}, \mathbf{X})$ which is a continuous* IP-evolution *on $(\mathcal{I}_H, d'_H)$. Furthermore the process $\mathbf{X}$ has a local time that is continuous in time and level. Let τ^0 be the right-continuous inverse of the local time of $\mathbf{X}$ at level 0. Then for $T = \tau^0(s)$, for some $s > 0$, $(\gamma^y, y \geq 0)$ is a simple Markov* IP-evolution *in an appropriate filtered probability space and if furthermore $\mathbf{Z}$ satisfies Strong Assumption B then it is an* IP-diffusion*.*

We prove Theorem 1.6 in Section 4.6. In Theorem 4.36 we show that the IP-diffusion $(\gamma^y, y \geq 0)$ of Theorem 1.6 can be started from a large class of initial states. We refer to $(\gamma^y, y \geq 0)$ as a *branching IP-diffusion* because the IP-diffusion has the following branching property. For independent copies of the IP-diffusion $(\gamma^y_1, y \geq 0)$, $(\gamma^y_2, y \geq 0)$ with initial states γ_1, γ_2, the process $(\gamma^y_1 \star \gamma^y_2, y \geq 0)$ (i.e. for each $y \geq 0$ we take the concatenation of interval partitions γ^y_1 and γ^y_2) is the IP-diffusion with initial state $\gamma_1 \star \gamma_2$.

We now state Assumption C which considers a disjoint set of diffusions to Assumption B.

Assumption C. A diffusion $\mathbf{Z}$ satisfies *Assumption C* if it is of the form $\mathbf{Z} = g(\mathbf{Y})$, where $\mathbf{Y}$ is a diffusion that satisfies $A2$, $A3$ of Assumption A with state space $E \subseteq [0, \infty)$ of the form $E = [0, h]$ or $E = [0, h)$ for some $h \leq \infty$ and furthermore is such that:

(1) $\sigma^2_{\mathbf{Y}}(x) = 4x$;

there exists $\varepsilon_1 \in (0, h)$ such that

(2) for some $0 < \beta^- \leq \beta^+ < 1$ we have

$$2\beta^- \leq \mu_{\mathbf{Y}}(x) \leq 2\beta^+ \text{ for all } x \in (0, \varepsilon_1];$$

(3) and $g : E \to g(E)$ is a strictly increasing continuous function with $g(0) = 0$, $g(E) \subseteq [0, \infty)$ and

$$\int_{y=0}^{\varepsilon_1} \frac{g(y^{1/(1-\beta^-)})}{y^2} dy < \infty.$$

The hypotheses $C1$, $C2$ and $C3$ are used together to prove continuity of IP-evolutions in Section 4.5. We show in Lemma 4.1 that $A1$ follows from Assumption C. The hypothesis $C1$ plays the corresponding role of $B2$ in Assumption B and the generality of the transformation g is what allows us to consider diffusions with different infinitesimal variances, $\sigma^2_{\mathbf{Z}}$. We discuss this in Section 1.3.

Theorem 1.7 is our second main result.

Theorem 1.7 (Branching IP-evolutions when 0 is a regular point)**.** *Consider a diffusion* $\mathbf{Z}$ *under Assumption C with excursion measure* ν *and let* $(\mathbf{N}, \mathbf{X})$ *be scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb* $\otimes\, \nu$ *stopped at an a.s. finite valued random variable* T*. Then* $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X})$ *is a continuous* IP-evolution *on* $(\mathcal{I}_H, d'_H)$*. Furthermore we have that* $\mathbf{X}$ *is a Lévy process of bounded variation and when* T *denotes the* n^{th} *hitting time for* $\mathbf{X}$ *of level* 0 *for some* $n \in \mathbb{N}$ *then* $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X})$ *is a simple-Markov* IP-evolution *in an appropriate filtered probability space.*

We prove Theorem 1.7 in Section 4.5. We believe in this setting that $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X})$ is an IP-diffusion but showing this is beyond the scope of the paper. We do not believe that Theorems 1.6 and 1.7 cover all the one-dimensional diffusions from which it is possible to construct continuous Markovian IP-evolutions by the construction of scaffolding and spindles described in Section 1.1, although we do believe that all such diffusions must satisfy either (1.7) or both (1.8) and (1.10).

As in Assumption B the untransformed diffusions in Assumption C can bounded between two squared Bessel processes processes with different dimensions. As squared Bessel processes with dimension $\delta \in \mathbb{R}$ satisfy $A1$ exactly when $\delta \in (-2, 2)$ we believe our ranges for the drift in $B3$ where $\mu_{\mathbf{Y}}(x)$ takes values in a closed subset of $(-2, 0)$ for x near 0 and in $C2$ where $\mu_{\mathbf{Y}}(x)$ takes values in a closed subset of $(0, 2)$ for x near 0 should encompass a large class of the diffusions for which it is possible to generate branching IP-diffusions. Indeed, this only leaves boundary cases open where $\mu_{\mathbf{Y}}$ is not bounded away from -2, 0, 2 as well as cases where the transformations g do not satisfy the conditions needed in $B4$, $B5$, $C3$.

We give examples in the next section of diffusions which satisfy the hypotheses in Assumptions A, B and C.

1.3. *Examples.* In this section we list a few examples of diffusions that satisfy the hypotheses in the previous section. The diffusions in our selected examples satisfy $A1$, $A2$, $A3$ and we begin with some consequences.

Consider a diffusion that satisfies $A1$, $A2$, $A3$. For any $b \in (0, h)$, we can define a scale function s with derivative s' and its speed measure M with density m by, for $x \in (0, h)$,

$$s'(x) = \exp\left(-\int_{z=b}^{x} \frac{2\mu(z)}{\sigma^2(z)} dz\right); \quad m(x) = \frac{2}{\sigma^2(x)s'(x)} = \frac{2}{\sigma^2(x)} \exp\left(\int_{z=b}^{x} \frac{2\mu(z)}{\sigma^2(z)} dz\right); \tag{1.11}$$

$s(x) = \int_{y=0}^{x} s'(y)dy$; $M((l, r)) = \int_{y=l}^{r} m(y)dy$, see e.g. Revuz and Yor (1991, Exercise VII.3.20). We were free to choose the lower limit b here and in Section 4.1 we take b to be ε_0 when under Assumption B and to be ε_1 when under Assumption C. Different values in the interior only change s', s, m by the same constant scale factor and such changes do not affect any of the conditions in our results.

If g is a twice differentiable function with continuous second derivative, and if $\mu_{\mathbf{Y}}$ and $\sigma^2_{\mathbf{Y}}$ are continuous on $(0, h)$, then we have, for $x \in (0, h)$, that

$$\mu_{\mathbf{Z}}(g(x)) = \frac{1}{2}\sigma^2_{\mathbf{Y}}(x)g''(x) + \mu_{\mathbf{Y}}(x)g'(x); \quad \sigma^2_{\mathbf{Z}}(g(x)) = \sigma^2_{\mathbf{Y}}(x)(g'(x))^2,$$

and we further have that

$$s_{\mathbf{Z}}(g(x)) = s_{\mathbf{Y}}(x); \quad s'_{\mathbf{Z}}(g(x)) = \frac{1}{g'(x)} s'_{\mathbf{Y}}(x); \quad m_{\mathbf{Z}}(g(x)) = \frac{1}{g'(x)} m_{\mathbf{Y}}(x), \tag{1.12}$$

see e.g. Karlin and Taylor (1981, Theorem 15.2.1). Similarly, the results in Theorem 1.3 hold for the diffusion $\mathbf{Z}$ if and only if they hold for the diffusion $\mathbf{Y}$. We give a heuristic method that can be used to check if certain diffusions satisfy Assumption B or C:

(1) First you take a diffusion under Assumption A with infinitesimal drift $\mu_{\mathbf{Z}}$ and infinitesimal variance $\sigma^2_{\mathbf{Z}}$.
(2) Then you should find a solution g to the equation $\sigma^2_{\mathbf{Z}}(g(x)) = 4x(g'(x))^2$ with $g(0) = 0$.
(3) With this solution g you can then find a solution $\mu_{\mathbf{Y}}(x)$ to the equation

$$\mu_{\mathbf{Z}}(g(x)) = 2xg''(x) + \mu_{\mathbf{Y}}(x)g'(x).$$

(4) The original diffusion satisfies (strong) Assumption B if $\mu_{\mathbf{Y}}$ satisfies $B3$ and if g satisfies $B4$ (and $B5$). On the other hand the original diffusion satisfies Assumption C if $\mu_{\mathbf{Y}}$ satisfies $C2$ and if g satisfies $C3$.

We used this heuristic method for the processes in Example 1.9 and we verified that they satisfy Assumption C in Proposition 1.10.

The first example concerns self-similar diffusions. Our main reference in this work, Forman et al. (2020a), determined all the self-similar diffusions for which it is possible to use the scaffolding and spindles construction for IP-diffusions and we state their results here.

Example 1.8 (Self-similar diffusions). This family of IP-diffusions has already been established in Forman et al. (2020a, Theorems 1.3, 1.4). Let $k > 0$, $0 < \alpha < 1$, $\alpha < q < \infty$ and consider an untransformed diffusion $\mathbf{Y}$ on $[0, \infty)$ and a transformation g with

$$\mu_{\mathbf{Y}}(y) = -2\alpha; \quad \sigma^2_{\mathbf{Y}}(y) = 4y; \quad g(y) = ky^q.$$

Here $\mathbf{Y}$ is a squared Bessel process with dimension -2α. Let $\mathbf{Z}$ be the diffusion $\mathbf{Z} = g(\mathbf{Y})$. Then $\mathbf{Z}$ satisfies Assumption B and can be used to generate IP-diffusions. These are all the self-similar diffusions for which the construction of IP-diffusions that we use in this paper is possible, see Lamperti (1972, Theorem 5.1) for a parametrisation of self-similar Markov processes.

The second example concerns Wright–Fisher diffusions, which are well-known across the literature; we quote the relevant information about them from Karlin and Taylor (1981, Example 15.6.8). The Wright–Fisher model is a gene frequency model with mutation parameters that describes a population with two different types, say type 1 and type 2. The diffusion measures the proportion of the population which is type 1, and therefore takes values in $[0, 1]$. There are two parameters, γ_1, $\gamma_2 \geq 0$ which determine the rate of mutation from one type to another (i.e. the parameter γ_1 determines the rates of mutation from type 1 to type 2). They have infinitesimal drift and infinitesimal variance given by

$$\mu_{\mathbf{Z}}(z) = \gamma_2(1 - z) - \gamma_1 z, \quad \sigma^2_{\mathbf{Z}}(z) = z(1 - z). \tag{1.13}$$

Example 1.9 (Wright–Fisher gene frequency model involving mutation parameters). The Wright–Fisher diffusion with mutation parameters $\gamma_1 \geq 0$ and $0 < \gamma_2 < 1/2$ for IP-diffusions can be used to generate continuous simple Markov IP-evolutions.

We prove the existence of IP-evolutions in this case in Proposition 1.10 below.

Proposition 1.10. *The Wright–Fisher diffusions as defined in Example 1.9 satisfy Assumption C and therefore can be used to generate continuous simple Markov IP-evolutions.*

Proof: Let $\mathbf{Y}$ be defined the diffusion that satisfies $A3$ on $[0, \pi^2]$ with

$$\mu_{\mathbf{Y}}(y) = 1 + \sqrt{y}\Big(2\gamma_2 \cot\left(\frac{\sqrt{y}}{2}\right) - \cot(\sqrt{y}) - 2\gamma_1 \tan(\sqrt{y})\Big); \quad \sigma^2_{\mathbf{Y}}(y) = 4y; \quad y \in (0, \pi^2).$$

With the transformation $g(y) = \sin^2\left(\sqrt{y}/2\right)$ we have the Wright–Fisher diffusion as described above. We now verify the diffusion satisfies Assumption C and therefore, by Theorem 1.7, there exist continuous simple Markov IP-evolutions. It is equivalent to check that $\mathbf{Z}$ satisfies $A1$, $A2$ rather than check that diffusion $\mathbf{Y}$ does. The diffusion $\mathbf{Z}$ has scale function and speed measure with the following asymptotic properties:

$$\begin{aligned} s'(z) &= O(z^{-2\gamma_2}), & m(z) &= O(z^{2\gamma_2-1}) & &\text{as } z \to 0; \\ s'(z) &= O((1-z)^{-2\gamma_1}), & m(z) &= O((1-z)^{1-2\gamma_1}) & &\text{as } z \to 1. \end{aligned}$$

Therefore

$$\int_{z=0}^{b} s(z)m(z)dz = O\Big(\int_{z=0}^{b} z^{1-2\gamma_2} z^{2\gamma_2-1} dz\Big) = O(b) \quad \text{as } b \to 0.$$

Similarly, the condition $\int_{z=b}^{1} m_{\mathbf{Z}}(z)dz < \infty$ can be checked.

The drift satisfies $\mu_{\mathbf{Y}}(y) = 4\gamma_2 + o(1)$ for y near 0 and therefore there exists $\varepsilon > 0$ such that $0 < 2\gamma_2 \leq \mu_{\mathbf{Y}}(y) \leq 2\gamma_2 + 1 < 2$ for $y \in (0, \varepsilon)$. Furthermore, $g(y) = O(y)$ near 0 and so

$$\int_{y=0}^{b} \frac{g(y^{1/(1-\gamma_2)})}{y^2} dy = O\Big(\int_{y=0}^{b} y^{-2+1/(1-\gamma_2)} dy\Big) = O(b^{\gamma_2/(1-\gamma_2)}) \quad \text{as } b \to 0$$

and this is again finite. $\square$

We believe that Wright–Fisher diffusions with $\gamma_2 = 0$ should also lead to continuous Markov IP-evolutions, but our methods were not powerful enough to prove this. On the other hand, the requirement that $\gamma_1 > 0$ is necessary here to satisfy $A2$.

The next example concerns Cox-Ingersoll-Ross processes, see Cox et al. (1985). This family of diffusions is an alternative gene frequency model with two mutation parameters in a scaling where the mutation likelihood is larger in magnitude in the scaling limit, and in this context they are known as Laguerre diffusions after the Laguerre orthogonal polynomials involved, see Karlin and Taylor (1981, Example 15.2.F(d)) for details. They also appears as a population growth model in the same reference, see Karlin and Taylor (1981, Example 15.13.C). Finally, they are also referred to in Cox et al. (1985, Equation (17)) and have financial applications.

Example 1.11 (Cox-Ingersoll-Ross processes/Laguerre Diffusions). For $a > 0$ and b, $c \in \mathbb{R}$ the infinitesimal drift and infinitesimal variance is given by

$$\mu_{\mathbf{Z}}(z) = bx + c; \qquad \sigma^2_{\mathbf{Z}}(z) = az.$$

The diffusion can be constructed with the transformation $g(y) = (a/4)y$ and the untransformed diffusion with $\mu_{\mathbf{Y}}(y) = by + 4c/a$ and $\sigma^2_{\mathbf{Y}}(y) = 4y$. This satisfies Assumption C when $0 < 2c < a$ and $b < 0$ and therefore in these cases, by Theorem 1.7, can be used to generate continuous simple Markov IP-evolutions.

This final example is something of a middle case between Assumptions B and C for which we have some results but no proof for the existence of continuous Markov branching IP-evolutions.

Example 1.12. Consider diffusions $\mathbf{Z}$ of the form $\mathbf{Z} = g(\mathbf{Y})$, where $\mathbf{Y}$ is a diffusion that satisfies $A1$, $A2$, $A3$ with state space $E \subseteq [0, \infty)$ of the form $E = [0, h]$ or $E = [0, h)$ for some $h \leq \infty$ and furthermore is such that:

(1) $\sigma^2_{\mathbf{Y}}(x) = 4x$;

there exists $\varepsilon_2 \in (0, h)$ such that

(2) $\mu_{\mathbf{Y}}(x) = 0$ for all $x \in (0, \varepsilon_2]$;

(where the subscript $\mathbf{Y}$ in μ and σ^2 denote that they are the infinitesimal drift and variance for $\mathbf{Y}$), and $g : E \to g(E)$ is a strictly increasing continuous function with $g(0) = 0$, $g(E) \subseteq [0, \infty)$ and

(3) there exists $q_0' > 0$ such that the transformation g is q-Hölder continuous on $[0,\varepsilon_0)$ for all $q < q_0'$, i.e. for all $q < q_0'$ there exists $C_q' \geq 0$ such that for all x, $y \in [0,\varepsilon_2)$ we have $|g(x) - g(y)| \leq C_q'|x-y|^q$.

Note that $\mathbf{Y}$ satisfies $A4$ with $b = \varepsilon_2$. We define a scale function $s_{\mathbf{Y}}$ and a speed measure $m_{\mathbf{Y}}$ by setting $b = \varepsilon_2$ in (1.11) so that

$$s_{\mathbf{Y}}'(x) = 1, \quad s_{\mathbf{Y}}(x) = x, \quad m_{\mathbf{Y}}(x) = \frac{1}{2x} \quad \text{for } x \in (0,\varepsilon_2). \tag{1.14}$$

Condition (1.7) is not satisfied: $M_{\mathbf{Y}}((0,\varepsilon_2)) = \infty$; furthermore we have that (1.8) is satisfied as

$$\int_{v=0}^{\varepsilon_2}\int_{y=0}^{v} s_{\mathbf{Y}}(y)m_{\mathbf{Y}}(y)dy m_{\mathbf{Y}}(v)dv = \int_{v=0}^{\varepsilon_2}\int_{y=0}^{v}\frac{1}{2}dy\frac{1}{2v}dv = \frac{\varepsilon_2}{4}.$$

Therefore the Pitman-Yor excursion measure satisfies (1.9) by Theorem 1.3. Furthermore, for $q < q_0'$

$$\int_{v=0}^{\varepsilon_2} g(v)m_{\mathbf{Y}}(v)dv \leq \int_{v=0}^{\varepsilon_2}\frac{C_q'}{2}v^{q-1}dv = \frac{C_q'}{2q} < \infty$$

and so $M^0_{\mathbf{N},\mathbf{X}}(T)$ is finite for a stopped scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$ by Theorem 1.4.

Further examples that satisfy Assumption A but not Assumptions B or C that could in theory generate branching IP-evolutions include reflected Brownian motion with or without a constant drift on $E = [0,h]$ for some finite h; reflected Brownian motion on $E = [0,\infty)$ with a constant negative drift; reflected Ornstein-Uhlenbeck process on $E = [0,\infty)$. All these processes satisfy (1.6) (as 0 is a regular boundary point for these processes) and so we have aggregate health summability by Theorem 1.4.

1.4. *Related work.* Interval partitions can be found in several different areas of probability. As noted in Forman et al. (2020a, Introduction), interval partitions are the ranges of subordinators Pitman and Yor (1992), the complements of the zero-set of a Brownian bridges Gnedin and Pitman (2005, Example 3), the products of stick-breaking schemes Gnedin and Pitman (2005, Example 2), as well as the limits of compositions of natural numbers Gnedin (1997). Furthermore, in our setting the class of interval partitions, $\mathcal{I}_H$ along with the metric d_H' in Definition 1.5 serve as a useful complete and separable metric space to explore branching processes with diffusively evolving characteristics and to produce continuous Markovian processes. That said, we feel that these diffusions on the space of interval partitions should be taken as objects of interest in their own right.

Using only the lifetimes of the individuals, and not the health excursions, the branching processes we study give rise to splitting trees, which can be found in the literature in e.g. Geiger and Kersting Geiger and Kersting (1997). A splitting tree is a rooted planar tree that represents the type of branching processes that we study. It is a rooted tree where branch points represent births, leaves represent deaths, and the edges of varying length represent the times between birth or death events. Starting with the ancestor at the root, one can trace the path along the edge until a branch or boundary point. In the case of a boundary point, the ancestor had no offspring and the tree is a single edge of length given by the ancestor's lifetime. In the case of a branch point, the left-most edge denotes the ancestor and the other edge denotes its first offspring. Continuing to follow the left-most edges at each branch point gives the collection of edges that represent the ancestor, and the sum of lengths is the lifetime. All the other edges at the branch points denote the offspring of the ancestor. Repeating this process for all descendants would return all the information about the branching process. The collection of individuals alive in the branching process at time t is therefore in bijection with the collection of points of distance t from the root.

The skewer process draws similarities to the Ray-Knight theorems, see Revuz and Yor (1991, Theorems XI.2.1,XI.2.3) for a reference, which show that the local time process, indexed by level, of certain stopped Brownian motions are squared Bessel processes. In our setting we have interval partitions instead of local times, although, we find in Lemma 3.16 that the local time at any fixed level is a function of the interval partition at that level a.s. In Forman et al. (2018) they show a stronger result for the self-similar cases: this same function applied to the interval partitions at all levels simultaneously gives the local times at those levels a.s. We still believe that this holds in non-self-similar cases but our methods were not powerful enough to prove it.

A motivation for our primary source, Forman et al. (2020a), and related papers by the same authors, Forman et al. (2018, 2021, 2023b), is to generate interval-partition-valued diffusions to explore diffusively evolving masses of the sub-trees of a spine of a continuum random tree. This is part of their monograph, Forman et al. (2023a), to solve the Aldous conjecture (Aldous, 1999), showing the existence of a diffusion on the space of continuum trees with the Brownian Continuum Random tree as the stationary distribution.

1.5. *Structure of this paper.* Section 2 explores the relevant literature we use to prove our results. In Section 2.1 we look at the general theory of SPLPs with reference Bertoin (1996), and state key results about excursions of SPLPs from Pardo et al. (2018). In Section 2.2 we give a summary of the Pitman-Yor excursion measure from Pitman and Yor (1982) as well as the Williams disintegration for the Pitman-Yor excursion measure in Theorem 2.7, and we give an overview of the theory of disintegration. The disintegration theory also allows us to consider conditional measures such as $\nu(\cdot|\zeta = t)$, for $t > 0$, which denote the probability distribution of a health excursion for an individual with a lifetime equal to t. In Section 2.3 we explore Green's functions as these along with the Williams decomposition are useful for calculations for all our main results.

Then in Section 3 we prove results for diffusions under Assumption A. In Section 3.1 we prove Theorem 1.3 which is necessary to have the SPLP "scaffolding". Then, in Section 3.2 we prove Theorem 1.4 which determines that the levels of health are summable at a fixed point in time, and this is necessary to produce interval partitions. Section 3.3 proves necessary disintegration results for the excursion measure of the SPLP. In Section 3.4 we restrict cases in Assumption A to find the possible initial states for an IP-evolution.

In Section 4.1 we consider the parameters for the untransformed diffusion under Assumption B, which we use in Section 4.2 where we show conditions for the recurrent SPLP of unbounded variation to have local time that is jointly continuous in time and space. This is required in our Kolmogorov–Čentsov continuity criterion argument in Section 4.4 along with some further calculations in Section 4.3. We do not use a Kolmogorov–Čentsov argument for diffusions under Assumption C as we do not have an equivalent to the continuity of local time result that was given in Section 4.2 that only holds for SPLPs of unbounded variation. We prove Theorems 1.6 (without the strong Markov property) and 1.7 in Section 4.5. We then show the strong Markov property for Theorem 1.6 under Strong Assumption B in Section 4.6 by showing continuity in the initial state.

2. Preliminaries

In this section we give an overview of the results we use from the existing literature. We explore the excursions of a spectrally one-sided Lévy process away from level 0 as described in Bertoin (1996) and Pardo et al. (2018). We also use the excursion measure of a one-dimensional diffusion from Pitman and Yor (1982); in particular we use the Williams decomposition (Pitman and Yor, 1982, Description (3.3)) which describes the Pitman-Yor excursion measure. We require a notion of a conditional expectation for σ-finite measures, and for this we use the concept of a disintegration from Chang and Pollard (1997). Finally, we look at Revuz and Yor (1991) for the Green's function for real-valued diffusions and use a result from Karlin and Taylor (1981, Section 15) which applies to such diffusions under Assumption $A3$.

We define path spaces and various functions on these spaces.

Definition 2.1. We define the following metric spaces of functions:

(1) Let $\mathcal{D}$ be the Skorokhod space of one-dimensional càdlàg functions $g : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{C}$ be the metric space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ under the uniform metric.
(2) Let $X : \mathcal{D} \to \mathcal{D}$ be the canonical process on $\mathcal{D}$, and let Y be its restriction to $\mathcal{C}$, given by
$$X_t(\omega) = \omega(t), \quad \omega \in \mathcal{D}; \qquad Y_t(\omega) = \omega(t), \quad \omega \in \mathcal{C}.$$
We have defined two processes here, X and Y for clarity later on.
(3) Let $\mathcal{D}_{\text{exc}} \subset \mathcal{D}$ be the subspace of $\mathcal{D}$ consisting of càdlàg excursions away from 0, i.e.
$$\mathcal{D}_{\text{exc}} := \left\{ g \in \mathcal{D} \middle| \exists z \in (0,\infty) : g|_{(-\infty,0]\cup[z,\infty)} = 0, g(x) > 0 \text{ for all } x \in (0,z) \right\}.$$
Let $\mathcal{E} \subset \mathcal{C}$ of be the subspace of $\mathcal{D}_{exc}$ of continuous positive excursions from 0, given by
$$\mathcal{E} = \left\{ f \in \mathcal{C} \middle| \exists z \in (0,\infty) : f|_{(-\infty,0]\cup[z,\infty)} = 0; f \text{ positive, continuous on } [0,z] \right\} = \mathcal{D}_{\text{exc}} \cap \mathcal{C}.$$

Definition 2.2. We define functions on the spaces given in Definition 2.1:
(1) For $g \in \mathcal{D}_{\text{exc}}$, we define $\zeta(g) := \sup\{s \geq 0 : g(s) > 0\}$, and we call $\zeta(g)$ the *lifetime* of g.
(2) For $g \in \mathcal{D}_{\text{exc}}$, we define $A(g) := \sup\{g(s), 0 \leq s \leq \zeta(g)\}$, and we call $A(g)$ the *amplitude* of g.
(3) For $g \in \mathcal{D}_{\text{exc}}$, we define $R(g) := \inf\{s > 0 : g(s) = A(g) \text{ or } g(s-) = A(g)\}$.
(4) For $g \in \mathcal{D}$, we define $T_0^+(g) := \inf\{t > 0 : g(t) \geq 0\}$. We call $T_0^+(g)$ the crossing time of 0 for g, and we call $g(T_0^+(g)-)$ and $g(T_0^+(g))$ the *undershoot* of g and the *overshoot* of g respectively.
(5) For $g \in \mathcal{D}$, and $E \subset \mathbb{R}$ measurable, we define $T_E(g) := \inf\{t > 0, g(t) \in E\} \in [0,\infty]$. For $y \in \mathbb{R}$, we define $T_y(g) := T_{\{y\}}(g)$ and we call $T_y(g)$ the *hitting time of* y for g.

Finally, we formally define the notion of concatenating interval partitions from Forman et al. (2020a, Section 2.2).

Definition 2.3. Let $(\beta_a)_{a\in\mathcal{A}}$ denote a family of interval partitions indexed by a totally ordered set $(A, \preceq)$. Let $S(a-) := \sum_{b\prec a} \|\beta_b\|$ for $a \in \mathcal{A}$. If $S(a-) < \infty$ for every $a \in \mathcal{A}$, then we define the concatenation
$$\bigstar_{a\in\mathcal{A}}\beta_a := \{(x + S(a-), y + S(a-)) : a \in \mathcal{A}, (x,y) \in \beta_a\}$$
and otherwise we define the concatenation as the empty interval partition $\emptyset$. When $\mathcal{A} = \{a_1, a_2\}$, $a_1 \prec a_2$ we denote this by $\beta_{a_1} \star \beta_{a_2}$. If $\sum_{a\in\mathcal{A}} \|\beta_a\| < \infty$, we call this collection $(\beta_a)_{a\in\mathcal{A}}$ summable.

2.1. *Itô excursion theory for recurrent SPLPs of bounded and of unbounded variation.* As discussed in Section 1.1, the scaffolding $\mathbf{X}$ is a recurrent SPLP either of bounded variation or of unbounded variation but with no Gaussian component. We discuss the excursions of the SPLPs in both cases.

Firstly, let $\mathbf{X}$ be a recurrent SPLP of bounded variation. For $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the distribution on $\mathcal{D}$ of $\mathbf{X}$ started at $\mathbf{X}_0 = x$, and let $\mathbb{E}_x$ denote the expectation operator. For all $y \in \mathbb{R}$, the level y is irregular ($\mathbb{P}_y(T_y(X) = 0) = 0$), see Bertoin (1996, Corollary VII.5). For any fixed level $y \in \mathbb{R}$ the number of returns to y after some time $t > 0$ is finite a.s. (Lambert, 2010, Lemma 4.2), but the set of levels for which the number of returns to a level is infinite is dense in the range of the SPLP a.s. (Lambert, 2010, Theorem 5.3).

Now instead consider a recurrent SPLP $\mathbf{X}$ of unbounded variation and with no Gaussian component. For all $y \in \mathbb{R}$,

the level y is regular ($\mathbb{P}_y(T_y(X) = 0) = 1$) and instantaneous ($\mathbb{P}_y(T_{\mathbb{R}\setminus y}(X) = 0) = 1$) for $\mathbf{X}$,

see Bertoin (1996, Chapter IV.1-4). Therefore we can define the local time ℓ^y for $\mathbf{X}$ at level y. The local time at all levels $y \in \mathbb{R}$ is an occupation density for the Lévy process such that for any bounded and measurable function f and any $t > 0$ we have

$$\int_{y=-\infty}^{\infty} f(y)\ell^y(t)dy = \int_{s=0}^{t} f(\mathbf{X}_s)ds \quad \text{a.s.} \tag{2.1}$$

The local time at level y is a continuous increasing function $\ell^y : [0,\infty) \to [0,\infty)$, and so we can define a right inverse τ^y. Define $\tau^y : [0,\infty) \to [0,\infty)$ as, for $s > 0$, $\tau^y(s) = \inf\{t > 0 : \ell^y(t) > s\}$. Note that τ^0 is a subordinator (Bertoin, 1996, Theorem IV.8). Then, following Bertoin (1996, Equation IV.6), we can define the excursion process $e := (e_s, s \geq 0 : \tau^0(s) - \tau^0(s-) > 0)$ for $\mathbf{X}$,

$$e_s = \left(\mathbf{X}_{\tau^0(s-)+r}, 0 \leq r < \tau^0(s) - \tau^0(s-)\right), \tag{2.2}$$

where each $e_s \in \mathcal{D}_{\text{exc}}$. There are only countably many points s such that $\tau^0(s) - \tau^0(s-) > 0$ a.s. From Bertoin (1996, Theorem IV.10) we note that e is a Poisson point process.

Define $\mathbf{S} = (\mathbf{S}_t)_{t\in[0,\infty)}$, where, for $t > 0$, $\mathbf{S}_t := \sup_{0\leq s\leq t} \mathbf{X}_s$, i.e. $\mathbf{S}$ is the running supremum of $\mathbf{X}$. The process $\mathbf{S} - \mathbf{X}$ is non-negative for all $t > 0$ and only hits the level 0 at the points where $\mathbf{X}$ attains new suprema. Note that the level 0 is also regular and instantaneous for $\mathbf{S} - \mathbf{X}$ and we can define $\overline{e}$ to be the excursion process for $\mathbf{S} - \mathbf{X}$ in the same way that we defined e for $\mathbf{X}$. Let n be the characteristic measure for e; i.e. the excursion measure for $\mathbf{X}$, and let $\overline{n}$ the characteristic measure for $\overline{e}$; i.e. the excursion measure for $\mathbf{S} - \mathbf{X}$. The Laplace exponent, $\psi : [0,\infty) \to \mathbb{R}$, for $\mathbf{X}$ is formally defined to be the function that satisfies

$$\mathbb{E}_0(e^{-\lambda X_t}) = e^{t\psi(\lambda)},$$

and in our setting is equal to

$$\psi(\lambda) = \int_{x=0}^{\infty} \left(e^{-\lambda x} - 1 + \lambda x\right) \Lambda(dx), \tag{2.3}$$

see Bertoin (1996, Chapter VII.1).

As the recurrent SPLP $\mathbf{X}$ has no Gaussian component we have that, up to a null set, all excursions under n have the following structure a.s.: initially the excursion takes values in $(-\infty, 0)$, before entering $(0,\infty)$ by a jump across 0, and then taking values in $(0,\infty)$ until hitting 0, see Pardo et al. (2018, Theorem 3).

We define a relation between the excursion measure n of $\mathbf{X}$ and the excursion measure $\overline{n}$ for $\mathbf{S} - \mathbf{X}$, with the scale function $\hat{W}$ for $\hat{\mathbf{X}} := -\mathbf{X}$, defined in Bertoin (1996, Section VII.2). The scale function $\hat{W} : [0,\infty) \to [0,\infty)$ is a continuous increasing function which relates to the Laplace exponent for $\hat{\mathbf{X}}$ in the following way: for all $\lambda > 0$,

$$\int_{x=0}^{\infty} e^{-\lambda x}\hat{W}(x)dx = \frac{1}{\psi(\lambda)}.$$

Let $\hat{\mathbb{P}}_x$ be the probability measure for the law of the dual process $\hat{\mathbf{X}}$ started at $\hat{\mathbf{X}}(0) = x$. The scale function $\hat{W}$ for $\hat{\mathbf{X}}$ solves the two-sided exit problem for $\hat{\mathbf{X}}$ (Bertoin, 1996, Theorem VII.8): for $x, y > 0$, the probability that $\hat{\mathbf{X}}$ makes its first exit from $[-x, y]$ at y is

$$\hat{\mathbb{P}}_0\Big(\inf_{0\leq s\leq T_y}(X_s) \geq -x\Big) = \frac{\hat{W}(x)}{\hat{W}(x+y)}.$$

In the following, we use notation such as

$$\overline{n}(g(t) \in A, \zeta > t) := \overline{n}(\{g \in \mathcal{D}_{\text{exc}} : g(t) \in A, \zeta(g) > t\}),$$

where A is a measurable set of $\mathbb{R}$ in this setting. To relate n to $\overline{n}$, we use an instance of a general result for recurrent SPLPs from Pardo et al. (2018, Theorem 3 (iii)): there exists $k_1 > 0$ such that for all $y,\, z > 0$

$$n\left(-g\left(T_0^+-\right)\in dy, g\left(T_0^+\right)\in dz, T_0^+ > t\right) = k_1 \int_{x=0}^{\infty} \big(\hat{W}(x) - \hat{W}(x-y)\big)\overline{n}(g(t)\in dx, \zeta > t)\Lambda(dz+y)dy, \tag{2.4}$$

where $\hat{W}(t) \coloneqq 0$ for $t < 0$.

For $x > 0$, when we consider events of $\hat{\mathbf{X}}$ up to a finite fixed time t (i.e. any event A such that $A \in \mathcal{F}_t$, the natural filtration for the process $\mathbf{X}$), $\hat{\mathbb{P}}_x^{\uparrow}$ is the distribution on $\mathcal{D}$ of $\hat{\mathbf{X}}$ when started from x and conditioned to be positive in $(0,t)$, which can be found in Bertoin (1996, Equation VII.6):

$$\hat{\mathbb{P}}_x^{\uparrow}(X_t \in dy) \coloneqq \frac{\hat{W}(y)}{\hat{W}(x)}\hat{\mathbb{P}}_x(X_t \in dy, T_{(-\infty,0)}(X) > t).$$

Furthermore, Bertoin (1996, Proposition VII.14) states that the probability measure for $\mathbf{X}$ conditioned to stay positive when started at level 0, $\mathbb{P}_0^{\uparrow}$, can be defined by the limit $\lim_{x\downarrow 0}\mathbb{P}_x^{\uparrow}$. Then, from Bertoin (1996, Proposition VII.15), we have that for some $k_2 > 0$ and for any $x,\, t > 0$,

$$\overline{n}(g(t)\in dx, \zeta(g) > t) = \frac{k_2}{\hat{W}(x)}\hat{\mathbb{P}}_0^{\uparrow}(X_t \in dx). \tag{2.5}$$

2.2. *Pitman-Yor excursion theory for one-dimensional diffusions and disintegration theory.* Consider a diffusion that satisfies Assumption $A1$. We construct an excursion measure following Pitman and Yor (1982, Section 3). This is equivalent to the Itô excursion measure when 0 is a regular boundary point, but is more general as it also applies in the case where 0 is an exit boundary point.

Before we define the excursion measure, we need to define two more diffusions – the 0-diffusion and the $\uparrow$-diffusion. For $x > 0$, the 0-diffusion started at x is simply the original diffusion started at x which we stop when it hits level 0. For $x > 0$, the $\uparrow$-diffusion started at the level x has law given by the limit, as $y \uparrow \infty$, of the conditional probability measure for the original diffusion (again started at x) conditioned on $\{T_y < T_0\}$; details of this can be found in Pitman and Yor (1982, Section 3). We can define the $\uparrow$-diffusion starting at 0 and in this case the process never returns to level 0. The scale function $s^{\uparrow}$ and the speed measure $M^{\uparrow}$ of the $\uparrow$-diffusion can be determined in terms of s and M:

$$s^{\uparrow}(x) = \frac{-1}{s(x)}; \qquad M^{\uparrow}(dx) = s(x)^2 M(dx). \tag{2.6}$$

The scale function $s^{\uparrow}$ result follows from the defining characteristic of a scale function in (1.2) and some simple calculations. The speed measure result is noted in the literature in e.g. Salminen et al. (2007). With the $\uparrow$-diffusion and the 0-diffusion we can define the Pitman-Yor excursion measure ν.

Definition 2.4 (Pitman-Yor excursion measure ν)**.** We define the σ-finite excursion measure ν on the space of positive continuous excursions $\mathcal{E}$ by the following conditions:

(1) $\nu(f \equiv 0) = \nu(f(0) \neq 0) = 0$;
(2) for all $y \in E\backslash\{0\}$ we have $\nu\left(T_y(f) < \infty\right) = 1/s(y)$;
(3) for all $y \in E\backslash\{0\}$, the conditional distribution $\nu\left(\cdot|T_y(f) < \infty\right)$ is the law of the following $\mathbf{V} = (\mathbf{V}_t)_{t\in[0,\infty)}$. The process $(\mathbf{V}_t)_{t\in[0,T_y)}$ has the law of an $\uparrow$-diffusion started at 0 and run until it hits y, and $\left(\mathbf{V}_{T_y+t}\right)_{t\in[0,\infty)}$ has the law of an independent 0-diffusion started at y.

The interpretation for ν is that the excursions come into $E\backslash\{0\}$ from the level 0 according to the $\uparrow$-diffusion and then evolve in $E\backslash\{0\}$ according to the 0-diffusion. Like the scale function, this measure is unique up to a normalising constant in the sense that if ν is an excursion measure and $k > 0$ then $k\nu$ is also an excursion measure (the normalising constant for s determines the normalising constant for ν).

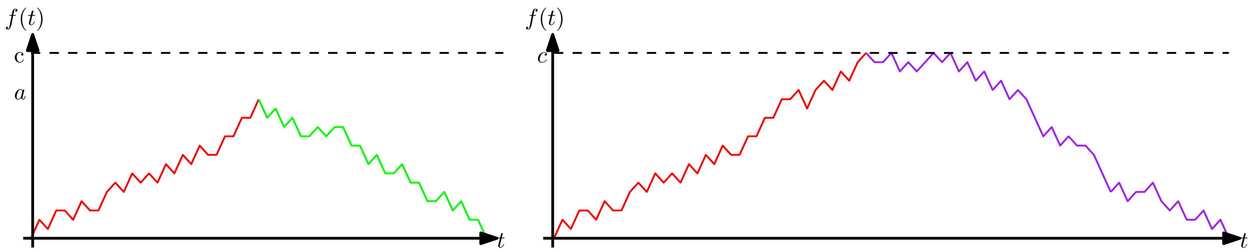

FIGURE 2.2. Left: an excursion with amplitude $a \in (0, h)$, constructed from an $\uparrow$-diffusion (red) and a time-reversed $\uparrow$-diffusion (green) both started at 0 and stopped at a. Right: an excursion with amplitude h, constructed from an $\uparrow$-diffusion stopped at h (red) and a 0-diffusion started at h and stopped at 0 (purple).

We give an overview of the results from Chang and Pollard (1997) on the theory of disintegration, which will allow us to condition over infinite dimensional measures such as the Pitman-Yor excursion measure ν.

Definition 2.5 (Disintegration)**.** Let λ be a σ-finite measure on a metric space $\mathcal{X}$ and let $S : \mathcal{X} \to \mathbb{R}$ be a measurable map. Then λ has a disintegration $\{\lambda(\cdot|S = t)\}_{t\in[0,\infty)}$ with respect to S, known as an S-disintegration, if

(1) $\lambda(\cdot|S = t)$ is a probability measure on $\mathcal{X}$ concentrated on $\{S = t\}$.
In other words, $\lambda(\{S \neq t\}|S = t) = 0$ for $\lambda(S \in \cdot)$-almost all t;
(2) for any $F : \mathcal{X} \to [0, \infty)$, $t \mapsto \int_{g\in\mathcal{X}} F(g)\lambda(dg|S = t)$ is measurable and
(3) $\int_{g\in\mathcal{X}} F(g)\lambda(dg) = \int_{t\in\mathbb{R}} \int_{g\in\mathcal{X}} F(g)\lambda(dg|S = t)\lambda(S \in dt)$.

The next theorem states sufficient conditions for a function $S : \mathcal{X} \to \mathbb{R}$ on a complete and separable space to disintegrate a measure λ.

Theorem 2.6 (Existence Theorem for a disintegration for λ)**.** *Suppose $(\mathcal{X}, \Sigma(\mathcal{X}), \lambda)$ is a complete and separable space and that S is a measurable map. Then λ has an S-disintegration. Furthermore, the measures $\lambda(\cdot|S = t)$ are uniquely determined up to an almost sure equality: that is to say that if λ_t is another S-disintegration then $\lambda(S^{-1}(t \in [0, \infty] : \lambda_t \neq \lambda(\cdot|S = t))) = 0$.*

Definition 2.5 and Theorem 2.6 are given in greater generality in Chang and Pollard (1997, Definition 1) and Chang and Pollard (1997, Theorem 1) respectively – what we refer to as an S-disintegration they refer to as a $(S, S\lambda)$-disintegration. The Williams decomposition in Theorem 2.7 satisfies the conditions of Definition 2.5 and so is an A-disintegration of ν. Furthermore, by Theorem 2.6 there exists an ζ-disintegration of ν.

We abuse notation and write $s(dw)$, for $w \in E\backslash\{0\}$, to denote the Lebesgue-Stieltjes measure on $E\backslash\{0\}$ as defined from the continuous increasing scale function s. Formally, $s(dw)$ is the Borel measure $\phi(dw)$ for which we have for any $0 \leq x \leq y < h$ that $\phi((x, y)) = s(y) - s(x)$. We remark that under Assumption $A3$ we have that $s(dw) = s'(w)dw$ but that s is not in general differentiable.

The following description of the Williams decomposition of ν depends on whether $s(h) < \infty$. If $s(h) < \infty$ then the level h is a regular boundary point for the diffusion, and therefore the diffusion can hit h in finite time. In this case the process reflects off the point h and the Williams decomposition of the excursion measure therefore includes a case where the amplitude A is equal to h which can occur with positive probability. The graph on the right in Figure 2.2 shows an excursion that hits level h in this setting.

Theorem 2.7 (Williams decomposition of ν)**.** *Consider a diffusion that satisfies $A1$ on a state space $[0, h]$ for some finite h or on a state space $[0, h)$ for some h which is either finite or infinity. Then its Pitman-Yor excursion measure ν is the unique measure that satisfies the following:*

(1) $\nu(A \in dw) = s(dw)/s(w)^2$ *for* $w \in (0,h)$.
If $s(h)$ *is finite, we further have that* $\nu(A = h) = 1/s(h)$.

(2) *For* $w \in (0,h)$, *a process* $\mathbf{V}$ *under* $\nu(\cdot|A = w)$ *has an a.s. unique time at which it attains its amplitude, given by* R, *so that* $\mathbf{V}_R = w$. *Furthermore, the processes*

$$(\mathbf{V}_t, 0 \le t \le R) \text{ and } (\mathbf{V}_{\zeta - t}, 0 \le t \le \zeta - R)$$

are independent $\uparrow$*-diffusions each started at the level* 0 *and then stopped at time* T_w. *See the graph on the left in Figure 2.2.*

(3) *In the case where* $s(h)$ *is finite the process* $\mathbf{V}$ *under* $\nu(\cdot|A = h)$ *is defined and evolves as the following. The process* $(\mathbf{V}_t)_{t\in[0,T_h)}$ *has the law of an* $\uparrow$*-diffusion started at* 0 *run until it hits* h, *and* $(\mathbf{V}_{T_h+t})_{t\in[0,\infty)}$ *has the law of an independent* 0*-diffusion started at* h. *See the graph on the right in Figure 2.2.*

Theorem 2.7 is proven in Pitman and Yor (1982, Section 3) in slightly less generality, but the generalisation in our version follows from this.

2.3. *Green's functions of diffusions.* In this section we explore the properties of Green's functions of diffusions that we use in this paper. For a probability measure $\mathbb{Q}$ we let $\mathbb{E}_{\mathbb{Q}}$ denote the expectation operator under $\mathbb{Q}$. For a diffusion on a state space E and for $x \in E$ let $\mathbb{Q}_x$ denote the probability measure of the diffusion started at x. This first result is from Revuz and Yor (1991, Corollary VII.3.8).

Theorem 2.8. *Consider a diffusion on a state space* $E \subseteq [0,\infty)$ *which is of the form* $E = [0,h)$ *or* $E = [0,h]$ *for some* $h \le \infty$. *Let* a, $w \in E$ *with* $a < w$, *and let* $F : \mathbb{R} \to \mathbb{R}$ *be a locally bounded Borel function. We have for* $x \in (a,w)$ *that*

$$\mathbb{E}_{\mathbb{Q}_x}\left(\int_{s=0}^{T_a\wedge T_w} F(Y_s)ds\right) = \int_{v=a}^{w} F(v)G_{(a,w)}(x,v)M(dv)$$

where $G_{(a,w)}$ *is the Green's function for the diffusion on* (a,w) *as in* (1.3).

The next result is from Revuz and Yor (1991, Proposition VII.3.10) and includes some immediate consequences that apply to our setting.

Lemma 2.9. *Consider a diffusion which satisfies A1 and has speed measure* M *such that* $M((0,b)) < \infty$ *for some* $b \in E$ *and let* $F : \mathbb{R} \to [0,\infty)$ *be a locally bounded Borel function. Then, for* $w \in E$, *we have*

$$\mathbb{E}_{\mathbb{Q}_0}\left(\int_{s=0}^{T_w} F(Y_s)ds\right) = \int_{v=0}^{w} F(v)(s(w) - s(v))M(dv).$$

In particular, if a diffusion $\mathbf{Y}$ *satisfies A1, A2 and* $s(h)$ *is finite then* h *is a regular boundary point for* $\mathbf{Y}$ *and we can define the diffusion* $\hat{\mathbf{Y}} := h - \mathbf{Y}$. *Also, using "hats" for anything associated with* $\hat{\mathbf{Y}}$, *we have that* $\hat{\mathbf{Y}}$ *satisfies A1,* $M_{\hat{\mathbf{Y}}}((0,b)) < \infty$ *for some* $b \in E$ *and*

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_h}\left(\int_{s=0}^{T_0} F(Y_s)ds\right) &= \mathbb{E}_{\hat{\mathbb{Q}}_0}\left(\int_{s=0}^{T_h} F(h - Y_s)ds\right)\\ &= \int_{v=0}^{h} F(h-v)(s_{\hat{\mathbf{Y}}}(h) - s_{\hat{\mathbf{Y}}}(v))M_{\hat{\mathbf{Y}}}(dv) = \int_{v=0}^{h} F(v)s_{\mathbf{Y}}(v)M_{\mathbf{Y}}(dv).\end{aligned}$$

We now look at results about $\uparrow$-diffusions. For $x \in E$, we define $\mathbb{Q}_x^{\uparrow}$ to be the distribution of the $\uparrow$-diffusion started from x. This final lemma uses results from Karlin and Taylor (1981, Section 15.3).

Lemma 2.10. *Consider a diffusion that satisfies A1, A3 and has infinitesimal drift μ and infinitesimal variance σ^2 that are continuous on E. Then we can define the $\uparrow$-diffusion in this setting and we have that the infinitesimal drift $\mu^\uparrow$ and infinitesimal variance $(\sigma^2)^\uparrow$ for the $\uparrow$-diffusion are the continuous functions*

$$\mu^\uparrow(x) = \mu(x) + \frac{s'(x)}{s(x)}\sigma^2(x); \qquad (\sigma^2)^\uparrow(x) = \sigma^2(x). \tag{2.7}$$

Furthermore, for $a > 0$, we define functions $U_{(a,w)}$ and $V_{(a,w)}$ from U and V in Karlin and Taylor (1981, Equation 15.3.37): for $x \in (a, w)$ let

$$U_{(a,w)}(x) \coloneqq \mathbb{E}_{\mathbb{Q}_x^\uparrow}\Big((T_a \wedge T_w)^2\Big), \qquad V_{(a,w)}(x) \coloneqq 2\mathbb{E}_{\mathbb{Q}_x^\uparrow}(T_a \wedge T_w). \tag{2.8}$$

Then $U_{(a,w)}(\cdot)$, $V_{(a,w)}(\cdot)$ satisfy:

$$\frac{1}{2}(\sigma^2)^\uparrow(x)U''_{(a,w)}(x) + \mu^\uparrow(x)U'_{(a,w)}(x) + V_{(a,w)}(x) = 0. \tag{2.9}$$

Furthermore, there exists a unique solution $(U_{(a,w)}(x))_{x\in(a,w)}$ to (2.9) with boundary conditions $U_{(a,w)}(a) = 0$ and $U_{(a,w)}(w) = 0$, which is given by

$$U_{(a,w)}(x) = k_{(a,w)}(s^\uparrow(w) - s^\uparrow(x)) + \int_{y=x}^{w}\int_{z=a}^{y}(s^\uparrow)'(y)V_{(a,w)}(z)m^\uparrow(z)dydz \text{ for } x \in (a, w),$$

where

$$k_{(a,w)} = \frac{-1}{s^\uparrow(w) - s^\uparrow(a)}\int_{y=a}^{w}\int_{z=a}^{y}(s^\uparrow)'(y)V_{(a,w)}(z)m^\uparrow(z)dzdy.$$

We prove the solution satisfies the differential equation (2.9) in Lemma 5.2 in the Appendix. The formulae for $\mu^\uparrow$ and $(\sigma^2)^\uparrow$, stated in (2.7), can be found in the literature, e.g. Pinsky (1985, p. 371). In (1.4) we have $V_{(a,w)}(x)$ in terms of the speed measure and the Green's function for the diffusion on (a, w) which is given in (1.3). We can then make substitutions for $s^\uparrow$ and $m^\uparrow$ using (2.6). We take monotonic limits as a decreases to 0 and then as x decreases to 0 to obtain

$$\mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w) = \int_{v=0}^{w}(s^\uparrow(w) - s^\uparrow(v))M^\uparrow(dv) = \int_{v=0}^{w}\Big(\frac{1}{s(v)} - \frac{1}{s(w)}\Big)s(v)^2M(dv) \text{ for } w \in E, \tag{2.10}$$

$$\mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w^2) = 2\int_{y=0}^{w}\int_{z=0}^{y}\int_{v=z}^{w}\left(\frac{1}{s(v)} - \frac{1}{s(w)}\right)s(v)^2m(v)dvs(z)^2m(z)dz\frac{s'(y)}{s(y)^2}dy \tag{2.11}$$

for $w \in E$. This ultimately follows as 0 is not a regular or exit boundary point for the $\uparrow$-diffusion and so we have $\lim_{a\downarrow 0}\mathbb{E}_{\mathbb{Q}_x^\uparrow}(T_a \wedge T_w) = \mathbb{E}_{\mathbb{Q}_x^\uparrow}(T_w)$ and $\lim_{x\downarrow 0}\mathbb{E}_{\mathbb{Q}_x^\uparrow}(T_w) = \mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w)$ and similarly for $\mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w^2)$.

3. Scaffolding and spindles – Assumption A

In this section we give results for the interval partition produced at a fixed level $y \in \mathbb{R}$ by our scaffolding-and-spindles construction.

3.1. *Proof of Theorem 1.3.* We use Green's functions to determine the Lévy measure condition in Theorem 1.3. We prove that with the conditions of Theorem 1.3 the lifetime measure under the Pitman-Yor excursion measure is the jump measure of a recurrent SPLP, and the conclusion of the theorem follows. Our diffusion takes values state space $E \subseteq [0, \infty)$ of the form $E = [0, h)$ or

$E=[0,h]$ where $h\leq\infty$. For $b\in(0,h)$ we can split the excursions $f\in\mathcal{E}$ of the diffusion based on whether or not they have amplitude $A(f)$ greater than b:

$$\int_{x=0}^{\infty}(x^2\wedge x)\nu(\zeta\in dx)=\int_{f\in\mathcal{E}}\big(\zeta(f)^2\wedge\zeta(f)\big)\,\nu(df)\leq\int_{f\in\mathcal{E}}\zeta(f)^2\mathbb{1}_{A(f)\leq b}\nu(df)+\int_{f\in\mathcal{E}}\zeta(f)\mathbb{1}_{A(f)>b}\nu(df). \tag{3.1}$$

Lemma 3.1. *Consider a diffusion that satisfies A1 and A2 with Pitman-Yor excursion measure ν. Then for $b\in(0,h)$ we have that*

$$\int_{f\in\mathcal{E}}\zeta(f)\mathbb{1}_{A(f)>b}\nu(df)<\infty.$$

Proof: The disintegration described in the Williams decomposition in Theorem 2.7 gives the excursion conditional on $A=w$ as the concatenation of two $\uparrow$-diffusions, the second one being time-reversed so that the concatenated pair form a non-negative excursion of amplitude w, see the left graph in Figure 2.2. Therefore for $w\in(0,h)$ the expectation of the lifetime under $\nu(\cdot|A=w)$ is given by

$$\int_{f\in\mathcal{E}}\zeta(f)\nu(df|A=w)=2\int_{f\in\mathcal{E}}R(f)\nu(df|A=w)=2\mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_w).$$

We have calculated the expectation of T_w under $\mathbb{Q}_0^\uparrow$ in (2.10), and so

$$\int_{f\in\mathcal{E}}\zeta(f)\nu(df|A=w)=4\int_{v=0}^{w}\left(\frac{1}{s(v)}-\frac{1}{s(w)}\right)s(v)^2M(dv).$$

We integrate over the amplitude $A=w\in(b,h)$ and take the upper bound of $1/s(v)$ for $1/s(v)-1/s(w)$ to obtain

$$\begin{aligned}\int_{f\in\mathcal{E}}\zeta(f)\mathbb{1}_{A(f)\in(b,h)}\nu(df)&=4\int_{w=b}^{h}\int_{v=0}^{w}\frac{s(v)^2}{s(w)^2}\left(\frac{1}{s(v)}-\frac{1}{s(w)}\right)M(dv)s(dw)\\&\leq 4\int_{w=b}^{h}\int_{v=0}^{w}s(v)M(dv)\frac{1}{s(w)^2}s(dw).\end{aligned}\tag{3.2}$$

Finally, we split the integral for $v\in[0,b]$ and for $v\in(b,w)$ to show the claim.

$$\int_{w=b}^{h}\int_{v=0}^{b}s(v)M(dv)\frac{1}{s(w)^2}s(dw)\leq\frac{1}{s(b)}\int_{v=0}^{b}s(v)M(dv),$$

which is finite by $A1$ and

$$\int_{w=b}^{h}\int_{v=b}^{w}s(v)M(dv)\frac{1}{s(w)^2}s(dw)=\int_{v=b}^{h}\int_{w=v}^{h}\frac{1}{s(w)^2}s(dw)s(v)M(dv)\leq M((b,h))$$

which is finite by $A2$.

The final part we need to consider are excursions that hit level h, and this occurs when $s(h)$ is finite (equivalently when h is a regular boundary point of the diffusion). In this case, as stated in the Williams decomposition in Theorem 2.7, we have that conditional on $\{A=h\}$ the excursion follows a probability distribution that consists of an $\uparrow$-diffusion started at 0 and stopped at T_h, the hitting time of h, followed by an independent 0-diffusion started at h, i.e. a diffusion started at h and stopped at T_0.

We calculate the contributions. The first term is given for $w=h$ in (2.10) and the second contribution is given in Lemma 2.9. The combination is

$$\int_{f\in\mathcal{E}}\zeta(f)\nu(df,A=h)=\nu(A=h)\Big(\mathbb{E}_{\mathbb{Q}_0^\uparrow}(T_h)+\mathbb{E}_h(T_0)\Big)=\frac{1}{s(h)}\int_{y\in(0,h]}\left(2-\frac{s(y)}{s(h)}\right)s(y)M(dy). \tag{3.3}$$

This is less than $(2/s(h))\int_{y=0}^{h} s(y)M(dy)$. The result follows as $\int_{y=0}^{b} s(y)M(dy)$ is finite by $A1$ and we have that $\int_{y=b}^{h} s(y)M(dy)$ is finite because $s(h)$ is finite and as $M((b,h))$ is finite by $A2$. □

Lemma 3.2. *Consider a diffusion that satisfies Assumption A and let ν be the Pitman-Yor measure. Let $b\in(0,h)$ be as in A4. Then we have*

$$\int_{f\in\mathcal{E}} \zeta(f)^2 \mathbb{1}_{A(f)\le b}\nu(df) \le 8\int_{v=0}^{b}\int_{z=0}^{v} s(z)m(z)dzm(v)dv. \tag{3.4}$$

Proof: As we assume $A3$ in this lemma we can write $M(dv)=m(v)dv$ and $s(dw)=s'(w)dw$, i.e. M has a density m with respect to the Lebesgue measure and s is differentiable. As in the previous lemma, we use from Theorem 2.7 that, for $w\in(0,b)$, R and $\zeta-R$ are independent and identically distributed under $\nu(\cdot|A=w)$ with the distribution of T_w under $\mathbb{Q}_0^{\uparrow}$. By expanding $\zeta^2=(R+(\zeta-R))^2$ we obtain the upper bound

$$\int_{f\in\mathcal{E}} \big(R(f)^2+2R(f)\big(\zeta(f)-R(f)\big)+(\zeta(f)-R(f))^2\big)\nu(df|A=w)\le 4\int_{f\in\mathcal{E}} R(f)^2\nu(df|A=w).$$

To find an upper bound for the right-hand-side we consider a different diffusion $\tilde{\mathbf{Z}}$ with infinitesimal drift $\tilde\mu$ and infinitesimal drift $\tilde\sigma^2$ given by

$$\tilde\mu(w):=\mu(w),\quad \tilde\sigma^2(w):=\sigma^2(w)\text{ for } w\in(0,b);\qquad \tilde\mu(w):=\mu(b),\quad \tilde\sigma^2(w):=\sigma^2(b)\text{ for } w\ge b.$$

Let $\tilde s$, $\tilde m$ and $\tilde\nu$ be the scale function, speed measure density and Pitman-Yor excursion measure for $\tilde{\mathbf{Z}}$. Then $\tilde{\mathbf{Z}}$ is a diffusion that satisfies $A1$ and $A3$ and $\tilde\mu$ and $\tilde\sigma^2$ are continuous on E and so we can apply Lemma 2.10 and, substituting the expectation for T_w^2, we obtain

$$\int_{f\in\mathcal{E}} R(f)^2\tilde\nu(df|A=w)=2\int_{y=0}^{w}\int_{z=0}^{y}\int_{v=z}^{w}\left(\frac{1}{\tilde s(v)}-\frac{1}{\tilde s(w)}\right)\tilde s(v)^2\tilde m(v)dv\tilde s(z)^2\tilde m(z)dz\frac{\tilde s'(y)}{\tilde s(y)^2}dy$$

for $w\in(0,b)$. However as, for $w\in(0,b)$, we have $\tilde s(w)=s(w)$, $\tilde m(w)=m(w)$ and also $\nu(\cdot|A=w)=\tilde\nu(\cdot|A=w)$ by the Williams decomposition in Theorem 2.7, we can substitute the terms in the equation above. We then integrate over $A=w\in(0,b)$ and take the same upper bound of $1/s(v)$ for $1/s(v)-1/s(w)$ as in the previous lemma to get

$$\int_{w=0}^{b}\int_{f\in\mathcal{E}}\zeta(f)^2\nu(df|A=w)dw\le 8\int_{w=0}^{b}\int_{y=0}^{w}\int_{z=0}^{y}\int_{v=z}^{w} s(v)m(v)dvs(z)^2m(z)dz\frac{s'(y)}{s(y)^2}dy\frac{s'(w)}{s(w)^2}dw$$

for $w\in(0,b)$. We swap the order of integration and then extend the range of the now inner y and w integrals to upper bound the integral above by

$$\int_{z=0}^{b}\int_{v=z}^{b}\int_{w=v}^{h}\int_{y=z}^{h}\frac{s'(y)}{s(y)^2}dy\frac{s'(w)}{s(w)^2}dws(v)m(v)dvs(z)^2m(z)dz\le\int_{z=0}^{b}\int_{v=z}^{b} m(v)dvs(z)m(z)dz,$$

where we again use the same upper bound of $1/s(v)$ for $1/s(v)-1/s(w)$ and also in z. Finally we swap the order of integration of z and v to give the upper bound in the claim. □

Proof of Theorem 1.3: Consider a diffusion under $A1$ and $A2$. Then as in (3.2), but now without the amplitude constraint of $A(f)>b$, we note that

$$\int_{f\in\mathcal{E}}\zeta(f)\nu(df)=4\int_{w=0}^{h}\int_{v=0}^{w}\frac{s(v)^2}{s(w)^2}\left(\frac{1}{s(v)}-\frac{1}{s(w)}\right)M(dv)s(dw)$$

which we can rearrange to give

$$4\int_{v=0}^{h}\int_{w=v}^{h}\Big(\frac{s(v)}{s(w)^2}-\frac{s(v)^2}{s(w)^3}\Big)s(dw)M(dv).$$

We split into two cases. When $s(h)$ is infinite the above term is equal to $2M((0,h))$ and therefore we have the Pitman-Yor excursion measure satisfies (1.6) if and only if (1.7) holds. When $s(h)$ is finite we have the above integral is equal to

$$2M((0,h)) - \frac{4}{s(h)} \int_{v=0}^{h} s(v)M(dv) + \frac{2}{s(h)^2} \int_{v=0}^{h} s(v)^2 M(dv).$$

To complete the proof in this case it suffices to show that $\int_{v=0}^{h} s(v)M(dv)$ is finite. We have $\int_{v=0}^{b} s(v)M(dv)$ is finite by $A1$ and furthermore we have that $\int_{v=b}^{h} s(v)M(dv)$ is finite as $M((b,h))$ and $s(h)$ are both finite.

For the second setting described in Theorem 1.3 (where the diffusion satisfies all of Assumption A) the result follows immediately from Lemmas 3.1 and 3.2. □

3.2. *Proof of Theorem 1.4.* Throughout this section we consider a diffusion that satisfies $A1$ and $A2$ and has a Pitman-Yor excursion measure ν that satisfies (1.9), which we recall means that for a scaffolding-and-spindles pair in this setting, the scaffolding $\mathbf{X}$ is an SPLP of unbounded variation with no Gaussian component (we deal with the bounded variation case in the final proof of Theorem 1.4 as the end of this section). With this in mind, we begin with a lemma that applies to such Lévy processes.

Lemma 3.3 (Excursion undershoot-overshoot measure)**.** *Let $\mathbf{X}$ be a recurrent SPLP of unbounded variation with Lévy measure Λ and with no Gaussian component. Let n be the excursion measure for the law of $\mathbf{X}$. For some constant $k_0 > 0$, the undershoot and overshoot joint measure under the excursion measure n is given by the following measure on* $(0,\infty)^2$

$$n(-g(T_0^+-) \in dy, g(T_0^+) \in dz) = k_0 \Lambda(dz+y)dy. \tag{3.5}$$

Proof: We substitute the expression for the excursion measure of $\mathbf{S}-\mathbf{X}$ from (2.5) into (2.4) to give

$$\begin{aligned} &n(-g(T_0^+-) \in dy, g(T_0^+) \in dz, T_0^+ > t) \\ &\qquad = k_1 \left(\int_{x=0}^{\infty} (\hat{W}(x) - \hat{W}(x-y)) \frac{k_2}{\hat{W}(x)} \hat{\mathbb{P}}^{\uparrow}(X_t \in dx) \right) \Lambda(dz+y)dy. \end{aligned}$$

For $x > 0$ and $y > 0$, $\left(\hat{W}(x) - \hat{W}(x-y)\right)/\hat{W}(x) \leq 1$. Also, $\lim_{x\downarrow 0} \left(\hat{W}(x) - \hat{W}(x-y)\right)/\hat{W}(x) = 1$ by monotonicity. Finally, for X with law $\mathbb{P}^{\uparrow}$, we have that $X_t \to 0$ as $t \downarrow 0$ a.s. Also we have $n(T_0^+ = 0) = 0$, see Pardo et al. (2018, Theorem 3), and so

$$\begin{aligned} n\left(-g(T_0^+-) \in dy, g(T_0^+) \in dz\right) &= \lim_{t\downarrow 0} n\left(-g(T_0^+-) \in dy, g(T_0^+) \in dz, T_0^+ > t\right) \\ &= k_1 k_2 \left(\lim_{t\downarrow 0} \int_{x=0}^{\infty} \frac{\hat{W}(x) - \hat{W}(x-y)}{\hat{W}(x)} \hat{\mathbb{P}}^{\uparrow}(X_t \in dx) \right) \Lambda(dz+y)dy \\ &= k_0 \Lambda(dz+y)dy \text{ where } k_0 := k_1 k_2, \end{aligned}$$

and where the limit of measures is a vague limit. □

Definition 3.4. For a complete and separable metric space S, let $\mathcal{N}(S)$ be the set of counting measures on S that are boundedly finite. We equip $\mathcal{N}(S)$ with the σ-algebra $\Sigma(\mathcal{N}(S))$ generated by the evaluation maps $N \mapsto N(B)$ for Borel subsets $B \subseteq S$. Let $N \in \mathcal{N}([0,\infty) \times \mathcal{E})$ and define

(1) $\operatorname{len}(N) := \inf\{t > 0 : N([t,\infty) \times \mathcal{E}) = 0\} \in [0,\infty]$;
(2) $N|_{[a,b]}$ is N restricted to the interval $[a,b]$: $N|_{[a,b]} := \sum_{t\in[a,b],(t,f_t) \text{ atom of } N} \delta_{(t,f_t)}$.
(3) $N|^{\leftarrow}_{[a,b]}$ is N with the time index shifted back by a: $N|^{\leftarrow}_{[a,b]} := \sum_{t\in[a,b],(t,f_t) \text{ atom of } N} \delta_{(t-a,f_t)}$;

(4) for an atom (t, f_t) of N we define, for $y \in (\xi_N(t-), \xi_N(t))$ (recall (1.1) and paragraph below the equation for definition of ξ_N), the decomposition of the spindle f_t between the component of f_t that falls below level y, denoted as $\check{f}_t \in \mathcal{E}$, and the component of f_t that is above level y, denoted as $\hat{f}_t \in \mathcal{E}$:
$$\check{f}_t^y(z) := f_t(z)\mathbb{1}_{z\in[0,y-\xi_N(t-))}, \qquad \hat{f}_t^y(z) := f_t(y-\xi_N(t-)+z)\mathbb{1}_{z\in[0,\xi_N(t)-y]},$$
and we write $\check{f}_t := \check{f}_t^0$ and $\hat{f}_t := \hat{f}_t^0$;

(5) we define a subspace of $\mathcal{N}([0,\infty)\times\mathcal{E})$. Let $\mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$ be the set of counting measures with the additional properties:
  (a) $N(\{t\}\times\mathcal{E}) \le 1$ for every $t \in [0,\infty)$,
  (b) $N([0,t]\times\{f\in\mathcal{E} : \zeta(f) > z\}) < \infty$ for every t, $z > 0$,
  (c) $\mathrm{len}(N) < \infty$ and limit $\xi_N(t)$ exists for each $t \in [0, \mathrm{len}(N)]$;

(6) we define the subspace $\mathcal{N}^{\mathrm{sp}} \subset \mathcal{N}$ by saying that $N \in \mathcal{N}^{\mathrm{sp}}$ if and only if $N|_{[0,t]} \in \mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$ for each $t \ge 0$.

With the above definitions, we can define the following sets that contain measures associated with càdlàg excursions away from 0. We define
$$\mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} := \{N \in \mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}} : \xi(N) \in \mathcal{D}_{\mathrm{exc}}\},$$
$$\mathcal{N}^{\mathrm{sp}}_{+\mathrm{cld}} := \{N \in \mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} : \inf_t \xi_N(t) = 0\}, \qquad \mathcal{N}^{\mathrm{sp}}_{-\mathrm{cld}} := \{N \in \mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} : \sup_t \xi_N(t) = 0\},$$
to classify the possible excursions of $\mathbf{X}$ marked with the points of $\mathbf{N}$. We split
$$N \in \mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} \backslash (\mathcal{N}^{\mathrm{sp}}_{-\mathrm{cld}} \cup \mathcal{N}^{\mathrm{sp}}_{+\mathrm{cld}})$$
into what we call its clade component $N^+ \in \mathcal{N}^{\mathrm{sp}}_{+\mathrm{cld}}$ and its anti-clade component $N^- \in \mathcal{N}^{\mathrm{sp}}_{-\mathrm{cld}}$ by
$$N^- := N|_{[0,T_0^+)} + \delta(T_0^+, \check{f}^0_{T_0^+}), \qquad N^+ := \delta(0, \hat{f}^0_{T_0^+}) + N|^{\leftarrow}_{(T_0^+,\infty)}, \tag{3.6}$$
(and we define (N^-, N^+) to be $(N, 0)$ for $N \in \mathcal{N}^{\mathrm{sp}}_{-\mathrm{cld}}$ and similarly for $N \in \mathcal{N}^{\mathrm{sp}}_{+\mathrm{cld}}$). The terminology is inherited from our main reference Forman et al. (2020a). The component N^+ represents all the descendents of an individual, and this is therefore a clade of the total population. The term anti-clade was then coined due to it being the inverse to the clade. Finally, we use the term bi-clade for the combination of the clade and the anti-clade. We also refer to elements of $\mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}$, $\mathcal{N}^{\mathrm{sp}}_{+\mathrm{cld}}$ and $\mathcal{N}^{\mathrm{sp}}_{-\mathrm{cld}}$ as bi-clades, clades and anti-clades respectively.

In Definition 3.5 we define V^y to be the set of all complete excursions about level y and we define V_0^y to be V^y along with the possibly incomplete first and last excursions in the following way from Forman et al. (2020a, Definition A.1).

Definition 3.5. For a càdlàg function $g \in \mathcal{D}$ and $y \in \mathbb{R}$ we define
$$V^y(g) := \left\{ [a,b] \subset [0, \mathrm{len}(g)] \;\middle|\; \begin{array}{c} a < b < \infty; g(t-) \neq y \neq g(t) \text{ for } t \in (a,b); \\ g(a-) = y \text{ or } g(a) = y; g(b-) = y \text{ or } g(b) = y \end{array} \right\}.$$
The set $V_0^y(g) \supset V^y(g)$ then includes any incomplete first or last excursions. To define this rigorously, let
$$T^y(g) := \inf(\{t \in [0,\mathrm{len}(g)] : g(t) = y \text{ or } g(t-) = y\} \cup \{\mathrm{len}(g)\}),$$
$$T^y_*(g) := \sup(\{t \in [0,\mathrm{len}(g)] : g(t) = y \text{ or } g(t-) = y\} \cup \{0\}).$$
Then, for $y \neq 0$, we include $[0, T^y(g)] \cap [0,\infty)$ in $V_0^y(g)$, even if $T^y(g)$=0. Also, if $T^y_*(g) < \mathrm{len}(g)$ or $g(\mathrm{len}(g)) \neq y$, then we include $[T^y_*(g), \mathrm{len}(g)] \cap [0,\infty)$ in $V_0^y(g)$. For $N \in \mathcal{N}^{\mathrm{sp}}$ we define
$$V^y(N) := V^y(\xi(N))$$
and similarly for V_0^y.

Finally, we need to take care with possible non-degenerate excursions and so we carefully define intervals $I_N^y(a,b)$ to be one of $[a,b]$, $(a,b]$, $[a,b)$, (a,b) as follows: we exclude the point a iff $a<b$ and $g(a-)<y=g(a)$; we exclude b iff both $a<b$ and $g(b-)=y<g(b)$.

With this definition we can consider the measure on excursions of the process $\mathbf{X}:=\xi(\mathbf{N})$ where the jumps are marked by the spindles that generate them. For $N\in\mathcal{N}^{\text{sp}}_{\text{fin}}$, let $V^y(N)$ for $y\in\mathbb{R}$, $N\in\mathcal{N}^{\text{sp}}$ be the set of intervals of complete excursions of $\mathbf{X}$, so that $[a,b]\in V^y(N)$ means that the path $(\xi_N(a+t), 0\le t\le b-a)$ is an excursion about level y – we define this rigorously in Proposition 5.4 in the Appendix. As $\mathbf{X}$ is of unbounded variation we have that, for each level $y\in\mathbb{R}$, there are no degenerate excursions of $\mathbf{X}$ a.s. and $\mathbf{X}$ has local times. Furthermore, the local times are injective a.s. for each y in the sense that for $[a,b]$, $[c,d]\in V^y$, $\ell^y(a)\neq\ell^y(c)$ unless $[a,b]=[c,d]$; this is shown in Millar (1973) which summarises whether Lévy processes either first jump across or first hit fixed levels. While it holds at any fixed level a.s., across all levels simultaneously this would be false. We define F^0 for $N\in\mathcal{N}^{\text{sp}}$ and $\mathbf{F}^0$ for $\mathbf{N}$ by

$$F^0(N):=\sum_{[a,b]\in V^0(N)}\delta_{\ell^0(a),N|^{\leftarrow}_{[a,b]}},\qquad \mathbf{F}^0:=F^0(\mathbf{N}). \tag{3.7}$$

We note that $\mathbf{F}^0$ is a Poisson random measure where each element is the collection of spindles that generate an excursion about level 0 for the process ξ_N. We define the measure ν_{cld} on $\mathcal{N}^{\text{sp}}_{\pm\text{cld}}$ by

$$\nu_{\text{cld}}(A):=\mathbf{E}(\mathbf{F}^0([0,1]\times A))\text{ for a measurable set }A\subset\mathcal{N}^{\text{sp}}_{\pm\text{cld}}.$$

The bi-clades here will have a unique jump that crosses zero, that is to say that the set of all bi-clades with no unique crossing jump has measure 0 under ν_{cld}, and the unique jump will have associated with it an excursion of our diffusion. Therefore we can discuss the crossing width m^0 of the unique crossing jump:

$$m^0(N):=\sum_{(r,f_r)\text{ atom of }N}\max\{f_r((-\xi_N(r-))-),f_r(-\xi_N(r-))\};\quad N\in\mathcal{N}^{\text{sp}}_{\pm\text{cld}}. \tag{3.8}$$

The reason that we define m^0 in this way is that it makes sense for $N\in\mathcal{N}^{\text{sp}}_{-\text{cld}}\cup\mathcal{N}^{\text{sp}}_{+\text{cld}}$. Indeed, we have

$$m^0(N)=f_{T_0^+(\xi_N)}(-\xi_N(T_0^+(\xi_N)-))\text{ for }N\in\mathcal{N}^{\text{sp}}_{\pm\text{cld}}\backslash(\mathcal{N}^{\text{sp}}_{-\text{cld}}\cup\mathcal{N}^{\text{sp}}_{+\text{cld}}).$$

This second formulation for the crossing width is more intuitive. We use the following notation, similar in style to that used in Section 2.1: take as an example, for a measurable set A of $[0,\infty)\times(-\infty,0]\times[0,\infty)$,

$$\nu_{\text{cld}}((m^0,\xi_N(T_0^+-),\xi_N(T_0^+))\in A):=\nu_{\text{cld}}(\{N\in\mathcal{N}^{\text{sp}}_{\pm\text{cld}}:(m^0(N),\xi_N(T_0^+(\xi_N)-),\xi_N(T_0^+(\xi_N)))\in A\}).$$

Proposition 3.6. *Consider a diffusion that satisfies A1 and A2 with Pitman-Yor excursion measure ν that satisfies (1.9). Let $(\mathbf{N},\mathbf{X})$ be a scaffolding-and-spindles pair for this diffusion. Then, in the setting described above, we have that*

$$\sum_{(t,N')\text{ atom of }F^0(\mathbf{N})}\delta_{t,m^0(N')}$$

is a Poisson random measure with intensity $\nu_{cld}(m^0\in dx)=k_0M(dx)$ where k_0 is the constant in Lemma 3.3 for the SPLP $\mathbf{X}$.

Proof: The fact that this measure is a Poisson random measure with intensity $\nu_{\text{cld}}(m^0\in dx)$ follows from the fact that $\mathbf{F}^0$ is a Poisson random measure with intensity ν_{cld}. It remains to show $\nu_{\text{cld}}(m^0\in dx)=k_0M(dx)$. We rearrange the result in Lemma 3.3 to obtain

$$\nu_{\text{cld}}(-\xi_N(T_0^+-)\in dv,\xi_N(T_0^+)-\xi_N(T_0^+-)\in du)=k_0dv\nu(\zeta\in du)\text{ for }k_0>0,\ \text{for }0<v<u,$$

where k is the constant stated in Lemma 3.3. Recall that Λ is the jump measure of $\mathbf{X}$ and therefore $\nu(\zeta \in \cdot) = \Lambda(\cdot)$ in this context. Let H be a positive measurable function $H : [0,\infty) \to [0,\infty)$. We claim that

$$\nu_{\mathrm{cld}}\big(H(m^0)\big) = \int_{N\in\mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}} H(m^0(N))\nu_{\mathrm{cld}}(dN) = k_0 \int_{y\in E} H(y)M(dy).$$

First note that

$$\begin{aligned}&\int_{N\in\mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}} H(m^0(N))\nu_{\mathrm{cld}}(dN)\\ &\qquad = \int_{u=0}^{\infty}\int_{v=0}^{u}\int_{x\in E} H(x)\nu_{\mathrm{cld}}(m^0\in dx, -\xi_N(T_0^+-)\in dv, \xi_N(T_0^+)-\xi_N(T_0^+-)\in du)\end{aligned}$$

and in the next step we disintegrate $\nu_{\mathrm{cld}}(m^0 \in \cdot)$ with respect to $(-\xi_N(T_0^+-), \xi_N(T_0^+))$. This is possible as the bi-clade measure ν_{cld} can be obtained from marking the jumps of an excursion of a càdlàg process under the excursion measure of the Lévy process $\mathbf{X}$, see Forman et al. (2020a, Proposition 4.9). We obtain

$$\begin{aligned}&k_0\int_{u=0}^{\infty}\int_{v=0}^{u}\int_{x\in E} H(x)\nu_{\mathrm{cld}}(m^0\in dx| -\xi_N(T_0^+-)=v, \xi_N(T_0^+)-\xi_N(T_0^+-)=u)dv\nu(\zeta\in du)\\ &\qquad = k_0\int_{u=0}^{\infty}\int_{v=0}^{u}\int_{f\in\mathcal{E}} H(f(v))\nu(df|\zeta=u)dv\nu(\zeta\in du),\end{aligned}$$

by (3.8) and the fact that $\nu_{\mathrm{cld}}(\mathcal{N}^{\mathrm{sp}}_{-\mathrm{cld}}\cup\mathcal{N}^{\mathrm{sp}}_{+\mathrm{cld}}) = 0$. We can rewrite this as

$$k_0\int_{u=0}^{\infty}\int_{f\in\mathcal{E}}\int_{v=0}^{\zeta(f)} H(f(v))dv\nu(df|\zeta=u)\nu(\zeta\in du),$$

where f represents the excursion associated with the unique crossing jump and ν is the Pitman-Yor excursion measure of our diffusion. We can express this in terms of an integral against the Pitman-Yor excursion measure as

$$k_0\int_{f\in\mathcal{E}}\Big(\int_{v=0}^{\zeta(f)} H(f(v))dv\Big)\nu(df).$$

From Theorem 2.7 we see this disintegrates with respect to the amplitude A to give

$$k_0\int_{w\in E\setminus\{0\}}\int_{f\in\mathcal{E}}\Big(\int_{v=0}^{\zeta(f)} H(f(v))dv\Big)\nu(df|A=w)\nu(A\in dw). \tag{3.9}$$

As shown in the left graph of Figure 2.2, for $w \in (0,h)$ the Williams decomposition result gives a disintegration with the conditional distribution $\nu(\cdot|A = w)$ being the concatenation of two $\uparrow$-diffusions both stopped at T_w. Therefore for the $w \in (0,h)$ in (3.9) we have that

$$\int_{w=0}^{h}\int_{f\in\mathcal{E}}\Big(\int_{v=0}^{\zeta(f)} H(f(v))dv\Big)\nu(df|A=w)\nu(A\in dw) = \int_{w=0}^{h} 2\mathbb{E}_{\mathbb{Q}_0^{\uparrow}}\Big(\int_{t=0}^{T_w} H(Y_t)dt\Big)\frac{s(dw)}{s(w)^2}.$$

which is equal to

$$\int_{w=0}^{h}\int_{y=0}^{w} 2H(y)(s^{\uparrow}(w)-s^{\uparrow}(y))M^{\uparrow}(dy)\frac{s(dw)}{s(w)^2}$$

by Theorem 2.8 (where we have taken monotonic limits $a \downarrow 0$ and $x \downarrow 0$ as we did in Lemma 2.9). With the equations for $s^{\uparrow}$ and $M^{\uparrow}$ measure from (2.6), we substitute these terms and swap the order of integration to obtain

$$2\int_{y=0}^{h}\int_{w=y}^{h}\left(\frac{1}{s(y)}-\frac{1}{s(w)}\right)\frac{1}{s(w)^2}s(dw)H(y)s(y)^2M(dy).$$

We split into cases depending on whether or not $s(h)$ is finite. If $s(h)$ is infinite then $E = [0, h)$ and the above integral is equal to $\int_{y\in E} H(y)M(dy)$. When $s(h)$ is finite we instead have that $E = [0, h]$ and the above integral is equal to

$$\int_{y\in E}\Big(1 - 2\frac{s(y)}{s(h)} + \frac{s(y)^2}{s(h)^2}\Big)H(y)M(dy).$$

Furthermore when $s(h)$ is finite we have that the measure $\nu(A \in \cdot)$ has an atom at h and so there is a positive contribution to the integral in (3.9) at $w = h$. We can calculate this from the Williams decomposition in Theorem 2.7 to obtain

$$\nu(A = h)\Big(\mathbb{E}_{\mathbb{Q}_0^\uparrow}\Big(\int_{y=0}^{T_h} H(Y_s)ds\Big) + \mathbb{E}_{\mathbb{Q}_h}\Big(\int_{s=0}^{T_0} H(Y_s)ds\Big)\Big).$$

We have that $\nu(A = h)$ is equal to $1/s(h)$ and

$$\mathbb{E}_{\mathbb{Q}_0^\uparrow}\Big(\int_{y=0}^{T_h} H(Y_s)ds\Big) + \mathbb{E}_{\mathbb{Q}_h}\Big(\int_{s=0}^{T_0} H(Y_s)ds\Big) = \int_{y\in E} H(y)\Big(\frac{1}{s(y)} - \frac{1}{s(h)}\Big)s(y)^2M(dy) + \int_{y\in E} H(y)s(y)M(dy)$$

which equals $\int_{y\in E} H(y)\Big(2s(y) - \frac{s(y)^2}{s(h)}\Big)M(dy)$. Therefore we have (3.9) is equal to $\int_{y\in E} H(y)M(dy)$ in both cases as required. $\square$

With Proposition 3.6, we can prove Theorem 1.4.

Proof of Theorem 1.4: Let $(\mathbf{N}, \mathbf{X})$ be a scaffolding-and-spindles pair stopped at an a.s. finite T for a diffusion $\mathbf{Z}$. In the setting where $\nu_{\mathbf{Z}}$ satisfies (1.6) for $\nu = \nu_{\mathbf{Z}}$, the scaffolding $\mathbf{X}$ is an SPLP of bounded variation. Therefore the set $\{r \in [0, t] : \mathbf{X}_{r-} < 0 \leq \mathbf{X}_r\}$ is finite a.s., i.e. there are only finitely many crossing widths to sum; therefore, the aggregate health $M^0_{\mathbf{N},\mathbf{X}}(T)$ at level 0 is finite as it is a finite sum of positive values.

In the setting where $\nu_{\mathbf{Z}}$ satisfies (1.9) for $\nu = \nu_{\mathbf{Z}}$ we can apply Proposition 3.6. This proposition states that, for fixed $t > 0$, the aggregate fitness process $M^0_{\mathbf{N},\mathbf{X}}(t)$ is a sum over a Poisson random measure with intensity $\mathrm{Leb} \otimes M_{\mathbf{Z}}$ on $[0, t] \times E\backslash\{0\}$. Let $b := 1 \wedge h/2 \in (0, h)$. We deal with the sum of crossing widths less than b and the crossing widths of size greater than b separately. The crossing widths of size less than b form a Poisson random measure with intensity $\mathrm{Leb} \otimes M_{\mathbf{Z}}$ on $[0, t] \times (0, b)$. Therefore, by Campbell's Theorem, see Kingman (1993, Section 3.2), these are summable if and only if

$$\int_{[0,t]\times(0,b)} z dt M_{\mathbf{Z}}(dz) = t\int_{z=0}^{b} zM_{\mathbf{Z}}(dz) < \infty$$

and this is finite by the assumption in the theorem. The crossing widths of size greater than b are from a Poisson random measure with intensity $\mathrm{Leb} \otimes M_{\mathbf{Z}}$ on $[0, t] \times [b, h]$. Note that

$$\int_{[0,t]\times[b,h]} dt M_{\mathbf{Z}}(dz) = tM_{\mathbf{Z}}([b, h])$$

and this is finite by $A2$ and therefore there are only finitely many such crossings so the sum is clearly finite. The result then follows for the aggregate fitness process evaluated at the a.s. finite time T.

To see the result in terms of an untransformed diffusion $\mathbf{Y}$ with transformation g, apply Proposition 3.6 to diffusion $\mathbf{Y}$ to obtain that the crossing widths in this case are a Poisson random measure with intensity $\mathrm{Leb} \otimes M_{\mathbf{Y}}$. We apply the transformation g to these crossing widths and this is equal in distribution to the crossing widths from the diffusion $\mathbf{Z}$. The condition for summability of the small crossing widths then becomes the condition in (1.10). $\square$

3.3. *Disintegration with respect to the crossing width.* In this section we disintegrate the bi-clade measure ν_{cld} with respect to the crossing width m^0. As in the previous section, we assume that our diffusion satisfies $A1$ and $A2$ and that the Pitman-Yor excursion measure ν satisfies (1.9). We define some operators that time-reverses excursions and point measures.

Definition 3.7. Let $f \in \mathcal{E}$ and $N \in \mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$. Then

(1) We define the time-reversal for f by

$$\mathcal{R}_{\mathrm{spdl}}(f) := (f((\zeta(f) - y)-), y \in \mathbb{R}), \tag{3.10}$$

following the definition given in Forman et al. (2020a, Equation (2.16)).

(2) We also define the time-reversal for N by

$$\mathcal{R}_{\mathrm{cld}}(N) := \sum_{(t,f) \text{ atom of } N} \delta(\mathrm{len}(N) - t, \mathcal{R}_{\mathrm{spdl}}(f)), \tag{3.11}$$

following the definition given in Forman et al. (2020a, Equation (4.19)).

The reversibility of the excursions under the Pitman-Yor excursion measure ν, see e.g. the Williams decomposition in Pitman and Yor (1982, Section 3), along with the independence between excursions away from 0 for the Lévy process $\mathbf{X}$, means that

$$\nu_{\mathrm{cld}}(A) = \nu_{\mathrm{cld}}(\mathcal{R}_{\mathrm{cld}}(A)) \text{ for a measurable } A \subset \mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}. \tag{3.12}$$

The next result is a generalisation of Forman et al. (2020a, Lemma 4.13), and it proves conditional independence between the clade and the anti-clade given the crossing width.

Lemma 3.8 (Mid-spindle Markov property)**.** *Consider a diffusion that satisfies $A1$ and $A2$ with Pitman-Yor excursion measure ν that satisfies (1.9), and let $(\mathbf{N}, \mathbf{X})$ be a scaffolding-and-spindles construction for this diffusion. Let T be a stopping time either of the form $T^{\geq y}$ for some $y > 0$ or of the form*

$$T^y_{n,j} := \inf\left\{ t > 0 \Big| \int_{[0,t]\times\mathcal{E}} \mathbb{1}_{f(y-\mathbf{X}(s-))>1/n} d\mathbf{N}(s,f) \geq j \right\} \text{ for } y \in \mathbb{R}, j,n \in \mathbb{N}, \tag{3.13}$$

and let f_T denote the spindle of $\mathbf{N}$ at this stopping time T. Then, conditionally given $f_T(y - \mathbf{X}(T-)) = x > 1/n$, we have that

$$(\mathbf{N}|_{[0,T)}, \check{f}^y_T) \text{ is independent of } (\mathbf{N}|^{\leftarrow}_{(T,\infty)}, \hat{f}^y_T)$$

(see Definition 3.4 for $\check{f}^y_T$ and $\hat{f}^y_T$). Under this conditional law, $\mathbf{N}|^{\leftarrow}_{(T,\infty)}$ is a Poisson random measure with intensity measure Leb $\otimes\, \nu$ independent of $\hat{f}_T$, which is a 0-diffusion started at x.

Proof: The proof given in Forman et al. (2020a, Lemma 4.13) applies in our setting as we have the following conditions: the Pitman-Yor description of an excursion that hits level $1/n$ is given by an $\uparrow$-diffusion started at 0 and stopped at $1/n$, followed by an independent 0-diffusion started at $1/n$; the Poisson random measure $\mathbf{N}$ has a strong Markov property; the scaffolding process $\mathbf{X}$ is of unbounded variation and so we have $T^{\geq y} = \inf_{n\geq 1} T^y_{n,1}$. □

For $x > 0$, let $\mathbb{Q}^0_x$ be the probability measure on $\mathcal{D}$ for the 0-diffusion started from x.

Lemma 3.9. *In the setting of Lemma 3.8, for $n > 1$ and $x > 1/n$ we have that*

$$\begin{aligned}&\mathbf{P}\Big(\hat{f}_{T^0_{n,1}} \in A, \check{f}_{T^0_{n,1}} \in B, \mathbf{N}|_{(\tau^0(\ell^0(T^0_{n,1})-),T^0_{n,1})} \in C, \mathbf{N}|^{\leftarrow}_{(T^0_{n,1},\tau^0(\ell^0(T^0_{n,1})))} \in D \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big)\\ &= \int_{f\in A} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f)})} \in D)\mathbb{Q}^0_x(df) \int_{f'\in\mathcal{R}_{spdl}(B)} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f')})} \in \mathcal{R}_{cld}(C))\mathbb{Q}^0_x(df').\end{aligned}$$

Proof: First note by the conditional independence property given in Lemma 3.8 we have, for $x > 0$, that

$$\mathbf{P}\Big(\hat{f}_{T^0_{n,1}} \in A, \check{f}_{T^0_{n,1}} \in B \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) = \mathbb{Q}^0_x(A)\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big).$$

Let $\lambda > 0$ and let e_λ be an exponential random variable with rate λ. The excursions of $\mathbf{X}$ away from 0 are indexed by its local time. Therefore $\mathbf{N}|_{[0,\tau^0(e_\lambda)]}$ consists of a collection of complete bi-clades. Therefore, by reversibility of the clade measure, noted in (3.12), we have

$$\mathbf{N}|_{[0,\tau^0(e_\lambda)]} \stackrel{d}{=} \mathcal{R}_{cld}\Big(\mathbf{N}|_{[0,\tau^0(e_\lambda)]}\Big). \tag{3.14}$$

Now, on the event $\{T^0_{n,1} < \tau^0(e_\lambda)\}$ there exists $j \in \mathbb{N}$ such that

$$T^0_{n,1}(\mathbf{N}|_{[0,e_\lambda]}) = \tau^0(e_\lambda) - T^0_{n,j}(\mathcal{R}_{\mathrm{cld}}(\mathbf{N}|_{[0,e_\lambda]})). \tag{3.15}$$

Furthermore, we have that

$$\begin{aligned}\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big)& \\ = \lim_{\lambda\downarrow 0}\sum_{j=1}^{\infty}\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B, T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1} \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big)&\end{aligned}$$

as $\lim_{\lambda\downarrow 0}\mathbf{P}(T^0_{n,1} < \tau^0(e_\lambda)) = 1$. We consider each term in the sum. We apply the time reversal and the mid-spindle Markov property (Lemma 3.8) to obtain

$$\begin{aligned}\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B \Big| T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1}, f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big)& \\ = \mathbf{P}\Big(\hat{f}_{T^0_{n,j}} \in \mathcal{R}_{\mathrm{spdl}}(B) \Big| f_{T^0_{n,j}}(-\mathbf{X}(T^0_{n,j}-)) = x\Big)&\end{aligned}$$

where the conditioning $T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1}$ was able to be removed due to the conditional independence given $f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x$ from the mid-spindle Markov property.

Let $p_\lambda = \mathbf{P}(\tau^0(e_\lambda) > T^0_{n,1})$. Then

$$\mathbf{P}\Big(T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1} \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) = \mathbf{P}\Big(T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1}\Big) = p^j_\lambda(1 - p_\lambda).$$

Pulling all this together we obtain

$$\lim_{\lambda\downarrow 0}\mathbf{P}\Big(\check{f}_{T^0_{n,1}} \in B, T^0_{n,j} < \tau^0(e_\lambda) < T^0_{n,j+1} \Big| f_{T^0_{n,1}}(-\mathbf{X}(T^0_{n,1}-)) = x\Big) = \lim_{\lambda\downarrow 0}\sum_{j=1}^{\infty} p^j_\lambda(1-p_\lambda)\mathbb{Q}^0_x(\mathcal{R}_{\mathrm{spdl}}(B)),$$

which equals $\mathbb{Q}^0_x(\mathcal{R}_{\mathrm{spdl}}(B))$ because $p_\lambda \to 1$ as $\lambda \to 0$. The generalisation to the statement in the proposition follows from the Lemma 3.8; the strong Markov property of $\mathbf{N}$ at $T^0_{n,1}$ leads to the independence of the Poisson random measures $\mathbf{N}|_{[0,T^0_{n,1})}$ and $\mathbf{N}|^{\leftarrow}_{(T^0_{n,1},\infty)}$. □

We now describe the regular conditional distribution of ν_{cld} disintegrated with respect to m^0. Let $x > 0$ denote the crossing width. Let $\hat{f}$ and $\check{f}$ be independent 0-diffusions started at x. Let $\check{\mathbf{N}}$ and $\hat{\mathbf{N}}$ be independent Poisson random measures with intensity $\mathrm{Leb} \otimes \nu$.

We define the idea of concatenating point measures in the analogous way to how we defined concatenating interval partitions in Definition 2.3. Let $N_i \in \mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$ be a collection of point measures where $i \in I$ is an ordered index set. We define $N := \bigstar_{i\in I} N_i$ to be the concatenation with respect to the ordering of I in the following way. On the condition that for all $i \in I$ we have $S(j-) := \sum_{j\in I: j<i}\mathrm{len}(N_j) < \infty$ we take N to be the point measure with length $\sum_{i\in I}\mathrm{len}(N_i) \le \infty$ such that $N|^{\leftarrow}_{(S(i-),S(i-)+\mathrm{len}(N_i))} = N_i$. If the condition does not hold we take $N = \emptyset$. When $I = \{1, 2, \ldots, n\}$ is a finite set we write $N_1 \star \cdots \star N_n$ for $\bigstar_{i\in I} N_i$.

We define a function to stick spindles together: for f_1, f_2 excursions we define $f_{1\to 2}$ by

$$f_{1\to 2}(y) := \begin{cases} f_1(\zeta(f_1) - y) & \text{for} \quad 0 < y < \zeta(f_1) \\ f_2(y - \zeta(f_1)) & \text{for} \quad \zeta(f_1) \le y \le \zeta(f_1) + \zeta(f_2) \\ 0 & \text{otherwise.} \end{cases}$$

Define the map for paths f_1, f_2 and for N_1, $N_2 \in \mathcal{N}^{\mathrm{sp}}$

$$H(f_1, f_2, N_1, N_2) := \mathcal{R}_{\mathrm{cld}}(N_1|_{[0,T^{-\zeta(f_1)})}) \star \delta_{(0,f_{1\to 2})} \star N_2|_{[0,T^{-\zeta(f_2)})},$$

and define the measure $\mathbf{M}$ by

$$\mathbf{M} := H(\check{f}, \hat{f}, \check{\mathbf{N}}, \hat{\mathbf{N}}),$$

and let ν_x be the distribution of $\mathbf{M}$ on $\mathcal{N}^{\mathrm{sp}}_{\mathrm{cld}}$. We note from the independence in the construction that for $x > 0$ and $G_1 \subseteq \mathcal{N}^{\mathrm{sp}}_{+\mathrm{cld}}$, $G_2 \subseteq \mathcal{N}^{\mathrm{sp}}_{-\mathrm{cld}}$ that

$$\nu_x(N^+ \in G_1, N^- \in G_2) = \nu_x(N^+ \in G_1)\nu_x(N^- \in G_2).$$

Furthermore, let $\hat{f}$ be a 0-diffusion started at x, independent of $\mathbf{N}$. Let

$$\hat{T}^0 := \inf\{t > 0 : \xi_{\mathbf{N}}(t) = -\zeta(\hat{f})\}.$$

Then

$$\overline{N}^+_x := \delta(0, \hat{f}) + \mathbf{N}|_{[0,\hat{T}^0)}$$

has the law $\nu_x(N^+ \in \cdot)$, and $\mathcal{R}_{\mathrm{cld}}(\overline{N}^+_x)$ has the law $\nu_x(N^- \in \cdot)$.

Proposition 3.10. *Consider a diffusion that satisfies A1 and A2 with Pitman-Yor excursion measure ν that satisfies (1.9). Then ν_x, $x \in E$, is a disintegration of ν_{cld} with respect to m^0.*

Proof: This proof concerns kernels as defined in Kallenberg (2021, Chapter I.3). Let μ_x be the distribution of $(\check{f}, \hat{f}, \check{\mathbf{N}}, \hat{\mathbf{N}})$. We first show that $x \mapsto \mu_x$ is a kernel. First note that $x \mapsto \mathbf{P}(\mathbf{N} \in \cdot)$ is a kernel and we also have that $x \mapsto \mathbb{Q}^0_x$ is a kernel. The first follows as the image is independent of x and the second follows as the measure $\mathbb{Q}^0_x$ is continuous in the weak sense and is therefore measurable. It therefore follows that $x \mapsto \mu_x$ is a kernel as we have that $\mu_x = \mathbb{Q}^0_x \otimes \mathbb{Q}^0_x \otimes \mathbf{P}(\mathbf{N} \in \cdot) \otimes \mathbf{P}(\mathbf{N} \in \cdot)$, see Kallenberg (2021, Lemma I.3.3). The result then follows as H is a measurable map and ν_x is $\mu_x(H^{-1}(\cdot))$, and therefore the map $x \mapsto \nu_x$ is measurable and therefore is a kernel, see Kallenberg (2021, Lemma I.3.1). Therefore the map on $\mathbb{R}_+ \to \mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}$ given by $x \mapsto \nu_x$ is a kernel because the map H from $(\check{f}, \hat{f}, \check{\mathbf{N}}, \hat{\mathbf{N}})$ to $\mathbf{M}$ is measurable.

We now verify that $(\nu_x, x > 0)$ do integrate to the bi-clade measure ν_{cld}. Let A, $B \subset \mathcal{E}$ measurable and let C, $D \subset \mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$ measurable. By definition of ν_{cld} we have that

$$\begin{aligned} &\nu_{\mathrm{cld}}(H(A \times B \times C \times D) \cap \{m^0 > 1/n\}) \\ &\quad = \nu_{\mathrm{cld}}(\hat{f}_{T_0^+} \in A, \check{f}_{T_0^+} \in B, N^-|_{[0,\mathrm{len}(N^-))} \in C, N^+|_{[0,\mathrm{len}(N^+))} \in D, m^0 > 1/n) \\ &\quad = k_0 M(E\backslash[0,1/n])\mathbf{P}(\hat{f}_{T^0_{n,1}} \in A, \check{f}_{T^0_{n,1}} \in B, \mathbf{N}|_{[\tau^0(\ell^0(T^0_{n,1})-),T^0_{n,1})} \in C, \mathbf{N}|_{\overleftarrow{(T^0_{n,1},\tau^0(\ell^0(T^0_{n,1})))}} \in D). \end{aligned}$$

Let $B' = \mathcal{R}_{\mathrm{spdl}}(B)$ and $C' = \mathcal{R}_{\mathrm{cld}}(C)$. Then with the result from Lemma 3.9 along with the crossing width intensity measure from Proposition 3.6 this is equal to

$$\int_{x \in E\backslash[0,1/n]} \int_{f \in B'} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f)})} \in C')\mathbb{Q}^0_x(df) \int_{f' \in A} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f')})} \in D)\mathbb{Q}^0_x(df')k_0 M(dx).$$

We take the limit as n tends to infinity to obtain

$$\begin{aligned} \nu_{\mathrm{cld}}(H(A \times B \times C \times D)) &= \nu_{\mathrm{cld}}(\hat{f}_{T_0^+} \in A, \check{f}_{T_0^+} \in B, N^-|_{[0,\mathrm{len}(N^-))} \in C, N^+|_{[0,\mathrm{len}(N^+))} \in D) \\ &= \int_{x \in E\backslash\{0\}} \int_{f \in B'} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f)})} \in C')\mathbb{Q}^0_x(df) \int_{f' \in A} \mathbf{P}(\mathbf{N}|_{[0,T^{-\zeta(f')})} \in D)\mathbb{Q}^0_x(df')2k_0 M(dx). \end{aligned}$$

By the definition of ν_x and μ_x this is equal to

$$\int_{x\in E\setminus\{0\}} \mu_x(A\times B\times C\times D)\nu_{\mathrm{cld}}(m^0\in dx) = \int_{x\in E\setminus\{0\}} \nu_x(H(A\times B\times C\times D))\nu_{\mathrm{cld}}(m^0\in dx).$$

The generality of $A,\ B\in\mathcal{E}$, and $C,\ D\in\mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$ is sufficient here to prove the proposition by the monotone class theorem. ☐

We also note from the proof of the proposition above that μ_x, $x\in E\setminus\{0\}$, is in fact a disintegration of $\nu_{\mathrm{cld}}(\check{f}_{T_0^+}\in\cdot, \hat{f}_{T_0^+}\in\cdot, N^-|_{[0,\mathrm{len}(N^-))}\in\cdot, N^+|_{[0,\mathrm{len}(N^+))}\in\cdot)$.

3.4. *Lévy process scaffolding from fixed initial interval partition.* In this section we give the construction of scaffolding-and-spindles pairs started from certain interval partitions $\beta\in\mathcal{I}_H$, which we denote as $(\mathbf{N}_\beta,\mathbf{X}_\beta)$. We give the construction of $\mathbf{N}_\beta$ in Definition 3.11 and we prove that this construction is possible under certain conditions in Proposition 3.12. We define random measures $\mathbf{N}_\beta$ on the space $\mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$.

Definition 3.11. Let $\mathcal{I}_H(E)\coloneqq\{\gamma\in\mathcal{I}_H : \mathrm{Leb}(U)\in E\ \forall U\in\gamma\}$ and let $\beta\in\mathcal{I}_H(E)$. If $\beta=\varnothing$ then define $\mathbf{N}_\beta$ to be the zero measure. Otherwise, for each interval U of β we construct each $\mathbf{N}_U$ independently in the following way. First let f_U have law $\mathbb{Q}^0_{\mathrm{Leb}(U)}$ and let $\mathbf{N}$ be a Poisson random measure with intensity $\mathrm{Leb}\otimes\nu$.

Let $T\coloneqq\inf\{t>0 : \xi_{\mathbf{N}}(t)=-\zeta(f_U)\}$ and let $\mathbf{N}_U\coloneqq\delta(0,f_U)+\mathbf{N}|_{[0,T]}$. Under the condition that $\sum_{U\in\beta}\mathrm{len}(\mathbf{N}_U)$ is finite, let $\mathbf{N}_\beta\coloneqq\bigstar_{U\in\beta}\mathbf{N}_U$ (and otherwise take $\mathbf{N}_\beta$ to be the zero measure). Let $\mathbf{X}_\beta\coloneqq\xi(\mathbf{N}_\beta)$ be the corresponding scaffolding constructed from $\mathbf{N}_\beta$. We write $\mathbf{P}_\beta$ to denote the law of $\mathbf{N}_\beta$ on $\mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$.

In all the results that follow we consider intervals partitions $\beta\in\mathcal{I}_H(E)$. We give sufficient conditions on an interval partition β for concatenating clades $(\mathbf{N}_U, U\in\beta)\subset\mathcal{N}^{\mathrm{sp}}_{+\mathrm{cld}}$ as in Definition 3.11.

Proposition 3.12. *Consider a diffusion that satisfies A1, A2 with a Pitman-Yor excursion measure ν that satisfies (1.9). Further suppose that*

$$\limsup_{x\downarrow 0}\frac{1}{x}\Big(\int_{v\in E} s(v\wedge x)M(dv)\Big) =: k_3<\infty. \tag{3.16}$$

Then, in the setting of Definition 3.11, for any interval partition $\beta\in\mathcal{I}_H(E)$, we have that $\sum_{U\in\beta}$ len$(\mathbf{N}_U)$ is finite a.s. and hence $(\mathbf{N}_\beta,\mathbf{X}_\beta)$ is a non-trival scaffolding-and-spindles pair (unless $\beta=\emptyset$).

Proof: We calculate the expected lifetime of an interval with initial width x, for $x\in E$, and we define the function $r(x)\coloneqq\mathbb{E}_{\mathbb{Q}^0_x}(\zeta)$ to be this expectation. We first assume that $s(h)$ is finite and we calculate $\mathbb{E}_{\mathbb{Q}^0_x}(T_0\wedge T_h)$. For $0<a<x<w<h$ we use $\mathbb{E}_{\mathbb{Q}^0_x}(T_a\wedge T_w)$ from (1.4) and we can take the monotonic limits of $a\downarrow 0$ and $w\uparrow h$ to find

$$\mathbb{E}_{\mathbb{Q}_x}(T_0\wedge T_h) = \int_{v\in E}\Big(\frac{s(x)}{s(h)}\Big(s(h)-s(v)\Big)\mathbb{1}_{x<v}+\frac{s(v)}{s(h)}\Big(s(h)-s(x)\Big)\mathbb{1}_{v\le x}\Big)M(dv).$$

To account for the path from h to 0 on the event that $T_h<T_0$ we have

$$\mathbb{E}_{\mathbb{Q}_x}\Big((T_0-T_h)\mathbb{1}_{T_h<T_0}\Big) = \mathbb{Q}_x(T_h<T_0)\mathbb{E}_{\mathbb{Q}_h}(T_0).$$

Firstly we have $\mathbb{Q}_x(T_h<T_0)=s(x)/s(h)$ (see (1.2)). The second term can be calculated from Lemma 2.9 to give

$$\mathbb{E}_{\mathbb{Q}_x}\Big((T_0-T_h)\mathbb{1}_{T_h<T_0}\Big) = \frac{s(x)}{s(h)}\int_{v\in E} s(v)M(dv).$$

We add the two contributions to obtain

$$r(x) = \int_{v \in E} s(v \wedge x) M(dv) \tag{3.17}$$

In the case where $s(h)$ is infinite we use $\mathbb{E}_{\mathbb{Q}_x^0}(T_a \wedge T_w)$ from (1.4) and we can take the monotonic limits of $a \downarrow 0$ and $w \uparrow h$ and this equals $\mathbb{E}_{\mathbb{Q}_x^0}(T_0)$ and this limit is as in (3.17).

Under the assumption given in (3.16), there exists $\varepsilon' > 0$ such that for all $x < \varepsilon'$ we have that $r(x) \leq (1 + k_3)x$. Therefore, for an interval partition $\beta \in \mathcal{I}_H(E)$ we have

$$\mathbb{E}\Bigg(\sum_{\substack{U \in \beta \\ \mathrm{Leb}(U) < \varepsilon'}} \zeta(f_U) \Bigg) = \sum_{\substack{U \in \beta \\ \mathrm{Leb}(U) < \varepsilon'}} r(\mathrm{Leb}(U)) \leq \sum_{\substack{U \in \beta \\ \mathrm{Leb}(U) < \varepsilon'}} (1 + k_3)\mathrm{Leb}(U) = (1 + k_3)\|\beta\| < \infty. \tag{3.18}$$

Therefore we have that $\sum_{U \in \beta \mathrm{Leb}(U) < \varepsilon'} \zeta(f_U) < \infty$ a.s. and as there are only finitely many intervals U of the interval partition β that have length greater than ε' we have that $\sum_{U \in \beta} \zeta(f_U) < \infty$ a.s. From here the result follows; to see this consider a scaffolding-and-spindles pair $(\mathbf{N}, \mathbf{X})$ stopped at $T_{-\sum_{U \in \beta} \zeta(f_U)}$, the hitting time of $-\sum_{U \in \beta} \zeta(f_U)$ by $\mathbf{X}$. Let $\preceq$ denote the total order of the intervals in the interval partition β and, for $U \in \beta$, define S_U, W_U by

$$S_U := T_{-\sum_{V \in \beta : V \prec U} \zeta(f_V)}; \qquad W_U = T_{-\sum_{V \in \beta : V \preceq U} \zeta(f_V)}.$$

Finally let $\mathbf{N}_U := \delta_{(0, f_U)} + \mathbf{N}|^{\leftarrow}_{[S_U, W_U]}$ and $\mathbf{N}_\beta = \bigstar_{U \in \beta} \mathbf{N}_U$. Then $\sum_{U \in \beta} \mathrm{len}(\mathbf{N}_U)$ is equal to $T_{-\sum_{U \in \beta} \zeta(f_U)}$ which is finite a.s. and by the Strong Markov property for the Poisson random measure $\mathbf{N}$ and we have that $\mathbf{N}_\beta$ has the distribution given in Definition 3.11. ☐

We show in Corollary 3.13 that (3.16) is satisfied for a large class of diffusions and in Remark 3.14 we highlight a class of diffusions that satisfy $A1$, $A2$ and (1.9) but that do not satisfy (3.16).

Corollary 3.13. *Consider a diffusion that satisfies $A1$, $A2$, $A3$ with Pitman-Yor excursion measure that satisfies (1.9) and suppose that there exists $k_4 < 0$ such that*

$$\limsup_{y \downarrow 0} \mu(y) = k_4 < 0.$$

Then the diffusion satisfies (3.16) and therefore for any interval partition $\beta \in \mathcal{I}_H(E)$ the construction of $(\mathbf{N}_\beta, \mathbf{X}_\beta)$ as in Definition 3.11 exists and len$(\mathbf{N}_\beta)$ is finite a.s. Furthermore we have that the function r, defined as $r(x) = \mathbb{E}_{\mathbb{Q}_x^0}(\zeta)$ in the proof of Proposition 3.12, has a continuous derivative r' on $(0, h)$ and $\limsup_{x \to 0} r'(x) < \infty$.

Proof: As our diffusion satisfies $A3$ we can use the equations for the scale function derivative and speed measure density from (1.11). This means that we can calculate $r'(x)$ and note that the condition in (3.16) is satisfied if $\limsup_{x \to 0} r'(x)$ is finite because this is greater than or equal to $\limsup_{x \to 0} r(x)/x$. We obtain

$$r'(x) = s'(x) \int_{v=x}^{h} m(v) dv$$

and this function is continuous on $(0, h)$. By the condition given in the corollary, there exists $\varepsilon > 0$ such that for $y < \varepsilon$ we have $\mu(y) < k_4/2$. Now, for $x < \varepsilon$, we have that

$$s'(x) \int_{v=x}^{\varepsilon} m(v) dv = \int_{v=x}^{\varepsilon} \frac{2}{\sigma^2(v)} \exp\Big(\int_{z=x}^{v} \frac{2\mu(z)}{\sigma^2(z)} dz \Big) dv \leq \int_{v=x}^{\varepsilon} \frac{2}{\sigma^2(v)} \exp\Big(\int_{z=x}^{v} \frac{k_4}{\sigma^2(z)} dz \Big) dv$$

and therefore

$$\begin{aligned} \limsup_{x \downarrow 0} s'(x) \int_{v=x}^{\varepsilon} m(v) dv &\leq \limsup_{x \downarrow 0} \int_{v=x}^{\varepsilon} \frac{2}{\sigma^2(v)} \exp\Big(\int_{z=x}^{v} \frac{k_4}{\sigma^2(z)} dz \Big) dv \\ &= \limsup_{x \downarrow 0} \frac{-2}{k_4} \Big(1 - \exp\Big(\int_{z=x}^{\varepsilon} \frac{k_4}{\sigma^2(z)} dz \Big) \Big) \leq \frac{-2}{k_4} < \infty. \end{aligned} \tag{3.19}$$

It remains to show that $\limsup_{x\downarrow 0} s'(x)\int_{v=\varepsilon}^{h} m(v)dv$ is finite. We have that $\int_{v=\varepsilon}^{h} m(v)dv$ is finite by $A2$. Also as μ is negative locally at 0 we have that the limit $\lim_{x\downarrow 0} s'(x)$ exists and equals

$$s'(0) := \exp\left(\int_{z=0}^{b} \frac{2\mu(z)}{\sigma^2(z)}dz\right)$$

where $b\in(0,h)$ is from (1.11) as the function $s'(x)$ is monotonic decreasing for $x\in(0,\varepsilon)$. Therefore we have an upper bound for $\limsup_{x\to 0} r(x)/x$ and for $\limsup_{x\to 0} r'(x)$ of $-2/k_4+s'(0)\int_{v=\varepsilon}^{h} m(v)dv$ and from this we see that the diffusion satisfies (3.16) by Proposition 3.12 and therefore a construction of $(\mathbf{N}_\beta,\mathbf{X}_\beta)$ exists. $\square$

We note that the untransformed diffusions under Assumption B satisfy the condition in Corollary 3.13 and so satisfy (3.16). This does not extend to all diffusions under Assumption B as (3.16) is not invariant under spatial transformations g (unlike our previous conditions with the speed measure and scale function in (1.7) and (1.8)).

Remark 3.14. The untransformed diffusions in Example 1.12 satisfy $A1$, $A2$ and (1.9) but do not satisfy (3.16): using (1.14) for the density m of the speed measure M and for the scale function s we have, for $x\in(0,\varepsilon_2)$,

$$\begin{aligned}\frac{1}{x}\Big(\int_{v\in E} s(v\wedge x)m(v)dv\Big) &= \frac{1}{x}\Big(\int_{v=0}^{x}\frac{1}{2}dv+\int_{v=x}^{\varepsilon_2} x\frac{1}{2v}dv + xM((\varepsilon_2,h])\Big)\\ &= \frac{-\log(x)+1+\log(\varepsilon_2)}{2}+M((\varepsilon_2,h])\end{aligned}$$

and this goes to infinity as x goes to zero.

The purpose of Proposition 3.12 and Corollary 3.13 is to give a framework from which we can develop a simple Markov property for the IP-evolutions for Theorem 1.6. For the remainder of propositions and lemmas in this section we assume (3.16) and so that for any interval partition $\beta\in\mathcal{I}_H(E)$ we have a non-trivial scaffolding-and-spindles pair $(\mathbf{N}_\beta,\mathbf{X}_\beta)$. We now define several functions that will allow us to construct suitable filtrations in space and in time, and then prove the simple Markov property in the setting of the Poisson random measure $\mathbf{N}$ on the probability space $(\Omega,\mathcal{A},\mathbf{P})$. We omit the exact proofs, rather, we give references to results from Forman et al. (2020a) and explain how they generalise to our setting. First we define M-diversity which is crucial for the arguments for the Markov property.

Definition 3.15. Let β be an interval partition and $t\geq 0$. Then we define the M-diversity of β at t as the following limit when it exists:

$$\mathcal{D}_\beta^M(t) := \lim_{x\to 0}\frac{1}{M(E\backslash[0,x])}\Big|\Big\{(a,b)\in\beta : |b-a|>x, b\leq t\Big\}\Big| \in [0,\infty), \tag{3.20}$$

and if the limit does not exist then we say that β at t does not have an M-diversity. Furthermore, for an interval $U=(a,b)\in\beta$ we define the limit $\mathcal{D}_\beta^M(U) := \mathcal{D}_\beta^M(b)$, when this limit exists. Finally, we also define the limsup-M-diversity by

$$\overline{\mathcal{D}}_\beta^M(t) := \limsup_{x\to 0}\frac{1}{M(E\backslash[0,x])}\Big|\Big\{(a,b)\in\beta : |b-a|>x, b\leq t\Big\}\Big| \in [0,\infty], \tag{3.21}$$

and the limit always exists for any interval partition β, and for an interval $U\in\beta$ we define $\overline{\mathcal{D}}_\beta^M(U) := \overline{\mathcal{D}}_\beta^M(b)$. Let $\mathcal{I}_M\subset\overline{\mathcal{I}}_M$ be

$$\mathcal{I}_M := \{\beta\in\mathcal{I}_H : \forall t>0, \mathcal{D}_\beta^M(t) \text{ exists}\}; \quad \overline{\mathcal{I}}_M := \{\beta\in\mathcal{I}_H : \forall t>0, \overline{\mathcal{D}}_\beta^M(t)<\infty\}, \tag{3.22}$$

and let $\mathcal{I}_M(E)=\mathcal{I}_H(E)\cap\mathcal{I}_M$ for a measurable subset $E\subseteq[0,\infty)$ and similarly for $\overline{\mathcal{I}}_M(E)$. We substitute the M for m and write $\mathcal{D}_\beta^m$ for $\mathcal{D}_\beta^M$ and similarly for the other terms when the speed measure M has a density m, and we call replace the M- for m- in their names.

The M-diversity defined here is a generalisation of the notion of α-diversity used in Forman et al. (2020a). In their work the α-diversity features in a metric d_α that they consider. While we only use the metric d'_H in this paper (a metric that does not have diversity in its definition) we use the M-diversity of interval partitions to determine the Markov property of the IP-evolutions. Specifically, the M-diversity is used as an index label to order the components $\mathbf{N}_U$ of $\mathbf{N}_\beta$ to define another random measure in (3.26), and we use this random measure to show a conditional independence result in Proposition 3.19.

Lemma 3.16. *Consider a diffusion that satisfies A1 and A2 of Assumption A with speed measure and scale function that satisfies (3.16), and for which the Pitman-Yor excursion measure ν satisfies (1.9). Let $(\mathbf{N}, \mathbf{X})$ be a stopped scaffolding-and-spindles pair for this diffusion with an a.s. finite T, defined on $(\Omega, \mathcal{A}, \mathbf{P})$. Then, for $y \in \mathbb{R}$ and $t > 0$, $\beta^y := skewer(y, \mathbf{N}, \mathbf{X})$ has an M-diversity at t a.s. We further have for all intervals $U \in \beta^y$ simultaneously that $\mathcal{D}^M_{\beta^y}(U) = k_0 \ell^y(U)$ a.s. (where k_0 is the constant given in Lemma 3.3, where ℓ^y is the local time for the scaffolding $\mathbf{X}$ at level y as defined in (2.1), and where, proceeding in the same manner as we did for diversities, for an interval $U = (a, b)$ we define $\ell^y(U)$ to be $\ell^y(b)$).*

Proof: As ν satisfies (1.9) it does not satisfy (1.6) and so, by the first part of Theorem 1.3, we have that $M(E) = \infty$. We first assume that $T = \tau^y(a)$ for some $a > 0$ where τ^y is the inverse local time at level y, see Section 2.1, and prove the case for general T after. Let $x \in E\backslash\{0\}$. By considering the Poisson random measure given in Proposition 3.6 we see that, for $u \le a$,

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N}|^{\leftarrow}_{[T_y, \tau^y(u)]})} \mathbb{1}_{\{m^0(N') > x\}} \tag{3.23}$$

is a Poisson random variable with mean $k_0 u M(E\backslash[0, x])$ (where T_y is the first hitting time of level y, see Definition 2.2, and F^0 is defined in (3.7)) as the sum is the number of atoms in an interval of the Poisson random measure over the interval $E\backslash[0, x]$. This sum is finite as the mean of the Poisson random variable is finite by $A2$ of Assumption A. Furthermore, for $0 < x' < x$, we have

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N}|^{\leftarrow}_{[T_y, \tau^y(u)]})} \mathbb{1}_{\{x' < m^0(N') \le x\}}$$

is a Poisson random variable with finite mean $k_0 u M((x', x])$ independent of (3.23) as this sum is the number of atoms over the interval $(x', x]$ of the Poisson random measure in Proposition 3.6. Finally

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N}|^{\leftarrow}_{[0, T_y]})} \mathbb{1}_{\{m^0(N') > 0\}}$$

is at most 1 (and so will not impact the diversity limit). As we similarly have independence for the Poisson random variables over disjoint intervals $(x', x]$, and as we have already noted $M(E) = \infty$, then by the strong law of large numbers $\mathcal{D}^M_{\beta^y}(\tau^y(u))$ exists and equals $k_0 u$ for all $0 \le u \le a$. Now, as for all $0 < u \le a$ we have

$$\sum_{(t,N') \text{ atom of } F^0(\mathbf{N}|^{\leftarrow}_{[\tau^y(u-), \tau^y(u)]})} \mathbb{1}_{\{m^0(N') > 0\}}$$

is at most one a.s. we have that $\mathcal{D}^M_{\beta^y}(t)$ exists for all $t \le \tau^y(a)$ where $\mathcal{D}^M_{\beta^y}(t) = \mathcal{D}^M_{\beta^y}(\tau^y(\ell^y(t)))$ which is equal to $k_0 \ell^y(t)$ a.s.

Finally, we note for the case of general finite T that $\mathbf{P}(\bigcup_{a>0}\{T < \tau^y(a)\}) = 1$ and so the result applies for general T. $\square$

Lemma 3.17. *Consider a diffusion that satisfies A1 and A2 of Assumption A with speed measure and scale function that satisfies (3.16), and for which the Pitman-Yor excursion measure ν satisfies*

(1.9). *Let* $\beta \in \mathcal{I}_H(E)$ *and let* $(\mathbf{N}_\beta, \mathbf{X}_\beta)$ *be a scaffolding-and-spindles pair started from* β *for this diffusion as in Definition 3.11.*

Then for $y > 0$ *fixed, the local times of the excursions about level* y *are distinct a.s. in the sense that for* $[a, b]$, $[c, d] \in V^y(\mathbf{N}_\beta)$, $\ell^y(a) \neq \ell^y(c)$ *unless* $[a, b] = [c, d]$, *where* ℓ *denotes the local time of* $\xi(\mathbf{N}_\beta)$ *and where* V^y *is defined in Definition 3.5.*

Furthermore, the crossing widths are summable, and the M*-diversities exist for all* $U \in \beta^y := \mathit{skewer}(y, \mathbf{N}_\beta, \mathbf{X}_\beta)$ *and are distinct in the sense that for* $U \neq V \in \beta^y$ *we have that* $\mathcal{D}^M_{\beta^y}(U) \neq \mathcal{D}^M_{\beta^y}(V)$.

Proof: First note that $\xi_{\mathbf{N}_U}$ is a Lévy process of unbounded variation started from a random level $\zeta(f_U)$, where f_U is the incomplete spindle with initial value $\mathrm{Leb}(U)$, so that $\xi_{\mathbf{N}_U}(0) \sim \mathbb{Q}^0_{\mathrm{Leb}(U)}(\zeta \in \cdot)$. For $y > 0$ at most finitely many of the $\xi_{\mathbf{N}_U}$ from $\mathbf{N}_\beta$ intersect level y. To see this, recall from the proof of Proposition 3.12 where we note that the process $\mathbf{X}_\beta$ with the jumps for the incomplete spindles $(\zeta(f_U), U \in \beta)$ removed produces a Lévy process $\mathbf{X}$ started at 0 and stopped at hitting time of $-\sum_{U\in\beta} \zeta(f_U)$. For every $\xi_{\mathbf{N}_U}$ that intersects level y we either have that $\zeta(f_U)$ has length of at least $y/2$ or we have an upcrossing of $\mathbf{X}$ of size $y/2$. As there are finitely many such $\zeta(f_U)$ and finitely many such upcrossings we have our assertion. From this, the local times claim follows from Millar (1973); see Proposition 5.4 in the Appendix for a full description.

The summability of the crossing widths at level y follows from Theorem 1.4 along with the fact that only finitely many of the $\mathbf{N}_U$ intersect level y. The M-diversities claim follows from the previous two results of this proposition along with Lemma 3.16. □

We now define cutoff functions which are used to define the filtrations from Forman et al. (2020a). The idea is to have a filtration indexed by level $y \in \mathbb{R}$ that can be used to define a Markov property for the skewer process.

$$\begin{aligned}
&\mathrm{cutoff}^{\leq y}_{\xi(N)}(s) := \xi_N(\sup\{t \geq 0 : \sigma^y_N(t) \leq s\}) - \min\{y, 0\} \quad \text{where } \sigma^y_N(t) := \mathrm{Leb}\{u \leq t : \xi_N(u) \leq y\},\\
&\mathrm{cutoff}^{\geq y}_{\xi(N)}(s) := \xi_N(\sup\{t \geq 0 : t - \sigma^y_N(t) \leq s\}) - \max\{y, 0\},
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\mathrm{cutoff}^{\leq y}_N &:= \sum_{(t,f)\in N} \mathbb{1}_{\xi_N(t-)<y<\xi_N(t)} \delta(\sigma^y_N(t), \check{f}^y_t) + \mathbb{1}_{\xi_N(t)\leq y} \delta(\sigma^y_N(t), f_t),\\
\mathrm{cutoff}^{\geq y}_N &:= \sum_{(t,f)\in N} \mathbb{1}_{\xi_N(t-)<y<\xi_N(t)} \delta(t - \sigma^y_N(t), \hat{f}^y_t) + \mathbb{1}_{\xi_N(t-)\geq y} \delta(t - \sigma^y_N(t), f_t),
\end{aligned} \tag{3.25}$$

Definition 3.18 (Filtrations)**.** We define a filtration indexed by level $y \geq 0$, $(\overline{\mathcal{F}}^y, y \geq 0)$, and a filtration, indexed by time, $(\overline{\mathcal{F}}_t, t \geq 0)$.

(1) We define $(\mathcal{F}^y, y \geq 0)$ to be the least right-continuous filtration under which the maps $N \mapsto \mathrm{cutoff}^{\leq y}_{\xi(N)}$ and $N \mapsto \mathrm{cutoff}^{\leq y}_N$ are $\mathcal{F}^y$-measurable for $y \geq 0$.

(2) We define $(\mathcal{F}_t, t \geq 0)$ to be the least right-continuous filtration on $\mathcal{N}^{\mathrm{sp}}$ under which $N \mapsto N|_{[0,t]}$ is $\mathcal{F}_t$-measurable for $t \geq 0$.

We define $(\overline{\mathcal{F}}_t, t \geq 0)$ and $(\overline{\mathcal{F}}^y, y \geq 0)$ to be the $\mathbf{P}$-completions of $(\mathcal{F}_t, t \geq 0)$ and $(\mathcal{F}^y, y \geq 0)$ respectively on $\mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$, where $(\Omega, \mathcal{A}, \mathbf{P})$ is the probability space on which the PRM $\mathbf{N}$ is defined.

For $\beta \in \mathcal{I}_H(E)$ recall $\mathbf{N}_\beta$, the $\mathbf{N}_U$ for $U \in \beta$ and the $\mathbf{P}_\beta$ probability measure from Definition 3.11. We define the collection of excursion intervals about a fixed level $y \geq 0$ for $N \in \mathcal{N}^{\mathrm{sp}}$ by

$$F^{\geq y}_0(N) = \sum_{[a,b]\in V^y_0(N)} \delta\Big(\ell^y_N(a), (N|^{\leftarrow}_{[a,b]})^+\Big).$$

By a slight abuse of notation with the above definition for $\beta \in \mathcal{I}_M(E)$ we define for a measure $\mathbf{N}_\beta$ as in Definition 3.11

$$F_0^{\geq 0}(\mathbf{N}_\beta) := \sum_{U \in \beta} \delta(k_0^{-1}\mathcal{D}_\beta^M(U), \mathbf{N}_U), \tag{3.26}$$

and we set $F_0^{\geq 0}(\mathbf{N}_\beta)$ to be the zero measure when $\beta \in \mathcal{I}_H(E) \backslash \mathcal{I}_M(E)$. We state results from Forman et al. (2020a, Section 5) that generalise to our setting.

Proposition 3.19. *Consider a diffusion that satisfies A1 and A2 of Assumption A with speed measure and scale function that satisfies* (3.16)*, and for which the Pitman-Yor excursion measure ν satisfies* (1.9)*. Let* $(\mathbf{N}, \mathbf{X})$ *be a scaffolding-and-spindles pair where* $\mathbf{N}$ *has intensity Leb $\otimes\, \nu$ stopped at T, an a.s. finite time T with the following properties:*

(1) $S^0 := \ell^0(T)$ is measurable in $\overline{\mathcal{F}}^0$;
(2) $\mathbf{X} < 0$ on the time interval $(\tau^0(S^0-), T)$.

Then for each $y \geq 0$, $F_0^{\geq y}(\mathbf{N})$ is conditionally independent of $\overline{\mathcal{F}}^y$ given $\beta^y :=$ skewer$(\mathbf{N}, \mathbf{X}, y)$, with the regular conditional distribution $\mathbf{P}_{\beta^y}(F_0^{\geq 0} \in \cdot)$.

Proof: The proof of Forman et al. (2020a, Proposition 5.6) applies in our setting as we have the following:

(1) The scaffolding $\mathbf{X}$ is a Lévy process of unbounded variation and therefore we can decompose $\mathbf{N}$ into biclades at every fixed level y of $\mathbf{X}$: $(\mathbf{N}|_I^{\leftarrow}, I \in V_0^y(\mathbf{N}))$.
(2) We have a mid-spindle Markov property (Lemma 3.8).
(3) We have a disintegration for the bi-clade excursion measure ν_{cld} with respect to the crossing width m^0 (Proposition 3.10).
(4) Local times of the excursions at a fixed positive level $y > 0$ are equal to the M-diversities of the interval partitions a.s. (Lemma 3.16). These are measurable functions of the skewer function at that level.

□

This result is sufficient to prove the simple Markov property with respect to the filtration generated by the skewer process. For a probability measure μ on the space of interval partitions $\mathcal{I}_H(E)$, we define the probability measure $\mathbf{P}_\mu$ on the space $\mathcal{N}^{\text{sp}}_{\text{fin}}$ by $\mathbf{P}_\mu(\cdot) := \int \mathbf{P}_\beta(\cdot)\mu(d\beta)$.

Proposition 3.20 (Simple Markov property of the skewer process under $\mathbf{P}_\mu$)**.** *Consider a diffusion that satisfies A1 and A2 of Assumption A and either with speed measure and scale function that satisfies* (3.16) *and for which the Pitman-Yor excursion measure ν satisfies* (1.9)*, or with Pitman-Yor excursion measure that satisfies* (1.6)*. Let μ be a probability measure on $\mathcal{I}_H(E)$. Let $z > 0$ and $0 \leq y_1 < \cdots < y_n$. Let $\eta : \mathcal{N}^{sp}_{fin} \to [0, \infty)$ be $\overline{\mathcal{F}}^z$ measurable, and let $f : (\mathcal{I}_H(E))^n \to [0, \infty)$ be measurable. Then*

$$\mathbf{P}_\mu[\eta f(\mathit{skewer}(z + y_j, \cdot, \xi(\cdot)), j \in [n])] = \int \eta(N)\mathbf{P}_{\mathit{skewer}(z,N)}[f(\mathit{skewer}(y_j, \cdot, \xi(\cdot)), j \in [n])]\mathbf{P}_\mu(dN).$$

Proof: We split the proof into two cases. If the Pitman-Yor excursion measure satisfies (1.6), the SPLP $\mathbf{X}$ is of bounded variation a.s. and therefore there are only finitely many crossings at level y a.s. Therefore the simple Markov property follows in this case as the concatenation of the simple Markov properties for each incomplete spindle and its SPLP extension added to it.

For the other case, the proof follows from the previous results and it uses the same argument as given in Forman et al. (2020a, Corollary 6.7) with the following conditions:

(1) For an interval partition $\beta \in \mathcal{I}_H(E)$, and fixed $y > 0$, the local times of the excursions of $\xi(\mathbf{N}_\beta)$ about level y are distinct a.s. (Lemma 3.17).

(2) We have conditional independence of $F_0^{\geq y}(\mathbf{N})$ and $\overline{\mathcal{F}}^y$ given skewer$(\mathbf{N}, \xi(\mathbf{N}), y)$, for certain stopping times T (Proposition 3.19). In particular, for an interval partition $\beta \in \mathcal{I}_H(E)$, and for $U \in \beta$, it is shown in Forman et al. (2020a, Proposition 5.10) that $F_0^{\geq y}(\mathbf{N}_U)$ is conditionally independent of $\overline{\mathcal{F}}^y$ given skewer$(y, \mathbf{N}_U, \xi_{\mathbf{N}_U})$. As only finitely many of the $\mathbf{N}_U$, $U \in \beta$, will contribute to the skewer process at levels above y, the result follows.

□

4. Continuity and Markov property – Assumptions B and C

In this section we prove Theorems 1.6 and 1.7. The majority of the section is dedicated to the proof of Theorem 1.6 which is completed in Section 4.6. The proof of Theorem 1.7 is simpler and is completed at the end of Section 4.4.

4.1. *Scale function and speed measure under Assumptions B' and C'.* We now define a modification of Assumption B which we will use to prove intermediate results in this paper.

Assumption B'. A diffusion $\mathbf{Y}$ with state space $E \subseteq [0, \infty)$ of the form $E = [0, h]$ or $E = [0, h)$ for some $h \leq \infty$ satisfies *Assumption B'* if it satisfies $A2$, $A3$, $B1$, $B2$, $B3$.

Note that this is Assumption B in the cases where the transformation g is the identity map. We define Assumption C' in the same way for Assumption C.

Assumption C'. A diffusion $\mathbf{Y}$ with state space $E \subseteq [0, \infty)$ of the form $E = [0, h]$ or $E = [0, h)$ for some $h \leq \infty$ satisfies *Assumption C'* if it satisfies $A2$, $A3$, $C1$, $C2$.

We use the definition of the derivative s' of a scale function s from (1.11) and we take b in (1.11) to be ε_0 when under Assumption B' and to be ε_1 when under Assumption C'. We obtain upper and lower bounds for the speed measure and scale function from the corresponding quantities in the squared Bessel case given above. We have under Assumption B', $x \in (0, \varepsilon_0)$

$$\varepsilon_0^{-\alpha^+} x^{\alpha^+} = \exp\left(\int_{z=x}^{\varepsilon_0} \frac{-\alpha^+}{z} dz\right) \leq s'(x) = \exp\left(\int_{z=x}^{\varepsilon_0} \frac{\mu(z)}{2z} dz\right) \leq \varepsilon_0^{-\alpha^-} x^{\alpha^-} \tag{4.1}$$

and under Assumption C', $x \in (0, \varepsilon_1)$

$$\varepsilon_1^{\beta^-} x^{-\beta^-} = \exp\left(\int_{z=x}^{\varepsilon_1} \frac{\beta^-}{z} dz\right) \leq s'(x) = \exp\left(\int_{z=x}^{\varepsilon_1} \frac{\mu(z)}{2z} dz\right) \leq \varepsilon_1^{\beta^+} x^{-\beta^+}. \tag{4.2}$$

For the density of the speed measure this results in under Assumption B', $x \in (0, \varepsilon_0)$

$$\frac{1}{2}\varepsilon_0^{\alpha^-} x^{-1-\alpha^-} \leq m(x) \leq \frac{1}{2}\varepsilon_0^{\alpha^+} x^{-1-\alpha^+} \tag{4.3}$$

and under Assumption C', $x \in (0, \varepsilon_1)$

$$\frac{1}{2}\varepsilon_1^{-\beta^+} x^{-1+\beta^+} \leq m(x) \leq \frac{1}{2}\varepsilon_1^{-\beta^-} x^{-1+\beta^-}. \tag{4.4}$$

For $s(x)$ we have under Assumption B'

$$\frac{\varepsilon_0^{-\alpha^+}}{1+\alpha^+} x^{1+\alpha^+} = \varepsilon_0^{-\alpha^+} \int_{y=0}^{x} y^{\alpha^+} dy \leq s(x) \leq \varepsilon_0^{-\alpha^-} \int_{y=0}^{x} y^{\alpha^-} dy = \frac{\varepsilon_0^{-\alpha^-}}{1+\alpha^-} x^{1+\alpha^-} \tag{4.5}$$

for $x \in (0, \varepsilon_0)$. Finally, under Assumption C', for $s(x)$ we have that

$$\frac{\varepsilon_1^{\beta^-}}{1-\beta^-} x^{1-\beta^-} = \varepsilon_1^{\beta^-} \int_{y=0}^{x} x^{-\beta^-} \leq s(x) \leq \varepsilon_1^{\beta^+} \int_{y=0}^{x} x^{-\beta^+} = \frac{\varepsilon_1^{\beta^+}}{1-\beta^+} x^{1-\beta^+} \tag{4.6}$$

for $x \in (0, \varepsilon_1)$.

Lemma 4.1 (Assumption B and Assumption C imply *A*1)**.** *Consider a diffusion* $\mathbf{Z}$ *with speed measure* $M_{\mathbf{Z}}$ *and scale function* $s_{\mathbf{Z}}$ *under Assumption B or Assumption C. Let* g *be the corresponding spatial transformation and* $\mathbf{Y}$ *the untransformed diffusion under Assumption B' or C' such that* $\mathbf{Z}$ *is equal in distribution to* $g(\mathbf{Y})$*. Then* $\mathbf{Z}$ *satisfies A*1*, i.e. for some (equivalently for all)* $x \in g(E)$ *we have that*

$$\int_{y=0}^{x} s_{\mathbf{Z}}(y) M_{\mathbf{Z}}(dy) < \infty.$$

.

In particular, if the diffusion satisfies Assumption B then $M_{\mathbf{Z}}(g(E))$ *is infinite and the boundary point* 0 *is an exit point for the diffusion. On the other hand, if the diffusion satisfies Assumption C then* $M_{\mathbf{Z}}(g(E))$ *is finite and so* 0 *is a regular point for the diffusion.*

Proof: We first note that without loss of generality we can consider a diffusion under Assumption B' or C'. This is because the integral is invariant under spatial transformation, i.e. we have

$$\int_{y=0}^{x} s_{\mathbf{Z}}(y) M_{\mathbf{Z}}(dy) = \int_{v=0}^{g^{-1}(x)} s_{\mathbf{Y}}(v) M_{\mathbf{Y}}(dv)$$

where $s_{\mathbf{Y}}$ and $M_{\mathbf{Y}}$ are the scale function and speed measure of $\mathbf{Y}$. This follows from Revuz and Yor (1991, Exercise VII.3.18) which states that the scale function $s_{\mathbf{Z}}(x)$ is equal to $s_{\mathbf{Y}}(g^{-1}(x))$ and that the speed measure $M_{\mathbf{Z}}$ is the image of the speed measure $M_{\mathbf{Y}}$ under g.

The remainder of the proof uses the bounds for scale functions and densities of speed measures for diffusions under Assumptions B' or C' from the start of this section. For a diffusion $\mathbf{Y}$ under Assumption B' we have the speed measure $M_{\mathbf{Y}}$ has a density $m_{\mathbf{Y}}$ and by the bounds in (4.3) and (4.5) we have that

$$\int_{v=0}^{\varepsilon_0} s_{\mathbf{Y}}(v) m_{\mathbf{Y}}(v) dv \le \int_{v=0}^{\varepsilon_0} \frac{\varepsilon_0^{\alpha^- + \alpha^+}}{2(1+\alpha^-)} v^{\alpha^- - \alpha^+} dv < \infty.$$

We could do the corresponding argument for a diffusion $\mathbf{Y}$ under Assumption C' with the bounds in (4.4) and (4.6). To show the stronger result under Assumption C that $M_{\mathbf{Z}}(g(E))$ is finite we first note from Revuz and Yor (1991, Exercise VII.3.18) that $M_{\mathbf{Y}}(E)$ is equal to $M_{\mathbf{Z}}(g(E))$. Then, by the bound in (4.4) and we have

$$M_{\mathbf{Z}}(0, g(\varepsilon_1)) = M_{\mathbf{Y}}(0, \varepsilon_1) \le \frac{1}{2} \varepsilon_1^{-\beta^-} \int_{v=0}^{\varepsilon_1} v^{-1+\beta^-} dv = \frac{1}{2\beta^-} < \infty$$

and it then follows by $A2$ that $M_{\mathbf{Z}}(g(E))$ is finite.

To show that under Assumption B that $M_{\mathbf{Z}}(g(E))$ is infinite we note that by the lower bound for $m_{\mathbf{Y}}(x)$ given in (4.3) that

$$M_{\mathbf{Z}}(g(E)) \ge M_{\mathbf{Z}}((0, g(\varepsilon_0)) = M_{\mathbf{Y}}((0, \varepsilon_0)) \ge \frac{1}{2} \varepsilon_0^{\alpha^-} \int_{x=0}^{\varepsilon_0} x^{-1-\alpha^-} dx = \infty.$$

□

For diffusions under Assumption B, if we further assume $A4$, then we can show $B1$, which implies that scaffoldings can be generated by this diffusion and that such scaffoldings are SPLPs of unbounded variation. We include this in Proposition 4.2 below for completeness.

Proposition 4.2. *Consider a diffusion* $\mathbf{Z}$ *that satisfies Assumption B and let* $(\mathbf{N}, \mathbf{X})$ *be a scaffolding-and-spindles pair for this diffusion which is stopped at some a.s. finite random time* T*. Then we have the aggregate health process* $M^0_{\mathbf{N},\mathbf{X}}(T)$ *is finite a.s.*

Furthermore, if instead we have a diffusion $\mathbf{Z}$ *that satisfies all of Assumption A and* $B2$, $B3$ *then the diffusion also satisfies* $B1$.

Proof: For the first result we consider an untransformed diffusion $\mathbf{Y}$ along with a transformation g which gives rise to our diffusion $\mathbf{Z}$ and we note $\mathbf{Y}$ satisfies Assumption B' so we can use the bound in (4.3) for its speed measure $m_{\mathbf{Y}}$. Let $q \in (\alpha^+, q_0)$ and let C_q be the Hölder constant for g as in from Assumption $B4$. Then we have

$$\int_{y=0}^{\varepsilon_0} g(y) m_{\mathbf{Y}}(y) dy \leq \frac{1}{2} C_q \varepsilon_0^{\alpha^+} \int_{y=0}^{\varepsilon_0} y^{-1-\alpha^+ + q} dy = \frac{C_q \varepsilon_0^{\alpha^+}}{2(q-\alpha^+)} < \infty.$$

For the second result we begin by noting that on $(0, \varepsilon_0)$ the scale function for $\mathbf{Y}$, $s_{\mathbf{Y}}$, is such that

$$s'_{\mathbf{Y}}(x) = \exp\left(\int_{z=x}^{\varepsilon_0} \frac{\mu_{\mathbf{Y}}(z)}{2z} dz \right),$$

and so as $\mu_{\mathbf{Y}}$ is negative on $(0, \varepsilon_0)$ by $B3$ we have $s'_{\mathbf{Y}}$ is increasing on $(0, \varepsilon_0)$, and so $s_{\mathbf{Y}}(x) \leq x s'_{\mathbf{Y}}(x)$ for $x \in (0, \varepsilon_0)$. Recall from (1.11) that $m_{\mathbf{Y}}(x) = 2/(\sigma^2_{\mathbf{Y}}(x) s'_{\mathbf{Y}}(x))$. Then, as $\sigma^2_{\mathbf{Y}}(x) = 4x$ by $B2$ we have

$$\int_{x=0}^{\varepsilon_0} m_{\mathbf{Y}}(x) \int_{z=0}^{x} s_{\mathbf{Y}}(z) m_{\mathbf{Y}}(z) dz dx \leq \frac{1}{2} \int_{x=0}^{\varepsilon_0} m_{\mathbf{Y}}(x) \int_{z=0}^{x} dz dx$$

which by the bounds for $m_{\mathbf{Y}}$ given in (4.3) (that follow from $B2$ and $B3$) is less than or equal to

$$\frac{\varepsilon_0^{\alpha^+}}{4} \int_{x=0}^{\varepsilon_0} x^{-\alpha^+} dx$$

which is finite, and so the condition in (1.8) is satisfied. Also, by Lemma 4.1 we have $M_{\mathbf{Y}}(E)$ is infinite and so the condition in (1.7) is not satisfied. Therefore by Theorem 1.3 we have that the Pitman-Yor excursion measure for $\mathbf{Y}$ (equivalently for $\mathbf{Z}$) satisfies (1.9) as required. $\square$

4.2. *Continuity of local time.* In our proof of Theorem 1.6 we use the continuity in space and time of the local time of the scaffolding process $\mathbf{X}$. To show continuity of the local time, we first prove some technical lemmas and we use them to show a necessary and sufficient condition for the continuity of local time on Laplace exponents of SPLPs with infinite variation, from Lambert and Simatos (2015, Lemma 2.2). We note that necessary and sufficient conditions for the space-time continuity of the local time of Lévy processes more generally had been determined by Barlow (1988) but the conditions in Lambert and Simatos (2015) are more accessible for us to verify in our setting. To begin, we define the measure of the overshoot (or equivalently of the undershoot) χ from the Lévy measure $\nu(\zeta \in \cdot)$ as

$$\chi(\cdot) \coloneqq \nu_{\mathrm{cld}}(\zeta(\hat{f}_{T_0^+}) \in \cdot) = \nu_{\mathrm{cld}}(\zeta(\check{f}_{T_0^+}) \in \cdot),$$

where the second equality is an immediate consequence of (3.12). We also have from Lemma 3.3 that

$$\chi(dz) = \nu_{\mathrm{cld}}(\zeta(\check{f}_{T_0^+}) \in dz) = \int_{y=0}^{\infty} \nu(\zeta \in dz + y) dy = \nu(\zeta \in (z, \infty)) dz.$$

In the next lemma we also use the stochastic integral equation representation for the diffusion based on the stochastic differential equation representation given in (1.5). Let $\mathbb{Q}_x$ denote the probability measure for the diffusion under Assumption B'. Recall that a diffusion under Assumption B' satisfies $A1$, see Lemma 4.1, and so we can define its Pitman-Yor excursion measure ν. For $\alpha \in (0,1)$, let ${}^{-2\alpha}\mathbb{Q}_u(\cdot)$ denote the law of a diffusion with infinitesimal drift $\mu(x) = -2\alpha$ and infinitesimal variance $\sigma^2(x) = 4x$ (and we again use a superscript 0 to denote a 0-diffusion and a superscript $\uparrow$ to denote an $\uparrow$-diffusion as before).

Lemma 4.3. *Consider a diffusion that satisfies Assumption B'. Let ν be its Pitman-Yor excursion measure and let χ be defined from ν as above. Then there exist $\varepsilon_2 > 0$ and $k_5 > 0$ such that*

$$\chi((x, \infty)) \geq k_5 x^{-\alpha^-} \text{ for all } x \in (0, \varepsilon_2).$$

Proof: First note that we have

$$\chi((x,\infty)) = \nu_{\mathrm{cld}}(\zeta(\hat{f}_{T_0^+}) > x) = k_0 \int_{u=0}^{\infty} m(u)\mathbb{Q}_u^0(\zeta > x)du, \tag{4.7}$$

where k_0 is the constant given in Lemma 3.3. We can bound the density of the speed measure m below by (4.3). Let $u \in (0,1)$, and let B be a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the stochastic processes Y, Y^+, Y^- on this space by the stochastic integral equations,

$$Y_t = u + \int_{s=0}^{t} \mu(Y_s)ds + 2\int_{s=0}^{t} \sqrt{|Y_s|}dB_s; \quad Y_t^{\pm} = u - 2\alpha^{\pm}t + 2\int_{s=0}^{t}\sqrt{|Y_s^{\pm}|}dB_s, \tag{4.8}$$

until they hit 0 when they are stopped. We have that

$$\mathbb{P}(Y \in \cdot) = \mathbb{Q}_u^0(\cdot); \quad \mathbb{P}(Y^+ \in \cdot) = {}^{-2\alpha^+}\mathbb{Q}_u^0(\cdot); \quad \mathbb{P}(Y^- \in \cdot) = {}^{-2\alpha^-}\mathbb{Q}_u^0(\cdot),$$

and furthermore that

$$Y_t^+ \leq Y_t \leq Y_t^- \text{ for } t \in (0, T_0(Y^+) \wedge T_{\varepsilon_0}(Y^-)) \;\; \mathbb{P}\text{-a.s.},$$

where the ε_0 is as in Assumption B'. Therefore, by using these inequalities along with the marginal distributions of Y, Y^+, Y^- under $\mathbb{P}$, we have that

$$\mathbb{Q}_u^0(\zeta > x) = \mathbb{P}(T_0(Y) > x) \geq \mathbb{P}\Big(T_0(Y^+) > x, \sup_{0 \leq t \leq T_0(Y^-)}\{Y_t^-\} < \varepsilon_0\Big) \geq {}^{-2\alpha^+}\mathbb{Q}_u^0(\zeta > x) - {}^{-2\alpha^-}\mathbb{Q}_u^0(A \geq \varepsilon_0).$$

Göing-Jaeschke and Yor (2003, Equation (13)) state ζ under ${}^{-2\alpha}\mathbb{Q}_u^0$ has distribution given by InverseGamma$(1+\alpha, u/2)$, i.e.

$${}^{-2\alpha}\mathbb{Q}_u^0(\zeta \in dx) = \frac{1}{\Gamma(1+\alpha)}\Big(\frac{u}{2}\Big)^{1+\alpha} x^{-2-\alpha}\exp\Big(-\frac{u}{2x}\Big). \tag{4.9}$$

By the two-sided exit problem and (4.5) we have ${}^{-2\alpha^-}\mathbb{Q}_u^0(A \geq \varepsilon_0) = (u/\varepsilon_0)^{1+\alpha^-}$. For $x < \varepsilon_0/4$, we now state an upper bound for $\chi((x,\infty))$ by using the upper bound for $\mathbb{Q}_u^0(\zeta > x)$ given above along with the upper bound for m given in (4.3). We also upper bound $\mathbb{Q}_u^0(\zeta > x)$ by $\mathbb{Q}_u^0(\zeta \in (x, \varepsilon_0))$ and obtain

$$\chi((x,\infty)) \geq \int_{u=0}^{\varepsilon_0} \frac{k_0 \varepsilon_0^{\alpha^-}}{2} u^{-1-\alpha^-}\Big(\int_{y=x}^{\varepsilon_0/2}\Big(\Big(\frac{1}{\Gamma(1+\alpha^+)}\frac{u^{1+\alpha^+}}{y^{2+\alpha^+}}\exp\Big(-\frac{u}{2y}\Big)dy - \Big(\frac{u}{\varepsilon_0}\Big)^{1+\alpha^-}\Big) \wedge 0\Big)\Big)du$$

which is greater than or equal to

$$\int_{u=0}^{\varepsilon_0} \frac{k_0\varepsilon_0^{\alpha^-}}{2}u^{-1-\alpha^-}\int_{y=x}^{\varepsilon_0/2}\frac{1}{\Gamma(1+\alpha^+)}\frac{u^{1+\alpha^+}}{y^{2+\alpha^+}}\exp\Big(-\frac{u}{2y}\Big)dydu - \int_{u=0}^{\varepsilon_0}\frac{k_0\varepsilon_0^{\alpha^-}}{2}u^{-1-\alpha^-}\int_{y=x}^{\varepsilon_0/2}\Big(\frac{u}{\varepsilon_0}\Big)^{1+\alpha^-}dydu.$$

We consider the two double integrals separately. For the first double integral we may swap the integrals to find and restrict $u \in (0, \varepsilon_0)$ to $u \in (0, 2y)$ as $y \in (x, \varepsilon_0/2)$. We get a lower bound of

$$\frac{k_0\varepsilon_0^{\alpha^-}}{2\Gamma(1+\alpha^+)}\int_{y=x}^{\varepsilon_0/2} y^{-2-\alpha^+}\int_{u=0}^{2y} u^{\alpha^+ - \alpha^-}\exp\Big(-\frac{u}{2y}\Big)dudy$$

and we can bound $\exp(-u/2y)$ from below by $1/e$ to get a lower bound of

$$\frac{k_0\varepsilon_0^{\alpha^-}}{2\Gamma(1+\alpha^+)e}\int_{y=x}^{\varepsilon_0/2} y^{-2-\alpha^+}\int_{u=0}^{2y} u^{\alpha^+-\alpha^-}dudy = \frac{2^{\alpha^+-\alpha^-}k_0\varepsilon_0^{\alpha^-}}{\Gamma(1+\alpha^+)e(1-\alpha^+ + \alpha^-)}\int_{y=x}^{\varepsilon_0/2} y^{-1-\alpha^-}dy$$

which is equal to

$$\frac{2^{\alpha^+-\alpha^-}k_0\varepsilon_0^{\alpha^-}}{\Gamma(1+\alpha^+)e(1-\alpha^++\alpha^-)\alpha^-}\Big(x^{-\alpha^-} - \Big(\frac{\varepsilon_0}{2}\Big)^{-\alpha^-}\Big).$$

For the second double integral we see the upper bound given by

$$\int_{u=0}^{\varepsilon_0} \frac{k_0\varepsilon_0^{\alpha^-}}{2} u^{-1-\alpha^-} \int_{y=0}^{\varepsilon_0/2} \Big(\frac{u}{\varepsilon_0}\Big)^{1+\alpha^-} dydu = \frac{k_0}{2\varepsilon_0}\int_{u=0}^{\varepsilon_0}\int_{y=0}^{\varepsilon_0/2} dydu = \frac{k_0\varepsilon_0}{4}.$$

Therefore the claim in the lemma is satisfied for some k_5 and ε_2. ☐

Lemma 4.4. *In the same setting as in Lemma 4.3, we have that, for some positive constant k_6,*

$$\chi((x,\infty)) \le k_6(x^{-\alpha^+}+1) \text{ for all } x\in(0,\infty),$$

Proof: We use the formulation of $\chi((x,\infty))$ as given in (4.7), as well as the coupled processes Y, Y^+ and Y^- as in Lemma 4.3. We note that

$$\begin{aligned}
\mathbb{Q}_u^0(\zeta > x) &= \mathbb{P}(T_0(Y) > x)\\
&= \mathbb{P}\Big(T_0(Y)>x, \sup_{0\le t\le T_0(Y^-)}\{Y_t^-\}\ge\varepsilon_0\Big) + \mathbb{P}\Big(T_0(Y)>x, \sup_{0\le t\le T_0(Y^-)}\{Y_t^-\}<\varepsilon_0\Big)\\
&\le \mathbb{P}(A(Y^-)\ge\varepsilon_0) + \mathbb{P}(\zeta(Y^-)>x) = {}^{-2\alpha^-}\mathbb{Q}_u^0(A\ge\varepsilon_0) + {}^{-2\alpha^-}\mathbb{Q}_u^0(\zeta>x),
\end{aligned}$$

where the ε_0 is as in Assumption B'. Therefore we have that $\chi((x,\infty))$ is less than or equal to

$$\frac{k_0\varepsilon_0^{\alpha^+}}{2}\int_{u=0}^{\varepsilon_0} u^{-1-\alpha^+}\Big({}^{-2\alpha^-}\mathbb{Q}_u^0(A\ge\varepsilon_0) + {}^{-2\alpha^-}\mathbb{Q}_u^0(\zeta>x)\Big)du + k_0\int_{u=\varepsilon_0}^{h}\mathbb{Q}_u^0(\zeta>x)m(u)du$$

and we bound this above by

$$\frac{k_0\varepsilon_0^{\alpha^+}}{2}\int_{u=0}^{\varepsilon_0} u^{-1-\alpha^+}\Big(u^{1+\alpha^-} + \int_{y=x}^{\infty}\frac{1}{\Gamma(1+\alpha^-)}\frac{u^{1+\alpha^-}}{y^{2+\alpha^-}}\exp\Big(-\frac{u}{2y}\Big)dy\Big)du + \frac{k_0}{2}M(E\backslash[0,\varepsilon_0]).$$

We deal with the two parts of the integral separately. We have

$$\begin{aligned}
&\frac{k_0\varepsilon_0^{\alpha^+}}{2}\int_{u=0}^{\varepsilon_0} u^{-1-\alpha^+}\int_{y=x}^{\infty}\frac{1}{\Gamma(1+\alpha^-)}\frac{u^{1+\alpha^-}}{y^{2+\alpha^-}}\exp\Big(-\frac{u}{2y}\Big)dydu\\
&\qquad\le \frac{k_0\varepsilon_0^{\alpha^+}}{2\Gamma(1+\alpha^-)}\int_{y=x}^{\infty} y^{-2-\alpha^-}\int_{u=0}^{\varepsilon_0} u^{\alpha^--\alpha^+}dudy
\end{aligned}$$

which equals

$$\frac{k_0\varepsilon_0^{1+\alpha^-}}{2\Gamma(1+\alpha^-)(1+\alpha^--\alpha^+)}\int_{y=x}^{\infty} y^{-1-\alpha^+}dy = \frac{k_0\varepsilon_0^{1+\alpha^-}}{2\Gamma(1+\alpha^-)(1+\alpha^--\alpha^+)\alpha^+}x^{-\alpha^+}.$$

For the second part of the integral we have that

$$\frac{k_0\varepsilon_0^{\alpha^+}}{2}\int_{u=0}^{\varepsilon_0} u^{\alpha^--\alpha^+}du = \frac{k_0\varepsilon_0^{1+\alpha^-}}{2(1+\alpha^--\alpha^+)},$$

and so

$$\chi((x,\infty)) \le \frac{k_0\varepsilon_0^{1+\alpha^-}}{2\Gamma(1+\alpha^-)(1+\alpha^--\alpha^+)\alpha^+}x^{-\alpha^+} + \frac{k_0\varepsilon_0^{1+\alpha^-}}{2(1+\alpha^--\alpha^+)} + \frac{k_0}{2}M(E\backslash[0,\varepsilon_0])$$

and the result follows. ☐

Lemma 4.5. *Consider a diffusion that satisfies Assumption B', and let ψ be the Laplace exponent for the scaffolding process* $\mathbf{X}$*, see (2.3). Then we have for $\lambda\in(0,\infty)$ that*

$$\psi(\lambda) = \int_{x=0}^{\infty}\lambda^2 e^{-\lambda x}\chi((x,\infty))dx. \tag{4.10}$$

Proof: This follows by integration-by-parts applied twice to $\psi(\lambda) = \int_{x=0}^{\infty}(e^{-\lambda x} - 1 + \lambda x)\Lambda(dx)$, where we recall that $\Lambda(\cdot) = \nu(\zeta \in \cdot)$, and we require the upper bound for $\chi((x,\infty))$ in Lemma 4.4 to do this. $\square$

We now use a necessary and sufficient condition from Lambert and Simatos (2015) to determine that the local time is a.s. jointly continuous in time and space.

Proposition 4.6. *Consider a diffusion* $\mathbf{Z}$ *under Assumption B and let* $(\mathbf{N}, \mathbf{X})$ *be a scaffolding-and-spindles construction. Then the local time* $(\ell_t^u : t, u \in \mathbb{R})$ *is jointly continuous a.s.*

Proof: Without loss of generality we can assume our diffusion satisfies Assumption B'. This is because, as discussed in the introduction after Theorem 1.3, the lifetime under the excursion measure is not affected by spatial transformations, i.e. $\nu_{\mathbf{Z}}(\zeta \in \cdot) = \nu_{\mathbf{Y}}(\zeta \in \cdot)$ where $\mathbf{Y}$ is an untransformed transformation that satisfies Assumption B'.

By Lambert and Simatos (2015, Lemma 2.2), as $\mathbf{X}$ is a SPLP that satisfies (1.9), a necessary and sufficient condition for the local time to be jointly continuous a.s. is

$$\int_c^{\infty} \frac{d\lambda}{\psi(\lambda)\sqrt{\log(\lambda)}} < \infty$$

for any (equivalently for all) $c > 0$. Now, with the lower bound for $\chi((x,\infty))$ from Lemma 4.3 we have that

$$\psi(\lambda) \geq k_5 \int_{x=0}^{\delta} \lambda^2 e^{-\lambda x} x^{-\alpha^-}\, dx.$$

We let $\lambda > 1/\delta$ and consider the Taylor expansion of $e^{-\lambda x}$ to obtain that

$$\psi(\lambda) \geq k_5 \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^{n+2}}{n!} \int_{x=0}^{1/\lambda} x^{n-\alpha^-}\, dx = k_5 \Big(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-\alpha^-)} \Big) \lambda^{1+\alpha^-}$$

and therefore

$$\int_{\lambda=1/\delta}^{\infty} \frac{d\lambda}{\psi(\lambda)\sqrt{\log(\lambda)}} \leq \Big(k_5 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1-\alpha^-)} \Big)^{-1} \int_{\lambda=1/\delta}^{\infty} \frac{d\lambda}{\lambda^{1+\alpha^-}\sqrt{\log(\lambda)}} < \infty.$$

$\square$

4.3. *Some further calculations.* In this section we give upper bounds of certain probabilities and expectations for diffusions with a state space $E = [0,\infty)$ and we use these results in our proof of Theorem 1.6. In Section 4.3.1 we prove results for diffusions conditioned to not hit 0 and we reference the literature on quasi-stationary distributions, see e.g. Méléard and Villemonais (2012). In Section 4.3.2 we consider squared Bessel cases of negative dimension.

4.3.1. *Expectation for a diffusion conditioned not to hit level 0.* In this section we calculate conditional moments for the deviation of a diffusion with state space $E \subseteq [0,\infty)$. As before, for $s > 0$, $\mathbb{Q}_s$ denotes the law of the diffusion started from s and $\mathbb{Q}_s^0$ denotes the law of the diffusion started from s and stopped at 0.

Lemma 4.7. *Consider a diffusion that satisfies A1 of Assumption A. Then for* r, $s > 0$, $0 < a < h$ *and* $0 < t \leq r$ *we have*

$$\mathbb{Q}_s^0(Y_t \in E\backslash[0,a)) \leq \mathbb{Q}_s^0(Y_t \in E\backslash[0,a) | \zeta > r).$$

Proof: Let $t \in (0,r)$. We first note that

$$b := \mathbb{Q}_s^0(\zeta > r | Y_t \in [0,a)) \leq \mathbb{Q}_a^0(\zeta > r - t) \leq \mathbb{Q}_s^0(\zeta > r | Y_t \in E\backslash[0,a)) =: c,$$

by the Markov property and almost-sure continuity of the sample paths. Let $p = \mathbb{Q}^0_s(Y_t \in E\backslash[0,a))$. We then have that

$$\mathbb{Q}^0_s(Y_t \in E\backslash[0,a))\mathbb{Q}^0_s(\zeta > r) = p(cp + b(1-p)) \leq cp = \mathbb{Q}^0_s(Y_t \in E\backslash[0,a), \zeta > r)$$

which implies that

$$\mathbb{Q}^0_s(Y_t \in E\backslash[0,a)) \leq \mathbb{Q}^0_s(Y_t \in E\backslash[0,a)|\zeta > r).$$

Taking $t \uparrow r$ we have by a.s. continuity of the sample paths that

$$\mathbb{Q}^0_s(Y_r \in E\backslash[0,a)) \leq \mathbb{Q}^0_s(Y_r \in E\backslash[0,a)|\zeta > r).$$

□

Lemma 4.8. *In the same setting as Lemma 4.7, for r, s, $u > 0$ and $0 < a < h$ we have*

$$\mathbb{Q}^0_s(Y_r \in E\backslash[0,a)|\zeta > r) \leq \mathbb{Q}^0_s(Y_r \in E\backslash[0,a)|\zeta > r+u).$$

Proof: In a very similar fashion to the previous lemma we first note that

$$b_u := \mathbb{Q}^0_s(\zeta > r+u|Y_r \in (0,a)) \leq \mathbb{Q}^0_a(\zeta > u) \leq \mathbb{Q}^0_s(\zeta > r+u|Y_r \in E\backslash[0,a)) =: c_u,$$

and let $p := \mathbb{Q}^0_s(Y_r \in (a,\infty)) \leq \mathbb{Q}^0_s(\zeta > y) =: q$. Then we have that

$$\mathbb{Q}^0_s(\zeta > r+u, Y_r \in E\backslash[0,a)) = c_u p.$$

Furthermore note that

$$\mathbb{Q}^0_s(\zeta > r+u|\zeta > r) = c_u\frac{p}{q} + b_u\frac{q-p}{q} \leq c_u, \quad \mathbb{Q}^0_s(Y_r \in E\backslash[0,a)|\zeta > r) = \frac{p}{q},$$

and therefore

$$\mathbb{Q}^0_s(Y_r \in E\backslash[0,a), \zeta > r+u|\zeta > r) = \frac{pc}{q} \geq \mathbb{Q}^0_s(Y_r \in E\backslash[0,a)|\zeta > r)\mathbb{Q}^0_s(\zeta > r+u|\zeta > r),$$

and the claim follows. □

This final lemma, Lemma 4.10, uses the following convergence result for the ↑-diffusion from Méléard and Villemonais (2012, Theorem 27).

Lemma 4.9. *In the same setting as the previous lemmas, for r, s, $a > 0$ we have*

$$\lim_{u\to\infty} \mathbb{Q}^0_s(Y_r \in E\backslash[0,a)|\zeta > r+u) = \mathbb{Q}^\uparrow_s(Y_r \in E\backslash[0,a)).$$

Lemma 4.10. *Let $F : [0,\infty) \to \mathbb{R}$ be an absolutely continuous increasing function. Then, in the same setting as the previous lemmas, for r, $s > 0$ we have*

$$\mathbb{E}_{\mathbb{Q}^0_s}(F(Y_r)) \leq \mathbb{E}_{\mathbb{Q}^0_s}(F(Y_r)|\zeta > r) \leq \mathbb{E}_{\mathbb{Q}^\uparrow_s}(F(Y_r)).$$

Proof: We first note F has derivative F' Lebesgue-almost everywhere, see e.g. Kallenberg (2021, Theorem 2.15). We therefore have that

$$\mathbb{E}_{\mathbb{Q}^0_s}(F(Y_r)) = \int_{z=0}^{\infty} \mathbb{Q}^0_s(Y_r \in E\backslash[0,z))F'(z)dz,$$

and similarly for $\mathbb{E}_{\mathbb{Q}^0_s}(F(Y_r)|\zeta > r)$ and $\mathbb{E}_{\mathbb{Q}^\uparrow_s}(F(Y_r))$. By Lemmas 4.7, 4.8 and 4.9 we have that for any $a \in (0,\infty)$

$$\begin{aligned}\mathbb{Q}^0_s(Y_r \in E\backslash[0,a)) \leq \mathbb{Q}^0_s(Y_r \in E\backslash[0,a)|\zeta > r) &\leq \lim_{u\to\infty} \mathbb{Q}^0_s(Y_r \in E\backslash[0,a)|\zeta > r+u) \\ &= \mathbb{Q}^\uparrow_s(Y_r \in E\backslash[0,a)),\end{aligned}$$

and the result follows. □

4.3.2. *Certain calculations concerning squared Bessel processes with negative dimension.* In this section we work with squared Bessel processes with dimension -2α, where $\alpha \in (0,1)$. We make a slight abuse of notation and write ${}^{-2\alpha}\mathbb{Q}^0_s$ in place of $\mathbb{E}_{{}^{-2\alpha}\mathbb{Q}^0_s}$ for readability.

Lemma 4.11. *Let $r,\ q > 0$, $0 < \alpha, s < 1$ and $n \in \mathbb{N}$. Then we have*

$$ {}^{-2\alpha}\mathbb{Q}^0_s(|Y_r - s|^{qn}|\zeta > r) \leq {}^{-2\alpha}\mathbb{Q}^0_s(|Y_r - s|^{qn}) + {}^{4+2\alpha}\mathbb{Q}^0_s(|Y_r - s|^{qn}). $$

Proof: We note that the functions

$$ |z-s|^{qn}\mathbb{1}_{z\geq s}; \qquad -|z-s|^{qn}\mathbb{1}_{z\leq s}, $$

are absolutely continuous increasing functions that are Lebesgue differentiable almost everywhere. Therefore, by Lemma 4.10 we have that

$$ \begin{aligned} {}^{-2\alpha}\mathbb{Q}^0_s(|Y_r - s|^{qn}|\zeta > r) &\leq {}^{-2\alpha}\mathbb{Q}^{\uparrow}_s(|Y_r - s|^{qn}\mathbb{1}_{Y_r\geq s}|\zeta > r) - {}^{-2\alpha}\mathbb{Q}^0_s(-|Y_r - s|^{qn}\mathbb{1}_{Y_r\leq s}|\zeta > r) \\ &\leq {}^{4+2\alpha}\mathbb{Q}^0_s(|Y_r - s|^{qn}) + {}^{-2\alpha}\mathbb{Q}^0_s(|Y_r - s|^{qn}). \end{aligned} $$

□

We establish upper and lower bounds for the probability that the lifetime ζ is greater than r under ${}^{-2\alpha}\mathbb{Q}^0_s$.

Lemma 4.12. *For $r,\ s > 0$, $0 < \alpha < 1$, we have*

$$ {}^{-2\alpha}\mathbb{Q}^0_s(\zeta \leq r) \leq \frac{\Gamma(1+\alpha+v)2^v}{\Gamma(1+\alpha)}\frac{r^v}{s^v}; \qquad {}^{-2\alpha}\mathbb{Q}^0_s(\zeta > r) \leq \frac{\Gamma(1+\alpha-v)}{\Gamma(1+\alpha)2^v}\frac{s^v}{r^v} \tag{4.11} $$

for $v > 0$ and for $0 < v < 1+\alpha$ respectively.

Proof: For the first inequality let $v > 0$ and apply Markov's inequality and (4.9) to see that

$$ {}^{-2\alpha}\mathbb{Q}^0_s(\zeta \leq r) = {}^{-2\alpha}\mathbb{Q}^0_s\Big(\frac{1}{\zeta^v} \geq \frac{1}{r^v}\Big) \leq {}^{-2\alpha}\mathbb{Q}^0_s\Big(\frac{1}{\zeta^v}\Big)r^v = r^v\int_{x=0}^{\infty}\frac{s^{1+\alpha}}{2^{1+\alpha}\Gamma(1+\alpha)}\frac{1}{x^{2+\alpha+v}}\exp\Big(-\frac{s}{2x}\Big)\,dx $$

and we integrate out the probability density function of an InverseGamma$(1+\alpha+v, s/2)$ to obtain

$$ \frac{\Gamma(1+\alpha+v)2^v}{\Gamma(1+\alpha)}\frac{r^v}{s^v}\int_{x=0}^{\infty}\frac{s^{1+\alpha+v}}{2^{1+\alpha+v}\Gamma(1+\alpha+v)}\frac{1}{x^{2+\alpha+v}}\exp\Big(-\frac{s}{2x}\Big)\,dx = \frac{\Gamma(1+\alpha+v)2^v}{\Gamma(1+\alpha)}\frac{r^v}{s^v}. $$

For the second inequality let $0 < v < 1+\alpha$, we use the same ideas:

$$ {}^{-2\alpha}\mathbb{Q}^0_s(\zeta > r) \leq \frac{1}{r^v}{}^{-2\alpha}\mathbb{Q}^0_s(\zeta^v) = \frac{1}{r^v}\int_{x=0}^{\infty}\frac{s^{1+\alpha}}{2^{1+\alpha}\Gamma(1+\alpha)}\frac{1}{x^{2+\alpha-v}}\exp\Big(-\frac{s}{2x}\Big)\,dx = \frac{\Gamma(1+\alpha-v)}{\Gamma(1+\alpha)2^v}\frac{s^v}{r^v}, $$

by integrating out the probability distribution function of an InverseGamma$(1+\alpha-v, s/2)$. □

We also establish a further lemma with some useful probability bounds for squared Bessel processes.

Lemma 4.13. *Let $\gamma \in (-\infty, 0) \cup [2,\infty)$, and $d > 0$. Then there exists a constant $C = C(\gamma, d)$ such that for any $r,\ s > 0$, we have*

$$ {}^{\gamma}\mathbb{Q}^0_s(|Y_r - s|^d) \leq Cr^{d/2}(r^{d/2} \vee s^{d/2}). $$

Proof: We use Forman et al. (2018, Equation (19)) to note that for $d \geq 2$ and $\gamma \in \mathbb{R}$ that we have

$$ \begin{aligned} \Big({}^{\gamma}\mathbb{Q}_s(|Y_r - s|^d)\Big)^{(1/d)} &\leq |\gamma|r + 2\sqrt{(d-1)(|s|r + r^2(|\gamma| + (d-2)))} \\ &\leq \Big(|\gamma| + 2\sqrt{(d-1)(1 + (|\gamma| + (d-2)))}\Big)r^{1/2}\Big(r^{1/2} \vee s^{1/2}\Big) \end{aligned} \tag{4.12} $$

Squared Bessel processes with negative dimension never return to the positive half-line after the first hitting time of 0 a.s. Also, squared Bessel processes with dimension greater than or equal to 2 that start positive never hit 0 a.s. Therefore, for $\gamma \in (-\infty, 0) \cup [2, \infty)$ we have that

$$\left({}^{\gamma}\mathbb{Q}_s^0(|Y_r - s|^d)\right)^{(1/d)} \leq \left(|\gamma| + 2\sqrt{(d-1)(1+(|\gamma|+(d-2)))}\right) r^{1/2}\left(r^{1/2} \vee s^{1/2}\right)$$

for $d \geq 2, \gamma \in (-\infty, 0) \cup [2, \infty)$ by (4.12). Now let $d < 2$. Then $4/d > 2$ and so we can apply Jensen's inequality to obtain that

$${}^{\gamma}\mathbb{Q}_s^0(|Y_r - s|^d) \leq ({}^{\gamma}\mathbb{Q}_s^0(|Y_r - s|^4))^{d/4} \leq C(\gamma, 4)^{d/4} r^{d/2}\left(r^{d/2} \vee s^{d/2}\right)$$

for $0 < d < 2, \gamma \in (-\infty, 0) \cup [2, \infty)$. □

Lemma 4.14. *Let $n \in \mathbb{N}$, $0 < \alpha, \gamma < 1$, $0 < \kappa < 1 - \alpha$, and $\alpha + \kappa < q < 1$. Then there exists a constant $C = C(n, q, \alpha, \gamma, \kappa) > 0$ such that for $0 < r \leq 1$ we have*

$$\int_{s=0}^{1} {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}) s^{-1-\alpha} ds \leq C r^{n((q-\alpha-\kappa)\wedge q/2)}.$$

Proof: We can apply Lemma 4.13 as

$${}^{-2\gamma}\mathbb{Q}_s^0(|Y_r^q - s^q|^n) \leq {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}) \text{ for } 0 < q < 1, n \geq 1.$$

We split the integral. We first consider the expectation on the event $\{\zeta \leq r\}$ and use the upper bound for ${}^{-2\gamma}\mathbb{Q}_s^0(\zeta \leq r)$ in (4.11) with $v = qn - \alpha - \kappa$:

$$\begin{aligned}\int_{s=0}^{1} {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}\mathbb{1}_{\zeta \leq r}) s^{-1-\alpha} ds &\leq \frac{\Gamma(1+\gamma+qn-\alpha-\kappa)2^{qn-\alpha-\kappa}}{\Gamma(1+\alpha)} \int_{s=0}^{1} s^{-1-\alpha} s^{qn} \frac{r^{qn-\alpha-\kappa}}{s^{qn-\alpha-\kappa}} ds \\ &= \frac{\Gamma(1+\gamma+qn-\alpha-\kappa)2^{qn-\alpha-\kappa}}{\Gamma(1+\alpha)\kappa} r^{qn-\alpha-\kappa}.\end{aligned}$$

We now consider the integral on the event $\{\zeta > r\}$ on the range $0 < s < r$. By Lemmas 4.11 and 4.13 we have that

$${}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn} | \zeta > r) \leq (C(-2\gamma, qn) + C(4+2\gamma, qn)) r^{qn/2}(s^{qn/2} \vee r^{qn/2}),$$

where $C(-2\gamma, qn)$ and $C(4+2\gamma, qn)$ are from Lemma 4.13. Let $K = (C(-2\gamma, qn) + C(4+2\gamma, qn))$. We use (4.11) with $v = \alpha + \kappa$ to bound above ${}^{-2\gamma}\mathbb{Q}_s^0(\zeta > r)$:

$$\begin{aligned}\int_{s=0}^{r\wedge 1} {}^{-2\gamma}\mathbb{Q}_s^0[|Y_r - s|^{qn}\mathbb{1}_{\zeta > r}] s^{-1-\alpha} ds &\leq \frac{K\Gamma(1+\gamma-\alpha-\kappa)}{\Gamma(1+\gamma)2^{\alpha+\kappa}} \int_{s=0}^{r\wedge 1} \frac{s^{\alpha+\kappa}}{r^{\alpha+\kappa}} s^{-1-\alpha} r^{qn} ds \\ &\leq \frac{K\Gamma(1+\gamma-\alpha-\kappa)}{\Gamma(1+\gamma)2^{\alpha+\kappa}} r^{qn-\alpha-\kappa} \int_{s=0}^{1} s^{-1+\kappa} ds \\ &= \frac{K\Gamma(1+\gamma-\alpha-\kappa)}{\Gamma(1+\gamma)2^{\alpha+\kappa}\kappa} r^{qn-\alpha}.\end{aligned}$$

We note that

$$r^{qn-\alpha-\kappa} \leq r^{n((q-\alpha-\kappa)\wedge q/2)} \text{ for } 0 < r < 1.$$

Finally, we consider the integral on the event $\{\zeta > r\}$ over the range $r \wedge 1 \leq s \leq 1$. We split this calculation into two parts to deal with the cases $q/2 \geq q - \alpha - \kappa$ and $q/2 < q - \alpha - \kappa$ separately. In the case where $q/2 \geq q - \alpha - \kappa$ we use the bound in (4.11) for ${}^{-2\gamma}\mathbb{Q}_s^0(\zeta > r)$ with

$0 < v = \alpha + \kappa - q/2 < 1 + \alpha$:

$$\begin{aligned}\int_{s=r\wedge 1}^{1} {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}\mathbb{1}_{\zeta > r})s^{-1-\alpha}ds &\leq \frac{K\Gamma(1+\gamma+q/2-\alpha-\kappa)}{\Gamma(1+\alpha)2^{\alpha+\kappa-q/2}}\int_{s=r\wedge 1}^{1}\frac{s^{\alpha+\kappa-q/2}}{r^{\alpha+\kappa-q/2}}s^{-1-\alpha}r^{qn/2}s^{qn/2}ds\\ &\leq \frac{K\Gamma(1+\gamma+q/2-\alpha-\kappa)}{\Gamma(1+\alpha)2^{\alpha+\kappa-q/2}}r^{qn/2+q/2-\alpha-\kappa}\int_{s=0}^{1}s^{-1+\kappa}ds\\ &\leq \frac{K\Gamma(1+\gamma+q/2-\alpha-\kappa)}{\Gamma(1+\alpha)2^{\alpha+\kappa-q/2}\kappa}r^{n(q-\alpha-\kappa)} \text{ for } q/2 \geq q-\alpha-\kappa.\end{aligned}$$

For the case where $q/2 < q - \alpha - \kappa$ we do not use (4.11) to bound ${}^{-2\gamma}\mathbb{Q}_s^0(\zeta > r)$ (we instead use the trivial bound of 1):

$$\begin{aligned}\int_{s=r\wedge 1}^{1} {}^{-2\gamma}\mathbb{Q}_s^0(|Y_r - s|^{qn}\mathbb{1}_{\zeta > r})s^{-1-\alpha}ds &\leq K\int_{s=r\wedge 1}^{1}s^{-1-\alpha}r^{qn/2}s^{qn/2}ds\\ &= K\int_{s=r\wedge 1}^{1}s^{-1-\alpha+qn/2}ds r^{qn/2} \leq Kr^{qn/2}\int_{s=0}^{1}s^{-1+\kappa}ds = \frac{K}{\kappa}r^{qn/2}.\end{aligned}$$

□

4.4. *Continuity of the skewer process.* In this section we prove the continuity results for Theorems 1.6 and 1.7. We begin with some definitions. Consider a diffusion under Assumption B' with Pitman-Yor excursion measure ν, and a transformation $g : E \to g(E)$ (where $E = [0, h)$ or $E = [0, h]$ for some $h \leq \infty$) that satisfies the conditions stated in Assumption B. Without loss of generality we can take the ε_0 in Assumption B to be less than 1 and we do this here. As before, let $\mathbf{N}$ be a stopped PRM with intensity $\mathrm{Leb} \otimes \nu$, where the stopping is at an a.s. finite random variable T. Finally, for M, $\varepsilon_0 > 0$ we define $\mathbf{N}_{M,\varepsilon_0}$ by

$$\mathbf{N}_{M,\varepsilon_0} := \mathbb{1}_{\sup_{u\in\mathbb{R}}\ell^u(T)<M}\sum_{\substack{(t,f)\text{ atom of }\mathbf{N}:\\ A(f)<\varepsilon_0}}\delta_{(t,f)}. \tag{4.13}$$

Lemma 4.15. *Let x, $y \in \mathbb{R}$, $z \in \{x, y\}$, and let $k \in \mathbb{N}$. Let π_k denote the set of compositions of k, i.e. $\pi_k := \{(n_1, \dots, n_a)|a, n_i \in \mathbb{N}\backslash\{0\}, \sum_i n_i = k\}$. Then we have that*

$$\begin{aligned}&\mathbf{E}\Bigg(\Bigg(\sum_{\substack{(t,f)\text{ atom of }\mathbf{N}_{M,\varepsilon_0}:\\ \mathbf{X}_{t-}<z<\mathbf{X}_t,\ell^z(t)>0}}\Big|g\big(f(y-\mathbf{X}_{t-})\big)-g\big(f(x-\mathbf{X}_{t-})\big)\Big|\Bigg)^k\Bigg)\\ &\quad\leq C_q^k\sum_{\pi_k}(k_0M)^a\prod_{i=1}^{a}\int_{s=0}^{\varepsilon_0}m(s)\Big({}^{-2\alpha^+}\mathbb{Q}_s^0(|Y_{|x-y|}-s|^{qn_i})+{}^{-2\alpha^-}\mathbb{Q}_s^0(|Y_{|x-y|}-s|^{qn_i})\Big)ds,\end{aligned} \tag{4.14}$$

where C_q is the Hölder constant for g as in Assumption B and k_0 is as in Proposition 3.6.

Proof: The bound is trivial for $x = y$ so assume $x \neq y$. First recall from Assumption B that there exists $C_q > 0$ such that $|g(a) - g(b)| \leq C_q|a - b|^q$ for $0 < a$, $b < \varepsilon_0$. We can bound the term on the left-hand-side of (4.14) by

$$C_q^k\mathbf{E}\Bigg(\Bigg(\sum_{\substack{(t,f)\text{ atom of }\mathbf{N}:A(f)<\varepsilon_0\\ 0<\ell^z(t)<M,\mathbf{X}_{t-}<z<\mathbf{X}_t}}\Big|f(y-\mathbf{X}_{t-})-f(x-\mathbf{X}_{t-})\Big|^q\Bigg)^k\Bigg). \tag{4.15}$$

As T^z is a stopping time for $\mathbf{X}$, we have by Lemma 3.9 that the process $\xi(\mathbf{N}|^{\leftarrow}_{[T^z,\infty)})$ can be split into excursions about level 0, and all these excursions a.s. include a single jump across level 0. We have a.s. a one-to-one correspondence between crossing across a level and excursions about the level. Therefore the sum in (4.15) less than or equal to a sum indexed by the Poisson point process of crossing widths at level z and is summing functionals of the clades or anti-clades associated with each of these crossing widths. Specifically, let $\mathbf{N}'$ be the unstopped Poisson random measure, then

$$\mathbf{J} := \sum_{\substack{t:0<\ell^z(t)<M,\\ A(f)<\varepsilon_0}} \delta_{(\ell^z(t), \mathbf{N}'|^{\leftarrow}_{(\ell^z(t)-,\ell^z(t))})}$$

is distributed as a Poisson random measure with intensity

$$\mathrm{Leb}|_{[0,M]} \otimes k_0 \int_{s=0}^{\varepsilon_0} m(s)\nu_{\mathrm{cld}}(\mathbb{1}_{A(f_{T_0^+})<\varepsilon_0} N \in \cdot | f_{T_0^+}(-\xi_N(T_0^+-)) = s)ds,$$

where k_0 is as in Proposition 3.6 and ε_0 is as in Assumption B. Let $w \in \{x,y\}\backslash\{z\}$. Then, we define

$$F(N) := |f_{T_0^+}(w - z - \xi_N(T_0^+-)) - f_{T_0^+}(-\xi_N(T_0^+-))|^q \text{ for } N \in \mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}}$$

and therefore the expectation in (4.15) is bounded above by

$$\mathbf{E}\Bigg(\Bigg(\sum_{\substack{(s,N) \text{ atom}\\ \text{of } \mathbf{J}}} F(N)\Bigg)^k\Bigg) = \sum_{\pi_k} \mathbf{E}\Bigg(\sum_{\substack{(s_1,N_1),\dots,(s_a,N_a)\\ \text{distinct atoms of } \mathbf{J}}} \prod_{i=1}^a F(N_i)^{n_i}\Bigg).$$

We apply Campbell's formula for Poisson random measures, see Kingman (1993, Equation (3.14)), to obtain, for a composition $(n_1,\dots,n_a)$ of k, that the expectation in the right-hand-side of the above equation equals

$$(k_0 M)^a \prod_{i=1}^a \int_{s=0}^{\varepsilon_0} ds\, m(s) \cdot$$
$$\nu_{\mathrm{cld}}\Big(|f_{T_0^+}(w - z - \xi_N(T_0^+-)) - f_{T_0^+}(-\xi_N(T_0^+-))|^{qn_i} \mathbb{1}_{A(f_{T_0^+})<\varepsilon_0} \Big| f_{T_0^+}(-\xi_N(T_0^+-)) = s\Big).$$

Now, let $\tilde{\mathbf{Y}}$ be a diffusion with infinitesimal parameters

$$\sigma^2_{\tilde{\mathbf{Y}}}(x) = 4x; \quad \mu_{\tilde{\mathbf{Y}}}(x) = \begin{cases} \mu(x) \text{ for } x \in (0,\varepsilon_0]; \\ \mu(\varepsilon_0) \text{ for } x > \varepsilon_0, \end{cases} \tag{4.16}$$

and let $\tilde{\nu}_{\mathrm{cld}}$ be the bi-clade excursion measure for $\tilde{\mathbf{Y}}$. Then for any measurable set $B \subseteq \mathcal{C}$ of continuous paths we have for $C := \{N \in \mathcal{N}^{\mathrm{sp}}_{\pm\mathrm{cld}} : f_{T_0^+} \in B\}$ that

$$\nu_{\mathrm{cld}}\Big(\{A(f_{T_0^+}) < \varepsilon_0\} \cap C \Big| f_{T_0^+}(-\xi_N(T_0^+-)) = s\Big) = \tilde{\nu}_{\mathrm{cld}}\Big(\{A(f_{T_0^+}) < \varepsilon_0\} \cap C \Big| f_{T_0^+}(-\xi_N(T_0^+-))) = s\Big)$$

because the two excursion measures differ only on excursions with amplitude greater than ε_0. Indeed, you can construct a Poisson random measure $\mathbf{N}$ with bi-clade measure ν_{cld} by first constructing a Poisson random measure $\tilde{\mathbf{N}}$ with bi-clade measure $\tilde{\nu}_{\mathrm{cld}}$, removing the finitely many bi-clades with spindles with amplitude greater than ε_0 (up to a finite time T) and adding finitely many bi-clades with spindles with amplitude greater than ε_0 since such bi-clades have finite intensity under the bi-clade measures.

In Proposition 3.10 we established that the bi-clade measure about a fixed level disintegrates with respect to the crossing width. Let $n \in \mathbb{N}$. The terms in the sum in (4.15) are the functionals of the clade (if $z = \min\{x,y\}$) or the anti-clade (if $z > \min\{x,y\}$) of excursions about level z. Therefore we have an upper bound given by

$$\tilde{\nu}_{\mathrm{cld}}\Big(|f_{T_0^+}(w - z - \xi_N(T_0^+-)) - f_{T_0^+}(-\xi_N(T_0^+-))|^{qn} \Big| f_{T_0^+}(-\xi_N(T_0^+-))) = s\Big) = \tilde{\mathbb{Q}}^0_s(|Y_{|x-y|} - s|^{qn}),$$

and as $-2\alpha^+ \le \mu_{\tilde{\mathbf{Y}}}(x) \le -2\alpha^-$ we have, through a coupling argument, that

$$\tilde{\mathbb{Q}}^0_s(|Y_{|x-y|} - s|^{qn}) \le {}^{-2\alpha^+}\mathbb{Q}^0_s(|Y_{|x-y|} - s|^{qn}) + {}^{-2\alpha^-}\mathbb{Q}^0_s(|Y_{|x-y|} - s|^{qn}), \tag{4.17}$$

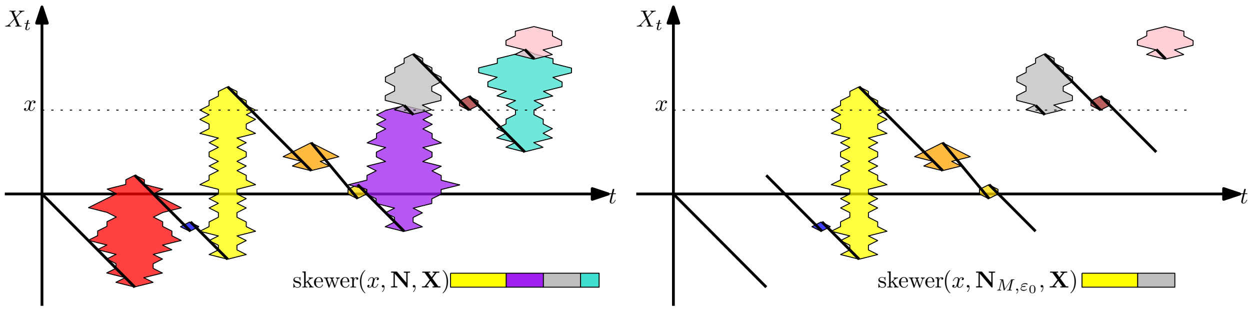

FIGURE 4.3. Left: The skewer process at level x constructed from the PRM $\mathbf{N}$ and scaffolding $\mathbf{X}$. Right: The skewer process at level x constructed from the reduced PRM $\mathbf{N}_{M,\varepsilon_0}$ from (4.13) and scaffolding $\mathbf{X}$.

which gives the bound stated in this lemma. □

The next lemma deals with the first spindle to cross level $z > 0$. In this case, we do not have a complete excursion about level z, and moreover the distribution of the crossing width is not clear. However, it is sufficient to know that some probability density function $p_z, z \in [0, \infty)$ gives the distribution for the crossing width where the anti-clade has infimum less than or equal to $-z$, i.e. the distribution of

$$\nu_{\text{cld}}(f_{T_0^+}(-\xi_N(T_0^+ -))) \in \cdot | \inf \xi_N \leq -z).$$

Lemma 4.16. *In the setting of Lemma 4.15 we have that*

$$\mathbf{E}\Bigg(\Bigg(\sum_{\substack{(t,f) \text{ atom of } \mathbf{N}: \\ t<T, \ell^z(t)=0, \\ A(f)<\varepsilon_0, \mathbf{X}_{t-}<z<\mathbf{X}_t}} \Big| g\big(f(y - \mathbf{X}_{t-})\big) - g\big(f(x - \mathbf{X}_{t-})\big)\Big|\Bigg)^k\Bigg)$$
$$\leq C_q^k \int_{s=0}^{\varepsilon_0} p_z(s)\Big({}^{-2\alpha^+}\mathbb{Q}_s^0(|Y_{|x-y|} - s|^{qk}) + {}^{-2\alpha^-}\mathbb{Q}_s^0(|Y_{|x-y|} - s|^{qk})\Big) ds \quad (4.18)$$

where p_z is the probability density function on $[0, \infty)$ that gives the distribution for

$$\nu_{cld}(m^0 \in \cdot | \zeta^+(N^-) > z)$$

(i.e. the distribution for the crossing width under ν_{cld} conditional on the anti-clade having depth at least $-z$) and where C_q is the Hölder constant given in Assumption B.

Proof: We use the same method as in the previous lemma. The sum on the left-hand-side consists of the first spindle to cross level z a.s. if it exists and has amplitude less than ε_0. We can then apply Lemma 3.8 with stopping time $T^{\geq z}$. This tells us that the spindle above level z evolves diffusively. We compare this to the diffusion $\tilde{\mathbf{Y}}$ to upper bound this expectation in the same way as in Lemma 4.15. □

Theorem 4.17. *Consider a diffusion that satisfies Assumption B. Let $(\mathbf{N}, \mathbf{X})$ be a scaffolding-and-spindles pair for this diffusion stopped at an a.s. finite random variable T on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Then there is a continuous IP-evolution that is a modification of (skewer$(y, \mathbf{N}, \mathbf{X}), y \in \mathbb{R}$) which takes values on $(\mathcal{I}_H(g(E)), d'_H)$.*

Proof: First assume $g(E)$ is closed and x, $y \geq 0$. Let $\kappa > 0$ be small enough so that $\alpha^+ + \kappa < q$. Let $\mathbf{N}_{M,g(\varepsilon_0)}$ be as in (4.13) but now with $(\mathbf{N}, \mathbf{X})$ a scaffolding-and-spindles pair for our diffusion under Assumption B. We first show that there exists $A \geq 0$ such that

$$\mathbf{E}\left(\left(\sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_{M,\varepsilon_0}: \\ \mathbf{X}_{t-}<x<\mathbf{X}_t \\ \text{or } \mathbf{X}_{t-}<y<\mathbf{X}_t}} \left|f(y-\mathbf{X}_{t-}) - f(x-\mathbf{X}_{t-})\right|\right)^k\right) \leq A|x-y|^{k(q/2\wedge(q-\alpha^+-\kappa))}. \tag{4.19}$$

Note we do not have the g transformations applied to the spindles f in the equation above as these spindles are from the Poisson random measure for the diffusion under Assumption B and not the untransformed diffusion under Assumption B' as in the previous lemmas. First we upper bound sum in the equation above by

$$\left(\sum_{\substack{(t,f) \text{ atom of} \\ \mathbf{N}_{M,\varepsilon_0}:t<T, \\ \mathbf{X}_{t-}<x<\mathbf{X}_t, \\ \ell^x(t)=0}} + \sum_{\substack{(t,f) \text{ atom of} \\ \mathbf{N}_{M,\varepsilon_0}:t<T, \\ \mathbf{X}_{t-}<y<\mathbf{X}_t, \\ \ell^y(t)=0}} + \sum_{\substack{(t,f) \text{ atom of} \\ \mathbf{N}_{M,\varepsilon_0}:t<T, \\ \mathbf{X}_{t-}<x<\mathbf{X}_t, \\ \ell^x(t)>0}} + \sum_{\substack{(t,f) \text{ atom of} \\ \mathbf{N}_{M,\varepsilon_0}:t<T, \\ \mathbf{X}_{t-}<y<\mathbf{X}_t, \\ \ell^y(t)>0}}\right)\left|f(y-\mathbf{X}_{t-}) - f(x-\mathbf{X}_{t-})\right|.$$

Now note that for any a, b, c, $d \geq 0$ we have that $(a+b+c+d)^k \leq 4^{k-1}(a^k+b^k+c^k+d^k)$ and therefore we can deal with each of the sums in the term above individually. We apply Lemmas 4.14 and 4.15 and the upper bound for m given in (4.3) to obtain an upper bound for the third sum in the equation above:

$$\mathbf{E}\left(\left(\sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_{M,\varepsilon_0}: \\ t<T,\mathbf{X}_{t-}<x<\mathbf{X}_t,\ell^x(t)>0}} \left|f(y-\mathbf{X}_{t-}) - f(x-\mathbf{X}_{t-})\right|\right)^k\right) \leq A'|x-y|^{k(q/2\wedge(q-\alpha^+-\kappa))},$$

for some $A' \geq 0$ and similarly for the fourth sum. We deal with the first two sums in the same way, using Lemma 4.16 and therefore obtain the upper bound given in (4.19). We now consider the $\mathbf{P}$-random times

$$S_0 := 0; \quad S_n := \inf\{t > S_{n-1} : (t,f) \text{ atom of } \mathbf{N}, \quad A(f) \geq \varepsilon_0\} \text{ for } n \geq 1. \tag{4.20}$$

As $\mathbf{N}$ is a stopped PRM stopped at random time T there exists $P \in \mathbb{N}$ such that $S_{P-1} \leq T < S_P$. For $1 \leq n \leq P$ we define the random measures

$$\mathbf{N}_{M,\varepsilon_0,n} := \sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_{M,\varepsilon_0}: \\ t\in[0,T]\cap(S_n,S_{n+1})}} \delta_{(t,f)} \text{ for } n \geq 0$$

so that $\mathbf{N}_{M,\varepsilon_0} = \sum_{n=0}^P \mathbf{N}_{M,\varepsilon_0,n}$. Let $n \in \mathbb{N}$. Consider the correspondence between

$$\text{skewer}(x, \mathbf{N}_{M,\varepsilon_0,n}, \mathbf{X}) \quad \text{and} \quad \text{skewer}(y, \mathbf{N}_{M,\varepsilon_0,n}, \mathbf{X})$$

given by pairing all the intervals generated by spindles that cross both levels x and y. This is a correspondence as we only have finitely many spindles cross level x and level y $\mathbf{P}$-a.s and has distortion

$$\sum_{\substack{(t,f) \text{ atom} \\ \text{of } \mathbf{N}_{M,\varepsilon_0,n}: \\ \mathbf{X}_{t-}<x\leq y<\mathbf{X}_t}} \left|f(y-\mathbf{X}_{t-}) - f(x-\mathbf{X}_{t-})\right| + \max\left\{\sum_{\substack{(t,f) \text{ atom} \\ \text{of } \mathbf{N}_{M,\varepsilon_0,n}: \\ x\leq\mathbf{X}_{t-}<y<\mathbf{X}_t}} f(y-\mathbf{X}_{t-}), \sum_{\substack{(t,f) \text{ atom} \\ \text{of } \mathbf{N}_{M,\varepsilon_0}: \\ \mathbf{X}_{t-}<x<\mathbf{X}_t\leq y}} f(x-\mathbf{X}_{t-})\right\}.$$

This sum is less than or equal to the sum in (4.19). Therefore we have $\overline{\text{skewer}}(\mathbf{N}_{M,\varepsilon_0,n}, \mathbf{X})$ satisfies the Kolmogorov-Čentsov continuity criterion and so there exists a continuous modification $(\gamma^y_{M,n}, y \geq 0)$. We can take M to infinity as by Proposition 4.6 we have that $\sup_{u\in\mathbb{R}} \ell^u(T)$ is finite $\mathbf{P}$-a.s. and so we have a continuous modification $(\gamma^y_n, y \geq 0)$ of $\overline{\text{skewer}}(\mathbf{N}|_{(S_n,S_{n+1})}, \mathbf{X})$.

From here we can construct a continuous modification of $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$. For $n\in\mathbb{N}$ let f_n be the n^{th} spindle of $\mathbf{N}$ with amplitude greater than or equal to ε_0, i.e. so that $\delta_{(S_n,f_n)}$ is an atom of $\mathbf{N}$. Then we define the IP-evolution $(\gamma^y, y\geq 0)$ by

$$\gamma^y := \bigstar_{n=1}^{P}\big(\gamma_n^y \star \{(0, f_n(y-\mathbf{X}_{S_n-}))\}\big), \tag{4.21}$$

where $\star$ and $\bigstar$ are the concatenate operations from Definition 2.3. Note there are only finitely many spindles with amplitude greater than or equal to ε_0 on $[0,T]$ $\mathbf{P}$-a.s. Therefore we have that $(\gamma^y, y\geq 0)$ is the concatenation of finitely many d'_H-continuous IP-evolutions (the IP-evolutions $(\{(0,f_n(y-\mathbf{X}_{S_n}))\}, y\geq 0)$ are clearly d'_H-continuous) and therefore $(\gamma^y, y\geq 0)$ is d'_H-continuous. Finally we note that concatenating modifications gives modifications and so $(\gamma^y, y\geq 0)$ is a modification of $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$ on $(\mathcal{I}_H, d'_H)$. The subspace $(\mathcal{I}_H(g(E)), d'_H)$ is closed and so this modification exists on the appropriate subspace. To finish the proof assume $g(E)$ is not closed. This only occurs if $g(h)$ is finite and $g(E)=[0,g(h))$. In this case however we still obtain our result as we do not take a modification of the diffusions with amplitude greater than ε_0 and as each of these diffusions with amplitude greater than ε_0 do not hit $g(h)$ then $(\gamma^y, y\geq 0)$ does take values on the subspace $(\mathcal{I}_H(g(E)), d'_H)$.

Finally, assume x, $y\leq 0$. Then this case follows ultimately by (3.12). If $x\leq 0\leq y$ then we have the result by the previous two cases and the triangle inequality. $\square$

We conclude this section with the analogous result for diffusions under Assumption C.

Proposition 4.18. *Let $\mathbf{Z}$ be a diffusion that satisfies Assumption C, and let $(\mathbf{N},\mathbf{X})$ be a scaffolding-and-spindles pair stopped at an a.s. finite stopping time. Then $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$ is a.s. continuous and takes values on $(\mathcal{I}_H(g(E)), d'_H)$.*

Proof: Let $\mathbf{Y}$ be an untransformed diffusion that satisfies Assumption C' and g be a transformation that satisfies $C3$ of Assumption C such that $\mathbf{Z}=g(\mathbf{Y})$.

By the bounds in (4.4) we have that $M((0,\varepsilon_0))$ is finite. Therefore by Theorem 1.3 the Pitman-Yor excursion measure for $\mathbf{Y}$ (equivalently $\mathbf{Z}$) satisfies (1.6). For the diffusion $\mathbf{Z}$ let $(\mathbf{N},\mathbf{X})$ be a scaffolding-and-spindles pair stopped at an a.s. finite stopping time.

To show continuity of the IP-evolution we first show that the sum of the amplitudes of all of the spindles is finite. From Theorem 2.7 we know that the intensity of spindles for unstopped $\mathbf{N}$ with amplitude greater than x is $1/s(x)$. Therefore, the amplitudes of the spindles form a Poisson point process with intensity $s'(x)/s(x)^2$ which, when $\mathbf{N}$ is stopped at an a.s. finite random time, is summable a.s. With the substitution $y=s(x)$ we have that it is equal to

$$\int_{x=0}^{b} g(x)\frac{s'(x)}{s(x)^2}dx = \int_{y=0}^{s(b)}\frac{g(s^{-1}(y))}{y^2}dy<\infty \tag{4.22}$$

for some (equivalently all) $b\in(0,h)$ (we have included the integral obtained with the substitution $y=s(x)$ here as we use it below). By the bound in (4.2) on s' we have

$$\frac{\varepsilon_1^{\beta^-}}{1-\beta^-}x^{1-\beta^-}\leq s(x) \text{ for } x\in(0,\varepsilon_1).$$

Therefore for $y=\big(\varepsilon_1^{-\beta^-}(1-\beta^-)\big)^{1/(1-\beta^-)}x^{1/(1-\beta^-)}$ we have

$$x=\frac{\varepsilon_1^{\beta^-}}{1-\beta^-}y^{(1-\beta^-)}\leq s(y) \text{ for } y\in(0,\varepsilon_1).$$

This means that

$$s^{-1}(x)\leq \big(\varepsilon_1^{-\beta^-}(1-\beta^-)\big)^{1/(1-\beta^-)}x^{1/(1-\beta^-)} \qquad \text{for } x<s(\varepsilon_1)$$

and so the second integral in (4.22) is less than or equal to

$$\int_{y=0}^{s(b)} \frac{g\big(\big(\varepsilon_1^{-\beta^-}(1-\beta^-)\big)^{1/(1-\beta^-)} y^{1/(1-\beta^-)}\big)}{y^2} dy \qquad \text{provided that } b < \varepsilon_1.$$

To put this into the form of $C3$ requires a linear transformation $z = ay$ for $a = \big(\varepsilon_1^{-\beta^-}(1-\beta^-)\big)$. Therefore, the integral in (4.22) is finite by $C3$ and the summability of the amplitudes follows.

From here we can show that $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$ is continuous a.s. with respect to the metric d'_H. Let I be an index set for the spindles of $\mathbf{N}$, $\varepsilon > 0$ and $C = \sum_{i\in I} A(f_i) < \infty$ be the sum of amplitudes Let $(t_1, f_1), \dots, (t_K, f_K)$ be atoms of $\mathbf{N}$ such that $\sum_{i=1}^K A(f_i) > C - \varepsilon/2$, and let $\tilde{\mathbf{N}}$ be the Poisson random measure $\mathbf{N}$ with the atoms $(t_1, f_1), \dots, (t_K, f_K)$ removed. The spindles f_i are continuous functions with finite lifetimes so are uniformly continuous. Let $\delta > 0$ such that for any $1 \le i \le K$ and any s, t with $|s-t| < \delta$ we have $|f_i(s) - f_i(t)| < \varepsilon/2K$. Let x, $y \in \mathbb{R}$ with $|x-y| < \delta$. Then, for the interval partitions $\text{skewer}(x,\mathbf{N},\mathbf{X})$ and $\text{skewer}(y,\mathbf{N},\mathbf{X})$ we can define the correspondence of size at most K that matches every interval generated by spindle f_i in $\text{skewer}(x,\mathbf{N},\mathbf{X})$ to the corresponding interval in $\text{skewer}(y,\mathbf{N},\mathbf{X})$. The remaining intervals in $\text{skewer}(x,\mathbf{N},\mathbf{X})$ or $\text{skewer}(y,\mathbf{N},\mathbf{X})$ have sum less than $\varepsilon/2$ and therefore the distortion is equal to

$$\sum_{j=1}^K |f_i(x-\mathbf{X}_{t_i-}) - f_i(y-\mathbf{X}_{t_i-})| + \max\left\{ \sum_{(t,f) \text{ atom of } \tilde{\mathbf{N}}} f(x-\mathbf{X}_{t-}), \sum_{(t,f) \text{ atom of } \tilde{\mathbf{N}}} f(y-\mathbf{X}_{t-}) \right\} \le K\frac{\varepsilon}{2K} + \frac{\varepsilon}{2}$$

and therefore $d'_H(\text{skewer}(x,\mathbf{N},\mathbf{X}), \text{skewer}(y,\mathbf{N},\mathbf{X})) \le \varepsilon$. As in the previous result the process takes values on the subspace $(\mathcal{I}_H(g(E)), d'_H)$ as required. $\square$

4.5. *Further continuity results and proof of Theorem 1.7.* For a measurable map $g : E \to g(E)$ (where as before $E = [0,h)$ or $E = [0,h]$ for some $h \le \infty$) we define $g^* : \mathcal{I}_H(E) \to g^*(\mathcal{I}_H(E))$ in the following way: for an interval partition $\beta = (U_i, i \in I)$ such that $\sum_{i\in I} g(\text{Leb}(U_i))$ is finite we define $g^*(\beta)$ to be the interval partition $\gamma := (V_i, i \in I)$ where we have $\text{Leb}(V_i) = g(\text{Leb}(U_i))$ and the order of the intervals in β and γ is consistent; when $\sum_{i\in I} g(\text{Leb}(U_i))$ is infinite we take $g^*(\beta) := \emptyset$, the empty interval partition. Note that, $g^*(\mathcal{I}_H(E)) \subseteq \mathcal{I}_H(g(E))$. Also, as g is an injection, g^* restricted to $\{\beta \in \mathcal{I}_H(E) : \sum_{U\in\beta} g(\text{Leb}(U)) < \infty\}$ has an inverse $(g^*)^{-1}$ and for any $\gamma \in g^*(\mathcal{I}_H(E))$ we have that $(g^*)^{-1}(\gamma) = (g^{-1})^*(\gamma)$. We also note that g^* is a measurable function. To see this, first define the functions $(g_n, n \in \mathbb{N})$ by $g_n(x) = g(x)$ if $x > 1/n$ and $g_n(x) = x$ otherwise and observe that g^* is the limit of the sequence of the functions $(g_n^*, n \in \mathbb{N})$. The measurability of g^* then follows by the fact that for all $n \in \mathbb{N}$ the functions g_n^* are continuous. The proof of continuity of the functions $(g_n^*, n \in \mathbb{N})$ follows from the continuity of g and that for any interval partition β the interval partition differs from its image $g_n^*(\beta)$ on only finitely many interval widths (see also Lemma 4.30).

The next step is to develop a simple Markov property. In Corollary 3.13 we have that for an interval partition $\beta \in \mathcal{I}_H(E)$ and a diffusion that satisfies Assumption B', we can construct a scaffolding-and-spindles pair $(\mathbf{N}_\beta, \mathbf{X}_\beta)$ as in Definition 3.11. We state a corollary here for diffusions under Assumption B.

Corollary 4.19. *Consider a diffusion $\mathbf{Z}$ that satisfies Assumption B and let $\gamma \in g^*(\mathcal{I}_H(E))$. Consider a scaffolding-and-spindles pair started from γ, denoted as $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ as in Definition 3.11. Then we have that len$(\mathbf{N}_\gamma) = \sum_{U\in\gamma}$ len$(\mathbf{N}_U)$ is finite a.s.*

Proof: Let $\mathbf{Y}$ be the untransformed diffusion that satisfies Assumption B', let $\gamma \in g^*(\mathcal{I}_H(E))$ and then let $\beta \in \mathcal{I}_H(E)$ such that $g^*(\beta) = \gamma$. As mentioned above, by Corollary 3.13 we can construct a scaffolding-and-spindles pair $(\mathbf{N}_\beta^{\mathbf{Y}}, \mathbf{X}_\beta^{\mathbf{Y}})$ for $\mathbf{Y}$ started from β and $\text{len}(\mathbf{N}_\beta)$ is finite a.s. Therefore we can define a scaffolding-and-spindles pair for diffusion $\mathbf{Z}$ and started from γ by

$$\mathbf{N}_\gamma^{\mathbf{Z}} = \sum_{(t,f) \text{ atom of } \mathbf{N}_\beta^{\mathbf{Y}}} \delta_{(t,g(f))},$$

where for a function $(f(x), x \in \mathbb{R})$ we write $g(f)$ to be the function $(g(f(x)), x \in \mathbb{R})$. Then define $\mathbf{X}_\gamma^{\mathbf{Z}} = \xi(\mathbf{N}_\gamma^{\mathbf{Z}})$ which is equal to $\mathbf{X}_\beta^{\mathbf{Y}}$ as the spatial transformation of the spindles does not change the scaffolding. □

We now state a continuity result for an IP-evolution with a fixed initial state. For $N \in \mathcal{N}_{\text{fin}}^{\text{sp}}$, let $\zeta^+(N) := \sup_{t \in \mathbb{R}} \xi_N(t)$.

Lemma 4.20. *Consider a diffusion* $\mathbf{Z}$ *that satisfies Assumption B and let* $\gamma \in g^*(\mathcal{I}_H(E))$*. Let* $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ *be a scaffolding-and-spindles pair* γ *as in Definition 3.11. Then there exists a modification of* $\overline{\text{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ *that we call* $(\gamma^y, y \geq 0)$ *that is continuous for* $y \in (0, \infty)$*. Furthermore, for* $y \in (0, \infty)$ *fixed, we have* $\gamma^y \in g^*(\mathcal{I}_H(E))$ *a.s.*

Proof: Let $n \in \mathbb{N}$ and $\gamma \in g^*(\mathcal{I}_H(E))$. From Corollary 4.19 we can construct $(\mathbf{N}_U, U \in \gamma)$ as in Definition 3.11. Recall that $\mathbf{N}_U := \delta_{(0,f_U)} + \mathbf{N}|_{[0,T^{-\zeta(f_U)}]}$ where $\mathbf{N}$ is a Poisson random measure. The process $(\text{skewer}(y, \mathbf{N}|_{[0,T^{-\zeta(f_U)}]}, \xi(\mathbf{N}|_{[0,T^{-\zeta(f_U)}]})), y \in \mathbb{R})$ has a continuous modification by Theorem 4.17 and it follows that $\overline{\text{skewer}}(\mathbf{N}_U, \mathbf{X}_U)$ has a continuous modification. Also, from the fact that $\sum_{U \in \gamma} \text{len}(\mathbf{N}_U)$ is finite a.s., from Corollary 4.19, we have that for only finitely many $U \in \gamma$ that $\zeta^+(\mathbf{N}_U) > 1/n$. Therefore we can conclude that there exists a continuous modification of $\overline{\text{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ on $(1/n, \infty)$. As this is true for all $n \in \mathbb{N}$, we have a continuous modification on $(0, \infty)$.

It remains to show that for $y \in (0, \infty)$ fixed, we have $\gamma^y \in g^*(\mathcal{I}_H(E))$ a.s. Let $y > 0$. For each $U \in \gamma$ we have that $\mathbf{N}_U$ is constructed by an initial incomplete spindle f_U and then a stopped Poisson random measure $\mathbf{N}$ with stopping time $T^{-\zeta(f_U)}$. We claim that $\text{skewer}(y, \mathbf{N}_U, \mathbf{X}_U) \in g^*(\mathcal{I}_H(E))$ a.s. By the spatial homogeneity of the scaffolding-and-spindles pair and the independence of f_U and $\mathbf{N}$ it is sufficient to show that $\text{skewer}(0, \mathbf{N}, \mathbf{X}) \in g^*(\mathcal{I}_H(E))$ a.s. We have that $\text{skewer}(0, \mathbf{N}, \mathbf{X})$ is in $\mathcal{I}_H(E)$ by the first part of Proposition 4.2 and we can conclude that it is in fact in $g^*(\mathcal{I}_H(E))$ by again using the first part of Proposition 4.2 with the identity transformation $g(x) = x$ (this tells us the untransformed spindles produce an interval partition and therefore the transformed spindles produce an interval partition in $g^*(\mathcal{I}_H(E))$ a.s.).

We concatenate the interval partitions $\text{skewer}(y, \mathbf{N}_U, \mathbf{X}_U)$ for the finitely many of the $U \in \gamma$ with $\zeta^+(\mathbf{N}_U) > 1/n$ to form $\text{skewer}(y, \mathbf{N}_\gamma, \mathbf{X}_\gamma)$ and conclude that $\text{skewer}(y, \mathbf{N}_\gamma, \mathbf{X}_\gamma) \in g^*(\mathcal{I}_H(E))$ a.s. and therefore the modification $\gamma^y \in g^*(\mathcal{I}_H(E))$ a.s. □

Define the filtration $(\mathcal{F}_{\mathcal{I}}^y, y \geq 0)$ to be the right-continuous filtration generated by $(\gamma^y, y \in (0, \infty))$ on $\Omega := \mathcal{C}((0, \infty), \mathcal{I}_H)$, the class of all paths in $\mathcal{I}_H$ continuous on $(0, \infty)$. We define a filtered probability space $(\Omega, \mathcal{F}_{\mathcal{I}}, (\mathcal{F}_{\mathcal{I}}^y, y \geq 0), \mathbb{P}_\gamma)$ where $\mathcal{F}_{\mathcal{I}} := \bigcup_{y \geq 0} \mathcal{F}_{\mathcal{I}}^y$, and $\mathbb{P}_\gamma(\cdot)$ is the probability measure

$$\mathbb{P}_\gamma(\cdot) = \mathbf{P}((\gamma^y, y \geq 0) \in \cdot) \tag{4.23}$$

where $(\gamma^y, y \geq 0)$ is the modification of $\overline{\text{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ that is continuous on $(0, \infty)$ that is given in the lemma above.

Corollary 4.21 (Simple Markov property under Assumption B)**.** *Consider a diffusion under Assumption B. Let* μ *be a probability distribution on interval partitions that is supported on* $g^*(\mathcal{I}_H(E))$*, and define the probability measure* $\mathbb{P}_\mu(\cdot) = \int_{\gamma \in g^*(\mathcal{I}_H(E))} \mathbb{P}_\gamma(\cdot) \mu(d\gamma)$*. For any* $y \in (0, \infty)$ *let* θ_y *be the shift operator, and let* $\eta, f : \mathcal{C}((0, \infty), \mathcal{I}_H) \to [0, \infty)$ *be measurable where* η *is measurable with respect to* $\mathcal{F}_{\mathcal{I}}^y$*. Then we have that*

$$\mathbb{P}_\mu[\eta f \circ \theta_y] = \mathbb{P}_\mu[\eta \mathbb{P}_{\gamma^y}[f]].$$

Proof: This proof uses the same arguments as the analogous result in Forman et al. (2020a, Corollary 6.13); specifically, by Lemma 4.20 we have that the sample paths a.s. are in Ω and the simple Markov property for the skewer process on a larger probability space is given in Proposition 3.20. $\square$

We now show the analogous result for diffusions under Assumption C and prove Theorem 1.7. We let $\mathcal{I}_{\text{fin}} \subset \mathcal{I}_H$ be the interval partitions that consist of finitely many intervals and $\mathcal{I}_{\text{fin}}(g(E)) = \mathcal{I}_{\text{fin}} \cap \mathcal{I}_H(g(E))$. For any interval partition $\gamma \in \mathcal{I}_{\text{fin}}(g(E))$ we define $\mathbb{P}_\gamma$ as before and note that the scaffolding-and-spindles pair $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ can always be constructed for such initial interval partitions γ. Let $\Omega_0 := \mathcal{C}([0,\infty), \mathcal{I}_H)$ be the space of continuous paths on $\mathcal{I}_H$ indexed by $[0,\infty)$.

Proof of Theorem 1.7: Let $\mathbf{Z}$ be a diffusion under Assumption C, and let $\gamma \in \mathcal{I}_{\text{fin}}(g(E))$. Let $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ be a scaffolding-and-spindles pair started at γ as defined in Definition 3.11. For $y \geq 0$ let $\gamma^y = \text{skewer}(y, \mathbf{N}_\gamma, \mathbf{X}_\gamma)$. Then $(\gamma^y, y \geq 0)$ is continuous by the same reasoning as in Proposition 4.18; for each y we have only concatenated finitely many interval partitions.

In this setting let $(\mathcal{F}^y_{\mathcal{I}}, y \geq 0)$ be the right-continuous filtration generated by $(\gamma^y, y \geq 0)$ on the smaller space Ω_0, rather than on Ω as we had done previously. The simple Markov property for the skewer process from Proposition 3.20 applies in this setting as for fixed $y > 0$ the interval partition $\gamma^y \in \mathcal{I}_{\text{fin}}(g(E))$ a.s. To see this, note that for each $U \in \gamma$ the SPLP $\mathbf{X}_U$ is of bounded variation and so only crosses a fixed level finitely many times a.s. (Bertoin, 1996, Corollary VII.5), and therefore γ^y consists of finitely many intervals as it is the concatenation of finitely many interval partitions that each consist of finitely many intervals. To conclude the proof for the setting described in Theorem 1.7 we note that, for $n \in \mathbb{N}$, if we stop the PRM $\mathbf{N}$ of a scaffolding-and-spindles pair $(\mathbf{N}, \mathbf{X})$ at the n^{th} hitting time of 0 then $\text{skewer}(0, \mathbf{N}, \mathbf{X}) \in \mathcal{I}_{\text{fin}}(g(E))$ a.s. $\square$

We used $\Omega_0 \subset \Omega$ under Assumption C whereas under Assumption B we were required to use Ω as we did not have a proof of continuity at time 0. However, if we restrict the class of initial interval partitions then we can use Ω_0 to state the simple Markov property under Assumption B, and we do this in the following subsection. This subsection is not necessary to prove Theorem 1.6 but we include it to show the class of initial interval partitions for which we have a proof of continuity at time 0.

4.5.1. *Continuity at time 0 under Assumption B.* To prove the results of this section we define certain classes of interval partitions. For any $q > 0$ and any class of interval partitions $\mathcal{I} \subseteq \mathcal{I}_H$ we define

$$\mathcal{I}^{(q)} := \{\beta = (U_i, i \in I) \in \mathcal{I} : \sum_{i \in I} \text{Leb}(U_i)^q < \infty\}.$$

Note that for $q \geq 1$ we have that $\mathcal{I}^{(q)} = \mathcal{I}$. We now state a corollary of Theorem 4.17.

Lemma 4.22. *Consider a diffusion that satisfies Assumption B' with a stopped scaffolding-and-spindles pair $(\mathbf{N}, \mathbf{X})$ on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$, and let $q > \alpha^+$. Then, for $\big(\text{skewer}(y, \mathbf{N}, \mathbf{X}), y \geq 0\big)$, there exists a modification $(\beta^y, y \geq 0)$ which is continuous on $(\mathcal{I}_H^{(q)}(E), d'_H)$.*

Proof: We first note that for $q \in (0,1)$ we can apply Theorem 4.17 with a diffusion $\mathbf{Z}$ constructed from an untransformed diffusion $\mathbf{Y}$ and transformation $f(x) = x^q$, to produce the continuous IP-evolution $(\phi^y, y \geq 0)$. Also, note that the map $(f^{-1})^*$ is continuous on the space of interval partitions $\mathcal{I}_H$ (this ultimately follows because the map f^{-1} is Lipschitz on $[0,1]$, see also Lemma 4.30). Therefore $((f^{-1})^*(\phi^y), y \geq 0)$ is $\mathbf{P}$-a.s. continuous in $\mathcal{I}_H$.

Now, $(\phi^y), y \geq 0)$ is a modification of $f^*(\overline{\text{skewer}}(\mathbf{N}, \mathbf{X}))$ and therefore $((f^{-1})^*(\phi^y), y \geq 0)$ is a modification of $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X})$. Also, again by Theorem 4.17, there exists a continuous process $(\beta^y, y \geq 0)$ which is a modification of $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X})$ and so is also a modification of $((f^{-1})^*(\phi^y), y \geq 0)$. As both $(\beta^y, y \geq 0)$ and $((f^{-1})^*(\phi^y), y \geq 0)$ are both continuous processes they are therefore indistinguishable from one another, and therefore $(\beta^y, y \geq 0)$ is a continuous process in $\mathcal{I}_H^{(q)}(E)$ a.s. $\square$

As in the previous sections, we consider a diffusion $\mathbf{Z}$ under Assumption B, which is equal in distribution to $g(\mathbf{Y})$ where $\mathbf{Y}$ is a diffusion under Assumption B' and $g : E \to g(E)$ is a continuous bijection as in Assumption B. Let q_0, α^+, α^- be the associated constants and let $q \in (\alpha^+, q_0)$. Finally, let $m_{\mathbf{Y}}$ and $m_{\mathbf{Z}}$ be the speed measure densities of $\mathbf{Y}$ and $\mathbf{Z}$. We now state a corollary of Lemma 3.16 and Lemma 4.22.

Corollary 4.23. *Consider a stopped scaffolding-and-spindles pair* $(\mathbf{N}, \mathbf{X})$ *for the diffusion* $\mathbf{Z}$ *which satisfies Assumption B, and let* $(\gamma^y, y \geq 0)$ *be a continuous modification of* $\overline{\text{skewer}}(\mathbf{N}, \mathbf{X})$*. Then for any fixed* $y \geq 0$ *we have that*

$$\gamma^y \in g^*(\mathcal{I}_{m_{\mathbf{Y}}}^{(q)}(E)) = \Big\{\gamma = (V_i, i \in I) \in g^*(\mathcal{I}_{m_{\mathbf{Y}}}(E)) : \sum_{i \in I} g^{-1}(Leb(V_i))^q < \infty\Big\} \ \mathbf{P} - a.s.$$

Proof: Let $\tilde{\mathbf{N}}$ be the Poisson random measure of untransformed spindles. The $m_{\mathbf{Y}}$-diversity of $\overline{\text{skewer}}(\tilde{\mathbf{N}}, \mathbf{X})$ at any fixed level has already been established in Lemma 3.16. This shows that $\text{skewer}(y, \mathbf{N}, \mathbf{X}) \in g^*(\mathcal{I}_{m_{\mathbf{Y}}}(E))$ a.s. and so $\gamma^y \in g^*(\mathcal{I}_{m_{\mathbf{Y}}}(E))$.

It remains to show that $\gamma^y \in g^*(\mathcal{I}_H^{(q)}(E))$. To do this we note that the process $(\gamma^y, y \geq 0)$ is a modification of $g^*(\overline{\text{skewer}}(\tilde{\mathbf{N}}, \mathbf{X}))$. Therefore $((g^{-1})^*(\gamma^y), y \geq 0)$ is a modification of $\overline{\text{skewer}}(\tilde{\mathbf{N}}, \mathbf{X})$. By Lemma 4.22 $\overline{\text{skewer}}(\tilde{\mathbf{N}}, \mathbf{X})$ has a continuous modification that takes values in $\mathcal{I}_H^{(q)}(E)$, and the result follows as for fixed $y \geq 0$ we have that $(g^{-1})^*(\gamma^y) = \text{skewer}(y, \tilde{\mathbf{N}}, \mathbf{X})$ a.s. $\square$

The following lemma concerns the initial spindles of the IP-evolution, i.e. the spindles that are not equal to 0 at time 0.

Lemma 4.24. *Consider a diffusion that satisfies Assumption B'. Let* $(a_i, i \in I)$ *be a potentially infinite sequence of positive numbers such that* $\sum_{i \in I} a_i^{q \wedge 1} < \infty$*. Consider a family of independent diffusions* $(f_i, i \in I)$ *distributed as independent diffusions started from* $(a_i, i \in I)$ *stopped at* 0 *on some common probability space* $(\Omega, \mathcal{A}, \mathbf{P})$*, i.e.* $f_i \sim \mathbb{Q}_{a_i}^0$*. Then we have that* $\sum_{i \in I} A(f_i)^q < \infty$ $\mathbf{P}$*-a.s.*

Proof: By the coupling given in (4.8) we have for $0 < a < y < \varepsilon_0$ that $\mathbb{Q}_a^0(A > y) \leq {}^{-2\alpha^-}\mathbb{Q}_a^0(A > y)$ which in turn is equal to $(a/y)^{1+\alpha^-}$ (by (1.2) and as $x^{1+\alpha^-}$ for $x \in [0, \infty)$ is a scale function for a squared Bessel process with dimension $-2\alpha^-$). Therefore we have

$$\sum_{i \in I : a_i < \varepsilon_0} \mathbf{P}(A(f_i) > \varepsilon_0) \leq \varepsilon_0^{-1-\alpha^-} \sum_{i \in I} a_i^{1+\alpha^-} < \infty,$$

and so by the Borel-Cantelli lemma and by the fact that only finitely many a_i are such that $a_i \geq \varepsilon_0$, we have that a.s. only finitely many of the incomplete spindles f_i will have amplitude greater than ε_0.

Furthermore, again with the upper bound for $\mathbb{Q}_a^0(A > y)$ stated above, we have

$$\mathbf{E}(A(f_i)\mathbb{1}_{A(f_i) < \varepsilon_0}) = \int_{y=0}^{\varepsilon_0} \mathbf{P}(y \leq A(f_i) < \varepsilon) dy.$$

This is less than or equal to

$$\int_{y=0}^{\varepsilon_0} \mathbb{Q}_{a_i}^0(A > y) dy = \int_{y=0}^{\varepsilon_0} \Big((a_i/y)^{1+\alpha^-} \wedge 1\Big) dy = \Big(\frac{1}{\alpha^-} + 1\Big) a_i - \frac{a_i^{1+\alpha^-}}{\alpha^- \varepsilon_0^{\alpha^-}} \leq \Big(\frac{1}{\alpha^-} + 1\Big) a_i.$$

Therefore, when $q < 1$, we have that

$$\mathbf{E}(A(f_i)^q \mathbb{1}_{A(f_i) < \varepsilon_0}) \leq \Big(\mathbf{E}(A(f_i)\mathbb{1}_{A(f_i) < \varepsilon_0})\Big)^q \leq C_q \Big(\frac{1}{\alpha^-} + 1\Big)^q a_i^q$$

where C_q is the Hölder constant from Assumption B. Therefore we have that

$$\sum_{i\in I}\mathbf{E}(A(f_i)^q : A(f_i) < \varepsilon_0) \text{ is finite, and so } \sum_{i\in I : A(f_i)<\varepsilon_0} A(f_i)^q \text{ is finite } \mathbf{P}\text{-a.s.} \qquad \square$$

In the next lemma we give a result that compares certain interval partitions with $\mathrm{skewer}(0,\mathbf{N},\mathbf{X})$ for a particular stopped scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$, which we then use in a coupling argument in the proposition after to prove continuity of a IP-evolutions with certain initial interval partitions.

Lemma 4.25 (Interval partition domination)**.** *Consider a diffusion $\mathbf{Z}$ that satisfies Assumption B and let $\gamma \in \overline{\mathcal{I}}_{m_\mathbf{Z}}(g(E))$. Recall the notation $\mathcal{D}$ and $\overline{\mathcal{D}}$ from Definition 3.15 and let $(\mathbf{N},\mathbf{X})$ be a scaffolding-and-spindles pair stopped at $\tau^0(k_0^{-1}(\overline{\mathcal{D}}_\gamma^{m_\mathbf{Z}}(\infty)+1))$ (where τ^0 is the inverse local time of $\mathbf{X}$ at level 0 and k_0 is the constant from Lemma 3.3), and let $\tilde{\gamma} := \mathrm{skewer}(0,\mathbf{N},\mathbf{X})$. Firstly we have that*

$$\mathcal{D}_{\tilde{\gamma}}^{m_\mathbf{Z}}(\infty) = \overline{\mathcal{D}}_\gamma^{m_\mathbf{Z}}(\infty) + 1 \text{ a.s.}$$

Also, with positive probability we have that there exists a matching between γ and $\tilde{\gamma}$ such that every interval of γ is matched with a larger interval of $\tilde{\gamma}$ (but that does not necessarily respect the total-order of the intervals). We say that $\tilde{\gamma}$ dominates γ by matching in this case. Finally we have that $g^*(\overline{\mathcal{I}}_{m_\mathbf{Y}}(E)) = \overline{\mathcal{I}}_{m_\mathbf{Z}}(g(E))$ and also $g^*(\mathcal{I}_{m_\mathbf{Y}}(E)) = \mathcal{I}_{m_\mathbf{Z}}(g(E))$.

Proof: The diversity of $\tilde{\gamma}$ follows from Lemma 3.16. The second part of this lemma generalises Forman et al. (2020a, Lemma 6.8) to interval partitions with finite $m_\mathbf{Z}$-diversity and the proof is the same. It requires the aggregate health summability under Assumption B which was shown in Proposition 4.2. The result further generalises to any interval partition with a finite limsup-$m_\mathbf{Z}$-diversity as intervals can be added to an interval partition with finite limsup-$m_\mathbf{Z}$-diversity to produce an interval partition with a finite $m_\mathbf{Z}$-diversity and, therefore, the domination result carries over.

For the final part first note that if a pair of interval partitions $\hat{\beta}$, $\hat{\gamma} \neq \emptyset$ satisfy $g^*(\hat{\beta}) = \hat{\gamma}$ then $\hat{\beta} \in \mathcal{I}_{m_\mathbf{Y}}(E)$ if and only if $\hat{\gamma} \in \mathcal{I}_{m_\mathbf{Z}}(g(E))$ and also $\mathcal{D}_{\hat{\beta}}^{m_\mathbf{Y}}(\infty) = \mathcal{D}_{\hat{\gamma}}^{m_\mathbf{Z}}(\infty)$ (and similarly for limsup diversities). Therefore for $\hat{\gamma} \in \overline{\mathcal{I}}_{m_\mathbf{Z}}(g(E))$, to show $\hat{\gamma} \in g^*(\overline{\mathcal{I}}_{m_\mathbf{Y}}(E))$ it suffices to show that $\sum_{V\in\hat{\gamma}} g^{-1}(\mathrm{Leb}(V))$ is finite.

Let $\hat{\gamma} \in \overline{\mathcal{I}}_{m_\mathbf{Z}}(g(E))$ and for the diffusion $\mathbf{Z}$ let $(\mathbf{N},\mathbf{X})$ be a scaffolding-and-spindles pair stopped at $\tau^0(k_0^{-1}(\overline{\mathcal{D}}_\gamma^{m_\mathbf{Z}}(\infty)+1))$. Let $\tilde{\gamma} := \mathrm{skewer}(0,\mathbf{N},\mathbf{X})$. As untransformed diffusions also satisfy Assumption B (as the identity transformation $g(x) = x$ satisfies the hypotheses in Assumption B) we also have the aggregate health summability result for the untransformed diffusions by Proposition 4.2. Therefore we have $\tilde{\beta} := (g^{-1})^*(\tilde{\gamma})$ is also a non-trivial interval partition (as well as $\tilde{\gamma}$) because $\tilde{\beta}$ is equal in distribution to $\mathrm{skewer}(0,\tilde{\mathbf{N}},\tilde{\mathbf{X}})$ for a stopped scaffolding-and-spindles pair $(\tilde{\mathbf{N}},\tilde{\mathbf{X}})$ for the untransformed diffusion. Furthermore as we have with positive probability that $\tilde{\gamma}$ dominates $\hat{\gamma}$ we can take a realisation of $\tilde{\gamma}$ and $\tilde{\beta}$ where $\tilde{\gamma}$ dominates $\hat{\gamma}$ and we find that

$$\sum_{V\in\hat{\gamma}} g^{-1}(\mathrm{Leb}(V)) \le \sum_{V\in\tilde{\gamma}} g^{-1}(\mathrm{Leb}(V)) = \sum_{U\in\tilde{\beta}} \mathrm{Leb}(U) = \|\tilde{\beta}\| < \infty$$

and therefore we have that $\hat{\gamma} \in g^*(\overline{\mathcal{I}}_{m_\mathbf{Y}}(E))$ as required.

Therefore we can conclude that $g^*(\overline{\mathcal{I}}_{m_\mathbf{Y}}(E)) = \overline{\mathcal{I}}_{m_\mathbf{Z}}(g(E))$ as it is immediate that

$$g^*(\overline{\mathcal{I}}_{m_\mathbf{Y}}(E)) \subseteq \overline{\mathcal{I}}_{m_\mathbf{Z}}(g(E)).$$

The other case $g^*(\mathcal{I}_{m_\mathbf{Y}}(E)) = \mathcal{I}_{m_\mathbf{Z}}(g(E))$ follows by the same reasoning. $\square$

Proposition 4.26. *Let $\gamma = (V_i, i \in I) \in g^*(\overline{\mathcal{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$ and let $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ be a scaffolding-and-spindles pair for γ as in Definition 3.11. Then there exists a modification of $\overline{\mathit{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ that is continuous on $[0,\infty)$.*

Proof: As in the proof of Theorem 4.17 first assume that $g(E)$ is closed. Let

$$\gamma \in g^*(\overline{\mathcal{I}}_{m_{\mathbf{Y}}}(E)) = \overline{\mathcal{I}}_{m_{\mathbf{Z}}}(g(E))$$

and let $\tilde{\gamma}$ be a random interval partition such that $\mathcal{D}^{m_{\mathbf{Z}}}_{\tilde{\gamma}}(\infty) = \overline{\mathcal{D}}^{m_{\mathbf{Z}}}_{\gamma}(\infty) + 1$ and that dominates γ as in Lemma 4.25. We describe a coupling for a scaffolding-and-spindles pair $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ and another stopped scaffolding-and-spindles pair $(\mathbf{N}, \mathbf{X})$.

Let $(\tilde{V}_i, i \in \tilde{I})$ be the intervals of $\tilde{\gamma}$ and let $I \subset \tilde{I}$ so that $(V_i, i \in I)$ denotes the matching intervals of γ. For each $i \in I$, we consider an independent Brownian motion $(B^i_y, y \geq 0)$ and we define $f_{\tilde{V}_i}$ and f_{V_i} to be strong solutions to the stochastic differential equations

$$\begin{aligned} df_{\tilde{V}_i}(y) &= \mathrm{Leb}(\tilde{V}_i) + \mu_{\mathbf{Z}}(f_{\tilde{V}_i}(y))dy + \sigma_{\mathbf{Z}}(f_{\tilde{V}_i}(y))dB^i_y; \\ df_{V_i}(y) &= \mathrm{Leb}(V_i) + \mu_{\mathbf{Z}}(f_{V_i}(y))dy + \sigma_{\mathbf{Z}}(f_{V_i}(y))dB^i_y, \end{aligned}$$

and we stop $f_{\tilde{V}_i}$ and f_{V_i} when they hit 0. With this construction we have that $(f_{\tilde{V}_i}, i \in I)$ are independent diffusions with the same dynamics and with initial states $(\tilde{V}_i, i \in I)$, and similarly for $(f_{V_i}, i \in I)$. We also have for each $i \in I$ that $\zeta(f_{V_i}) \leq \zeta(f_{\tilde{V}_i})$ and for each $y > 0$ that $f_{V_i}(y) \leq f_{\tilde{V}_i}(y)$. We then define a pair of point measures $(\mathbf{N}, \mathbf{N}_\gamma)$. For $z > 0$, let $\mathbb{Q}^0_z$ denote the law of the diffusion $\mathbf{Z}$ started from z, and let ν be the Pitman-Yor excursion measure of $\mathbf{Z}$. Consider independent Poisson random measures $(\mathbf{N}_i, i \in I)$ with intensity $\mathrm{Leb} \otimes \nu$ and scaffolding $\xi(\mathbf{N}_i) = \mathbf{X}_i$ for $i \in I$; we use this to define

$$\mathbf{N}_{\tilde{V}_i} := \delta(0, f_{\tilde{V}_i}) + \mathbf{N}_i|_{[0,\tilde{T}_i]} \quad \text{where} \quad \tilde{T}_i := \inf\{t > 0 : \mathbf{X}_i(t) = -\zeta(f_{\tilde{V}_i})\}.$$

We further define $\mathbf{N}_\gamma := \bigstar_{i\in I}\mathbf{N}_{V_i}$ where $\mathbf{N}_{V_i}$ is given by

$$\mathbf{N}_{V_i} := \delta(0, f_{V_i}) + \mathbf{N}_i|^{\leftarrow}_{(T_i,\tilde{T}_i]} \quad \text{with} \quad T_i := \inf\{t \geq 0 : \mathbf{X}_i(t) = \zeta(f_{V_i}) - \zeta(f_{\tilde{V}_i})\} \tag{4.24}$$

and $\tilde{T}_i$ is as stated above. Let $T = \sum_{i\in I} T_i$ and note T is a $\mathbf{P}$-a.s. finite random variable. Finally let $\mathbf{N} := \bigstar_{i\in I}\mathbf{N}_{\tilde{V}_i}$ and $\mathbf{N}_\gamma := \bigstar_{i\in I}\mathbf{N}_{V_i}$.

The rest of the proof uses the same ideas as the proof of Theorem 4.17. Let q, $\kappa > 0$ such that $\alpha^+ + \kappa < q < q_0$ where α^+, q_0 are as in Assumption B and let $\rho := (q - \alpha^+ - \kappa) \wedge q/2$. We recall $\mathbf{N}_{M,\varepsilon_0}$ from (4.13) and define $\mathbf{N}_{\gamma,M,\varepsilon_0}$:

$$\mathbf{N}_{M,\varepsilon_0} := \mathbb{1}_{\sup_{u\in\mathbb{R}}\ell^u(T)<M} \sum_{\substack{(t,f)\text{ atom of}\\ \mathbf{N}: A(f)<\varepsilon_0}} \delta_{(t,f)}; \quad \mathbf{N}_{\gamma,M,\varepsilon_0} := \mathbb{1}_{\sup_{u\in\mathbb{R}}\ell^u(T)<M} \sum_{\substack{(t,f)\text{ atom of }\mathbf{N}_\gamma:\\ A(f)<\varepsilon_0, f(0)=0}} \delta_{(t,f)}$$

where ℓ^u for $u \in \mathbb{R}$ denotes the local time of $\xi(\mathbf{N})$. We note that the set of atoms of $\mathbf{N}_{\gamma,M,\varepsilon_0}$ is a subset of the set of atoms of $\mathbf{N}_{M,\varepsilon_0}$ and so we can use the bound in (4.19) as $\mathbf{N}$ is equal in distribution to the concatenation of the clade components of a Poisson random measure stopped at $\tau^0(k_0^{-1}\overline{\mathcal{D}}^{m_{\mathbf{Z}}}_{\gamma}(\infty) + 1)$, see (3.6). The constant in front of the bound in (4.19) will change from A to A/p where $p := \mathbf{P}(\tilde{\gamma}\text{ dominates }\gamma) > 0$ and the definition of domination is given in Lemma 4.25 (note that the fact that the domination of γ by $\tilde{\gamma}$ does not respect the order of the intervals in the interval partition does not impact the upper bound for the sum).

At this point we need to incorporate the initial spindles and the spindles with amplitude greater than ε_0. In the proof of Theorem 4.17, we only had the spindles with amplitude greater than ε_0 to add in, and as there were only finitely many of these we could concatenate finitely many continuous IP-evolutions to finish the proof. In this setting however as we have potentially infinitely many initial spindles we need to take care to incorporate these.

We adapt the proof of the Kolmogorov-Čentsov continuity criterion from Revuz and Yor (1991, Theorem 1.2.1). We define

$$D_m := \{i2^{-m} : i \in \mathbb{N} \cap [0, 2^m)\} \text{ for } m \in \mathbb{N}; \qquad D := \bigcup_{m \in \mathbb{N}} D_m$$

and

$$M_\rho := \sup_{x,y \in D, x \neq y} |x-y|^{-\rho} \Bigg\{ \sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_{M,\varepsilon_0}:,\\ \mathbf{X}_{t-} < x < \mathbf{X}_t \text{ or}\\ \mathbf{X}_{t-} < y < \mathbf{X}_t}} \Big| g\big(f(y - \mathbf{X}_{t-})\big) - g\big(f(x - \mathbf{X}_{t-})\big) \Big| \Bigg\}.$$

Then it follows from the proof in Revuz and Yor (1991, Theorem 1.2.1) that M_ρ is finite $\mathbf{P}$-a.s. as we have shown the bound in (4.19) with $\mathbf{N}_{\gamma,M,\varepsilon_0}$ in place of $\mathbf{N}_{M,\varepsilon_0}$. Let $\mathbf{N}_{\gamma,M} := \mathbb{1}_{\sup_{u\in\mathbb{R}} \ell^u(T) < M} \mathbf{N}_\gamma$. From this point we show that $\overline{\text{skewer}}(\mathbf{N}_{\gamma,M}, \mathbf{X}_\gamma)$ is d'_H-continuous on D. Let $\varepsilon > 0$, let $K \in \mathbb{N}$, $i_1, \ldots, i_K \in I$ such that $\sum_{i \in I : i \neq i_1, \ldots, i_K} A(f_{V_i}) < \varepsilon/4$, and let $K' \in \mathbb{N}$ and $(t_1, f_1), \ldots, (t_{K'}, f_{K'})$ be the atoms of $\mathbf{N}_\gamma$ with amplitude greater than or equal to ε_0. There exists $\delta_1 > 0$ such that for x, $y \in D$ with $|x - y| < \delta_1$ we have

$$\sum_{j=1}^{K} \Big| g(f_{V_j}(y)) - g(f_{V_j}(x)) \Big| < \frac{\varepsilon}{4}; \qquad \sum_{j=1}^{K'} \Big| g\big(f_j(y - \mathbf{X}_{t_j-})\big) - g\big(f_j(x - \mathbf{X}_{t_j-})\big) \Big| < \frac{\varepsilon}{4}.$$

Furthermore, for x, $y \in D$ with $|x - y| < \delta_2 := (\varepsilon/4M_\rho)^{1/\rho}$ we have

$$\sum_{\substack{(t,f) \text{ atom of } \mathbf{N}_{M,\varepsilon_0}:,\\ \mathbf{X}_{t-} < x < \mathbf{X}_t \text{ or}\\ \mathbf{X}_{t-} < y < \mathbf{X}_t}} \Big| g\big(f(y - \mathbf{X}_{t-})\big) - g\big(f(x - \mathbf{X}_{t-})\big) \Big| < \frac{\varepsilon}{4}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all x, $y \in D$ with $|x-y| < \delta$ the sum of $|g(f(y - \mathbf{X}_{t-})) - g(f(x - \mathbf{X}_{t-}))|$ over all atoms (t, f) in $\mathbf{N}_{\gamma,M}$ is less than ε $\mathbf{P}$-a.s. This is an upper bound for the distortion between skewer$(x, \mathbf{N}_{\gamma,M}, \mathbf{X}_\gamma)$ and skewer$(y, \mathbf{N}_{\gamma,M}, \mathbf{X}_\gamma)$ for the correspondence that matches intervals with each other if they both are generated by the same spindle. This is a correspondence as we have only finitely many spindles with lifetime greater than $|x - y|$; it is the same correspondence used in Theorem 4.17. We can take M to infinity at this stage as by Proposition 4.6 the local time of $\mathbf{X}$ is continuous and therefore bounded. Therefore we have that $\overline{\text{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ is continuous on D $\mathbf{P}$-a.s. To finish, we can define for $y \in [0, \infty)$

$$\gamma^y := \lim_{\substack{s \to y\\ s \in D}} \text{skewer}(s, \mathbf{N}_\gamma, \mathbf{X}_\gamma)$$

and $(\gamma^y, y \geq 0)$ is a continuous modification of $\overline{\text{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$. This modification is in the appropriate subspace $(\mathcal{I}_H(g(E)), d'_H)$ as this subspace is closed. Now assume $g(E)$ is not closed. To complete the argument in this case we avoid taking modifications of IP-evolutions with intervals great than ε_0. As in Theorem 4.17 there are only finitely many spindles with amplitude greater than ε_0 and so we can consider stopping times in (4.20) with $\mathbf{N}_\gamma$ in place of $\mathbf{N}$. Denoting these stopping times by $(S_n^\gamma, n \in \mathbb{N})$ we can obtain finitely many skewer processes $(\overline{\text{skewer}}(\mathbf{N}_\gamma|_{(S_n^\gamma, S_{n+1}^\gamma)}), n \leq P)$ for some $P \in \mathbb{N}$ and concatenate these with the skewer processes generated by the individual spindles with amplitude larger than ε_0 as in (4.21). □

Remark 4.27. The previous proposition shows path continuity at 0 for certain initial states but not for all possibles ones. This is a partial response to the claim just after Forman et al. (2020a, Corollary 6.19) that states "We believe that d'_H-path-continuity extends to time 0" which is in reference to their IP-evolutions generated from self-similar diffusions. As their self-similar diffusions satisfy our Assumption B we have confirmed that this belief is correct for at least some initial states.

Proposition 4.28 (Simple Markov property under Assumption B on Ω_0)**.** *Consider a diffusion under Assumption B. Let $\gamma \in g^*(\overline{\mathcal{I}}^{(q)}_{m\mathbf{Y}}(E))$ and let $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ be a scaffolding-and-spindles pair with initial state* $\overline{\textit{skewer}}(0, \mathbf{N}_\gamma, \mathbf{X}_\gamma) = \gamma$ *a.s. as in Definition 3.11. Let $(\gamma^y, y \geq 0)$ be a modification of* $\overline{\textit{skewer}}(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ *that is continuous on $[0,\infty)$ which exists by Proposition 4.26. Recall the filtration $(\mathcal{F}^y_{\mathcal{I}}, y \geq 0)$ from Corollary 4.21. We define $(\Omega_0, \mathcal{F}, (\mathcal{F}^y_{\mathcal{I}}, y \geq 0), \mathbb{P}_\gamma)$, a filtered probability space, where Ω_0 is the set of all paths over $\mathcal{I}_H$ that are continuous on $[0,\infty)$ and with initial states in $g^*(\overline{\mathcal{I}}^{(q)}_{m\mathbf{Y}})$, $\mathcal{F} := \bigcup_{y\geq 0} \mathcal{F}^y_{\mathcal{I}}$, and also $\mathbb{P}_\gamma(\cdot) := \mathbf{P}((\gamma^y, y \geq 0) \in \cdot)$. Finally, let μ be a probability distribution on interval partitions that is supported on $g^*(\overline{\mathcal{I}}^{(q)}_{m\mathbf{Y}}(E))$, and define the probability measure $\mathbb{P}_\mu(\cdot) = \int_{\gamma\in g^*(\mathcal{I}_H(E))} \mathbb{P}_\gamma(\cdot)\mu(d\gamma)$. Then $(\gamma^y, y \geq 0)$ is a simple Markov process on $(\Omega_0, \mathcal{F}, (\mathcal{F}^y_{\mathcal{I}}, y \geq 0), \mathbb{P}_\mu)$.*

Proof: This proof uses the same arguments as the analogous result Forman et al. (2020a, Corollary 6.13). In particular, we have continuity of the sample paths from Proposition 4.26 with initial states in $g^*(\overline{\mathcal{I}}^{(q)}_{m\mathbf{Y}}(E))$ as well as the simple Markov property for the skewer process given in Proposition 3.20. Furthermore, at each level we have, for $y \in (0,\infty)$, that the

$$\text{skewer}(y, \mathbf{N}_\gamma, \mathbf{X}_\gamma) \in g^*(\mathcal{I}^{(q)}_{m\mathbf{Y}}(E)) \subset g^*(\overline{\mathcal{I}}^{(q)}_{m\mathbf{Y}}(E)) \text{ a.s.}$$

by Corollary 4.23. Therefore, for any fixed $y \in (0,\infty)$, the continuous modification takes values in $g^*(\overline{\mathcal{I}}^{(q)}_{m\mathbf{Y}}(E))$ a.s. as required. □

4.6. *Strong Markov property and proof of Theorem 1.6.* In this section we assume that the diffusion satisfies Strong Assumption B, that is Assumption B along with the assumption that either $\liminf_{x\to 0} g(x)/x > 0$ or $\limsup_{x\to 0} g(x)/x < \infty$. Note that the identity transformation $g(x) = x$ satisfies these conditions, and therefore Strong Assumption B is more restrictive than Assumption B but more general than Assumption B'. Before we begin the proof we give a class of transformations g that do not satisfy $B5$ of Strong Assumption B to show that this really is a restriction.

Example 4.29 (Transformation g that does not satisfy $B5$).
Consider a function $g : [0, 1/32] \to [0, 1/32]$ such that for all $n \geq 3$ we have:

$$g(2^{-2^n}) = 2^{-2^n}; \quad g(2^{-2^n-1}) = 2^{-2^n-n}; \quad g(2^{-2^{n+1}+1}) = 2^{-2^{n+1}+n},$$

and where for all $n \geq 3$, $x \in [2^{-2^{n+1}}, 2^{-2^{n+1}+1}]$, $[2^{-2^{n+1}+1}, 2^{-2^n-1}]$, $[2^{-2^n-1}, 2^{-2^n}]$ that g is smooth and strictly increasing and $g(0) = 0$. Then g is a smooth continuous strictly increasing bijection and moreover

$$\liminf_{x\downarrow 0} \frac{g(x)}{x} \leq \lim_{n\to\infty} \frac{g(2^{-2^n-1})}{2^{-2^n-1}} = \lim_{n\to\infty} 2^{-n+1} = 0;$$

$$\limsup_{x\downarrow 0} \frac{g(x)}{x} \geq \lim_{n\to\infty} \frac{g(2^{-2^{n+1}+1})}{2^{-2^{n+1}+1}} = \lim_{n\to\infty} 2^{n-1} = \infty.$$

We state key results about the transformations g^* and $(g^{-1})^*$ which apply under Strong Assumption B.

Lemma 4.30. *Let $g : E \to g(E)$ be a strictly increasing continuous function with $g(0) = 0$ where $E \subseteq [0,\infty)$ is of the form $E = [0,h)$ or $E = [0,h]$ for some $h > 0$ and $g(E) \subseteq [0,\infty)$. If $\limsup_{x\to 0} g(x)/x < \infty$ then the map $g^* : \mathcal{I}_H(E) \to \mathcal{I}_H(g(E))$ is injective and continuous. If instead $\liminf_{x\to 0} g(x)/x > 0$ then the map $(g^{-1})^* : \mathcal{I}_H(g(E)) \to \mathcal{I}_H(E)$ is injective and continuous.*

Proof: First suppose that $\limsup_{x\to 0} g(x)/x < \infty$. Let $\varepsilon > 0$, let $\beta \in \mathcal{I}_H(E)$ and let $(a_j)_{j\in\mathbb{N}}$ be the interval lengths of β in a non-decreasing order and let $(U_j, j \in \mathbb{N})$ be the corresponding intervals of β. If $g(h)$ is finite let $D = (a_1+g(h))/2$ and if $g(h)$ is infinite let $D = a_1+1$. Let $C := \sup_{x\in(0,D]} g(x)/x$.

We note that $\sum_{j\in\mathbb{N}} g(a_j) \le C\sum_{j\in\mathbb{N}} a_j$ which is finite so $g^*(\beta)$ is a non-trivial interval partition. The fact that g^* is injective then follows immediately from g being injective.

Let $K \in \mathbb{N}$ be large enough so that $\sum_{j>K} a_j < \varepsilon/4C$. As g is continuous, then g is uniformly continuous on $[0, D]$, and therefore let $\tilde{\varepsilon} > 0$ be small enough so that for $x, y \in [0, D]$ with $|x-y| < \tilde{\varepsilon}$ we have $|g(x) - g(y)| < \varepsilon/2K$.

Let $\delta \coloneqq \frac{1}{2}\min\{\tilde{\varepsilon}, a_K/2, \varepsilon/4C, D - a_1\}$. Let $\gamma \in \mathcal{I}_H(E)$ such that $d'_H(\beta,\gamma) < \delta$. Consider a correspondence $(\tilde{U}_j, \tilde{V}_j)_{1\le j\le\tilde{K}}$ from β to γ with distortion less than δ. Since $\delta < a_K$, we must have that $\tilde{K} \ge K$ and $\{U_j\}_{1\le j\le K} \subseteq \{\tilde{U}_j\}_{1\le j\le\tilde{K}}$. Let $(V_j)_{1\le j\le K}$ denote the intervals of γ paired in the correspondence with the respective U_j, and let $b_j \coloneqq \mathrm{Leb}(V_j)$.

Consider the correspondence of $g^*(\beta)$ and $g^*(\gamma)$ that uses the corresponding interval pairs to $(U_j, V_j)_{1\le j\le K}$. This has distortion

$$\sum_{j=1}^{K} |g(a_j) - g(b_j)| + \max\left\{\|g^*(\beta)\| - \sum_{j=1}^{K} g(a_j), \|g^*(\gamma)\| - \sum_{j=1}^{K} g(b_j)\right\}.$$

First, let $1 \le j \le K$. Then $|a_j - b_j| < \tilde{\varepsilon}$ and therefore we have $|g(a_j) - g(b_j)| < \varepsilon/2K$. Also note that $\|\beta\| - \sum_{j=1}^{K} a_j < \varepsilon/4C$, and therefore we have

$$\|\gamma\| - \sum_{j=1}^{K} b_j \le d'_H(\beta,\gamma) + \|\beta\| - \sum_{j=1}^{K} a_j < \frac{\varepsilon}{2C}.$$

From this we can conclude that

$$\max\left\{\|g^*(\beta)\| - \sum_{j=1}^{K} g(a_j), \|g^*(\gamma)\| - \sum_{j=1}^{K} g(b_j)\right\} < \frac{\varepsilon}{2}$$

and the first result follows. To prove the second result, we show that if $\liminf_{x\to 0} g(x)/x > 0$ then

$$\limsup_{x\to 0} \frac{g^{-1}(x)}{x} \le \left(\liminf_{y\to 0} \frac{g(y)}{y}\right)^{-1} < \infty$$

and the result then follows as previously. To see this let $(x_n, n \ge 1)$ be a positive sequence that converges to 0. Then $(g^{-1}(x_n), n \ge 1)$ is also a sequence that converges to 0. Therefore we have that

$$\liminf_{n\to\infty} \frac{x_n}{g^{-1}(x_n)} = \liminf_{n\to\infty} \frac{g(g^{-1}(x_n))}{g^{-1}(x_n)} \ge \liminf_{y\to 0} \frac{g(y)}{y} > 0.$$

Therefore $\limsup_{n\to\infty} g^{-1}(x_n)/x_n \le 1/(\liminf_{y\to 0} g(y)/y) < \infty$ and the second result follows in the same way as the first result. □

Lemma 4.30 states that under the condition $\liminf_{x\to 0} g(x)/x > 0$ we have the equality of sets $g^*(\mathcal{I}_H(E))$ and $\mathcal{I}_H(g(E))$. We recall a similar result in Lemma 4.25 which applies to smaller spaces of interval partitions where interval partitions have diversities; in that setting, under Assumption B, we have the equality of $g^*(\mathcal{I}_{m_{\mathbf{Y}}}(E))$ and $\mathcal{I}_{m_{\mathbf{Z}}}(g(E))$.

Remark 4.31. In fact the conditions in Lemma 4.30 are necessary as well as sufficient in the sense that if $\limsup_{x\to 0} g(x)/x = \infty$ then the map g^* is not injective and if $\liminf_{x\to 0} g(x)/x = 0$ then the map $(g^{-1})^*$ is not injective (they both map certain interval partitions to $\emptyset$ as the transformed intervals are not summable).

We now give certain bounds for the diffusion under Strong Assumption B which we will use to prove continuity in the initial state in Proposition 4.35.

Lemma 4.32. *Consider a diffusion $\mathbf{Z}$ that satisfies Strong Assumption B with untransformed diffusion $\mathbf{Y}$ and spatial transformation g such that $\liminf_{x\to 0} g(x)/x > 0$, and let $z > 0$ and let $\varepsilon > 0$. Then there exists $k_7 = k_7(\varepsilon, z) > 0$ such that for any interval partition $\gamma \in \mathcal{I}_H(g(E))$ such that $\|\gamma\| < k_7$ we have that*

$$\mathbf{P}(\zeta^+(\mathbf{X}_\gamma) < z) > 1 - \varepsilon,$$

where $\mathbf{X}_\gamma$ is constructed as in Definition 3.11 and $\zeta^+(X) := \sup_{t\in[0,len(X)]} X(t)$ for $X \in \mathcal{D}$.

Proof: Let ε, $z > 0$. The injectivity of $(g^{-1})^*$ from Lemma 4.30 leads to the surjectivity of g^* and so $g^*(\mathcal{I}_H(E)) = \mathcal{I}_H(g(E))$. Also we note that the untransformed diffusion satisfies Corollary 3.13. Therefore we can create a scaffolding-and-spindles pair $(\mathbf{N}_\gamma, \mathbf{X}_\gamma)$ with an initial interval partition $\gamma \in \mathcal{I}_H(g(E))$ as in Definition 3.11 where we construct the scaffolding by adding $\zeta(f_{V_i})$ jumps for each interval V_i of the initial interval partition γ. We will compare this to an SPLP with a single initial jump $P = \sum_{U\in\gamma} \zeta(f_U)$ at the beginning of the process which we run until it hits level 0 and we will denote this process as $\mathbf{X}_P$. By a natural coupling we have $\mathbf{X}_P(t) \geq \mathbf{X}_\gamma(t)$ for all $t \geq 0$ and therefore the event $\{\zeta^+(\mathbf{X}_\gamma) > z\}$ is a subset of by $\{\zeta^+(\mathbf{X}_P) > z\}$. We construct $\mathbf{X}_P$ by considering a Lévy process started at $\sum_{U\in\beta} \zeta(f_U)$ such that $\mathbf{X}_P - \sum_{U\in\beta} \zeta(f_U)$ has the same distribution as an SPLP $\mathbf{X}$ with Lévy measure $\nu(\zeta \in \cdot)$.

Let W be the scale function for the Lévy process $\mathbf{X}$. This function is continuous and increasing with $W(0) = 0$ (Bertoin, 1996, Theorem VII.8), and therefore there exists $x > 0$ such that $W(x)/W(z) \leq \varepsilon/2$. Also note that for the empty interval partition $\emptyset$ we have $d'_H(\gamma, \emptyset) = \|\gamma\|$. Recall that we can apply Proposition 3.12 to the untransformed diffusion by Corollary 3.13. By Lemma 4.30 $(g^{-1})^*$ is continuous at $\emptyset$ and therefore there exists $k_7 > 0$ such that for all $\gamma \in \mathcal{I}_H(g(E))$ with $\|\gamma\| < k_7$ we have $\|(g^{-1})^*(\gamma)\| < \varepsilon x/2(1 + k_3)$ where k_3 is constant from Proposition 3.12. Recall ε' from the same proposition and take $k_7 < g(\varepsilon')$. Then, by (3.18), we therefore have that

$$\mathbf{E}\Big(\sum_{i\in I} \zeta(f_{V_i})\Big) \leq (1 + k_3) \sum_{i\in I} g^{-1}(\mathrm{Leb}(V_i)) = (1 + k_3)\|(g^{-1})^*(\gamma)\|.$$

Therefore, for $\gamma \in \mathcal{I}_H(g(E))$ such that $||\gamma|| < k_7$, we have that

$$\mathbf{P}\big(\zeta^+(\mathbf{X}_\gamma) > z\big) \leq \mathbf{P}\big(\zeta^+(\mathbf{X}_P) > z\big) \leq \mathbf{P}\Big(\sum_{i\in I} \zeta(f_{V_i}) > x\Big) + \frac{W(x)}{W(z)} \leq \varepsilon.$$

□

In the following lemma, for $x \in g(E)$, let $\mathbb{Q}^0_x(\cdot)$ denote the law of the diffusion under Assumption B.

Lemma 4.33. *Consider a diffusion under Assumption B, let $D \in g(E)$ and let d_0, $d_1 > 0$. Then there exists $d_2 > 0$ such that for all a, $b \in [0, D]$ with $a < b < a + d_2$ we have*

$$\mathbb{Q}^0_b(T_a < d_1) > 1 - d_0.$$

Proof: This result uses the Markov inequality. We first note that

$$\mathbb{Q}^0_b(T_a \geq d_1) \leq \frac{\mathbb{E}_{\mathbb{Q}^0_b}(T_a)}{d_1}$$

and recall from Corollary 3.13 that the function $r(x) := \mathbb{E}_{\mathbb{Q}^0_{g^{-1}(x)}}(\zeta)$ has a derivative that is bounded above near 0 (we have the transformation g^{-1} here as the corollary we use only applies to untransformed diffusions). Therefore, setting $C := \sup_{x\in(0,g^{-1}(D))} r'(x)$ we have for all $0 \leq a < b \leq D$

$$\mathbb{E}_{\mathbb{Q}^0_b}(T_a) = r(g^{-1}(b)) - r(g^{-1}(a)) \leq C|g^{-1}(b) - g^{-1}(a)|.$$

Now, g^{-1} is uniformly continuous on $[0, D]$ and so there exists $d_2 > 0$ such that for a, $b \in [0, D]$ with $b > a$ and $b - a < d_2$ we have $g^{-1}(b) - g^{-1}(a) < d_0 d_1/C$ and the result follows. □

Lemma 4.34 (Continuity in the initial state for $\overline{\text{skewer}}(\mathbf{N}_U, \mathbf{X}_U)$)**.** *Let d_3, d_4, $z > 0$, $D \in g(E)$ and let $a \in [0, D]$. Then there exists $d_5 > 0$ such that for $b \in [0, D]$ with $|a - b| < d_5$ we have the following. Let $U = (0, a)$ and $V = (0, b)$. Then, on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$, there exists a coupled pair of scaffolding and spindles $(\mathbf{N}_U, \mathbf{X}_U)$, $(\mathbf{N}_V, \mathbf{X}_V)$ started from U and V as in Definition 3.11 on which we have*

$$\mathbf{P}(d'_H(\mathit{skewer}(z, \mathbf{N}_U, \mathbf{X}_U), \mathit{skewer}(z, \mathbf{N}_V, \mathbf{X}_V)) > d_3) < d_4.$$

Proof: Let d_3, d_4, $z > 0$ and let $a \le D \in g(E)$. Let $d_6 > 0$ be small enough that

$$W(z - 2d_6)/W(z) > 1 - d_4/5$$

where W is the scale function for the SPLP from Lemma 4.32. Let $d_7 = d_2/2$ where d_2 is the constant obtained from Lemma 4.33 when applied with $d_0 = d_4/5$ and $d_1 = d_6$. Let f be a 0-diffusion started at $(a + d_7) \wedge D$ and note that

$$\mathbf{P}\Big(G_0 := \Big\{T_{(a-d_7)\vee 0}(f) < d_6\Big\}\Big) > 1 - \frac{d_4}{5}$$

where $T_{(a-d_7)\vee 0}(f)$ is the hitting time of $(a - d_7) \vee 0$ by f. The diffusion f has a finite lifetime and so is uniformly continuous and therefore there exists $d_8 > 0$ such that

$$\mathbf{P}\Big(G_1 := \Big\{|f(x) - f(y)| < \frac{d_3}{2} \;\; \forall x, y \in \mathbb{R} : |x - y| < d_8\Big\}\Big) > 1 - \frac{d_4}{5}.$$

Now consider an independent Poisson random measure $\mathbf{N}$ with scaffolding $\mathbf{X}$ and which is stopped at $T_{-\zeta(f)}$. First note that

$$\mathbf{P}\Big(G_2 := \Big\{\sup_{t \in [T_{-\zeta(f)+2d_6}, T_{-\zeta(f)}]} \mathbf{X}_t \le z - \zeta(f)\Big\}\Big) \ge \frac{W(z - 2d_6)}{W(z)} > 1 - \frac{d_4}{5}$$

by the independence of f and $\mathbf{N}$ and the strong Markov property of $\mathbf{X}$ and where we follow convention that the supremum over the empty set is $-\infty$. Furthermore note that $(\text{skewer}(y, \mathbf{N}, \mathbf{X}), y \in \mathbb{R})$ has a uniformly continuous modification $(\gamma^y, y \in \mathbb{R})$ by Theorem 4.17. Let $d_9 > 0$ be small enough so that

$$\mathbf{P}\Big(G_3 := \Big\{d'_H(\gamma^x, \gamma^y) < d_3/2 \text{ for all } x, y \in \mathbb{R} : |x - y| < d_9\Big\}\Big) > 1 - \frac{d_4}{5}.$$

Now, let $d_{10} = d_2/2$ where d_2 is now the constant obtained from Lemma 4.33 when applied with $d_1 = d_8 \wedge d_9$ and $d_0 = d_4/5$ and let $d_5 := d_7 \wedge d_{10}$. Then we have

$$\mathbf{P}\Big(G_4 := \Big\{T_{(a-d_5)\vee 0}(f) - T_{(a+d_5)\wedge D}(f) < d_8 \wedge d_9\Big\}\Big) > 1 - \frac{d_4}{5}.$$

Now let $b \in [0, D]$ with $|a - b| < d_5$. Let $f_U(\cdot) := f(T_a + \cdot)$ and let $f_V(\cdot) := f(T_b + \cdot)$. We define

$$\tilde{\mathbf{N}}_U := \mathbf{N}|_{[0, T_{-\zeta(f_U)}(\mathbf{N})]}; \quad \mathbf{N}_U := \delta_{(0, f_U)} + \tilde{\mathbf{N}}_U; \quad \tilde{\mathbf{N}}_V := \mathbf{N}|_{[0, T_{-\zeta(f_V)}(\mathbf{N})]}; \quad \mathbf{N}_V := \delta_{(0, f_V)} + \tilde{\mathbf{N}}_V.$$

Note that

$$\mathbf{P}\Big(G_5 := \Big\{\gamma^{z-\zeta(f_U)} = \text{skewer}(z - \zeta(f_U), \mathbf{N}, \mathbf{X}), \quad \gamma^{z-\zeta(f_V)} = \text{skewer}(z - \zeta(f_V), \mathbf{N}, \mathbf{X})\Big\}\Big) = 1$$

by the spatial homogeneity of the scaffolding-and-spindles pair and the independence of f_U and $\mathbf{N}$ and of f_V and $\mathbf{N}$. Therefore we have

$$\text{skewer}(z, \tilde{\mathbf{N}}_U, \mathbf{X}_U) = \gamma^{z-\zeta(f_U)}, \quad \text{skewer}(z, \tilde{\mathbf{N}}_V, \mathbf{X}_V) = \gamma^{z-\zeta(f_V)} \text{ on } G_0 \cap G_2 \cap G_5.$$

Furthermore we have that

$$|f_U(z) - f_V(z)| < d_3/2 \text{ on } G_1 \cap G_4; \quad \text{and} \quad d'_H(\gamma^{z-\zeta(f_U)}, \gamma^{z-\zeta(f_V)}) < d_3/2 \text{ on } G_3 \cap G_4.$$

Therefore, as

$$d'_H(\text{skewer}(z,\mathbf{N}_U,\mathbf{X}_U),\text{skewer}(z,\mathbf{N}_V,\mathbf{X}_V)) \le |f_U(z)-f_V(z)| + d'_H(\text{skewer}(z,\tilde{\mathbf{N}}_U,\mathbf{X}_U),\text{skewer}(z,\tilde{\mathbf{N}}_V,\mathbf{X}_V))$$

then it follows that

$$d'_H(\text{skewer}(z,\mathbf{N}_U,\mathbf{X}_U),\text{skewer}(z,\mathbf{N}_V,\mathbf{X}_V)) < d_3 \text{ on } G := G_0\cap G_1\cap G_2\cap G_3\cap G_4\cap G_5,$$

and as $\mathbf{P}(G) > 1-d_4$ the result follows. □

Proposition 4.35 (Continuity in the initial state)**.** *Consider a diffusion under Strong Assumption B for which* $\liminf_{x\to 0} g(x)/x > 0$*. For* $\gamma \in g^*(I_H(E))$ *recall* $\mathbb{P}_\gamma(\cdot) := \mathbf{P}((\gamma^y, y\ge 0)\in\cdot)$ *from* (4.23)*, where* $(\gamma^y,y\ge 0)$ *is a modification of* $\overline{\text{skewer}}(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ *that is continuous for* $y\in(0,\infty)$*. Let* $\mathbb{E}_\gamma$ *be the expectation operator for* $\mathbb{P}_\gamma$*. Let* $z>0$ *and let* $f:\mathcal{I}_H\to[0,\infty)$ *be a bounded and continuous function. Then the map* $\gamma\mapsto\mathbb{E}_\gamma(f(\gamma^z))$ *is continuous on* $(\mathcal{I}_H(g(E)),d'_H)$*.*

Proof: This was shown for self-similar diffusions in Forman et al. (2020a, Proposition 6.15). We adapt the proof to generalise to our setting.

As in Lemma 4.32 we are under the assumption $\liminf_{x\to 0} g(x)/x > 0$ and so

$$g^*(\mathcal{I}_H(E)) = \mathcal{I}_H(g(E)).$$

Let $\gamma\in\mathcal{I}_H(g(E))$, ε, $z>0$, and let $(\Omega,\mathcal{F}_\mathcal{I},\mathbb{P}_\gamma)$ be the probability space as defined in Section 4.5, where $\Omega := \mathcal{C}((0,\infty),\mathcal{I}_H)$, $\mathbb{P}_\gamma$ is the probability measure for $(\gamma^y,y\ge 0)$ and $\mathcal{F}_\mathcal{I}$ is the natural filtration for $(\gamma^y,y\ge 0)$. As f is continuous at γ^z we can define a $(0,\infty)$-valued random variable $A_\varepsilon(\gamma^z)$ on $(\Omega,\mathcal{F}_\mathcal{I},\mathbb{P}_\gamma)$ for which we have that for $\beta\in\mathcal{I}_H(g(E))$ we have $d'_H(\gamma^z,\beta) < A_\varepsilon(\gamma^z)$ implies that $|f(\gamma^z)-f(\beta)| < \varepsilon/3$. Let $M := \sup_{\beta\in\mathcal{I}_H(g(E))} f(\beta)$. Then there is $\tilde\varepsilon > 0$ such that $\mathbb{P}_\gamma(A_\varepsilon(\gamma^z) < \tilde\varepsilon) < \varepsilon/3M$. We take $\hat\gamma\in\mathcal{I}_H(g(E))$ and consider a probability space $(\tilde\Omega,\mathcal{A},\mathbf{P})$ for a pair of processes $(\gamma^y,y\ge 0)$ and $(\hat\gamma^y,y\ge 0)$ which have marginal distributions $\mathbb{P}_\gamma$ and $\mathbb{P}_{\hat\gamma}$ respectively; we will later specify an explicit coupling. By Lemma 4.30 note that g^* is surjective so as the untransformed diffusion satisfies Corollary 3.13 it is possible to construct IP-evolutions $(\gamma^y,y\ge 0)$ and $(\hat\gamma^y,y\ge 0)$ with initial states γ and $\hat\gamma$. We first note that

$$\Big|\mathbb{E}_\gamma(f(\gamma^z)) - \mathbb{E}_{\hat\gamma}(f(\hat\gamma^z))\Big| \le \mathbf{E}\Big(|f(\gamma^z)-f(\hat\gamma^z)|\Big).$$

Then we can further bound the expectation above by

$$\mathbf{E}\Big(|f(\gamma^z)-f(\hat\gamma^z)|\Big(\mathbb{1}_{d'_H(\gamma^z,\hat\gamma^z)<\tilde\varepsilon, A_\varepsilon(\gamma^z)\ge\tilde\varepsilon} + \mathbb{1}_{d'_H(\gamma^z,\hat\gamma^z)\ge\tilde\varepsilon} + \mathbb{1}_{A_\varepsilon(\gamma^z)<\tilde\varepsilon}\Big)\Big).$$

The first of the three terms is less than or equal to $\varepsilon/3$; the second is less than or equal to $M\mathbf{P}(d'_H(\gamma^z,\hat\gamma^z)\ge\tilde\varepsilon)$; and the third is less than or equal to $\varepsilon/3$. Therefore it suffices to show that there is some $\delta>0$ such that for $\hat\gamma\in\mathcal{I}_H(g(E))$ with $d'_H(\gamma,\hat\gamma)<\delta$ we have a coupled pair of IP-evolutions $(\gamma^y,y\ge 0)$ and $(\hat\gamma^y,y\ge 0)$ starting from these two initial states, with

$$\mathbf{P}(d'_H(\gamma^z,\hat\gamma^z) > \tilde\varepsilon) < \frac{\varepsilon}{3M}. \tag{4.25}$$

To this end, let U_1, U_2, … denote the blocks of γ, listed in non-increasing order by mass and let $a_j := \text{Leb}(U_j)$.

Let $k_7(\varepsilon/9M,z)$ be the constant from Lemma 4.32 and let K be large enough that $\sum_{j>K} a_j < \frac{1}{2}k_7$. If $a_1 < g(h)$ then take $D\in g(E)$ such that $a_1 < D < g(h)$ and if $a_1 = g(h)$ then take $D = a_1 = g(h)$. For $1\le j\le K$ let $\delta_j = d_5$ as in Lemma 4.34 with $d_3 = \tilde\varepsilon/K$, $d_4 = \varepsilon/9MK$, D as above and $a = a_j$. Now let

$$\delta < \begin{cases} \min\{\delta_1,\dots,\delta_K,a_K,\frac{k_7}{2},D-a_1\} \text{ if } a_1 < D < g(h),\\ \min\{\delta_1,\dots,\delta_K,a_K,\frac{k_7}{2}\} \text{ if instead } a_1 = D = g(h).\end{cases}$$

Let $\hat{\gamma}\in\mathcal{I}_H(g(E))$ with $d'_H(\gamma,\hat{\gamma})<\delta$. By definition of d'_H, there exists a correspondence $(\tilde{U}_j,\tilde{V}_j)_{1\le j\le\tilde{K}}$ from γ to $\hat{\gamma}$ with distortion less than δ. Since $\delta<a_K$, we get $\tilde{K}\ge K$ and $\{U_j\}_{1\le j\le K}\subset\{\tilde{U}_j\}_{1\le j\le\tilde{K}}$. Let $(V_j)_{1\le j\le K}$ denote the terms paired with the respective U_j in the correspondence. For $1\le j\le K$, let $b_j:=\mathrm{Leb}(V_j)$. Let $(V_j,j>K)$ be the remaining intervals of $\hat{\gamma}$. Note that necessarily we have $|a_j-b_j|<\delta$ for $1\le j\le K$. Consider coupled scaffolding and spindles pairs $(\mathbf{N}_{U_j},\mathbf{X}_{U_j})$, $(\mathbf{N}_{V_j},\mathbf{X}_{V_j})$ independently for each $1\le j\le K$ as in Lemma 4.34. For $j>K$ we consider independent scaffolding and spindles $(\mathbf{N}_{U_j},\mathbf{X}_{U_j})$ and $(\mathbf{N}_{V_j},\mathbf{X}_{V_j})$. We construct coupled scaffolding and spindles $(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ and $(\mathbf{N}_{\hat{\gamma}},\mathbf{X}_{\hat{\gamma}})$ constructed from coupled scaffolding and spindles $(\mathbf{N}_{U_j},\mathbf{X}_{U_j})$ and $(\mathbf{N}_{V_j},\mathbf{X}_{V_j})$ for $j\in\mathbb{N}$ as in Definition 3.11. Let $(\gamma^y,y\ge0)$ and $(\hat{\gamma}^y,y\ge0)$ be the modifications of $\overline{\mathrm{skewer}}(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ and $\overline{\mathrm{skewer}}(\mathbf{N}_{\hat{\gamma}},\mathbf{X}_{\hat{\gamma}})$ that are continuous on $(0,\infty)$ which are shown to exist in Lemma 4.20. First we make the obvious note that

$$\mathbf{P}\Big(F_0:=\Big\{\gamma^z=\mathrm{skewer}(z,\mathbf{N}_\gamma,\mathbf{X}_\gamma);\quad \hat{\gamma}^z=\mathrm{skewer}(z,\mathbf{N}_{\hat{\gamma}},\mathbf{X}_{\hat{\gamma}})\Big\}\Big)=1$$

as $(\gamma^y,y\ge0)$ and $(\hat{\gamma}^y,y\ge0)$ are modifcations of the skewer processes. Also, for $1\le j\le K$, we have $|a_j-b_j|<\delta_j$ and so

$$\mathbf{P}\Big(F_1^j:=\Big\{d'_H(\mathrm{skewer}(z,\mathbf{N}_{U_j},\mathbf{X}_{U_j}),\mathrm{skewer}(z,\mathbf{N}_{V_j},\mathbf{X}_{V_j}))<\frac{\tilde{\varepsilon}}{K}\Big\}\Big)>1-\frac{\varepsilon}{9MK}\text{ for }1\le j\le K.$$

Furthermore we have

$$\mathbf{P}\Big(F_2:=\Big\{\forall j>K,\zeta^+(\mathbf{N}_{U_j})<z\Big\}\Big)\ge1-\frac{\varepsilon}{9M}\text{ by Lemma 4.32.}$$

Also we have that

$$||\hat{\gamma}||-\sum_{j=1}^K b_j\le d'_H(\gamma,\hat{\gamma})+||\gamma||-\sum_{j=1}^K a_j<\delta+\frac{1}{2}k_7<k_7.$$

Therefore

$$\mathbf{P}\Big(F_3:=\Big\{\zeta^+(\mathbf{N}_{V_j})<z,\forall j>K\Big\}\Big)\ge1-\frac{\varepsilon}{9M}\text{ again by Lemma 4.32.}$$

Pulling all these together we conclude

$$\mathbf{P}\Big(F:=F_0\cap\Big(\bigcap_{j=1}^K F_1^j\Big)\cap F_2\cap F_3\Big)\ge1-\frac{\varepsilon}{3M},$$

and we now show that $d'_H(\gamma^z,\hat{\gamma}^z)<\tilde{\varepsilon}$ on F to finish the proof. First we note that on F_2 γ^z consists only of intervals from spindles from $\mathbf{N}_{U_j}$ for $1\le j\le K$ and similarly for $\hat{\gamma}^z$ on F_3 and therefore

$$d'_H(\gamma^z,\hat{\gamma}^z)\le\sum_{j=1}^K d'_H(\mathrm{skewer}(z,\mathbf{N}_{U_j},\mathbf{X}_{U_j}),\ \mathrm{skewer}(z,\mathbf{N}_{V_j},\mathbf{X}_{V_j}))\text{ on }F_0\cap F_2\cap F_3,$$

and so $d'_H(\gamma^z,\hat{\gamma}^z)\le\tilde{\varepsilon}$ on F and the result follows. □

Theorem 4.36 (Strong Markov property)**.** *Consider a diffusion under Strong Assumption B and then let $\gamma\in g^*(\mathcal{I}_H(E))$. Let $(\mathbf{N}_\gamma,\mathbf{X}_\gamma)$ be a scaffolding-and-spindles pair started from γ as in Definition 3.11 and let $(\gamma^y,y\ge0)$ be a modification that is continuous on $(0,\infty)$ as in Lemma 4.20. Recall the appropriate filtered probability space a filtered probability space $(\Omega,\mathcal{F}_\mathcal{I},(\mathcal{F}^y_\mathcal{I},y\ge0),\mathbb{P}_\gamma)$ as used in Corollary 4.21. Then we have that $(\gamma^y,y\ge0)$ is a strong Markov process on this filtered probability space.*

Furthermore, in the case where $\gamma\in g^*(\overline{\mathcal{I}}^{(q)}_{m_\mathbf{Y}}(E))$ we have that $(\gamma^y,y\ge0)$ is continuous at 0 and so $(\gamma^y,y\ge0)$ is an IP-diffusion.

Proof: We first consider a diffusion that satisfies Strong Assumption B and we begin with the case where $\liminf_{x\to 0} g(x)/x > 0$. As in Forman et al. (2020a, Proposition 6.17) we extend the result in Proposition 4.35 to the claim of continuity of the process

$$\gamma \mapsto \mathbb{E}_\gamma\Big(\prod_{i=1}^m f_i(\gamma^{y_i})\Big),$$

where $f_1,\ldots,f_m : \mathcal{I}_H \to [0,\infty)$ are bounded and continuous. This can be shown by induction on m using the simple Markov property from Corollary 4.21. Then we show the strong Markov property from this by the method given by Kallenberg (2021, Theorem 17.17) where the continuity in the initial state stated above then works in place of the Feller property that is used in the result in Forman et al. (2020a, Proposition 6.17).

Next we consider the case where $\limsup_{x\to 0} g(x)/x < \infty$. We first note that Proposition 4.35 applies to the identity transformation $g(x) = x$. Also we have the map $g^* : \mathcal{I}_H(E) \to g^*(\mathcal{I}_H(E))$ is continuous and bijective by Lemma 4.30 which means that the Strong Markov property in the untransformed case leads to the strong Markov property in the transformed case.

Finally for the case where $\gamma \in g^*(\overline{\mathcal{I}}^{(q)}_{m_{\mathbf{Y}}}(E))$ we use Proposition 4.28 which states that $(\gamma^y, y \geq 0)$ is continuous on $[0,\infty)$ as required. □

We can now prove Theorem 1.6.

Proof of Theorem 1.6: First consider the case where we have a diffusion under Assumption B and we have a scaffolding-and-spindles pair $(\mathbf{N},\mathbf{X})$ for this diffusion stopped at an a.s. finite random time T. Then there exists a continuous modification of $\overline{\text{skewer}}(\mathbf{N},\mathbf{X})$ by Theorem 4.17 and the furthermore $\mathbf{X}$ has a local time that is continuous in time and space by Proposition 4.6.

Now, instead consider an untransformed diffusion under Assumption B' and let $s > 0$. Let $(\tilde{\mathbf{N}},\tilde{\mathbf{X}})$ be a scaffolding-and-spindles pair for this untransformed diffusion that is stopped at $\tau^0(s)$. Then by Corollary 3.13 the diffusion satisfies (3.16). Therefore, as it also satisfies $B1$ we can apply Proposition 3.19 as $\tau^0(s)$ satisfies the conditions of the proposition. This gives a regular conditional distribution of the clade component for level 0 given $\beta^0 := \text{skewer}(0,\tilde{\mathbf{N}},\tilde{\mathbf{X}})$. By Proposition 3.20 we have a simple Markov property for $\overline{\text{skewer}}(\tilde{\mathbf{N}},\tilde{\mathbf{X}})$ on the probability space $(\Omega,\mathcal{A},\mathbf{P})$ for $\tilde{\mathbf{N}}$.

We have aggregate health summability at level 0 of the transformed diffusion by Proposition 4.2 and Theorem 1.4 and so we also have the simple Markov property in the transformed case. Moreover, under Strong Assumption B, by Theorem 4.36 we have a strong Markov property for a modification of the transformed skewer process on the space of paths continuous on $(0,\infty)$ which we denote as $(\Omega,\mathcal{F},\mathbb{P}_\mu)$ where μ is the probability distribution on $g^*(\mathcal{I}_H(E))$ of $g^*(\beta^0)$. □

5. Appendix

Lemma 5.1. *Let ν be a σ-finite measure on $\mathcal{E}$. Then, for any measurable map $S : \mathcal{E} \to [0,\infty]$, there exists an S-disintegration $(\nu_t)_{t\in[0,\infty]}$. This disintegration is uniquely determined up to an almost sure equivalence, i.e. if $(\nu^*_t)_{t\in[0,\infty]}$ is another S-disintegration of ν then $\nu(S^{-1}(t\in[0,\infty] : \nu_t \neq \nu^*_t)) = 0$.*

Lemma 5.2 restates Lemma 2.10. The Green's function for the $\uparrow$-diffusion is given by

$$G^\uparrow_{(a,w)}(x,v) = 2\frac{s^\uparrow(x)-s^\uparrow(a)}{s^\uparrow(w)-s^\uparrow(a)}\Big(s^\uparrow(w)-s^\uparrow(v)\Big)m^\uparrow(v)\mathbb{1}_{x<v<w} + 2\frac{s^\uparrow(w)-s^\uparrow(x)}{s^\uparrow(w)-s^\uparrow(a)}\Big(s^\uparrow(v)-s^\uparrow(a)\Big)m^\uparrow(v)\mathbb{1}_{a<v<x}.$$

Recall the functions $U_{(a,w)}$ and $V_{(a,w)}$ from (2.8):

$$U_{(a,w)}(x) = \mathbb{E}_{\mathbb{Q}^\uparrow_x}\big((T_a\wedge T_w)^2\big), \quad V_{(a,w)}(x) = 2\mathbb{E}_{\mathbb{Q}^\uparrow_x}\big(T_a\wedge T_w\big).$$

We have from Section 2.3 that for $0 < a < w < h$ that $U_{(a,w)}$ is the solution to the following differential equation, from (2.9), in $x \in (a,w)$

$$\frac{1}{2}\sigma^2(x)U''_{(a,w)}(x) + \mu^{\uparrow}(x)U'_{(a,w)}(x) + V_{(a,w)}(x) = 0,$$

with boundary conditions $U_{(a,w)}(a) = U_{(a,w)}(w) = 0$.

Lemma 5.2 (Proof of Lemma 2.10)**.** *We claim that the solution to the differential equation in* (2.9)*, with boundary conditions given by $U(a) = U(w) = 0$, is the following:*

$$U_{(a,w)}(x) := \mathbb{E}_{\mathbb{Q}_x^{\uparrow}}\left(\left(T_a \wedge T_w\right)^2\right) = k(a,w)(s^{\uparrow}(w) - s^{\uparrow}(x)) + \int_{y=x}^{w}(s^{\uparrow})'(y)\int_{z=a}^{y}\frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)}dzdy,$$

where $k(a,w)$ is a constant given by

$$k(a,w) = \frac{-1}{s^{\uparrow}(w) - s^{\uparrow}(a)}\int_{y=a}^{w}(s^{\uparrow})'(y)\int_{z=a}^{y}\frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)}dzdy,$$

and where $V_{(a,w)}(x)$ is

$$V_{(a,w)}(x) = 2\mathbb{E}_{\mathbb{Q}_x^{\uparrow}}\left(T_a \wedge T_w\right) = 2\int_{v=0}^{w}G^{\uparrow}_{(a,w)}(x,v)dv,$$

where $G^{\uparrow}$ is the Green's function for the $\uparrow$-diffusion.

Proof: We first note that the $\uparrow$-diffusion satisfies $A3$ and that the infinitesimal drift $\mu^{\uparrow}$ and infinitesimal variance $(\sigma^2)^{\uparrow}$ are continuous functions on $(0,h)$. Therefore the differential equation in (2.9), from Karlin and Taylor (1981, Equation 15.3.37), has solution given in (2.8).

We now check that the boundary conditions are satisfied:

$$U_{(a,w)}(w) = k(a,w)(s^{\uparrow}(w) - s^{\uparrow}(w)) + \int_{y=w}^{w}(s^{\uparrow})'(y)\int_{z=a}^{y}\frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)}dzdy = 0,$$

$$\begin{aligned} U_{(a,w)}(a) =& \frac{-1}{s^{\uparrow}(w) - s^{\uparrow}(a)}\int_{y=a}^{w}(s^{\uparrow})'(y)\int_{z=a}^{y}\frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)}dzdy(s^{\uparrow}(w) - s^{\uparrow}(a)) \\ &+ \int_{y=a}^{w}(s^{\uparrow})'(y)\int_{z=a}^{y}\frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)}dzdy = 0. \end{aligned}$$

Now we show that the differential equation itself is solved by this solution. Taking a derivative with respect to x gives

$$U'_{(a,w)}(x) = -k(a,w)(s^{\uparrow})'(x) - (s^{\uparrow})'(x)\int_{z=0}^{x}\frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)}dz.$$

Then multiplying by $1/(s^{\uparrow})'(x) = \exp\left(\int_{t=b}^{x}\frac{2\mu^{\uparrow}(t)}{\sigma^2(t)}dt\right)$ gives

$$\exp\left(\int_{t=1}^{x}\frac{2\mu^{\uparrow}(t)}{\sigma^2(t)}dt\right)U'_{(a,w)}(x) = \frac{1}{(s^{\uparrow})'(x)}U'_{(a,w)}(x) = -k(a,w) - \int_{z=0}^{x}\frac{2V_{(a,w)}(z)}{(s^{\uparrow})'(z)\sigma^2(z)}dz.$$

Finally taking another a derivative with respect to x gives

$$\exp\left(\int_{t=b}^{x}\frac{2\mu^{\uparrow}(t)}{\sigma^2(t)}dt\right)\left(U''_{(a,w)}(x) + \frac{2\mu^{\uparrow}(x)}{\sigma^2(x)}U'_{(a,w)}(x)\right) = \frac{-2V_{(a,w)}(x)}{(s^{\uparrow})'(x)\sigma^2(x)},$$

and multiplying by $(s^{\uparrow})'(x) = \exp\left(\int_{t=b}^{x}\frac{-2\mu^{\uparrow}(t)}{\sigma^2(t)}dt\right)$, adding $\frac{2V_{(a,w)}(x)}{\sigma^2(x)}$, then multiplying by $\frac{1}{2}\sigma^2(x)$ gives the differential equation in the form of (2.9). Therefore the result follows by standard results for the uniqueness of solutions to differential equations. $\square$

This final result generalises Forman et al. (2020c, Lemma 8) to our setting. The other relevant results from this supplementary paper apply without any modification.

Lemma 5.3. *For fixed $y \in \mathbb{R}$, the map $N \mapsto skewer(y, N, \xi_N)$ is measurable on the set $\mathcal{N}^{sp}_{fin}$. Recall that we map (y, N, ξ_N) to the empty interval partition $\emptyset$ in the event that the aggregate mass is infinite; i.e. when $M^y_{N,\xi(N)}(\infty) = \infty$, see Definition 1.2.*

Proof: The proof in Forman et al. (2020c, Lemma 8) that the map restricted to the finite aggregate mass summability case is measurable generalises to our setting because the σ-algebra of $(\mathcal{I}_H, d'_H)$ is the same as $(\mathcal{I}_H, d_H)$ (Forman et al., 2020b, Theorem 2.3(b)), where for β and $\gamma \in \mathcal{I}_H$ we define $d_H(\beta, \gamma)$ to be the Hausdorff distance between $[0, \|\beta\|]\backslash\beta$ and $[0, \|\gamma\|]\backslash\gamma$. The extension to the map to include the case where the aggregate mass is infinite follows as the subset $\mathcal{N}^{\mathrm{sp}}_{\mathrm{fin}}$ for which the aggregate mass at level y is finite is measurable. $\square$

5.1. *Excursion intervals.* This section follows Forman et al. (2020a, A Excursion intervals) very closely. All their measure theoretic results hold in our setting, and we state the results and state why they generalise to our case.

Proposition 5.4. *Let $\mathbf{X}$ be an SPLP with jump measure that satisfies (1.9). For $y \in \mathbb{R}$ recall the concept of the sets of excursion intervals $V^y(\mathbf{X})$ and $V^y_0(\mathbf{X})$ from Definition 3.5. Let $N \in \mathcal{N}^{sp}$ and $y \in \mathbb{R}$. Then we have the following a.s.:*

(1) $V^y(\mathbf{X}) = \Big\{[a, b] \subset [0, \infty) | a < b; \mathbf{X}_{a-} = y = \mathbf{X}_b;$ and $\mathbf{X}_t \neq 0$ for $t \in (a, b)\Big\}$.
(2) For I, $J \in V^y_0(\mathbf{X})$, $I \neq J$, the set $I \cap J$ is either empty or a single shared endpoint.
(3) If two interval $[a, b]$, $[b, c] \in V^y_0(\mathbf{X})$ share an endpoint b then $\mathbf{X}$ does not have a jump at b.
(4) For every $t \notin \bigcup_{I \in V^y_0(\mathbf{X})} I$, we have that $\mathbf{X}_{t-} = \mathbf{X}_t = y$.
(5) $Leb\Big([0, \infty)\backslash \bigcup_{I \in V^y_0(\mathbf{X})} I\Big) = 0$.
(6) There are no degenerate excursions of $\mathbf{X}$ about level y, that is to say that for all excursions we have $0 < T^+_0 < \zeta$ a.s.
(7) Local times $(\ell^y(t), t \geq 0)$ of $\mathbf{X}$ exist and are injective with respect to the excursion intervals in the sense that, for $[a, b]$, $[c, d] \in V^y_0$, $\ell^y(a) \neq \ell^y(c)$ unless $[a, b] = [c, d]$.
(8) If $y > 0$, we have $T^y > T^{\geq y} := \inf\{t \geq 0 : \mathbf{X}_t \geq y\}$.
(9) For $[a, b] \in V^y(N)$, the process $N|^{\leftarrow}_{I^y_N(a,b)}$ is a bi-clade.
Furthermore, the set $\{N|^{\leftarrow}_{I^y_N(a,b)} : [a, b] \in V^y_0(N)\}$ partitions the spindles of N.

Proof: These results are given in the self-similar cases in Forman et al. (2020a, Proposition A.2-4). The proofs in Forman et al. (2020a, Proposition A.2) immediately generalise. As for the results of Forman et al. (2020a, Proposition A.3), they rely on the exit properties of Lévy processes characterised by Millar (1973) which applies to our general setting. The final part, from Forman et al. (2020a, Proposition A.4), is a result of the careful constructions of the intervals $I^y_N(a, b)$ and also applies in our general setting. $\square$

We conclude this section with definitions, specifically concerning incomplete excursions, to formally define the cutoff processes stated in (3.24) and (3.25). We first define $T^{\geq y}$ and $T^{\geq y}_*$ in the same way as we have defined T^y and T^y_* and then with these define $N^{\leq y}_{\text{first}}$, $N^{\leq y}_{\text{last}}$, $N^{\geq y}_{\text{first}}$, $N^{\geq y}_{\text{last}}$:

$$T^{\geq y} := \inf(\{t \in [0, \mathrm{len}(N)] : \xi_N(t) \geq y\} \cup \{\mathrm{len}(N)\});$$
$$T^{\geq y}_* := \sup(\{t \in [0, \mathrm{len}(N)] : \xi_N(t-) \leq y\} \cup \{0\});$$

$$\begin{aligned}
N^{\leq y}_{\mathrm{first}} &= \left(N|_{[0,T^{\geq y})} + \mathbb{1}_{\xi_N(T^{\geq y}-)<y}\delta_{(T^{\geq y},\check{f}^y_{T^{\geq y}})}\right)\mathbb{1}_{T^{\geq y}\neq T^{\geq y}_*};\\
N^{\geq y}_{\mathrm{last}} &= N|^{\leftarrow}_{(T^{\geq y}_*,\mathrm{len}(N)]} + \mathbb{1}_{\xi_N(T^{\geq y}_*-)>y}\delta_{(0,\hat{f}^y_{T^{\geq y}_*})};\\
N^{\leq y}_{\mathrm{last}} &= N|^{\leftarrow}_{[T^y_*,T^{\geq y}_*)} + \mathbb{1}_{y\neq\xi_N(\mathrm{len}(N));\xi_N(T^{\geq y}_*-)<(y\wedge\xi_N(T^{\geq y}_*))}\delta_{(T^{\geq y}_*-T^y_*,\check{f}^y_{T^{\geq y}_*})};\\
N^{\geq y}_{\mathrm{first}} &= \left(N|^{\leftarrow}_{(T^{\geq y},T^y]} + \mathbb{1}_{y\neq 0;\xi_N(T^{\geq y}-)>y\vee 0}\delta_{(0,\hat{f}^y_{T^{\geq y}})}\right)\mathbb{1}_{T^{\geq y}\neq T^{\geq y}_*}.
\end{aligned}$$

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References

Aldous, D. J. Exchangeability and related topics. In *École d'été de probabilités de Saint-Flour, XIII—1983*, volume 1117 of *Lecture Notes in Math.*, pp. 1–198. Springer, Berlin (1985). MR883646.

Aldous, D. J. Wright–Fisher diffusions with negative mutation rate (1999). Available at `https://www.stat.berkeley.edu/~aldous/Research/OP/fw.html`.

Barlow, M. T. Necessary and sufficient conditions for the continuity of local time of Lévy processes. *Ann. Probab.*, **16** (4), 1389–1427 (1988). MR958195.

Bertoin, J. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge (1996). ISBN 0-521-56243-0. MR1406564.

Chang, J. T. and Pollard, D. Conditioning as disintegration. *Statist. Neerlandica*, **51** (3), 287–317 (1997). MR1484954.

Cox, J. C., Ingersoll, J. E., Jr., and Ross, S. A. A theory of the term structure of interest rates. *Econometrica*, **53** (2), 385–407 (1985). MR785475.

Forman, N., Pal, S., Rizzolo, D., and Winkel, M. Uniform control of local times of spectrally positive stable processes. *Ann. Appl. Probab.*, **28** (4), 2592–2634 (2018). DOI: 10.1214/17-AAP1370.

Forman, N., Pal, S., Rizzolo, D., and Winkel, M. Diffusions on a space of interval partitions: construction from marked Lévy processes. *Electron. J. Probab.*, **25**, 1–46 (2020a). DOI: 10.1214/20-EJP521.

Forman, N., Pal, S., Rizzolo, D., and Winkel, M. Metrics on sets of interval partitions with diversity. *Electron. Commun. Probab.*, **25**, Paper No. 38, 16 (2020b). MR4112769.

Forman, N., Pal, S., Rizzolo, D., and Winkel, M. Supplement to Diffusions on a space of interval partitions: construction from marked Lévy processes. *Electron. J. Probab.*, **25**, 1–46 (2020c). DOI: 10.1214/20-EJP521SUPPA.

Forman, N., Pal, S., Rizzolo, D., and Winkel, M. Diffusions on a space of interval partitions: Poisson-Dirichlet stationary distributions. *Ann. Probab.*, **49** (2), 793–831 (2021). MR4255131.

Forman, N., Pal, S., Rizzolo, D., and Winkel, M. The Aldous diffusion: a stationary evolution of the Brownian CRT. *ArXiv Mathematics e-prints* (2023a). arXiv: 2305.17269.

Forman, N., Rizzolo, D., Shi, Q., and Winkel, M. Diffusions on a space of interval partitions: the two-parameter model. *Electron. J. Probab.*, **28**, Paper No. 61, 46 (2023b). MR4583068.

Geiger, J. and Kersting, G. Depth-first search of random trees, and Poisson point processes. In *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *IMA Vol. Math. Appl.*, pp. 111–126. Springer, New York (1997). MR1601713.

Gnedin, A. and Pitman, J. Regenerative composition structures. *Ann. Probab.*, **33** (2), 445–479 (2005). MR2122798.

Gnedin, A. V. The representation of composition structures. *Ann. Probab.*, **25** (3), 1437–1450 (1997). DOI: 10.1214/aop/1024404519.

Göing-Jaeschke, A. and Yor, M. A survey and some generalizations of Bessel processes. *Bernoulli*, **9** (2), 313–349 (2003). MR1997032.

Jagers, P. *Branching processes with biological applications*. Wiley Series in Probability and Mathematical Statistics—Applied Probability and Statistics. Wiley-Interscience, London-New York-Sydney (1975). ISBN 0-471-43652-6. MR488341.

Kallenberg, O. *Foundations of modern probability*, volume 99 of *Probability Theory and Stochastic Modelling*. Springer, Cham, third edition (2021). ISBN 978-3-030-61871-1; 978-3-030-61870-4. MR4226142.

Karlin, S. and Taylor, H. M. *A second course in stochastic processes*. Academic Press, Inc. , New York-London (1981). ISBN 0-12-398650-8. MR611513.

Kingman, J. F. C. *Poisson processes*, volume 3 of *Oxford Studies in Probability*. The Clarendon Press, Oxford University Press, New York (1993). ISBN 0-19-853693-3. MR1207584.

Lambert, A. The contour of splitting trees is a Lévy process. *Ann. Probab.*, **38** (1), 348–395 (2010). MR2599603.

Lambert, A. and Simatos, F. Asymptotic behavior of local times of compound Poisson processes with drift in the infinite variance case. *J. Theoret. Probab.*, **28** (1), 41–91 (2015). MR3320960.

Lambert, A. and Uribe Bravo, G. Totally ordered measured trees and splitting trees with infinite variation. *Electron. J. Probab.*, **23**, Paper No. 120, 41 (2018). MR3896857.

Lamperti, J. Semi-stable Markov processes. I. *Z. Wahrsch. Verw. Gebiete*, **22** (3), 205–225 (1972). DOI: 10.1007/BF00536091.

Méléard, S. and Villemonais, D. Quasi-stationary distributions and population processes. *Probab. Surv.*, **9**, 340–410 (2012). MR2994898.

Millar, P. W. Exit properties of stochastic processes with stationary independent increments. *Trans. Amer. Math. Soc.*, **178**, 459–479 (1973). MR321198.

Pardo, J. C., Pérez, J. L., and Rivero, V. M. The excursion measure away from zero for spectrally negative Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, **54** (1), 75–99 (2018). DOI: 10.1214/16-AIHP795.

Pinsky, R. G. On the convergence of diffusion processes conditioned to remain in a bounded region for large time to limiting positive recurrent diffusion processes. *Ann. Probab.*, **13** (2), 363–378 (1985). MR781410.

Pitman, J. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (2006). ISBN 978-3-540-30990-1; 3-540-30990-X. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, MR2245368.

Pitman, J. and Yor, M. A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete*, **59** (4), 425–457 (1982). MR656509.

Pitman, J. and Yor, M. Arcsine laws and interval partitions derived from a stable subordinator. *Proc. London Math. Soc. (3)*, **65** (2), 326–356 (1992). MR1168191.

Revuz, D. and Yor, M. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin (1991). ISBN 3-540-52167-4. MR1083357.

Salminen, P., Vallois, P., and Yor, M. On the excursion theory for linear diffusions. *Jpn. J. Math.*, **2** (1), 97–127 (2007). MR2295612.