ALEA, Lat. Am. J. Probab. Math. Stat. **23**, 155–219 (2026)
DOI: 10.30757/ALEA.v23-08

Aging and sub-aging for Bouchaud trap models on resistance metric spaces

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Abstract. In this paper, we prove that if a sequence of electrical networks converges in the local Gromov–Hausdorff topology and satisfies a non-explosion condition, then the associated Bouchaud trap models (BTMs) also converge and exhibit aging. Moreover, when local structures of electrical networks converge, we prove sub-aging. Our results are applicable to a wide class of low-dimensional graphs, including the two-dimensional Sierpiński gasket, critical Galton-Watson trees, and the critical Erdős-Rényi random graph. The proof consists of two main steps: Polish metrization of the vague-and-point-process topology and showing the precompactness of transition densities of BTMs.

1. Introduction

Aging refers to the phenomenon in which a system never reaches equilibrium on laboratory time scales, and is typically observed in disordered media such as spin glasses at low temperatures. This out-of-equilibrium physical behavior has been of great interest in condensed matter physics for over thirty years and has been much discussed in the literature; for the physical background see Bouchaud et al. (1997), for example.

Research on aging in mathematics began about twenty years ago, and aging has been proven in several spin glass models: see Ben Arous (2002); Mathieu and Mourrat (2015) and the references therein. To understand the mechanism of aging, Bouchaud (1992) proposed a toy model, now called the Bouchaud trap model (BTM). Over several papers, Ben Arous and Černý studied aging in the Bouchaud trap model (BTM), and their results are summarized in Ben Arous and Černý (2006). The BTM treated in this article is set out in Definition 1.4 below, but, for discussion, here we introduce a simplified version of it. The (symmetric and simplified) Bouchaud trap model refers to a Markov chain on a randomly weighted graph defined as follows: fix a connected, simple, undirected graph $G = (V, E)$ with finite vertex set V and edge set E; let $\nu = (\xi_x)_{x \in V}$ be a family of i.i.d. positive random variables built on a probability space with probability measure P such that $u^\alpha \mathsf{P}(\xi_x > u) \to 1$ as $u \to \infty$ for some $\alpha \in (0, 1)$; conditional on ν, the BTM is the continuous-time

Received by the editors December 11th, 2024; accepted December 2nd, 2025; published under license CC BY 4.0.

2010 *Mathematics Subject Classification.* 60K37, 28A33, 60J27, 60J35, 82B41, 82D30.

Key words and phrases. Bouchaud trap model, aging and sub-aging, resistance metric space, vague-and-point-process topology.

The author has been supported by JSPS KAKENHI Grant Number JP 24KJ1447 and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

Markov chain $(X^\nu, \{P^\nu_x\}_{x\in V})$ on V whose jump rate w_{xy} is given by $w_{xy} \coloneqq \xi_x^{-1}$ if $\{x,y\} \in E$ and otherwise $w_{xy} \coloneqq 0$. The idea to quantify aging is to consider a suitably chosen two-point function $C(t_w, t_w+t)$ of X^ν between times t_w and t_w+t. It has been observed that good two-point functions are given by, for example,

$$C_1(t_w, t_w+t) \coloneqq \mathsf{E}\left[P^\nu\big(X^\nu(t_w) = X^\nu(t_w+t)\big)\right],$$
$$C_2(t_w, t_w+t) \coloneqq \mathsf{E}\left[P^\nu\big(X^\nu(t_w) = X^\nu(t_w+s),\ \forall s \in [0,t]\big)\right],$$

where E denotes the expectation with respect to P. Physical experiments that are represented by this model suggest that $C(t_w, t_w+t)$ only depends on $t/h(t_w)$ (for sufficiently large t_w), where h is an increasing function. Therefore, aging is proven by showing the existence of the following non-trivial limit:

$$R(\theta) = \lim_{t_w\to\infty} C(t_w, t_w + \theta h(t_w)).$$

If $h(t_w) = t_w$, then it is called (full) aging, and if $h(t_w) = o(t_w)$, then it is called sub-aging.

In Ben Arous and Černý (2005, 2008); Ben Arous et al. (2006), Ben Arous, Černý and Mountford studied the BTM for $G = \mathbb{Z}^d$ with nearest neighbor edges, proving aging for C_1 and sub-aging for C_2, as defined above. In particular, it was found that the descriptions of the limits of the two-point functions are significantly different between $d = 1$ and $d \geq 2$. This is due to the difference in the scaling limits of the BTMs. When $d = 1$, the scaling limit of the BTM is a Markov process called Fontes–Isopi–Newman (F.I.N.) diffusion, firstly proven in Fontes et al. (2002). On the other hand, when $d \geq 2$, it converges to a non-Markovian process called the fractional-kinetics process Ben Arous and Černý (2007). This suggests that the BTMs can be divided into low-dimensional and high-dimensional regimes. Ben Arous and Černý generalized their discussions of the BTM on $\mathbb{Z}^d$ with $d \geq 2$ and obtained a method to prove (sub-)aging for a class of high-dimensional graphs, including complete graphs. Example graphs in the low-dimensional regime other than $\mathbb{Z}$ were found by Croydon, Hambly, and Kumagai (2017). Using the theory of resistance forms developed by Kigami (2001, 2012), they obtained that if a sequence of graphs converges in the local Gromov–Hausdorff-vague topology (introduced in Section 3) as resistance metric spaces equipped with the counting measures and if it satisfies the uniform volume doubling (UVD) condition (see Croydon et al., 2017, Definition 1.1), then the associated BTMs converge. In particular, their results are applicable to the (two-dimensional) Sierpiński gasket. However, (sub-)aging results were left open. In this paper, we improve their results by replacing the UVD condition with a weaker condition, the non-explosion condition introduced in Croydon (2018). Moreover, we show (sub-)aging for the associated BTMs. Our results are applicable to a wide class of low-dimensional graphs, including the Sierpiński gasket, the critical Galton-Watson tree, and the critical Erdős-Rényi random graph.

To present our main results, we begin by introducing several pieces of notation. We write $\mathbb{R}_{\geq 0} \coloneqq [0,\infty)$, $\mathbb{R}_{>0} \coloneqq (0,\infty)$ and $\mathbb{Z}_{\geq 0} \coloneqq \mathbb{Z} \cap \mathbb{R}_{\geq 0}$. For $a, b \in \mathbb{R}$, we write $a \vee b \coloneqq \max\{a,b\}$ and $a \wedge b \coloneqq \min\{a,b\}$. Given a metric space (S,d), we set, for each $x \in S$ and $r > 0$,

$$B_S(x,r) = B_d(x,r) \coloneqq \{y \in S \mid d(x,y) < r\}, \quad D_S(x,r) = D_d(x,r) \coloneqq \{y \in S \mid d(x,y) \leq r\}. \tag{1.1}$$

We say that (S,d) is *boundedly compact* if and only if $D_S(x,r)$ is compact for all $x \in S$ and $r > 0$. Note that a boundedly-compact metric space is complete, separable and locally compact. A tuple (S,d,ρ,μ) is said to be a *rooted-and-measured boundedly-compact metric space* if and only if (S,d) is a boundedly-compact metric space, ρ is a distinguished element of S called the *root*, and μ is a Radon measure on S, that is, μ is a Borel measure on S such that $\mu(K) < \infty$ for every compact subset K. Given a rooted-and-measured boundedly-compact metric space $G = (S,d,\rho,\mu)$, for each $r > 0$, we define a rooted-and-measured compact metric space $G^{(r)} = (S^{(r)}, d^{(r)}, \rho^{(r)}, \mu^{(r)})$ by setting

$$S^{(r)} \coloneqq \mathrm{cl}(B_d(\rho,r)), \quad d^{(r)} \coloneqq d|_{S^{(r)}\times S^{(r)}}, \quad \rho^{(r)} \coloneqq \rho, \quad \mu^{(r)}(\cdot) \coloneqq \mu(\cdot \cap S^{(r)}), \tag{1.2}$$

where cl(·) denotes the closure of a set. We write $\mathbb{G}$ for the collection of rooted-and-measured isometric equivalence classes of rooted-and-measured boundedly-compact metric spaces and equip $\mathbb{G}$ with the local Gromov–Hausdorff-vague topology. (See Section 3 for details).

Our argument relies on the theory of resistance forms. Here we will prepare the minimum necessary information on resistance forms in order to state our main results. See Section 4 for details. Let (F, R) be a regular resistance metric space and write $(\mathcal{E}, \mathcal{F})$ for the corresponding regular resistance form. Given a Radon measure μ on F of full support, there exists a related regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(F, \mu)$ and also an associated strong Markov process $(X^\mu = (X^\mu_t)_{t\geq 0}, \{P^\mu_x\}_{x\in F})$, which we call the *process associated with* (F, R, μ) (it is known that the process X^μ can be chosen so that it is a Hunt process). We note that if the resistance metric R is recurrent in the sense of Definition 4.7 below, then it is automatically regular (Noda, 2024b, Corollary 3.26).

We next introduce electrical networks and associated resistance forms.

Definition 1.1 (Electrical network)**.** Let (V, E) be a connected, simple, undirected graph with finite or countably many vertices, where V denotes the vertex set and E denotes the edge set. (NB. A graph being simple means that it has no loops and no multiple edges.) For $x, y \in V$, we write $x \sim y$ if and only if $\{x, y\} \in E$. Let $\{\mu(x, y)\}_{x,y\in V}$ be a family of non-negative real numbers such that $\mu(x, y) = \mu(y, x)$ for all $x, y \in V$, $\mu(x, y) > 0$ if and only if $x \sim y$, and

$$\mu(x) \coloneqq \sum_{y\in V} \mu(x, y) < \infty, \quad \forall x \in V.$$

We call $\mu(x, y)$ the *conductance* on the edge $\{x, y\}$ and (V, E, μ) an *electrical network*. Note that the edge set E is uniquely determined by conductances. We equip V with the discrete topology and define a Radon measure $\mu^{\#}$ as the *counting measure* on V, which is a Radon measure on V given by

$$\mu^{\#}(A) \coloneqq \#A = \sum_{x\in V} \delta_x(A) \quad A \subseteq V,$$

where δ_x is the Dirac measure putting mass 1 at x. Given an electrical network G, we write V_G, E_G, $\{\mu_G(x, y)\}_{x,y\in V_G}$, and $\mu^{\#}_G$ for the vertex set, the edge set, the conductances, and the counting measure, respectively. When we say that G is a *rooted electrical network*, there exists a distinguished vertex, which we denote by $\rho_G \in V_G$.

Definition 1.2 (Resistance form associated with electrical network)**.** Let G be an electrical network. For functions $f, g \colon V_G \to \mathbb{R}$, we set

$$\mathcal{E}_G(f, g) \coloneqq \frac{1}{2} \sum_{x,y\in V_G} \mu_G(x, y)(f(x) - f(y))(g(x) - g(y))$$

(if the right-hand side is well-defined). We then set $\mathcal{F}_G \coloneqq \{f \in \mathbb{R}^{V_G} \mid \mathcal{E}(f, f) < \infty\}$ and, for each $x, y \in F$,

$$R_G(x, y) \coloneqq \sup\{\mathcal{E}(f, f)^{-1} \mid f \in \mathcal{F},\ f(x) = 1,\ f(y) = 0\},$$

where we define $\sup \emptyset \coloneqq 0$.

We note that, for any electrical network G, the pair $(\mathcal{E}_G, \mathcal{F}_G)$ is a regular resistance form, R_G is the corresponding resistance metric, and the topology on V_G induced from R_G is the discrete topology (see Noda, 2024b, Theorem 4.1). We say that G is a *recurrent electrical network* if R_G is a recurrent resistance metric in the sense of Definition 4.7. Given an electrical network G and a measure ν on V_G of full support, we write $X^\nu_G = (X^\nu_G(t))_{t\geq 0}$ for the process associated with (V_G, R_G, ν). By Noda (2024b, Theorem 4.1), X^ν_G is the minimal continuous-time Markov chain on V_G with generator

$$(\Delta^\nu_G f)(x) \coloneqq \sum_{y\in V_G} \frac{\mu_G(x, y)}{\nu(\{x\})}(f(y) - f(x)).$$

Now, we define the (symmetric) Bouchaud trap model on an electrical network. Throughout this paper, we fix a constant $\alpha \in (0,1)$.

Definition 1.3 (The random variable ξ)**.** Let ξ be a positive random variable built on a probability space with probability measure P_ξ such that there exists a slowly varying function ℓ satisfying $P_\xi(\xi \geq u) = u^{-\alpha}\ell(u)$ for all $u > 0$.

Definition 1.4 (The Bouchaud trap model)**.** Fix an electrical network G. Let $\{\xi_x^G\}_{x \in V_G}$ be i.i.d. random variables with $\xi_x^G \stackrel{\mathrm{d}}{=} \xi$. Set $\nu_G := \sum_{x \in V_G} \xi_x^G \delta_x$, which is a random measure on V_G. Conditional on ν_G, the (symmetric) Bouchaud trap model (BTM) is defined as the continuous-time Markov chain $X_G^{\nu_G}$. The random measure ν_G is called a *trap*.

Our first result concerns BTMs for a convergent sequence of deterministic (scaled) electrical networks. For each $n \in \mathbb{N}$, let G_n be a rooted recurrent electrical network such that (V_{G_n}, R_{G_n}) is boundedly compact. We simply write

$$V_n := V_{G_n}, \quad \mu_n := \mu_{G_n}, \quad \mu_n^\# := \mu_{G_n}^\#, \quad \rho_n := \rho_{G_n}, \quad R_n := R_{G_n}, \quad \nu_n := \nu_{G_n}, \quad X_n^{\nu_n} := X_{G_n}^{\nu_{G_n}}$$

We write P_n for the underlying probability measure of the random measure ν_n. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences of positive real numbers with $a_n \wedge b_n \to \infty$. We then define

$$c_n := \inf\{u > 0 \mid P_\xi(\xi > u) < b_n^{-1}\}.$$

Assumption 1.5.

(i) *It holds that*

$$(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1}\mu_n^\#) \to (F, R, \rho, \mu)$$

in the local Gromov–Hausdorff-vague topology for some $(F, R, \rho, \mu) \in \mathbb{G}$*, where* μ *is of full support and non-atomic, that is,* $\mu(\{x\}) = 0$ *for all* $x \in F$.

(ii) *It holds that*

$$\lim_{r\to\infty} \liminf_{n\to\infty} a_n^{-1} R_n(\rho_n, B_{R_n}(\rho_n, a_n r)^c) = \infty.$$

Under Assumption 1.5, the limiting metric space (F, R) is a recurrent resistance metric space (see Noda, 2024b, Theorem 5.1). Given a Radon measure ν on F of full support, we write $X^\nu = (X^\nu(t))_{t\geq 0}$ for the process associated with (F, R, ν). For the description of the limit of Bouchaud trap models $X_n^{\nu_n}$, we define a random measure ν on F as follows. Let π be a Poisson random measure on $F \times \mathbb{R}_{>0}$ with intensity measure $\mu(dx)\, \alpha v^{-1-\alpha} dv$ defined on a probability space equipped with probability measure P. We then define a random measure ν on F by setting

$$\nu(A) := \int 1_A(x) v \, \pi(dx\, dv), \quad \forall A \in \mathcal{B}(F),$$

where $\mathcal{B}(F)$ denotes the collection of Borel subsets of F. The random measure ν is a fully-supported Radon measure almost surely (see Lemma 6.1 below) and is an analogue of the F.I.N. measure, the speed measure of the F.I.N. diffusion. The random measures π and ν are the limits of traps. More precisely, in Section 6, it will be proven that, under Assumption 1.5, $(\pi_n, c_n^{-1}\nu_n) \xrightarrow{\mathrm{d}} (\pi, \nu)$, where π_n is given by

$$\pi_n := \sum_{x \in V_n} \delta_{(x, c_n^{-1}\nu_n(\{x\}))}.$$

Note that the convergence of π_n to π contains information of atoms of traps, which the convergence of $c_n^{-1}\nu_n$ to ν does not guarantee.

Write $\tilde{X}_n^{\nu_n}(t) := X_n^{\nu_n}(a_n c_n t)$ and define $\tilde{\mathcal{L}}_n^{\nu_n}$ to be the law of $\tilde{X}_n^{\nu_n}$ under $P_{\rho_n}^{\nu_n}$, i.e.,

$$\tilde{\mathcal{L}}_n^{\nu_n}(\cdot) := P_{\rho_n}^{\nu_n}\big((\tilde{X}_n^{\nu_n}(t))_{t\geq 0} \in \cdot\big).$$

Note that conditional on ν_n, $\tilde{\mathcal{L}}_n^{\nu_n}$ is a probability measure on $D(\mathbb{R}_{\geq 0}, V_n)$, which denotes the space of cadlag functions with values in V_n equipped with the usual J_1-Skorohod topology. Since $P_{\rho_n}^{\nu_n}(X_n^{\nu_n} \in \cdot)$ is measurable with respect to ν_n (see Noda, 2025+, Proposition 6.1), $\tilde{\mathcal{L}}_n^{\nu_n}$ is a random element of $\mathcal{P}(D(\mathbb{R}_{\geq 0}, V_n))$, which is defined to be the space of the probability measures on $D(\mathbb{R}_{\geq 0}, V_n)$ equipped with the weak topology. Similarly, we define

$$\mathcal{L}^\nu(\cdot) := P_\rho^\nu(X^\nu \in \cdot),$$

which is a random element of $\mathcal{P}(D(\mathbb{R}_{\geq 0}, F))$.

For aging, we consider the following two-point functions:

$$\tilde{\Phi}_n^{\nu_n}(s,t) := P_{\rho_n}^{\nu_n}\big(\tilde{X}_n^{\nu_n}(s) = \tilde{X}_n^{\nu_n}(t)\big), \quad \Phi^\nu(s,t) := P_\rho^\nu\big(X^\nu(s) = X^\nu(t)\big)$$

for $s, t > 0$. We note that $\tilde{\Phi}_n^{\nu_n}$ and Φ^ν are continuous (see Section 5). In the first result below, we obtain that the BTMs $\tilde{X}_n^{\nu_n}$ and the associated aging functions $\tilde{\Phi}_n^{\nu_n}$ converge to X^ν and Φ^ν, respectively.

Theorem 1.6. *Under Assumption 1.5, it holds that*

$$\Big(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}\mu_n^\#, \mathsf{P}_n\big((c_n^{-1}\nu_n, \tilde{\mathcal{L}}_n^{\nu_n}, \tilde{\Phi}_n^{\nu_n}) \in \cdot\big)\Big) \to \Big(F, R, \rho, \mu, \mathsf{P}\big((\nu, \mathcal{L}^\nu, \Phi^\nu) \in \cdot\big)\Big) \tag{1.3}$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}} \times \tau_{\mathcal{P}}(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau_{\mathcal{P}}(\tau^{J_1}) \times \tau^{C(\mathbb{R}_{>0}^2, \mathbb{R}_{\geq 0})}))$ (defined in Section 3 below). In particular, $\tilde{\Phi}_n^{\nu_n} \xrightarrow{\mathrm{d}} \Phi^\nu$ in $C(\mathbb{R}_{>0}^2, \mathbb{R}_{\geq 0})$ with respect to the compact-convergence topology, where the limit is positive with probability 1.

Remark 1.7. Fix $s, t > 0$. Since $\tilde{\Phi}_n^{\nu_n}(s,t) \leq 1$, the family $(\tilde{\Phi}_n^{\nu_n}(s,t))_{n \geq 1}$ of random variables is uniformly integrable. Combining this with Theorem 1.6, we obtain that

$$\lim_{n \to \infty} \mathsf{E}_n\big[\tilde{\Phi}_n^{\nu_n}(s,t)\big] = \mathsf{E}\big[\Phi^\nu(s,t)\big],$$

where E_n and E denote the expectations with respect to P_n and P, respectively. This recovers the aging result of Fontes et al. (2002).

For sub-aging, we additionally assume convergence of local structures of electrical networks. Roughly speaking, we assume that a uniformly chosen vertex and the total conductance at the vertex converge jointly; this assumption is precisely stated below. Given an electrical network G, we define a map $\psi_G \colon V_G \to V_G \times \mathbb{R}_{\geq 0}$ by setting

$$\psi_G(x) := (x, \mu_G(x)), \quad x \in V_G. \tag{1.4}$$

Write $\dot{\mu}_G^\#$ for the pushforward measure of $\mu_G^\#$ by ψ_G, which is a Radon measure on $V_G \times \mathbb{R}_{\geq 0}$. We emphasize that $\dot{\mu}_G^\#$ is a natural object in terms of graph theory. Indeed, if G is a simple electrical network, that is, $\mu_G(x,y) = 1$ if $\mu_G(x,y) > 0$, then $\dot{\mu}_G^\#(A \times \{k\})$ is the number of vertices in $A \subseteq V_G$ whose degree is k. From this observation, one can see that the measure $\dot{\mu}^\#$ carries more information than the counting measure. We simply write $\dot{\mu}_n^\# := \dot{\mu}_{G_n}^\#$, and, for sub-aging, we assume the convergence of $\dot{\mu}_n^\#$ instead of that of the counting measures $\mu_n^\#$.

Assumption 1.8. *Assumption 1.5(ii) is satisfied, and*

$$(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}\dot{\mu}_n^\#) \to (F, R, \rho, \dot{\mu})$$

for some $(F, R, \rho, \dot{\mu}) \in \mathfrak{M}_\bullet(\tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0})})$ in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0})})$ (defined in Section 3 below). Moreover, the measure μ on F defined by

$$\mu(A) := \dot{\mu}(A \times \mathbb{R}_{\geq 0}), \quad \forall A \in \mathcal{B}(F),$$

is of full support and non-atomic.

Under Assumption 1.8, we let $\dot{\pi}(dx\,dw\,dv)$ be the Poisson random measure on $F \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ with intensity measure $\dot{\mu}(dx\,dw)\alpha v^{-1-\alpha}dv$ defined on a probability space with probability measure P. Define a random measure ν on F by setting

$$\nu(A) := \int 1_A(x) v\, \dot{\pi}(dx\,dw\,dv), \quad \forall A \in \mathcal{B}(F).$$

Then, ν is a fully-support Radon measure almost surely (see Lemma 6.4 below). In Section 6, it will be proven that, under Assumption 1.8, $(\dot{\pi}_n, c_n^{-1}\nu_n) \to (\dot{\pi}, \nu)$, where $\dot{\pi}_n$ is given by

$$\dot{\pi}_n := \sum_{x \in V_n} \delta_{(x, \mu_n(x), c_n^{-1}\nu_n(\{x\}))}.$$

Note that $\dot{\pi}_n$ contains information of local structure: the total conductance at every vertex.

For sub-aging, we consider the following two-point function:

$$\tilde{\Psi}_n^{\nu_n}(s,t) := P_{\rho_n}^{\nu_n}\left(X_n^{\nu_n}(a_n c_n t) = X_n^{\nu_n}(a_n c_n t + t'),\ \forall t' \in [0, c_n s]\right)$$

for $s \geq 0, t > 0$. By the Markov property and the fact that the waiting time of $X_n^{\nu_n}$ at x has the exponential distribution with mean $\nu_n(\{x\})/\mu_n(x)$, we deduce that

$$\begin{aligned}\tilde{\Psi}_n^{\nu_n}(s,t) &= E_{\rho_n}^{\nu_n}\left[\exp(-\mu_n(\tilde{X}_n^{\nu_n}(t))s/c_n^{-1}\nu_n(\tilde{X}_n^{\nu_n}(t)))\right] \\ &= \int e^{-ws/v} P_{\rho_n}^{\nu_n}(\tilde{X}_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\,dw\,dv).\end{aligned}$$

Following this expression, we define

$$\Psi^\nu(s,t) := \int e^{-ws/v} P_\rho^\nu(X^\nu(t) = x)\, \dot{\pi}(dx\,dw\,dv)$$

for $s \geq 0, t > 0$. We note that the functions $\tilde{\Psi}_n^{\nu_n}$ and Ψ^ν are continuous on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ (see Section 5). Under Assumption 1.8, we obtain not only the same aging result as Theorem 1.6, but also a sub-aging result, that is, $\tilde{\Psi}_n^{\nu_n}$ converges to Ψ^ν.

Theorem 1.9. *Under Assumption 1.8, it holds that*

$$\begin{aligned}&\Big(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}\dot{\mu}_n^{\#}, \mathsf{P}_n\big((c_n^{-1}\nu_n, \tilde{\mathcal{L}}_n^{\nu_n}, \tilde{\Phi}_n^{\nu_n}, \tilde{\Psi}_n^{\nu_n}) \in \cdot\big)\Big) \\ \to &\Big(F, R, \rho, \dot{\mu}, \mathsf{P}\big((\nu, \mathcal{L}^\nu, \Phi^\nu, \Psi^\nu) \in \cdot\big)\Big) \qquad (1.5)\end{aligned}$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0})} \times \tau_{\mathcal{P}}(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau_{\mathcal{P}}(\tau^{J_1}) \times \tau^{C(\mathbb{R}^2_{>0}, \mathbb{R}_{\geq 0})} \times \tau^{C(\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}, \mathbb{R}_{\geq 0})}))$*. In particular,* $\tilde{\Psi}_n^{\nu_n} \xrightarrow{\mathrm{d}} \Psi^\nu$ *in* $C(\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}, \mathbb{R}_{\geq 0})$*, where the limit is positive with probability* 1.

Remark 1.10. Similarly to Remark 1.7, from Theorem 1.9, we obtain the convergence of the annealed sub-aging functions, i.e., for any $s \geq 0$ and $t > 0$,

$$\lim_{n \to \infty} \mathsf{E}_n\big[\tilde{\Psi}_n^{\nu_n}(s,t)\big] = \mathsf{E}\big[\Psi^\nu(s,t)\big].$$

Remark 1.11. By definition, the waiting time of the BTM $X_G^{\nu_G}$ at a vertex x is determined by the trap $\nu_G(\{x\})$ and a 1-neighbor local structure of G, that is, the total conductance $\mu_G(x) = \sum_{y \sim x} \mu_G(x,y)$. This is the reason why we assume the convergence of $\dot{\mu}_n^{\#}$ in Assumption 1.8. Even when one considers other trap models, it seems possible to derive similar results by assuming convergence of corresponding local structures. (For a generalized trap model, see Ben Arous et al., 2015.)

Next, we consider random electrical networks. Namely, we assume that $(V_n, R_n, \rho_n, \mu_n)$ is a random element of $\mathbb{G}$ and we denote its underlying probability measure by $\mathbf{P}_n$. For aging, we consider a random version of Assumption 1.5, given below.

Assumption 1.12.

(i) *It holds that*

$$(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1} \mu_n^{\#}) \xrightarrow{\mathrm{d}} (F, R, \rho, \mu)$$

in the local Gromov–Hausdorff-vague topology for some random element (F, R, ρ, μ) *of* $\mathbb{G}$*, where* μ *is of full support and non-atomic with probability* 1.

(ii) *It holds that*

$$\lim_{r\to\infty} \liminf_{n\to\infty} \mathbf{P}_n\bigl(a_n^{-1} R_n(\rho_n, B_{R_n}(\rho_n, a_n r)^c) > \lambda\bigr) = 1, \quad \forall \lambda > 0.$$

Theorem 1.13. *Under Assumption 1.12, the convergence (1.3) holds in distribution.*

Remark 1.14. Similarly to Remark 1.7, under Assumption 1.12, we deduce from Theorem 1.13 that, for any $s, t > 0$, $\mathsf{E}_n\bigl[\tilde{\Phi}_n^{\nu_n}(s,t)\bigr] \xrightarrow{\mathrm{d}} \mathsf{E}\bigl[\Phi^{\nu}(s,t)\bigr]$ and

$$\lim_{n\to\infty} \mathbf{E}_n\bigl[\mathsf{E}_n\bigl[\tilde{\Phi}_n^{\nu_n}(s,t)\bigr]\bigr] = \mathbf{E}\bigl[\mathsf{E}\bigl[\Phi^{\nu}(s,t)\bigr]\bigr].$$

where $\mathbf{E}_n$ and $\mathbf{E}$ denote the expectations with respect to $\mathbf{P}_n$ and $\mathbf{P}$, respectively.

For sub-aging, we consider a random version of Assumption 1.8, given below.

Assumption 1.15. *Assumption 1.12(ii) is satisfied, and there exists a random element* $(F, R, \rho, \dot{\mu})$ *of* $\mathfrak{M}_{\bullet}(\tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0})})$ *such that*

$$(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1} \dot{\mu}_n^{\#}) \xrightarrow{\mathrm{d}} (F, R, \rho, \dot{\mu}) \tag{1.6}$$

in the space $\mathfrak{M}_{\bullet}(\tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0})})$*. Moreover, a measure* μ *on* F *defined by*

$$\mu(A) := \dot{\mu}(A \times \mathbb{R}_{\geq 0}), \quad \forall A \in \mathcal{B}(F)$$

is of full support and non-atomic with probability 1.

Theorem 1.16. *Under Assumption 1.15, the convergence (1.5) holds in distribution.*

Remark 1.17. Similarly to Remark 1.7, under Assumption 1.15, we deduce from Theorem 1.16 that, for any $s \geq 0$ and $t > 0$, $\mathsf{E}_n\bigl[\tilde{\Psi}_n^{\nu_n}(s,t)\bigr] \xrightarrow{\mathrm{d}} \mathsf{E}\bigl[\Psi^{\nu}(s,t)\bigr]$ and

$$\lim_{n\to\infty} \mathbf{E}_n\bigl[\mathsf{E}_n\bigl[\tilde{\Psi}_n^{\nu_n}(s,t)\bigr]\bigr] = \mathbf{E}\bigl[\mathsf{E}\bigl[\Psi^{\nu}(s,t)\bigr]\bigr].$$

Remark 1.18. More generally than our BTMs, one can consider non-symmetric BTMs (cf. Ben Arous and Černý, 2006, Definition 2.1). Fix $a \in (0, 1]$ and an electrical network G. Conditional on ν_G, the non-symmetric BTM with parameter a on G refers to the continuous-time Markov chain X^{ν_G} with generator

$$(\Delta^{\nu_G} f)(x) := \sum_{y \in V_G} \frac{\mu_G(x,y)(\nu_G(\{x\})\nu_G(\{y\}))^a}{\nu_G(\{x\})}(f(y) - f(x)).$$

If we define an electrical network G' by setting $V_{G'} := V_G$ and $\mu_{G'}(x,y) := \mu_G(x,y)(\nu_G(\{x\})\nu_G(\{y\}))^a$, then X^{ν_G} is the process associated with $(V_{G'}, R_{G'}, \nu_G)$. Thus, when one considers a non-symmetric BTM, the associated resistance metric becomes random and does not coincide with the resistance metric on G. Our arguments work even for these non-symmetric BTMs, and similar results hold once the corresponding assumptions with respect to associated resistance metrics are verified. However, it is not easy to check in general, and so we only consider symmetric BTMs in this article.

In the proof of our main results, it is crucial to find a coupling of traps ν_n and ν so that $c_n^{-1}\nu_n \to \nu$ almost surely vaguely and in the point process sense, where we recall that the convergence in the point process sense is notion of convergence of discrete measures introduced in Fontes et al. (2002, Definition 2.2) and means convergence of atoms. When discrete measures converge both vaguely and in the point process sense, we say that they converge in the *vague-and-point-process topology*. In that paper, where they prove the convergence of the scaled BTM on $\mathbb{Z}$ to the F.I.N. diffusion, they constructed such a coupling by using Lévy processes and the usual J_1-Skorohod topology. However, their argument cannot be applied to general graphs. In this paper, we construct the coupling by the Skorohod representation theorem (cf. Kallenberg, 2021, Theorem 5.31). Specifically, we first show that the vague-and-point-process topology is Polish, that is, it is separable and completely metrizable, which can be seen as an extension of the usual J_1-Skorohod topology. Then it is an immediate consequence of the Skorohod representation theorem that there exists a desired coupling once the convergence $c_n^{-1}\nu_n \to \nu$ in distribution is verified.

The remainder of the article is organized as follows. In Section 2, we prove that the vague-and-point-process topology introduced above is a Polish topology. In Section 3, we introduce the Gromov–Hausdorff-type topologies which are used to discuss convergence of objects on different metric spaces. In Section 4, we recall some fundamental results about the theory of resistance forms and study transition densities of processes on measured resistance metric spaces. In particular, it is proven that if a family of measured resistance metric spaces is precompact in the local Gromov–Hausdorff-vague topology, then the family of the transition densities of associated processes is precompact. This result is used to prove the precompactness of two-point functions. In Section 5, we prove that if deterministic traps converge in the vague-and-point-process topology, then the (sub-)aging functions converge. Combining this result with the above-mentioned coupling of traps, we establish the main results in Section 6. Finally, in Section 7, we present some examples to which our main results are applicable.

2. The vague-and-point-process topology

Convergence of discrete measures in the vague-and-point-process topology means the convergence both in the vague topology and in the point process sense (Fontes et al., 2002, Definition 2.2). In Section 2.1, we recall fundamental results on the vague topology, and then we introduce and study the vague-and-point-process topology in Sections 2.2 and 2.3.

For the following discussions, we introduce several pieces of notation. Given a topological space S, $\mathcal{B}(S)$ denotes the totality of Borel subsets of S, id_S denotes the identity map from S to itself, and, for a subset A of S, $\partial A = \partial_S A$ denotes the boundary of A in S. For a function $f\colon S \to \mathbb{R}$, we write $\|f\|_\infty := \sup\{|f(x)| \mid x \in S\}$. Given two maps $f\colon S_1 \to S_2$ and $g\colon T_1 \to T_2$, we write $f \times g\colon S_1 \times T_1 \to S_2 \times T_2$ by setting

$$(f \times g)(x, y) := (f(x), g(y)), \quad (x, y) \in S_1 \times T_1.$$

2.1. *The vague metric.* In this subsection, we introduce the vague metric, which induces the vague topology on the set of measures. Let (S, d^S, ρ_S) be a rooted boundedly-compact metric space. Write $\mathcal{M}_{\mathrm{fin}}(S)$ for the set of finite Borel measures on S, which we equip with the weak topology. Recall that the weak topology is induced from the *Prohorov metric* d_P^S given by

$$d_P^S(\mu, \nu) := \inf\{\varepsilon > 0 \mid \mu(A) \le \nu(A^{\langle\varepsilon\rangle}) + \varepsilon,\ \nu(A) \le \mu(A^{\langle\varepsilon\rangle}) + \varepsilon,\ \forall A \subseteq \mathcal{B}(S)\}, \tag{2.1}$$

where we set

$$A^{\langle\varepsilon\rangle} := \{x \in S \mid \exists y \in A \text{ such that } d^S(x, y) \le \varepsilon\}. \tag{2.2}$$

Definition 2.1 (The vague metric d_V^{S,ρ_S})**.** We denote the set of Radon measures on S by $\mathcal{M}(S)$. For $\mu \in \mathcal{M}(S)$, we write $\mu^{(r)}$ for the restriction of μ to $S^{(r)} := \mathrm{cl}(B_S(\rho, r))$, that is, $\mu^{(r)}$ is a finite Borel measure given by

$$\mu^{(r)}(\cdot) := \mu|_{S^{(r)}}(\cdot) = \mu\left(\cdot \cap S^{(r)}\right).$$

We then define, for each $\mu, \nu \in \mathcal{M}(S)$,

$$d_V^{S,\rho_S}(\mu, \nu) := \int_0^\infty e^{-r} \left(1 \wedge d_P^S(\mu^{(r)}, \nu^{(r)})\right) dr. \tag{2.3}$$

Theorem 2.2 (Noda, 2024a, Theorems 3.19 and 3.20)**.** *The function d_V^{S,ρ_S} is a complete, separable metric on $\mathcal{M}(S)$. Let $\mu, \mu_1, \mu_2, \dots$ be Radon measures on S. Then the following conditions are equivalent:*

(i) *μ_n converges to μ with respect to d_V^{S,ρ_S};*
(ii) *$\mu_n^{(r)}$ converges weakly to $\mu^{(r)}$ for all but countably many $r > 0$;*
(iii) *μ_n converges vaguely to μ, that is, for all continuous functions $f\colon S \to \mathbb{R}$ with compact support, it holds that*

$$\lim_{n\to\infty} \int_S f(x)\, \mu_n(dx) = \int_S f(x)\, \mu(dx).$$

We call d_V^{S,ρ_S} the *vague metric (with the root ρ_S)*. The distance between Radon measures with respect to the vague metric is preserved under pushforward by root-and-distance-preserving maps, as shown in Proposition 2.3 below. This is important for metrization of Gromov–Hausdorff-type topologies in Section 3. Here, we note that, for two rooted boundedly-compact metric spaces $(S_i, d^{S_i}, \rho_{S_i})$, a map $f\colon S_1 \to S_2$ is said to be *root-preserving* if $f(\rho_{S_1}) = \rho_{S_2}$, and *distance-preserving* if $d^{S_2}(f(x), f(y)) = d^{S_1}(x, y)$ for all $x, y \in S_1$.

Proposition 2.3 (Noda, 2024a, Proposition 3.24)**.** *Let (S_i, d^{S_i}, ρ_i), $i = 1, 2$, be rooted boundedly-compact metric spaces and $f\colon S_1 \to S_2$ be a root-and-distance-preserving map. Then, the map from $(\mathcal{M}(S_1), d_V^{S_1,\rho_{S_1}})$ to $(\mathcal{M}(S_2), d_V^{S_2,\rho_{S_2}})$ given by $\mu \mapsto \mu \circ f^{-1}$ is distance-preserving.*

2.2. *The space $\mathcal{M}^{\mathrm{dis}}(S)$.* In this subsection, we define the vague-and-point-process topology, which yields jointly vague convergence and convergence in the point process sense introduced in Fontes et al. (2002, Definition 2.2). In particular, various characterizations of this topology in terms of convergence are given in Theorem 2.9.

Let (S, d^S, ρ_S) be a rooted boundedly-compact metric space. Recall that a Radon measure ν is called a *discrete measure* if it is written in the following form:

$$\nu = \sum_{i\in I} w_i \delta_{x_i}, \tag{2.4}$$

where I is a countable set, w_i is a positive number, x_i is an element of S such that $x_i \neq x_j$ if $i \neq j$, and δ_{x_i} denotes the Dirac measure putting mass 1 at x_i. The representation of (2.4) is called an *atomic decomposition* of ν and is unique up to the order of terms (cf. Kallenberg, 2017, Lemma 1.6). We say that ν is *simple* if $w_i = 1$ for all $i \in I$. We write $\mathrm{At}(\nu) = \{x_i\}_{i\in I}$ for the set of the atoms of ν.

Proposition 2.4. *Let $\nu, \nu_1, \nu_2, \cdots$ be simple measures on S. For each compact subset K of S, consider the following condition.*

(VC) *Write $\{x_i\}_{i\in I_K} = \mathrm{At}(\nu) \cap K$ for the atoms lying in K. (NB. I_K is a finite set.) Then, for all sufficiently large n, we can write $\mathrm{At}(\nu_n) \cap K = \{x_i^{(n)} \mid i \in I_K\}$ in such a way that $x_i^{(n)} \to x_i$ for each $i \in I_K$.*

Then, the statements below are equivalent with each other.

(i) *The measures ν_n converge to ν vaguely.*
(ii) *Any compact subset K of S with $\nu(\partial K) = 0$ satisfies (VC).*
(iii) *There exists an increasing sequence $(D_k)_{k\geq 1}$ of relatively compact open subsets of S such that $\bigcup_{k\geq 1} D_k = S$, $\mu(\partial D_K) = 0$ for each k, and each closure $\mathrm{cl}(D_k)$ satisfies (VC).*

Proof: Assume that (i) is satisfied. Fix a compact subset of F with $\nu(\partial K) = 0$. Since $\nu_n|_K \to \nu|_K$ weakly, we have that $\nu_n(K) \to \nu(K) = \#I_K$. Hence, for all sufficiently large n, $\#(\mathrm{At}(\nu_n) \cap K) = \#I_K$. Fix a small $\varepsilon \in (0,1)$ such that $D_S(x_i, 2\varepsilon) \subseteq K$ for each $i \in I_K$ and $D_S(x_i, 2\varepsilon) \cap D_S(x_j, 2\varepsilon) = \emptyset$ if $i \neq j$. Since we have that $d_P^S(\nu_n|_K, \nu|_K) < \varepsilon$ for all sufficiently large n, we deduce from the definition of the Prohorov metric that

$$1 - \varepsilon \leq \nu_n(\{x_i\}^{\langle \varepsilon \rangle}) \leq 1 + \varepsilon,$$

which implies that there is exactly one atom of ν_n in $D_S(x_i, \varepsilon)$, which is denoted by $x_i^{(n)}$. We then have that $\mathrm{At}(\nu_n) \cap K = \{x_i^{(n)} \mid i \in I_K\}$ for all sufficiently large n. Fix $i \in I_K$. Given $\eta \in (0, \varepsilon)$, by the same argument, one can check that for all sufficiently large n there is exactly one atom of ν_n in $D_S(x_i, \eta)$. Since $D_S(x_i, \eta) \subseteq D_S(x_i, \varepsilon)$, the atom must be $x_i^{(n)}$. Therefore, we obtain that $x_i^{(n)} \to x_i$.

The implication (ii) $\Rightarrow$ (iii) is straightforward. Assume that (iii) is satisfied. Fix a compactly supported continuous function f on S. Let D_k be such that D_k contains the support of f. Write $\mathrm{At}(\nu) \cap D_k = \{x_i\}_{i \in I_{D_k}}$ and $\mathrm{At}(\nu_n) = \{x_i^{(n)}\}_{i \in I_{D_k}}$ in such a way that $x_i^{(n)} \to x_i$ for each $i \in I_{D_k}$. Then, since I_K is finite, we deduce that

$$\lim_{n \to \infty} \int f(x)\, \nu_n(dx) = \lim_{n \to \infty} \sum_{i \in I_{D_k}} f(x_i^{(n)}) = \sum_{i \in I_{D_k}} f(x_i) = \int f(x)\, \nu(dx),$$

which establishes (i). □

Definition 2.5 (The space $\mathcal{M}^{\mathrm{dis}}(S)$)**.** We define $\mathcal{M}^{\mathrm{dis}}(S)$ to be the collection of discrete Radon measures μ on S.

Given $\nu \in \mathcal{M}^{\mathrm{dis}}(S)$ written in the form (2.4), we define

$$\mathfrak{p}(\nu) := \sum_{i \in I} \delta_{(x_i, w_i)}. \tag{2.5}$$

It is easy to check that $\mathfrak{p}(\nu)$ is a discrete Radon measure on $S \times \mathbb{R}_{>0}$ by observing that there are only finitely many atoms of $\mathfrak{p}(\nu)$ in any compact subset of $S \times \mathbb{R}_{>0}$.

Before proceeding with the discussion of discrete measures, we make some technical remarks regarding the space $\mathbb{R}_{>0}$. The space $\mathbb{R}_{>0}$ equipped with the usual Euclidean metric $d^{\mathbb{R}}(v,w) = |v - w|$ is not a boundedly-compact metric space. Indeed, $(0,1]$ is bounded and closed in $\mathbb{R}_{>0}$ but not compact. Throughout this paper, we equip $\mathbb{R}_{>0}$ with another metric $d^{\mathbb{R}_{>0}}$ given by

$$d^{\mathbb{R}_{>0}}(v,w) := |\log v - \log w|, \quad v, w \in \mathbb{R}_{>0}.$$

It is elementary to check that the topology on $\mathbb{R}_{>0}$ induced from $d^{\mathbb{R}_{>0}}$ coincides with the Euclidean topology and $(\mathbb{R}_{>0}, d^{\mathbb{R}_{>0}})$ is boundedly compact. We set $1 \in \mathbb{R}_{>0}$ to be the root of $\mathbb{R}_{>0}$. Note that the closed ball with radius r centered at 1 is $[e^{-r}, e^r]$. We then think the product space $S \times \mathbb{R}_{>0}$ as a rooted boundedly-compact metric space by equipping it with the root $\tilde{\rho}_S := (\rho_S, 1)$ and the *max product metric* $d^{S \times \mathbb{R}_{>0}}$ given by

$$d^{S \times \mathbb{R}_{>0}}((x,v),(y,w)) := d^S(x,y) \vee d^{\mathbb{R}_{>0}}(v,w), \quad (x,v),(y,w) \in S \times \mathbb{R}_{>0}. \tag{2.6}$$

For $\nu_1, \nu_2 \in \mathcal{M}^{\mathrm{dis}}(S)$, we define

$$d_{\mathcal{M}^{\mathrm{dis}}}^{S,\rho_S}(\nu_1, \nu_2) := d_V^{S,\rho_S}(\nu_1, \nu_2) \vee d_V^{S \times \mathbb{R}_{>0}, \tilde{\rho}_S}(\mathfrak{p}(\nu_1), \mathfrak{p}(\nu_2)). \tag{2.7}$$

Proposition 2.6. *The function $d_{\mathcal{M}^{\mathrm{dis}}}^{S,\rho_S}$ is a metric on $\mathcal{M}^{\mathrm{dis}}(S)$.*

Proof: By Theorem 2.2, d_V^{S,ρ_S} is a metric on $\mathcal{M}^{\mathrm{dis}}(S)$, which implies that $d_{\mathcal{M}^{\mathrm{dis}}}^{S,\rho_S}$ is positive definite. Symmetry and the triangle inequality are obvious. □

Definition 2.7 (The vague-and-point-process topology)**.** We call the topology on $\mathcal{M}^{\mathrm{dis}}(S)$ induced by $d^{S,\rho_S}_{\mathcal{M}^{\mathrm{dis}}}$ the *vague-and-point-process topology.*

Below, we study the vague-and-point-process topology in terms of convergence.

Proposition 2.8. *Let $\nu, \nu_1, \nu_2, \dots$ be elements of $\mathcal{M}^{\mathrm{dis}}(S)$ such that $\nu_n \to \nu$ vaguely. Fix $x_n, x \in S$ with $x_n \to x$. Then $\nu_n(\{x_n\}) \to \nu(\{x\})$ if and only if $\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \nu_n(B_S(x_n, \delta) \setminus \{x_n\}) = 0$.*

Proof: Since $\nu_n \to \nu$ vaguely and $x_n \to x$, we have that, for each $\delta > 0$,

$$\nu(D_S(x,\delta)) \geq \limsup_{n\to\infty} \nu_n(D_S(x,\delta)) \geq \limsup_{n\to\infty} \nu_n(\{x_n\}).$$

Letting $\delta \downarrow 0$ yields that $\nu(\{x\}) \geq \limsup_{n\to\infty} \nu_n(\{x_n\})$. Similarly, we deduce that

$$\begin{aligned}\nu(\{x\}) &= \lim_{\delta\downarrow 0} \nu(B_S(x,\delta)) \\ &\leq \lim_{\delta\downarrow 0} \liminf_{n\to\infty} \nu_n(B_S(x,\delta)) \\ &\leq \lim_{\delta\downarrow 0} \liminf_{n\to\infty} \nu_n(B_S(x_n,2\delta)).\end{aligned} \tag{2.8}$$

Moreover, we have that, for each $\delta > 0$,

$$\liminf_{n\to\infty} \nu_n(B_S(x_n,2\delta)) \leq \limsup_{n\to\infty} \nu_n(B_S(x_n,2\delta) \setminus \{x_n\}) + \liminf_{n\to\infty} \nu_n(\{x_n\}). \tag{2.9}$$

Thus, from (2.8) and (2.9), we obtain the "if" part of the assertion. For the other direction, suppose that $\nu_n(\{x_n\}) \to \nu(\{x\})$. We then have that, for each $\delta > 0$,

$$\begin{aligned}\limsup_{n\to\infty} \nu_n(B_S(x_n,\delta) \setminus \{x_n\}) &= \limsup_{n\to\infty} \{\nu_n(B_S(x_n,\delta)) - \nu_n(\{x_n\})\} \\ &\leq \limsup_{n\to\infty} \nu_n(D_S(x_n,\delta)) - \liminf_{n\to\infty} \nu_n(\{x_n\}) \\ &\leq \limsup_{n\to\infty} \nu_n(D_S(x,2\delta)) - \liminf_{n\to\infty} \nu_n(\{x_n\}) \\ &\leq \nu(D_S(x,2\delta)) - \nu(\{x\}),\end{aligned}$$

where we use the vague convergence $\nu_n \to \nu$ to establish the last inequality. Letting $\delta \downarrow 0$ yields the result. □

Theorem 2.9. *Let $\nu, \nu_1, \nu_2, \dots$ be elements of $\mathcal{M}^{\mathrm{dis}}(S)$. The following statements are equivalent:*

(i) *$\nu_n \to \nu$ in the vague-and-point-process topology;*
(ii) *$\nu_n \to \nu$ vaguely and $\mathfrak{p}(\nu_n) \to \mathfrak{p}(\nu)$ vaguely;*
(iii) *$\nu_n \to \nu$ vaguely and, for any $x \in \mathrm{At}(\nu)$, there exist atoms $x_n \in \mathrm{At}(\nu_n)$ such that $x_n \to x$ and $\nu_n(\{x_n\}) \to \nu(\{x\})$;*
(iv) *$\nu_n \to \nu$ vaguely and, for any $x \in \mathrm{At}(\nu)$, there exist atoms $x_n \in \mathrm{At}(\nu_n)$ such that $x_n \to x$ and $\lim_{\delta\to 0} \limsup_{n\to\infty} \nu_n(B_S(x_n,\delta) \setminus \{x_n\}) = 0$.*

Proof: The equivalence of (i) and (ii) follows from the definition of $d^{S,\rho_S}_{\mathcal{M}^{\mathrm{dis}}}$. The implication (ii) ⇒ (iii) follows by applying Proposition 2.4 to the convergence of $\mathfrak{p}(\nu_n)$ to $\mathfrak{p}(\nu)$. Assume that (iii) is satisfied. Let $r > 0$ be such that the boundary of $K := D_S(\rho_S, r) \times [e^{-r}, e^r]$ does not contain any atoms of $\mathfrak{p}(\nu)$. Write $\{(x_i, w_i)\}_{i\in I_K}$ for the atoms of $\mathfrak{p}(\nu)$ lying in K. For each $i \in I_K$, we let $x_i^{(n)} \in \mathrm{At}(\nu_n)$ be such that $x_i^{(n)} \to x_i$ and $w_i^{(n)} := \nu_n(\{x_i^{(n)}\}) \to w_i$. Since (x_i, w_i) is in the interior of K, we have that $(x_i^{(n)}, w_i^{(n)}) \in K$ for all $i \in I_K$ (at least for all sufficiently large n). By Proposition 2.4, it remains to prove that $\mathrm{At}(\mathfrak{p}(\nu_n)) \cap K = \{(x_i^{(n)}, w_i^{(n)})\}_{i\in I_K}$ for all sufficiently large n. Suppose that it is not the case. Then, there exist a subsequence $(n_k)_{k\geq 1}$ and $(x^{(n_k)}, w^{(n_k)}) \in \mathrm{At}(\mathfrak{p}(\nu_{n_k})) \cap K$ such that $(x^{(n_k)}, w^{(n_k)}) \notin \{(x_i^{(n_k)}, w_i^{(n_k)})\}_{i\in I_K}$. Since $D_S(\rho_S, r)$ is compact and $\nu_n \to \nu$ vaguely, we have that $\limsup_{k\to\infty} \nu_{n_k}(D_S(\rho_S, r)) \leq \nu(D_S(\rho_S, r))$. This, combined with $e^{-r} \leq w^{(n_k)}$ and $w_i^{(n_k)} \to w_i$, yields that

$$\begin{aligned} e^{-r} + \sum_{i\in I_K} w_i &\leq \limsup_{k\to\infty} \left(w^{(n_k)} + \sum_{i\in I_K} w_i^{(n_k)} \right) \\ &\leq \limsup_{k\to\infty} \nu_{n_k}(D_S(\rho_S, r)) \\ &\leq \nu(D_S(\rho_S, r)) \\ &= \sum_{i\in I_K} w_i, \end{aligned}$$

which is a contradiction. Therefore, we obtain (ii). The equivalence of (iii) and (iv) follows from Proposition 2.8. □

The following is an analogue of Proposition 2.3, and asserts that the distance between discrete measures with respect to the metric $d^{S,\rho_S}_{\mathcal{M}^{\mathrm{dis}}}$ is preserved under pushforward by root-and-distance-preserving maps. This fact is important for metrization of Gromov–Hausdorff-type topologies in Section 3.

Proposition 2.10. *Let (S_i, d^{S_i}, ρ_i), $i = 1, 2$, be rooted boundedly-compact metric spaces and $f : S_1 \to S_2$ be a root-and-distance-preserving map. Then the map from $(\mathcal{M}^{\mathrm{dis}}(S_1), d^{S_1,\rho_{S_1}}_{\mathcal{M}^{\mathrm{dis}}})$ to $(\mathcal{M}^{\mathrm{dis}}(S_2), d^{S_2,\rho_{S_2}}_{\mathcal{M}^{\mathrm{dis}}})$ given by $\nu \mapsto \nu \circ f^{-1}$ is distance-preserving.*

Proof: It is easy to check that $\mathfrak{p}(\nu \circ f^{-1}) = \mathfrak{p}(\nu) \circ (f \times \mathrm{id}^{\mathbb{R}_{>0}})^{-1}$ and $f \times \mathrm{id}^{\mathbb{R}_{>0}}$ is a root-and-distance preserving map from $S_1 \times \mathbb{R}_{>0}$ to $S_2 \times \mathbb{R}_{>0}$. Since the Prohorov metrics and the vague metrics are preserved by root-and-distance-preserving maps (see Noda, 2024a, Lemma 3.17 and Proposition 2.3), we obtain the desired result. □

Although the vague metrics d^{S,ρ_S}_V and $d^{S\times\mathbb{R}_{>0},\tilde{\rho}_S}_V$ are complete, $d^{S,\rho_S}_{\mathcal{M}^{\mathrm{dis}}}$ is not complete in general. To see this, consider a sequence of measures whose atoms collide. For example, let $S = \mathbb{R}$ and $\nu_n := \delta_0 + \delta_{n^{-1}}$. It is elementary to check that ν_n converges vaguely to the measure $2\delta_0$ on $\mathbb{R}$, and $\mathfrak{p}(\nu_n) = \delta_{(0,1)} + \delta_{(n^{-1},1)}$ converges vaguely to the measure $2\delta_{(0,1)}$ on $\mathbb{R} \times \mathbb{R}_{>0}$. Thus, $(\nu_n)_{n\geq 1}$ is Cauchy with respect to $d^{\mathbb{R},0}_{\mathcal{M}^{\mathrm{dis}}}$, but it does not converge in $\mathcal{M}^{\mathrm{dis}}(\mathbb{R})$. However, the vague-and-point-process topology is Polish, that is, there exists another metric that is complete, separable and induces the same topology (see Theorem 2.21 below). For further study of the vague-and-point-process topology, such as Polishness and a precompactness criterion, it is convenient to introduce a larger space $\mathscr{P}(S)$, into which $\mathcal{M}^{\mathrm{dis}}(S)$ is topologically embedded. This is the main aim of the following subsection.

2.3. *The space $\mathscr{P}(S)$.* As already explained above, in this subsection, we introduce a space $\mathscr{P}(S)$, into which $\mathcal{M}^{\mathrm{dis}}(S)$ is topologically embedded, and study its topological properties. In particular, we prove that the vague-and-point-process topology is Polish (Theorem 2.21) and provide precompact and tightness criteria (Theorems 2.22 and 2.26).

Fix a rooted bounded-compact metric space (S, d^S, ρ_S). We denote by $\mathcal{N}(S \times \mathbb{R}_{>0})$ the collection of integer-valued Radon measures π on $S \times \mathbb{R}_{>0}$, i.e., $\pi(E) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ for any Borel subset E of $S \times \mathbb{R}_{>0}$. Note that any $\pi \in \mathcal{N}(S \times \mathbb{R}_{>0})$ is a discrete measure and if an atomic decomposition of π is given by $\pi = \sum_{i\in I} \beta_i \delta_{(x_i,w_i)}$, then β_i is a positive integer (cf. Kallenberg, 2021, Theorem 2.18). We associate π with a Borel measure $\mathfrak{v}(\pi)$ on S by setting

$$\mathfrak{v}(\pi)(A) := \int_{A\times\mathbb{R}_{>0}} w\, \pi(dx\, dw) = \sum_{i\in I} \beta_i w_i \delta_{x_i}(A), \quad A \in \mathcal{B}(S). \tag{2.10}$$

Definition 2.11 (The space $\mathscr{P}(S)$)**.** We define

$$\mathscr{P}(S) := \{\pi \in \mathcal{N}(S \times \mathbb{R}_{>0}) \mid \mathfrak{v}(\pi) \text{ is a Radon measure on } S\}.$$

For each $\pi = \sum_{i\in I} \beta_i \delta_{(x_i,w_i)} \in \mathscr{P}(S)$ and $r > 0$, we define a Borel measure $\mathfrak{m}^{(r)}(\pi)$ on $\mathbb{R}_{>0}$ by setting

$$\mathfrak{m}^{(r)}(\pi)(A) := \int_{S^{(r)}\times A} w\,\pi(dx\,dw) = \sum_{\substack{x_i\in S^{(r)}\\ w_i\in A}} \beta_i w_i, \quad \forall A \in \mathcal{B}(\mathbb{R}_{>0}).$$

Note that $\mathfrak{m}^{(r)}(\pi)$ is a finite measure. Indeed, we have that $\mathfrak{m}^{(r)}(\mathbb{R}_{>0}) = \mathfrak{v}(\pi)(S^{(r)})$, which is finite since $\mathfrak{v}(\pi)$ is a Radon measure. We write

$$M^{(r)}_\varepsilon(\pi) := \mathfrak{m}^{(r)}(\pi)((0,\varepsilon]) = \int_{S^{(r)}\times(0,\varepsilon]} w\,\pi(dx\,dw) = \sum_{\substack{x_i\in S^{(r)}\\ w_i\leq\varepsilon,}} \beta_i w_i, \quad \varepsilon > 0, \tag{2.11}$$

$$W^{(r)}(\pi) := \sup\{w_i \mid x_i \in S^{(r)}\} = \inf\bigl\{l > 0 \mid \mathfrak{m}^{(r)}(\pi)([l,\infty)) = 0\bigr\}. \tag{2.12}$$

Below, we discuss some basic properties of $\mathfrak{m}^{(r)}(\pi)$, $M^{(r)}_\varepsilon$ and $W^{(r)}(\pi)$.

Lemma 2.12. *Fix $\pi \in \mathscr{P}(S)$. Then the following statements hold.*

(i) *For any $r > s > 0$, $d_P^{\mathbb{R}_{>0}}(\mathfrak{m}^{(r)}(\pi), \mathfrak{m}^{(s)}(\pi)) \leq \mathfrak{v}(\pi)(S^{(r)} \setminus S^{(s)})$.*
(ii) *The map $r \mapsto \mathfrak{m}^{(r)}(\pi) \in (\mathcal{M}_{\mathrm{fin}}(\mathbb{R}_{>0}), d_P^{\mathbb{R}_{>0}})$ is continuous for all but countably many r.*
(iii) *For each $r > 0$, the function $\varepsilon \mapsto M^{(r)}_\varepsilon(\pi)$ is increasing and $\lim_{\varepsilon\to 0} M^{(r)}_\varepsilon(\pi) = 0$.*
(iv) *For each $r > 0$, $W^{(r)}(\pi) \leq \mathfrak{v}(\pi)(S^{(r)}) < \infty$.*

Proof: (i). We have that, for any Borel subset $A \subseteq \mathbb{R}_{>0}$,

$$\begin{aligned}\mathfrak{m}^{(r)}(\pi)(A) &= \int_{S^{(r)}\times A} w\,\pi(dx\,dw)\\ &\leq \int_{S^{(s)}\times A} w\,\pi(dx\,dw) + \int_{(S^{(r)}\setminus S^{(s)})\times\mathbb{R}_{>0}} w\,\pi(dx\,dw)\\ &= \mathfrak{m}^{(s)}(\pi)(A) + \mathfrak{v}(\pi)(S^{(r)} \setminus S^{(s)}),\end{aligned}$$

and $\mathfrak{m}^{(s)}(\pi)(A) \leq \mathfrak{m}^{(r)}(\pi)(A)$. Thus, we obtain the desired result.

(ii). Since $\mathfrak{v}(\pi)$ is a Radon measure on S, for all but countably many $r > 0$, we have $\mathfrak{v}(\pi)(\{x \in S \mid d_S(\rho, x) = r\}) = 0$. By Lemma 2.12(i), it is then straightforward to verify that the map is continuous at such values of r.

(iii) and (iv). These are obvious by their definitions. □

We now introduce a metric on $\mathscr{P}(S)$. For $\pi_1, \pi_2 \in \mathscr{P}(S)$, we set

$$\begin{aligned}d_{\mathscr{P}}^{S,\rho_S}(\pi_1,\pi_2) = \int_0^\infty e^{-r}\bigl\{1 \wedge d_P^{\mathbb{R}_{>0}}(\mathfrak{m}^{(r)}(\pi_1), \mathfrak{m}^{(r)}(\pi_2))\bigr\}\,dr&\\ \vee\, d_V^{S\times\mathbb{R}_{>0},\tilde{\rho}_S}(\pi_1,\pi_2) \vee d_V^{S,\rho_S}(\mathfrak{v}(\pi_1), \mathfrak{v}(\pi_2)).&\end{aligned} \tag{2.13}$$

Note that the integrals are well-defined by Lemma 2.12(ii) (cf. Noda, 2024a, Proof of Proposition 2.4).

Proposition 2.13. *The function $d_{\mathscr{P}}^{S,\rho_S}$ is a metric on $\mathscr{P}(S)$.*

Proof: By Theorem 2.2, $d_V^{S\times\mathbb{R}_{>0},\tilde{\rho}_S}$ is a metric on $\mathscr{P}(S)$, which implies that $d_{\mathscr{P}}^{S,\rho_S}$ is positive definite. Symmetry and the triangle inequality are obvious. □

Below, we provide some characterizations of the topology on $\mathscr{P}(S)$ in terms of convergence.

Theorem 2.14 (Convergence in $\mathscr{P}(S)$). *Let $\pi, \pi_1, \pi_2, \ldots$ be elements of $\mathscr{P}(S)$. The following statements are equivalent:*

(i) *$\pi_n \to \pi$ with respect to $d_{\mathscr{P}}^{S,\rho_S}$;*

(ii) *$\pi_n \to \pi$ vaguely as measures on $S \times \mathbb{R}_{>0}$, and $\mathfrak{v}(\pi_n) \to \mathfrak{v}(\pi)$ vaguely as measures on S;*

(iii) *$\pi_n \to \pi$ vaguely as measures on $S \times \mathbb{R}_{>0}$, and, for each $r > 0$, $\limsup_{n\to\infty} W^{(r)}(\pi_n) < \infty$ and $\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} M_\varepsilon^{(r)}(\pi_n) = 0$.*

Proof: The implication (i) $\Rightarrow$ (ii) is obvious. Suppose that (ii) is satisfied. The vague convergence $\mathfrak{v}(\pi_n) \to \mathfrak{v}(\pi)$ implies that $\sup_{n\geq 1} \mathfrak{v}(\pi_n)(S^{(r)}) < \infty$ for each $r > 0$. From Lemma 2.12(iv), we obtain that $\sup_{n\geq 1} W^{(r)}(\pi_n) < \infty$ for each $r > 0$. Let $\varepsilon, r > 0$ be such that π has no atoms on the boundary of $S^{(r)} \times (\varepsilon, \infty)$. We then choose $l > 0$ so that π has no atoms on the boundary of $S^{(r)} \times (\varepsilon, l)$ and $\sup_{n\geq 1} W^{(r)}(\pi_n) < l$. We have that

$$\begin{aligned} |M_\varepsilon^{(r)}(\pi_n) - M_\varepsilon^{(r)}(\pi)| &= \left| \int_{S^{(r)} \times (0,\varepsilon]} w\, \pi_n(dx\, dw) - \int_{S^{(r)} \times (0,\varepsilon]} w\, \pi(dx\, dw) \right| \\ &\leq \left| \mathfrak{v}(\pi_n)(S^{(r)}) - \mathfrak{v}(\pi)(S^{(r)}) \right| \\ &\quad + \left| \int_{S^{(r)} \times (\varepsilon,l)} w\, \pi_n(dx\, dw) - \int_{S^{(r)} \times (\varepsilon,l)} w\, \pi(dx\, dw) \right|. \end{aligned}$$

Since $\mathfrak{v}(\pi_n)|_{S^{(r)}} \to \mathfrak{v}(\pi)|_{S^{(r)}}$ weakly and $\pi_n|_{S^{(r)}\times(\varepsilon,l)} \to \pi|_{S^{(r)}\times(\varepsilon,l)}$ weakly, we deduce that $M_\varepsilon^{(r)}(\pi_n) \to M_\varepsilon^{(r)}(\pi)$. From the dominated convergence theorem and Lemma 2.12(iii), we obtain (iii).

Assume that (iii) is satisfied. We will prove (i). We note that, by Lemma 2.12(iv), $\sup_{n\geq 1} W^{(r)}(\pi_n)$ is finite for each $r > 0$. Fix a continuous function f on S whose support is contained in $S^{(r)}$, where we assume that $\mathfrak{v}(\pi)$ has no atoms on the boundary of $S^{(r)}$. Let $\varepsilon, l > 0$ be such that π has no atoms on the boundary of $S^{(r)} \times (\varepsilon, l)$ and $\sup_{n\geq 1} W^{(r)}(\pi_n) < l$. We then have that

$$\begin{aligned} &\left| \int f(x)\, \mathfrak{v}(\pi_n)(dx) - \int f(x)\, \mathfrak{v}(\pi)(dx) \right| \\ &= \left| \int_{S^{(r)} \times (0,\infty)} w f(x)\, \pi_n(dx\, dw) - \int_{S^{(r)} \times (0,\infty)} w f(x)\, \pi(dx\, dw) \right| \\ &\leq \left| \int_{S^{(r)} \times (\varepsilon,l)} w f(x)\, \pi_n(dx\, dw) - \int_{S^{(r)} \times (\varepsilon,l)} w f(x)\, \pi(dx\, dw) \right| + \|f\|_\infty \big(M_\varepsilon^{(r)}(\pi_n) + M_\varepsilon^{(r)}(\pi) \big). \end{aligned}$$

By Lemma 2.12(iii), (iii), and the weak convergence $\pi_n|_{S^{(r)}\times(\varepsilon,l)} \to \pi|_{S^{(r)}\times(\varepsilon,l)}$, we obtain that $\mathfrak{v}(\pi_n) \to \mathfrak{v}(\pi)$ vaguely. It remains to show that $\mathfrak{m}^{(r)}(\pi_n) \to \mathfrak{m}^{(r)}(\pi)$ weakly for all but countably many $r > 0$. Fix $r > 0$ satisfying $\mathfrak{v}(\pi)(\partial S^{(r)}) = 0$. Let $\varepsilon, l > 0$ be such that π has no atoms on the boundary of $S^{(r)} \times (\varepsilon, l)$ and $\sup_{n\geq 1} W^{(r)}(\pi_n) < l$. For a bounded continuous function g on $\mathbb{R}_{>0}$, we have that

$$\begin{aligned} &\left| \int g(w)\, \mathfrak{m}^{(r)}(\pi_n)(dw) - \int g(w)\, \mathfrak{m}^{(r)}(\pi)(dw) \right| \\ &= \left| \int_{S^{(r)} \times \mathbb{R}_{>0}} g(w) w\, \pi_n(dx\, dw) - \int_{S^{(r)} \times \mathbb{R}_{>0}} g(w) w\, \pi(dx\, dw) \right| \\ &\leq \left| \int_{S^{(r)} \times (\varepsilon,l)} g(w) w\, \pi_n(dx\, dw) - \int_{S^{(r)} \times (\varepsilon,l)} g(w) w\, \pi(dx\, dw) \right| + \|g\|_\infty \big(M_\varepsilon^{(r)}(\pi_n) + M_\varepsilon^{(r)}(\pi) \big) \end{aligned}$$

Hence, similarly as before, we obtain that $\mathfrak{m}^{(r)}(\pi_n) \to \mathfrak{m}^{(r)}(\pi)$ weakly. □

Recall the map $\mathfrak{p}$ from (2.5). The following corollary asserts that, via this map, the space $\mathcal{M}^{\mathrm{dis}}(S)$ equipped with the vague-and-point-process topology is topologically embedded into $\mathscr{P}(S)$.

Corollary 2.15. *The map $\mathfrak{p} : \mathcal{M}^{\mathrm{dis}}(S) \to \mathscr{P}(S)$ is a topological embedding, i.e., a homeomorphism onto its image. If we write $\mathscr{P}^*(S)$ for the image, then*

$$\mathscr{P}^*(S) = \Big\{\pi \in \mathscr{P}(S) \mid \pi = \sum_{i \in I} \delta_{(x_i,w_i)} \text{ with } x_i \neq x_j \text{ if } i \neq j\Big\}, \tag{2.14}$$

and the inverse map is $\mathfrak{v}|_{\mathscr{P}^*(S)}$, i.e., the restriction of the map $\mathfrak{v}$ to $\mathscr{P}^*(S)$.

Proof: The assertion (2.14) follows from definition. Using (2.14), one can check that $\mathfrak{v}|_{\mathscr{P}^*(S)}$ is the inverse map. The continuity of $\mathfrak{p}$ and $\mathfrak{v}|_{\mathscr{P}^*(S)}$ is obvious by Theorems 2.9 and 2.14. □

The following result is an immediate consequence of the above result. From this result, convergence in the vague-and-point-process topology is proven by convergence in $\mathscr{P}(S)$.

Corollary 2.16. *Let $\pi, \pi_1, \pi_2, \dots$ be elements of $\mathscr{P}^*(S)$. If $\pi_n \to \pi$ in $\mathscr{P}(S)$, then $\mathfrak{v}(\pi_n) \to \mathfrak{v}(\pi)$ in the vague-and-point-process topology (as elements in $\mathcal{M}^{\mathrm{dis}}(S)$).*

Henceforth, for each $\pi \in \mathscr{P}(S)$ and $r > 0$, we write $\pi^{[r]}$ for the restriction of π to $D_S(\rho_S, r) \times [e^{-r}, e^r]$.

Lemma 2.17. *For any $\pi \in \mathscr{P}(S)$, we have that $\pi^{[s]} \to \pi$ in $\mathscr{P}(S)$ as $s \to \infty$.*

Proof: By Theorem 2.2, we have that $\pi^{[s]} \to \pi$ vaguely. It is easy to check that $W^{(r)}(\pi^{[s]}) \leq W^{(r)}(\pi)$ and $M_\varepsilon^{(r)}(\pi^{[s]}) \leq M_\varepsilon^{(r)}(\pi)$ for each $r > 0$. It then follows from Lemma 2.12(iii) and (iv) and Theorem 2.14 that $\pi^{[s]} \to \pi$ in $\mathscr{P}(S)$. □

Theorem 2.18 (Polishness of $\mathscr{P}(S)$)**.** *The metric $d_{\mathscr{P}}^{S,\rho_S}$ is complete and separable.*

Proof: We first show the separability. Let S_d be a countable dense subset of S. Write $\mathscr{D}$ for the collection of $\pi \in \mathscr{P}(S)$ such that

$$\pi = \sum_{i=1}^{n} \beta_i \delta_{(x_i,w_i)},$$

where $n \in \mathbb{N}$, $\beta_i \in \mathbb{N}$, $x_i \in S_d$, and $w_i \in \mathbb{Q} \cap \mathbb{R}_{>0}$. Note that $\mathscr{D}$ is countable. It is easy to check that any $\pi \in \mathscr{P}(S)$ with finitely many atoms is approximated by a sequence in $\mathscr{D}$. For each $\pi \in \mathscr{P}(S)$, $\pi^{[r]}$ has only finitely many atoms. Hence, by Lemma 2.17, we deduce that $\mathscr{D}$ is dense in $\mathscr{P}(S)$.

Next, we show the completeness. Let $(\pi_n)_{n\geq 1}$ be a Cauchy sequence with respect to $d_{\mathscr{P}}^{S,\rho_S}$. Since $d_V^{S\times\mathbb{R}_{>0},\tilde{\rho}_S}$ is complete, there exists a Radon measure π on $S \times \mathbb{R}_{>0}$ such that $\pi_n \to \pi$ vaguely. Then, for any bounded measurable set E with $\pi(\partial E) = 0$, we have that $\pi_n(E) \to \pi(E)$, which implies that $\pi(E) \in \mathbb{Z}_{\geq 0}$. Hence $\pi \in \mathcal{N}(S \times \mathbb{R}_{>0})$. For each $r > 0$, let $g_r \colon S \to [0, 1]$ be a continuous function such that $g_r|_{S^{(r)}} \equiv 1$ and $g_r|_{S \setminus S^{(r+1)}} \equiv 0$ and let $(f_k)_{k\geq 1}$ be non-negative continuous functions on $S \times \mathbb{R}_{>0}$ with compact support increasing to the constant function $1_{S\times\mathbb{R}_{>0}}$. Using the vague convergence $\pi_n \to \pi$ and the monotone convergence theorem, we deduce that

$$\begin{aligned}
\mathfrak{v}(\pi)(S^{(r)}) &\leq \int_{S\times\mathbb{R}_{>0}} g_r(x) w \, \pi(dx\, dw) \\
&= \lim_{k\to\infty} \int f_k(x,w) g_r(x) w \, \pi(dx\, dw) \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \int f_k(x,w) g_r(x) w \, \pi_n(dx\, dw) \\
&\leq \limsup_{n\to\infty} \int g_r(x) w \, \pi_n(dx\, dw) \\
&\leq \limsup_{n\to\infty} \mathfrak{v}(\pi_n)(S^{(r+1)}),
\end{aligned}$$

Since $(\mathfrak{v}(\pi_n))_{n\geq 1}$ is tight in the vague topology, we have that $\sup_{n\geq 1} \mathfrak{v}(\pi_n)(S^{(r+1)}) < \infty$. Hence, $\pi \in \mathscr{P}(S)$. Moreover, from Lemma 2.12(iv), it holds that $\sup_{n\geq 1} W^{(r)}(\pi_n) < \infty$ for each $r > 0$. Thus, Theorem 2.14 implies that it is enough to show that, for all but countably many $r > 0$,

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} M_\varepsilon^{(r)}(\pi_n) = 0. \tag{2.15}$$

Since d_V^{S,ρ_S} is complete, there exists a Radon measure μ on S such that $\mathfrak{v}(\pi_n) \to \mu$ vaguely on S. Fix $r > 0$ such that $\mu(D_S(\rho_S, r) \setminus B_S(\rho_S, r)) = 0$. We then have that

$$\begin{aligned}
\limsup_{\delta\to 0} \limsup_{n\to\infty} \mathfrak{v}(\pi_n)(S^{(r+\delta)} \setminus S^{(r-\delta)}) &\leq \limsup_{\delta\to 0} \limsup_{n\to\infty} \mathfrak{v}(\pi_n)(S^{(r+\delta)} \setminus B_S(\rho_S, r-\delta)) \\
&\leq \limsup_{\delta\to 0} \mu(S^{(r+\delta)} \setminus B_S(\rho_S, r-\delta)) \\
&= 0.
\end{aligned} \tag{2.16}$$

Fix $\delta > 0$. For any $s \in (r-\delta, r+\delta)$, we have from Lemma 2.12(i) that

$$d_P^{\mathbb{R}_{>0}}(\mathfrak{m}^{(s)}(\pi_n), \mathfrak{m}^{(r)}(\pi_n)) \leq \mathfrak{v}(\pi_n)(S^{(r+\delta)} \setminus S^{(r-\delta)}), \quad \forall n \geq 1.$$

We thus deduce that

$$\begin{aligned}
1 \wedge d_P^{\mathbb{R}_{>0}}(\mathfrak{m}^{(r)}(\pi_m), \mathfrak{m}^{(r)}(\pi_n)) &= (2\delta)^{-1} \int_{r-\delta}^{r+\delta} (1 \wedge d_P^{\mathbb{R}_{>0}}(\mathfrak{m}^{(r)}(\pi_m), \mathfrak{m}^{(r)}(\pi_n)))\, ds \\
&\leq \mathfrak{v}(\pi_m)(S^{(r+\delta)} \setminus S^{(r-\delta)}) + \mathfrak{v}(\pi_n)(S^{(r+\delta)} \setminus S^{(r-\delta)}) \\
&\quad + (2\delta)^{-1} \int_{r-\delta}^{r+\delta} (1 \wedge d_P^{\mathbb{R}_{>0}}(\mathfrak{m}^{(s)}(\pi_m), \mathfrak{m}^{(s)}(\pi_n)))\, ds \\
&\leq \mathfrak{v}(\pi_m)(S^{(r+\delta)} \setminus S^{(r-\delta)}) + \mathfrak{v}(\pi_n)(S^{(r+\delta)} \setminus S^{(r-\delta)}) \\
&\quad + (2\delta)^{-1} e^{r+\delta} \, d_{\mathscr{P}}^{S,\rho_S}(\pi_m, \pi_n),
\end{aligned}$$

which, combined with (2.16), implies that $(\mathfrak{m}^{(r)}(\pi_n))_{n\geq 1}$ is a Cauchy sequence with respect to $d_P^{\mathbb{R}_{>0}}$. In particular, it is tight and so we have that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathfrak{m}^{(r)}(\pi_n)((0,\varepsilon)) = 0.$$

Since $M_{\varepsilon/2}^{(r)}(\pi_n) \leq \mathfrak{m}^{(r)}(\pi_n)((0,\varepsilon))$ by the definition of $M_{\varepsilon/2}^{(r)}(\pi_n)$, we obtain (2.15). □

Now it is possible to prove that the vague-and-point-process topology is Polish. By Propositions 2.15 and 2.18, it suffices to show that the set $\mathscr{P}^*(S)$ is an intersection of countably many open subsets of $\mathscr{P}(S)$, and this follows from two lemmas below. For each $n \in \mathbb{N}$, we set

$$\mathscr{P}^{(n)}(S) := \Big\{\pi \in \mathscr{P}(S) \mid \text{for some } r \in (n - n^{-1}, n),\ \pi^{[r]} = \sum_{i \in I} \delta_{(x_i, w_i)} \text{ with } x_i \neq x_j \text{ if } i \neq j\Big\}. \tag{2.17}$$

Lemma 2.19. *The set $\mathscr{P}^{(n)}(S)$ is open in $\mathscr{P}(S)$.*

Proof: Fix $\pi \in \mathscr{P}^{(n)}(S)$. Let $r \in (n - n^{-1}, n)$ be such that $\pi^{[r]} = \sum_{i \in I} \delta_{(x_i, w_i)}$ with $x_i \neq x_j$ if $i \neq j$. Since the index set I is finite, we may choose $\varepsilon \in (0,1)$ so that $n < \varepsilon^{-1}$, $n - n^{-1} + 2\varepsilon < r$, and

$$D_S(x_i, 2\varepsilon) \cap D_S(x_j, 2\varepsilon) = \emptyset \quad \text{if} \quad i \neq j. \tag{2.18}$$

Fix $\tilde{\pi} \in \mathscr{P}(S)$ with $d_{\mathscr{P}}^{S, \rho_S}(\pi, \tilde{\pi}) < \varepsilon e^{-1/\varepsilon}$. It suffices to show that $\tilde{\pi} \in \mathscr{P}^{(n)}(S)$. We can find $r' > \varepsilon^{-1}$ such that $d_P^{S \times \mathbb{R}_{>0}}(\pi^{[r']}, \tilde{\pi}^{[r']}) < \varepsilon$. Let $\{(y_j, v_j)\}_{j \in J}$ be the atoms of $\tilde{\pi}^{[r-2\varepsilon]}$. By the definition of the Prohorov metric, we have that

$$1 \leq \tilde{\pi}(\{(y_j, v_j)\}) \leq \pi\big(D_{S \times \mathbb{R}_{>0}}((y_j, v_j), \varepsilon)\big) + \varepsilon.$$

This, combined with (2.18), implies that there is exactly one atom of π lying in $D_{S \times \mathbb{R}_{>0}}((y_j, v_j), \varepsilon)$. Thus, $\tilde{\pi}(\{(y_j, v_j)\}) = 1$. Assume that there exist $j \neq j'$ such that $y_j = y_{j'} = y$. It is then the case that

$$2 = \tilde{\pi}(\{(y, v_j), (y, v_{j'})\}) \leq \pi\big(D_{S \times \mathbb{R}_{>0}}((y, v_j), \varepsilon) \cup D_{S \times \mathbb{R}_{>0}}((y, v_{j'}), \varepsilon)\big) + \varepsilon$$

Since $C := D_{S \times \mathbb{R}_{>0}}((y, v_j), \varepsilon) \cup D_{S \times \mathbb{R}_{>0}}((y, v_{j'}), \varepsilon)$ is contained in $D_S(\rho_S, r) \times [e^{-r}, e^r]$, there are at least two atoms of π lying in C, which contradicts (2.18). Therefore, we deduce that $\tilde{\pi} \in \mathscr{P}^{(n)}(S)$. □

Lemma 2.20. *It holds that $\mathscr{P}^*(S) = \bigcap_{n=1}^{\infty} \mathscr{P}^{(n)}(S)$.*

Proof: This is immediate from the definition of $\mathscr{P}^{(n)}(S)$ and (2.14). □

Theorem 2.21 (Polishness of $\mathcal{M}^{\mathrm{dis}}(S)$)**.** *In general, $d_{\mathcal{M}^{\mathrm{dis}}}^{S, \rho_S}$ is not complete. However, the vague-and-point-process topology is Polish.*

Proof: At the end of Section 2, we checked that $d_{\mathcal{M}^{\mathrm{dis}}}^{S, \rho_S}$ is not complete in general. However, by Corollary 2.15 and Lemmas 2.19 and 2.20, we can apply Alexandrov's theorem (see Srivastava, 1998, Theorem 2.2.1) to conclude that the vague-and-point-process topology on $\mathcal{M}^{\mathrm{dis}}(S)$ is Polish. □

We further study topological properties of $\mathscr{P}(S)$: precompactness, the Borel σ-algebra, and tightness.

Theorem 2.22 (Precompactness in $\mathscr{P}(S)$)**.** *A non-empty subset $\{\pi_j\}_{j \in J}$ of $\mathscr{P}(S)$ is precompact in $\mathscr{P}(S)$ if and only if the following conditions are satisfied.*

(i) *The subset $\{\pi_j\}_{j \in J}$ is precompact in the vague topology as measures on $S \times \mathbb{R}_{>0}$.*

(ii) *For each $r > 0$, $\sup_{j \in J} W^{(r)}(\pi_j) < \infty$ and $\lim_{\varepsilon \downarrow 0} \sup_{j \in J} M_\varepsilon^{(r)}(\pi_j) = 0$.*

Proof: Suppose that $\{\pi_j\}_{j \in J}$ is precompact in $\mathscr{P}(S)$. Since the topology on $\mathscr{P}(S)$ is finer than the vague topology, we have (i). If (ii) is not satisfied, then for some $r > 0$ there exists a sequence $(j_n)_{n \geq 1}$ in J such that $W^{(r)}(\pi_{j_n}) \to \infty$ or $M_{\varepsilon_n}^{(r)}(\pi_{j_n}) > \delta$ for some $\delta > 0$ and $\varepsilon_n \downarrow 0$. However, since $(\pi_{j_n})_{n \geq 1}$ has a convergent subsequence, by Theorem 2.14, we obtain a contradiction, which yields (ii). The converse assertion immediately follows from Theorem 2.14. □

To consider random element of $\mathscr{P}(S)$, we first identify the Borel σ-algebra on $\mathscr{P}(S)$. In particular, we show that it coincides with the one generated from the vague topology on $\mathscr{P}(S)$ in Proposition 2.25 below.

Lemma 2.23. *Let $\pi, \pi_1, \pi_2, \dots$ be elements in $\mathscr{P}(S)$ such that $\pi_n \to \pi$ vaguely. Fix $r > 0$ such that π has no atoms on the boundary of $D_S(\rho_S, r) \times [e^{-r}, e^r]$. Then, $\pi_n^{[r]} \to \pi^{[r]}$ in $\mathscr{P}(S)$.*

Proof: It is elementary to check that $\pi_n^{[r]} \to \pi^{[r]}$ vaguely (and even weakly). If (x, w) is an atom of $\pi_n^{[r]}$, then we have that $w \in [e^{-r}, e^r]$. Thus, by Theorem 2.14, we obtain the desired result. □

Proposition 2.24. *The set $\mathscr{P}(S)$ is a Borel subset of $\mathcal{M}(S\times\mathbb{R}_{>0})$ equipped with the vague topology.*

Proof: By Kallenberg (2021, Theorem 2.19), the set $\mathcal{N}(S \times \mathbb{R}_{>0})$ of integer valued Radon measures is a Borel subset of $\mathcal{M}(S)$. Since $\mathcal{N}(S \times \mathbb{R}_{>0}) \ni \pi \mapsto \mathfrak{v}(\pi)(S^{(r)})$ is measurable with respect to the vague topology for each $r > 0$, we obtain the desired result. □

Proposition 2.25 (The Borel σ-algebra on $\mathscr{P}(S)$)**.** *The Borel σ-algebra on $\mathscr{P}(S)$ coincides with the Borel σ-algebra generated from the vague topology.*

Proof: To distinguish between the two topological spaces, we rewrite $\mathscr{P}(S)'$ for the topological space $\mathscr{P}(S)$ equipped with the vague topology, and let $\mathscr{P}(S)$ represent the topological space $\mathscr{P}(S)$ with the topology induced from $d_{\mathscr{P}}^{S,\rho_S}$ (which we have considered so far). Write Σ and Σ' for the Borel σ-algebras on $\mathscr{P}(S)$ and on $\mathscr{P}(S)'$, respectively. Since the topology on $\mathscr{P}(S)$ is finer than the vague topology, we have that $\Sigma \supseteq \Sigma'$. To show the converse relation, we let f be a bounded continuous function on $\mathscr{P}(S)$. It suffices to show that f is vaguely measurable, i.e., Σ'-measurable. For $\pi \in \mathscr{P}(S)'$ and $r > 0$, if π has no atoms on the boundary of $D_S(\rho_S, r)\times[e^{-r}, e^r]$, then $\pi^{[r]} = \pi^{[s]}$ for all $s > 0$ sufficiently close to r. Hence, for each $\pi \in \mathscr{P}(S)'$, $f(\pi^{[r]})$ is continuous for all but countably many $r > 0$ and so we can define a map $f_r \colon \mathscr{P}(S)' \to \mathbb{R}$ by setting

$$f_r(\pi) := \int_0^1 f(\pi^{(r+s)})\, ds.$$

Suppose that $\pi_n \to \pi$ in the vague topology. Lemma 2.23 and the continuity of f imply that $f(\pi_n^{(s)}) \to f(\pi^{[s]})$ for all but countably many $s > 0$. Thus, the dominated convergence theorem yields that f_r is continuous on $\mathscr{P}(S)'$. In particular, f_r is Σ'-measurable. By Lemma 2.17, the continuity of f and the dominated convergence theorem, we have that $f_r(\pi) \to f(\pi)$ as $r \to \infty$ for each $\pi \in \mathscr{P}(S)'$. Hence, f is Σ'-measurable. □

Now, we provide a tightness criterion for random elements of $\mathscr{P}(S)$.

Theorem 2.26 (Tightness in $\mathscr{P}(S)$)**.** *Let $(\pi_n)_{n\geq1}$ be a sequence of random elements of $\mathscr{P}(S)$. Write P_n for the underlying probability measure of π_n. Then the sequence $(\pi_n)_{n\geq1}$ is tight if and only if the following conditions are satisfied.*

(i) *The sequence $(\pi_n)_{n\geq1}$ is tight with respect to the vague topology.*
(ii) *For each $r > 0$, $\lim_{l\to\infty} \limsup_{n\to\infty} P_n(W^{(r)}(\pi_n) > l) = 0$.*
(iii) *For each $r, \delta > 0$, $\lim_{\varepsilon\downarrow0} \limsup_{n\to\infty} P_n(M_\varepsilon^{(r)}(\pi_n) > \delta) = 0$.*

Proof: Assume that $(\pi_n)_{n\geq1}$ is tight. The condition (i) is obvious. Fix $r, \delta, \eta > 0$. By tightness, there exists a compact subset $\mathcal{A}$ of $\mathscr{P}(S)$ such that $P_n(\pi_n \notin \mathcal{A}) < \eta$ for all $n \geq 1$. Theorem 2.22 yields that $l := \sup_{\pi\in\mathcal{A}} W^{(r)}(\pi) < \infty$ and $\sup_{\pi\in\mathcal{A}} M_{\varepsilon_0}^{(r)}(\pi) < \delta$ for some $\varepsilon_0 > 0$. We then deduce that $\sup_{n\geq1} P_n(W^{(r)}(\pi_n) > l) < \eta$ and $\sup_{n\geq1} P_n(M_{\varepsilon_0}^{(r)}(\pi_n) > \delta) < \eta$. Using the monotonicity of $M_\cdot^{(r)}(\pi_n)$ stated in Lemma 2.12(iii), we obtain (ii) and (iii). Conversely, assume that (i), (ii) and (iii) are satisfied. Fix $\varepsilon > 0$. By (i), there exists a subset $\mathcal{A}$ of $\mathscr{P}(S)$ such that $\mathcal{A}$ is vaguely compact and $P_n(\pi_n \notin \mathcal{A}) < \varepsilon$ for all $n \geq 1$. By Lemma 2.12(iii) and (iv), we note that "$\limsup_{n\to\infty}$" in the statement of (ii) and (iii) can be replaced by "$\sup_{n\geq1}$". Then, for each $k, m \in \mathbb{N}$, we can find $l_k > 0$ and $\varepsilon_{k,m} > 0$ such that $\sup_{n\geq1} P_n(W^{(k)}(\pi_n) > l_k) < 2^{-k}\varepsilon$ and $\sup_{n\geq1} P_n(M_{\varepsilon_{k,m}}^{(k)}(\pi_n) > m^{-1}) < 2^{-k-m}\varepsilon$. Define $\mathcal{A}'$ be a collection of $\pi \in \mathscr{P}(S)$ such that $\pi \in \mathcal{A}$ and, for all $k, m \in \mathbb{N}$, $W^{(k)}(\pi) \leq l_k$

and $M^{(k)}_{\varepsilon_{k,m}}(\pi) \leq m^{-1}$. By Theorem 2.22, $\mathcal{A}'$ is precompact in $\mathscr{P}(S)$. Moreover, we have that

$$P_n(\pi_n \notin \mathcal{A}') \leq P_n(\pi_n \notin \mathcal{A}) + \sum_{k \in \mathbb{N}} P_n(W^{(k)}(\pi_n) > l_k) + \sum_{k,m \in \mathbb{N}} P_n(M_{\varepsilon^{(k)}_{k,m}}(\pi_n) > m^{-1}) \leq 3\varepsilon.$$

Therefore, $(\pi_n)_{n\geq 1}$ is tight. □

Distributional convergence of random measures in the vague topology is well-studied in Kallenberg (2017, Section 4.2). In the following result, we provide a useful condition for strengthening distributional convergence of random measures in the vague topology to distributional convergence in $\mathscr{P}(S)$.

Corollary 2.27. *Let $\pi_1, \pi_2, \ldots$ be random elements of $\mathscr{P}(S)$. If $\pi_n \xrightarrow{\mathrm{d}} \pi$ in the vague topology for some random element π of $\mathcal{M}(S \times \mathbb{R}_{>0})$ and the conditions (ii) and (iii) of Theorem 2.26 are satisfied, then π is a random element of $\mathscr{P}(S)$ and $\pi_n \xrightarrow{\mathrm{d}} \pi$ in $\mathscr{P}(S)$.*

Proof: By Theorem 2.26, $(\pi_n)_{n\geq 1}$ is tight in $\mathscr{P}(S)$. Let $(\pi_{n_k})_{k\geq 1}$ be a sequence such that $\pi_{n_k} \xrightarrow{\mathrm{d}} \pi'$ in $\mathscr{P}(S)$ for some random element π' of $\mathscr{P}(S)$. Then, by Theorem 2.14, we have that $\pi_{n_k} \xrightarrow{\mathrm{d}} \pi$ in the vague topology. Hence, π and π' give the same probability distribution on $\mathcal{M}(S \times \mathbb{R}_{>0})$ equipped with the vague topology. It then follows from Propositions 2.24 and 2.25 that $\pi \overset{\mathrm{d}}{=} \pi'$ as random elements of $\mathscr{P}(S)$, which implies that $\pi_{n_k} \xrightarrow{\mathrm{d}} \pi$ in $\mathscr{P}(S)$. This completes the proof. □

3. Gromov–Hausdorff-type topologies

Gromov–Hausdorff-type topologies are topologies on the set of (equivalence classes) of metric spaces equipped with additional objects such as points, measures, and laws of stochastic processes. Recently, the author established a general framework for the metrization of these topologies (Noda, 2024a), which we follow in this article. (NB. There is a related work by Khezeli, 2023, but the framework in Noda, 2024a relies on milder assumptions. See Noda, 2024a, Section 1 for a detailed comparison of the two approaches.) In Section 3.1, we recall the Fell topology, a natural topology on the space of closed subsets of a given topological space, with a particular focus on its metrization. We then introduce the framework presented in Noda (2024a) in Section 3.2. In Section 3.3, we collect the Gromov–Hausdorff-type topologies used in this article.

3.1. *The Fell topology.* The Fell topology is a commonly used topology on the space of closed subsets (cf. Molchanov, 2017, Appendix C). The purpose of this subsection is to present a metrization of this topology. For detailed properties of the metrization, see Noda (2024a, Section 2.1), and for further topological background, see Molchanov (2017, Appendix C).

Fix a rooted boundedly-compact metric space (S, d^S, ρ_S). Recall that, for a subset $A \subseteq S$, the *(closed) ε-neighborhood* of A in (S, d^S) is given by

$$A^{\langle \varepsilon \rangle} := \{x \in S \mid \exists y \in A \text{ such that } d^S(x, y) \leq \varepsilon\}.$$

Let $\mathcal{C}(S)$ be the set of closed subsets in S and $\mathcal{C}_c(S)$ be the set of compact subsets in S (containing the empty set). The *Hausdorff metric* d^S_H on $\mathcal{C}_c(S)$ is defined by setting

$$d^S_H(A, B) := \inf\{\varepsilon \geq 0 \mid A \subseteq B^{\langle \varepsilon \rangle}, B \subseteq A^{\langle \varepsilon \rangle}\},$$

where the infimum over the empty set is defined to be ∞. It is known that d^S_H is indeed a metric (allowed to take the value ∞ due to the empty set) on $\mathcal{C}_c(S)$ (see Burago et al., 2001, Section 7.3.1), and the induced topology is called the *Hausdorff topology.* To deal with non-compact sets, we introduce a metric on $\mathcal{C}(S)$.

Definition 3.1. For $A \in \mathcal{C}(S)$ and $r > 0$, we write

$$A^{(r)} \coloneqq \mathrm{cl}(A \cap B_S(\rho, r)), \tag{3.1}$$

where we recall that $\mathrm{cl}(\cdot)$ denotes the closure of a set. We then define, for $A, B \in \mathcal{C}(S)$,

$$d_{\bar{H}}^{S,\rho_S}(A, B) \coloneqq \int_0^\infty e^{-r} \left(1 \wedge d_H^S(A^{(r)}, B^{(r)})\right) dr. \tag{3.2}$$

The function $d_{\bar{H}}^{S,\rho_S}$ is a metric on $\mathcal{C}(S)$ and a natural extension of the Hausdorff metric for non-compact sets. The following is a basic property of $d_{\bar{H}}^{S,\rho_S}$.

Theorem 3.2 (Noda, 2024a, Theorems 3.8, 3.9, and 3.11)**.** *The function $d_{\bar{H}}^{S,\rho_S}$ is a metric on $\mathcal{C}(S)$ and the metric space $(\mathcal{C}(S), d_{\bar{H}}^{S,\rho_S})$ is compact. A sequence $(A_n)_{n\geq 1}$ converges to A with respect to $d_{\bar{H}}^{S,\rho_S}$ if and only if $A_n^{(r)}$ converges to $A^{(r)}$ in the Hausdorff topology for all but countably many $r > 0$. Moreover, the topology on $\mathcal{C}(S)$ induced from $d_{\bar{H}}^{S,\rho_S}$ is independent of the root ρ_S.*

In particular, the above result implies that the induced topology on $\mathcal{C}(S)$ by $d_{\bar{H}}^{S,\rho_S}$ coincides with the Fell topology (see Molchanov, 2017, Appendix C for its definition).

Remark 3.3. In Noda (2024a), the restriction $A^{(r)}$ of a closed set A is defined differently from (3.1), namely as

$$A^{(r)} = A \cap D_S(\rho_S, r).$$

This difference makes no impact on the definition of the metric $d_{\bar{H}}^{S,\rho_S}$, since, by using the fact that S is boundedly compact, one can verify that, for all but countably many r,

$$\mathrm{cl}(A \cap B_S(\rho, r)) = A \cap D_S(\rho_S, r).$$

We adopt the definition via closure rather than closed balls, because this version of restriction is more compatible with resistance metric spaces that we treat in this paper (see Lemma 4.8, for example).

3.2. *The Gromov–Hausdorff-type topologies generated by structures.* In this subsection, we recall the general theory on metrization of Gromov–Hausdorff-type topologies developed in Noda (2024a).

We first introduce a rule that determines structures to be added to metric spaces.

Definition 3.4 (Structure)**.** We call τ a *structure* (on boundedly-compact metric spaces) if it satisfies the following.

(i) For every boundedly-compact metric space (S, d^S), there is a corresponding topological space $\tau(S, d^S)$, which we simply write as $\tau(S)$.
(ii) For every distance preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, there is a corresponding topological embedding $\tau_f\colon \tau(S_1) \to \tau(S_2)$, i.e., τ_f is a homeomorphism onto its image.
(iii) For any two distance-preserving maps $f\colon S_1 \to S_2$ and $g\colon S_2 \to S_3$ between boundedly-compact metric spaces, it holds that $\tau_{g\circ f} = \tau_g \circ \tau_f$.
(iv) For any boundedly-compact metric space (S, d^S), it holds that $\tau_{\mathrm{id}_S} = \mathrm{id}_{\tau(S)}$.

We say that τ is *separable* if each $\tau(S)$ is separable.

Throughout this subsection, we fix a separable structure τ on boundedly-compact metric spaces. Given $\mathcal{S}_i = (S_i, d^{S_i}, \rho_{S_i}, a_{S_i})$, $i = 1, 2$, such that $(S_i, d^{S_i}, \rho_{S_i})$ is a rooted boundedly-compact metric space and $a_{S_i} \in \tau(S_i)$, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *rooted-τ-isometric* if and only if there exists a root-preserving isometry $f\colon S_1 \to S_2$ such that $\tau_f(a_{S_1}) = a_{S_2}$. Note that f being an isometry means that f is a surjective distance-preserving map (and hence f is bijective). We denote the collection of rooted-τ-isometric equivalence classes by $\mathfrak{M}_\bullet(\tau)$.

Remark 3.5. From a rigorous point of view of set theory, $\mathfrak{M}_\bullet(\tau)$ is not a set. However, it can be regarded as a set. This is because one can construct a legitimate set $\mathscr{M}(\tau)$ by appropriately choosing representatives, such that every (S, d^S, ρ_S, a_S) is rooted-τ-isometric to a unique element of $\mathscr{M}(\tau)$. For details, refer to Noda (2024a, Proposition 6.2). Therefore, in this article, we will proceed with the discussion by treating $\mathfrak{M}_\bullet(\tau)$ as a set to avoid repeatedly referring to this set-theoretic formality concerning the choice of representatives.

To define a metric on $\mathfrak{M}_\bullet(\tau)$, we introduce the notion of metrization of structure.

Definition 3.6. We say that τ admits a *metrization* if and only if, for every rooted boundedly-compact metric space (S, d_S, ρ_S), there exists a metric d_τ^{S,ρ_S} on $\tau(S)$ inducing the given topology such that the following condition holds.

(v) Let $(S_1, d_{S_1}, \rho_{S_1})$ and $(S_2, d_{S_2}, \rho_{S_2})$ be rooted boundedly-compact metric spaces. For every root-preserving distance preserving map $f\colon S_1 \to S_2$, the map $\tau_f\colon \tau(S_1) \to \tau(S_2)$ is distance-preserving with respect to $d_\tau^{S_1,\rho_{S_1}}$ and $d_\tau^{S_2,\rho_{S_2}}$.

If each d_τ^{S,ρ_S} is complete, we say that τ admits a *complete metrization.*

Henceforth, we assume that τ admits a metrization. Then the metric on $\mathfrak{M}_\bullet(\tau)$ is defined by generalizing the Gromov–Hausdorff metric Burago et al. (2001). To read the following definition, recall the metric for the Fell topology defined in (3.2).

Definition 3.7. For $\mathcal{S} = (S, d^S, \rho_S, a_S)$ and $\mathcal{T} = (T, d^T, \rho_T, a_T)$ in $\mathfrak{M}_\bullet(\tau)$, we define

$$d_{\mathfrak{M}_\bullet}^\tau(\mathcal{S}, \mathcal{T}) := \inf_{f,g,M} \left\{ d_{\bar{H}}^{M,\rho_M}\big(f(S), g(T)\big) \vee d_\tau^{M,\rho_M}\big(\tau_f(a_S), \tau_g(a_T)\big) \right\},$$

where the infimum is taken over all rooted boundedly-compact metric spaces (M, d^M, ρ_M) and root-and-distance-preserving maps $f\colon S \to M$ and $g\colon T \to M$.

To ensure that $d_{\mathfrak{M}_\bullet}^\tau$ is a metric, we assume the following continuity condition on τ.

Definition 3.8 (Embedding-continuity)**.** We say that a structure τ is *embedding-continuous* if τ satisfies the following condition.

(EC) Fix boundedly-compact metric spaces S and T. Let $f_n\colon S \to T$, $n \in \mathbb{N} \cup \{\infty\}$ be root-and-distance-preserving maps. If $f_n \to f_\infty$ in the compact-convergence topology, i.e., uniformly on every compact subset, then $\tau_{f_n}(a) \to \tau_{f_\infty}(a)$ in $\tau(T)$ for all $a \in \tau(S)$.

Theorem 3.9 (Noda, 2024a, Theorems 6.18 and 6.19)**.** *Assume that τ is embedding-continuous. Then the function $d_{\mathfrak{M}_\bullet}^\tau$ is a metric on $\mathfrak{M}_\bullet(\tau)$. Moreover, a sequence $(S_n, d^{S_n}, \rho_{S_n}, a_{S_n})$, $n \in \mathbb{N}$, in $\mathfrak{M}_\bullet(\tau)$ converges to (S, d^S, ρ_S, a_S) with respect to $d_{\mathfrak{M}_\bullet}^\tau$ if and only if there exist a rooted boundedly-compact metric space (M, d^M, ρ_M) and root-and-distance-preserving maps $f_n\colon S_n \to M$ and $f\colon S \to M$ such that $f_n(S_n) \to f(S)$ in the Fell topology as subsets of M and $\tau_{f_n}(a_{S_n}) \to \tau_f(a_S)$ in $\tau(M)$.*

Remark 3.10. In Theorem 3.9, the roots ρ_{S_n} of S_n are mapped to a common root ρ_M of M. By relaxing this requirement, one can introduce another notion of convergence in $\mathfrak{M}_\bullet(\tau)$. Namely, we say that $\mathcal{S}_n$ converges to $\mathcal{S}_\infty$ if and only if

> *there exist a boundedly-compact space M and isometric embeddings $f_n\colon S_n \to M$ and $f\colon S \to M$ such that $f_n(S_n) \to f(S)$ in the Fell topology, $f_n(\rho_{S_n}) \to f_(\rho_S)$ in M, and $\tau_{f_n}(a_{S_n}) \to \tau_f(a_S)$ in $\tau(M)$.*

In general, this convergence is weaker than the convergence induced by $d_{\mathfrak{M}_\bullet}^\tau$ defined above. However, for most structures of interest the two notions of convergence coincide. See Noda (2024a, Section 6.3) for details.

Remark 3.11 (The local Gromov–Hausdorff topology). If one follows the above framework without introducing the structure τ, one obtains an extension of the (pointed) Gromov–Hausdorff topology (cf. Burago et al., 2001, Section 7.3) to non-compact metric spaces, which have been studied in Abraham et al. (2013); Khezeli (2020). We briefly recall the construction below; for further details, see Noda (2024a, Section 4). Let $\mathfrak{M}_\bullet$ be the collection of rooted-isometric equivalence classes of rooted boundedly-compact metric spaces. For each rooted boundedly-compact metric space $\mathcal{S} = (S, d^S, \rho_S)$ and $\mathcal{T} = (T, d^T, \rho_T)$, set

$$d_{\mathfrak{M}_\bullet}(\mathcal{S}, \mathcal{T}) \coloneqq \inf_{f,g,M} d_{\bar{H}}^{M,\rho_M}\big(f(S), g(T)\big),$$

then $d_{\mathfrak{M}_\bullet}$ defines a metric on $\mathfrak{M}_\bullet$. We call the induced topology the *local Gromov–Hausdorff topology* (see Noda, 2024a, Definition 4.8).

To discuss convergence of probability measures on $\mathfrak{M}_\bullet(\tau)$, it is desirable that the space be Polish. This will be ensured by verifying that τ itself is *Polish* (see Definition 3.16 below). To formulate this notion, we introduce several auxiliary definitions concerning τ below; however, they will not be used in the proofs of our main results. The essential statement required for later purposes is Theorem 3.17 below.

For rooted boundedly-compact metric spaces $\mathcal{S} = (S, d^S, \rho_S)$ and $\mathcal{T} = (T, d^T, \rho_T)$, we write $\mathcal{S} \preceq \mathcal{T}$ if and only if $S \subseteq T$, $d^T|_{S\times S} = d^S$, and $\rho_S = \rho_T$. In this case, we often regard $\tau(S)$ as a subset of $\tau(T)$ via the topological embedding τ_ι, where ι denotes the inclusion map from S to T.

Assumption 3.12. *Let $\mathcal{S}_n = (S_n, d^{S_n}, \rho_{S_n})$, $n \in \mathbb{N} \cup \{\infty\}$, and $\mathcal{M} = (M, d^M, \rho_M)$ be rooted boundedly-compact metric spaces such that $\mathcal{S}_n \preceq \mathcal{M}$ for all $n \in \mathbb{N} \cup \{\infty\}$ and S_n converges to S_∞ in the Fell topology as subsets M.*

(i) *If $a \in \tau(M)$ and $a_{S_n} \in \tau(S_n)$ are such that $a_{S_n} \to a$ in $\tau(M)$, then $a \in \tau(S_\infty)$.*
(ii) *For every $a \in \tau(S_\infty)$, there exists a sequence $a_n \in \tau(S_n)$ such that $a_n \to a_\infty$ in $\tau(M)$.*

Definition 3.13 (Semicontinuity and continuity)**.** We say that τ is *upper* (resp. *lower*) *semicontinuous* if and only if it satisfies Assumption 3.12(i) (resp. (ii)). We say that τ is *continuous* if and only if it is embedding-continuous and both upper and lower semicontinuous.

Theorem 3.14 (Noda, 2024a, Theorems 6.40 and 6.41)**.** *If τ is separable, continuous, and admits a complete metrization, then the function $d^\tau_{\mathfrak{M}_\bullet}$ is a complete and separable metric on $\mathfrak{M}_\bullet(\tau)$.*

Although the above result already covers a wide class of structures, some important structures fail to satisfy semicontinuity, and hence the theorem does not apply. Even in such cases, the Polishness of the space $\mathfrak{M}_\bullet(\tau)$ can still be established, by verifying that the space is a G_δ-subset of a larger Polish space, as we did in Section 2.3. With this background, we introduce the following notion.

Definition 3.15 (Topological embedding)**.** Recall that we have a fixed structure τ. Let $\tilde{\tau}$ be another structure. A *topological embedding* $\eta\colon \tau \Rightarrow \tilde{\tau}$ is a family of $\{\eta_S\colon \tau(S) \to \tilde{\tau}(S)\}_S$ indexed by boundedly-compact metric spaces S, satisfying the following conditions.

(TE1) Each map $\eta_S\colon \tau(S) \to \tilde{\tau}(S)$ is a topological embedding.
(TE2) For any rooted boundedly-compact metric spaces $(S_1, d^{S_1}, \rho_{S_1})$ and $(S_2, d^{S_2}, \rho_{S_2})$ and any root-and-distance-preserving map $f\colon S_1 \to S_2$, it holds that $\tilde{\tau}_f \circ \eta_{S_1} = \eta_{S_2} \circ \tau_f$.

As the terminology suggests, a topological embedding $\eta\colon \tau \Rightarrow \tilde{\tau}$ induces a topological embedding of the associated Gromov–Hausdorff type spaces as follows:

$$\mathfrak{M}_\bullet(\tau) \ni (S, d^S, \rho_S, a_S) \longmapsto (S, d^S, \rho_S, \eta_S(a_S)) \in \mathfrak{M}_\bullet(\tilde{\tau}),$$

see Noda (2024a, Lemma 6.44). Through this embedding, topological properties of $\mathfrak{M}_\bullet(\tau)$ can be inferred from those of the larger space $\mathfrak{M}_\bullet(\tilde{\tau})$.

Definition 3.16 (Polish structure)**.** We say that τ is *Polish* if there exist another structure $\tilde{\tau}$, a topological embedding $\eta \colon \tau \Rightarrow \tilde{\tau}$, and, for each rooted boundedly-compact metric space (S, d^S, ρ_S), a sequence $(\tilde{\tau}_k(S, \rho_S))_{k=1}^{\infty}$ of open subsets of $\tilde{\tau}(S)$, satisfying the following conditions.

(P1) The structure $\tilde{\tau}$ is continuous, separable, and admits a complete metrization.
(P2) For each rooted boundedly-compact metric space (S, d^S, ρ_S), $\eta_S(\tau(S)) = \bigcap_{k \geq 1} \tilde{\tau}_k(S, \rho_S)$.
(P3) Let $(S_1, d^{S_1}, \rho_{S_1})$ and $(S_2, d^{S_2}, \rho_{S_2})$ be rooted boundedly-compact metric spaces. For every root-and-distance-preserving map $f \colon S_1 \to S_2$, it holds that $\tilde{\tau}_f^{-1}(\tilde{\tau}_k(S_2, \rho_{S_2})) = \tilde{\tau}_k(S_1)$ for each $k \geq 1$.

We call $(\tilde{\tau}, (\tilde{\tau}_k)_{k \geq 1})$ a *Polish system* of τ.

Theorem 3.17 (Noda, 2024a, Theorem 6.46)**.** *If τ is a Polish structure, then the topology on $\mathfrak{M}_\bullet(\tau)$ induced from $d^\tau_{\mathfrak{M}_\bullet}$ is Polish. (NB. The metric $d^\tau_{\mathfrak{M}_\bullet}$ is not necessarily a complete metric.)*

So far, we have considered boundedly-compact spaces. When one considers only compact underlying spaces, it is more natural to use the Hausdorff topology rather than the Fell topology to describe the convergence of the underlying spaces. In what follows, we briefly introduce a corresponding modification of the above framework. See Noda (2024a, Appendix B) for details.

Recall that we have a fixed embedding-continuous structure τ that admits a metrization. We define $\mathfrak{K}_\bullet(\tau)$ as the collection of $(S, d_S, \rho_S, a_S) \in \mathfrak{K}_\bullet(\tau)$ such that S is compact. We define a metric on $\mathfrak{K}_\bullet(\tau)$ as follows: for $\mathcal{S}_1 = (S_1, d_{S_1}, \rho_{S_1}, a_{S_1})$ and $\mathcal{S}_2 = (S_2, d_{S_2}, \rho_{S_2}, a_{S_2})$ in $\mathfrak{K}_\bullet(\tau)$, set

$$d^\tau_{\mathfrak{K}_\bullet}(\mathcal{S}_1, \mathcal{S}_2) \coloneqq \inf_{f,g,M} \Bigl\{ d^M_H(f(S_1), g(S_2)) \vee d^\tau_{M,\rho_M}\bigl(\tau_f(a_{S_1}), \tau_g(a_{S_2})\bigr) \Bigr\},$$

where the infimum is taken over all rooted compact metric spaces (M, d^M, ρ_M) and all root-preserving isometric embeddings $f \colon S_1 \to M$ and $g \colon S_2 \to M$.

In the above definition, the Hausdorff metric d^M_H is employed instead of the Fell metric. Accordingly, the characterization of convergence in $\mathfrak{K}_\bullet(\tau)$ is given in the same manner as in Theorem 3.9, with the Fell topology replaced by the Hausdorff topology. It should also be noted that, if τ is Polish, then the resulting topology on $\mathfrak{K}_\bullet(\tau)$ is again Polish.

3.3. *Structures used in the present paper.* Thanks to the theory introduced in Section 3.2, we can easily handle various Gromov–Hausdorff-type topologies by defining corresponding structures. In this subsection, we provide structures used for Gromov–Hausdorff-type topologies in our discussions.

(S1) **Fixed spaces.** Let Ξ be a Polish space. We define a structure τ^Ξ as follows.
- For each boundedly-compact metric space (S, d^S), set $\tau^\Xi(S) \coloneqq \Xi$.
- For each distance-preserving map $f \colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau^\Xi_f \coloneqq \mathrm{id}_\Xi$.

Let d^Ξ be a metric on Ξ inducing the given topology. We define a metrization of τ by equipping $\tau^\Xi(S) = \Xi$ with the metric d^Ξ. The structure τ^Ξ is Polish (see Noda, 2024a, Section 8.1).

(S2) **Points.** We define a structure τ^{pt} as follows.
- For each boundedly-compact metric space (S, d^S), set $\tau^\Xi(S) \coloneqq S$.
- For each distance-preserving map $f \colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau^{\mathrm{pt}}_f \coloneqq f$.

We define a metrization of τ^{pt} by equipping $\tau^{\mathrm{pt}}(S) = S$ with the the associated metric d^S. The structure τ^{pt} is Polish (see Noda, 2024a, Section 8.2).

(S3) **Measures.** Recall from Section 2.1 that $\mathcal{M}(S)$ denotes the space of Radon measures on S equipped with the vague topology. We define a structure $\tau^{\mathcal{M}}$ as follows.
- For each boundedly-compact metric space (S, d^S), set $\tau^{\mathcal{M}}(S) \coloneqq \mathcal{M}(S)$.

- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau^{\mathcal{M}}_f(\mu) \coloneqq \mu \circ f^{-1}$ for each $\mu \in \mathcal{M}(S)$, i.e., $\tau^{\mathcal{M}}_f(\mu)$ is the pushforward measure of μ by f.

We define a metrization of $\tau^{\mathcal{M}}$ by equipping, for each rooted boundedly-compact metric space (S, d^S, ρ_S), $\tau^{\mathcal{M}}(S) = \mathcal{M}(S)$ with the the vague metric d_V^{S,ρ_S}, as recalled from (2.3).

We also introduce a structure $\tau^{\mathcal{M}_{\mathrm{fin}}}$ for finite Borel measures, defined analogously but using the weak topology instead of the vague topology. Recall that $\mathcal{M}_{\mathrm{fin}}(S)$ denotes the space of finite Borel measures on S, equipped with the weak topology.

- For each boundedly-compact metric space (S, d^S), set $\tau^{\mathcal{M}_{\mathrm{fin}}}(S) \coloneqq \mathcal{M}_{\mathrm{fin}}(S)$.
- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau^{\mathcal{M}_{\mathrm{fin}}}_f(\mu) \coloneqq \mu \circ f^{-1}$ for each $\mu \in \mathcal{M}_{\mathrm{fin}}(S)$.

We define a metrization of $\tau^{\mathcal{M}_{\mathrm{fin}}}$ by equipping $\tau^{\mathcal{M}_{\mathrm{fin}}}(S) = \mathcal{M}_{\mathrm{fin}}(S)$ with the Prohorov metric d_P^S, as recalled from (2.1).

Both structures $\tau^{\mathcal{M}}$ and $\tau^{\mathcal{M}_{\mathrm{fin}}}$ are Polish (see Noda, 2024a, Section 8.4). We call the topology on $\mathbb{G}_c \coloneqq \mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}})$ the *(pointed) Gromov–Hausdorff–Prohorov topology*, which was firstly introduced in Abraham et al. (2013). The topology on $\mathbb{G} \coloneqq \mathfrak{M}_\bullet(\tau^{\mathcal{M}})$ is an extension of the Gromov–Hausdorff–Prohorov topology, which we call the *local Gromov–Hausdorff-vague topology*. It is a consequence of Theorem 3.14 that $\mathbb{G}_c$ and $\mathbb{G}$ are Polish. (Moreover, the associated metrics are complete. This is a consequence of Theorem 3.14; see Noda, 2024a, Section 8.4 for details.)

(S4) **Measures on marked spaces.** Fix a rooted boundedly-compact metric space (Ξ, d^Ξ, ρ_Ξ). Recall from Section 2.1 that $\mathcal{M}(S)$ denotes the space of Radon measures on S equipped with the vague topology. We define a structure $\tau^{\mathcal{M}(\cdot\times\Xi)}$ as follows.

- For each boundedly-compact metric space (S, d^S), set $\tau^{\mathcal{M}(\cdot\times\Xi)}(S) \coloneqq \mathcal{M}(S \times \Xi)$.
- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau^{\mathcal{M}(\cdot\times\Xi)}_f(\mu) \coloneqq \mu\circ(f\times \mathrm{id}_\Xi)^{-1}$ for each $\mu \in \mathcal{M}(S\times\Xi)$, i.e., $\tau^{\mathcal{M}(\cdot\times\Xi)}_f(\mu)$ is the pushforward measure of μ by $f \times \mathrm{id}_\Xi$.

We define a metrization of $\tau^{\mathcal{M}(\cdot\times\Xi)}$ by equipping, for each rooted boundedly-compact metric space (S, d^S, ρ_S), $\tau^{\mathcal{M}(\cdot\times\Xi)}(S) = \mathcal{M}(S \times \Xi)$ with the the vague metric $d_V^{S\times\Xi,(\rho_S,\rho_\Xi)}$.

Similarly, we define a structure $\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\Xi)}$ as follows.

- For each boundedly-compact metric space (S, d^S), set $\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\Xi)}(S) \coloneqq \mathcal{M}_{\mathrm{fin}}(S \times \Xi)$.
- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\Xi)}_f(\mu) \coloneqq \mu \circ (f \times \mathrm{id}_\Xi)^{-1}$ for each $\mu \in \mathcal{M}_{\mathrm{fin}}(S \times \Xi)$.

We define a metrization of $\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\Xi)}$ by equipping $\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\Xi)}(S) = \mathcal{M}(S \times \Xi)$ with the the Prohorov metric $d_P^{S\times\Xi}$.

Both structures $\tau^{\mathcal{M}(\cdot\times\Xi)}$ and $\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\Xi)}$ are Polish. This can be verified in the same manner as the proof of the Polishness of the structures $\tau^{\mathcal{M}}$ and $\tau^{\mathcal{M}_{\mathrm{fin}}}$ introduced in (S3). Alternatively, one may apply the notion of space transformation and composition introduced in Noda (2024a, Section 7.2). Indeed, the structure $\tau^{\mathcal{M}(\cdot\times\Xi)}$ (resp. $\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\Xi)}$) can be expressed as a composition of $\tau^{\mathcal{M}}$ (resp. $\tau^{\mathcal{M}_{\mathrm{fin}}}$) with a suitable space transformation, as shown in Noda (2024a, Examples 7.7 and 7.15). Hence, by Noda (2024a, Theorem 7.14), we conclude that both $\tau^{\mathcal{M}(\cdot\times\Xi)}$ and $\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\Xi)}$ are Polish.

(S5) **Discrete measures.** Recall from Section 2.2 the space $\mathcal{M}^{\mathrm{dis}}(S)$ for discrete Radon measures on S and the vague-and-point-process topology on it. We define a structure $\tau^{\mathcal{M}^{\mathrm{dis}}}$ as follows.

- For each boundedly-compact metric space (S, d^S), set $\tau^{\mathcal{M}^{\mathrm{dis}}}(S) \coloneqq \mathcal{M}^{\mathrm{dis}}(S)$.

- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau_f^{\mathcal{M}^{\mathrm{dis}}}(\nu) := \nu \circ f^{-1}$ for each $\nu \in \mathcal{M}^{\mathrm{dis}}(S)$.

We define a metrization of $\tau^{\mathcal{M}^{\mathrm{dis}}}$ by equipping, for each rooted boundedly-compact metric space (S, d^S, ρ_S), $\tau^{\mathcal{M}^{\mathrm{dis}}}(S) = \mathcal{M}^{\mathrm{dis}}(S)$ with the the metric $d^{S,\rho_S}_{\mathcal{M}^{\mathrm{dis}}}$, as recalled from (2.7).

Proposition 3.18. *The structure $\tau^{\mathcal{M}^{\mathrm{dis}}}$ is Polish.*

Proof: Recall the space $\mathscr{P}(S)$ from Section 2.3. Define a structure τ as follows.

- For each boundedly-compact metric space (S, d^S), set $\tau(S) := \mathscr{P}(S)$.
- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau_f(\pi) := \pi \circ (f \times \mathrm{id}_{\mathbb{R}_{>0}})^{-1}$ for each $\pi \in \mathscr{P}(S)$.

We define a metrization of τ by equipping $\tau(S) = \mathscr{P}(S)$ with the metric $d^{S,\rho_S}_{\mathscr{P}}$, as recalled from (2.13). By Theorem 2.18, τ is separable and its metrization is complete. Moreover, by the same argument as that of Noda (2024a, Proof of Theorem 8.9), one can check that τ is continuous in the sense of Definition 3.13.

For each $k \in \mathbb{N}$ and rooted boundedly-compact metric space (S, d^S, ρ_S), we set $\tau_k(S, \rho_S) := \mathscr{P}^{(k)}(S)$ (recall this space from (2.17)). We then obtain a Polish system $(\tau, (\tau_k)_{k\geq 1})$ of $\tau^{\mathcal{M}^{\mathrm{dis}}}$ by Corollary 2.15 and Lemmas 2.19 and 2.20. Therefore, the desired result follows from Theorem 3.17. □

(S6) **Cadlag curves.** Given a boundedly-compact metric space (S, d^S) and an interval I of $\mathbb{R}_{\geq 0}$, we write $D(I, S)$ for the set of cadlag functions from I to S. For every $t > 0$, we write $d^S_{J_1,t}$ for the metric on $D([0,t], S)$ given by Billingsley (1999, Equation (12.13)), which induces the usual J_1-Skorohod topology. Then the Skorohod metric on $D(\mathbb{R}_{\geq 0}, S)$ is defined by setting, for $\xi, \eta \in D(\mathbb{R}_{\geq 0}, S)$,

$$d^S_{J_1}(\xi, \eta) := \int_0^\infty e^{-t}\big(1 \wedge d^S_{J_1,t}(\xi|_{[0,t]}, \eta|_{[0,t]})\big)\, dt. \tag{3.3}$$

The function $d^S_{J_1}$ is indeed a metric on $D(\mathbb{R}_{\geq 0}, S)$ inducing the usual J_1-Skorohod topology (Whitt, 1980, Theorem 2.6). Define a structure τ^{J_1} as follows.

- For each boundedly-compact metric space (S, d^S), set $\tau^{J_1}(S) := D(\mathbb{R}_{\geq 0}, S)$ equipped with the usual J_1-Skorohod topology.
- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau_f^{J_1}(\xi) := f \circ \xi$ for each $f \in D(\mathbb{R}_{\geq 0}, S)$.

We define a metrization of τ by equipping $\tau^{J_1}(S) = D(\mathbb{R}_{\geq 0}, S)$ with the metric $d^S_{J_1}$. The structure τ^{J_1} is Polish (see Noda, 2024a, Theorem 8.14).

In Section 7.3, we will use another structure τ^{Unif} to deal with cadlag curves in the compact-convergence topology.

- For each boundedly-compact metric space (S, d^S), set $\tau^{\mathrm{Unif}}(S) := D([0,1], S)$ equipped with the compact-convergence topology.
- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau_f^{\mathrm{Unif}}(\xi) := f \circ \xi$ for each $f \in D([0,1], S)$.

We define a metrization of τ by equipping $\tau^{\mathrm{Unif}}(S) = D([0,1], S)$ with the uniform metric $d^S_{\tau^{\mathrm{Unif}}}$, i.e., $d^S_{\tau^{\mathrm{Unif}}}(f,g) := \sup_{0\leq t\leq 1} |f(t) - g(t)|$ for each $f, g \in D([0,1], S)$. The structure τ^{Unif} is embedding-continuous, but the resulting Gromov–Hausdorff-type topology is not Polish because the compact-convergence topology on the set of cadlag functions is not Polish.

(S7) **Product structures.** In this framework, it is fairly easy to consider multiple objects. Fix $N \in \mathbb{N}$. Let $(\tau^{(k)})_{k=1}^N$ be a sequence of structures. The *product structure* $\tau = \prod_{k=1}^N \tau^{(k)}$ is defined as follows.

- For each boundedly-compact metric space (S, d^S), set $\tau(S) \coloneqq \prod_{k=1}^N \tau^{(k)}(S)$, equipped with the product topology.
- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau_f \coloneqq \prod_{k=1}^N \tau_f^{(k)}$, that is, $\tau_f\colon \tau(S_1) \to \tau(S_2)$ is a topological embedding given by

$$\tau_f\bigl((a_k)_{k=1}^N\bigr) \coloneqq \bigl(\tau_f^{(k)}(a_k)\bigr)_{k=1}^N.$$

Given a metrization of each structure $\tau^{(k)}$, we define a metrization of the product structure τ by equipping, for each rooted boundedly-compact metric space (S, d_S, ρ_S), $\tau(S) = \prod_{k=1}^N \tau^{(k)}(S)$ with the associated max product metric (cf. (2.6)). Properties of the component structures $\tau^{(k)}$, such as continuity, are naturally inherited by τ, as shown in Noda (2024a, Section 7.1). In particular, if all the structures $\tau^{(k)}$ are Polish, then so is the product structure τ (see Noda, 2024a, Theorem 7.5).

(S8) **Laws of structures.** Fix a Polish structure σ. We define a structure $\tau = \tau_{\mathcal{P}}(\sigma)$ as follows.
- For each boundedly-compact metric space (S, d^S), set $\tau(S) \coloneqq \mathcal{P}(\sigma(S))$ to be the space of probability measures on $\sigma(S)$ equipped with the weak topology.
- For each distance-preserving map $f\colon S_1 \to S_2$ between boundedly-compact metric spaces, set $\tau_f(P) \coloneqq P \circ \sigma_f^{-1}$ for each probability measure P on $\sigma(S_1)$, that is, $\tau_f(P)$ is the pushforward probability measure of P by the topological embedding $\sigma_f\colon \sigma(S_1) \to \sigma(S_2)$.

Since σ is assumed to be Polish, the structure $\tau_{\mathcal{P}}(\sigma)$ is Polish by Noda (2024a, Theorem 8.35).

4. Resistance forms and transition densities

This section is divided into three subsections. In Section 4.1, we recall some basics of the theory of resistance forms and resistance metrics. In Section 4.2, we introduce recurrent resistance metrics, which are assumed for electrical networks in the main results, and present some auxiliary results. Then, in Section 4.3, we prove the precompactness of transition densities of stochastic processes on measured resistance metric spaces, which plays a crucial role in the proof of our main results.

4.1. *Preliminaries.* Following Croydon (2018), in this subsection we recall some basic properties of resistance forms, starting with their definition. The reader is referred to Kigami (2012) for further background. Also, for further study of resistance forms and their extended Dirichlet spaces, see Noda (2024b, Section 3).

Definition 4.1 (Resistance form and resistance metric, Kigami, 2012, Definition 3.1)**.** Let F be a non-empty set. A pair $(\mathcal{E}, \mathcal{F})$ is called a *resistance form* on F if it satisfies the following conditions.

(RF1) $\mathcal{F}$ is a linear subspace of the collection of functions $\{f\colon F \to \mathbb{R}\}$ containing constants, and $\mathcal{E}$ is a non-negative symmetric bilinear form on $\mathcal{F}$ such that $\mathcal{E}(f, f) = 0$ if and only if f is constant on F.

(RF2) Let $\sim$ be the equivalence relation on $\mathcal{F}$ defined by saying $f \sim g$ if and only if $f - g$ is constant on F. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.

(RF3) If $x \neq y$, then there exists a function $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

(RF4) For any $x, y \in F$,

$$R_{(\mathcal{E},\mathcal{F})}(x, y) \coloneqq \sup\left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} \,\middle|\, f \in \mathcal{F},\ \mathcal{E}(f, f) > 0 \right\} < \infty.$$

(RF5) If $\bar{f} \coloneqq (f \wedge 1) \vee 0$, then $\bar{f} \in \mathcal{F}$ and $\mathcal{E}(\bar{f}, \bar{f}) \leq \mathcal{E}(f, f)$ for any $f \in \mathcal{F}$.

For the following definition, recall the effective resistance on an electrical network with a finite vertex set from Levin and Peres (2017, Section 9.4) (see also Kigami, 2001, Section 2.1).

Definition 4.2 (Resistance metric, Kigami, 2001, Definition 2.3.2)**.** A metric R on a non-empty set F is called a *resistance metric* if and only if, for any non-empty finite subset $V \subseteq F$, there exists an electrical network G with the vertex set V such that the effective resistance on G coincides with $R|_{V\times V}$.

Theorem 4.3 (Kigami, 2001, Theorem 2.3.6)**.** *Fix a non-empty subset F. There exists a one-to-one correspondence between resistance forms $(\mathcal{E},\mathcal{F})$ on F and resistance metrics R on F via $R = R_{(\mathcal{E},\mathcal{F})}$. In other words, a resistance form $(\mathcal{E},\mathcal{F})$ is characterized by $R_{(\mathcal{E},\mathcal{F})}$ given in (RF4).*

In the assumptions for the main results of this article, we consider effective resistance between sets. This is precisely defined below.

Definition 4.4 (Effective resistance between sets)**.** Fix a resistance form $(\mathcal{E},\mathcal{F})$ on F and write R for the corresponding resistance metric. For sets $A, B \subseteq F$, we define

$$R(A,B) := \left(\inf\{\mathcal{E}(f,f) : f \in \mathcal{F},\ f|_A \equiv 1,\ f|_B \equiv 0\}\right)^{-1},$$

which is defined to be zero if the infimum is taken over the empty set. Note that by (RF4) we clearly have $R(\{x\},\{y\}) = R(x,y)$.

A simple lower bound on the effective resistance between a point and a subset is given by the metric entropy as described below. Note that, for a metric space (S, d^S) and $\delta > 0$, we write

$$N_{d^S}(S,\delta) := \inf\left\{ \#A \;\middle|\; A \subseteq S,\, S \subseteq \bigcup_{x\in A} D_S(x,\delta) \right\},$$

where we recall from (1.1) that $D_S(x,\delta)$ denotes the closed ball with radius δ centered at x. The family $(N_{d^S}(S,\delta))_{\delta>0}$ is called the metric entropy of S (cf. Marcus and Rosen, 2006).

Lemma 4.5 (Kigami, 2012, Theorem 5.3)**.** *For any $x \in F$ and $r > 0$,*

$$R(x, B_R(x,r)^c) \geq \frac{r}{4N_R(F, r/2)}$$

We will henceforth assume that we have a non-empty set F equipped with a resistance form $(\mathcal{E},\mathcal{F})$, and denote the corresponding resistance metric by R. Furthermore, we assume that (F,R) is locally compact and separable, and the resistance form $(\mathcal{E},\mathcal{F})$ is regular, as described by the following.

Definition 4.6 (Regular resistance form, Kigami, 2012, Definition 6.2)**.** Let $C_c(F)$ be the collection of compactly supported, continuous functions on (F,R) equipped with the compact-convergence topology. A resistance form $(\mathcal{E},\mathcal{F})$ on F is called *regular* if and only if $\mathcal{F}\cap C_c(F)$ is dense in $C_c(F)$.

We next introduce related Dirichlet forms and stochastic processes. First, suppose that we have a Radon measure μ of full support on (F,R). Let $\mathcal{B}(F)$ be the Borel σ-algebra on (F,R) and $\mathcal{B}^\mu(F)$ be the completion of $\mathcal{B}(F)$ with respect to μ. Two extended real-valued functions are said to be μ-equivalent if they coincide outside a μ-null set. The space $L^2(F,\mu)$ consists of μ-equivalence classes of square-integrable $\mathcal{B}^\mu(F)$-measurable extended real-valued functions on F. Now, we define a bilinear form $\mathcal{E}_1$ on $\mathcal{F}\cap L^2(F,\mu)$ by setting

$$\mathcal{E}_1(f,\, g) := \mathcal{E}(f,\, g) + \int_F fg\, d\mu.$$

Then $(\mathcal{F}\cap L^2(F,\mu), \mathcal{E}_1)$ is a Hilbert space (see Kigami, 2001, Theorem 2.4.1). We write $\mathcal{D}$ to be the closure of $\mathcal{F}\cap C_c(F)$ with respect to $\mathcal{E}_1$. Under the assumption that $(\mathcal{E},\mathcal{F})$ is regular, we then have from Kigami (2012, Theorem 9.4) that $(\mathcal{E},\mathcal{D})$ is a regular Dirichlet form on $L^2(F,\mu)$ (see Fukushima et al., 2011 for the definition of a regular Dirichlet form). Moreover, standard theory gives us the existence of an associated Hunt process $((X_t)_{t\geq 0}, \{P_x\}_{x\in F})$ (e.g. Fukushima et al., 2011, Theorem 7.2.1). We refer to this Hunt process as the *(Hunt) process associated with* (F, R, μ). Note that such a process is, in general, only specified uniquely for starting points outside a set of zero capacity. However, in this setting, every point has strictly positive capacity (see Kigami, 2012, Theorem 9.9), and so the process is defined uniquely everywhere. From Kigami (2012, Theorem 10.4), X admits a (unique) jointly continuous transition density with respect to μ.

4.2. *Recurrent resistance metrics and auxiliary results.* In this subsection, we introduce recurrent resistance metrics, which we consider throughout this article. We then present some auxiliary results that are used in the proofs of the main results.

Definition 4.7 (Recurrent resistance metric)**.** Let (F, R) be a boundedly-compact resistance metric space. We say that R is recurrent if and only if $\lim_{r\to\infty} R(\rho, B_R(\rho, r)^c) = \infty$ for some (or, equivalently, any) $\rho \in F$.

Henceforth, we write $\mathbb{F}$ for the collection of $(F, R, \rho, \mu) \in \mathbb{G}$ such that (F, R) is a recurrent resistance metric space and μ is of full support. Fix $(F, R, \rho, \mu) \in \mathbb{F}$. We note that the resistance form $(\mathcal{E}, \mathcal{F})$ associated with (F, R) is regular and the Dirichlet form $(\mathcal{E}, \mathcal{D})$ associated with (F, R, μ) is recurrent (see Noda, 2024b, Corollary 3.22 and Croydon, 2018, Lemma 2.3). Write $(X = (X_t)_{t\geq 0}, \{P_x\}_{x\in F})$ for the process associated with (F, R, μ).

The first Lemma regards traces of X onto subsets. For further details, the reader is referred to Fukushima et al. (2011); Noda (2024b). For a non-empty closed subset B of F, we define a positive continuous additive functional (PCAF) A of X by setting $A(t) := \int_0^t 1_B(X(s))\, ds$ and γ to be the right-continuous inverse of A, i.e., $\gamma(t) := \inf\{s > 0 : A(s) > t\}$. Then the *trace* of X onto B is given by setting $\operatorname{tr}_B X := X \circ \gamma$.

Lemma 4.8. *Fix a non-empty open subset U of F. Set $B := \operatorname{cl}(U)$, i.e., the closure of U in F. Let $(Y, \{Q_x\}_{x\in B})$ be the Hunt process associated with $(B, R|_{B\times B}, \mu|_B)$. For any $x \in B$, it holds that $P_x(\operatorname{tr}_B X \in \cdot) = Q_x(Y \in \cdot)$ as probability measures on $D(\mathbb{R}_{\geq 0}, B)$ equipped with the usual J_1-Skorohod topology.*

Proof: The metric $R_{B\times B}$ is a regular resistance metric on B by Kigami (2012, Theorem 8.4). Since μ is of full support on F, one can check that $\mu|_B$ is of full support on B. Hence, the result follows immediately from Noda (2024b, Theorem 3.34). ☐

By combining the trace technique described above with the following estimates of exit times of X from balls, various analyses of processes on recurrent resistance metric spaces essentially reduce to analyses of processes on compact resistance metric spaces. For a subset $A \subseteq F$, we denote by τ_A the first exit time of X from A, i.e.,

$$\tau_A := \inf\{t > 0 \mid X(t) \notin A\}.$$

Lemma 4.9 (Croydon, 2018, Lemma 4.2)**.** *For any $x \in F$, $\delta \in (0, R(x, B_R(x, r)^c))$ and $T \geq 0$, it holds that*

$$P_x(\tau_{B_R(x,r)} \leq T) \leq \frac{4\delta}{R(x, B_R(x, r)^c)} + \frac{4T}{\mu(B_R(x, \delta))(R(x, B_R(x, r)^c) - \delta)}.$$

We now prove new results, which provide lower bounds for a probability that X is at a point at a fixed time.

Lemma 4.10. *Assume that (F, R) is compact. Then, for any $x \in F$ and $t \geq 0$,*

$$P_x(X(t) = x) \geq \frac{\mu(\{x\})}{\mu(F)}.$$

Proof: Fix $x \in F$ and $t \geq 0$. When F is a finite set, then X is simply a Markov chain on an electrical network with vertex set F (see Noda, 2024b, Theorem 4.2). So, the result is proven by the same argument as the proof of Fontes et al. (2002, Lemma 2.5). To extend the result to a general compact resistance metric space (F, R), we approximate the space by finite subsets. To this end, we first show that the following statement holds for each $n \in \mathbb{N}$.

(A) There exist $k_n \in \mathbb{N}$ and a finite collection of disjoint non-empty Borel subsets $\{F_{i,n}\}_{i=1}^{k_n}$ such that $F = \bigcup_{i=1}^{k_n} F_{i,n}^{\langle n^{-1} \rangle}$, $F_{1,n} = B_R(x, n^{-1})$, $\mu(F_{i,n}) > 0$, $\mu(\bigcup_{i=1}^{k_n} F_{i,n}) = \mu(F)$, and $\operatorname{diam} F_{i,n} \leq 2/n$, where $\operatorname{diam} A$ denotes the diameter of a subset A. (Recall the closed neighborhood $\cdot^{\langle \varepsilon \rangle}$ from (2.2).)

Let $\{x_i\}_{i=1}^{k}$ be a finite subset of F such that $F = \bigcup_{i=1}^{k} B_R(x_i, n^{-1})$ and $x_1 = x$. We then set $F_1 \coloneqq B_R(x_1, n^{-1})$ and inductively, for $i \geq 2$, $F_i \coloneqq B_R(x_i, n^{-1}) \setminus \bigcup_{j=1}^{i-1} F_i$. Define I_0 to be the collection of i such that $\mu(F_i) = 0$, and define $I_+ \coloneqq \{1, \dots, k\} \setminus I_0$. It remains to prove that $F = \bigcup_{i \in I_+} F_i^{\langle n^{-1} \rangle}$. Suppose that there exists an $x \in F \setminus \bigcup_{i \in I_+} F_i^{\langle n^{-1} \rangle}$. This implies that $B_R(x, n^{-1}) \cap \bigcup_{i \in I_+} F_i = \emptyset$. Hence, it is the case that $B_R(x, n^{-1}) \subseteq \bigcup_{i \in I_0} F_i$. Since μ is of full support, it follows that $\mu(\bigcup_{i \in I_0} F_i) > 0$, which is a contradiction. Thus, we obtain that $F = \bigcup_{i \in I_+} F_i^{\langle n^{-1} \rangle}$.

For each $n \in \mathbb{N}$, we let $\{F_{i,n}\}_{i=1}^{k_n}$ be a finite collection of disjoint Borel subsets satisfying (A). Choose an element $x_{i,n} \in F_{i,n}$ for each i with $x_{1,n} = x$. We then write $F_n \coloneqq \{x_{i,n}\}_{i=1}^{k_n}$, $R_n \coloneqq R|_{F_n \times F_n}$, and $x_n \coloneqq x$. We define a fully-supported Radon measure μ_n on F_n by setting $\mu_n(\{x_{i,n}\}) \coloneqq \mu(F_{i,n})$. Noting that (F_n, R_n) is a recurrent resistance metric space, we let $(X_n, \{P_y^{(n)}\}_{y \in F_n})$ be the process associated with (F_n, R_n, μ_n). It is not difficult to check that

$$(F_n, R_n, x_n, \mu_n) \to (F, R, x, \mu)$$

in $\mathbb{G}_c$ with respect to the (pointed) Gromov–Hausdorff–Prohorov topology (recall this topology from Section 3.3). Thus, by Croydon (2018, Theorem 1.2) and Theorem 3.9, it is possible to embed (F_n, R_n) and (F, R) isometrically into a common rooted compact metric space (K, d^K, x_K) in such a way that $x_n = x = x_K$ as elements of K, $F_n \to F$ in the Hausdorff topology as subsets of K, $\mu_n \to \mu$ weakly as measures on K, and $P_{x_n}^{(n)}(X_n \in \cdot) \xrightarrow{\mathrm{d}} P_x(X \in \cdot)$ as probability measures on $D(\mathbb{R}_{\geq 0}, K)$. By the quasi-left-continuity of X, X is continuous at the fixed time t almost surely. Hence, we have that $X_n(t) \xrightarrow{\mathrm{d}} X(t)$. Noting that $\{x\} = \{x_n\} = \{x_K\} \subseteq K$ is closed, we deduce that

$$P_x(X(t) = x) \geq \limsup_{n \to \infty} P_{x_n}^{(n)}(X_n(t) = x_n). \tag{4.1}$$

As noted at the beginning, the desired result holds for any finite resistance metric space. Thus, it holds that

$$P_{x_n}^{(n)}(X_n(t) = x_n) \geq \mu_n(\{x_n\})/\mu_n(F_n) = \mu(B_R(x, n^{-1}))/\mu(F)$$

Combining this with (4.1), we obtain the desired result. □

Using the trace technique, the lower bound given above is improved as follows.

Proposition 4.11. *For any $x \in F$, $t \geq 0$, and $\varepsilon > 0$,*

$$P_x(X(t) = x) \geq \frac{\mu(\{x\})}{\mu(D_R(x, \varepsilon))} - P_x(\tau_{B_R(x,\varepsilon)} \leq t).$$

Proof: Fix $x \in F$ and $\varepsilon > 0$, and write $B \coloneqq \operatorname{cl}(B_R(x, \varepsilon))$. Let $(Y, \{Q_x\}_{x \in B})$ be the Hunt process associated with $(B, R|_{B \times B}, \mu|_B)$. If $\tau_B > t$, then we have that $\operatorname{tr}_B X(t) = X(t)$. Thus, by Lemma 4.8, we deduce that

$$\begin{aligned} P_x(X(t) = x) &\geq P_x(X(t) = x,\, \tau_B > t) \\ &= P_x(\operatorname{tr}_B X(t) = x,\, \tau_B > t) \\ &\geq P_x(\operatorname{tr}_B X(t) = x) - P_x(\tau_B \leq t) \\ &= Q_x(Y(t) = x) - P_x(\tau_B \leq t) \\ &\geq \frac{\mu(\{x\})}{\mu(B)} - P_x(\tau_B \leq t), \end{aligned}$$

where we apply Lemma 4.10 to obtain the last inequality. Since $B_R(x, \varepsilon) \subseteq B \subseteq D_R(x, \varepsilon)$, we obtain the desired result. □

As mentioned in the last paragraph in Section 4.1, the process X admits a unique jointly continuous transition density $p\colon \mathbb{R}_{>0} \times F \times F \to \mathbb{R}_{\geq 0}$ with respect to μ. The following result is used later to show the non-triviality of the (sub-)aging function.

Proposition 4.12. *For any $x \in F$ and $t > 0$, $p(t, x, x) > 0$.*

Proof: Fix $t > 0$ and $x \in F$. Write $(\mathcal{E}, \mathcal{D})$ for the Dirichlet form associated with (F, R, μ). Suppose that $p(t, x, \cdot)$ is constant. Since $(\mathcal{E}, \mathcal{D})$ is recurrent, X is conservative (see Fukushima et al., 2011, Lemma 1.6.5 and Excercise 4.5.1). It follows that $p(t, x, y) > 0$ for some $y \in F$. Hence, we obtain that $p(t, x, x) > 0$. Suppose that $p(t, x, \cdot)$ is not constant. Since $p(t, x, \cdot) \in \mathcal{D}$ (see Kigami, 2012, Theorem 10.4), by Fukushima et al. (2011, Lemma 1.3.3) and the Chapman-Kolmogorov equation, we deduce that

$$t\, \mathcal{E}(p(t, x, \cdot), p(t, x, \cdot)) \leq \int_F p\bigl(t/2, x, y\bigr)^2 \mu(dy) = p(t, x, x).$$

Since $p(t, x, \cdot)$ is not constant, it holds from (RF1) that $\mathcal{E}(p(t, x, \cdot), p(t, x, \cdot)) > 0$. Hence, we complete the proof. □

4.3. *Transition densities of processes on resistance metric spaces.* In this subsection, we prove that when measured resistance metric spaces converge in the local Gromov–Hausdorff-vague topology, the family of the transition densities of the associated processes is precompact (Proposition 4.13) in the sense that it is uniformly bounded and equicontinuous on every compact subset.

Fix $(F_n, R_n, \rho_n, \mu_n) \in \mathbb{F}$ for each $n \in \mathbb{N}$ and $(F, R, \rho, \mu) \in \mathbb{F}$. We define $(p_n(t, x, y))_{t>0, x, y \in F_n}$ to be the jointly continuous transition density of the process associated with (F_n, R_n, μ_n). Similarly, we define $(p(t, x, y))_{t>0, x, y \in F}$. To state the result, we introduce new notation. Given a rooted boundedly-compact metric space (M, d^M, ρ_M), we set, for $T > 0$ and $r > 0$, $K_{d^M}(T, r) := [T, \infty) \times D_M(\rho_S, r) \times D_M(\rho_S, r)$. For a function $f\colon \mathbb{R}_{>0} \times M \times M \to \mathbb{R}$, we define

$$\mathfrak{s}_M(f, T, r) := \sup_{(t,x,y) \in K_{d^M}(T,r)} f(t, x, y), \tag{4.2}$$

and, for each $\delta > 0$,

$$\mathfrak{d}_M(f, T, r, \delta) := \sup \left\{ |f(t, x, y) - f(t', x', y')| \;\middle|\; \begin{array}{l} (t, x, y), (t', x', y') \in K_{d^M}(T, r), \\ |t - t'| \vee d^M(x, x') \vee d^M(y, y') \leq \delta \end{array} \right\}. \tag{4.3}$$

Proposition 4.13. *Assume that $(F_n, R_n, \rho_n, \mu_n)$ converges to (F, R, ρ, μ) in the local Gromov–Hausdorff-vague topology. Then, for any $T > 0$ and $r > 0$,*

$$\sup_{n \geq 1} \mathfrak{s}_{F_n}(p_n, T, r) < \infty, \quad \lim_{\delta \downarrow 0} \sup_{n \geq 1} \mathfrak{d}_{F_n}(p_n, T, r, \delta) = 0.$$

Proof: Fix $T > 0$ and $r > 0$. Note that the local Gromov–Hausdorff-vague convergence and μ being of full support imply that

$$c_1 := \inf_{n\geq 1} \inf_{x \in D_{R_n}(\rho_n, r)} \mu_n(D_{R_n}(x,r)) > 0$$

(cf. Athreya et al., 2016, Corollary 5.7). By Kigami (2012, Equation (10.4)), it holds that, for any $t \geq T$ and $x \in D_{R_n}(\rho_n, r)$,

$$p_n(t,x,x) \leq 2rt^{-1} + \sqrt{2}\mu_n(D_{R_n}(\rho_n,r))^{-1} \leq 2rT^{-1} + \sqrt{2}c_1^{-1}.$$

Using the Chapman-Kolmogorov equation and the Cauchy-Schwarz inequality, we deduce that, for any $(t,x,y) \in K_{R_n}(T,r)$,

$$\begin{aligned} p_n(t,x,y) &= \int p_n(t/2,x,z)p_n(t/2,z,y)\,\mu_n(dz) \\ &\leq \left(\int p_n(t/2,x,z)^2\,\mu(dz)\right)^{1/2}\left(\int p_n(t/2,z,y)^2\,\mu(dz)\right)^{1/2} \\ &= p_n(t,x,x)^{1/2}p_n(t,y,y)^{1/2} \\ &\leq 2rT^{-1} + \sqrt{2}c_1^{-1}, \end{aligned}$$

which shows the first result. By following the proof of Kigami (2012, Theorem 10.4) (specifically, the top of page 44 of Kigami, 2012), we obtain that

$$\begin{aligned} |p_n(t,x,y) - p_n(t',x',y')| \leq{}& \sqrt{\frac{p_n(t,x,x)R_n(y,y')}{t}} + \sqrt{\frac{p_n(t,y',y')R_n(x,x')}{t}} \\ &+ 2|t-t'|\frac{\sqrt{p_n(s/2,x',x')p_n(s/2,y',y')}}{s}, \end{aligned}$$

where s is a value between t and t'. Thus, for $(t,x,y),(t',x',y') \in K_{R_n}(T,r)$ with $|t-t'| \vee R_n(x,x') \vee R_n(y,y') \leq \delta$, we deduce that

$$|p_n(t,x,y) - p_n(t',x',y')| \leq 2\sqrt{T^{-1}\mathfrak{s}_{F_n}(p_n,T,r)\delta} + 2\delta T^{-1}\mathfrak{s}_{F_n}(p_n,T/2,r).$$

This, combined with the first result, yields the second result. □

5. Aging and sub-aging for deterministic traps

In this section, we prove aging and sub-aging results for processes on resistance metric spaces associated with deterministic traps. Throughout this section, we fix a sequence $(F_n,R_n,\rho_n)_{n\geq 1}$ of rooted recurrent resistance metric spaces and a rooted recurrent resistance metric space (F,R,ρ).

5.1. *Aging result.* We first prove an aging result, Theorem 5.7. Due to the length of the proof, we divide this subsection into two smaller sections.

We let $\nu_n = \sum_{i \in I_n} v_i^{(n)} \delta_{x_i^{(n)}}$ be a fully-supported discrete measure on F_n and $\nu = \sum_{i\in I} v_i \delta_{x_i}$ be a fully-supported discrete measure on F. We write $(X_n^{\nu_n}, \{P_x^{\nu_n}\}_{x\in F_n})$ for the Hunt process associated with (F_n,R_n,ν_n) and $p_n^{\nu_n}$ for the jointly continuous transition density of $X_n^{\nu_n}$ with respect to ν_n. Similarly, we write $(X^\nu, \{P_x^\nu\}_{x\in F})$ for the Hunt process associated with (F,R,ν) and p^ν for the jointly continuous transition density of X^ν with respect to ν. For a subset $A \subseteq F$, we denote by τ_A^ν the first exit time of X^ν from A, i.e.,

$$\tau_A^\nu := \inf\{t > 0 \mid X^\nu(t) \notin A\}.$$

We similarly define $\tau_A^{\nu_n}$ for the first exit time of $X_n^{\nu_n}$ from $A \subseteq F_n$. We define an aging function Φ^ν associated with X^ν and ρ by setting

$$\Phi^\nu(s,t) := P_\rho^\nu(X^\nu(s) = X^\nu(t)).$$

Similarly, we define $\Phi_n^{\nu_n}$ to be the aging function associated with $X_n^{\nu_n}$ and ρ_n.

Throughout this subsection, we suppose that the following condition is satisfied.

Assumption 5.1. *It holds that*

$$\big(F_n, R_n, \rho_n, \nu_n, P_{\rho_n}^{\nu_n}(X_n^{\nu_n} \in \cdot)\big) \to \big(F, R, \rho, \nu, P_\rho^{\nu}(X^{\nu} \in \cdot)\big).$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau_{\mathcal{P}}(\tau^{J_1}))$

Under Assumption 5.1, by Theorem 3.9, we may assume that (F_n, R_n, ρ_n) and (F, R, ρ) are embedded isometrically into a common rooted boundedly-compact metric space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements of M, $F_n \to F$ in the Fell topology as closed subsets in M, and $\nu_n \to \nu$ in the vaguely-and-point-process topology as discrete measures on M, and $X_n^{\nu_n} \xrightarrow{\mathrm{d}} X^{\nu}$ in $D(\mathbb{R}_{\geq 0}, M)$. In Sections 5.1.1 and 5.1.2 below, we assume this embedding.

5.1.1. *Precompactness of the aging functions.* Here, we prove that the family $(\Phi_n^{\nu_n})_{n\geq 1}$ of the aging functions is precompact (Proposition 5.5).

Lemma 5.2. *The following statements hold.*

(i) *For each* $l > 0$,

$$\lim_{r\to\infty} \sup_{n\geq 1} P_{\rho_n}^{\nu_n}\Big(\tau_{B_{R_n}(\rho_n, r)}^{\nu_n} \leq l\Big) = 0. \tag{5.1}$$

(ii) *For any* $x_n \in F_n$ *and* $x \in F$ *such that* $x_n \to x$ *in* M, *and any* $\delta > 0$,

$$\lim_{\eta\to 0} \limsup_{n\to\infty} P_{x_n}^{\nu_n}(\tau_{B_{R_n}(x_n,\delta)}^{\nu_n} \leq \eta) = 0.$$

Proof: (i). This is a consequence of the weak convergence of $X_n^{\nu_n}$ to X^{ν} in the usual J_1-Skorohod topology (see Kallenberg, 2021, Theorem 23.8).

(ii). If $x_n \to x$ in M, then we have from Croydon (2018, Theorem 1.2) that $P_{x_n}^{\nu_n}(X_n^{\nu_n} \in \cdot) \to P_x^{\nu}(X^{\nu} \in \cdot)$ as probability measures on $D(\mathbb{R}_{\geq 0}, M)$. This yields that

$$\big(F_n, R_n, x_n, \nu_n, P_{x_n}^{\nu_n}(X_n^{\nu_n} \in \cdot)\big) \to \big(F, R, x, \nu, P_x^{\nu}(X^{\nu} \in \cdot)\big).$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau_{\mathcal{P}}(\tau^{J_1}))$. Hence, it is enough to show the result for $x_n = \rho_n$ and $x = \rho$. Fix $\delta > 0$. For $r > 0$, we write $(X_n^{\nu_n^{(r)}}, \{P_x^{\nu_n^{(r)}}\}_{x\in F_n^{(r)}})$ for the Hunt process associated with $(F_n^{(r)}, R_n^{(r)}, \nu_n^{(r)})$, where we recall the restriction operator $\cdot^{(r)}$ from (1.2). We write $\tau_A^{\nu_n^{(r)}}$ for the first exit time of $X_n^{\nu_n^{(r)}}$ from a set $A \subseteq F_n$. Using Lemma 4.8, we deduce that, for any $l > \eta$ and $r > \delta$,

$$P_{\rho_n}^{\nu_n}\big(\tau_{B_{R_n}(\rho_n,\delta)}^{\nu_n} \leq \eta\big) \leq P_{\rho_n}^{\nu_n}\big(\tau_{B_{R_n}(\rho_n,r)}^{\nu_n} \leq l\big) + P_{\rho_n}^{\nu_n^{(r)}}\Big(\tau_{B_{R_n^{(r)}}(\rho_n,\delta)}^{\nu_n^{(r)}} \leq \eta\Big). \tag{5.2}$$

Note that the convergence of F_n to F in the Fell topology implies that

$$c_r := \sup_{n\geq 1} N_{R_n^{(r)}}(F_n^{(r)}, \delta/2) < \infty$$

(cf. Noda, 2024a, Theorem 3.13). Lemma 4.5 yields that $R_n^{(r)}(\rho_n, B_{R_n^{(r)}}(\rho_n, \delta)^c) \geq \delta/(4c_r) =: c_r'$ for all $n \geq 1$. By Lemma 4.9, we obtain that, for any $\varepsilon < c_r'$,

$$P_{\rho_n}^{\nu_n^{(r)}}\Big(\tau_{B_{R_n^{(r)}}(\rho_n,\delta)}^{\nu_n^{(r)}} \leq \eta\Big) \leq \frac{4\varepsilon}{c_r'} + \frac{4\eta}{\nu_n(B_{R_n}(\rho_n, \varepsilon))(c_r' - \varepsilon)}. \tag{5.3}$$

By the vague convergence of ν_n to ν and the assumption that ν is of full support, we have that $\inf_{n\geq 1} \nu_n(B_{R_n}(\rho_n, \varepsilon)) > 0$. Therefore, using (i), (5.2), and (5.3), we deduce the desired result. □

Note that $([l^{-1}, l]^2)_{l \geq 1}$ is a sequence of compact subsets increasing to $\mathbb{R}^2_{>0}$. To prove the equicontinuity of $(\Phi_n^{\nu_n})_{n \geq 1}$ on $[l^{-1}, l]^2$, we divide $[l^{-1}, l]^2$ into a part near the diagonal and the other part as follows: for $l \in \mathbb{N}$ and $\eta > 0$,

$$T_1(l, \eta) := \{(s, t) \in [l^{-1}, l]^2 \mid |t - s| \leq \eta\},$$
$$T_2(l, \eta) := \{(s, t) \in [l^{-1}, l]^2 \mid \eta \leq |t - s|\}.$$

In Lemmas 5.3 and 5.4 below, we prove the equicontinuity of $(\Phi_n^{\nu_n})_{n \geq 1}$ on $T_1(l, \eta)$ and $T_2(l, \eta)$, respectively.

Lemma 5.3. *For every $l \in \mathbb{N}$ and $\eta > 0$,*

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \sup_{(s,t) \in T_1(l,\eta)} |\Phi_n^{\nu_n}(s, t) - 1| = 0.$$

Proof: Fix $(s, t) \in T_1(l, \eta)$ with $s \leq t$. Since $1 = P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) \in F_n)$, it holds that

$$\begin{aligned}|\Phi_n^{\nu_n}(s, t) - 1| &= |P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) = X_n^{\nu_n}(t)) - P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) \in F_n)| \\ &\leq 2P_{\rho_n}^{\nu_n}(\tau^{\nu_n}_{B_{R_n}(\rho_n, r)} \leq l) + \left|P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) = X_n^{\nu_n}(t) \in F_n^{(r)}) - P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) \in F_n^{(r)})\right|.\end{aligned}$$

By Lemma 5.2(i), it is enough to show that, for all but countably many $r > 0$,

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \sup_{(s,t) \in T_1(l,\eta)} \left|P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) = X_n^{\nu_n}(t) \in F_n^{(r)}) - P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) \in F_n^{(r)})\right| = 0.$$

Fix $r > 0$ such that the boundary of $F^{(r)}$ contains no atoms of ν. We have from Proposition 4.13 that

$$A := \sup_{n \geq 1} \sup\{p_n^{\nu_n}(t, x, y) \mid l^{-1} \leq t \leq l,\, x, y \in F_n^{(r)}\} < \infty.$$

Using the transition density $p_n^{\nu_n}$ and the Markov property, we deduce that, for any $\varepsilon > 0$,

$$\begin{aligned}&\left|P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) = X_n^{\nu_n}(t) \in F_n^{(r)}) - P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(s) \in F_n^{(r)})\right| \\ &= \int_{F_n^{(r)}} p_n^{\nu_n}(s, \rho_n, x)\,(1 - P_x^{\nu_n}(X_n^{\nu_n}(t - s) = x))\,\nu_n(dx) \\ &\leq A\,M_\varepsilon^{(r)}(\mathfrak{p}(\nu_n)) + A \sum_{i \in I_n(\varepsilon)} \left(1 - P^{\nu_n}_{x_i^{(n)}}(X_n^{\nu_n}(t - s) = x_i^{(n)})\right) v_i^{(n)},\end{aligned}$$

where we define $I_n(\varepsilon) := \{i \in I_n \mid x_i^{(n)} \in F_n^{(r)},\, v_i^{(n)} \geq \varepsilon\}$ and recall $M_\varepsilon^{(r)}$ from (2.11). By Theorem 2.14, it holds that $\lim_{\varepsilon \to 0} \limsup_{n \to \infty} M_\varepsilon^{(r)}(\mathfrak{p}(\nu_n)) = 0$. Thus, it suffices to show that, for all but countably many $\varepsilon > 0$,

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \sup_{(s,t) \in T_1(l,\eta)} \sum_{i \in I_n(\varepsilon)} \left(1 - P^{\nu_n}_{x_i^{(n)}}(X_n^{\nu_n}(t - s) = x_i^{(n)})\right) v_i^{(n)} = 0.$$

Proposition 4.11 yields that, for any $(s, t) \in T_1(l, \eta)$ and $\varepsilon, \varepsilon' > 0$,

$$\begin{aligned}&\sum_{i \in I_n(\varepsilon)} \left(1 - P^{\nu_n}_{x_i^{(n)}}(X_n^{\nu_n}(t - s) = x_i^{(n)})\right) v_i^{(n)} \\ &\leq \sum_{i \in I_n(\varepsilon)} \left(1 - \frac{v_i^{(n)}}{\nu_n(D_{R_n}(x_i^{(n)}, \varepsilon'))} + P^{\nu_n}_{x_i^{(n)}}\left(\tau^{\nu_n}_{B_{R_n}(x_i^{(n)}, \varepsilon')} \leq \eta\right)\right) v_i^{(n)} \\ &\leq \sum_{i \in I_n(\varepsilon)} \nu_n(D_{R_n}(x_i^{(n)}, \varepsilon') \setminus \{x_i^{(n)}\}) + W^{(r)}(\mathfrak{p}(\nu_n)) \sum_{i \in I_n(\varepsilon)} P^{\nu_n}_{x_i^{(n)}}\left(\tau^{\nu_n}_{B_{R_n}(x_i^{(n)}, \varepsilon')} \leq \eta\right),\end{aligned}$$

where we recall $W^{(r)}$ from (2.12). Set $W := \sup_{n\geq 1} W^{(r)}(\mathfrak{p}(\nu_n)) + W^{(r)}(\mathfrak{p}(\nu)) + 1$, which is finite by Theorem 2.22. Note that, by the definition of $W^{(r)}$, there are no atoms of $\mathfrak{p}(\nu_n)$ in $F_n^{(r)} \times [W, \infty)$. We choose $\varepsilon > 0$ so that the boundary of $K := F^{(r)} \times [\varepsilon, W]$ contains no atoms of $\mathfrak{p}(\nu)$. It is then the case that $\mathfrak{p}(\nu_n)|_K \to \mathfrak{p}(\nu)|_K$ weakly. Hence, if we write $I(\varepsilon) := \{i \in I \mid (x_i, v_i) \in K\}$, then, by Proposition 2.4, there exists a bijective map $f_n \colon I(\varepsilon) \to I_n(\varepsilon)$ (at least, for all sufficiently large n) such that $x^{(n)}_{f_n(i)} \to x_i$ and $v^{(n)}_{f_n(i)} \to v_i^{(n)}$ for each $i \in I(\varepsilon)$. Since $I(\varepsilon)$ is a finite set, we deduce from Lemma 5.2(ii) that, for each $\varepsilon' > 0$,

$$\begin{aligned}
&\lim_{\eta\to 0}\limsup_{n\to\infty} W^{(r)}(\mathfrak{p}(\nu_n)) \sum_{i\in I_n(\varepsilon)} P^{\nu_n}_{x_i^{(n)}}\left(\tau^{\nu_n}_{B_{R_n}(x_i^{(n)},\varepsilon')} \leq \eta\right)\\
&\leq W \sum_{i\in I(\varepsilon)} \lim_{\eta\to 0}\limsup_{n\to\infty} P^{\nu_n}_{x^{(n)}_{f_n(i)}}\left(\tau^{\nu_n}_{B_{R_n}(x^{(n)}_{f_n(i)},\varepsilon')} \leq \eta\right)\\
&= 0.
\end{aligned}$$

Using Proposition 2.4, we obtain that

$$\begin{aligned}
&\lim_{\varepsilon'\to 0}\limsup_{n\to\infty} \sum_{i\in I_n(\varepsilon)} \nu_n\big(D_{R_n}(x_i^{(n)},\varepsilon') \setminus \{x_i^{(n)}\}\big)\\
&= \sum_{i\in I(\varepsilon)} \lim_{\varepsilon'\to 0}\limsup_{n\to\infty} \nu_n\big(D_{R_n}(x^{(n)}_{f_n(i)},\varepsilon') \setminus \{x^{(n)}_{f_n(i)}\}\big)\\
&= 0.
\end{aligned}$$

Therefore, we complete the proof. □

Lemma 5.4. *For every $l \in \mathbb{N}$ and $\eta > 0$,*

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \sup_{\substack{(s,t)\in T_2(l,\eta)\\ |t-t'|\vee|s-s'|<\delta}} |\Phi_n^{\nu_n}(s,t) - \Phi_n^{\nu_n}(s',t')| = 0.$$

Proof: Note that the vague convergence $\nu_n \to \nu$ implies that, for each $r > 0$,

$$A_r := \sup_{n\geq 1} \nu_n(F_n^{(r)}) < \infty.$$

Fix $(s,t), (s',t') \in T_2(l,\eta)$ with $|s-s'| \vee |t-t'| \leq \delta$, $s < t$, and $s' < t'$. Using the transition density, we obtain that

$$\begin{aligned}
&|\Phi_n^{\nu_n}(s,t) - \Phi_n^{\nu_n}(s',t')|\\
&\leq 2P^{\nu_n}_{\rho_n}(\tau^{\nu_n}_{B_{R_n}(\rho_n,r)} \leq l)\\
&\qquad + \left|P^{\nu_n}_{\rho_n}(X_n^{\nu_n}(s) = X_n^{\nu_n}(t) \in F_n^{(r)}) - P^{\nu_n}_{\rho_n}(X_n^{\nu_n}(s') = X_n^{\nu_n}(t') \in F_n^{(r)})\right|\\
&\leq 2P^{\nu_n}_{\rho_n}(\tau^{\nu_n}_{B_{R_n}(\rho_n,r)} \leq l)\\
&\qquad + \int_{F_n^{(r)}} \nu_n(\{x\})\big|p_n^{\nu_n}(s,\rho_n,x)p_n^{\nu_n}(t-s,x,x) - p_n^{\nu_n}(s',\rho_n,x)p_n^{\nu_n}(t'-s',x,x)\big|\,\nu_n(dx)\\
&\leq 2P^{\nu_n}_{\rho_n}(\tau^{\nu_n}_{B_{R_n}(\rho_n,r)} \leq l)\\
&\qquad + A_r^2 \sup_{x\in F_n^{(r)}} \big|p_n^{\nu_n}(s,\rho_n,x)p_n^{\nu_n}(t-s,x,x) - p_n^{\nu_n}(s',\rho_n,x)p_n^{\nu_n}(t'-s',x,x)\big|.
\end{aligned} \tag{5.4}$$

For any $x \in F_n^{(r)}$, we have that

$$\begin{aligned}
&\left|p_n^{\nu_n}(s,\rho_n,x)p_n^{\nu_n}(t-s,x,x) - p_n^{\nu_n}(s',\rho_n,x)p_n^{\nu_n}(t'-s',x,x)\right| \\
&\leq p_n^{\nu_n}(s,\rho_n,x)\left|p_n^{\nu_n}(t-s,x,x) - p_n^{\nu_n}(t'-s',x,x)\right| + p_n^{\nu_n}(t'-s',x,x)\left|p_n^{\nu_n}(s,\rho_n,x) - p_n^{\nu_n}(s',\rho_n,x)\right| \\
&\leq \mathfrak{s}_{F_n}(p_n^{\nu_n}, l^{-1}, r)\mathfrak{d}_{F_n}(p_n^{\nu_n}, \eta, r, 2\delta) + \mathfrak{s}_{F_n}(p_n^{\nu_n}, \eta, r)\mathfrak{d}_{F_n}(p_n^{\nu_n}, l^{-1}, r, \delta),
\end{aligned}$$

where we recall the notation $\mathfrak{s}_{F_n}$ and $\mathfrak{d}_{F_n}$ from (4.2) and (4.3). Thus, we deduce from Proposition 4.13 that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{x\in F_n^{(r)}}\left|p_n^{\nu_n}(s,\rho_n,x)p_n^{\nu_n}(t-s,x,x) - p_n^{\nu_n}(s',\rho_n,x)p_n^{\nu_n}(t'-s',x,x)\right| = 0.$$

From Lemma 5.2(i), (5.4), and the above convergence, we establish the desired result. □

From Lemmas 5.3 and 5.4 above, we deduce the precompactness of $(\Phi_n^{\nu_n})_{n\geq 1}$ as follows. Below, we write $C(\mathbb{R}^2_{\geq 0}, \mathbb{R}_{\geq 0})$ for the space of continuous functions from $\mathbb{R}^2_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ equipped with the compact-convergence topology.

Proposition 5.5. *The family $\{\Phi_n^{\nu_n}\}_{n\geq 1}$ is precompact in $C(\mathbb{R}^2_{>0}, \mathbb{R}_{\geq 0})$.*

Proof: It is enough to show that the family is uniformly bounded and equicontinuous on each compact subset of $\mathbb{R}^2_{>0}$. (See Kallenberg, 2021, Theorem A5.2 for a necessarily and sufficient condition for precompactness in $C(\mathbb{R}^2_{>0}, \mathbb{R}_{\geq 0})$). Obviously, we have that $\sup_{n\geq 1}\sup_{s,t}\Phi_n^{\nu_n}(s,t) \leq 1$. So, it remains to prove that, for every $l \in \mathbb{N}$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\substack{(s,t),(s',t')\in[l^{-1},l]^2\\|t-t'|\vee|s-s'|<\delta}}|\Phi_n^{\nu_n}(s,t) - \Phi_n^{\nu_n}(s',t')| = 0.$$

Fix $\delta, \eta > 0$ with $\eta > 4\delta$. Fix also $(s,t),(s',t') \in [l^{-1},l]^2$ with $|t-t'| \vee |s-s'| \leq \delta$. If $|t-s| \leq \eta$,

$$|t'-s'| \leq |t-s| + |t-t'| + |s-s'| \leq \eta + 2\delta \leq 2\eta.$$

Otherwise,

$$|t'-s'| \geq |t-s| - |t-t'| - |s-s'| \geq \eta - 2\delta \geq \eta/2.$$

Thus, we deduce that

$$\begin{aligned}
&\sup_{\substack{(s,t),(s',t')\in[l^{-1},l]^2\\|t-t'|\vee|s-s'|<\delta}}|\Phi_n^{\nu_n}(s,t) - \Phi_n^{\nu_n}(s',t')| \\
&\leq \sup_{(s,t),(s',t')\in T_1(l,2\eta)}|\Phi_n^{\nu_n}(s,t) - \Phi_n^{\nu_n}(s',t')| + \sup_{\substack{(s,t),(s',t')\in T_2(l,\eta/2)\\|t-t'|\vee|s-s'|<\delta}}|\Phi_n^{\nu_n}(s,t) - \Phi_n^{\nu_n}(s',t')| \\
&\leq 2\sup_{(s,t)\in T_1(l,2\eta)}|\Phi_n^{\nu_n}(s,t) - 1| + \sup_{\substack{(s,t),(s',t')\in T_2(l,\eta/2)\\|t-t'|\vee|s-s'|<\delta}}|\Phi_n^{\nu_n}(s,t) - \Phi_n^{\nu_n}(s',t')|.
\end{aligned}$$

This, combined with Lemmas 5.3 and 5.4, yields the desired result. □

5.1.2. *Convergence of aging functions.* Here, we prove the convergence of the aging functions (Theorem 5.7). Since we already showed the precompactness of the aging functions in Proposition 5.5, it remains to prove the pointwise convergence of the aging functions. This is derived from the following result.

Lemma 5.6. *Fix $k \in \mathbb{N}$, $\boldsymbol{x}_n = (x_1^{(n)}, \dots, x_k^{(n)}) \in F_n^k$ for each n, and $\boldsymbol{x} = (x_1, \dots, x_k) \in F^k$. If $x_i^{(n)} \to x_i$ in M and $\nu_n(\{x_i^{(n)}\}) \to \nu(\{x_i\})$ for each i, then*

$$P_{\rho_n}^{\nu_n}\left(X_n^{\nu_n}(t_1) = x_1^{(n)}, \dots, X_n^{\nu_n}(t_k) = x_k^{(n)}\right) \to P_\rho\left(X^\nu(t_1) = x_1, \dots, X^\nu(t_k) = x_k\right)$$

for any $t_1, \dots, t_k \in (0,\infty)$.

Proof: Since we have the convergence of finite-dimensional distributions of $X_n^{\nu_n}$ to those of X^ν, by Proposition 2.8, it suffices to show that

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} P_{\rho_n}^{\nu_n}\big((X_n^{\nu_n}(t_1), \dots, X_n^{\nu_n}(t_k)) \in B_{M^k}(\boldsymbol{x}_n, \delta) \setminus \{\boldsymbol{x}_n\}\big) = 0, \tag{5.5}$$

where we equip the product space M^k with the max product metric (cf. (2.6)). We write $A(n,\delta)$ for the above probability. Using the transition density, we obtain that, for each $\delta \in (0,1)$,

$$\begin{aligned} A(n,\delta) &\le \sum_{i=1}^k P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(t_i) \in B_{R_n}(x_i^{(n)}, \delta) \setminus \{x_i^{(n)}\}\big) \\ &= \sum_{i=1}^k \int_{B_{R_n}(x_i^{(n)},\delta)\setminus\{x_i^{(n)}\}} p_n^{\nu_n}(t_i, \rho_n, y)\, \nu_n(dy) \\ &\le \sum_{i=1}^k \nu_n(B_{R_n}(x_i^{(n)}, \delta) \setminus \{x_i^{(n)}\}) \sup_{y \in B_{R_n}(x_i^{(n)},1)} p_n^{\nu_n}(t_i, \rho_n, y) \end{aligned}$$

By Proposition 4.13, we have that

$$\sup_{n\ge 1} \sup_{y\in B_{R_n}(x_i^{(n)},1)} p_n^{\nu_n}(t_i, \rho_n, y) < \infty.$$

Moreover, Proposition 2.8 and the convergence $\nu_n \to \nu$ in the vague-and-point-process topology yield that

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \nu_n(B_{R_n}(x_i^{(n)}, \delta) \setminus \{x_i^{(n)}\}) = 0.$$

Therefore, we obtain (5.5). □

Theorem 5.7. *Under Assumption 5.1, it holds that*

$$\big(V_n, R_n, \rho_n, \nu_n, P_{\rho_n}^{\nu_n}(X_n^{\nu_n} \in \cdot), \Phi_n^{\nu_n}\big) \to \big(F, R, \rho, \nu, P_\rho^\nu(X^\nu \in \cdot), \Phi^\nu\big)$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathrm{dis}} \times \tau_{\mathcal{P}}(\tau^{J_1}) \times \tau^{C(\mathbb{R}^2_{>0}, \mathbb{R}_{\ge 0})})$. *Moreover,* $\Phi^\nu(s,t) > 0$ *for all* $s, t \in \mathbb{R}_{>0}$.

Proof: By Lemma 5.5, it suffices to show that $\Phi_n^{\nu_n}(s,t) \to \Phi^\nu(s,t)$ for every $(s,t) \in \mathbb{R}^2_{>0}$. Fix $(s,t) \in T_2$. Since $X_n^{\nu_n} \xrightarrow{\mathrm{d}} X^\nu$ in the usual J_1-Skorohod topology and X^ν is continuous at s and t almost surely by its quasi-left-continuity, we have that $(X_n^{\nu_n}(s), X_n^{\nu_n}(t)) \xrightarrow{\mathrm{d}} (X^\nu(s), X^\nu(t))$. Noting that the diagonal set in a product metric space is closed, we obtain that

$$\limsup_{n\to\infty} P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(s) = X_n^{\nu_n}(t)\big) \le P_\rho^\nu\big(X^\nu(s) = X^\nu(t)\big). \tag{5.6}$$

Let $\{x_j\}_{j\in J}$ be the set of elements of F satisfying $P_\rho\big(X^\nu(s) = X^\nu(t) = x_i\big) > 0$. For each $i \in J$, by the convergence $\nu_n \to \nu$ in the vague-and-point-process topology and Theorem 2.9, there exists $x_i^{(n)} \in F_n$ such that $x_i^{(n)} \to x_i$ and $\nu_n(\{x_i^{(n)}\}) \to \nu(\{x_i\})$. Lemma 5.6 immediately yields that

$$P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(s) = X_n^{\nu_n}(t) = x_i^{(n)}\big) \to P_\rho^\nu\big(X^\nu(s) = X^\nu(t) = x_i\big).$$

Let $(J_l)_{l\ge 1}$ be an increasing sequence of finite subsets of J such that $\bigcup_{l\ge 1} J_l = J$. Then, the convergence $x_i^{(n)} \to x_i$ for each i implies that $x_i^{(n)} \neq x_j^{(n)}$ if $i \neq j$ with $i, j \in J_k$ for all sufficiently large n. Hence, we deduce that

$$\begin{aligned} \sum_{i\in J_k} P_\rho^\nu\big(X^\nu(s) = X^\nu(t) = x_i\big) &= \lim_{n\to\infty} \sum_{i \in J_k} P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(s) = X_n^{\nu_n}(t) = x_i^{(n)}\big) \\ &\le \liminf_{n\to\infty} P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(s) = X_n^{\nu_n}(t)\big). \end{aligned}$$

By letting $k \to \infty$ in the above inequality and (5.6), we obtain that

$$\Phi_n^{\nu_n}(s,t) = P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(s) = X_n^{\nu_n}(t)\big) \to P_\rho^\nu\big(X^\nu(s) = X^\nu(t)\big) = \Phi^\nu(s,t).$$

Checking the last assertion for $s = t$ is easy as it holds that $\Phi^\nu(s,s) = 1$. Suppose that $t > s$. We can find an $x \in F$ such that $p^\nu(s,\rho,x)\nu(\{x\}) > 0$. (Otherwise, one has that $P_\rho^\nu(X(s) \in F) = 0$.) Using the transition density p^ν, we obtain that

$$\Phi^\nu(s,t) = \int p^\nu(s,\rho,y)p^\nu(t-s,y,y)\nu(\{y\})\,\nu(dy) \geq p^\nu(s,\rho,x)p^\nu(t-s,x,x)\nu(\{x\})^2$$

Thus, we deduce that $\Phi^\nu(s,t) > 0$ from Proposition 4.12. □

5.2. *Sub-aging result.* We next prove a sub-aging result, Theorem 5.11. To do this, we extend the framework of Section 5.1. For each $n \geq 1$, we let $\dot\pi_n = \sum_{i \in I_n} \delta_{(x_i^{(n)}, w_i^{(n)}, v_i^{(n)})}$ be a simple measure on $F_n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$, and let $\dot\pi = \sum_{i \in I} \delta_{(x_i, w_i, v_i)}$ be a simple measure on $F \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. For each $n \geq 1$, we assume that $x_i^{(n)} \neq x_j^{(n)}$ if $i \neq j$, and the measure ν_n on F_n given below is of full support:

$$\nu_n(A) = \int 1_A(x) v\,\dot\pi_n(dx\,dw\,dv) = \sum_{i \in I_n} v_i^{(n)} \delta_{x_i^{(n)}}(A), \quad A \in \mathcal{B}(F_n).$$

Similarly, we assume that $x_i \neq x_j$ if $i \neq j$, and the measure ν on F given below is of full support:

$$\nu(A) = \int 1_A(x) v\,\dot\pi(dx\,dw\,dv) = \sum_{i \in I} v_i \delta_{x_i}(A), \quad A \in \mathcal{B}(F).$$

As in the previous section, we write $(X_n^{\nu_n}, \{P_x^{\nu_n}\}_{x \in F_n})$ for the Hunt process associated with (F_n, R_n, ν_n) and $p_n^{\nu_n}$ for the jointly continuous transition density of $X_n^{\nu_n}$ with respect to ν_n. Similarly, we write $(X^\nu, \{P_x^\nu\}_{x \in F})$ for the Hunt process associated with (F, R, ν) and p^ν for the jointly continuous transition density of X^ν with respect to ν. We define a sub-aging function associated with X^ν and ρ by setting

$$\Psi^\nu(s,t) := \int e^{-ws/v} P_\rho^\nu(X^\nu(t) = x)\,\dot\pi(dx\,dw\,dv).$$

We similarly define $\Psi_n^{\nu_n}$ to be the sub-aging function associated with $X_n^{\nu_n}$ and ρ_n. Since $\nu(\{x_i\}) = v_i$ and $P_\rho^\nu(X^\nu(t) = x) = p^\nu(t,\rho,x)\nu(\{x\})$, we note that

$$P_\rho^\nu(X^\nu(t) = x)\,\dot\pi(dx\,dw\,dv) = p^\nu(t,\rho,x) v\,\dot\pi(dx\,dw\,dv). \tag{5.7}$$

Assumption 5.8. *It holds that*

$$\big(F_n, R_n, \rho_n, \nu_n, \dot\pi_n, P_{\rho_n}^{\nu_n}(X_n^{\nu_n} \in \cdot)\big) \to \big(F, R, \rho, \nu, \dot\pi, P_\rho^\nu(X^\nu \in \cdot)\big)$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0})} \times \tau_{\mathcal{P}}(\tau^{J_1}))$.

Henceforth, we assume that Assumption 5.8 is satisfied. Under this assumption, by Theorem 3.9, we may assume that (F_n, R_n, ρ_n) and (F, R, ρ) are embedded isometrically into a common rooted boundedly-compact metric space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements of M, $F_n \to F$ in the Fell topology as closed subsets in M, $\nu_n \to \nu$ in the vaguely-and-point-process topology as discrete measures on M, $\dot\pi_n \to \dot\pi$ vaguely as measures on $M \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$, and $X_n^{\nu_n} \xrightarrow{\mathrm{d}} X^\nu$ in $D(\mathbb{R}_{\geq 0}, M)$.

For $l \in \mathbb{N}$, we set

$$T_3(l) := [0,l] \times [l^{-1}, l].$$

Note that $(T_3(l))_{l \geq 1}$ is a sequence of compact subsets increasing to $\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. Below, we show some technical results used to prove the precompactness of $(\Psi_n^{\nu_n})_{n \geq 1}$.

Lemma 5.9. *Fix $l \in \mathbb{N}$. The following statements hold.*

(i) *For every $r > 0$,*

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \limsup_{(s,t) \in T_3(l)} \int_{D_{R_n}(\rho_n, r) \times \mathbb{R}_{\geq 0} \times (0,\eta]} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) = 0.$$

(ii) *It holds that*

$$\lim_{r \to \infty} \limsup_{n \to \infty} \limsup_{(s,t) \in T_3(l)} \int_{D_{R_n}(\rho_n, r)^c \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) = 0.$$

(iii) *For every $r, \eta > 0$, it holds that*

$$\lim_{W \to \infty} \limsup_{n \to \infty} \sup_{(s,t) \in T_3(l)} \int_{D_{R_n}(\rho_n, r)^c \times [W,\infty) \times [\eta,\infty)} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) = 0.$$

Proof: (i). For $(s,t) \in T_3(l)$, we deduce by (5.7) that

$$\begin{aligned}
&\int_{D_{R_n}(\rho_n, r) \times \mathbb{R}_{\geq 0} \times (0,\eta]} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) \\
&\leq \int_{D_{R_n}(\rho_n, r) \times \mathbb{R}_{\geq 0} \times (0,\eta]} p_n^{\nu_n}(t, \rho_n, x) v\, \dot{\pi}_n(dx\, dw\, dv) \\
&\leq \mathfrak{s}_{F_n}(p_n^{\nu_n}, l^{-1}, r) \sum_{\substack{x_i^{(n)} \in D_{R_n}(\rho_n, r) \\ v_i^{(n)} \leq \eta}} v_i^{(n)} \\
&\leq \mathfrak{s}_{F_n}(p_n^{\nu_n}, l^{-1}, r) M_\eta^{(r)}(\mathfrak{p}(\nu_n)).
\end{aligned}$$

Hence, the result follows from Theorem 2.14 and Proposition 4.13.

(ii). For $(s,t) \in T_3(l)$, we have that

$$\begin{aligned}
\int_{D_{R_n}(\rho_n, r)^c \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) &\leq \sum_{x_i \in D_{R_n}(\rho_n, r)^c} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x_i) \\
&\leq P_{\rho_n}^{\nu_n}(\tau_{D_{R_n}(\rho_n, r)}^{\nu_n} \leq l).
\end{aligned}$$

Hence, the desired result follows from (5.1).

(iii). Fix $r, \eta > 0$ and let $r' > r$ and $\eta' < \eta$ be such that the boundary of $D_M(\rho, r') \times [\eta', \infty)$ does not contain the atoms of $\mathfrak{p}(\nu)$. By Theorem 2.22, it holds that

$$V := W^{(r')}(\mathfrak{p}(\nu)) + \sup_{n \geq 1} W^{(r')}(\mathfrak{p}(\nu_n)) + 1 < \infty.$$

Noting that $F_n \cap D_M(\rho_M, r) = D_{R_n}(\rho_n, r)$, we deduce that, for any $W > 0$,

$$\begin{aligned}
&\sup_{(s,t) \in T_3(l)} \int_{D_{R_n}(\rho_n, r) \times [W,\infty) \times [\eta,\infty)} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) \\
&\leq \int_{D_{R_n}(\rho_n, r) \times [W,\infty) \times [\eta,\infty)} \dot{\pi}_n(dx\, dw\, dv) \\
&= \#\big(\mathrm{At}(\dot{\pi}_n) \cap (D_{R_n}(\rho_n, r) \times [W, \infty) \times [\eta, \infty))\big) \\
&\leq \#\big(\mathrm{At}(\dot{\pi}_n) \cap (D_M(\rho_M, r') \times [W, \infty) \times [\eta', V])\big) \qquad (5.8)
\end{aligned}$$

We define a finite subset $J \subseteq I$ so that $\{(x_j, v_j)\}_{j \in J}$ are the atoms of $\mathfrak{p}(\nu)$ lying in $K := D_M(\rho_M, r') \times [\eta', V]$. Proposition 2.4 yields that $\#(\mathrm{At}(\mathfrak{p}(\nu_n)) \cap K) = \#J$ for all sufficiently large n, which implies that

$$\#\big(\mathrm{At}(\dot{\pi}_n) \cap (D_M(\rho_M, r') \times \mathbb{R}_{\geq 0} \times [\eta', V])\big) = \#J. \qquad (5.9)$$

Now, fix W sufficiently large so that $W > \max_{j\in J} w_j + 1$. Since $\dot{\pi}_n$ converges to $\dot{\pi}$ vaguely, Proposition 2.4 again yields that, for all sufficiently large n,

$$\begin{aligned}\#\big(\mathrm{At}(\dot{\pi}_n)\cap(D_M(\rho_M,r')\times[0,W)\times[\eta',V])\big) &= \#\big(\mathrm{At}(\dot{\pi})\cap(D_M(\rho_M,r')\times[0,W)\times[\eta',V])\big)\\ &= \#J.\end{aligned}$$

Combining this with (5.9), we obtain that

$$\lim_{n\to\infty}\#\big(\mathrm{At}(\dot{\pi}_n)\cap(D_M(\rho_M,r')\times[W,\infty)\times[\eta',V])\big)=0.$$

From this and (5.8), we obtain the desired result. □

Lemma 5.10. *The family* $\{\Psi_n^{\nu_n}\}_{n\ge1}$ *is precompact in* $C(\mathbb{R}_{\ge0}\times\mathbb{R}_{>0},\mathbb{R}_{\ge0})$.

Proof: Noting that, for all $s,t\in\mathbb{R}_{\ge0}\times\mathbb{R}_{>0}$ and $n\ge1$,

$$\Psi_n^{\nu_n}(s,t)\le\int P_n^{\nu_n}(X_n^{\nu_n}=x)\,\dot{\pi}_n(dx\,dw\,dv)=\sum_{i\in I_n}P_n^{\nu_n}(X_n^{\nu_n}=x_i^{(n)})=1,$$

it suffices to show that, for each $l\ge1$,

$$\lim_{\delta\to0}\limsup_{n\to\infty}\sup_{\substack{(s,t),(s',t')\in T_3(l)\\ |s-s'|\vee|t-t'|\le\delta}}|\Psi^{\nu}(s,t)-\Psi^{\nu}(s',t')|=0.$$

Fix $l\ge1$. By Lemma 5.9, this reduces to proving that, for any $r,W,\eta>0$,

$$\lim_{\delta\to0}\limsup_{n\to\infty}\sup_{\substack{(s,t),(s',t')\in T_3(l)\\ |s-s'|\vee|t-t'|\le\delta}}A_n(r,W,\eta)=0, \tag{5.10}$$

where $A_n=A_n(r,W,\eta)$ is given by

$$A_n:=\int_{D_{R_n}(\rho_n,r)\times[0,W]\times[\eta,\infty)}\left|e^{-ws/v}P_n^{\nu_n}(X_n^{\nu_n}(t)=x)-e^{-ws'/v}P_n^{\nu_n}(X_n^{\nu_n}(t')=x)\right|\dot{\pi}(dx\,dw\,dv).$$

Fix $(s,t),(s',t')\in T_3(l)$ with $|s-s'|\vee|t-t'|\le\delta$. Using (5.7), we can write

$$A_n=\int_{D_{R_n}(\rho_n,r)\times[0,W]\times[\eta,\infty)}v\left|e^{-ws/v}p_n^{\nu_n}(t,\rho_n,x)-e^{-ws'/v}p_n^{\nu_n}(t',\rho_n,x)\right|\dot{\pi}(dx\,dw\,dv).$$

By the triangle inequality, we have that

$$\begin{aligned}&\left|e^{-ws/v}p_n^{\nu_n}(t,\rho_n,x)-e^{-ws'/v}p_n^{\nu_n}(t',\rho_n,x)\right|\\ &\le|p_n^{\nu_n}(t,\rho_n,x)-p_n^{\nu_n}(t',\rho_n,x)|+p_n^{\nu_n}(t',\rho_n,x)|e^{-ws/v}-e^{-ws'/v}|\\ &\le\mathfrak{d}_{F_n}(p_n^{\nu_n},l^{-1},r,\delta)+\mathfrak{s}_{F_n}(p_n^{\nu_n},l^{-1},r)|e^{-ws/v}-e^{-ws'/v}|.\end{aligned}$$

For $(x,w,v)\in D_{R_n}(\rho_n,r)\times[0,W]\times[\eta,\infty)$, we deduce by the mean value theorem that

$$|e^{-ws/v}-e^{-ws'/v}|\le\frac{w}{v}|s-s'|\le\frac{W}{\eta}\delta.$$

Hence, it follows that

$$\begin{aligned}A_n&\le\left\{\mathfrak{d}_{F_n}(p_n^{\nu_n},l^{-1},r,\delta)+\mathfrak{s}_{F_n}(p_n^{\nu_n},l^{-1},r)W\eta^{-1}\delta\right\}\sum_{x_i^{(n)}\in D_{R_n(\rho_n,r)}}v_i^{(n)}\\ &=\left\{\mathfrak{d}_{F_n}(p_n^{\nu_n},l^{-1},r,\delta)+\mathfrak{s}_{F_n}(p_n^{\nu_n},l^{-1},r)W\eta^{-1}\delta\right\}\nu_n(D_{R_n}(\rho_n,r)).\end{aligned}\tag{5.11}$$

Since the vague convergence $\nu_n\to\nu$ implies that $\sup_{n\ge1}\nu_n(D_{R_n}(\rho_n,r))<\infty$, we obtain (5.10) from Proposition 4.13 and (5.11). □

Theorem 5.11. *Under Assumption 5.8, It holds that*

$$\big(F_n, R_n, \rho_n, \nu_n, \pi_n, P_{\rho_n}^{\nu_n}(X_n^{\nu_n} \in \cdot), \Psi_n^{\nu_n}\big) \to \big(F, R, \rho, \nu, \pi, P_\rho^\nu(X^\nu \in \cdot), \Psi^\nu\big)$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0})} \times \tau_{\mathcal{P}}(\tau^{J_1}) \times \tau^{C(\mathbb{R}^2_{>0}, \mathbb{R}_{\geq 0})})$. *Moreover,* $\Psi^\nu(s,t) > 0$ *for any* $s \geq 0$ *and* $t > 0$.

Proof: The second assertion is straightforward. Indeed, for some $i \in I$, we have that $P_\rho^\nu(X^\nu(t) = x_i) > 0$, which implies that

$$\Psi^\nu(s,t) \geq e^{-w_i s/v_i} P_\rho^\nu(X^\nu(t) = x_i) > 0.$$

We next show the first assertion. By Lemma 5.10, it suffices to show that $\Psi_n^{\nu_n}(s,t) \to \Psi^\nu(s,t)$ for every $(s,t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. Fix $(s,t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. By Lemma 5.9, it is enough to show that, for some sequences $(r_k)_{k \geq 1}$, $(W_l)_{l \geq 1}$, and $(\eta_m)_{m \geq 1}$ with $r_k \uparrow \infty$, $W_l \uparrow \infty$, and $\eta_m \downarrow 0$,

$$\begin{aligned}
&\lim_{n \to \infty} \int_{D_M(\rho_M, r_k) \times [0, W_l] \times [\eta_m, \infty)} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) \\
&= \int_{D_M(\rho_M, r_k) \times [0, W_l] \times [\eta_m, \infty)} e^{-ws/v} P_\rho^\nu(X^\nu(t) = x)\, \dot{\pi}(dx\, dw\, dv).
\end{aligned}$$

We choose $(r_k)_{k \geq 1}$ so that $r_k \uparrow \infty$ and $\nu(\partial D_M(\rho_M, r_k)) = 0$. We then choose $(W_l)_{l \geq 1}$ and $(\eta_m)_{m \geq 1}$ so that $W_l \uparrow \infty$, $\eta_m \downarrow 0$, and $\dot{\pi}(\partial(D_M(\rho_M, r_k) \times [0, W_l] \times [\eta_m, \infty))) = 0$ for all l, m. Fix k, l, and m, and simply write $r = r_k$, $W = W_l$, and $\eta = \eta_m$. Set

$$V := \left\{ W^{(r)}(\mathfrak{p}(\nu)) \vee \sup_{n \geq 1} W^{(r)}(\mathfrak{p}(\nu_n)) \right\} + 1 < \infty.$$

Then there are no atoms of π_n nor π in $D_M(\rho_M, r) \times [0, W] \times (V, \infty)$. We define a finite subset $J \subseteq I$ so that $\{(x_j, w_j, v_j)\}_{j \in J}$ are the atoms of $\dot{\pi}$ lying in $K := D_M(\rho_M, r) \times [0, W] \times [\eta, V]$. By Proposition 2.4, there exist injections $f_n \colon J \to I_n$ (at least for all sufficiently large n) such that

$$\mathrm{At}(\dot{\pi}_n) \cap K = \{(x_{f_n(j)}^{(n)}, w_{f_n(j)}^{(n)}, v_{f_n(j)}^{(n)})\}_{j \in J}$$

and $(x_{f_n(j)}^{(n)}, w_{f_n(j)}^{(n)}, v_{f_n(j)}^{(n)}) \to (x_j, w_j, v_j)$ for each $j \in J$. It then follows from Lemma 5.6 that

$$P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x_i^{(n)}) \to P_\rho^\nu(X^\nu(t) = x_i), \quad \forall j \in J.$$

Therefore, we deduce that

$$\begin{aligned}
&\lim_{n \to \infty} \int_{D_M(\rho_M, r) \times [0, W] \times [\eta, \infty)} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) \\
&= \lim_{n \to \infty} \int_{D_M(\rho_M, r) \times [0, W] \times [\eta, V]} e^{-ws/v} P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x)\, \dot{\pi}_n(dx\, dw\, dv) \\
&= \lim_{n \to \infty} \sum_{j \in J} \exp\big(-w_{f_n(j)}^{(n)} s / v_{f_n(j)}^{(n)}\big) P_{\rho_n}^{\nu_n}(X_n^{\nu_n}(t) = x_{f_n(j)}^{(n)}) \\
&= \sum_{j \in J} e^{-w_j s/v_j} P_\rho^\nu(X^\nu(t) = x_j) \\
&= \int_{D_M(\rho_M, r) \times [0, W] \times [\eta, \infty)} e^{-ws/v} P_\rho^\nu(X^\nu(t) = x)\, \dot{\pi}(dx\, dw\, dv),
\end{aligned}$$

which completes the proof. □

6. Proof of the main results

In this section, we prove the main results. Since we already obtained the results in the previous section when the traps are deterministic and converge, thanks to the Skorohod representation theorem, it remains to show the convergence of traps in distribution.

6.1. *Proof of Theorem 1.6.* We prove the aging result, Theorem 1.6. Suppose that Assumption 1.5 is satisfied. We first study some properties of the limiting trap environment. Recall that $\pi(dx\,dv)$ is the Poisson random measure with intensity measure $\mu(dx)\,\alpha v^{-1-\alpha}dv$ and $\nu = \mathfrak{v}(\pi)$, where the map $\mathfrak{v}$ is introduced in Section 2.3 above.

Lemma 6.1.

(i) *The set* $\mathrm{At}(\nu)$ *of the atoms of* ν *is dense in* F *almost surely. In particular,* ν *is of full support almost surely.*

(ii) *It holds that* $\pi \in \mathscr{P}^*(F)$ *almost surely (recall* $\mathscr{P}^*(F)$ *from (2.14)).*

Proof: (i). Since μ is of full support, it holds that, for any $x \in F$ and $r > 0$,

$$\mathsf{E}[\pi(B_R(x,r)\times\mathbb{R}_{>0})] = \mu(B_R(x,r))\int_0^\infty \alpha u^{-1-\alpha}\,du = \infty,$$

where E denotes the expectation with respect to P, the underlying probability measure of π. Hence, $\mathrm{At}(\nu)\cap B_R(x,r) \neq \emptyset$ almost surely. Let $\{x_i\}_{i\in I}$ be a countable dense subset of F. Then, almost surely, we have that $\mathrm{At}(\nu)\cap B_R(x_i,q)\neq\emptyset$ for all $i \in I$ and positive rational numbers q. This implies the desired result.

(ii). Write $\pi = \sum_{i\in I}\delta_{(x_i,v_i)}$. For each $\varepsilon > 0$, we set $I_\varepsilon \coloneqq \{i \in I \mid v_i \geq \varepsilon\}$. We then define a random measure ν'_ε on F by setting

$$\nu'_\varepsilon(A) \coloneqq \sum_{i\in I_\varepsilon}\delta_{x_i}(A) = \pi(A\times[\varepsilon,\infty)).$$

It is easy to check that ν'_ε is a Poisson random measure with intensity measure $\varepsilon^{-\alpha}\mu(dx)$. Since μ is non-atomic, it follows from Kallenberg (2017, Lemma 3.6(i)) that $x_i \neq x_j$ for any $i, j \in I_\varepsilon$ with $i \neq j$, almost surely. Since I_ε increases to I as $\varepsilon \to 0$, we deduce the desired result. ☐

For the following result, recall the random variable ξ from Definition 1.3.

Lemma 6.2. *For every* $u > 0$, $\lim_{n\to\infty} b_n P_\xi(c_n^{-1}\xi > u) = u^{-\alpha}$.

Proof: Define $f(u) \coloneqq P_\xi(\xi > u)^{-1}$ and $g(u) \coloneqq \inf\{s \geq 0 \mid f(s) > u\}$. Then f is a regularly varying function with index α and g is an asymptotic inverse of f, that is, $f(g(u)) \sim g(f(u)) \sim u$ as $u \to \infty$ (Bingham et al., 1987, Theorem 1.5.12). (NB. For functions h and H, we write $h \sim H$ when $h(u)/H(u) \to 1$.) Using the relation $c_n = g(b_n)$, we deduce that

$$P_\xi(c_n^{-1}\xi > u) = (uc_n)^{-\alpha}\ell(u) = u^{-\alpha}\frac{\ell(c_n u)}{\ell(c_n)}P_\xi(\xi > c_n) = u^{-\alpha}\frac{\ell(c_n u)}{\ell(c_n)}f(g(b_n))^{-1}.$$

Since ℓ is slowly varying and $c_n \to \infty$, we obtain the desired result. ☐

We now prove the distributional convergence of traps in the vague-and-point-process topology.

Lemma 6.3. *It holds that*

$$\bigl(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}\mu_n^{\#}, \mathsf{P}_n(c_n^{-1}\nu_n \in \cdot)\bigr) \to \bigl(F, R, \rho, \mu, \mathsf{P}(\nu \in \cdot)\bigr)$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}} \times \tau_{\mathcal{P}}(\tau^{\mathcal{M}^{\mathrm{dis}}}))$.

Proof: By Assumption 1.5(i) and Theorem 3.9, we may assume that $(V_n, a_n^{-1}R_n, \rho_n)$ and (F, R, ρ) are embedded isometrically into a common rooted boundedly-compact metric space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements in M, $V_n \to F$ in the Fell topology as closed subsets of M, and $b_n^{-1}\mu_n^{\#} \to \mu$ vaguely as measures on M. It suffices to show that $c_n^{-1}\nu_n \xrightarrow{\mathrm{d}} \nu$ in the vague-and-point-process topology as measures on M. Recall that $\pi_n \coloneqq \sum_{x \in V_n} \delta_{(x, c_n^{-1}\nu_n(\{x\}))}$. Fix a bounded subset A of M such that $\mu(\partial A) = 0$. We have that, for every $u > 0$,

$$\mathsf{P}_n(\pi_n(A \times (u, \infty)) = 0) = \mathsf{P}_n(c_n^{-1}\xi_x^{(n)} \le u \text{ for all } x \in A) = \left(1 - P_\xi(c_n^{-1}\xi > u)\right)^{\mu_n^{\#}(A)}$$

and $\mathsf{E}_n[\pi_n(A \times (u, \infty))] = \mu_n^{\#}(A) P_\xi(c_n^{-1}\xi > u)$. It then follows from Lemma 6.2 that

$$\begin{aligned} \lim_{n\to\infty} \mathsf{P}_n(\pi_n(A \times (u, \infty)) = 0) &= e^{-\mu(A)u^{-\alpha}} = \mathsf{P}(\pi(A \times (u, \infty)) = 0), \\ \lim_{n\to\infty} \mathsf{E}_n[\pi_n(A \times (u, \infty))] &= \mu(A)u^{-\alpha} = \mathsf{E}[\pi(A \times (u, \infty))]. \end{aligned} \tag{6.1}$$

By Kallenberg (2017, Theorem 4.18), $\pi_n \xrightarrow{\mathrm{d}} \pi$ vaguely. We will check condition (ii) of Theorem 2.26. Fix $r > 0$ satisfying $\mu(M^{(r)}) = 0$. Since we have that

$$p(n, l) \coloneqq \mathsf{P}_n(W^{(r)}(\pi_n) > l) = \mathsf{P}_n(\pi_n(M^{(r)} \times (l, \infty)) > 0),$$

equation (6.1) yields that $p(n, l) \to \mathsf{P}(\pi(M^{(r)} \times (l, \infty)) > 0) = 1 - e^{-\mu(M^{(r)})l^{-\alpha}}$. Hence, we obtain that $\lim_{l\to\infty} \limsup_{n\to\infty} p(n, l) = 0$. The Markov inequality yields that

$$\mathsf{P}_n(M_\varepsilon^{(r)}(\pi_n) > \delta) \le \delta^{-1}\mu_n^{\#}(M^{(r)}) E_\xi[c_n^{-1}\xi \cdot 1_{(0, c_n\varepsilon)}(\xi)] = -\delta^{-1}\mu_n^{\#}(M^{(r)}) c_n^{-1} \int_0^{\varepsilon c_n} u \, df(u),$$

where we set $f(u) \coloneqq P(\xi > u)$. Since f is a regularly varying function with index $-\alpha$, we have by Bingham et al. (1987, Theorem 1.6.4) that, as $n \to \infty$,

$$\int_0^{\varepsilon c_n} u \, df(u) \sim \frac{-\alpha}{1 - \alpha} \varepsilon c_n f(\varepsilon c_n) = \frac{-\alpha}{1 - \alpha} \varepsilon c_n P_\xi(c_n^{-1}\xi > \varepsilon).$$

It then follows from Lemma 6.2 that

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} \mathsf{P}_n(M_\varepsilon^{(r)}(\pi_n) > \delta) \le \lim_{\varepsilon \downarrow 0} \delta^{-1}\alpha(1 - \alpha)^{-1}\mu(M^{(r)})\varepsilon^{1-\alpha} = 0.$$

By Corollary 2.27, $\pi_n \xrightarrow{\mathrm{d}} \pi$ in $\mathscr{P}(M)$. It follows from Corollary 2.16 and Lemma 6.1(ii) that $c_n^{-1}\nu_n = \mathfrak{v}(\pi_n) \xrightarrow{\mathrm{d}} \mathfrak{v}(\pi) = \nu$ in the vague-and-point-process topology. □

Now, we prove Theorem 1.6.

Proof of Theorem 1.6: By Lemma 6.3, we may assume that $(V_n, a_n^{-1}R_n, \rho_n)$ and (F, R, ρ) are embedded isometrically into a common rooted boundedly-compact metric space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements in M, $V_n \to F$ in the Fell topology as closed subsets of M, $b_n^{-1}\mu_n^{\#} \to \mu$ vaguely as measures on M, and $c_n^{-1}\nu_n \xrightarrow{\mathrm{d}} \nu$ in $\mathcal{M}^{\mathrm{dis}}(M)$. Using the Skorohod representation theorem, we may further assume that $c_n^{-1}\nu_n \to \nu$ almost surely on some probability space. Then, by Croydon (2018, Theorem 1.2), we obtain that $P_{\rho_n}^{\nu_n}(\tilde{X}_n^{\nu_n} \in \cdot) \xrightarrow{\mathrm{d}} P_\rho^\nu(X^\nu \in \cdot)$. Hence, the desired result follows from Theorem 5.7. □

6.2. *Proof of Theorem 1.9.* Next, we prove the sub-aging result, Theorem 1.9. Suppose that Assumption 1.8 is satisfied. Recall that $\dot{\pi}(dx\, dw\, dv)$ is a Poisson random measure on $F \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{>0}$ with the intensity measure $\dot{\mu}(dx\, dw)\, \alpha v^{-1-\alpha}\, dv$ and ν is a random measure on F defined by

$$\nu(A) \coloneqq \int 1_A(x) v \, \dot{\pi}(dx\, dw\, dv), \quad \forall A \in \mathcal{B}(F). \tag{6.2}$$

For convenience, we write π for the pushforward measure of $\dot{\pi}$ by the map $(x, w, v) \mapsto (x, v)$. It is easy to show that π is a Poisson random measure with intensity $\mu(dx)\, \alpha v^{-1-\alpha} dv$ and $\nu = \mathfrak{v}(\pi)$, where we recall the map $\mathfrak{v}$ from (2.10).

Lemma 6.4. (i) *The random measure ν is of full support almost surely.*
(ii) *It holds that $\pi \in \mathscr{P}^*(F)$ almost surely.*

Proof: These results are proven similarly to Lemma 6.1. □

We set

$$\dot{\pi}_n := \sum_{x \in V_n} \delta_{(x, \mu_n(x), c_n^{-1} \nu_n(\{x\}))}.$$

The following result is a version of Lemma 6.3 for the sub-aging result.

Lemma 6.5. *It holds that*

$$\big(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1} \dot{\mu}_n^{\#}, \mathsf{P}_n((\nu_n, \dot{\pi}_n) \in \cdot)\big) \to \big(F, R, \rho, \dot{\mu}, \mathsf{P}((\nu, \dot{\pi}) \in \cdot)\big),$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0})} \times \tau_{\mathcal{P}}(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau^{\mathcal{M}(\cdot \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0})}))$.

Proof: Under Assumption 1.8, we may assume that $(V_n, a_n^{-1} R_n, \rho_n)$ and (F, R, ρ) are embedded isometrically into a common rooted boundedly-compact metric space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements in M, $V_n \to F$ in the Fell topology as closed subsets of M, and $b_n^{-1} \dot{\mu}_n^{\#} \to \dot{\mu}$ vaguely as measures on $M \times \mathbb{R}_{\geq 0}$. From the vague convergence $b_n^{-1} \dot{\mu}_n^{\#} \to \dot{\mu}$ and the continuous mapping theorem, we have that $b_n^{-1} \mu_n^{\#} \to \mu$ vaguely. Therefore, by following the proof of Lemma 6.3, we deduce that $c_n^{-1} \nu_n \xrightarrow{\mathrm{d}} \nu$ in the vague-and-point-process topology.

We next prove that $\dot{\pi}_n \to \dot{\pi}$ vaguely as measures on $M \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ in the same way as before. Fix a bounded subset $A \subseteq M$ and a $u_1 > 0$ such that $\mu(\partial(A \times (u_1, \infty))) = 0$. Recall from (1.4) that $\psi_n(x) := \psi_{G_n}(x) = (x, c_n(x))$ and that $\dot{\mu}_n^{\#} = \mu_n^{\#} \circ \psi_n^{-1}$. We have that, for every $u_2 > 0$,

$$\begin{aligned}\mathsf{P}_n\big(\dot{\pi}_n(A \times (u_1, \infty) \times (u_2, \infty)) = 0\big) &= \mathsf{P}_n\big(c_n^{-1} \xi_x^{(n)} \leq u_2 \text{ for all } x \in \psi_n^{-1}(A \times (u_1, \infty))\big) \\ &= \big(1 - P_\xi(c_n^{-1} \xi > u_2)\big)^{\dot{\mu}_n^{\#}(A \times (u_1, \infty))}\end{aligned}$$

and $\mathsf{E}_n[\dot{\pi}_n(A \times (u_1, \infty) \times (u_2, \infty))] = \dot{\mu}_n^{\#}(A \times (u_1, \infty)) P_\xi(c_n^{-1} \xi > u_2)$. It follows from Lemma 6.2 and the convergence $\dot{\mu}_n^{\#}(A \times (u_1, \infty)) \to \dot{\mu}(A \times (u_1, \infty))$ that

$$\lim_{n \to \infty} \mathsf{P}_n\big(\dot{\pi}_n(A \times (u_1, \infty) \times (u_2, \infty)) = 0\big) = \mathsf{P}(\dot{\pi}(A \times (u_1, \infty) \times (u_2, \infty)) = 0),$$

$$\lim_{n \to \infty} \mathsf{E}_n[\dot{\pi}_n(A \times (u_1, \infty) \times (u_2, \infty))] = \mathsf{E}[\dot{\pi}(A \times (u_1, \infty) \times (u_2, \infty))].$$

Thus, by Kallenberg (2017, Theorem 4.18), we obtain that $\dot{\pi}_n \xrightarrow{\mathrm{d}} \dot{\pi}$ vaguely.

From the above arguments, it is the case that $(c_n^{-1} \nu_n, \dot{\pi}_n)_{n \geq 1}$ is tight as a sequence of random elements of $\mathcal{M}^{\mathrm{dis}}(M) \times \mathcal{M}(M \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0})$. Let $(n_k)_{k \geq 1}$ be a subsequence such that $(c_{n_k}^{-1} \nu_{n_k}, \dot{\pi}_{n_k})$ converges to some random element $(\nu', \dot{\pi}')$. We then have that $\dot{\pi}' \stackrel{\mathrm{d}}{=} \dot{\pi}$. Using the Skorohod representation theorem, we may assume that $(c_{n_k}^{-1} \nu_{n_k}, \dot{\pi}_{n_k}) \to (\nu', \dot{\pi}')$ almost surely on some probability space. Define $p : M \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to M \times \mathbb{R}_{>0}$ by supposing $(x, w, v) \mapsto (x, v)$. Let $\nu' = \sum_{i \in I} v_i \delta_{x_i}$ and $\dot{\pi}' = \sum_{j \in J} \delta_{(x'_j, w'_j, v'_j)}$ be the atomic decompositions of ν' and $\dot{\pi}'$, respectively. The continuous mapping theorem yields that

$$\mathfrak{p}(c_{n_k}^{-1} \nu_{n_k}) = \sum_{x \in V_{n_k}} \delta_{(x, c_{n_k}^{-1} \nu_{n_k}(\{x\}))} = \dot{\pi}_{n_k} \circ p^{-1} \to \dot{\pi}' \circ p^{-1} = \sum_{j \in J} \delta_{(x'_j, v'_j)}.$$

On the other hand, from the convergence $c_{n_k}^{-1} \nu_{n_k} \to \nu'$ in the vague-and-point-process topology and Theorem 2.9, we have that $\mathfrak{p}(c_n^{-1} \nu_n) \to \mathfrak{p}(\nu') = \sum_{i \in I} \delta_{(x_i, v_i)}$ vaguely. Hence, we obtain that

$\sum_{i\in I}\delta_{(x_i,v_i)} = \sum_{j\in J}\delta_{(x'_j,v'_j)}$, which implies that $\{(x_i,v_i) \mid i \in I\} = \{(x'_j,v'_j) \mid j \in J\}$. This yields that, for any $A \in \mathcal{B}(F)$,

$$\nu'(A) = \sum_{i\in I} v_i \delta_{x_i}(A) = \sum_{j\in J} v'_j \delta_{x'_j}(A) = \int 1_A(x) v\, \dot\pi'(dx\, dw\, dv).$$

Since $\dot\pi \overset{\mathrm{d}}{=} \dot\pi'$ and ν is given by (6.2), we obtain that $(\nu', \dot\pi') \overset{\mathrm{d}}{=} (\nu, \dot\pi)$. Therefore, we deduce that $(c_n^{-1}\nu_n, \dot\pi_n) \xrightarrow{\mathrm{d}} (\nu, \dot\pi)$, which completes the proof. □

Using Theorem 5.11 and Lemma 6.5, Theorem 1.9 is proven similarly to Theorem 1.6 as follows.

Proof of Theorem 1.9: By Lemma 6.5, we may assume that $(V_n, a_n^{-1}R_n, \rho_n)$ and (F, R, ρ) are embedded isometrically into a common rooted boundedly-compact metric space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements in M, $V_n \to F$ in the Fell topology as closed subsets of M, and $b_n^{-1}\dot\mu_n^{\#} \to \dot\mu$ vaguely as measures on $M \times \mathbb{R}_{\geq 0}$, $\mathsf{P}_n((c_n^{-1}\nu_n, \dot\pi_n) \in \cdot) \to \mathsf{P}((\nu, \dot\pi) \in \cdot)$ as probability measures on $\mathcal{M}^{\mathrm{dis}}(M) \times \mathcal{M}(M \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0})$. By the Skorohod representation theorem, we may further assume that $(c_n^{-1}\nu_n, \dot\pi_n) \to (\nu, \dot\pi)$ almost surely on some probability space. Applying Croydon (2018, Theorem 1.2), we obtain that $P_n^{\nu_n}(\tilde X_n^{\nu_n} \in \cdot) \to P_\rho^{\nu}(X^\nu \in \cdot)$ as probability measures on $D(\mathbb{R}_{\geq 0}, M)$. Hence, from Theorems 5.7 and 5.11, the desired result follows. □

6.3. *Proof of Theorems 1.13 and 1.16.* Finally, we prove the aging and sub-aging results for random electrical networks, Theorems 1.13 and 1.16. We start with a basic result regarding the Prohorov metric (recall this from (2.1)).

Lemma 6.6. *Let X and Y be random elements of a separable metric space (S, d^S) defined on a common probability space with probability measure P. Suppose that on an event E, we have that $d^S(X,Y) \leq \varepsilon$ almost surely. It then holds that*

$$d_P^S\big(P(X \in \cdot), P(Y \in \cdot)\big) \leq P(E^c) + \varepsilon.$$

Proof: Recall the definition of the ε-neighborhood $\cdot^{\langle\varepsilon\rangle}$ from (2.2). Fix a Borel subset A of S. We deduce that

$$P(X \in A) \leq P(X \in A,\, d^S(X,Y) \leq \varepsilon) + P(E^c) \leq P(Y \in A^{\langle\varepsilon\rangle}) + P(E^c),$$

and similarly, $P(Y \in A) \leq P(X \in A^{\langle\varepsilon\rangle}) + P(E^c)$. Hence, we obtain the desired result. □

Thanks to the Skorohod representation theorem, Theorem 1.13 (resp. Theorem 1.16) is obtained similarly to Theorem 1.6 (resp. Theorem 1.9) by showing a version of Lemma 6.3 (resp. Lemma 6.5) for random electrical networks. To do this, we will use the following technical results. Recall the space $\mathbb{F}$ from Section 4.2.

Lemma 6.7. *Fix $(F, R, \rho, \nu) \in \mathbb{F}$. Assume that, for some $\lambda > 1$, $r > 0$, and $\delta > 0$,*

$$R(\rho, B_R(\rho, r)^c) \geq \lambda, \qquad \nu(B_R(\rho, 1)) \geq \delta.$$

Write $(X^\nu, \{P_x^\nu\}_{x\in F})$ (resp. $(X^{\nu^{(r)}}, \{P_x^{\nu^{(r)}}\}_{x \in F^{(r)}})$) for the process associated with (F, R, ν) (resp. $(F^{(r)}, R^{(r)}, \rho^{(r)})$). It then holds that, for any $T > 0$,

$$d_P^{D(\mathbb{R}_{\geq 0}, F)}\big(P_\rho^\nu(X^\nu \in \cdot), P_\rho^{\nu^{(r)}}(X^{\nu^{(r)}} \in \cdot)\big) \leq e^{-T} + \frac{4}{\lambda} + \frac{4T}{\delta(\lambda - 1)},$$

where we recall that $d_P^{D(\mathbb{R}_{\geq 0}, F)}$ is the Prohorov metric on $\mathcal{P}(D(\mathbb{R}_{\geq 0}, F))$ induced by the Skorohod metric d_{J_1} on $D(\mathbb{R}_{\geq 0}, F)$ defined as (3.3).

Proof: Let $\tau_{F^{(r)}}$ be the first exit time of X from $F^{(r)}$. Fix $T > 0$ and define an event $E \coloneqq \{\tau_{F^{(r)}} > T\}$. On the event E, by the definition of the trace of X (recall it from Section 4.2), we have that $X(t) = \mathrm{tr}_{F^{(r)}} X(t)$ for all $t < T$. Thus,

$$d_{J_1}(X, \mathrm{tr}_{F^{(r)}} X) \leq \int_T^\infty e^{-s}\, ds = e^{-T}.$$

Thus, the desired result follows from Lemmas 4.8, 4.9, and 6.6. □

Lemma 6.8. *Under Assumption 1.12 or 1.15, it holds that*

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \mathbf{P}_n\bigl(\mathsf{P}_n\bigl(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) < \delta\bigr) > \eta\bigr) = 0, \quad \forall \eta > 0.$$

Proof: Suppose that Assumption 1.12 (resp. Assumption 1.15) is satisfied. Using the Skorohod representation theorem, we may assume that the convergence in Assumption 1.12(i) (resp. the convergence (1.6)) takes place almost surely on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Fix $\omega \in \Omega$. From Lemma 6.3 (resp. Lemma 6.5), we have that

$$\bigl(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}\mu_n^{\#}, \mathsf{P}_n(c_n^{-1}\nu_n \in \cdot)\bigr) \to \bigl(F, R, \rho, \mu, \mathsf{P}(\nu \in \cdot)\bigr)$$

in $\mathfrak{M}_\bullet(\tau^{\mathcal{M}} \times \tau_{\mathcal{P}}(\tau^{\mathcal{M}^{\mathrm{dis}}}))$. Then, by Theorem 3.9, we may assume that $(V_n, a_n^{-1}R_n, \rho_n)$ and (F, R, ρ) are embedded isometrically into a common boundedly-compact metric space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements in M, $V_n \to F$ in the Fell topology, $b_n^{-1}\mu_n^{\#} \to \mu$ vaguely, and $c_n^{-1}\nu_n \xrightarrow{\mathrm{d}} \nu$ in the vague-and-point-process topology. Then, for some $r \in (0,1)$, we have that $c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n r)) \xrightarrow{\mathrm{d}} \nu(B_R(\rho, r))$ (see Kallenberg, 2017, Theorem 4.11). It is then the case that, for all but countably many $\delta > 0$,

$$\lim_{n\to\infty} \mathsf{P}_n\bigl(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n r)) < \delta\bigr) = \mathsf{P}\bigl(\nu(B_R(\rho, r)) < \delta\bigr).$$

Since ν is of full support with probability 1, we deduce that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \mathsf{P}_n\bigl(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) < \delta\bigr) \leq \lim_{\delta\downarrow 0}\limsup_{n\to\infty} \mathsf{P}_n\bigl(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n r)) < \delta\bigr) = 0.$$

Hence, by (reverse) Fatou's lemma, we obtain that

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \mathbf{P}_n\bigl(\mathsf{P}_n\bigl(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) < \delta\bigr) > \eta\bigr) \\
&\leq \mathbb{P}\bigl(\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \mathsf{P}_n\bigl(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) < \delta\bigr) > \eta\bigr) \\
&= 0,
\end{aligned}$$

which completes the proof. □

The following results correspond to Lemmas 6.3 and 6.5 for random electrical networks.

Lemma 6.9. *Under Assumption 1.12, it holds that*

$$\Bigl(V_n, a_n^{-1}R_n, \rho_n, b_n\mu_n^{\#}, \mathsf{P}_n\bigl((c_n^{-1}\nu_n, \tilde{\mathcal{L}}_n^{\nu_n}) \in \cdot\bigr)\Bigr) \xrightarrow{\mathrm{d}} \Bigl(F, R, \rho, \mu, \mathsf{P}\bigl((\nu, \mathcal{L}^\nu) \in \cdot\bigr)\Bigr) \tag{6.3}$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}} \times \tau_{\mathcal{P}}(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau_{\mathcal{P}}(\tau^{J_1})))$.

Proof: This is proven similarly to Lemma 6.10 below. So, we omit the proof. □

Lemma 6.10. *Under Assumption 1.15, it holds that*

$$\Bigl(V_n, a_n^{-1}R_n, \rho_n, b_n\dot{\mu}_n^{\#}, \mathsf{P}_n\bigl((c_n^{-1}\nu_n, \dot{\pi}_n, \tilde{\mathcal{L}}_n^{\nu_n}) \in \cdot\bigr)\Bigr) \xrightarrow{\mathrm{d}} \Bigl(F, R, \rho, \dot{\mu}, \mathsf{P}\bigl((\nu, \dot{\pi}, \mathcal{L}^\nu) \in \cdot\bigr)\Bigr)$$

in the space $\mathfrak{M}_\bullet(\tau^{\mathcal{M}(\cdot\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0})} \times \tau_{\mathcal{P}}(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau^{\mathcal{M}(\cdot\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0})} \times \tau_{\mathcal{P}}(\tau^{J_1})))$.

Proof: For simplicity, we write

$$\mathfrak{M}_\bullet^1 := \mathfrak{M}_\bullet(\tau^{\mathcal{M}(\cdot\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0})} \times \tau_{\mathcal{P}}(\tau^{\mathcal{M}^{\mathrm{dis}}} \times \tau^{\mathcal{M}(\cdot\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0})} \times \tau_{\mathcal{P}}(\tau^{J_1}))).$$

For each $r > 0$, given ν_n, we write $(X_n^{\nu_n^{(a_n r)}}, \{P_x^{\nu_n^{(a_n r)}}\}_{x\in V_n^{(a_n r)}})$ for the process associated with $(V_n^{(a_n r)}, R_n^{(a_n r)}, \nu_n^{(a_n r)})$. Note that by Lemma 4.8, $X_n^{\nu_n^{(a_n r)}}$ has the same distribution as the trace of $X_n^{\nu_n}$ onto $V_n^{(a_n r)}$. For every $r > 0$, we set $(\dot{\mu}_n^{\#})^{\langle a_n r\rangle} = \dot{\mu}_n^{\#}|_{V_n^{(a_n r)}\times\mathbb{R}_{\geq 0}}$, $\dot{\pi}_n^{\langle a_n r\rangle} := \dot{\pi}_n|_{V_n^{(a_n r)}\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}}$, and

$$\mathcal{X}_n := \Big(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}\dot{\mu}_n^{\#}, \mathsf{P}_n\big((c_n^{-1}\nu_n, \dot{\pi}_n, \tilde{\mathcal{L}}_n^{\nu_n}) \in \cdot\big)\Big),$$
$$\tilde{\mathcal{X}}_n^{(r)} := \Big(V_n^{(a_n r)}, a_n^{-1}R_n^{(a_n r)}, \rho_n, b_n^{-1}(\dot{\mu}_n^{\#})^{(a_n r)}, \mathsf{P}_n\big((c_n^{-1}\nu_n^{(a_n r)}, \dot{\pi}_n^{\langle a_n r\rangle}, \tilde{\mathcal{L}}_n^{\nu_n^{(a_n r)}}) \in \cdot\big)\Big),$$

where we note that

$$\tilde{\mathcal{L}}_n^{\nu_n^{(a_n r)}} := P_{\rho_n}^{\nu_n^{(a_n r)}}\Big((X_n^{\nu_n^{(a_n r)}}(a_n b_n t))_{t\geq 0} \in \cdot\Big).$$

Similarly, we set $\dot{\mu}^{\langle r\rangle} = \dot{\mu}|_{F^{(r)}\times\mathbb{R}_{\geq 0}}$, $\dot{\pi}^{\langle r\rangle} := \dot{\pi}|_{F^{(r)}\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}}$, and

$$\mathcal{X} := \Big(F, R, \rho, \dot{\mu}, \mathsf{P}\big((\nu, \dot{\pi}, \mathcal{L}^{\nu}) \in \cdot\big)\Big),$$
$$\mathcal{X}^{(r)} := \Big(F^{(r)}, R^{(r)}, \rho^{(r)}, \dot{\mu}^{\langle r\rangle}, \mathsf{P}\big((\nu^{(r)}, \dot{\pi}^{\langle r\rangle}, \mathcal{L}^{\nu^{(r)}}) \in \cdot\big)\Big).$$

Define

$$Q_n^{(r)} := \int_r^{r+1} \mathbf{P}_n(\tilde{\mathcal{X}}_n^{(s)} \in \cdot)\, ds, \quad Q^{(r)} := \int_r^{r+1} \mathbf{P}(\mathcal{X}^{(s)} \in \cdot)\, ds,$$

which are probability measures on $\mathfrak{M}_\bullet^1$. (NB. The measurability of integrands can be verified by a similar argument to Noda, 2025+, Lemma 6.3.) Using the Skorohod representation theorem, we may assume that the convergence (1.6) takes place almost surely on some probability space with probability measure $\mathbb{P}$. Fix a realization. For $r > 0$ such that

$$(V_n^{(a_n r)}, a_n^{-1}R_n^{(a_n r)}, \rho_n, b_n^{-1}(\dot{\mu}_n^{\#})^{\langle a_n r\rangle}) \to (F^{(r)}, R^{(r)}, \rho^{(r)}, \dot{\mu}^{\langle r\rangle}),$$

we have from Lemma 6.5 that

$$\begin{aligned}&\big(V_n^{(a_n r)}, a_n^{-1}R_n^{(a_n r)}, \rho_n^{(a_n r)}, b_n^{-1}(\dot{\mu}_n^{\#})^{\langle a_n r\rangle}, \mathsf{P}_n((c_n^{-1}\nu_n^{(a_n r)}, \dot{\pi}_n^{\langle a_n r\rangle}) \in \cdot)\big)\\ \to&\big(F^{(r)}, R^{(r)}, \rho^{(r)}, \dot{\mu}^{\langle r\rangle}, \mathsf{P}((\nu^{(r)}, \dot{\pi}^{\langle r\rangle}) \in \cdot)\big).\end{aligned}$$

Note that the rooted resistance metric spaces $(V_n^{(r)}, a_n^{-1}R_n^{(a_n r)}, \rho_n^{(a_n r)})$ satisfy Assumption 1.5(ii). Thus, following the proof of Theorem 1.6, we obtain that $\tilde{\mathcal{X}}_n^{(r)} \to \mathcal{X}^{(r)}$ almost surely under $\mathbb{P}$. Hence, we deduce that $Q_n^{(r)} \to Q^{(r)}$ weakly for every $r > 0$.

Next, we estimate the Prohorov distance between $Q_n^{(r)}$ and $\mathbf{P}_n(\mathcal{X}_n \in \cdot)$. Define an event

$$E_n(s, \lambda, \delta, \eta) := \big\{a_n^{-1}R_n(\rho_n, B_{R_n}(\rho_n, a_n s)) \geq \lambda\big\} \cap \big\{\mathsf{P}_n\big(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) < \delta\big) \leq \eta\big\}$$

Fix $r, \delta, \eta > 0$, $s \in (r, r+1)$ and $\lambda > 1$, and suppose that the event $E_n(s, \lambda, \delta, \eta)$ occurs. To estimate the distance between $\tilde{\mathcal{X}}_n^{(s)}$ and $\mathcal{X}_n$, we think that $(V_n^{(a_n r)}, a_n^{-1}R_n^{(a_n r)}, \rho_n^{(a_n r)})$ are embedded into $(V_n, a_n^{-1}R_n, \rho_n)$ in the obvious way. On the event $\{c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) \geq \delta\}$, we have from Lemma 6.7 that

$$d_P^{D(\mathbb{R}_{\geq 0}, V_n)}\big(\tilde{\mathcal{L}}_n^{\nu_n^{(a_n s)}}, \tilde{\mathcal{L}}_n^{\nu_n}\big) \leq e^{-T} + \frac{4}{\lambda} + \frac{4T}{\delta(\lambda - 1)} =: c(\lambda, \delta, T),$$

where we note that the Skorohod metric on $D(\mathbb{R}_{\geq 0}, V_n)$ is induced from the scaled metric $a_n^{-1}R_n$. By definition, it is easy to check that

$$d_{\mathcal{M}^{\mathrm{dis}}}^{V_n,\rho_n}(c_n^{-1}\nu_n^{(a_n s)}, c_n^{-1}\nu_n) \leq e^{-s}, \quad d_V^{V_n\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0},(\rho_n,0,1)}(\dot{\pi}_n^{\langle a_n s\rangle}, \dot{\pi}_n) \leq e^{-s}$$

(recall these metrics from Sections 2.1 and 2.2). Thus, if we write d_P for the Prohorov metric on $\mathcal{M}^{\mathrm{dis}}(V_n) \times \mathcal{M}(V_n \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}) \times \mathcal{P}(D(\mathbb{R}_{\geq 0}, V_n))$, then we deduce from Lemma 6.6 that

$$\begin{aligned}
&d_P\Big(\mathsf{P}_n\big((c_n^{-1}\nu_n^{(a_n s)}, \tilde{\mathcal{L}}_n^{\nu_n^{(a_n s)}}) \in \cdot\big), \mathsf{P}_n\big((c_n^{-1}\nu_n, \tilde{\mathcal{L}}_n^{\nu_n}) \in \cdot\big)\Big)\\
&\leq \mathsf{P}_n\big(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) < \delta\big) + e^{-s} + c(\lambda, \delta, T)\\
&\leq \eta + e^{-s} + c(\lambda, \delta, T),
\end{aligned}$$

where the last inequality follows from the definition of $E_n(s, \lambda, \delta, \eta)$. By definition, we have that

$$d_{\bar{H}}^{V_n, \rho_n}(V_n^{(a_n s)}, V_n) \leq e^{-s}, \quad d_V^{V_n \times \mathbb{R}_{\geq 0}, (\rho_n, 0)}(b_n^{-1}(\dot{\mu}_n^{\#})^{\langle a_n s\rangle}, b_n^{-1}\dot{\mu}_n^{\#}) \leq e^{-s}.$$

Thus, on the event $E_n(s, \lambda, \delta, \eta)$, the distance between $\tilde{\mathcal{X}}_n^{(s)}$ and $\mathcal{X}_n$ in $\mathfrak{M}_\bullet^1$ is bounded above by $\eta + e^{-s} + c(\lambda, \delta, T)$. If we simply write d_P' for the Prohorov metric on the space of the probability measures on $\mathfrak{M}_\bullet^1$, then it follows from Lemma 6.6 that

$$\begin{aligned}
&d_P'\big(\mathbf{P}_n(\tilde{\mathcal{X}}_n^{(s)} \in \cdot), \mathbf{P}_n(\mathcal{X}_n \in \cdot)\big)\\
&\leq \mathbf{P}_n\big(a_n^{-1}R_n(\rho_n, B_{R_n}(\rho_n, a_n s)^c) < \lambda\big) + \mathbf{P}_n\big(\mathsf{P}_n\big(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) < \delta\big) > \eta\big)\\
&\quad + \eta + e^{-s} + c(\lambda, \delta, T)\\
&\leq \mathbf{P}_n\big(a_n^{-1}R_n(\rho_n, B_{R_n}(\rho_n, a_n r)^c) < \lambda\big) + \mathbf{P}_n\big(\mathsf{P}_n\big(c_n^{-1}\nu_n(B_{R_n}(\rho_n, a_n)) < \delta\big) > \eta\big)\\
&\quad + \eta + e^{-r} + c(\lambda, \delta, T)\\
&=: c'(n, r, \lambda, \delta, \eta, T).
\end{aligned}$$

where we use that $s > r$ at the second inequality. The above inequality implies that

$$d_P'\big(Q_n^{(r)}, \mathbf{P}_n(\mathcal{X}_n \in \cdot)\big) \leq c'(n, r, \lambda, \delta, \eta, T).$$

Combining this with Assumption 1.12(ii) and Lemma 6.8, by letting $n \to \infty$, $r \to \infty$, $\lambda \to \infty$, $T \to \infty$, $\delta \to 0$, and then $\eta \to 0$, we obtain that

$$\lim_{r \to \infty} \limsup_{n \to \infty} d_P'\big(Q_n^{(r)}, \mathbf{P}_n(\mathcal{X}_n \in \cdot)\big) = 0.$$

A similar argument yields that $Q^{(r)} \to \mathbf{P}(\mathcal{X} \in \cdot)$ weakly. Recalling that $Q_n^{(r)} \to Q^{(r)}$ weakly, we deduce that $\mathcal{X}_n \xrightarrow{\mathrm{d}} \mathcal{X}$, which completes the proof. □

By Lemma 6.9 (resp. Lemma 6.10), Theorem 1.13 (resp. 1.16) is proven similarly to Theorem 1.6 (resp. Theorem 1.9) as follows.

Proof of Theorems 1.13 and 1.16: We only give the proof of Theorem 1.13. Theorem 1.16 is proven similarly by using Theorem 5.11 and Lemma 6.10. By Lemma 6.9 and the Skorohod representation theorem, we may assume that the convergence (6.3) takes place almost surely on some probability space. Fix such a realization. Then, by Theorem 3.9, it is possible to embed $(V_n, a_n^{-1}R_n, \rho_n)$ and (F, R, ρ) isometrically into a common boundedly-compact metric space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements in M, $V_n \to F$ in the Fell topology, $b_n^{-1}\mu_n^{\#} \to \mu$ vaguely, and $(c_n^{-1}\nu_n, \tilde{\mathcal{L}}_n^{\nu_n}) \xrightarrow{\mathrm{d}} (\nu, \mathcal{L}^{\nu})$. Using the Skorohod representation theorem again, we may assume that $(c_n^{-1}\nu_n, \tilde{\mathcal{L}}_n^{\nu_n}) \to (\nu, \mathcal{L}^{\nu})$ almost surely. Then, from Theorem 5.7, we deduce the desired result. □

7. Applications

In this section, we apply the main results to several examples. Throughout this section, for simplicity, we assume that $\ell(u) = 1$ for all sufficiently large u (recall the slowly varying function ℓ from Definition 1.3).

7.1. *The Sierpiński gasket.* The Sierpiński gasket is a well-studied self-similar fractal. For details on the probabilistic analysis of fractals see Barlow (1998); Kigami (2012), for example. Here, we briefly confirm that our results apply to a sequence of electrical networks that converge to the Sierpiński gasket.

Set $\rho_0 := (0,0) \in \mathbb{R}^2$. Let $V_0 := \{x_1, x_2, x_3\} \subseteq \mathbb{R}^2$ consist of the vertices of an equilateral triangle of side length 1 with $x_1 = \rho_0$. Write $\psi_i(x) := (x + x_i)/2$ for $i = 1,2,3$. We then inductively define $V_n := \bigcup_{i=1}^3 \psi_i(V_{n-1})$. The electrical network $G_n = (V_n, E_n, c_n)$ is defined by setting E_n to be the set of pairs of elements of V_n at a Euclidean distance 2^{-n} apart and $c_n(x,y) := 1$ if $\{x,y\} \in E_n$. Write $\rho_n := \rho_0$, R_n for the resistance metric on V_n associated with G_n, and $\mu_n^{\#}$ for the counting measure on V_n. Then, it is elementary to check that, for some $(F, R, \rho, \mu) \in \mathbb{F}$,

$$(V_n, (3/5)^n R_n, \rho, 3^{-n}\mu_n^{\#}) \to (F, R, \rho, \mu)$$

in $\mathbb{G}_c$, i.e., the pointed Gromov–Hausdorff–Prohorov topology (recall this from Section 3.3). The set F is the Sierpiński gasket and μ is a self-similar measure corresponding to the $\log_2 3$ Hausdorff measure in the Euclidean metric. Moreover, since the degree of any vertex in $V_n \setminus V_0$ is 4, we deduce that

$$(V_n, (3/5)^n R_n, \rho_n, 3^{-n}\dot{\mu}_n^{\#}) \to (F, R, \rho, \mu \otimes \delta_{\{4\}}),$$

which implies that $(G_n)_{n\geq 1}$ satisfies Assumption 1.8. Thus, by Theorem 1.9, we obtain the aging and sub-aging results as follows. Let $\{(x_i, v_i)\}_{i\in I}$ be a Poisson point process on $F \times \mathbb{R}_{>0}$ with intensity $\mu(dx)\,\alpha v^{-1-\alpha}dv$ and define $\nu(dx) := \sum_{i\in I} v_i \delta_{x_i}(dx)$. Given ν, we write $(X^\nu, \{P_x^\nu\}_{x\in F})$ for the process associated with (F, R, ν). We simply write $X_n^{\nu_n} := X_{G_n}^{\nu_{G_n}}$, which is the BTM on G_n (see Definition 1.4). We denote by $P_{\rho_n}^{\nu_n}$ for the underlying probability measure for $X_n^{\nu_n}$ started at ρ_n. Set $\tilde{c}_n := (5/3)^n \cdot 3^{n/\alpha}$. As a consequence of Theorem 1.9, we obtain that

$$P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(\tilde{c}_n s) = X_n^{\nu_n}(\tilde{c}_n t)\big) \to P_\rho^\nu\big(X^\nu(s) = X^\nu(t)\big),$$
$$P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(\tilde{c}_n t) = X_n^{\nu_n}(\tilde{c}_n t + t'),\ \forall t' \in [0, 3^{n/\alpha}s]\big) \to \sum_{i\in I} e^{-4s/v_i} P_\rho^\nu(X^\nu(t) = x_i).$$

7.2. *The random conductance model.* Given a connected simple graph, the random conductance model is defined by placing random conductances on the edges. So, it is a random electrical network. Here, we consider a very simple model, the random conductance model on $\mathbb{Z}$, but similar results hold for other graphs such as the Sierpiński gasket graph. Note that the BTM we consider here is not a usual random walk on the random conductance model because the speed of the BTM is determined by random traps, which are irrelevant to random conductances. However, when the random conductances are heavy tailed, it is natural to think (sub-)aging occurs because edges with very heavy conductances (resp. very light conductances) serve as traps (resp. walls) for the random walk. Indeed, Croydon, Kious, and Scali (2025) confirmed this for the constant speed random walk on the random conductance model on $\mathbb{Z}$. For details regarding application of resistance form theory to the random conductance model, see Croydon et al. (2017, Section 6) and Noda (2024b, Appendix A).

Fix i.i.d. positive random variables $\{\zeta_i\}_{i\in\mathbb{Z}}$ built on a probability space with probability measure P such that $c < \zeta_i < c'$ almost surely for some deterministic constants $c, c' \in (0,\infty)$. By scaling, without loss of generality, we may assume that $E[\zeta_0^{-1}] = 1$. Define a random electrical network G_n as follows. Set $V_n := 2^{-n}\mathbb{Z}$ and E_n to be the set of the pairs of elements of V_n at a Euclidean distance 2^{-n} apart. We place the random conductance ζ_i on the edge $\{i/2^n, (i+1)/2^n\}$. We equip G_n with the root $\rho_{G_n} := 0$. Then, from Noda (2024b, Theorem A.3), we deduce that $(V_{G_n}, 2^{-n}R_{G_n}, \rho_{G_n})_{n\geq 1}$ satisfies Assumption 1.12(ii) and

$$(V_{G_n}, 2^{-n}R_{G_n}, \rho_{G_n}, 2^{-n}\mu_{G_n}^{\#}) \xrightarrow{\mathrm{p}} (\mathbb{R}, d^{\mathbb{R}}, 0, \mathrm{Leb}),$$

where $d^{\mathbb{R}}$ and Leb denote the Euclidean metric on $\mathbb{R}$ and the Lebesgue measure, respectively. Moreover, since the total conductance at $i/2^n \in V_{G_n}$ is $\zeta_{i-1} + \zeta_i$, it is possible to show that

$$(V_{G_n}, 2^{-n} R_{G_n}, \rho_{G_n}, 2^{-n} \dot{\mu}^{\#}_{G_n}) \xrightarrow{\mathrm{p}} (\mathbb{R}, d^{\mathbb{R}}, 0, \mathrm{Leb} \otimes P(\zeta_0 + \zeta_1 \in \cdot)).$$

Hence, we can apply Theorem 1.16 and obtain the aging and sub-aging results as follows. We let $\{(x_i, w_i, v_i)\}_{i \in I}$ be a Poisson point process on $F \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ with intensity $dx\, P(\zeta_0 + \zeta_1 \in dw)\, \alpha v^{-1-\alpha} dv$ and define $\nu(dx) := \sum_{i \in I} v_i \delta_{x_i}(dx)$. Given ν, we write $(X^\nu, \{P^\nu_x\}_{x \in \mathbb{R}})$ for the process associated with $(\mathbb{R}, d^{\mathbb{R}}, \nu)$. We simply write $X^{\nu_n}_n := X^{\nu_{G_n}}_{G_n}$, which is the BTM on G_n (see Definition 1.4). We denote by $P^{\nu_n}_{\rho_n}$ for the underlying probability measure for $X^{\nu_n}_n$ started at ρ_n. Set $\tilde{c}_n := 2^n \cdot 2^{n/\alpha}$. As a consequence of Theorem 1.9, we obtain that

$$P^{\nu_n}_{\rho_n}\big(X^{\nu_n}_n(\tilde{c}_n s) = X^{\nu_n}_n(\tilde{c}_n t)\big) \to P^\nu_\rho\big(X^\nu(s) = X^\nu(t)\big),$$
$$P^{\nu_n}_{\rho_n}\big(X^{\nu_n}_n(\tilde{c}_n t) = X^{\nu_n}_n(\tilde{c}_n t + t'),\ \forall t' \in [0, 2^{n/\alpha} s]\big) \to \sum_{i \in I} e^{-w_i s / v_i} P^\nu_\rho(X^\nu(t) = x_i).$$

As seen above, the effect of random conductances disappears in the scaling limit of the aging functions. This is called homogenization in the study of the random conductance model. On the other hand, the scaling limit of the sub-aging functions is affected by random conductances. This is because the sub-aging functions capture the behaviors of the BTMs on a time scale shorter than that of homogenization.

7.3. *The critical Galton-Watson tree.* We apply our results to the critical Galton-Watson tree conditioned on its size, which is naturally regarded as an electrical network by placing conductance 1 on each edge. It is well-known that the suitably conditioned scaled critical Galton-Watson trees converge to the continuum random tree in the Gromov–Hausdorff–Prohorov topology (Abraham et al., 2013; Aldous, 1993; Noda, 2025+). Thus, Theorem 1.13 immediately shows that there is aging for the BTMs on the critical Galton-Watson trees. To apply the sub-aging result (Theorem 1.13), we need to prove that a uniformly chosen random vertex and its degree jointly converge. The former convergence is related to the global properties of the graphs, while the latter convergence is related to the local properties of the graphs. Usually, these global and local properties are studied independently of each other, and their respective convergences are already known. The convergence of uniformly chosen random vertices is an immediate consequence of the Gromov–Hausdorff–Prohorov convergence, and the convergence of their degrees were shown in Janson (2012, Theorem 7.11). In this section, we prove the joint convergence of uniformly chosen random vertices and their degrees, Corollary 7.4 below. The proof of this result relies on Thévenin (2020), where the joint convergence of two associated functions was shown: the contour functions and functions that count the number of vertices having certain outdegrees. For details of notion regarding plane trees, we refer to Thévenin (2020).

Let T be a plane tree T with n vertices. We write ρ_T for the root and d_T for the graph metric on T. By declaring that $\{u, v\} \subseteq V$ is an edge if and only if $d_T(u, v) = 1$ and placing conductance 1 on each edge, we regard T as an electrical network. We then define $\mu_T(x)$ to be the total conductance at x, which is exactly the degree of x, $\mu^{\#}_T$ to be the counting measure, and $\dot{\mu}^{\#}_T$ to be the pushforward of $\mu^{\#}_T$ by the map $\psi_T(x) = (x, \mu_T(x))$. Let $v_0 = \rho_T, v_1, \ldots, v_{n-1}$ be the vertices of T in the lexicographical order (also known as the depth first order). We define a map $p_T \colon [0, n-1] \to T$ by setting $p_T(i) := v_i$ for $i \in \mathbb{Z}$ and $p_T(t) := p_T(\lfloor t \rfloor)$. We also define the height function $H_T \colon [0, n-1] \to \mathbb{R}_{\geq 0}$ by setting $H_T(i) := d_T(v_i, v_0)$ and linearly interpolating between integers. We next introduce functions $N^{(k)}_T, D^{(k)}_T : [0, n-1] \to \mathbb{R}_{\geq 0}$ that record local structures of T. Recall that the *outdegree* of a vertex is the number of its children. For every non-negative integer

k and $t \in [0, n-1]$, we define $N_T^{(k)}(t)$ by

$$N_T^{(k)}(t) := \#\{p_T(s) \in T \mid s \le t \text{ and the outdegree of } p_T(s) \text{ is } k\}. \tag{7.1}$$

We similarly define $D_T^{(k)}$ by

$$D_T^{(k)}(t) := \#\{p_T(s) \in T \mid s \le t \text{ and the degree of } p_T(s) \text{ is } k\}. \tag{7.2}$$

To describe scaling limits of plane trees, we introduce real trees. For details, see Le Gall (2006), for example. Write $\mathscr{E}$ for the space of excursions, that is,

$$\mathscr{E} := \{f \in C(\mathbb{R}_{\ge 0}, \mathbb{R}_{\ge 0}) : f(0) = 0, \exists \sigma_f < \infty \text{ such that } f(x) > 0\ \forall x \in (0, \sigma_f), f(x) = 0\ \forall x \ge \sigma_f\}.$$

Given a function $f \in C([0, \sigma], \mathbb{R}_{\ge 0})$ with $f(0) = f(\sigma) = 0$ and $f(x) > 0$ for all $x \in (0, \sigma)$, we will abuse notation by identifying f with the function $g \in \mathscr{E}$ which has $g(x) = f(x)$, $0 \le x \le \sigma$ and $g(x) = 0$, $x \ge \sigma$. We equip $\mathscr{E}$ with the metric induced by the supremum norm $\|\cdot\|_\infty$. Given an excursion $f \in \mathscr{E}$, we define a pseudometric $\bar{d}_f$ on $[0, \sigma_f]$ by setting

$$\bar{d}_f(s, t) := f(s) + f(t) - 2 \inf_{u \in [s \wedge t, s \vee t]} f(u).$$

Then, we use the equivalence

$$s \sim t \qquad \Leftrightarrow \qquad \bar{d}_f(s, t) = 0$$

to define $T_f := [0, \sigma_f]/\sim$. Let $p_f : [0, \sigma_f] \to T_f$ be the canonical projection. It is then elementary to check that

$$d_f(p_f(s), p_f(t)) := \bar{d}_f(s, t)$$

defines a metric on T_f. The metric space (T_f, d_f) is called a *real tree* coded by f. The canonical Radon measure on T_f is given by $\mu_f := \mathrm{Leb} \circ (p_f)^{-1}$, where Leb stands for the one-dimensional Lebesgue measure. We define the root ρ_f by setting $\rho_f := p_f(0)$.

In Theorem 7.3 below, we show that if scaled height functions $\tilde{H}_T$ and scaled out-degree-counting functions $\tilde{D}_T^{(k)}$ converge, then the associated plane trees converge and moreover uniformly chosen random vertices and their degrees also converge jointly. To do this, we use the following result on convergence of Lebesgue-Stieltjes integrals in terms of convergence of integrators. Given a non-decreasing cadlag function g on $[0, 1]$, we denote by s_g the Lebesgue-Stieltjes measure associated with g.

Lemma 7.1. *Fix a metric space (M, d^M). Let $g_1, g_2, \dots$ be non-decreasing functions in $D([0, 1], \mathbb{R})$ such that $g_n \to g$ in the usual J_1-Skorohod topology for some strictly increasing function $g \in D([0, 1], \mathbb{R})$, and let $q, q_1, q_2, \dots$ be measurable functions from $[0, 1]$ to M such that $q_n \to q$ in the supremum norm $\|\cdot\|_\infty$. Then it holds that $s_{g_n} \circ q_n^{-1} \to s_g \circ q^{-1}$ weakly as measures on M.*

Proof: Define $\tilde{g}_n \in D(\mathbb{R}_{\ge 0}, \mathbb{R}_{\ge 0})$ by setting

$$\tilde{g}_n(t) := \begin{cases} g_n(t) - g_n(0), & t \in [0, 1], \\ (t - 1) + g_n(1) - g_n(0), & t > 1. \end{cases}$$

Similarly, we define $\tilde{g}$. Noting that $g_n(0) \to g(0)$ and $g_n(1) \to g(1)$, it is easy to check that $\tilde{g}_n \to \tilde{g}$ in the usual J_1-Skorohod topology. Let $\tilde{\tau}_n \in D(\mathbb{R}_{\ge 0}, \mathbb{R}_{\ge 0})$ be the right-continuous inverse of $\tilde{g}_n$ given by

$$\tilde{\tau}_n(t) := \inf\{s \ge 0 \mid \tilde{g}_n(s) > t\}.$$

Similarly, we let $\tilde{\tau}$ be the right-continuous inverse of $\tilde{g}$. Since $\tilde{g}$ is strictly increasing, we deduce from Whitt (2002, Corollary 13.6.4) that $\tilde{\tau}_n \to \tilde{\tau}$ with respect to $\|\cdot\|_\infty$. Fix a bounded continuous function F on M. By Chen and Fukushima (2012, Lemma A.3.7), we have that

$$\int_{(0,1]} F(q_n(t))\, s_{\tilde{g}_n}(dt) = \int_0^\infty 1_{(0,1]}(\tilde{\tau}_n(t)) F(q_n \circ \tilde{\tau}_n(t))\, dt. \tag{7.3}$$

If $\tilde{\tau}_n(t) < 2$, then it is the case that $t < \tilde{g}_n(2)$. Since $(\tilde{g}_n(2))_{n\geq 1}$ is bounded, for some $K > 0$, it holds that

$$1_{(0,1]}(\tilde{\tau}_n(t)) \leq 1_{(0,2)}(\tilde{\tau}_n(t)) \leq 1_{(0,K)}(t).$$

Thus, we can apply the dominated convergence theorem to (7.3), and we obtain that

$$\lim_{n\to\infty} \int_{(0,1]} F(q_n(t))\, s_{\tilde{g}_n}(dt) = \int_0^\infty 1_{(0,1]}(\tilde{\tau}(t)) F(q \circ \tilde{\tau}(t))\, dt = \int_{(0,1]} F(q(t))\, s_{\tilde{g}}(dt),$$

where we use Chen and Fukushima (2012, Lemma A.3.7) to deduce the second equality. By the definitions of $\tilde{g}_n$ and $\tilde{g}$, we have that $s_{\tilde{g}_n} = s_{g_n}$ and $s_{\tilde{g}} = s_g$ on $[0,1]$. Therefore, the proof is completed. □

Convergence of scaled plane trees in the Gromov–Hausdorff–Prohorov topology follows from convergence of coding functions (Abraham et al., 2013, Proposition 3.3). Furthermore, the convergence of the coding functions of plane trees T_n leads to the convergence of the projections p_{T_n} to the projection of the limiting real tree. This seems to be a basic and well-known fact, but it is asserted as follows for the first time in a rigorous form by using the general theory of Gromov–Hausdorff-type topologies introduced in Section 3.

Lemma 7.2. *For each $n \geq 1$, let T_n be a plane trees with n vertices. Let $(\alpha_n)_{n\geq 1}$ be a sequence of positive real numbers with $\alpha_n \to \infty$. Assume that there exists an $f \in \mathscr{E}$ with $\sigma_f = 1$ such that*

$$\frac{1}{\alpha_n} H_{T_n}((n-1)\cdot) \to f \tag{7.4}$$

in $C([0,1], \mathbb{R}_{\geq 0})$. Then it holds that

$$(T_n, \alpha_n^{-1} d_{T_n}, \rho_{T_n}, n^{-1}\mu^{\#}_{T_n}, p_{T_n}((n-1)\cdot)) \to (T_f, d_f, \rho_f, \mu_f, p_f)$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}} \times \tau^{\mathrm{Unif}})$.

Proof: Let $\mathcal{C}_n$ be a correspondence between T_n and T_f given by

$$\mathcal{C}_n := \{(v, x) \in T_n \times T_f \mid v = p_{T_n}((n-1)t) \text{ and } x = p_f(t) \text{ for some } t \in [0,1]\}.$$

(see Burago et al., 2001 for the definitions of a correspondence between sets). Then one can check, by applying the same argument as that used to prove Noda (2025+, Proposition 8.3), that the distortion $\operatorname{dis}\mathcal{C}_n$ of $\mathcal{C}_n$ given below converges to 0:

$$\operatorname{dis}\mathcal{C}_n := \sup\{|\alpha_n^{-1} d_{T_n}(u, u') - d_f(x, x')| \mid (u,x), (u',x') \in \mathcal{C}_n\}$$

(see also Burago et al., 2001 for the notion of distortion). Using the correspondence $\mathcal{C}_n$, we define a metric d_n on the disjoint union $S_n := T_n \sqcup T_f$, extending $\alpha_n^{-1} d_{T_n}$ and d_f, by setting, for $u \in T_n$ and $x \in T_f$,

$$d_n(u, x) := \inf\left\{\alpha_n^{-1} d_{T_n}(u, u') + \frac{1}{2}\operatorname{dis}\mathcal{C}_n + n^{-1} + d_f(x', x) \mid (u', x') \in \mathcal{C}_n\right\}.$$

In the obvious way, we regard $\mu^{\#}_{T_n}$ and μ_f as measures on S_n and regard $p_{T_n}((n-1)\cdot)$ and p as maps from $[0,1]$ to S_n. It is then the case that

$$d_n(p_{T_n}((n-1)t), p(t)) \leq \frac{1}{2}\operatorname{dis}\mathcal{C}_n + n^{-1},$$

and so we obtain that

$$(T_n, a_n^{-1} d_{T_n}, \rho_{T_n}, p_{T_n}((n-1)\cdot)) \to (T_f, d_f, \rho_f, p).$$

Thus, by Theorem 3.9, we may assume that (T_n, α_n^{-1}) and (T_f, d_f) are embedded isometrically into a common (compact) metric space (M, d^M) in such a way that $T_n \to T_f$ in the Hausdorff topology,

$\rho_{T_n} = \rho_f$ as elements in M, and $p_{T_n}((n-1)\cdot) \to p_f$ with respect to $\|\cdot\|_\infty$. This immediately yields that $\mathrm{Leb} \circ p_{T_n}((n-1)\cdot)^{-1} \to \mathrm{Leb} \circ p_f^{-1} = \mu_f$ weakly. For any $u \in T_n \setminus \{\rho_{T_n}\}$, we have that

$$\mathrm{Leb} \circ p_{T_n}((n-1)\cdot)^{-1}(\{u\}) = (n-1)^{-1}.$$

Hence, for any $A \subseteq T_n$,

$$|\,\mathrm{Leb} \circ p_{T_n}((n-1)\cdot)^{-1}(A) - (n-1)^{-1}\mu^{\#}_{T_n}(A)| \le (n-1)^{-1}.$$

This, combined with $\mathrm{Leb} \circ p_{T_n}((n-1)\cdot)^{-1} \to \mu_f$, implies that $n^{-1}\mu^{\#}_{T_n} \to \mu_f$ weakly. This completes the proof. □

We now prove the main result.

Theorem 7.3. *Suppose that we are in the same setting as Lemma 7.2. For each $n \ge 1$ and $k \ge 1$, we define Set $I(t) := t$ for $t \in [0,1]$. In addition to (7.4), assume that there exists a probability measure $p = (p_k)_{k\ge 1}$ on $\mathbb{Z}_{\ge 0}$ such that*

$$\frac{1}{n} N^{(k)}_{T_n}((n-1)\cdot) \to p_k\, I \tag{7.5}$$

in $D([0,1], \mathbb{R}_{\ge 0})$ for every $k \ge 0$, where we recall $N^{(k)}_{T_n}$ from (7.1). Then it holds that

$$(T_n, \alpha_n^{-1} d_{T_n}, \rho_{T_n}, n^{-1}\dot{\mu}^{\#}_{T_n}) \to (T_f, d_f, \rho_f, \mu_f \otimes \tilde{p})$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\mathbb{R}_{\ge 0})})$, where $\tilde{p} = (\tilde{p}_k)_{k\ge 1}$ is a probability measure on $\mathbb{N}$ given by $\tilde{p}_k := p_{k-1}$.

Proof: Recall the function $D^{(k)}_{T_n}$ from (7.2). Since the degree of a vertex, except for the root, is the outdegree plus 1, it is easy to deduce from (7.5) that

$$\frac{1}{n} D^{(k)}_{T_n}((n-1)\cdot) \to p_k\, I$$

in $D([0,1], \mathbb{R}_{\ge 0})$ for every $k \ge 1$. By Theorem 3.9 and Lemma 7.2, we may assume that $(T_n, \alpha_n^{-1} d_{T_n}, \rho_{T_n})$ and (T_f, d_f, ρ_f) are embedded isometrically into a common rooted compact metric space (M, d^M, ρ_M) in such a way that $T_n \to T_f$ in the Hausdorff topology, $\rho_{T_n} = \rho_f = \rho_M$ as elements in M, $n^{-1}\mu^{\#}_{T_n} \to \mu_f$ weakly, and $\tilde{p}_{T_n} \to p_f$ with respect to $\|\cdot\|_\infty$. For each $k \ge 1$, we write $T_n^{(k)}$ for the set of vertices of T_n whose degree is k. Then one can check that, for any subset $A \subseteq T_n$,

$$\mu^{\#}_{T_n}|_{T_n^{(k)}}(A) = D^{(k)}_{T_n}(0) \cdot \delta_{\{0\}}(p_{T_n}^{-1}(A)) + s_{D^{(k)}_{T_n}}(p_{T_n}^{-1}(A)).$$

This, combined with Lemma 7.1, yields that $n^{-1}\mu^{\#}_{T_n}|_{T_n^{(k)}} \to p_{k-1}\,\mu_f$ weakly for every $k \ge 1$. Fix compactly supported continuous functions $f_1 \colon M \to \mathbb{R}$ and $f_2 \colon \mathbb{R}_{\ge 0} \to \mathbb{R}$. We have that

$$\begin{aligned}
\lim_{n\to\infty} \int f_1(x) f_2(w)\, n^{-1}\dot{\mu}^{\#}_{T_n}(dx\,dw) &= \lim_{n\to\infty} n^{-1} \sum_{v\in T_n} f_1(v) f_2(\mu_{T_n}(v)) \\
&= \lim_{n\to\infty} n^{-1} \sum_{k\ge 1} f_2(k) \sum_{v\in T_n^{(k)}} f_1(v) \\
&= \sum_{k\ge 1} f_2(k) \int f_1(x)\, n^{-1}\mu^{\#}_n|_{T_n^{(k)}}(dx) \\
&= \sum_{k\ge 1} f_2(k) \int f_1(x) p_{k-1}\, \mu_f(dx) \\
&= \int f_1(x) f_2(w)\, \mu_f \otimes \tilde{p}(dx\,dw).
\end{aligned}$$

This implies that $n^{-1}\dot{\mu}^{\#}_{T_n} \to \mu_f \otimes \tilde{p}$ vaguely. Hence, it remains to prove the tightness of $(n^{-1}\dot{\mu}^{\#}_{T_n})_{n\geq 1}$ in the weak topology, and this can be verified by the following:

$$\begin{aligned}
\lim_{L\to\infty}\limsup_{n\to\infty} n^{-1}\dot{\mu}^{\#}_{T_n}\bigl(M\times(L,\infty)\bigr) &= 1-\lim_{L\to\infty}\liminf_{n\to\infty} n^{-1}\dot{\mu}^{\#}_{T_n}\bigl(M\times[1,L]\bigr)\\
&= 1-\lim_{L\to\infty}\liminf_{n\to\infty}\sum_{i=1}^{L} n^{-1}\mu^{\#}_{T_n}|_{T_n^{(k)}}(M)\\
&= 1-\lim_{L\to\infty}\sum_{i=1}^{L} p_{k-1}\,\mu_f(M)\\
&= 1-\lim_{L\to\infty}\sum_{i=1}^{L} p_{k-1}\\
&= 0.
\end{aligned}$$

□

We apply the above result to the critical Galton-Watson tree conditioned on its size. Let $p = (p_k)_{k\geq 0}$ be a probability measure on $\mathbb{Z}_{\geq 0}$ such that $\sum_{k\geq 1} kp_k = 1$, $p_1 < 1$, and its variance is 1. For simplicity, we assume that p is aperiodic, that is, the greatest common divisor of the set $\{k \mid p_k > 0\}$ is 1. We write T^{GW} for the Galton-Watson tree with the offspring distribution p, and write T^{GW}_n for a random plane tree having the same distribution as T^{GW} conditioned to have exactly n vertices, which is well-defined for all sufficiently large n by the aperiodicity of p. We define $e = (e(t))_{t\in[0,1]} \in \mathscr{E}$ to be the normalized Brownian excursion.

Corollary 7.4. *Define a probability measure $\tilde{p} = (\tilde{p}_k)_{k\geq 1}$ on $\mathbb{N}$ by setting $\tilde{p}_k := p_{k-1}$. Then it holds that*

$$(T^{GW}_n, n^{-1/2}d_{T^{GW}_n}, \rho_{T^{GW}_n}, n^{-1}\dot{\mu}^{\#}_{T^{GW}_n}) \xrightarrow{\mathrm{d}} (T_{2e}, d_{2e}, \rho_{2e}, \mu_{2e}\otimes\tilde{p})$$

in the space $\mathfrak{K}_{\bullet}(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\mathbb{R}_{\geq 0})})$.

Proof: From Duquesne (2003, Theorem 3.1) and Thévenin (2020, Theorem 1.1), we have that

$$\begin{aligned}
\bigl(n^{-1/2}H_{T^{GW}_n}((n-1)t)\bigr)_{t\in[0,1]} &\xrightarrow{\mathrm{d}} (2e(t))_{t\in[0,1]},\\
\bigl(n^{-1}N^{(k)}_{T^{GW}_n}((n-1)t)\bigr)_{t\in[0,1]} &\xrightarrow{\mathrm{p}} (p_k I(t))_{t\in[0,1]}, \quad \forall k\geq 0.
\end{aligned}$$

Hence, we obtain the desired result by Theorem 7.3 and the Skorohod representation theorem. □

By Corollary 7.4, we can apply Theorem 1.16 and we obtain the aging and sub-aging results as follows. Let $\{(x_i, w_i, v_i)\}_{i\in I}$ be a Poisson point process on $T_{2e}\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}$ with intensity $\mu_{2e}(dx)\,\tilde{p}(dw)\,\alpha v^{-1-\alpha}dv$ and define $\nu(dx) := \sum_{i\in I} v_i\delta_{x_i}(dx)$. Given ν, we write $(X^{\nu}, \{P^{\nu}_x\}_{x\in T_{2e}})$ for the process associated with (T_{2e}, d_{2e}, ν). We simply write $X^{\nu_n}_n := X^{\nu_{T^{GW}_n}}_{T^{GW}_n}$, which is the BTM on T^{GW}_n (see Definition 1.4). We denote by $P^{\nu_n}_{\rho_n}$ for the underlying probability measure for $X^{\nu_n}_n$ started at $\rho_n := \rho_{T^{GW}_n}$. Set $\tilde{c}_n := n^{1/2}\cdot n^{1/\alpha}$. As a consequence of Corollary 7.4 and Theorem 1.16, we obtain that

$$\begin{aligned}
P^{\nu_n}_{\rho_n}\bigl(X^{\nu_n}_n(\tilde{c}_n s) = X^{\nu_n}_n(\tilde{c}_n t)\bigr) &\to P^{\nu}_{\rho_{2e}}\bigl(X^{\nu}(s) = X^{\nu}(t)\bigr),\\
P^{\nu_n}_{\rho_n}\bigl(X^{\nu_n}_n(\tilde{c}_n t) = X^{\nu_n}_n(\tilde{c}_n t + t'),\ \forall t'\in[0, n^{1/\alpha}s]\bigr) &\to \sum_{i\in I} e^{-w_i s/v_i} P^{\nu}_{\rho_{2e}}(X^{\nu}(t) = x_i).
\end{aligned}$$

7.4. *The critical Erdős-Rényi random graph.* In this section, we consider the Erdős-Rényi random graph $G(n,p)$, which is a graph on n labeled vertices $[n] \coloneqq \{1,2,\dots,n\}$ chosen randomly by joining any two distinct vertices by an edge with probability p, independently for different pairs of vertices. This model exhibits a phase transition in its structure for large n. Let $p = c/n$ for some $c > 0$. When $c < 1$, the largest connected component has size $O(\log n)$. On the other hand, when $c > 1$, we see the emergence of a giant component that contains a positive proportion of the vertices. In the critical case $c = 1$, the largest connected components have sizes of order $n^{2/3}$. We will focus here on the critical case $c = 1$, and more specifically, on the critical window $p = n^{-1} + \lambda n^{-4/3}$, $\lambda \in \mathbb{R}$. We fix $\lambda \in \mathbb{R}$ and write $p_n = n^{-1} + \lambda n^{-4/3}$. Let $\mathcal{C}_i^n$ be the i-th largest connected component of $G(n, p_n)$.

One of the most significant results about random graphs in the above-mentioned critical regime was proved by Aldous (1997). Write Z_i^n and S_i^n for the size (that is, the number of vertices) and surplus (that is, the number of edges that would need to be removed in order to obtain a tree) of $\mathcal{C}_i^n$. Set $\boldsymbol{Z}^n \coloneqq (Z_1^n, Z_2^n, \dots)$ and $\boldsymbol{S}^n \coloneqq (S_1^n, S_2^n, \dots)$.

Theorem 7.5 (Aldous, 1997, Folk Theorem 1 and Corollary 2)**.** *As $n \to \infty$, it holds that*

$$(n^{-2/3}\boldsymbol{Z}^n, \boldsymbol{S}^n) \to (\boldsymbol{Z}, \boldsymbol{S})$$

in distribution, where the convergence of the first coordinate takes place in $l^2_{\searrow}$, the set of infinite sequences $(x_1, x_2, \dots)$ with $x_1 \geq x_2 \geq \cdots \geq 0$ and $\sum_i x_i^2 < \infty$, equipped with the usual l^2-norm.

The limits $\boldsymbol{Z} = (Z_1, Z_2, \dots)$ and $\boldsymbol{S} = (S_1, S_2, \dots)$ are constructed as follows. Consider a Brownian motion with parabolic drift, $(W_t^\lambda)_{t \geq 0}$, where

$$W_t^\lambda \coloneqq W_t + \lambda t - \frac{t^2}{2}$$

and $(W_t)_{t\geq 0}$ is a standard Brownian motion. Then the limit $\boldsymbol{Z}$ has the distribution of the ordered sequence of lengths of excursions of the reflected process $W_t^\lambda - \min_{0 \leq s \leq t} W_s^\lambda$ above 0, while $\boldsymbol{S}$ is the sequence of numbers of points of a Poisson point process with rate one in $\mathbb{R}^2_{\geq 0}$ lying under the corresponding excursions, where the Poisson point process is assumed to be independent of $(W_t)_{t\geq 0}$.

The scaling limit of $\mathcal{C}_1^n$ is given by fusing a tilted Brownian continuum random tree. For fused resistance metric spaces, see Appendix A. Recall the space $\mathscr{E}$ of excursions from the previous section, where we equip $\mathscr{E}$ with the metric induced by the supremum norm $\|\cdot\|_\infty$. Let $e^{(\sigma)} = (e^{(\sigma)}(t), 0 \leq t \leq \sigma)$ be a Brownian excursion of length σ. Note that, by Brownian scaling, the distribution of $e^{(\sigma)}$ coincides with that of $(\sqrt{\sigma}\, e(t/\sigma))_{t \in [0,\sigma]}$, where $(e(t))_{t\in[0,1]}$ denotes the standard Brownian excursion on $[0, 1]$. The *tilted excursion* of length σ, $\tilde{e}^{(\sigma)} = (\tilde{e}^{(\sigma)}(t), 0 \leq t \leq \sigma) \in \mathscr{E}$, is defined to be an excursion whose distribution is characterized by

$$\mathbf{P}\Big(\tilde{e}^{(\sigma)} \in \mathcal{B}\Big) = \frac{\mathbf{E}\Big[1_{\{e^{(\sigma)} \in \mathcal{B}\}} \exp(\int_0^\sigma e^{(\sigma)}(t)dt)\Big]}{\mathbf{E}\big[\exp(\int_0^\sigma e^{(\sigma)}(t)dt)\big]}$$

for $\mathcal{B} \subseteq \mathscr{E}$ a Borel set. For $f \in \mathscr{E}$ and $S \subseteq \mathbb{R}^2_{\geq 0}$, define

$$S \cap f \coloneqq \{(x,y) \in S \mid 0 \leq y < f(x)\}.$$

For $u = (u_x, u_y) \in S \cap f$, we define $u' = (u_x, u'_x)$ by setting $u'_x \coloneqq \inf\{x \geq u_x \mid f(x) = u_y\}$. We write

$$\mathscr{T}(S, f) \coloneqq \{u' \in \mathbb{R}^2_{\geq 0} \mid u \in S \cap f\}. \tag{7.6}$$

Let $\mathcal{P} \subseteq \mathbb{R}^2_{\geq 0}$ be a Poisson point process with rate one, independent of $(\tilde{e}^{(\sigma)})_{\sigma > 0}$ and $(W_t)_{t\geq 0}$. Assume that $\mathcal{P} \cap \tilde{e}^{(\sigma)}$ is non-empty and write $\mathscr{T}(\mathcal{P}, \tilde{e}^{(\sigma)}) = \{(\xi_l, \xi'_l) \mid 1 \leq l \leq s\}$. Define $a_l \coloneqq p_{2\tilde{e}^{(\sigma)}}(\xi_l)$ and $b_l \coloneqq p_{2\tilde{e}^{(\sigma)}}(\xi'_l)$, where we recall from Section 7.3 that $p_{2\tilde{e}^{(\sigma)}}$ denotes the canonical projection from $[0, \sigma]$ onto the real tree $T_{2\tilde{e}^{(\sigma)}}$ coded by $2\tilde{e}^{(\sigma)}$. Recall also that $d_{2\tilde{e}^{(\sigma)}}$ denotes the metric on $T_{2\tilde{e}^{(\sigma)}}$, $\rho_{2\tilde{e}^{(\sigma)}}$ denotes the root of $T_{2\tilde{e}^{(\sigma)}}$, and $\mu_{2\tilde{e}^{(\sigma)}}$ denotes the canonical measure on $T_{2\tilde{e}^{(\sigma)}}$. Define $(M^{(\sigma)}, R_{M^{(\sigma)}})$ to be the resistance metric space $(T_{2\tilde{e}^{(\sigma)}}, d_{2\tilde{e}^{(\sigma)}})$ fused over $(\{a_l, b_l\})_{l=1}^s$ (see Definition A.2). Let $\pi\colon T_{2\tilde{e}^{(\sigma)}} \to M^{(\sigma)}$ be the canonical map and set $\rho_{M^{(\sigma)}} := \pi(\rho_{2\tilde{e}^{(\sigma)}})$ and $\mu_{M^{(\sigma)}} := \mu_{2\tilde{e}^{(\sigma)}} \circ \pi^{-1}$. If $\mathcal{P} \cap \tilde{e}^{(\sigma)}$ is empty, then we define $(M^{(\sigma)}, R_{M^{(\sigma)}}, \rho_{M^{(\sigma)}}, \mu_{M^{(\sigma)}})$ to be equal to $(T_{2\tilde{e}^{(\sigma)}}, d_{2\tilde{e}^{(\sigma)}}, \rho_{2\tilde{e}^{(\sigma)}}, \mu_{2\tilde{e}^{(\sigma)}})$. We let $(M^{(Z_1)}, R_{M^{(Z_1)}}, \rho_{M^{(Z_1)}}, \mu_{M^{(Z_1)}})$ be a random element of $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}\mathbb{R}_{\geq 0}})$ such that, conditional on $Z_1 = \sigma$, its distribution is given by $(M^{(\sigma)}, R_{M^{(\sigma)}}, \rho_{M^{(\sigma)}}, \mu_{M^{(\sigma)}})$.

Given a finite connected graph G with labeled vertices, we regard G as a rooted electrical network by placing conductance 1 on each edge and set ρ_G to be the smallest-labeled vertex of G. The following result shows that Assumption 1.15 holds. As a consequence of Theorem 1.16, we obtain the aging and sub-aging results. In the remainder of this section, we set $(p_k)_{k\geq 0}$ to be the Poisson distribution with mean 1, i.e., $p_k := e^{-k}/k!$ for each $k \geq 0$. We then define a probability measure $\tilde{p} = (\tilde{p}_k)_{k\geq 0}$ on $\mathbb{Z}_{\geq 0}$ by setting $\tilde{p}_k := p_{k-1}$.

Theorem 7.6. *It holds that*

$$(V(\mathcal{C}_1^n), n^{-1/3}R_{\mathcal{C}_1^n}, \rho_{\mathcal{C}_1^n}, n^{-2/3}\dot{\mu}^{\#}_{\mathcal{C}_1^n}) \xrightarrow{\mathrm{d}} (M^{(Z_1)}, R_{M^{(Z_1)}}, \rho_{M^{(Z_1)}}, \mu_{M^{(Z_1)}} \otimes \tilde{p})$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\mathbb{R}_{\geq 0})})$.

To prove the above result, we let G_m^p be a random graph with the distribution of $G(m,p)$ conditioned to be connected. We assume that Z_1^n and $(G_m^{p_n})_{m\geq 1}$ are all independent (recall Z_1^n from the discussion above Theorem 7.5). It is then an easy exercise to check that the random graph $G^{p_n}_{Z_1^n}$ has the same distribution as the random graph $\mathcal{C}_1^n$ with relabeled vertices. Combining this with Theorem 7.5, we obtain Theorem 7.6, once the following Lemma is established.

Lemma 7.7. *If a sequence* $(m_n)_{n\geq 1}$ *of natural numbers satisfies* $n^{-2/3}m_n \to \sigma \in (0,\infty)$, *then it holds that*

$$(V(G^{p_n}_{m_n}), n^{-1/3}R_{G^{p_n}_{m_n}}, \rho_{G^{p_n}_{m_n}}, n^{-2/3}\dot{\mu}^{\#}_{G^{p_n}_{m_n}}) \xrightarrow{\mathrm{d}} (M^{(\sigma)}, R_{M^{(\sigma)}}, \rho_{M^{(\sigma)}}, \mu_{M^{(\sigma)}} \otimes \tilde{p})$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\mathbb{R}_{\geq 0})})$.

To prove the above lemma, we prepare some pieces of notation. Let $\mathbb{T}_m$ be the set of trees with the vertex set $[m]$, and T be an element of $\mathbb{T}_m$. We regard T as a plane tree by using the depth-first search (cf. Addario-Berry et al., 2012, Section 2). We write $H_T = (H_T(t), 0 \leq t \leq m-1)$ for the height function of T and $N_T^{(k)} = (N_T^{(k)}(t))_{t\in[0,m-1]}$ for the function counting the number of vertices whose outdegree is k (recall these functions from Section 7.3). We set $H_T(t) := 0$, $t \in [m-1, m]$ for convenience. Let $v_0, v_1, \ldots, v_{m-1}$ be the vertices of T as a plane tree in the depth-first order. Write k_i for the number of children of v_i. Then the depth-first walk $X_T = (X_T(t))_{t\in[0,m]}$ of T is given by setting $X_T(0) := 0$, $X_T(i) := \sum_{j=0}^{i-1}(k_j - 1)$ for $i \in \{0, 1, \ldots, m\}$, and $X_T(t) := X_T(\lfloor t \rfloor)$. We then set

$$a(T) := \sum_{i=1}^{m-1} X_T(i) = m\int_0^1 X_T(mt)\,dt.$$

Given $p \in (0,1)$, we define a random tree $\tilde{T}_m^p$ on $\mathbb{T}_m$ that has a "tilted" distribution given by

$$\mathbf{P}(\tilde{T}_m^p = T) \propto (1-p)^{-a(T)}, \quad T \in \mathbb{T}_m.$$

For $p \in (0,1)$, a *binomial pointset* $\mathcal{Q}^p \subseteq \mathbb{Z}^2_{\geq 0}$ of intensity p is defined to be a random subset of $\mathbb{Z}^2_{\geq 0}$ in which each point is present independently with probability p. In Addario-Berry et al. (2012), it is shown that G_m^p is recovered by attaching extra edges on $\tilde{T}_m^p$.

Lemma 7.8 (Addario-Berry et al., 2012, Lemma 18)**.** *Fix $p \in (0,1)$. Let $\tilde{T}^p_m$ to be a tilted tree as defined above and $\mathcal{Q}^p$ be a binomial pointset of intensity p, independent of $\tilde{T}^p_m$. Let $v_0, v_1, \ldots, v_{m-1}$ be the vertices of $\tilde{T}^p_m$ in depth-first order. Write $\mathscr{T}(\mathcal{Q}^p, X_{\tilde{T}^p_m}) = \{(x_i, x'_i) \mid 1 \le i \le s\}$ (recall its definition from (7.6)). We define a graph $G(\tilde{T}^p_m, \mathcal{Q}^p)$ by attaching an edge between v_{x_i} and $v_{x'_i}$ on $\tilde{T}^p_m$. (If $\mathscr{T}(\mathcal{Q}^p, X_{\tilde{T}^p_m})$ is empty, we define $G(\tilde{T}^p_m, \mathcal{Q}^p) := \tilde{T}^p_m$.) Then $G(\tilde{T}^p_m, \mathcal{Q}^p)$ has the same distribution as G^p_m.*

The following result is also proven in Addario-Berry et al. (2012), which provide the convergence of coding functions of $\tilde{T}^p_m$ and vertices where new edges are attached.

Lemma 7.9 (Addario-Berry et al., 2012, Lemma 19)**.** *Assume that a sequence $(m_n)_{n\ge1}$ satisfies $n^{-2/3}m_n \to \sigma \in (0,\infty)$. Let $\mathcal{Q}^{p_n}$ be a binomial pointset of intensity p_n, independent of $\tilde{T}^{p_n}_{m_n}$, and define $\mathcal{P}_n := \{((\sigma/m_n)i, (\sigma/m_n)^{1/2}j) \mid (i,j) \in \mathcal{Q}^{p_n}\}$. Then it holds that*

$$\left(\sqrt{\frac{\sigma}{m_n}} H_{\tilde{T}^{p_n}_{m_n}}\left(\left\lfloor \frac{m_n}{\sigma}\cdot\right\rfloor\right), \sqrt{\frac{\sigma}{m_n}} X_{\tilde{T}^{p_n}_{m_n}}\left(\left\lfloor \frac{m_n}{\sigma}\cdot\right\rfloor\right), \mathcal{P}_n \cap \left(\sqrt{\frac{\sigma}{m_n}} X_{\tilde{T}^{p_n}_{m_n}}\left(\left\lfloor \frac{m_n}{\sigma}\cdot\right\rfloor\right)\right)\right) \xrightarrow[n\to\infty]{\mathrm{d}} (2\tilde{e}^{(\sigma)}, \tilde{e}^{(\sigma)}, \mathcal{P} \cap \tilde{e}^{(\sigma)}), \tag{7.7}$$

where the convergence of the first and second coordinate takes place in $D([0,\sigma], \mathbb{R}_{\ge0})$ equipped with the usual J_1-Skorohod topology and the convergence of the third coordinate takes place with respect to the Hausdorff metric.

Below, we prove the convergence of functions $N^{(k)}_{\tilde{T}^p_m}$ defined at (7.1).

Lemma 7.10. *Write I for the identity map from $[0,1]$ to itself. Assume that a sequence $(m_n)_{n\ge1}$ satisfies $n^{-2/3}m_n \to \sigma \in (0,\infty)$. Then, for any $k \in \mathbb{Z}_{\ge0}$,*

$$m_n^{-1} N^{(k)}_{\tilde{T}^{p_n}_{m_n}}((m_n-1)\cdot) \xrightarrow{\mathrm{p}} p_k\, I(\cdot)$$

in $D([0,1], \mathbb{R}_{\ge0})$.

Proof: Let $T^U_{m_n}$ be a random tree uniformly chosen from $\mathbb{T}_{m_n}$. If we think of $T^U_{m_n}$ as a random plane tree, then it has the same distribution as the conditional Galton-Watson tree $T^{GW}_{m_n}$ with offspring distribution that is Poisson with mean 1. We write

$$\tilde{N}^{(k)}_n(t) := m_n^{-1} N^{(k)}_{\tilde{T}^{p_n}_{m_n}}((m_n-1)t),$$
$$N^{(k)}_n(t) := m_n^{-1} N^{(k)}_{T^U_{m_n}}((m_n-1)t),$$
$$X_n(t) := m_n^{-1/2} X_{T^U_{m_n}}(m_n t).$$

From Thévenin (2020, Theorem 1.1), we have that $(X_n, N^{(k)}_n) \to (e, p_{k-1} I)$. Let f be a bounded continuous function on $D([0,1], \mathbb{R})$. By the definition of $T^{p_n}_{m_n}$. we obtain that

$$\mathbf{E}[f(\tilde{N}^{(k)}_n)] = \frac{\mathbf{E}\left[f(N^{(k)}_n)(1-p_n)^{-m_n^{3/2}\int_0^1 X_n(t)\,dt}\right]}{\mathbf{E}\left[(1-p_n)^{-m_n^{3/2}\int_0^1 X_n(t)\,dt}\right]}.$$

Moreover, in Addario-Berry et al. (2012, Proof of Theorem 12), it is shown that the following family of random variables is uniformly integrable:

$$(1-p_n)^{-m_n^{3/2}\int_0^1 X_n(t)\,dt}, \quad n \ge 1.$$

Combining these with the scaling relation $e^{(\sigma)}(\cdot) \stackrel{\mathrm{d}}{=} \sqrt{\sigma} e(\cdot/\sigma)$, we deduce that

$$\lim_{n\to\infty} \mathbf{E}\Big[f(N_n^{(k)})(1-p_n)^{-m_n^{3/2}\int_0^1 X_n(t)\,dt}\Big] = \mathbf{E}\Big[f(p_{k-1}\,I)\exp\Big(\int_0^\sigma e^{(\sigma)}(t)\,dt\Big)\Big],$$

$$\lim_{n\to\infty} \mathbf{E}\Big[(1-p_n)^{-m_n^{3/2}\int_0^1 X_n(t)\,dt}\Big] = \mathbf{E}\Big[\exp\Big(\int_0^\sigma e^{(\sigma)}(t)\,dt\Big)\Big].$$

Therefore, the desired result follows. □

Combining the above results with a technical result regarding fused resistance metric space shown in Appendix A, we can prove Theorem 7.6 as follows.

Proof of Theorem 7.6: We proceed with the proof in the setting of Lemma 7.9. By the Skorohod representation theorem, we may assume that the convergence (7.7) takes place almost surely on some probability space. Assume that $\mathcal{P} \cap \tilde{e}^{(\sigma)}$ is non-empty and write $\mathscr{T}(\mathcal{P}, \tilde{e}^{(\sigma)}) = \{(\xi_l, \xi_l') \mid 1 \le l \le s\}$. For all sufficiently large n, we can write $\mathscr{T}(\mathcal{P}_n, \sqrt{\sigma/m_n} X_{\tilde{T}_{m_n}^{p_n}}(\lfloor (m_n/\sigma)\cdot \rfloor)) = \{(i_l^n, j_l^n) \mid 1 \le l \le s\}$ in such a way that

$$\max_{1\le l\le s}(|i_l^n - \xi_l| \vee |j_l^n - \xi_l'|) \to 0.$$

Let $v_0^n, v_1^n, \dots, v_{m_n-1}^n$ be the vertices of $\tilde{T}_{m_n}^{p_n}$ in depth-first order, and define $a_l^n := v_{m_n i_l^n/\sigma}^n$, $b_l^n := v_{m_n j_l^n/\sigma}^n$. Here, we note that we have

$$\{(m_n i_l^n/\sigma, m_n j_l^n/\sigma) \mid 1 \le l \le s\} = \mathscr{T}(\mathcal{Q}^{p_n}, X_{\tilde{T}_{m_n}^{p_n}})$$

and in particular the indices $m_n i_l^n/\sigma$ and $m_n j_l^n/\sigma$ are integers. Define $a_l, b_l \in T_{2\tilde{e}^{(\sigma)}}$ by setting $a_l := p_{2\tilde{e}^\sigma}(\xi_l)$ and $b_l := p_{2\tilde{e}^\sigma}(\xi_l')$, where we recall that $p_{2\tilde{e}^\sigma}$ is the canonical projection from $[0,\sigma]$ onto the real tree $T_{2\tilde{e}^{(\sigma)}}$. Using Lemma 7.10 and following the proof of Theorem 7.3, we deduce that

$$\begin{aligned}(V(\tilde{T}_{m_n}^{p_n}), n^{-1/3} d_{\tilde{T}_{m_n}^{p_n}}, \rho_{\tilde{T}_{m_n}^{p_n}}, m_n^{-1}\mu^{\#}_{\tilde{T}_{m_n}^{p_n}}, a_1^n, b_1^n, \dots, a_s^n, b_s^n)\\ \to (T_{2\tilde{e}^{(\sigma)}}, d_{2\tilde{e}^{(\sigma)}}, \rho_{2\tilde{e}^{(\sigma)}}, \sigma^{-1}\mu_{2\tilde{e}^{(\sigma)}} \otimes \tilde{p}, a_1, b_1, \dots, a_s, b_s)\end{aligned}$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\mathbb{R}_{\ge 0})} \times \tau^{2s\text{-pts}})$, where $\tau^{2s\text{-pts}}$ denotes the product structure of $2s$ copies of τ^{pt}, as recalled from (S2) and (S7). (See also Noda, 2025+, Proof of Lemma 8.42). This, combined with Lemma 7.8 and Theorem A.13 below, yields the desired result. □

By Theorem 7.6, we can apply Theorem 1.16 and we obtain the aging and sub-aging results as follows. Let $\{(x_i, w_i, v_i)\}_{i\in I}$ be a Poisson point process on $M^{(Z_1)} \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{>0}$ with intensity $\mu_{M^{(Z_1)}}(dx)\,\tilde{p}(dw)\,\alpha v^{-1-\alpha} dv$ and define $\nu(dx) := \sum_{i\in I} v_i \delta_{x_i}(dx)$. Given ν, we write $(X^\nu, \{P_x^\nu\}_{x\in M^{(Z_1)}})$ for the process associated with $(M^{(Z_1)}, R_{M^{(Z_1)}}, \nu)$. We simply write $X_n^{\nu_n} := X_{\mathcal{C}_1^n}^{\nu_{\mathcal{C}_1^n}}$, which is the BTM on $\mathcal{C}_1^n$ (see Definition 1.4). We denote by $P_{\rho_n}^{\nu_n}$ for the underlying probability measure for $X_n^{\nu_n}$ started at $\rho_n := \rho_{M^{(Z_1)}}$. Set $\tilde{c}_n := n^{1/3} \cdot n^{2/(3\alpha)}$. As a consequence of Theorems 1.16 and 7.6, we obtain that

$$P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(\tilde{c}_n s) = X_n^{\nu_n}(\tilde{c}_n t)\big) \to P^\nu_{\rho_{M^{(Z_1)}}}\big(X^\nu(s) = X^\nu(t)\big),$$

$$P_{\rho_n}^{\nu_n}\big(X_n^{\nu_n}(\tilde{c}_n t) = X_n^{\nu_n}(\tilde{c}_n t + t'),\ \forall t' \in [0, n^{2/(3\alpha)} s]\big) \to \sum_{i\in I} e^{-w_i s/v_i} P^\nu_{M^{(Z_1)}}(X^\nu(t) = x_i).$$

Appendix A. Convergence of fused spaces

In this appendix, we introduce the operation of fusing resistance metric spaces at disjoint pairs of subsets, which are used to describe the scaling limit of the Erdős-Rényi graph in Section 7.4. We note that this operation is considered in Croydon (2018); Kigami (2012). Our aim in this appendix is to formalize the topological aspects of fusing. In particular, we prove some convergence results regarding fused resistance metric spaces (Theorems A.9 and A.13). Throughout this appendix, we fix $N \in \mathbb{N}$.

Let (F, R) be a compact resistance metric space and $(\mathcal{E}, \mathcal{F})$ be the corresponding resistance form. Fix a collection $\Gamma = \{V_i\}_{i=1}^N$ of non-empty disjoint compact subsets of F and write

$$F^\Gamma \coloneqq \left(F \setminus \bigcup_{i=1}^N V_i\right) \cup \bigcup_{i=1}^N \{V_i\},$$

i.e., we consider each subset V_i as a single point. Let $\pi^\Gamma \colon F \to F^\Gamma$ be the canonical map, that is, $\pi^\Gamma(x) \coloneqq x$ for $x \in F \setminus \bigcup_{i=1}^N V_i$ and $\pi^\Gamma(x) \coloneqq V_i$ for $x \in V_i$. Define $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ by setting

$$\mathcal{F}^\Gamma \coloneqq \{f \colon F^\Gamma \to \mathbb{R} \mid f \circ \pi^\Gamma \in \mathcal{F}\},$$
$$\mathcal{E}^\Gamma(f, f) \coloneqq \mathcal{E}(f \circ \pi^\Gamma, f \circ \pi^\Gamma), \quad \forall f \in \mathcal{F}^\Gamma.$$

Theorem A.1 (Croydon, 2018, Lemma 8.3)**.** *The pair $(\mathcal{E}^\Gamma, \mathcal{F}^\Gamma)$ is a resistance form. If we write R^Γ for the associated resistance metric, then (F^Γ, R^Γ) is compact.*

Definition A.2 (Fused resistance metric space)**.** In the above setting, we refer to (F^Γ, R^Γ) as the resistance metric space (F, R) fused over Γ and $\pi^\Gamma \colon F \to F^\Gamma$ as the associated canonical map.

Remark A.3. The fusing operation can also be defined when $\Gamma = \{V_i\}_{i=1}^N$ is a family of compact subsets that are not necessarily disjoint. In that case, we consider an equivalence relation $\sim$ on $\bigcup_{i=1}^N V_i$ given by $x \sim y$ if and only if there exist $l \in \mathbb{N}$, $i_1, \ldots, i_l \in \{1, \ldots, N\}$, and $x = x_0, x_1, \ldots, x_{l-1}, x_l = y$ such that $\{x_{k-1}, x_k\} \subseteq V_{i_k}$ for each k. We let $\Gamma' \coloneqq \{V_i'\}_{i=1}^{N'}$ be the collection of equivalence classes $\Gamma' \coloneqq \{V_i'\}_{i=1}^{N'}$. Then we refer to $(F^{\Gamma'}, R^{\Gamma'})$ as the resistance metric space (F, R) fused over Γ.

It is easy to see that

$$R^\Gamma(\pi(x), \pi(y)) \leq R(x, y), \quad \forall x, y \in F, \tag{A.1}$$

which implies that $\pi^\Gamma \colon (F, R) \to (F^\Gamma, R^\Gamma)$ is continuous. We will show that when resistance metric spaces and collections of fusing points converge, then the associated fused resistance metric spaces and the canonical maps also converge. To describe this precisely, we introduce a suitable topology for convergence of functions with different domains. Fix a compact metric space (M, d^M) and a complete, separable metric space (Ξ, d^Ξ).

Definition A.4. Define

$$\widehat{C}_c(M, \Xi) \coloneqq \bigcup_{X \in \mathcal{C}_c(M)} C(X, \Xi),$$

where we recall that $\mathcal{C}_c(M)$ is the collection of compact subsets of M and $C(X, \Xi)$ denotes the set of continuous functions from X to Ξ. Note that $\widehat{C}_c(M, \Xi)$ contains the empty map $\emptyset_\Xi : \emptyset \to \Xi$. For $f \in \widehat{C}_c(M, \Xi)$, we write $\mathrm{dom}(f)$ for its domain.

Definition A.5 (The metric $d^M_{\widehat{C}_c(\cdot,\Xi)}$)**.** For $f, g \in \widehat{C}_c(M, \Xi)$ and $\varepsilon > 0$, consider the following condition.

($\widehat{C}_c(\varepsilon)$) For any $x \in \mathrm{dom}(f)$, there exists an element $y \in \mathrm{dom}(g)$ such that

$$d^M(x, y) \vee d^\Xi(f(x), g(y)) \leq \varepsilon.$$

Similarly, for any $y \in \mathrm{dom}(g)$, there exists an element $x \in \mathrm{dom}(g)$ such that

$$d^M(x, y) \vee d^\Xi(f(x), g(y)) \leq \varepsilon.$$

We define

$$d^M_{\hat{C}_c(\cdot,\Xi)}(f, g) \coloneqq \inf\{\varepsilon > 0 \mid \varepsilon \text{ satisfies } (\widehat{C}_c(\varepsilon)) \text{ with respect to } f, g\} \wedge 1,$$

where the infimum over the empty set is defined to be ∞.

Theorem A.6 (Noda, 2024a, Lemma 3.27 and Theorem 3.28). *The function $d^M_{\widehat{C}_c(\cdot,\Xi)}$ is a well-defined metric on $\widehat{C}_c(M,\Xi)$. The induced topology on $\widehat{C}_c(M,\Xi)$ is Polish.*

Definition A.7 (The compact-convergence topology with variable domains). We call the topology on $\widehat{C}_c(M,\Xi)$ induced by $d^M_{\widehat{C}_c(\cdot,\Xi)}$ the *compact-convergence topology with variable domains*.

Theorem A.8 (Convergence in $\widehat{C}_c(M,\Xi)$, Noda, 2024a, Theorem 3.30). *Let $f, f_1, f_2, \dots$ be elements of $\widehat{C}_c(M,\Xi)$. The following conditions are equivalent.*

(i) *The functions f_n converge to f in the compact-convergence topology with variable domains.*
(ii) *The sets $\mathrm{dom}(f_n)$ converge to $\mathrm{dom}(f)$ in the Hausdorff topology in M, and there exist functions $g_n, g \in C(M,\Xi)$ such that $g_n|_{\mathrm{dom}(f_n)} = f_n$, $g|_{\mathrm{dom}(f)} = f$ and $g_n \to g$ in the compact-convergence topology.*
(iii) *The sets $\mathrm{dom}(f_n)$ converge to $\mathrm{dom}(f)$ in the Hausdorff topology in M, and, for any $x_n \in \mathrm{dom}(f_n)$ and $x \in \mathrm{dom}(f)$ with $x_n \to x$, it holds that $f_n(x_n) \to f(x)$.*

Using the compact-convergence topology with variable domains, we can state rigorously the convergence of canonical maps associated with fused resistance metric spaces as follows. For notational convenience, we write $\tau^{2N\text{-pts}}$ for the product structure of $2N$ copies of τ^{pt}, as recalled from (S2) and (S7).

Theorem A.9. *Let (F_n, R_n, ρ_n), $n \geq 1$ and (F, R, ρ) be rooted compact resistance metric spaces. Let $(a_n^{(i)}, b_n^{(i)})_{i=1}^N$ and $(a^{(i)}, b^{(i)})_{i=1}^N$ be distinct elements of F_n and F, respectively. Assume that*

$$(F_n, R_n, \rho_n, (a_n^{(i)}, b_n^{(i)})_{i=1}^N) \to (F, R, \rho, (a^{(i)}, b^{(i)})_{i=1}^N)$$

in $\mathfrak{K}_\bullet(\tau^{2N\text{-pts}})$. Write $(\tilde{F}_n, \tilde{R}_n)$ and $(\tilde{F}, \tilde{R})$ for the resistance metric spaces fused over $\{\{a_n^{(i)}, b_n^{(i)}\}\}_{i=1}^N$ and $\{\{a^{(i)}, b^{(i)}\}\}_{i=1}^N$, respectively. Let $\pi_n \colon F_n \to \tilde{F}_n$ and $\pi \colon F \to \tilde{F}$ be the associated canonical maps, and set $\tilde{\rho}_n := \pi_n(\rho_n)$ and $\tilde{\rho} := \pi(\rho)$. Then there exist rooted compact metric spaces (M, d^M, ρ_M) and $(\tilde{M}, d^{\tilde{M}}, \rho_{\tilde{M}})$ satisfying the following:

(i) *(F_n, R_n, ρ_n) and (F, R, ρ) are embedded isometrically into (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements of M, $F_n \to F$ in the Hausdorff topology on M, and $a_n^{(i)} \to a^{(i)}$ and $b_n^{(i)} \to b^{(i)}$ in M for each i;*
(ii) *$(\tilde{F}_n, \tilde{R}_n, \tilde{\rho}_n)$ and $(\tilde{F}, \tilde{R}, \tilde{\rho})$ are embedded isometrically into $(\tilde{M}, d^{\tilde{M}}, \rho_{\tilde{M}})$ in such a way that $\tilde{\rho}_n = \tilde{\rho} = \rho_{\tilde{M}}$ as elements of $\tilde{M}$ and $\tilde{F}_n \to \tilde{F}$ in the Hausdorff topology on $\tilde{M}$;*
(iii) *if we regard π_n and π as elements of $\widehat{C}(M, \tilde{M})$ by the above embeddings, then $\pi_n \to \pi$ in $\widehat{C}(M, \tilde{M})$.*

To prove the above result, for each boundedly-compact metric space M, we define a structure $\tau^{\widehat{C}_c(M,\cdot)}$ as follows.

- For each boundedly-compact metric space (S, d^S), set $\tau^{\widehat{C}_c(M,\cdot)}(S) := \widehat{C}_c(M, S)$.
- For each distance-preserving map $f \colon S_1 \to S_2$ between boundedly-compact metric spaces, $\tau_f^{\widehat{C}_c(M,\cdot)}(g) := f \circ g$ for each $g \in \widehat{C}_c(M, S_1)$.

Following the proof of Noda (2024a, Theorem 8.27), one can readily verify that $\tau^{\widehat{C}_c(M,\cdot)}$ is Polish.

Lemma A.10. *For each boundedly-compact metric space M, the structure $\tau^{\widehat{C}_c(M,\cdot)}$ is Polish.*

Proof: Following the proof of Noda (2024a, Theorem 8.27), one can readily verify that $\tau^{\widehat{C}_c(M,\cdot)}$ is Polish. (Indeed, the desired result is deduced by replacing $\Psi(X)$ with the fixed space M in that proof.) □

Below, we provide a precompactness criterion for the space $\mathfrak{K}_\bullet(\tau^{\widehat{C}_c(M,\cdot)})$.

Lemma A.11. *A non-empty subset* $\{\mathcal{X}_\alpha = (S_\alpha, d^\alpha, \rho_\alpha, f_\alpha) \mid \alpha \in \mathcal{A}\}$ *of* $\mathfrak{K}_\bullet(\tau^{\widehat{C}_c(M,\cdot)})$ *is precompact if and only if the following conditions are satisfied.*

(i) *The subset* $\{(S_\alpha, d^\alpha, \rho_\alpha) \mid \alpha \in \mathcal{A}\}$ *is precompact in pointed the Gromov–Hausdorff topology (recall this from Remark 3.11).*

(ii) *It holds that*

$$\lim_{\delta\to\infty} \sup_{\alpha\in\mathcal{A}} \sup_{\substack{x,y\in \mathrm{dom}(f_\alpha)\\ d^M(x,y)\le\delta}} d^\alpha(f_\alpha(x), f_\alpha(y)) = 0.$$

Proof: Similarly to Lemma A.10, following the proof of Noda (2024a, Theorem 8.29), we deduce the desired result. □

Proof of Theorem A.9: By the assumption and Theorem 3.9, we can find a rooted compact metric space (M, d^M, ρ_M) where (F_n, R_n, ρ_n) and (F, R, ρ) are embedded isometrically in such a way that $\rho_n = \rho = \rho_M$ as elements of M, $F_n \to F$ in the Hausdorff topology, and $(a_n^{(i)}, b_n^{(i)}) \to (a^{(i)}, b^{(i)})$ in $M \times M$ for all i. We have from (A.1) that

$$\lim_{\delta\to 0} \sup_{n\ge 1} \sup_{\substack{x,y\in F_n\\ R_n(x,y)\le\delta}} \tilde{R}_n(\pi_n(x), \pi_n(y)) \le \lim_{\delta\to 0} \delta = 0.$$

By Lemma A.11, $\{(\tilde{F}_n, \tilde{R}_n, \tilde{\rho}_n, \pi_n)\}_{n\ge 1}$ is precompact in $\mathfrak{K}_\bullet(\tau^{\widehat{C}_c(M,\cdot)})$. It remains to show that the the limit of any convergent subsequence is $(\tilde{F}, \tilde{R}, \tilde{\rho}, \pi)$. So, we assume that $(\tilde{F}_n, \tilde{R}_n, \tilde{\rho}_n, \pi_n)$ converges to (K, d^K, ρ_K, π_K). It is enough to prove that (K, d^K, ρ_K, π_K) is equivalent to $(\tilde{F}, \tilde{R}, \tilde{\rho}, \pi)$. We may assume that $(\tilde{F}_n, \tilde{R}_n, \tilde{\rho}_n)$ and (K, d^K, ρ_K) are embedded isometrically into a common rooted compact metric space $(\tilde{M}, d^{\tilde{M}}, \rho_{\tilde{M}})$ in such a way that $\tilde{\rho}_n = \rho_K = \rho_{\tilde{M}}$ as elements of $\tilde{M}$, $\tilde{F}_n \to K$ in the Hausdorff topology in M, and $\pi_n \to \pi_K$ in $\widehat{C}(M, \tilde{M})$. Since $\mathrm{dom}(\pi_n) = F_n \to F$, we have from Theorem A.8 that $\mathrm{dom}(\pi) = F$. Moreover, the convergences of π_n to π_K and of ρ_n to ρ in M imply that $\pi_n(\rho_n) \to \pi_K(\rho)$. However, we have that $\pi_n(\rho_n) = \tilde{\rho}_n = \rho_K$, and so it holds that $\pi_K(\rho) = \rho_K$. Fix $x, y \in F$. Since $F_n \to F$ in the Hausdorff topology in M, there exist $x_n, y_n \in F_n$ such that $x_n \to x$ and $y_n \to y$ in M. Then, from Croydon (2018, Proof of Proposition 8.4), we deduce that

$$\lim_{n\to\infty} \tilde{R}_n(\pi_n(x_n), \pi_n(y_n)) = \tilde{R}(\pi(x), \pi(y)).$$

On the other hand, by Theorem A.8, we have that

$$\lim_{n\to\infty} d^{\tilde{M}}(\pi_n(x_n), \pi_n(y_n)) = d^{\tilde{M}}(\pi_K(x), \pi_K(y)).$$

It follows that

$$\tilde{R}(\pi(x), \pi(y)) = d^K(\pi_K(x), \pi_K(y)).$$

Thus, there exists a unique map $f\colon \tilde{F} \to K$ such that $f \circ \pi = \pi_K$. From the above equation, it is easy to check that f is distance-preserving. Recalling that $\pi_K(\rho) = \rho_K$, we deduce that f is root-preserving, i.e., $f(\tilde{\rho}) = \rho_K$. It remains to prove that f is surjective, which is equivalent to showing that π_K is surjective. Fix $y \in K$. We choose $y_n \in \tilde{F}_n$ so that $y_n \to y$ in $\tilde{M}$. Let $x_n \in F_n$ be such that $\pi_n(x_n) = y_n$. By the compactness of M, we can find a subsequence $(n_{k_l})_{l\ge 1}$ satisfying $x_{n_k} \to x$ in M for some $x \in F$. From Theorem A.8, it holds that $\pi_{n_k}(x_{n_k}) \to \pi_K(x)$. Thus, $\pi_K(x) = y$, which shows that π_K is surjective. □

We next consider fused electrical networks. Let G be an electrical network with finite vertex set V_G. Fix a collection $\Gamma = \{V_i\}_{i=1}^N$ of non-empty disjoint subsets of V_G and write

$$V_G^\Gamma := \left(V_G \setminus \bigcup_{i=1}^N V_i\right) \cup \bigcup_{i=1}^N \{V_i\}.$$

Define an electrical network $\tilde{G}$ with vertex set $V_{\tilde{G}} := V_G^{\Gamma}$ by setting the conductance $\mu_{\tilde{G}}$ as follows:

$$\mu_{\tilde{G}}(x,y) := \mu_G(x,y), \quad x,y \in V_G \setminus \bigcup_{i=1}^N V_i;$$

$$\mu_{\tilde{G}}(x,V_i) := \sum_{y \in V_i} \mu_G(x,y), \quad x \in V_G \setminus \bigcup_{i=1}^N V_i;$$

$$\mu_{\tilde{G}}(V_i,V_j) := \sum_{x \in V_i} \sum_{y \in V_j} \mu_G(x,y), \quad i \neq j.$$

The canonical map $\pi_{\tilde{G}} \colon V_G \to V_{\tilde{G}}$ is given by $\pi_{\tilde{G}}(x) := x$ for $x \in V_G \setminus \bigcup_{i=1}^N V_i$ and $\pi_{\tilde{G}}(x) := V_i$ for $x \in V_i$. We refer to $\tilde{G}$ as the electrical network G fused over Γ. It is easy to check that $(V_{\tilde{G}}, R_{\tilde{G}})$ coincides with the resistance metric space (V_G, R_G) fused over Γ. (Indeed, the associated resistance forms coincide.)

Proposition A.12. *Let G be an electrical network with finite vertex set V_G. Fix two distinct vertices $a,b \in V_G$ such that $\mu(a,b) > 0$. Write $\tilde{G}$ for the electrical network G fused over $V_0 := \{a,b\}$. Let $\pi \colon V_G \to V_{\tilde{G}}$ be the canonical map. Then, it holds that, for any $x,y \in V_G$,*

$$R_{\tilde{G}}(\pi(x),\pi(y)) \leq R_G(x,y) \leq R_{\tilde{G}}(\pi(x),\pi(y)) + \mu_G(a,b)^{-1}. \tag{A.2}$$

Proof: Fix $x,y \in V_G$ with $x \neq y$. If $\pi(x) = \pi(y)$, then $x = y$ or $\{x,y\} = \{a,b\}$. Thus, the assertion is straightforward as we have that $R_G(x,y) \leq \mu_G(a,b)^{-1}$. Henceforth, we assume that $\pi(x) \neq \pi(y)$. Let $\tilde{f} \colon V_{\tilde{G}} \to \mathbb{R}$ be such that $R_{\tilde{G}}(\pi(x),\pi(y)) = \mathcal{E}_{\tilde{G}}(\tilde{f},\tilde{f})^{-1}$, $\tilde{f}(\pi(x)) = 1$, and $\tilde{f}(\pi(y)) = 0$. Define $f \colon V_G \to \mathbb{R}$ by setting $f|_{V \setminus V_0} = \tilde{f}|_{V \setminus \{V_0\}}$ and $f|_{V_0} := \tilde{f}(V_0)$. It is the case that

$$\begin{aligned} R_G(x,y)^{-1} &\leq \mathcal{E}_G(f,f) \\ &= \frac{1}{2} \sum_{z,w \in V_G \setminus V_0} \mu_G(z,w)(f(z)-f(w))^2 + \sum_{z \in V_G \setminus V_0} \sum_{w \in V_0} \mu_G(z,w)(f(z)-f(w))^2 \\ &= \frac{1}{2} \sum_{z,w \in V_G \setminus \{V_0\}} \mu_{\tilde{G}}(z,w)(\tilde{f}(z)-\tilde{f}(w))^2 + \sum_{z \in V_G \setminus \{V_0\}} \mu_{\tilde{G}}(z,V_0)(\tilde{f}(z)-\tilde{f}(V_0))^2 \\ &= \mathcal{E}_{\tilde{G}}(\tilde{f},\tilde{f}) \\ &= R_{\tilde{G}}(\pi(x),\pi(y))^{-1}. \end{aligned}$$

Therefore, the first inequality of (A.2) follows.

To prove the second inequality, we use Thomson's principle. For details, see Levin and Peres (2017), for example. We first consider the case where $x,y \notin V_0$. Let $\tilde{\imath}$ be the unit current flow from x to y on $\tilde{G}$. We then define a flow i from x to y on G as follows:

$$\begin{aligned} i(z,w) &:= \tilde{\imath}(z,w), \quad z,w \in V_G \setminus V_0 \text{ such that } z \sim w, \\ i(z,a) &:= \frac{\mu_G(z,a)}{\mu_G(z,a)+\mu_G(z,b)} \tilde{\imath}(z,V_0), \quad z \in V_G \setminus V_0 \text{ such that } z \sim a, \\ i(z,b) &:= \frac{\mu_G(z,b)}{\mu_G(z,a)+\mu_G(z,b)} \tilde{\imath}(z,V_0), \quad z \in V_G \setminus V_0 \text{ such that } z \sim b, \\ i(a,b) &:= -\sum_{z \in V_G \setminus V_0} i(a,z). \end{aligned}$$

For non-negative real numbers s, t, we have that $|s-t| \le s \vee t$. This yields that

$$|i(a,b)| \le \max\left\{ \sum_{\tilde{\imath}(V_0,z)\ge 0} \tilde{\imath}(V_0,z),\, -\sum_{\tilde{\imath}(V_0,z)\le 0} \tilde{\imath}(V_0,z) \right\} \le 1.$$

By Thomson's principle, we deduce that

$$\begin{aligned} R_G(x,y) &\le \frac{1}{2} \sum_{\substack{z,w\in V\setminus V_0 \\ z\sim w}} \mu_G(z,w)^{-1} i(z,w)^2 + \sum_{\substack{z\in V\setminus V_0, w\in V_0 \\ z\sim w}} \mu_G(z,w)^{-1} i(z,w)^2 + \mu_G(a,b)^{-1} i(a,b)^2 \\ &\le \frac{1}{2} \sum_{\substack{z,w\in V\setminus V_0 \\ z\sim w}} \mu_{\tilde{G}}(z,w)^{-1} \tilde{\imath}(z,w)^2 + \sum_{\substack{z\in V\setminus V_0 \\ z\sim V_0}} \mu_{\tilde{G}}(z,V_0)^{-1} \tilde{\imath}(z,V_0)^2 + \mu_G(a,b)^{-1} i(a,b)^2 \\ &= R_{\tilde{G}}(x,y) + \mu_G(a,b)^{-1}. \end{aligned}$$

Next, we consider the case where $x \notin V_0$ and $y \in V_0$. We may assume that $y = a$. Let $\tilde{\imath}$ be the unit current flow from V_0 to x on $\tilde{G}$. We then define a flow i from a to x on G as follows:

$$\begin{aligned} i(z,w) &\coloneqq \tilde{\imath}(z,w), \quad z,w \in V_G \setminus V_0 \text{ such that } z \sim w, \\ i(a,z) &\coloneqq \frac{\mu_G(a,z)}{\mu_G(a,z)+\mu_G(b,z)} \tilde{\imath}(V_0,z), \quad z \in V_G \setminus V_0 \text{ such that } z \sim a, \\ i(b,z) &\coloneqq \frac{\mu_G(b,z)}{\mu_G(a,z)+\mu_G(b,z)} \tilde{\imath}(V_0,z), \quad z \in V_G \setminus V_0 \text{ such that } z \sim b, \\ i(a,b) &\coloneqq \sum_{z\in V_G\setminus V_0} i(b,z). \end{aligned}$$

Then, by the same argument as before, one can check that $R_G(a,x) \le R_{\tilde{G}}(x,y) + \mu_G(a,b)^{-1}$. Hence, we complete the proof. □

For electrical networks, besides fusing, another natural operation can be considered: adding edges. For each $n \ge 1$, let G_n be an electrical network with finite vertex set such that $\mu_{G_n}(x,y) = 1$ if $\mu_{G_n}(x,y) > 0$ (i.e., any positive conductance is 1). Fix $N \in \mathbb{N}$ and $a_i^{(n)}, b_i^{(n)} \in V_{G_n}$ such that $\{a_1^{(n)}, b_1^{(n)}\}, \ldots, \{a_N^{(n)}, b_N^{(n)}\}$ are distinct subsets of V_{G_n}. Define an electrical network $\bar{G}_n$ with vertex set $V_{\bar{G}_n} \coloneqq V_{G_n}$ by setting conductances as follows:

$$\mu_{\bar{G}_n}(x,y) \coloneqq \begin{cases} \mu_{G_n}(a_i^{(n)}, b_i^{(n)}) + 1, & \{x,y\} = \{a_i^{(n)}, b_i^{(n)}\}, \\ \mu_{G_n}(x,y), & \text{otherwise.} \end{cases}$$

In other words, $\bar{G}_n$ is obtained by attaching a new edge between $a_i^{(n)}$ and $b_i^{(n)}$ (if there are multiple edges, then those edges are replaced by a single edge with conductance 2). We let $\rho_{\bar{G}_n} \coloneqq \rho_G$.

Theorem A.13. *Assume the above setting. Let (F, R, ρ) be a rooted compact resistance metric, a_1, $b_1, \ldots, a_N$, b_N be distinct elements of F, and $\dot{\mu}$ be a Radon measure on $F \times \mathbb{R}_{\ge 0}$. Assume that*

$$\left(V_{G_n}, \alpha_n^{-1} R_{G_n}, \rho_{G_n}, \beta_n^{-1} \dot{\mu}_{G_n}^{\#}, (a_i^{(n)}, b_i^{(n)})_{i=1}^N\right) \to \left(F, R, \rho, \dot{\mu}, (a_i, b_i)_{i=1}^N\right)$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\mathbb{R}_{\ge 0})} \times \tau^{2N\text{-pts}})$, where $(\alpha_n)_{n\ge 1}$ and $(\beta_n)_{n\ge 1}$ are sequences of positive numbers with $\alpha_n \wedge \beta_n \to \infty$. Then, it holds that

$$\left(V_{\bar{G}_n}, \alpha_n^{-1} R_{\bar{G}_n}, \rho_{\bar{G}_n}, \beta_n^{-1} \dot{\mu}_{\bar{G}_n}^{\#}\right) \to \left(\tilde{F}, \tilde{R}, \tilde{\rho}, \dot{\mu} \circ (\pi \times \mathrm{id}_{\mathbb{R}_{\ge 0}})^{-1}\right)$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot\times\mathbb{R}_{\ge 0})})$, where $(\tilde{F}, \tilde{R})$ is the resistance metric space (F, R) fused over $\{\{a_i, b_i\}\}_{i=1}^N$, $\pi \colon F \to \tilde{F}$ is the canonical map, and we set $\tilde{\rho} = \pi(\rho)$.

Proof: Write $\tilde{G}_n$ for the electrical network G_n fused over $\{\{a_i^{(n)}, b_i^{(n)}\}\}_{i=1}^N$, and $\pi_n\colon V_{G_n} \to V_{\tilde{G}_n}$ for the canonical map. Set $\rho_{\tilde{G}_n} \coloneqq \pi_n(\rho_{G_n})$. By Theorem A.9, we may assume the following:

- $(V_{G_n}, \alpha_n^{-1} R_{G_n}, \rho_{G_n})$ and (F, R, ρ) are embedded isometrically into a common rooted compact metric space (M, d^M, ρ_M) in such a way that $V_{G_n} \to F$ in the Hausdorff topology, $\rho_{G_n} = \rho = \rho_M$ as elements of M, $\beta_n^{-1}\dot{\mu}_{G_n}^{\#} \to \dot{\mu}$ weakly, and $a_i^{(n)} \to a_i$ and $b_i^{(n)} \to b_i$ in M for each i;
- $(V_{\tilde{G}_n}, \alpha_n^{-1} R_{\tilde{G}_n}, \rho_{\tilde{G}_n})$ and $(\tilde{F}, \tilde{R}, \tilde{\rho})$ are embedded isometrically into a common rooted compact metric space $(\tilde{M}, d^{\tilde{M}}, \rho_{\tilde{M}})$ in such a way that $\rho_{\tilde{G}_n} = \tilde{\rho} = \rho_{\tilde{M}}$ as elements of $\tilde{M}$, and $V_{\tilde{G}_n} \to \tilde{F}$ in the Hausdorff topology;
- if we think of π_n and π as elements of $\widehat{C}(M, \tilde{M})$ by the above embeddings, then $\pi_n \to \pi$ in $\widehat{C}(M, \tilde{M})$.

Since $\pi_n \times \mathrm{id}_{\mathbb{R}_{\geq 0}} \to \pi \times \mathrm{id}_{\mathbb{R}_{\geq 0}}$ in $\widehat{C}(M \times \mathbb{R}_{\geq 0}, \tilde{M} \times \mathbb{R}_{\geq 0})$, we deduce that $\beta_n^{-1}\dot{\mu}_{G_n}^{\#} \circ (\pi_n \times \mathrm{id}_{\mathbb{R}_{\geq 0}})^{-1} \to \dot{\mu} \circ (\pi \times \mathrm{id}_{\mathbb{R}_{\geq 0}})^{-1}$ weakly. Hence, it holds that

$$\left(V_{\tilde{G}_n}, \alpha_n^{-1} R_{\tilde{G}_n}, \rho_{\tilde{G}_n}, \beta_n^{-1}\dot{\mu}_{G_n}^{\#} \circ (\pi_n \times \mathrm{id}_{\mathbb{R}_{\geq 0}})^{-1}\right) \to \left(\tilde{F}, \tilde{R}, \tilde{\rho}, \dot{\mu} \circ (\pi \times \mathrm{id}_{\mathbb{R}_{\geq 0}})^{-1}\right)$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot \times \mathbb{R}_{\geq 0})})$. Hence, it suffices to show that the distance between

$$\left(V_{\tilde{G}_n}, \alpha_n^{-1} R_{\tilde{G}_n}, \rho_{\tilde{G}_n}, \beta_n^{-1}\dot{\mu}_{G_n}^{\#} \circ (\pi_n \times \mathrm{id}_{\mathbb{R}_{\geq 0}})^{-1}\right) \quad \text{and} \quad \left(V_{\tilde{G}_n}, \alpha_n^{-1} R_{\tilde{G}_n}, \rho_{\tilde{G}_n}, \beta_n^{-1}\dot{\mu}_{\tilde{G}_n}^{\#}\right)$$

in the space $\mathfrak{K}_\bullet(\tau^{\mathcal{M}_{\mathrm{fin}}(\cdot \times \mathbb{R}_{\geq 0})})$ converges to 0. This is easily proven by Proposition A.12 and the following:

$$\limsup_{n \to \infty} \alpha_n^{-1} \sup_{1 \leq i \leq N} \mu_{\tilde{G}_n}(a_i^{(n)}, b_i^{(n)})^{-1} \leq \limsup_{n \to \infty} \alpha_n^{-1} = 0.$$

□

Acknowledgements

I would like to thank my supervisor Dr David Croydon for his support and fruitful discussions, and an anonymous referee for pointing out several mistakes in a preliminary version.

References

Abraham, R., Delmas, J.-F., and Hoscheit, P. A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces. *Electron. J. Probab.*, **18**, no. 14, 21 (2013). DOI: 10.1214/EJP.v18-2116.

Addario-Berry, L., Broutin, N., and Goldschmidt, C. The continuum limit of critical random graphs. *Probab. Theory Related Fields*, **152** (3-4), 367–406 (2012). DOI: 10.1007/s00440-010-0325-4.

Aldous, D. The continuum random tree. III. *Ann. Probab.*, **21** (1), 248–289 (1993). DOI: 10.1214/aop/1176989404.

Aldous, D. Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.*, **25** (2), 812–854 (1997). DOI: 10.1214/aop/1024404421.

Athreya, S., Löhr, W., and Winter, A. The gap between Gromov-vague and Gromov-Hausdorff-vague topology. *Stochastic Process. Appl.*, **126** (9), 2527–2553 (2016). DOI: 10.1016/j.spa.2016.02.009.

Barlow, M. T. Diffusions on fractals. In *Lectures on probability theory and statistics (Saint-Flour, 1995)*, volume 1690 of *Lecture Notes in Math.*, pp. 1–121. Springer, Berlin (1998). ISBN 3-540-64620-5. DOI: 10.1007/BFb0092537.

Ben Arous, G. Aging and spin-glass dynamics. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pp. 3–14. Higher Ed. Press, Beijing (2002). ISBN 7-04-008690-5. MR1957514.

Ben Arous, G., Cabezas, M., Černý, J., and Royfman, R. Randomly trapped random walks. *Ann. Probab.*, **43** (5), 2405–2457 (2015). DOI: 10.1214/14-AOP939.

Ben Arous, G. and Černý, J. Bouchaud's model exhibits two different aging regimes in dimension one. *Ann. Appl. Probab.*, **15** (2), 1161–1192 (2005). DOI: 10.1214/105051605000000124.

Ben Arous, G. and Černý, J. Dynamics of trap models. In *Mathematical statistical physics*, pp. 331–394. Elsevier B. V., Amsterdam (2006). ISBN 978-0-444-52813-1; 0-444-52813-X. DOI: 10.1016/S0924-8099(06)80045-4.

Ben Arous, G. and Černý, J. Scaling limit for trap models on $\mathbb{Z}^d$. *Ann. Probab.*, **35** (6), 2356–2384 (2007). DOI: 10.1214/009117907000000024.

Ben Arous, G. and Černý, J. The arcsine law as a universal aging scheme for trap models. *Comm. Pure Appl. Math.*, **61** (3), 289–329 (2008). DOI: 10.1002/cpa.20177.

Ben Arous, G., Černý, J., and Mountford, T. Aging in two-dimensional Bouchaud's model. *Probab. Theory Related Fields*, **134** (1), 1–43 (2006). DOI: 10.1007/s00440-004-0408-1.

Billingsley, P. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition (1999). ISBN 0-471-19745-9. DOI: 10.1002/9780470316962.

Bingham, N. H., Goldie, C. M., and Teugels, J. L. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (1987). ISBN 0-521-30787-2. DOI: 10.1017/CBO9780511721434.

Bouchaud, J.-P. Weak ergodicity breaking and aging in disordered systems. *J. Phys. I France*, **2** (9), 1705–1713 (1992). DOI: 10.1051/jp1:1992238.

Bouchaud, J.-P., Cugliandolo, L. F., Kurchan, J., and Mézard, M. *Out of Equilibrium dynamics in spin-glasses and other glassy systems*, volume 12 of *Series on Directions in Condensed Matther Physics*, pp. 161–223 (1997). DOI: 10.1142/9789812819437_0006.

Burago, D., Burago, Y., and Ivanov, S. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI (2001). ISBN 0-8218-2129-6. DOI: 10.1090/gsm/033.

Chen, Z.-Q. and Fukushima, M. *Symmetric Markov processes, time change, and boundary theory*, volume 35 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ (2012). ISBN 978-0-691-13605-9. MR2849840.

Croydon, D. A. Scaling limits of stochastic processes associated with resistance forms. *Ann. Inst. Henri Poincaré Probab. Stat.*, **54** (4), 1939–1968 (2018). DOI: 10.1214/17-AIHP861.

Croydon, D. A., Hambly, B., and Kumagai, T. Time-changes of stochastic processes associated with resistance forms. *Electron. J. Probab.*, **22**, Paper No. 82, 41 (2017). DOI: 10.1214/17-EJP99.

Croydon, D. A., Kious, D., and Scali, C. Aging and sub-aging for one-dimensional random walks amongst random conductances. *Stochastic Process. Appl.*, **182**, Paper No. 104562, 38 (2025). DOI: 10.1016/j.spa.2025.104562.

Duquesne, T. A limit theorem for the contour process of conditioned Galton-Watson trees. *Ann. Probab.*, **31** (2), 996–1027 (2003). DOI: 10.1214/aop/1048516543.

Fontes, L. R. G., Isopi, M., and Newman, C. M. Random walks with strongly inhomogeneous rates and singular diffusions: convergence, localization and aging in one dimension. *Ann. Probab.*, **30** (2), 579–604 (2002). DOI: 10.1214/aop/1023481003.

Fukushima, M., Oshima, Y., and Takeda, M. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition (2011). ISBN 978-3-11-021808-4. MR2778606.

Janson, S. Simply generated trees, conditioned Galton-Watson trees, random allocations and condensation. *Probab. Surv.*, **9**, 103–252 (2012). DOI: 10.1214/11-PS188.

Kallenberg, O. *Random measures, theory and applications*, volume 77 of *Probability Theory and Stochastic Modelling*. Springer, Cham (2017). ISBN 978-3-319-41596-3; 978-3-319-41598-7. DOI: 10.1007/978-3-319-41598-7.

Kallenberg, O. *Foundations of modern probability*, volume 99 of *Probability Theory and Stochastic Modelling*. Springer, Cham, third edition (2021). ISBN 978-3-030-61871-1; 978-3-030-61870-4. DOI: 10.1007/978-3-030-61871-1.

Khezeli, A. Metrization of the Gromov-Hausdorff (-Prokhorov) topology for boundedly-compact metric spaces. *Stochastic Process. Appl.*, **130** (6), 3842–3864 (2020). DOI: 10.1016/j.spa.2019.11.001.

Khezeli, A. A unified framework for generalizing the Gromov-Hausdorff metric. *Probab. Surv.*, **20**, 837–896 (2023). DOI: 10.1214/20-ps340.

Kigami, J. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge (2001). ISBN 0-521-79321-1. DOI: 10.1017/CBO9780511470943.

Kigami, J. Resistance forms, quasisymmetric maps and heat kernel estimates. *Mem. Amer. Math. Soc.*, **216** (1015), vi+132 (2012). DOI: 10.1090/S0065-9266-2011-00632-5.

Le Gall, J.-F. Random real trees. *Ann. Fac. Sci. Toulouse Math. (6)*, **15** (1), 35–62 (2006). MR2225746.

Levin, D. A. and Peres, Y. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, second edition (2017). ISBN 978-1-4704-2962-1. DOI: 10.1090/mbk/107.

Marcus, M. B. and Rosen, J. *Markov processes, Gaussian processes, and local times*, volume 100 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2006). ISBN 978-0-521-86300-1; 0-521-86300-7. DOI: 10.1017/CBO9780511617997.

Mathieu, P. and Mourrat, J.-C. Aging of asymmetric dynamics on the random energy model. *Probab. Theory Related Fields*, **161** (1-2), 351–427 (2015). DOI: 10.1007/s00440-014-0551-2.

Molchanov, I. *Theory of random sets*, volume 87 of *Probability Theory and Stochastic Modelling*. Springer-Verlag, London, second edition (2017). ISBN 978-1-4471-7347-2; 978-1-4471-7349-6. MR3751326.

Noda, R. Metrization of Gromov–Hausdorff-type topologies on boundedly-compact metric spaces. *ArXiv Mathematics e-prints* (2024a). arXiv: 2404.19681.

Noda, R. Scaling limits of discrete-time Markov chains and their local times on electrical networks. *ArXiv Mathematics e-prints* (2024b). arXiv: 2405.01871.

Noda, R. Convergence of local times of stochastic processes associated with resistance forms (2025+). To appear in *Ann. Inst. Henri Poincaré Probab. Stat.*

Srivastava, S. M. *A course on Borel sets*, volume 180 of *Graduate Texts in Mathematics*. Springer-Verlag, New York (1998). ISBN 0-387-98412-7. DOI: 10.1007/978-3-642-85473-6.

Thévenin, P. Vertices with fixed outdegrees in large Galton-Watson trees. *Electron. J. Probab.*, **25**, Paper No. 64, 25 (2020). DOI: 10.1214/20-ejp465.

Whitt, W. Some useful functions for functional limit theorems. *Math. Oper. Res.*, **5** (1), 67–85 (1980). DOI: 10.1287/moor.5.1.67.

Whitt, W. *Stochastic-process limits. An introduction to stochastic-process limits and their application to queues*. Springer Series in Operations Research. Springer-Verlag, New York (2002). ISBN 0-387-95358-2. MR1876437.