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On Berry-Esseen type estimates for randomized Martingales in the non stationary setting

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Abstract. In this paper, we consider partial sums of triangular martingale differences weighted by random variables drawn uniformly on the sphere, and globally independent of the martingale differences. Starting from the so-called principle of conditioning and using some arguments developed by Klartag-Sodin and Bobkov-Chistyakov-Götze, we give some upper bounds for the Kolmogorov distance between the distribution of these weighted sums and a Normal distribution. Under some conditions on the conditional variances of the martingale differences, the obtained rates are always faster than those obtained in case of usual partial sums.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(d_{j,n})_{1\le j\le n, n\ge 1}$ be a triangular array of martingale differences in $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a triangular array of filtrations $(\mathcal{F}_{j,n})_{1\le j\le n}$ (so $\mathcal{F}_{j,n} \subset \mathcal{F}_{j+1,n}$ and $\mathbb{E}(d_{j,n}|\mathcal{F}_{j-1,n}) = 0$) and such that

$$\sum_{k=1}^{n} \mathbb{E}(d_{k,n}^2) = n\,. \tag{1.1}$$

In this paper, we are interested in the rates of convergence in the CLT for randomized weighted sums associated with $(d_{j,n})_{1\le j\le n}$ in the following sense. Let $\theta = (\theta_1, \dots, \theta_n)^t$ be a random vector defined on $(\Omega, \mathcal{F}, \mathbb{P})$, independent of $(\mathcal{F}_{j,n})_{1\le j\le n, n\ge 1}$ and with uniform distribution μ_{n-1} on the unit sphere S^{n-1} of $\mathbb{R}^n$ $(n \ge 2)$. Let $S_n(\theta) = \sum_{j=1}^{n} \theta_j d_{j,n}$. By independence, taking the conditional

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expectation of $f(\theta,(d_{j,n})_{1\le j\le n})$ with respect to θ means keeping θ fixed and integrating with respect to the distribution of $(d_{j,n})_{1\le j\le n}$. We will designate by $\mathbb{P}_{|\theta}$, the conditional probability given θ. When we use conditional expectations, the equalities and inequalities are understood in the almost sure sense.

We shall give convergence rates in terms of the Kolmogorov distance, and more precisely for the quantity

$$\mathbb{E}(\kappa_\theta(P_{S_n(\theta)},P_{N_\theta})) \quad \text{with} \quad \kappa_\theta(P_{S_n(\theta)},P_{N_\theta})=\sup_{t\in\mathbb{R}}|\mathbb{P}_{|\theta}(S_n(\theta)\le t)-\mathbb{P}_{|\theta}(N_\theta\le t)|\,,$$

where, conditionally to θ, the r.v. N_θ is Gaussian with mean 0 and variance $\sum_{k=1}^n\theta_k^2\mathbb{E}(d_{k,n}^2)$. We will also denote by N a centered Gaussian random variable with variance one.

When $(d_k)_{k\in\mathbb{Z}}$ forms a sequence of independent centered random variables in $\mathbb{L}^4$ with $\mathbb{E}(d_k^2)=1$, Corollary 3.2 in Klartag and Sodin (2011) states that

$$\mathbb{E}\Big(\kappa_\theta(P_{S_n(\theta)},P_N)\Big)\le\frac{c}{n^2}\sum_{k=1}^n\mathbb{E}(d_k^4)\,. \tag{1.2}$$

Therefore, in the iid setting, the upper bound in this Berry-Esseen type estimate is of order n^{-1} whereas, the classical Berry-Esseen theorem (meaning that we take $\theta_i=1/\sqrt{n}$ for any integer i in $[1,n]$), gives a rate of order $n^{-1/2}$. As quoted in Section 3 of Dedecker et al. (2025) this type of randomized partial sums appears naturally when we study the ordinary least square estimator of the slope in a linear regression model with Gaussian design. As a consequence a super fast rate can be achieved for the Kolmogorov distance between the centered and normalized least square estimator and its Gaussian limit, when the random design is Gaussian and the errors are iid and in $\mathbb{L}^4$. In fact, this also holds when the errors form a stationary sequence of martingale differences satisfying some weak dependence conditions, see Dedecker et al. (2025, Corollary 2.3).

This type of phenomena concerning the Berry-Essen bound for randomized partial sums has also been studied in a series of papers by Bobkov et al. (2018, 2020b,a) where the independence assumption has been relaxed. Let us describe the result given in Theorem 1.1 of Bobkov et al. (2020a). Let $(d_k)_{k\in\mathbb{Z}}$ be a sequence of uncorrelated centered random variables with variance one, satisfying the following second order correlation condition: there exists a constant Λ such that, for any $n\ge1$ and any collection $a_{ij}\in\mathbb{R}$,

$$\mathrm{Var}\Big(\sum_{i,j=1}^n a_{ij}d_id_j\Big)\le\Lambda\sum_{i,j=1}^n a_{ij}^2\,. \tag{1.3}$$

If moreover the random vector $(d_1,\dots,d_n)$ has a symmetric distribution then

$$\mathbb{E}\Big(\kappa_\theta(P_{S_n(\theta)},P_N)\Big)\le\frac{c\log n}{n}\Lambda\,. \tag{1.4}$$

The main restriction in this result is probably the fact that the distribution of $(d_1,\dots,d_n)$ is assumed to be symmetric. In Bobkov et al. (2023, Chapter 17.4) the authors consider the case of non-symmetric distributions, but this leads to an additional term in the upper bound (1.4) (see Bobkov et al. (2023, Proposition 17.4.1)). Let us comment now on the second order condition (1.3). As shown in Bobkov et al. (2020a), it can be verified for random vectors satisfying a Poincaré-type inequality with positive constant (see Bobkov et al. (2020a, Proposition 3.4)). On another hand, as quoted in Dedecker et al. (2025), if $(d_k)_{k\in\mathbb{Z}}$ is a sequence of martingale differences such that $\sup_{i\ge1}\mathbb{E}(d_i^4)<\infty$, then condition (1.3) is satisfied under some natural dependence conditions.

In the recent paper Dedecker et al. (2025), the authors have provided a new method allowing to show that stationary sequences $(d_k)_{k\in\mathbb{Z}}$ of martingale differences satisfy an upper bound of the type (1.2) (up to some logarithmic terms) without requiring that the law of the vector $(d_1,\dots,d_n)$ is symmetric, provided that the sequence $(d_k)_{k\in\mathbb{Z}}$ satisfies a weak dependence condition. Their proof

is based on Lindeberg's method combined with a variant of Berry-Esseen's smoothing inequality, which is adapted to the type of randomisation we use (see Lemma 5.2 in Bobkov et al. (2018)). Note, however, that the arguments they developed for Lindeberg's method to work require that the sequence $(d_k)_{k\in\mathbb{Z}}$ is close to a stationary regime.

In this paper we are interested in the extension of the result of Klartag-Sodin (up to some additional logarithmic terms) to triangular array of martingale differences $(d_{i,n})_{1\le i\le n}$ that are not necessarily stationary. As a starting point, we shall use the Principle of Conditioning saying heuristically that starting with a limit theorem for independent random variables, if one replaces the expectations by conditional expectations with respect to the past, summation to constants by summation to stopping times, and the convergence of numbers by convergence in probability, one obtains a correct statement in the dependent setting. This heuristic was expressed as a rigorous theorem by Jakubowski (1986), and appears for instance in the books by de la Peña and Giné (1999, Chap. 7.1) and Merlevède et al. (2019, Section 2.3). This principle was used to rigorously transport several limit theorems such as the central limit theorem or the moderate deviation principle, from the independent setting to the martingale one.

The paper is organized as follows. In Section 2 we present our main result concerning the approximation of $S_n(\theta)$ by a Gaussian random variable in terms of the Kolmogorov distance, in the case where $(d_{i,n})_{1\le i\le n}$ is a triangular array of (not necessarily stationary) martingale differences. In Subsection 2.1 we consider an example of a nonstationary sequence of martingale differences given in Dedecker et al. (2022), for which the rates in the usual Berry-Esseen theorem is at best of order $n^{-1/3}$, while the rate for randomized sums is at most of order $n^{-1}(\log n)^2$. In Section 2.2, we give an example of a nonstationary ARCH model to which our results apply. The proofs are postponed to Section 3.

We shall use most of the times the notation $\ll$ instead of the Vinogradov symbol O. Note that all our results hold if we replace the random vector $\theta=(\theta_1,\dots,\theta_n)$ by $\tilde\xi=\|\xi\|_e^{-1}\xi$ where $\xi=(\xi_1,\dots,\xi_n)$ with $(\xi_i)_{1\le i\le n}$ iid centered and standard Gaussian r.v.'s independent of $(d_{j,n})_{1\le j\le n}$ and $\|\xi\|_e^2=\sum_{i=1}^n\xi_i^2$ stands for the Euclidean norm. Indeed, it is well-known that $\tilde\xi$ has uniform distribution on the unit sphere S^{n-1} of $\mathbb{R}^n$.

2. Main results

Recall that $S_n(\theta)=\sum_{j=1}^n\theta_j d_{j,n}$, where $\theta=(\theta_1,\dots,\theta_n)$ is independent of $(d_{j,n})_{1\le j\le n}$ and has uniform distribution on the unit sphere S^{n-1} of $\mathbb{R}^n$ ($n\ge 2$). To soothe the notation we shall denote $d_j=d_{j,n}$ and $\mathcal{F}_j=\mathcal{F}_{j,n}$. Let $\mathbb{E}_j$ be the conditional expectation with respect to $\mathcal{F}_j$. We introduce also the following notations:

$$\sup_{1\le k\le n}\mathbb{E}(d_k^4)=\alpha_n \quad\text{and}\quad \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^n\mathbb{E}_{j-1}^2(d_j^3)\right)^{1/2}=\beta_n\,, \tag{2.1}$$

$$\sigma_4^2(n)=\frac{1}{n}\mathrm{var}\left(\sum_{k=1}^n d_k^2\right)\,, \tag{2.2}$$

and

$$\sum_{j=1}^n\|\mathbb{E}_{j-1}(d_j^2)-\mathbb{E}(d_j^2)\|_1=\gamma_n\,. \tag{2.3}$$

Our main result is the following:

Theorem 2.1. *Assume that $(d_i)_{1\le i\le n}$ is a vector of martingale differences in $\mathbb{L}^4$, adapted to an array $(\mathcal{F}_i)_{0\le i\le n}$ of increasing sigma fields and satisfying the condition (1.1). Then*

$$\mathbb{E}(\kappa_\theta(P_{S_n(\theta)}, P_{N_\theta})) \ll \frac{1+v_n}{n},$$

where the conditional distribution of N_θ given θ is a normal distribution with mean 0 and variance $\sum_{i=1}^n \theta_i^2 \mathbb{E}(d_i^2)$ and

$$v_n = \gamma_n(\log n) + \beta_n(\log n)^{3/2} + \alpha_n(\log n)^2 + \sigma_4^2(n)(\log n)\,.$$

Remark 2.2. Let N be a standard centered Gaussian random variable. If we are interested in giving an upper bound for the quantity

$$\mathbb{E}\sup_{t\in\mathbb{R}} |\mathbb{P}_{|\theta}(S_n(\theta)\le t) - \mathbb{P}(N\le t)|\,,$$

we can use Theorem 2.1 together with the following upper bound whose proof is given in Subsection 3.5: for any $n\ge 2$,

$$\mathbb{E}\sup_{t\in\mathbb{R}} |\mathbb{P}_{|\theta}(N_\theta\le t) - \mathbb{P}(N\le t)| \le \frac{35\sqrt{5}}{4}\frac{\left(\sum_{k=1}^n (b_{k,n}^2-1)^2\right)^{1/2}}{n-1}\,, \tag{2.4}$$

where $b_{k,n}^2 = \mathbb{E}(d_{k,n}^2)$. Now, if we are interested in $\sup_{t\in\mathbb{R}} |\mathbb{P}(S_n(\theta)\le t) - \mathbb{P}(N\le t)|$, note that

$$\sup_{t\in\mathbb{R}} |\mathbb{E}(\mathbb{P}_{|\theta}(N_\theta\le t)) - \mathbb{P}(N\le t)| \ll \frac{1+\alpha_n(\log n)^2}{n}\,. \tag{2.5}$$

In Theorem 2.1 the upper bound makes use of the quantity γ_n defined in (2.3), and to get an interesting upper bound, we need to have $\mathbb{E}_{j-1}(d_j^2)$ "not too far" in $\mathbb{L}^1$ from $\mathbb{E}(d_j^2)$. Alternatively, we can introduce the following additional notation:

$$c_2^2(n) = \frac{1}{n}\sum_{k=1}^n\sum_{\ell=1}^n |\mathrm{Cov}(d_k^2, d_\ell^2)|\,, \tag{2.6}$$

and ask this quantity to be "not too big". Requiring $c_2^2(n)$ to be "not too big" is usually less stringent than imposing γ_n to be small because $c_2^2(n)$ only requires fast enough decorrelation between d_k^2 and d_ℓ^2 when $|k-\ell|$ is large enough. Note also that $\sigma_4^2(n)\le c_2^2(n)$. Using the quantity $c_2^2(n)$ rather than γ_n gives the following Berry-Esseen bound:

Theorem 2.3. *Assume that $(d_i)_{1\le i\le n}$ is a vector of martingale differences in $\mathbb{L}^4$, adapted to an array $(\mathcal{F}_i)_{0\le i\le n}$ of increasing sigma fields and satisfying the condition (1.1). Then*

$$\mathbb{E}\sup_{t\in\mathbb{R}} |\mathbb{P}_{|\theta}(S_n(\theta)\le t) - \mathbb{P}(N\le t)| \ll \frac{1+v_n'}{\sqrt{n}},$$

where N is a centered standard normal variable and

$$v_n' = (c_2(n)\sigma_4^2(n))\log n + \beta_n(\log n)^{3/2} + \alpha_n(\log n)^2\,.$$

As we can see, if the quantities $c_2(n)$, β_n and $\sigma_4^2(n)$ are uniformly bounded in n by a finite constant, the bound in the Berry-Esseen theorem for this type of randomized martingales is of order $(\log n)^2/\sqrt{n}$. This rate is slower than the one we could obtain by applying Theorem 2.1 but as a counterpart asking the quantity $c_2(n)$ to be uniformly bounded in n is less stringent than making a similar assumption on γ_n.

2.1. *Comparison with rates in the usual Berry-Essen bound for non stationary Martingales.* For a non stationary sequence of martingale differences, even in case of constant conditional variances and uniformly bounded third moments, the upper bound in the classical Berry-Esseen inequality may not be better than $n^{-1/4}$ (see Bolthausen (1982)). In the same context, if there is a uniformly bounded moment of order $p > 2$, Dedecker et al. (2022) proved the following lower bound.

Proposition 2.4 (Dedecker et al. (2022))**.** *Let $p > 2$ and $n \geq 20$. There exists $(d_1, \ldots, d_n)$ such that*

(1) $\mathbb{E}(d_k|\sigma(d_1, \ldots, d_{k-1})) = 0$ *and* $\mathbb{E}(d_k^2|\sigma(d_1, \ldots, d_{k-1})) = 1$ *a.s.,*
(2) $\sup_{1\leq k\leq n} \mathbb{E}(|d_k|^p) \leq \mathbb{E}(|Y|^p) + 5^{p-2}$ *where* $Y \sim \mathcal{N}(0, 1)$*,*
(3) $\sup_{t\in\mathbb{R}} \left|\mathbb{P}(S_n \leq t\sqrt{n}) - \mathbb{P}(N \leq t)\right| \geq 0.06\, n^{-(p-2)/(2p-2)}$*, where* $S_n = \sum_{k=1}^n d_k$ *and* N *is a standard centered Gaussian random variable.*

In particular, if we take $p = 4$ in the above proposition, the lower bound in the classical Berry-Esseen inequality is of order $n^{-1/3}$. Now, for the vector $(d_1, \ldots, d_n)$ constructed in the proof of the above proposition, we have that $\gamma_n = 0$ and the other quantities (α_n, β_n and $\sigma_4^2(n)$) are uniformly bounded. In addition, since $\sum_{i=1}^n \theta_i^2 \mathbb{E}(d_i^2) = 1$, it follows that

$$\mathbb{E} \sup_{t\in\mathbb{R}} |\mathbb{P}_{|\theta}(S_n(\theta) \leq t) - \mathbb{P}(N \leq t)| \ll \frac{(\log n)^2}{n}\,.$$

2.2. *Example.* Let us consider $(\varepsilon_k)_{k\geq 1}$ a sequence of independent r.v.'s such that $\mathbb{P}(\varepsilon_k = 0) = 1-2p_k$ and $\mathbb{P}(\varepsilon_k = a_k) = \mathbb{P}(\varepsilon_k = -a_k) = p_k$ where $(a_k)_{k\geq 1}$ and $(p_k)_{k\geq 1}$ are sequences of positive reals such that $2a_k^2 p_k = 1$, $2p_k \leq 1$ and $\sum_{k\geq 1} p_k = 1$. For any $i \geq 1$, we consider the ARCH model

$$d_i = \eta_{i-1}\varepsilon_i\,,$$

where $\eta_0 = 1$ and, for any $k \geq 1$,

$$\eta_k^2 = \alpha_k^{-1} \sum_{\ell=1}^k u_\ell \varepsilon_{k+1-\ell}^2\,,$$

with $(u_\ell)_{\ell\geq 1}$ a sequence of positive reals and $\alpha_k = \sum_{\ell=1}^k u_\ell$.

We note that $(d_k)_{k\geq 1}$ is a sequence of martingale differences adapted to the filtration $(\mathcal{F}_k)_{k\geq 1}$ where $\mathcal{F}_k = \sigma(\varepsilon_i, 1 \leq i \leq k)$. Moreover, since $\mathbb{E}(\varepsilon_i^2) = 1$, we have $\mathbb{E}(d_i^2) = 1$. In addition, for any $k \geq 1$ and any $\ell \geq k$,

$$\begin{aligned}\mathbb{E}_k(d_\ell^2) - 1 &= \alpha_{\ell-1}^{-1} \sum_{j=1}^{\ell-1} u_j \mathbb{E}_k(\varepsilon_{\ell-j}^2 \varepsilon_\ell^2) - 1 \\ &= \alpha_{\ell-1}^{-1} \sum_{j=1}^{\ell-k-1} u_j \mathbb{E}(\varepsilon_{\ell-j}^2 \varepsilon_\ell^2) + \alpha_{\ell-1}^{-1} \sum_{j=\ell-k}^{\ell-1} u_j \varepsilon_{\ell-j}^2 \mathbb{E}(\varepsilon_\ell^2) - 1 = \alpha_{\ell-1}^{-1} \sum_{j=\ell-k}^{\ell-1} u_j \big(\varepsilon_{\ell-j}^2 - 1\big)\,.\end{aligned}$$

Using the fact that $a_k^2 = 1/(2p_k) \geq 1$ for any $k \geq 1$, we get

$$\|\mathbb{E}_k(d_\ell^2) - 1\|_1 \leq \alpha_{\ell-1}^{-1} \sum_{j=\ell-k}^{\ell-1} u_j \{(1 - 2p_{\ell-j}) + 2(a_{\ell-j}^2 - 1)p_{\ell-j}\} = 2\alpha_{\ell-1}^{-1} \sum_{j=\ell-k}^{\ell-1} u_j (1 - 2p_{\ell-j})\,. \quad (2.7)$$

We take now, for any $i \geq 1$,

$$p_i = \frac{1}{2}\Big(1 - \frac{1}{i+1}\Big) \text{ and } u_i = (i+1)^{-\gamma} \text{with } \gamma > 2\,.$$

It follows from (2.7) that, for any $\ell \geq 2$,

$$\|\mathbb{E}_{\ell-1}(d_\ell^2) - 1\|_1 \ll \ell^{-1}\,,$$

proving that

$$\gamma_n = n^{-1}\sum_{\ell=1}^{n} \|\mathbb{E}_{\ell-1}(d_\ell^2) - 1\|_1 \le (\log n)/n\,.$$

Note also that since, for any $k \ge 1$, $2a_k^2 p_k = 1$, by the selection of p_k, we have that $\sup_{k\ge1} a_k^2 \le 2$ implying that $\sup_{k\ge1}\|d_k\|_\infty \le 2$. It follows that the sequences $(\alpha_n)_{n\ge1}$ and $(\beta_n)_{n\ge1}$ defined in (2.1) satisfy $\alpha_n = 2^4$ and $\beta_n = 2^3$. It remains to give an upper bound for $\sigma_4^2(n)$. We have, by (2.7),

$$\sigma_4^2(n) \le \frac{2}{n}\sum_{k=1}^{n}\sum_{\ell=k}^{n}\|d_k^2\mathbb{E}_k(d_\ell^2-1)\|_1 \le \frac{4}{n}\sum_{k=1}^{n}\sum_{\ell=k}^{n}\|\mathbb{E}_k(d_\ell^2-1)\|_1$$
$$\le \frac{8}{n}\sum_{k=1}^{n}\sum_{\ell=k}^{n}\alpha_{\ell-1}^{-1}\sum_{j=\ell-k}^{\ell-1}\frac{1}{(j+1)^\gamma}\frac{1}{\ell-j+1} \le C\,,$$

where C does not depend on n. Hence, applying Theorem 2.1, we derive that

$$\mathbb{E}\sup_{t\in\mathbb{R}}|\mathbb{P}_{|\theta}(S_n(\theta)\le t) - \mathbb{P}(N\le t)| \ll \frac{(\log n)^2}{n}\,.$$

3. Proofs

3.1. *A preliminary result.* Since our method of proof is based on the Principle of Conditioning, we start by recalling some useful results in the independent setting. In what follows, $(Y_{j,n})_{1\le j\le n, n\ge1}$ is a triangular array of independent random variables that are centered and in $\mathbb{L}^4$. We shall use the following notations:

$$R_2^2 = R_2^2(\theta,n) = \sum_{j=1}^{n}\theta_j^2\mathbb{E}(Y_{j,n}^2),$$

$$R_3^3 = R_3^3(\theta,n) = \sum_{j=1}^{n}\theta_j^3\mathbb{E}(Y_{j,n}^3),$$

$$R_4^4 = R_4^4(\theta,n) = \sum_{j=1}^{n}\theta_j^4\mathbb{E}(Y_{j,n}^4),$$

and

$$f_j(s) = \mathbb{E}(\mathrm{e}^{\mathrm{i}sY_{j,n}}).$$

Lemma 3.1 (Klartag and Sodin (2011)). *For all θ and any $t \le \min(R_4^{-1}(\theta,n), |R_3(\theta,n)|^{-1})$ we have*

$$\Big|\prod_{j=1}^{n} f_j(\theta_j t) - \mathrm{e}^{-R_2^2(\theta,n)t^2/2}\Big| \le c\mathrm{e}^{-R_2^2(\theta,n)t^2/2}\left(|R_3^3(\theta,n)|\cdot|t|^3 + R_4^4(\theta,n)t^4\right),$$

where c is a universal constant.

The proof follows, with small changes, the initial steps of the proof of Lemma 2.1 in Klartag and Sodin (2011) before integrating with t, and is therefore left to the reader.

3.2. *The Principle of Conditioning for randomized partial sums.* The proof of Theorems 2.1 and 2.3 is based on the principle of conditioning going back to Jakubowski (1986) (see also de la Peña and Giné (1999, Chap. 7.1) and Merlevède et al. (2019, Section 2.3)). The idea of the proof is to reduce the proof for the martingale case to the independent case.

Let $\mathcal{F}_{j,\theta} = \mathcal{F}_j \vee \sigma(\theta)$. Assuming that Ω is rich enough, there are a sigma algebra $\mathcal{G}$ and random variables $(U_j)_{1\le j\le n}$ such that: the conditional distribution of U_j given $\mathcal{G}$ is the conditional distribution of $\theta_j d_j$ given $\mathcal{F}_{j-1,\theta}$, and $U_1,\dots,U_n$ are conditionally independent given $\mathcal{G}$ (see for instance Proposition 6.1.5 in de la Peña and Giné (1999) for the existence of such a sequence). Denote by $E_{\mathcal{G}}$ the conditional expectation with respect to $\mathcal{G}$. Note that

$$f_{n,\theta}(t) := E_{\mathcal{G}}\left(\mathrm{e}^{it\sum_{j=1}^n U_j}\right) = \prod_{j=1}^n E_{\mathcal{G}}\left(\mathrm{e}^{itU_j}\right) = \prod_{j=1}^n \mathbb{E}(\mathrm{e}^{\mathrm{i}t\theta_j d_j}|\mathcal{F}_{j-1,\theta}) \tag{3.1}$$

(see for instance the last line of page 328 in de la Peña and Giné (1999).) For any martingale differences (d_j) adapted to $(\mathcal{F}_j)$ denote

$$T_2^2 = T_2^2(\theta,n) = \sum_{j=1}^n \theta_j^2 \mathbb{E}_{j-1}\left(d_j^2\right),\ T_3^3 = T_3^3(\theta,n) = \sum_{j=1}^n \theta_j^3 \mathbb{E}_{j-1}\left(d_j^3\right) \tag{3.2}$$

and

$$T_4^4 = T_4^4(\theta,n) = \sum_{j=1}^n \theta_j^4 \mathbb{E}_{j-1}\left(d_j^4\right). \tag{3.3}$$

Lemma 3.1 applied to the conditionally independent random variables $(U_j)_{1\le j\le n}$, leads to the following corollary.

Corollary 3.2. *Assume that $(d_i)_{1\le i\le n}$ is a vector of martingale differences adapted to an array $(\mathcal{F}_i)_{0\le i\le n}$ of increasing sigma fields. Then, for any $t \le \min(T_4^{-1}, |T_3|^{-1})$,*

$$\left|\prod_{j=1}^n \mathbb{E}(\mathrm{e}^{\mathrm{i}t\theta_j d_j}|\mathcal{F}_{j-1,\theta}) - \mathrm{e}^{-T_2^2t^2/2}\right| \le c\mathrm{e}^{-T_2^2t^2/2}\left(|t|^3|T_3^3| + t^4T_4^4\right),$$

where $c > 1$ is a universal constant.

Remark 3.3. As a consequence of Corollary 3.2 we can find a universal constant $K \le 1$ such that for $t \le K\min(T_4^{-1}, |T_3|^{-1})$,we have

$$\prod_{j=1}^n |\mathbb{E}(\mathrm{e}^{\mathrm{i}t\theta_j d_j}|\mathcal{F}_{j-1,\theta})| \ge \frac{1}{2}\mathrm{e}^{-T_2^2t^2/2}. \tag{3.4}$$

Now, if $\prod_{j=1}^n |\mathbb{E}(\mathrm{e}^{\mathrm{i}t\theta_j d_j}|\mathcal{F}_{j-1})| \ne 0$, we could use the well-known remarkable martingale property (it is relation (6.2.4) in de la Peña and Giné (1999)):

$$\mathbb{E}\left(\frac{\mathrm{e}^{\mathrm{i}tS_n(\theta)}}{\prod_{j=1}^n \mathbb{E}(\mathrm{e}^{\mathrm{i}t\theta_j d_j}|\mathcal{F}_{j-1,\theta})}\Big|\theta\right) = 1, \tag{3.5}$$

and use Corollary 3.2 to find Berry Esseen bounds for $\mathbb{E}(\mathrm{e}^{\mathrm{i}tS_n(\theta)})$.

To be able to rigorously use this approach, we shall construct an additional vector of martingale differences $\left(d'_j\right)$ adapted to $(\mathcal{F}_j)_{0\le j\le n}$ satisfying

$$|f'_{n,\theta}(t)| > \frac{1}{2}\mathrm{e}^{-t^2(V_n^2(\theta)+t^{-2})/2} = \frac{1}{2\sqrt{\mathrm{e}}}\mathrm{e}^{-t^2V_n^2(\theta)/2}, \tag{3.6}$$

where

$$f'_{n,\theta}(t) = \prod_{j=1}^n \mathbb{E}(\mathrm{e}^{\mathrm{i}t\theta_j d'_j}|\mathcal{F}_{j-1,\theta}) \text{ and } V_n^2(\theta) := \sum_{k=1}^n \theta_k^2\mathbb{E}(d_k^2). \tag{3.7}$$

For this new sequence of martingale differences we find Berry-Esseen bounds for $S'_n(\theta)=\sum_{j=1}^n \theta_j d'_j$ and then compare $S'_n(\theta)$ with $S_n(\theta)$. Here are the details.

Construction of the auxiliary sequence $(d'_j)_{1\le j\le n}$ of martingale differences. Define

$$A_j = A_j(\theta) = \Big\{ |f_{j,\theta}(t)| > \frac{1}{2\sqrt{e}} \mathrm{e}^{-t^2 V_n^2(\theta)/2} \Big\} \text{ and } d'_j = d_j I_{A_j}. \tag{3.8}$$

Note that the A_j's are $\mathcal{F}_{j-1,\theta}$ measurable and non-increasing in j (for the inclusion). Also define for $(d'_j)_{1\le j\le n}$ the quantities $T'_k = T'_k(\theta,n)$, $k=2,3,4$ as in (3.2) and (3.3).

This construction was used in Jakubowski (1986); de la Peña and Giné (1999); Merlevède et al. (2019) where condition (3.6) was verified. For an easy reference see relation (2.35) in Merlevède et al. (2019). Note that (3.6) implies that

$$|f'_{k,\theta}(t)| > \frac{1}{2}\mathrm{e}^{-t^2(V_n^2(\theta)+t^{-2})/2} \text{ for all } k\le n. \tag{3.9}$$

In addition, the following estimate will be needed in what follows:

$$\mathbb{P}_{|\theta}(A_n^c) \ll |t|^3 \mathbb{E}_{|\theta}|T_3^3| + t^4 \mathbb{E}_{|\theta}(T_4^4) + t^2 \mathbb{E}_{|\theta}\Big| \sum_{k=1}^n \theta_k^2 \big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2)\big)\Big|. \tag{3.10}$$

In case $t=0$, it is clear that $\mathbb{P}_{|\theta}(A_n^c)=0$, so that (3.10) is true. Now to prove (3.10) in case $t\neq 0$, we first write

$$\mathbb{P}_{|\theta}(A_n^c) \le \mathbb{P}_{|\theta}(H_{n,K}^c) + \mathbb{P}_{|\theta}\left(|f_{n,\theta}(t)| < \frac{1}{2}\mathrm{e}^{-t^2(V_n^2(\theta)+t^{-2})/2}, H_{n,K}(\theta)\right),$$

where

$$H_{n,K}(\theta) = \big\{ t \le K \min(|T_3^{-1}|, T_4^{-1})\big\}, \tag{3.11}$$

with K defined in Remark 3.3. By the Markov inequality,

$$\begin{aligned}\mathbb{P}_{|\theta}(H_{n,K}^c(\theta)) &\le \mathbb{P}_{|\theta}(|t|^3 > K^3 |T_3|^{-3}) + \mathbb{P}_{|\theta}(|t|^4 > K^4 T_4^{-4}) \\ &= \mathbb{P}_{|\theta}(T_3^3 > K^3 t^{-3}) + \mathbb{P}_{|\theta}(T_4^4 \ge K^4 t^{-4}) \\ &\le |t|^3 K^{-3} \mathbb{E}_{|\theta}|T_3^3| + |t|^4 K^{-4} \mathbb{E}_{|\theta}|T_4^4|. \end{aligned} \tag{3.12}$$

Next noting that, on $H_{n,K}(\theta)$, (3.4) holds, we have, for any $t\neq 0$,

$$\mathbb{P}_{|\theta}\left(|f_{n,\theta}(t)| < \frac{1}{2}\mathrm{e}^{-t^2(V_n^2(\theta)+t^{-2})/2}, H_{n,K}(\theta)\right) \le \mathbb{P}_{|\theta}(T_2^2 > V_n^2(\theta) + t^{-2}).$$

The estimate (3.10) follows from the previous considerations and Markov's inequality.

In the next lemma, we analyze the characteristic function of $S_n(\theta)$.

Lemma 3.4. *For any $t\in\mathbb{R}$,*

$$\Big|\mathbb{E}_{|\theta}(\mathrm{e}^{itS'_n(\theta)}) - \mathrm{e}^{-t^2V_n^2(\theta)/2}\Big| \ll B_n(\theta,t) \text{ and } \Big|\mathbb{E}_{|\theta}(\mathrm{e}^{itS_n(\theta)}) - \mathrm{e}^{-t^2V_n^2(\theta)/2}\Big| \ll B_n(\theta,t),$$

where

$$B_n(\theta,t) = |t|^3 \mathbb{E}_{|\theta}(|T_3^3|) + t^4 \mathbb{E}_{|\theta}(|T_4^4|) + t^2 \mathbb{E}_{|\theta}\Big| \sum_{k=1}^n \theta_k^2 \big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2)\big)\Big|.$$

Proof of Lemma 3.4. Clearly we can assume that $t\neq 0$. First we analyze $\mathbb{E}_{|\theta}\big(\mathrm{e}^{itS'_n(\theta)}\big)$. By (3.6), $f'_{n,\theta}(t)\neq 0$. Hence we can start from (3.5) where we use this time the random variables (d'_k), namely:

$$\mathbb{E}_{|\theta}\left(\frac{\mathrm{e}^{itS'_n(\theta)}}{f'_{n,\theta}(t)}\right) = 1. \tag{3.13}$$

Denote

$$\Delta_n(\theta) = \mathrm{e}^{itS'_n(\theta)} - \mathrm{e}^{-t^2V_n^2(\theta)/2}\left(\frac{\mathrm{e}^{itS'_n(\theta)}}{f'_{n,\theta}(t)}\right).$$

Using (3.13), we evaluate

$$\mathbb{E}_{|\theta}\big(\mathrm{e}^{itS'_n(\theta)}\big) - \mathrm{e}^{-t^2V_n^2(\theta)/2} = \mathbb{E}_{|\theta}(\Delta_n(\theta)).$$

Let

$$H'_{n,K}(\theta) = \{t \le K\min(|T_3'^{-1}|, T_4'^{-1})\},$$

with K defined in Remark 3.3. So, by (3.4), on $H'_{n,K}(\theta)$, we have $|f'_{n,\theta}(t)| \ge \frac{1}{2}\mathrm{e}^{-t^2T_2'^2(\theta,n)/2}$, which combined with (3.9) implies that, on $H'_{n,K}(\theta)$,

$$|f'_{n,\theta}(t)| \ge \frac{1}{2}\max\left(\mathrm{e}^{-t^2T_2'^2(\theta,n)/2}, \mathrm{e}^{-t^2(V_n^2(\theta)+t^{-2})/2}\right). \tag{3.14}$$

With the above notations, we use the following decomposition:

$$\mathbb{E}_{|\theta}\big(\mathrm{e}^{itS'_n(\theta)} - \mathrm{e}^{-t^2V_n^2(\theta)/2}\big) = \mathbb{E}_{|\theta}\big(\Delta_n(\theta)I_{H'_{n,K}(\theta)}\big) + \mathbb{E}_{|\theta}\big(\Delta_n(\theta)I_{H'^c_{n,K}(\theta)}\big). \tag{3.15}$$

By (3.6) we first notice that

$$\begin{aligned}\big|\mathbb{E}_{|\theta}\big(\Delta_n(\theta)I_{H'^c_{n,K}(\theta)}\big)\big| &\le \mathbb{E}_{|\theta}\left(\left\{|\mathrm{e}^{itS'_n(\theta)}| + \mathrm{e}^{-t^2V_n^2(\theta)/2}\left|\frac{\mathrm{e}^{itS'_n(\theta)}}{f'_{n,\theta}(t)}\right|\right\}I_{H'^c_{n,K}(\theta)}\right)\\ &\le (1+2\sqrt{\mathrm{e}})\mathbb{P}_{|\theta}(H'^c_{n,K}(\theta)).\end{aligned}$$

Now

$$\mathbb{P}_{|\theta}(H'^c_{n,K}(\theta)) \le \mathbb{P}_{|\theta}(H'^c_{n,K}(\theta)\cap A_n) + \mathbb{P}_{|\theta}(A_n^c).$$

Note that on A_n, $H'^c_{n,K}(\theta) = H^c_{n,K}(\theta)$ (since the A_j's are non increasing in j). Therefore, taking into account (3.12) and (3.10) we get

$$\big|\mathbb{E}_{|\theta}\big(\Delta_n(\theta)I_{H'^c_{n,K}(\theta)}\big)\big| \ll |t|^3\mathbb{E}_{|\theta}(|T_3^3|) + t^4\mathbb{E}_{|\theta}(T_4^4) + t^2\mathbb{E}_{|\theta}\Big|\sum_{k=1}^n\theta_k^2\big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2)\big)\Big|. \tag{3.16}$$

On another hand, note that

$$\begin{aligned}\big|\mathbb{E}_{|\theta}\big(\Delta_n(\theta)I_{H'_{n,K}(\theta)}\big)\big| \le{}& \mathbb{E}_{|\theta}\left(\frac{\left|f'_{n,\theta}(t) - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right|}{|f'_{n,\theta}(t)|}I_{H'_{n,K}(\theta)\cap A_n}\right)\\ &+ \mathbb{E}_{|\theta}\left(\frac{\left|f'_{n,\theta}(t) - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right|}{|f'_{n,\theta}(t)|}I_{A_n^c}\right).\end{aligned} \tag{3.17}$$

By (3.6),

$$\mathbb{E}_{|\theta}\left(\frac{\left|f'_{n,\theta}(t) - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right|}{|f'_{n,\theta}(t)|}I_{A_n^c}\right) \le (1+2\sqrt{\mathrm{e}})\mathbb{P}_{|\theta}(A_n^c). \tag{3.18}$$

Now, on $H'_{n,K}(\theta)$ we can use (3.14) and therefore,

$$\begin{aligned}\big|\mathbb{E}_{|\theta}\big(\Delta_n(\theta)I_{H'_{n,K}(\theta)\cap A_n}\big)\big| &\le \mathbb{E}_{|\theta}\left(\frac{\left|f'_{n,\theta}(t) - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right|}{|f'_{n,\theta}(t)|}I_{H'_{n,K}(\theta)\cap A_n}\right)\\ &\le 2\mathbb{E}_{|\theta}\left(\mathrm{e}^{t^2\min(T_2'^2(\theta,n),V_n^2(\theta)+t^{-2})/2}\left|f'_{n,\theta}(t) - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right|I_{H'_{n,K}(\theta)\cap A_n}\right).\end{aligned}$$

We continue the computations by introducing an intermediate term and use the triangle inequality to obtain:

$$\left|\mathbb{E}_{|\theta}\big(\Delta_n(\theta) I_{H'_{n,K}(\theta)\cap A_n}\big)\right| \le 2I_n(\theta) + 2I\!I_n(\theta)\,,$$

where

$$I_n(\theta) = \mathbb{E}\left(\mathrm{e}^{t^2T_2'^2(\theta,n)/2}\left|f'_{n,\theta}(t) - \mathrm{e}^{-t^2T_2'^2(\theta,n)/2}\right| I_{H'_{n,K}(\theta)\cap A_n}\right)$$

and

$$I\!I_n(\theta) = \mathbb{E}_{|\theta}\left(\mathrm{e}^{t^2\min(T_2'^2(\theta,n),V_n^2(\theta)+t^{-2})/2}\left|\mathrm{e}^{-t^2T_2'^2(\theta,n)/2} - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right| I_{A_n}\right)\,.$$

By Corollary 3.2 applied to (d'_j), we get

$$I_n(\theta) \ll \mathbb{E}_{|\theta}\big[(|t|^3|T_3'^3| + t^4T_4'^4)I_{A_n}\big] \ll \mathbb{E}_{|\theta}\big[(|t|^3|T_3^3| + t^4T_4^4)\big]\,, \tag{3.19}$$

since on A_n, $T'_3 = T_3$ and $T'_4 = T_4$.

We give now an estimate for the quantity $I\!I_n(\theta)$. With this aim, we introduce the following sets:

$$B_1(\theta) = \{T_2'^2(\theta,n) \le V_n^2(\theta)\}\,,\ B_2(\theta) = \{V_n^2(\theta) < T_2'^2(\theta,n) \le V_n^2(\theta) + t^{-2}\}$$

and

$$B_3(\theta) = \{T_2'^2(\theta,n) > V_n^2(\theta) + t^{-2}\}\,.$$

With these notations and setting

$$Q'_n(\theta) = \mathrm{e}^{t^2\min(T_2'^2(\theta,n),V_n^2(\theta)+t^{-2})/2}\left|\mathrm{e}^{-t^2T_2'^2(\theta,n)/2} - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right|\,,$$

we have

$$I\!I_n(\theta) = \sum_{i=1}^{3}\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_i(\theta)\cap A_n})\,.$$

Note first that

$$\begin{aligned}\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_1(\theta)\cap A_n}) &= \mathbb{E}_{|\theta}\left\{\mathrm{e}^{t^2T_2'^2(\theta,n)/2}\left(\mathrm{e}^{-t^2T_2'^2(\theta,n)/2} - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right) I_{B_1(\theta)\cap A_n}\right\}\\ &= \mathbb{E}_{|\theta}\left\{\left(1 - \mathrm{e}^{-t^2(V_n^2(\theta)-T_2'^2(\theta,n)/2}\right) I_{A_n\cap B_1(\theta)}\right\}\,.\end{aligned}$$

Now, on $B_1(\theta)$, $V_n^2(\theta) - T_2'^2(\theta,n) \ge 0$ and on A_n, $T_2'^2(\theta,n) = T_2^2(\theta)$. Hence, using the fact that, for $x \ge 0$, $1 - \mathrm{e}^{-x} \le x$, it follows that

$$\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_1(\theta)\cap A_n}) \ll t^2\mathbb{E}_{|\theta}\left|\sum_{k=1}^{n}\theta_k^2\big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2)\big)\right|\,. \tag{3.20}$$

We give now an estimate of $\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_2(\theta)\cap A_n})$. We have

$$\begin{aligned}\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_2(\theta)\cap A_n}) &= \mathbb{E}_{|\theta}\left\{\mathrm{e}^{t^2T_2'^2(\theta,n)/2}\left(\mathrm{e}^{-t^2V_n^2(\theta)/2} - \mathrm{e}^{-t^2T_2'^2(\theta,n)/2}\right) I_{B_2(\theta)\cap A_n}\right\}\\ &\le \sqrt{\mathrm{e}}\mathbb{E}_{|\theta}\left\{\mathrm{e}^{t^2V_n^2(\theta)/2}\left(\mathrm{e}^{-t^2V_n^2(\theta)/2} - \mathrm{e}^{-t^2T_2'^2(\theta,n)/2}\right) I_{A_n\cap B_2(\theta)}\right\}\,.\end{aligned}$$

Proceeding as above, we get that

$$\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_2(\theta)\cap A_n}) \ll t^2\mathbb{E}_{|\theta}\left|\sum_{k=1}^{n}\theta_k^2\big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2)\big)\right|\,. \tag{3.21}$$

It remains to deal with $\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_3(\theta)\cap A_n})$. We first write

$$\begin{aligned}\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_3(\theta)\cap A_n}) &= \mathbb{E}_{|\theta}\left\{\mathrm{e}^{t^2(V_n^2(\theta)+t^{-2})/2}\left(\mathrm{e}^{-t^2V_n^2(\theta)/2} - \mathrm{e}^{-t^2T_2'^2(\theta,n)/2}\right) I_{B_3(\theta)\cap A_n}\right\}\\ &\le \sqrt{\mathrm{e}}\mathbb{E}_{|\theta}\left\{\left(1 - \mathrm{e}^{-t^2(T_2'^2(\theta,n)-V_n^2(\theta))/2}\right) I_{A_n\cap B_3(\theta)}\right\}\,.\end{aligned}$$

Again, using the previous arguments, we get that

$$\mathbb{E}_{|\theta}(Q'_n(\theta)I_{B_3(\theta)\cap A_n}) \ll t^2\mathbb{E}_{|\theta}\Big|\sum_{k=1}^n \theta_k^2\big(\mathbb{E}_{k-1}(d_k^2)-\mathbb{E}(d_k^2)\big)\Big|. \tag{3.22}$$

So, overall, taking into account (3.20)-(3.22), we get that

$$I\!I_n(\theta) \ll t^2\mathbb{E}_{|\theta}\Big|\sum_{k=1}^n \theta_k^2\big(\mathbb{E}_{k-1}(d_k^2)-\mathbb{E}(d_k^2)\big)\Big|, \tag{3.23}$$

which combined with (3.19) entails that

$$\big|\mathbb{E}_{|\theta}\big(\Delta_n(\theta)I_{H'_{n,K}(\theta)\cap A_n}\big)\big| \ll |t|^3\mathbb{E}_{|\theta}(|T_3^3|)+t^4\mathbb{E}_{|\theta}(T_4^4)+t^2\mathbb{E}_{|\theta}\Big|\sum_{k=1}^n \theta_k^2\big(\mathbb{E}_{k-1}(d_k^2)-\mathbb{E}(d_k^2)\big)\Big|. \tag{3.24}$$

Starting from (3.17) and taking into account (3.18) (combined with (3.10)) and (3.24), it follows that

$$\big|\mathbb{E}_{|\theta}\big(\Delta_n(\theta)I_{H'_{n,K}(\theta)}\big)\big| \ll |t|^3\mathbb{E}_{|\theta}(|T_3^3|)+t^4\mathbb{E}_{|\theta}(T_4^4)+t^2\mathbb{E}_{|\theta}\Big|\sum_{k=1}^n \theta_k^2\big(\mathbb{E}_{k-1}(d_k^2)-\mathbb{E}(d_k^2)\big)\Big|. \tag{3.25}$$

Overall, starting from (3.15) and combining (3.16) and (3.25), the first part of the Lemma 3.4 follows.

To prove the second part we use the following decomposition (see for instance (2.32) in Merlevède et al. (2019)):

$$\Big|\mathbb{E}_{|\theta}\left(\mathrm{e}^{itS_n(\theta)}-\mathrm{e}^{-t^2V_n^2(\theta)/2}\right)\Big| \le 2\mathbb{P}_{|\theta}(A_n^c)+\Big|\mathbb{E}_{|\theta}\left(\mathrm{e}^{itS'_n(\theta)}-\mathrm{e}^{-t^2V_n^2(\theta)/2}\right)\Big|.$$

Then the second part of Lemma 3.4 follows from the first part of Lemma 3.4 and the upper bound (3.10). □

3.3. *Proof of Theorems 2.1 and 2.3.* We start with the following lemma concerning the vector θ:

Lemma 3.5. *Assume that θ is uniformly distributed on the sphere S^{n-1} of $\mathbb{R}^n$ and $(a_j)_{1\le j\le n}$ is a vector in $\mathbb{R}^n$. Then, there exists $c>0$ such that for any positive integer n,*

$$\mathbb{E}\left(\sum_{j=1}^n \theta_j^4|a_j|\right) \le c\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^n |a_j|\right),$$

and

$$\mathbb{E}\left|\sum_{j=1}^n \theta_j^3 a_j\right| \le c\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^n a_j^2\right)^{1/2}.$$

Proof. Let us introduce $Z=(Z_1,\dots,Z_n)$ i.i.d. standard normal variables independent of θ. Let γ be a chi-square random variable with n degrees of freedom independent of θ. Observe that the vector Z has the same distribution as $\sqrt{\gamma}\theta$. Since $\mathbb{E}(Z_1)=0$, by the independence of the Z_i's,

$$\mathbb{E}\left|\sum_{j=1}^n Z_j^3 a_j\right| = \mathbb{E}\left|\sum_{j=1}^n \gamma^{3/2}\theta_j^3 a_j\right| = \mathbb{E}(\gamma^{3/2})\mathbb{E}\left|\sum_{j=1}^n \theta_j^3 a_j\right|.$$

So

$$\mathbb{E}\left|\sum_{j=1}^n \theta_j^3 a_j\right| \ll \frac{1}{n^{3/2}}\mathbb{E}\left|\sum_{j=1}^n Z_j^3 a_j\right| \le \frac{1}{n^{3/2}}\left\|\sum_{j=1}^n Z_j^3 a_j\right\|_2 \ll \frac{1}{n^{3/2}}\left(\sum_{j=1}^n a_j^2\right)^{1/2},$$

and the second inequality is proved.

Also, we can use the distribution of θ_j which implies that there exists a positive constant C such that $\sup_{1\le i\le n}\mathbb{E}(\theta_i^4) \le Cn^{-2}$. This implies directly the first part of the lemma. □

We combine now Lemma 3.4 with Lemma 3.5 and use the fact that $\mathbb{E}(\theta_k^2) = 1/n$ for any $1 \le k \le n$. This gives Proposition 3.6 below, that will be used to prove Theorem 2.1.

Proposition 3.6. *For any real t, we have*

$$\mathbb{E}\left|\mathbb{E}_{|\theta}\left(\mathrm{e}^{itS_n(\theta)} - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right)\right| \ll \frac{1}{n}|t|^3\mathbb{E}\left(\frac{1}{n}\sum_{j=1}^n\left(\mathbb{E}_{j-1}(d_j^3)\right)^2\right)^{1/2} + t^4\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^n\mathbb{E}(d_j^4)\right) + t^2\frac{1}{n}\sum_{k=1}^n\|\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2)\|_1 .$$

Remark 3.7. If $\mathbb{E}_{j-1}(d_j^2) = \mathbb{E}(d_j^2) = 1$, then

$$\left(\mathbb{E}_{j-1}(d_j^3)\right)^2 \le \left(\mathbb{E}_{j-1}(d_j^2)\right)\left(\mathbb{E}_{j-1}(d_j^4)\right) \le \mathbb{E}_{j-1}(d_j^4),$$

implying that

$$\mathbb{E}\left(\frac{1}{n}\sum_{j=1}^n\left(\mathbb{E}_{j-1}(d_j^3)\right)^2\right)^{1/2} \le \left(\frac{1}{n}\sum_{j=1}^n\mathbb{E}(d_j^4)\right)^{1/2}.$$

Remark 3.8. Using the notation (2.1), it follows from Proposition 3.6 that

$$\mathbb{E}\left|\mathbb{E}_{|\theta}\left(\mathrm{e}^{itS_n(\theta)} - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right)\right| \ll \frac{1}{n}|t|^3(\alpha_n|t| + \beta_n) + t^2\frac{1}{n}\sum_{k=1}^n\|\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2)\|_1 .$$

For the proof of Theorem 2.3, we shall rather use Proposition 3.9 below.

Proposition 3.9. *We have*

$$\mathbb{E}\left|\mathbb{E}_{|\theta}\left(\mathrm{e}^{itS_n(\theta)} - \mathrm{e}^{-t^2V_n^2(\theta)/2}\right)\right| \ll \frac{1}{n}|t|^3\mathbb{E}\left(\frac{1}{n}\sum_{j=1}^n\left(\mathbb{E}_{j-1}(d_j^3)\right)^2\right)^{1/2} + t^4\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^n\mathbb{E}(d_j^4)\right) + \frac{t^2}{\sqrt{n}}\left(\frac{1}{n}\sum_{j=1}^n\mathbb{E}(d_j^4)\right)^{1/2} + \frac{t^2}{\sqrt{n}}c_2(n) .$$

where $c_2(n)$ is defined in (2.6)

Proof. We start again from Lemma 3.4 combined with Lemma 3.5. Here, compared to the statement of Proposition 3.6, we only have to show that

$$\Big\| \sum_{k=1}^{n} \theta_k^2 \big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2) \big) \Big\|_1 \ll n^{-1/2} \Big\{ \Big(\frac{1}{n} \sum_{j=1}^{n} \mathbb{E}(d_j^4) \Big)^{1/2} + c_2(n) \Big\} . \tag{3.26}$$

With this aim, we first write

$$\Big\| \sum_{k=1}^{n} \theta_k^2 \big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2) \big) \Big\|_1 \le \Big\| \sum_{k=1}^{n} \theta_k^2 \big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2) \big) \Big\|_2 .$$

Next,

$$\mathbb{E}_{|\theta} \Big(\sum_{k=1}^{n} \theta_k^2 \big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2) \big) \Big)^2 = \mathrm{Var}_{|\theta} \Big(\sum_{k=1}^{n} \theta_k^2 \mathbb{E}_{k-1}(d_k^2) \Big)$$
$$\le 2 \mathrm{Var}_{|\theta} \Big(\sum_{k=1}^{n} \theta_k^2 \big(\mathbb{E}_{k-1}(d_k^2) - d_k^2 \big) \Big) + 2 \mathrm{Var}_{|\theta} \Big(\sum_{k=1}^{n} \theta_k^2 d_k^2 \Big) .$$

Now

$$\mathrm{Var}_{|\theta} \Big(\sum_{k=1}^{n} \theta_k^2 \big(d_k^2 - \mathbb{E}_{k-1}(d_k^2) \big) \Big) = \sum_{k=1}^{n} \theta_k^4 \mathrm{Var}_{|\theta} (d_k^2 - \mathbb{E}_{k-1}(d_k^2)) \le 4 \sum_{k=1}^{n} \theta_k^4 \mathbb{E}(d_k^4) .$$

So, overall,

$$\mathbb{E}_{|\theta} \Big(\sum_{k=1}^{n} \theta_k^2 \big(\mathbb{E}_{k-1}(d_k^2) - \mathbb{E}(d_k^2) \big) \Big)^2 \le 4 \sum_{k=1}^{n} \theta_k^4 \mathbb{E}(d_k^4) + 2 \sum_{k,\ell=1}^{n} \theta_k^4 |\mathrm{cov}(d_k^2, d_\ell^2)| .$$

Next we integrate with respect to θ. Using the fact that, for any $1 \le k \le n$, $\mathbb{E}(\theta_k^4) \ll n^{-2}$, inequality (3.26) follows. □

Now we have to relate Propositions 3.6 and 3.9 to the Kolmogorov distance. With this aim, we use the approach in Bobkov et al. (2020a) (see their Lemma 6.1).

We have the following proposition:

Proposition 3.10. *Let $(d_k)_{1 \le k \le n}$ be a vector of martingale differences in $\mathbb{L}^4$ and N_θ be a r.v. such that its conditional distribution given θ is a normal distribution with mean 0 and variance $V_n^2(\theta) = \sum_{i=1}^{n} \theta_i^2 \mathbb{E}(d_i^2)$. If (1.1) is satisfied then, for all $T \ge T_0 > 1$,*

$$\mathbb{E} \left(\kappa_\theta(P_{S_n(\theta)}, P_{N_\theta}) \right) \ll \int_0^{T_0} \mathbb{E} |\mathbb{E}_{|\theta}(\mathrm{e}^{itS_n(\theta)}) - \mathrm{e}^{-t^2 V_n^2(\theta)/2}| \frac{dt}{t}$$
$$+ (\alpha_n + \sigma_4^2(n)) \frac{(1 + \log(T/T_0))}{n} + \mathrm{e}^{-T_0^2/16} + \frac{1}{T} . \tag{3.27}$$

Proof. By the usual Berry-Esseen smoothing inequality (see for instance inequality (6.2) in Bobkov et al. (2018)), there exists $c > 0$ such that, for any $T > 0$,

$$c \mathbb{E} \Big(\kappa_\theta(P_{S_n(\theta)}, P_{N_\theta}) \Big) \le \int_0^T \mathbb{E} \big| \mathbb{E}_{|\theta}(\mathrm{e}^{itS_n(\theta)}) - n_\theta(t) \big| \frac{dt}{t} + \frac{1}{T} \int_0^T \mathbb{E} |n_\theta(t)| dt, \tag{3.28}$$

where $n_\theta(t) = \mathbb{E}_{|\theta}(\mathrm{e}^{itN_\theta}) = \mathrm{e}^{-t^2 \sum_{i=1}^{n} \theta_i^2 \mathbb{E}(d_i^2)/2} = \mathrm{e}^{-t^2 V_n^2(\theta)/2}$. Since (1.1) is assumed, Lemma 5.2 together with Corollary 2.3 in Bobkov et al. (2018) imply that for all $T \ge T_0 > 1$,

$$\int_{T_0}^{T} t^{-1} \mathbb{E} |\mathbb{E}_{|\theta}(\mathrm{e}^{itS_n(\theta)})| dt \ll (M_4^4(n) + \sigma_4^2(n)) \frac{(1 + \log(T/T_0))}{n} + \mathrm{e}^{-T_0^2/16} ,$$

where

$$M_4^4(n) = \left\| \mathbb{E}_{|\theta} \left(\Big(\sum_{k=1}^n \theta_k d_k \Big)^4 \right) \right\|_\infty .$$

For martingales, by the Burkholder's inequality as given by Rio (2009) (see also Merlevède et al. (2019), Theorem 2.13), we have

$$\mathbb{E}_{|\theta} \left(\Big(\sum_{k=1}^n \theta_k d_k \Big)^4 \right) \le 3^2 \left(\sum_{k=1}^n \big(\mathbb{E}_{|\theta}(\theta_k^4 d_k^4) \big)^{1/2} \right)^2 \le 9 \sup_{1 \le k \le n} \mathbb{E}(d_k^4) = 9\alpha_n .$$

Therefore, if (1.1) holds, we have, for all $T \ge T_0 > 1$,

$$\int_{T_0}^T t^{-1} \mathbb{E} |\mathbb{E}_{|\theta}(\mathrm{e}^{itS_n(\theta)})| dt \ll (\alpha_n + \sigma_4^2(n)) \frac{(1 + \log(T/T_0))}{n} + \mathrm{e}^{-T_0^2/16} . \tag{3.29}$$

Next, note that we can also write $n_\theta(t) = \mathbb{E}_{|\theta}(\mathrm{e}^{itU_n(\theta)})$ where $U_n(\theta) = \sum_{i=1}^n \theta_i Y_i$ and the Y_i's are independent and such that $Y_i \sim \mathcal{N}(0, \mathbb{E}(d_i^2))$. Now

$$\sup_\theta \mathbb{E}_{|\theta}(U_n(\theta)^4) = 3 \sup_\theta \big(\mathbb{E}_{|\theta}(U_n(\theta)^2) \big)^2 \le 3 \left(\sup_{1 \le k \le n} \mathbb{E}(d_k^2) \right)^2 \le 3\alpha_n ,$$

and

$$n^{-1} \mathrm{Var} \left(\sum_{k=1}^n Y_k^2 \right) = n^{-1} \sum_{k=1}^n \mathrm{Var}(Y_k^2) \le 3n^{-1} \sum_{k=1}^n (\mathbb{E}(Y_k^2))^2 \le 3\alpha_n .$$

Therefore, taking into account (1.1), Lemma 5.2 together with Corollary 2.3 in Bobkov et al. (2018) imply that

$$|n_\theta(t)| \ll n^{-1}\alpha_n + e^{-t^2/16} ,$$

and consequently, for all $T \ge T_0 > 1$,

$$\int_{T_0}^T t^{-1} \mathbb{E}|n_\theta(t)| dt + T^{-1} \int_0^T \mathbb{E}|n_\theta(t)| dt \ll \alpha_n \frac{(1 + \log(T/T_0))}{n} + \mathrm{e}^{-T_0^2/16} + \frac{1}{T} . \tag{3.30}$$

Starting from (3.28) and taking into account (3.29) and (3.30), the proposition follows. □

3.4. *End of the proof of Theorems 2.1 and 2.3.* We shall apply Proposition 3.10 with $T = n$ and $T_0 = 4\sqrt{\log n}$. Therefore, to derive an upper bound for $\mathbb{E}\left(\kappa_\theta(P_{S_n(\theta)}, P_{N_\theta})\right)$ of order $1/n$ modulo an extra term $(\log n)^2$, one needs to prove that

$$\int_0^{T_0} \mathbb{E} |\mathbb{E}_{|\theta}(\mathrm{e}^{itS_n(\theta)}) - \mathrm{e}^{-t^2 V_n^2(\theta)/2}| \frac{dt}{t} \ll \frac{1}{n} \left(\gamma_n (\log n) + \beta_n (\log n)^{3/2} + \alpha_n (\log n)^2 \right) .$$

By Proposition 3.6,

$$\begin{aligned} \int_0^{T_0} \mathbb{E} |\mathbb{E}_{|\theta}(\mathrm{e}^{itS_n(\theta)}) - \mathrm{e}^{-t^2 V_n^2(\theta)/2}| \frac{dt}{t} &\ll \frac{1}{n} \int_0^{T_0} (t^2\beta_n + t^3\alpha_n + t\gamma_n) dt \ll \frac{1}{n} \left(|T_0|^3 + |T_0|^4 \right) \\ &\ll \frac{1}{n} \left(\gamma_n (\log n) + \beta_n (\log n)^{3/2} + \alpha_n (\log n)^2 \right) , \end{aligned}$$

and Theorem 2.1 follows. □

The proof of Theorem 2.3 is similar to the one of Theorem 2.1 above. The only difference is that we use Proposition 3.9 instead of Proposition 3.6 and we take into account Remark 2.2. □

3.5. *Proof of Remark 2.2.* We start by proving (2.4). With this aim, we recall Proposition 2.6 in Johnston and Prochno (2022). Let G_σ be a centered Gaussian r.v. with variance σ^2. Let α, β in $(0,\infty)$. If $\beta/\alpha > 1/2$, then

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}(G_\alpha \le t) - \mathbb{P}(G_\beta \le t)\right| \le \frac{3}{8}\frac{|\alpha^2-\beta^2|}{\alpha^2}\,. \tag{3.31}$$

Denote $V_n^2(\theta) = \theta_1^2 b_1^2 + \cdots + \theta_n^2 b_n^2$ with $b_j^2 = \mathbb{E}(d_j^2)$ for all j. Using inequality (3.31) with $\alpha = 1$ and $\beta^2 = V_n^2(\theta)$, we get

$$\begin{aligned}\mathbb{E}\big(\kappa_\theta(P_{N_\theta}, P_N)\big) &= \mathbb{E}\big(\kappa_\theta(P_{N_\theta}, P_N)\mathbf{1}_{\{V_n(\theta)>1/2\}}\big) + \mathbb{E}\big(\kappa_\theta(P_{N_\theta}, P_N)\mathbf{1}_{\{V_n(\theta)\le 1/2\}}\big)\\ &\le \frac{3}{8}\mathbb{E}\big|V_n^2(\theta) - 1\big| + \mathbb{P}(V_n(\theta)\le 1/2)\,.\end{aligned}$$

Now

$$\mathbb{P}(V_n(\theta)\le 1/2) = \mathbb{P}(1 - V_n^2(\theta) \ge 1/4) \le \mathbb{P}(|1-V_n^2(\theta)|\ge 1/4) \le 4\mathbb{E}\big|V_n^2(\theta)-1\big|\,.$$

Hence

$$\mathbb{E}\big(\kappa_\theta(P_{N_\theta}, P_N)\big) \le \frac{35}{8}\mathbb{E}\big|V_n^2(\theta)-1\big|\,. \tag{3.32}$$

To give an upper bound of the term in the right-hand side, we are going to use a second order concentration on the sphere developed in Bobkov et al. (2017) and also given in Proposition 4.1 in Bobkov et al. (2020b). Let

$$v(\theta) = V_n^2(\theta) - 1 = \sum_{k=1}^n \theta_k^2 b_k^2 - 1.$$

Notice that $v(\theta)$ has mean 0 because $\sum_{k=1}^n b_k^2 = n$ and $\mathbb{E}(\theta_k^2) = 1/n$ for all k. Moreover it is an even function ($v(\theta) = v(-\theta)$), so it is orthogonal in the Hilbert space $\mathbb{L}^2(\mu_{n-1})$ to all linear functions on the sphere. In addition, it is continuous and infinitely differentiable for all θ and the derivatives are continuous on an open interval containing θ. Proposition 4.1 in Bobkov et al. (2020b) then asserts that, for any a in $\mathbb{R}$,

$$\mathbb{E}\left(|v(\theta)|^2\right) \le \frac{5}{(n-1)^2}\mathbb{E}\left(\|\nabla^2 v(\theta) - aI_n\|_{HS}^2\right), \tag{3.33}$$

where HS stands for the Hilbert-Schmidt norm and I_n denotes the identity matrix of size n. We have

$$\frac{\partial^2 v(\theta)}{\partial\theta_k \partial\theta_j} = 2b_k^2\delta_{j,k},$$

with $\delta_{j,k} = 1$, for $j = k$ and 0 in rest. So, taking $a = 2$,

$$\mathbb{E}\left(\|\nabla^2 v(\theta) - aI_n\|_{HS}^2\right) = 4\sum_{k=1}^n (b_k^2 - 1)^2\,.$$

This implies that, for any $n \ge 2$,

$$\left\|\sum_{k=1}^n \theta_k^2 b_k^2 - 1\right\|_1 \le \left\|\sum_{k=1}^n \theta_k^2 b_k^2 - 1\right\|_2 \le 2\sqrt{5}\frac{\left(\sum_{k=1}^n (b_k^2-1)^2\right)^{1/2}}{n-1}\,. \tag{3.34}$$

Starting from (3.32) and considering the upper bound above, inequality (2.4) follows.

To prove (2.5), we start with the usual Berry-Esseen smoothing inequality which implies that there exists a $c > 0$ such that, for any $T > 0$,

$$c\sup_{t\in\mathbb{R}} |\mathbb{P}(N_\theta \le t) - \mathbb{P}(N \le t)| \le \int_0^T \left|\mathbb{E}(e^{itU_n(\theta)}) - e^{-t^2/2}\right|\frac{dt}{t} + \frac{1}{T}\int_0^T e^{-t^2/2}dt\,, \tag{3.35}$$

where $U_n(\theta) = \sum_{i=1}^n \theta_i Y_i$ and the Y_i's are independent, independent of θ and such that $Y_i \sim \mathcal{N}(0, \mathbb{E}(d_i^2))$.

Taking into account (3.29) with $S_n(\theta)$ replaced by $U_n(\theta)$ and setting $T_0 = 4\sqrt{\log n}$, we infer that the upper bound (2.5) follows from (3.35) if one can prove that

$$\int_0^{T_0} \left| \mathbb{E}\left(\mathrm{e}^{-t^2 V_n^2(\theta)/2}\right) - \mathrm{e}^{-t^2/2} \right| \frac{dt}{t} \ll \frac{\alpha_n (\log n)^2}{n} . \tag{3.36}$$

With this aim note that

$$\mathbb{E}\left(\mathrm{e}^{-t^2 V_n^2(\theta)/2}\right) - \mathrm{e}^{-t^2/2} = -\frac{t^2}{2}\mathbb{E}\big(V_n^2(\theta) - 1\big)\mathrm{e}^{-t^2/2} + \frac{t^4}{8}\mathbb{E}\Big[\big(V_n^2(\theta) - 1\big)^2 \int_0^1 (1-s)\mathrm{e}^{-t^2[1+s(V_n^2(\theta)-1)]/2} ds\Big] .$$

Now $\mathbb{E}\left(V_n^2(\theta)\right) = \sum_{i=1}^n \mathbb{E}(\theta_i^2)\mathbb{E}(d_i^2) = n^{-1}\sum_{i=1}^n \mathbb{E}(d_i^2) = 1$. Hence

$$\left| \mathbb{E}\left(\mathrm{e}^{-t^2 V_n^2(\theta)/2}\right) - \mathrm{e}^{-t^2/2} \right| \le \frac{t^4}{8}\mathbb{E}\left[\big(V_n^2(\theta) - 1\big)^2\right] ,$$

which combined with (3.34) implies that, for any $n \ge 2$,

$$\left| \mathbb{E}\left(\mathrm{e}^{-t^2 V_n^2(\theta)/2}\right) - \mathrm{e}^{-t^2/2} \right| \le \frac{5t^4}{2}\frac{\sum_{k=1}^n (b_k^2 - 1)^2}{(n-1)^2} \ll t^4 \frac{\alpha_n}{n} .$$

This ends the proof of (3.36). □

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