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The intertwining property for β-Laguerre processes and integral operators for Jack polynomials

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Abstract. The aim of this paper is to study intertwining relations for Laguerre processes with inverse temperature $\beta \geq 1$ and parameter $\alpha > -1$. We introduce a Markov kernel that depends on both β and α, and establish new intertwining relations for the β-Laguerre processes using this kernel. A key observation is that Jack symmetric polynomials are eigenfunctions of our Markov kernel, which allows us to apply a method established by Ramanan and Shkolnikov. Additionally, as a by-product, we derive an integral formula for multivariate Laguerre polynomials and multivariate hypergeometric functions associated with Jack polynomials.

1. Introduction

1.1. *Markov kernels and Laguerre processes.* Our purpose is to give intertwining relations for Laguerre processes for general inverse temperature β. Whilst intertwining relations for the β-Laguerre process were shown in Assiotis (2019), we establish new relations by using different Markov kernels, which are a β-extension of the kernel used in Bufetov and Kawamoto (2025b).

Throughout this paper, we set $\theta = \beta/2$. Set the closure of the Weyl chamber $W^N = \{\mathbf{x}_N = (x_1, \dots, x_N) \in \mathbb{R}^N\, ;\, x_1 \leq \cdots \leq x_N\}$. We will write $\mathbf{x}_N$ simply as $\mathbf{x}$ when no confusion can arise. The interior of W^N is denoted by $\mathring{W}^N := \{\mathbf{x} \in W^N\, ;\, x_1 < \cdots < x_N\}$. We write the set of interlaced configurations as

$$W^{N,N+1}(\mathbf{x}) := \{\mathbf{y} \in W^N\, ;\, x_1 \leq y_1 \leq x_2 \leq \cdots \leq y_N \leq x_{N+1}\} \quad \text{for } \mathbf{x} = (x_1, \dots, x_{N+1}) \in W^{N+1}.$$

The following formula is useful to introduce Markov kernels.

Lemma 1.1. *(Forrester, 2010, Proposition 4.2.1) Let $(w_1, \dots, w_{N+1})$ be a random variable distributed according to the Dirichlet distribution with parameters $(s_1, \dots, s_{N+1})$ and let $\mathbf{x} \in \mathring{W}^{N+1}$.*

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Then, the roots $\mathbf{y} \in W^N$ *of the random rational function*

$$y \mapsto \sum_{i=1}^{N+1} \frac{w_i}{y - x_i} \left(= \frac{\sum_{i=1}^{N+1} \left(w_i \prod_{1\le j\le N+1, j\ne i}(y - x_j)\right)}{\prod_{i=1}^{N+1}(y - x_i)} \right)$$

has the probability density function

$$\frac{\Gamma(s_1 + \cdots + s_{N+1})}{\Gamma(s_1)\cdots\Gamma(s_{N+1})} \frac{\prod_{1\le i<j\le N}(y_j - y_i)}{\prod_{1\le i<j\le N+1}(x_j - x_i)^{s_j+s_i-1}} \prod_{i=1}^{N+1}\prod_{j=1}^{N} |x_i - y_j|^{s_i-1} \mathbf{1}_{W^{N,N+1}(\mathbf{x})}(\mathbf{y}).$$

Following Assiotis and Najnudel (2021), we introduce the Markov kernel $\Lambda_{\theta,N}^{N+1}$ via a random polynomial.

Definition 1.2. For $\theta > 0$ and $\mathbf{x} = (x_1, \dots, x_{N+1}) \in W^{N+1}$, let $\Lambda_{\theta,N}^{N+1}(\mathbf{x}, \cdot)$ be the distribution on W^N of the roots of the random polynomial

$$y \mapsto \sum_{i=1}^{N+1} \Big(w_i \prod_{1\le j\le N+1, j\ne i} (y - x_j) \Big),$$

where $(w_1, \dots, w_{N+1})$ is distributed according to the Dirichlet distribution with all parameters being equal to θ.

The kernel $\Lambda_{\theta,N}^{N+1}$ is an integral operator on W^N depending on a parameter $\mathbf{x} \in W^{N+1}$. In this sense, we write $\Lambda_{\theta,N}^{N+1} : W^{N+1} \dashrightarrow W^N$. We can describe the probability density of the kernel $\Lambda_{\theta,N}^{N+1}$ due to Lemma 1.1.

Proposition 1.3. *For any* $\mathbf{x} \in \mathring{W}^{N+1}$, *we have*

$$\Lambda_{\theta,N}^{N+1}(\mathbf{x}, d\mathbf{y}) = \frac{1}{Z_\theta^N} \frac{\prod_{1\le i<j\le N}(y_j - y_i)}{\prod_{1\le i<j\le N+1}(x_j - x_i)^{2\theta-1}} \prod_{i=1}^{N+1}\prod_{j=1}^{N} |x_i - y_j|^{\theta-1} \mathbf{1}_{W^{N,N+1}(\mathbf{x})}(\mathbf{y}) d\mathbf{y}, \tag{1.1}$$

where the normalisation constant is given by

$$\frac{1}{Z_\theta^N} := \frac{\Gamma((N+1)\theta)}{\Gamma(\theta)^{N+1}}.$$

The distribution (1.1) is referred to as the Dixon–Anderson conditional distribution because this probability density was found in the work by Dixon (1905), and considered by Anderson in the context of the Selberg integral (Anderson, 1991). The formulation of Dixon–Anderson distribution by Definition 1.2 is used in Assiotis and Najnudel (2021).

We discuss intertwining relations between Markov kernels and semigroups of stochastic processes related to random matrices. Several diffusions intertwined by $\Lambda_{\theta,N}^{N+1}$ have been identified. The first example intertwined by this kernel is the non-intersecting Brownian motions, also known as the Dyson Brownian motions for $\beta = 2$ (Warren, 2007). Other examples for $\beta = 2$ were studied in Assiotis (2020); Assiotis et al. (2019). For general β, the intertwining relations for the β-Dyson Brownian motions were established (Gorin and Shkolnikov, 2015; Ramanan and Shkolnikov, 2018), and those for the β-Laguerre and β-Jacobi processes were obtained (Assiotis, 2019).

Here, we focus on intertwining relations for the β-Laguerre processes. Set $W_{\ge}^N = W^N \cap [0, \infty)^N$. The N-dimensional Laguerre process for the inverse temperature $\beta \ge 1$ is the process on $W_{\ge}^N$ defined by the stochastic differential equation

$$dX_t^{N,i} = \sqrt{2X_t^{N,i}} dB_t^i + \frac{\beta}{2}\Big(\alpha + 1 + \sum_{1\le j\le N, j\ne i} \frac{2X_t^{N,i}}{X_t^{N,i} - X_t^{N,j}}\Big) dt, \qquad 1 \le i \le N, \tag{1.2}$$

where $(B^i)_{i=1}^N$ is the N-dimensional Brownian motion and $\alpha > -1$. For any starting point $\mathbf{x}_N \in W_{\geq}^N$, the equation (1.2) has a unique strong solution with no collisions and no explosions (Graczyk and Małecki, 2014, Corollary 6.5). The Laguerre processes for general β were introduced and studied in Demni (2007b). When $\beta = 1$, the Laguerre process is the squared singular values process of a matrix with Brownian entries (Bru, 1991). The Laguerre process for $\beta = 2$ is identical to the non-colliding square Bessel process, which describes the squared singular value process of a matrix with complex Brownian entries (Demni, 2007a; König and O'Connell, 2001).

Set $\theta = \beta/2$ as before. Let $\{T_{\theta,\alpha,t}^N\}_{t\geq 0}$ be the Markov semigroup associated with the solution to (1.2). The following intertwining relation for Laguerre processes was established in Assiotis (2019) (see also Assiotis et al., 2019, Section 3.7 for $\theta = 1$).

Proposition 1.4 (Assiotis, 2019)**.** *Assume that $\theta \geq 1/2$ and $\alpha > -1$. Then, for any $N \in \mathbb{N}$ and $t \geq 0$, we have the equality of Markov kernels*

$$T_{\theta,\alpha,t}^{N+1} \Lambda_{\theta,N}^{N+1} = \Lambda_{\theta,N}^{N+1} T_{\theta,\alpha+1,t}^N. \tag{1.3}$$

Note that, in (1.1), the α-parameters of the Laguerre processes on the left- and right-hand sides are different. For this reason, we refer to the equation (1.3) as the shifted intertwining relation.

In this paper, we establish intertwining relations with fixed parameters, analogous to those for the Ornstein–Uhlenbeck counterpart of the Laguerre process for $\theta = 1$ given in Bufetov and Kawamoto (2025b). Specifically, we will identify a kernel $\Lambda_{\theta,\alpha,N}^{N+1}$, depending on θ and α, such that

$$T_{\theta,\alpha,t}^{N+1} \Lambda_{\theta,\alpha,N}^{N+1} = \Lambda_{\theta,\alpha,N}^{N+1} T_{\theta,\alpha,t}^N. \tag{1.4}$$

Unlike in (1.3), the α-parameters on both sides of (1.4) are the same.

1.2. *Main results.* To obtain new intertwining relations for the Laguerre processes, we introduce a kernel $\Lambda_{\theta,\alpha,N}^N : W_{\geq}^N \dashrightarrow W_{\geq}^N$ depending on θ and α.

Definition 1.5. Suppose $\theta > 0$ and $\alpha > -1$. For $\mathbf{x} = (x_1, \ldots, x_N) \in W_{\geq}^N$, let $\Lambda_{\theta,\alpha,N}^N(\mathbf{x}, \cdot)$ be the distribution on $W_{\geq}^N$ of the roots of the random polynomial

$$y \mapsto \sum_{i=0}^N \Big(w_i \prod_{0 \leq j \leq N, j \neq i} (y - x_j) \Big),$$

where $(w_0, w_1, \ldots, w_N)$ is distributed according to the Dirichlet distribution with parameters $(s_0, s_1, \ldots, s_N)$ given by $s_0 = (\alpha + 1)\theta$ and $s_1 = \cdots = s_N = \theta$. Here, we use the symbol $x_0 = 0$ for notational convenience.

The probability density of the kernel $\Lambda_{\theta,\alpha,N}^N$ is also explicitly given from Lemma 1.1. We set $\mathring{W}_{\geq}^N = \{\mathbf{x} \in W_{\geq}^N \,;\, 0 < x_1 < \cdots < x_N\}$ and

$$W_{\geq}^{N,N}(\mathbf{x}) = \{\mathbf{y} \in W_{\geq}^N \,;\, 0 \leq y_1 \leq x_1 \leq y_2 \leq \cdots \leq y_N \leq x_N\} \quad \text{for } \mathbf{x} = (x_1, \ldots, x_N) \in W_{\geq}^N.$$

Proposition 1.6. *Suppose $\theta > 0$ and $\alpha > -1$. For any $\mathbf{x} \in \mathring{W}_{\geq}^N$, we have*

$$\Lambda_{\theta,\alpha,N}^N(\mathbf{x}, d\mathbf{y}) = \frac{1}{Z_{\theta,\alpha}^N} \frac{\prod_{1\leq i<j\leq N}(y_j - y_i)}{\prod_{1\leq i<j\leq N}(x_j - x_i)^{2\theta-1}} \prod_{i,j=1}^N |x_i - y_j|^{\theta-1} \prod_{i=1}^N \frac{y_i^{\theta(\alpha+1)-1}}{x_i^{\theta(\alpha+2)-1}} \mathbf{1}_{W_{\geq}^{N,N}(\mathbf{x})}(\mathbf{y}) d\mathbf{y}, \tag{1.5}$$

where the normalisation constant is given by

$$\frac{1}{Z_{\theta,\alpha}^N} := \frac{\Gamma((N+\alpha+1)\theta)}{\Gamma(\theta)^N \Gamma((\alpha+1)\theta)}.$$

When $\alpha \in \{0\} \cup \mathbb{N}$, the probability density (1.5) for $\theta = 1/2, 1, 2$ was established as the distribution of the squared singular values of a fixed matrix multiplied with a truncated orthogonal, unitary, symplectic random matrix, respectively (Ahn and Strahov, 2022; Kieburg et al., 2016). This fact will be used in Section 5.

For $\theta > 0$ and $\alpha > -1$, we define the Markov kernel $\Lambda^{N+1}_{\theta,\alpha,N} : W^{N+1}_{\geq} \dashrightarrow W^{N}_{\geq}$ as

$$\Lambda^{N+1}_{\theta,\alpha,N} = \Lambda^{N+1}_{\theta,N} \Lambda^{N}_{\theta,\alpha,N}. \tag{1.6}$$

The kernel $\Lambda^{N+1}_{\theta,\alpha,N}$ gives an intertwining relation for β-Laguerre processes with a fixed parameter α.

Theorem 1.7. *Suppose $\theta \geq 1/2$ and $\alpha > -1$. Then, for any $N \in \mathbb{N}$ and $t \geq 0$, we have the equality of Markov kernels*

$$T^{N+1}_{\theta,\alpha,t} \Lambda^{N+1}_{\theta,\alpha,N} = \Lambda^{N+1}_{\theta,\alpha,N} T^{N}_{\theta,\alpha,t}. \tag{1.7}$$

Theorem 1.7 is demonstrated by employing new shifted intertwining relations.

Theorem 1.8. *Suppose $\theta \geq 1/2$ and $\alpha > -1$. Then, for any $N \in \mathbb{N}$ and $t \geq 0$, we have the equality of Markov kernels*

$$T^{N}_{\theta,\alpha+1,t} \Lambda^{N}_{\theta,\alpha,N} = \Lambda^{N}_{\theta,\alpha,N} T^{N}_{\theta,\alpha,t}. \tag{1.8}$$

Assuming Theorem 1.8, we can easily prove Theorem 1.7 as follows.

Proof of Theorem 1.7 The equations (1.3) and (1.8) imply that

$$T^{N+1}_{\theta,\alpha,t} \Lambda^{N+1}_{\theta,N} \Lambda^{N}_{\theta,\alpha,N} = \Lambda^{N+1}_{\theta,N} T^{N}_{\theta,\alpha+1,t} \Lambda^{N}_{\theta,\alpha,N} = \Lambda^{N+1}_{\theta,N} \Lambda^{N}_{\theta,\alpha,N} T^{N}_{\theta,\alpha,t}. \tag{1.9}$$

Therefore, combining (1.9) with (1.6), we complete the proof. □

Thus, it remains to establish Theorem 1.8, and we apply the method used in Ramanan and Shkolnikov (2018) to achieve this. The first step in their approach involves establishing intertwining relations at the level of generators by using Jack symmetric polynomials. We introduce key relations for Jack polynomials in the next subsection.

Furthermore, by the same strategy, we can show that Ornstein–Uhlenbeck counterparts of Laguerre processes satisfy intertwining relations similar to (1.7) and (1.8). We will discuss this in Appendix B.

Remark 1.9. The intuition behind Theorem 1.7 and Theorem 1.8 comes from an interpretation of $\Lambda^{N}_{\theta,\alpha,N}$ and $\Lambda^{N+1}_{\theta,\alpha,N}$ in the context of random matrices for $\theta = 1/2, 1, 2$ and $\alpha \in \{0\} \cup \mathbb{N}$ (see Remark 5.5), together with an extrapolation to general θ and α.

Intertwining relations have fruitful applications in probability theory. For example, they were used to prove convergence to stationary measures for Markov chains (Diaconis and Fill, 1990). More recently, intertwining relations have been studied in the area of integrable probability. In particular, they are closely related to multilevel processes on interlacing arrays, so-called interlacing processes. The construction of multilevel Dyson Brownian motions, initiated by Warren (2007) for $\beta = 2$ and extended to general β by Gorin and Shkolnikov (2015), also establishes intertwining relations for Dyson's Brownian motions. For $\beta = 2$, interlacing Laguerre processes corresponding to (1.3) were obtained in Assiotis et al. (2019), and more recently, interlacing processes for eigenvalues of Laguerre-type matrix processes were also constructed (Assiotis, 2023). It would be of interest to construct interlacing processes corresponding to intertwining relations (1.7). The construction of multilevel processes in Gorin and Shkolnikov (2015) is based on a limit from discrete dynamics defined via Jack polynomials. It is also an interesting problem to derive our intertwining relations via such a discrete approximation.

Furthermore, intertwining relations are key ingredients in constructing a limit process of a coherent family of finite-dimensional processes by the method of intertwiners. This method was introduced in Borodin and Olshanski (2012) to construct Feller processes on the boundary of the Gelfand-Tsetlin graph, and has since been applied in several other settings (Assiotis, 2020; Assiotis and Mirsajjadi, 2024; Borodin and Olshanski, 2013; Cuenca, 2018; Olshanski, 2016). In the case where $\beta = 2$ and $\alpha \in \{0\} \cup \mathbb{N}$, the method of intertwiners was applied to the projective system $\{W^N_{\geq}, \Lambda^{N+1}_{\theta,\alpha,N}\}_{N\in\mathbb{N}}$, and a limit Laguerre (Ornstein–Uhlenbeck) process was obtained by Bufetov and Kawamoto (2025a). For a general inverse temperature, the coherent family obtained from Theorem 1.7 is expected to give rise to a limit process.

1.3. *Integral operators and Jack polynomials.* In what follows, let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition of an integer and define $l(\lambda)$ as the length of λ. Let $P_\lambda(\mathbf{x}_N; \theta) = x_1^{\lambda_1} \cdots x_N^{\lambda_N} + \cdots$ be the Jack symmetric polynomial parametrised by a partition λ. See Section 2 for the precise definition of Jack polynomials. Hereafter, we use $P^\theta_\lambda(\mathbf{x}_N) := P_\lambda(\mathbf{x}_N; \theta)$ for simplicity.

For $z \in \mathbb{C}$, the shifted factorial is defined by $(x)_z = \Gamma(x + z)/\Gamma(x)$. We set

$$\begin{aligned} c(\lambda, N, \theta\,; \alpha) &:= \frac{\Gamma((N + \alpha + 1)\theta)}{\Gamma((\alpha + 1)\theta)} \prod_{i=1}^{N} \frac{\Gamma((N + \alpha + 1 - i)\theta + \lambda_i)}{\Gamma((N + \alpha + 2 - i)\theta + \lambda_i)} \\ &= \prod_{i=1}^{N} \frac{((N + \alpha + 1 - i)\theta)_{\lambda_i}}{((N + \alpha + 2 - i)\theta)_{\lambda_i}}. \end{aligned} \tag{1.10}$$

Note that $c(\lambda, N, \theta\,; \alpha) \in \mathbb{Q}(\theta, \alpha)$ for fixed λ and N. Furthermore, we use $c(\lambda, N, \theta) := c(\lambda, N, \theta\,; 0)$.

Jack polynomials satisfy the following relation in terms of $\Lambda^{N+1}_{\theta,N}$, which is the key to showing the intertwining relation at the level of generators in Assiotis (2019); Ramanan and Shkolnikov (2018).

Lemma 1.10. *(Okounkov, 1998; Okounkov and Olshanski, 1997) Suppose $\theta > 0$. Then, for any partition λ with $l(\lambda) \leq N$ and $\mathbf{x}_{N+1} \in W^{N+1}$, we have*

$$[\Lambda^{N+1}_{\theta,N} P^\theta_\lambda](\mathbf{x}_{N+1}) = c(\lambda, N, \theta) P^\theta_\lambda(\mathbf{x}_{N+1}), \tag{1.11}$$

where we write $[\Lambda^{N+1}_{\theta,N} P^\theta_\lambda](\mathbf{x}_{N+1}) := \int P^\theta_\lambda(\mathbf{y}_N) \Lambda^{N+1}_{\theta,N}(\mathbf{x}_{N+1}, d\mathbf{y}_N)$.

The equation (1.11) suggests that Jack polynomials are eigenfunctions of $\Lambda^{N+1}_{\theta,N}$. However, strictly speaking, this is not correct: whilst P^θ_λ on the left-hand side of (1.11) is in N variables, on the right-hand side, it is in $N+1$ variables. On the other hand, we establish that Jack polynomials are indeed eigenfunctions of $\Lambda^N_{\theta,\alpha,N}$.

Theorem 1.11. *Suppose $\theta > 0$ and $\alpha > -1$. Then, for any λ with $l(\lambda) \leq N$ and $\mathbf{x}_N \in W^N_{\geq}$, we have*

$$[\Lambda^N_{\theta,\alpha,N} P^\theta_\lambda](\mathbf{x}_N) = c(\lambda, N, \theta\,; \alpha) P^\theta_\lambda(\mathbf{x}_N). \tag{1.12}$$

Furthermore, using Theorem 1.11, we can compute the action of $\Lambda^N_{\theta,\alpha,N}$ on multivariate Laguerre polynomials and multivariate hypergeometric functions, as demonstrated in the appendix.

1.4. *Organisation of this paper.* The present paper is organised as follows. We collect some facts about Jack polynomials in Section 2. In Section 3, we give the proof of Theorem 1.11. Section 4 is devoted to the proof of Theorem 1.8. In Section 5, we observe that the kernel $\Lambda^{N+1}_{\theta,\alpha,N}$ has an interpretation in terms of squared singular values of invariant random matrices for classical parameter $\theta = 1/2, 1, 2$. In Appendix A, we apply Theorem 1.11 to obtain a formula for symmetric functions. In Appendix B, we prove that Laguerre Ornstein–Uhlenbeck processes satisfy intertwining relations similar to Theorem 1.7 and Theorem 1.8.

2. Preliminaries on Jack symmetric polynomials

We begin by introducing Jack polynomials. See Vilenkin and Klimyk (1995, Chapter 2) for further details. We introduce the differential operators in N-variables

$$E_k^N = \sum_{i=1}^N x_i^k \frac{\partial}{\partial x_i},$$
$$D_k^N = \sum_{i=1}^N x_i^k \frac{\partial^2}{\partial x_i^2} + 2\theta \sum_{i=1}^N \sum_{1\le j\le N, j\ne i} \frac{x_i^k}{x_i - x_j}\frac{\partial}{\partial x_i}.$$

The Jack polynomial parametrised by a partition λ is defined as a symmetric eigenfunction of D_2^N:

$$D_2^N P_\lambda^\theta(\mathbf{x}_N) = e(\lambda, N, \theta) P_\lambda^\theta(\mathbf{x}_N), \tag{2.1}$$

where P_λ^θ has the form

$$P_\lambda^\theta(\mathbf{x}_N) := P_\lambda(\mathbf{x}_N;\theta) = m_\lambda(\mathbf{x}_N) + (\text{lower terms with respect to the dominance order}). \tag{2.2}$$

Here, $m_\lambda(\mathbf{x}_N) = x_1^{\lambda_1}\cdots x_N^{\lambda_N} + \cdots$ denotes the monomial symmetric polynomial, and

$$e(\lambda, N, \theta) := 2B(\lambda') - 2\theta B(\lambda) + 2\theta(N-1)|\lambda|,$$

where $|\lambda| = \sum \lambda_i$, $B(\lambda) = \sum (i-1)\lambda_i$, and λ' is the conjugate partition of λ. A symmetric polynomial satisfying (2.1) and (2.2) is unique up to a constant factor, and we take normalisation such that the leading coefficient of P_λ^θ is 1. Specifically,

$$P_\lambda^\theta(1_N) = \prod_{1\le i<j\le N} (\lambda_i - \lambda_j + \theta(j-i))_\theta \prod_{k=1}^N \frac{\Gamma(\theta)}{\Gamma(\theta k)}, \tag{2.3}$$

where 1_N denotes the N-dimensional vector whose components are all 1 (see Okounkov and Olshanski, 1997, (6.4)).

From Kaneko (1993, (34), (35), (36)), Jack polynomials satisfy the following relations:

$$E_0^N P_\lambda^\theta(\mathbf{x}_N) = P_\lambda^\theta(1_N) \sum_{i=1}^{l(\lambda)} \binom{\lambda}{\lambda_{(i)}}_\theta \frac{P_{\lambda_{(i)}}^\theta(\mathbf{x}_N)}{P_{\lambda_{(i)}}^\theta(1_N)}, \tag{2.4}$$
$$E_1^N P_\lambda^\theta(\mathbf{x}_N) = |\lambda| P_\lambda^\theta(\mathbf{x}_N), \tag{2.5}$$
$$D_1^N P_\lambda^\theta(\mathbf{x}_N) = P_\lambda^\theta(1_N) \sum_{i=1}^{l(\lambda)} \binom{\lambda}{\lambda_{(i)}}_\theta (\lambda_i - 1 + (N-i)\theta) \frac{P_{\lambda_{(i)}}^\theta(\mathbf{x}_N)}{P_{\lambda_{(i)}}^\theta(1_N)}. \tag{2.6}$$

Here, $\lambda_{(i)}$ denotes the partition given by $\lambda_{(i)} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots)$, and $\binom{\lambda}{\rho}_\theta$ is the generalised binomial coefficient introduced in Lassalle (1990), determined by the expansion

$$\frac{P_\lambda^\theta(1_N + \mathbf{x}_N)}{P_\lambda^\theta(1_N)} = \sum_{m=0}^{|\lambda|} \sum_{|\rho|=m} \binom{\lambda}{\rho}_\theta \frac{P_\rho^\theta(\mathbf{x}_N)}{P_\rho^\theta(1_N)}. \tag{2.7}$$

We remark that $\binom{\lambda}{\rho}_\theta = 0$ unless $\rho \subseteq \lambda$ (Vilenkin and Klimyk, 1995, Section 2.5.2), where we write $\rho \subseteq \lambda$ if $\rho_i \le \lambda_i$ holds for all i.

3. Integral operators for Jack polynomials

We now prove Theorem 1.11 using Lemma 1.10. We fix $\theta > 0$, a partition λ, and $\mathbf{x}_N \in \mathring{W}^N_{\geq}$. For $z \in \mathbb{C}$ satisfying $\Re z > -\theta$, we set the function

$$f(z) := \int_0^{x_1} \cdots \int_{x_{N-1}}^{x_N} \frac{\prod_{1\leq i<j\leq N}(y_j - y_i)}{\prod_{1\leq i<j\leq N}(x_j - x_i)^{2\theta-1}} \prod_{i,j=1}^{N} |x_i - y_j|^{\theta-1} \prod_{i=1}^{N} \frac{y_i^{z+\theta-1}}{x_i^{z+2\theta-1}} P_\lambda^\theta(\mathbf{y}) d\mathbf{y}$$
$$- Z_\theta^N \prod_{i=1}^{N} \frac{((N+1-i)\theta)_{\lambda_i+z}}{((N+2-i)\theta)_{\lambda_i+z}} P_\lambda^\theta(\mathbf{x}_N).$$

By the definition of $c(\lambda, N, \theta; \alpha)$, we observe that

$$\begin{aligned} \frac{1}{Z^N_{\theta,\alpha}} f(\theta\alpha) &= \Lambda^N_{\theta,\alpha,N} P^\theta_\lambda(\mathbf{x}_N) - \frac{Z^N_\theta}{Z^N_{\theta,\alpha}} \prod_{i=1}^{N} \frac{((N+1-i)\theta)_{\lambda_i+\theta\alpha}}{((N+2-i)\theta)_{\lambda_i+\theta\alpha}} P^\theta_\lambda(\mathbf{x}_N) \\ &= \Lambda^N_{\theta,\alpha,N} P^\theta_\lambda(\mathbf{x}_N) - c(\lambda, N, \theta; \alpha) P^\theta_\lambda(\mathbf{x}_N). \end{aligned} \tag{3.1}$$

Hence, to prove Theorem 1.11, it suffices to show that $f \equiv 0$ on $\{z\,;\, \Re z > -\theta\}$. To establish this via Carlson's theorem, we first derive the following estimate for f.

Lemma 3.1. *For fixed $\theta > 0$, a partition λ, and $\mathbf{x}_N \in \mathring{W}^N_{\geq}$, the function f is bounded on the set $\{z\,;\, \Re z \geq 0\}$.*

Proof: For $\mathbf{y} \in W^{N,N}_{\geq}(\mathbf{x}_N)$, observe that $\prod_{i=1}^N |y_i^z/x_i^z| = \prod_{i=1}^N y_i^{\Re z}/x_i^{\Re z} \leq 1$ on $\{z\,;\, \Re z > -\theta\}$. Hence, the first term of f is bounded. Since we have

$$\prod_{i=1}^{N} \frac{((N+1-i)\theta)_{\lambda_i+z}}{((N+2-i)\theta)_{\lambda_i+z}} = \frac{\Gamma((N+1)\theta)}{\Gamma(\theta)} \prod_{i=1}^{N} \frac{\Gamma((N+1-i)\theta + \lambda_i + z)}{\Gamma((N+2-i)\theta + \lambda_i + z)},$$

the second term of f is also bounded from the formula $\Gamma(z+a)/\Gamma(z+b) = z^{a-b}(1 + o(z^{-1}))$ as $z \to \infty$ in $-\pi < \arg z < \pi$ (Erdélyi et al., 1953, equation (4) in Section 1.18, Vol 1). Thus, we obtain the desired result. □

Proof of Theorem 1.11

Let us first show that $f(n) = 0$ for any non-negative integer n. Let $\mathbf{x}_N = (x_1, \ldots, x_N) \in \mathring{W}^N_{\geq}$. Then, applying (1.11) for $(0, \mathbf{x}_N) \in \mathring{W}^{N+1}$, we obtain

$$\begin{aligned} [\Lambda^{N+1}_{\theta,N} P^\theta_{\lambda+n1_N}]((0,\mathbf{x}_N)) &= c(\lambda + n1_N, N, \theta) P^\theta_{\lambda+n1_N}((0,\mathbf{x}_N)) \\ &= \Big(\prod_{i=1}^{N} x_i^n\Big) \prod_{i=1}^{N} \frac{((N+1-i)\theta)_{\lambda_i+n}}{((N+2-i)\theta)_{\lambda_i+n}} P^\theta_\lambda(\mathbf{x}_N). \end{aligned} \tag{3.2}$$

Here, we used the formula $P^\theta_{\lambda+1_N}(\mathbf{x}_N) = \big(\prod_{i=1}^N x_i\big) P^\theta_\lambda(\mathbf{x}_N)$ (Okounkov and Olshanski, 1997, (1.2)). On the other hand, the left-hand side of (3.2) turns out to be

$$\begin{aligned} &\frac{1}{Z^N_\theta} \int_0^{x_1} \cdots \int_{x_{N-1}}^{x_N} \frac{\prod_{1\leq i<j\leq N}(y_j - y_i)}{\prod_{1\leq i<j\leq N}(x_j - x_i)^{2\theta-1}} \prod_{i,j=1}^{N} |x_i - y_j|^{\theta-1} \prod_{i=1}^{N} \frac{y_i^{\theta-1}}{x_i^{2\theta-1}} P^\theta_{\lambda+n1_N}(\mathbf{y}) d\mathbf{y} \\ &= \frac{\prod_{i=1}^N x_i^n}{Z^N_\theta} \int_0^{x_1} \cdots \int_{x_{N-1}}^{x_N} \frac{\prod_{1\leq i<j\leq N}(y_j - y_i)}{\prod_{1\leq i<j\leq N}(x_j - x_i)^{2\theta-1}} \prod_{i,j=1}^{N} |x_i - y_j|^{\theta-1} \prod_{i=1}^{N} \frac{y_i^{n+\theta-1}}{x_i^{n+2\theta-1}} P^\theta_\lambda(\mathbf{y}) d\mathbf{y}. \end{aligned} \tag{3.3}$$

Therefore, equations (3.2) and (3.3) imply that $f(n) = 0$ by the definition of f.

Here we recall Carlson's theorem (see Titchmarsh, 1958, Section 5.81 for example). Assume that g is analytic on $\{z\,;\,\Re z>0\}$ and continuous on $\{z\,;\,\Re z\geq 0\}$, and that it satisfies

$$|g(z)|\leq Ce^{c|z|}\text{ for all }z\text{ with }\Re z\geq 0$$

for some constants $C>0$ and $c<\pi$. Furthermore, if $g(n)=0$ for any non-negative integer n, then $g\equiv 0$ on $\{z\,;\,\Re z\geq 0\}$.

Carlson's theorem with Lemma 3.1 implies that $f(z)\equiv 0$ on $\{z\,;\,\Re z\geq 0\}$. Furthermore, since f is analytic on $\{z\,;\,\Re z>-\theta\}$, by analytic continuation (identity theorem) we have $f\equiv 0$ on $\{z\,;\,\Re z>-\theta\}$. Hence, we obtain $f(\theta\alpha)=0$ for any $\theta>0$ and $\alpha>-1$. Combining this with (3.1), we obtain (1.12) for all $\mathbf{x}_N\in\mathring{W}^N_{\geq}$.

We next prove (1.12) for all $\mathbf{x}_N\in W^N_{\geq}$. For any $\mathbf{x}_N\in W^N_{\geq}$, there exists a sequence $\{\mathbf{x}^n_N\}_{n\in\mathbb{N}}\subset\mathring{W}^N_{\geq}$ such that $\lim_{n\to\infty}\mathbf{x}^n_N=\mathbf{x}_N$. Then, by Definition 1.5 and the fact that the roots of a polynomial are continuous function of its coefficients, we have the weak convergence of probability measures $\lim_{n\to\infty}\Lambda^N_{\theta,\alpha,N}(\mathbf{x}^n_N,\cdot)=\Lambda^N_{\theta,\alpha,N}(\mathbf{x}_N,\cdot)$. Furthermore, since $\Lambda^N_{\theta,\alpha,N}(\mathbf{x}^n_N,\cdot)$ has compact support, it follows that $\lim_{n\to\infty}[\Lambda^N_{\theta,\alpha,N}P^\theta_\lambda](\mathbf{x}^n_N)=[\Lambda^N_{\theta,\alpha,N}P^\theta_\lambda](\mathbf{x}_N)$. Therefore, taking the limit $n\to\infty$ of the equation $[\Lambda^N_{\theta,\alpha,N}P^\theta_\lambda](\mathbf{x}^n_N)=c(\lambda,N,\theta\,;\alpha)P^\theta_\lambda(\mathbf{x}^n_N)$, we obtain (1.12) for $\mathbf{x}_N\in W^N_{\geq}$. Thus, we complete the proof of Theorem 1.11. □

We illustrate Theorem 1.11 with examples. When $N=1$, for a non-negative integer λ, the equation (1.12) is rephrased as

$$\frac{\Gamma((\alpha+2)\theta)}{\Gamma(\theta)\Gamma((\alpha+1)\theta)}\int_0^x(x-y)^{\theta-1}\frac{y^{\theta(\alpha+1)-1}}{x^{\theta(\alpha+2)-1}}\frac{y^\lambda}{((\alpha+1)\theta)_\lambda}dy=\frac{x^\lambda}{((\alpha+2)\theta)_\lambda}\tag{3.4}$$

from $P^\theta_\lambda(x)=x^\lambda$. The equation (3.4) can be shown also by straightforward computation as follows:

$$\int_0^x(x-y)^{\theta-1}\frac{y^{\theta(\alpha+1)-1}}{x^{\theta(\alpha+2)-1}}y^\lambda dy=x^\lambda\int_0^1(1-t)^{\theta-1}t^{\theta(\alpha+1)-1+\lambda}dt=x^\lambda\frac{\Gamma(\theta)\Gamma(\theta(\alpha+1)+\lambda)}{\Gamma(\theta(\alpha+2)+\lambda)}.$$

When $N=2$ and $\lambda=(\lambda_1,\lambda_2)$, the equation (1.12) is written as

$$\begin{aligned}\frac{\Gamma((\alpha+3)\theta)}{\Gamma(\theta)^2\Gamma((\alpha+1)\theta)}\int_0^{x_1}\int_{x_1}^{x_2}\frac{(y_2-y_1)}{(x_2-x_1)^{2\theta-1}}\prod_{i,j=1,2}|x_i-y_j|^{\theta-1}\frac{(y_1y_2)^{\theta(\alpha+1)-1}}{(x_1x_2)^{\theta(\alpha+2)-1}}P^\theta_\lambda(y_1,y_2)dy_1dy_2\\=\prod_{i=1,2}\frac{((\alpha+3-i)\theta)_{\lambda_i}}{((\alpha+4-i)\theta)_{\lambda_i}}P^\theta_\lambda(x_1,x_2).\end{aligned}\tag{3.5}$$

The Jack polynomial in two-variables has the following representation (Koornwinder, 2014/15, (10.15)):

$$\begin{aligned}P^\theta_\lambda(x_1,x_2)&=x_1^{\lambda_1}x_2^{\lambda_2}{}_2F_1\left(\begin{matrix}-\lambda_1+\lambda_2,\,\theta\\1-\lambda_1+\lambda_2-\theta\end{matrix};\frac{x_2}{x_1}\right)\\&=\frac{(\lambda_1-\lambda_2)!}{(\theta)_{\lambda_1-\lambda_2}}(x_1x_2)^{\frac{\lambda_1+\lambda_2}{2}}C^\theta_{\lambda_1-\lambda_2}\left(\frac{x_1+x_2}{2(x_1x_2)^{\frac{1}{2}}}\right),\end{aligned}$$

where ${}_2F_1$ is the hypergeometric function and C^θ_m is the Gegenbauer polynomial. Therefore, the equation (3.5) gives an integral representation of these special functions. To the best of our knowledge, a direct derivation of the equation (3.5) using hypergeometric functions or Gegenbauer polynomials is not known.

The equation (1.12) for a rectangular partition $\lambda = (m, \dots, m)$ with $l(\lambda) = N$ is

$$\frac{\Gamma(\theta(N+\alpha+1))}{\Gamma(\theta)^N \Gamma(\theta(\alpha+1))} \int_0^{x_1} \cdots \int_{x_{N-1}}^{x_N} \frac{\prod_{1\le i<j\le N}(y_j - y_i)}{\prod_{1\le i<j\le N}(x_j - x_i)^{2\theta-1}} \prod_{i,j=1}^N |x_i - y_j|^{\theta-1} \prod_{i=1}^N \frac{y_i^{\theta(\alpha+1)-1+m}}{x_i^{\theta(\alpha+2)-1+m}} d\mathbf{y}$$
$$= \prod_{i=1}^N \frac{((N+\alpha+1-i)\theta)_m}{((N+\alpha+2-i)\theta)_m}, \tag{3.6}$$

which is just a paraphrase of $\int \Lambda^N_{\theta,\alpha+m/\theta,N}(\mathbf{x}, d\mathbf{y}) = 1$.

4. Proof of Theorem 1.8

We observed the actions of differential operators E^N_k, D^N_k on Jack polynomials in Section 2. Combining these with Theorem 1.11, we obtain intertwining relations between these differential operators and $\Lambda^N_{\theta,\alpha,N}$.

Lemma 4.1. *Assume that $\theta > 0$ and $\alpha > -1$. Then, for any partition λ with $l(\lambda) \le N$, the following holds for $\mathbf{x}_N \in W^N_{\ge}$:*

$$[E^N_1 \Lambda^N_{\theta,\alpha,N} P^\theta_\lambda](\mathbf{x}_N) = [\Lambda^N_{\theta,\alpha,N} E^N_1 P^\theta_\lambda](\mathbf{x}_N) \tag{4.1}$$
$$[\big(D^N_1 + \theta(\alpha+2)E^N_0\big)\Lambda^N_{\theta,\alpha,N} P^\theta_\lambda](\mathbf{x}_N) = [\Lambda^N_{\theta,\alpha,N}\big(D^N_1 + \theta(\alpha+1)E^N_0\big)P^\theta_\lambda](\mathbf{x}_N). \tag{4.2}$$

Proof: The equation (4.1) results from (1.12) and (2.5). From (1.12), (2.4), and (2.6), we have the equations

$$[\big(D^N_1 + \theta(\alpha+2)E^N_0\big)\Lambda^N_{\theta,\alpha,N} P^\theta_\lambda](\mathbf{x}_N) \tag{4.3}$$
$$= c(\lambda, N, \theta\,; \alpha) P^\theta_\lambda(1_N)\Bigg[\sum_{i=1}^{l(\lambda)} \binom{\lambda}{\lambda_{(i)}}_\theta \{(N-i+\alpha+2)\theta + \lambda_i - 1\} \frac{P^\theta_{\lambda_{(i)}}(\mathbf{x}_N)}{P^\theta_{\lambda_{(i)}}(1_N)}\Bigg],$$
$$[\Lambda^N_{\theta,\alpha,N}\big(D^N_1 + \theta(\alpha+1)E^N_0\big)P^\theta_\lambda](\mathbf{x}_N) \tag{4.4}$$
$$= P^\theta_\lambda(1_N)\sum_{i=1}^{l(\lambda)} \binom{\lambda}{\lambda_{(i)}}_\theta c(\lambda_{(i)}, N, \theta\,; \alpha)\big\{(N-i+\alpha+1)\theta + \lambda_i - 1\big\} \frac{P^\theta_{\lambda_{(i)}}(\mathbf{x}_N)}{P^\theta_{\lambda_{(i)}}(1_N)}.$$

A direct computation yields, for any i,

$$c(\lambda, N, \theta\,; \alpha)\big\{(N-i+\alpha+2)\theta + \lambda_i - 1\big\} = c(\lambda_{(i)}, N, \theta\,; \alpha)\big\{(N-i+\alpha+1)\theta + \lambda_i - 1\big\},$$

which implies the coefficients of $P_{\lambda_{(i)}}$ in (4.3) and (4.4) are the same. Thus, we have proved (4.2). □

Let $A^N_{\theta,\alpha}$ be the generator associated with the stochastic differential equations (1.2). Specifically,

$$A^N_{\theta,\alpha} = D^N_1 + \theta(\alpha+1)E^N_0. \tag{4.5}$$

Lemma 4.2. *Assume that $\theta > 0$ and $\alpha > -1$. Then, for any partition λ with $l(\lambda) \le N$, the following holds for any $\mathbf{x}_N \in W^N_{\ge}$:*

$$[A^N_{\theta,\alpha+1}\Lambda^N_{\theta,\alpha,N} P^\theta_\lambda](\mathbf{x}_N) = [\Lambda^N_{\theta,\alpha,N} A^N_{\theta,\alpha} P^\theta_\lambda](\mathbf{x}_N). \tag{4.6}$$

Proof: The equation (4.6) results from (4.1) and (4.2) with (4.5). □

We next show that the intertwining relation for generators Lemma 4.2 gives rise to that for semigroups Theorem 1.8. The following argument is almost identical to that of Assiotis (2019); Ramanan and Shkolnikov (2018); however, for the convenience of the reader, a concise proof will be provided.

We write $\mathbf{X}^N$ as the solution to (1.2). The following exponential moment estimate of $\mathbf{X}^N$ follows from Step 2 in Assiotis (2019, pp.1888–1889). Hereafter, let $||\cdot||$ denote the l_1-norm on $\mathbb{R}^N$.

Lemma 4.3. *For some $\varepsilon > 0$, we have $E_{\mathbf{x}_N}[e^{\varepsilon||\mathbf{X}_t^N||}] < \infty$ for any $t \geq 0$.*

For a fixed partition λ, define $L(\{P_\kappa^\theta\}_{\kappa\subseteq\lambda})$ as the finite-dimensional vector space spanned by $\{P_\kappa^\theta\}_{\kappa\subseteq\lambda}$. The kernel $\Lambda_{\theta,\alpha,N}^N$ can be regarded as an operator on $L(\{P_\kappa^\theta\}_{\kappa\subseteq\lambda})$ by Theorem 1.11. Thus, $\Lambda_{\theta,\alpha,N}^N$ acts on the space as a matrix, denoted by $M_1 := [M_1(\kappa,\nu)]_{\kappa,\nu\subseteq\lambda}$. Then, by the definition of M_1, we have

$$\Lambda_{\theta,\alpha,N}^N P_\kappa^\theta(\mathbf{x}_N) = \sum_{\nu\subseteq\lambda} M_1(\kappa,\nu) P_\nu^\theta(\mathbf{x}_N) \tag{4.7}$$

for any $\kappa \subseteq \lambda$ and $\mathbf{x}_N \in W_\geq^N$. Similarly, from Lemma 1.10, the kernel $\Lambda_{\theta,N}^{N+1}$ acts on $L(\{P_\kappa^\theta\}_{\kappa\subseteq\lambda})$ as a matrix, denoted by M_1'.

Lemma 4.4. *Assume that $\theta \geq 1/2$ and $\alpha > -1$. Then, for any partition λ with $l(\lambda) \leq N$, the following holds for any $\mathbf{x}_N \in W_\geq^N$:*

$$[T_{\theta,\alpha+1,t}^N \Lambda_{\theta,\alpha,N}^N P_\lambda^\theta](\mathbf{x}_N) = [\Lambda_{\theta,\alpha,N}^N T_{\theta,\alpha,t}^N P_\lambda^\theta](\mathbf{x}_N). \tag{4.8}$$

Proof: Because of (4.5) with (2.4) and (2.6), the generator $A_{\theta,\alpha}^N$ can be regarded as an operator on $L(\{P_\kappa^\theta\}_{\kappa\subseteq\lambda})$. We define $M_2 := [M_2(\kappa,\nu)]_{\kappa,\nu\subseteq\lambda}$ as the finite-dimensional matrix describing an action of $A_{\theta,\alpha}^N$ on this vector space. That is, the matrix M_2 is determined by

$$A_{\theta,\alpha}^N P_\kappa^\theta = \sum_{\nu\subseteq\lambda} M_2(\kappa,\nu) P_\nu^\theta \tag{4.9}$$

for any $\kappa \subseteq \lambda$. Similarly, let M_3 be the matrix describing an action of $A_{\theta,\alpha+1}^N$ on $L(\{P_\kappa^\theta\}_{\kappa\subseteq\lambda})$. Then, we derive $M_1 M_3 = M_2 M_1$ from (4.6). Therefore, for any $t \geq 0$, we have

$$M_1 e^{tM_3} = e^{tM_2} M_1. \tag{4.10}$$

Recall that the exponential estimates for $\mathbf{X}^N$ are given in Lemma 4.3. Therefore, by Itô's formula for $P_\lambda^\theta(\mathbf{X}_t^N)$, and taking expectations, we have

$$T_{\theta,\alpha,t}^N P_\lambda^\theta(\mathbf{x}_N) = P_\lambda^\theta(\mathbf{x}_N) + \int_0^t T_{\theta,\alpha,s}^N A_{\theta,\alpha}^N P_\lambda^\theta(\mathbf{x}_N) ds \tag{4.11}$$

for any $\mathbf{x}_N \in W_\geq^N$. Fix $\mathbf{x}_N \in W_\geq^N$ and set $f_\kappa(t) := T_{\theta,\alpha,t}^N P_\kappa^\theta(\mathbf{x}_N)$. Because (4.11) holds for any κ with $\kappa \subseteq \lambda$, we obtain

$$f_\kappa(t) = f_\kappa(0) + \sum_{\nu\subseteq\lambda} M_2(\kappa,\nu) \int_0^t f_\nu(s) ds$$

from (4.9), which has a unique solution

$$f_\kappa(t) = \sum_{\nu\subseteq\lambda} e^{tM_2}(\kappa,\nu) f_\nu(0). \tag{4.12}$$

From (4.10) with (4.12) and its analogue for M_3 instead of M_2, we obtain (4.8). □

A probability measure m on $W_\geq^N$ gives rise to a symmetrised probability measure m^{sym} on $[0,\infty)^N$ via

$$m^{\mathrm{sym}}(dz_1,\ldots,dz_N) = \frac{1}{N!} m(dz_{(1)},\ldots,dz_{(N)}),$$

where $z_{(1)} \le z_{(2)} \le \cdots \le z_{(N)}$ are the ordered statistics of $(z_1, z_2, \ldots, z_N)$. Furthermore, for a (not necessarily symmetric) function f on $[0,\infty)^N$, we have

$$\int_{[0,\infty)^N} f(\mathbf{z}) m^{\mathrm{sym}}(d\mathbf{z}) = \int_{[0,\infty)^N} f^{\mathrm{sym}}(\mathbf{z}) m^{\mathrm{sym}}(d\mathbf{z}) = \int_{W^N_{\geq}} f^{\mathrm{sym}}(\mathbf{z}) m(d\mathbf{z}), \tag{4.13}$$

where $f^{\mathrm{sym}}(\mathbf{z})$ is a symmetric function given by $f^{\mathrm{sym}}(\mathbf{z}) = \frac{1}{N!}\sum_{\sigma\in S_N} f(z_{\sigma(1)}, \ldots, z_{\sigma(N)})$.

Proof of Theorem 1.8 Fix $\mathbf{x} \in W^N_{\geq}$. Recall that Jack polynomials form a basis for the space of symmetric polynomials. Therefore, from (4.8), we obtain

$$[T^N_{\theta,\alpha+1,t}\Lambda^N_{\theta,\alpha,N} p](\mathbf{x}_N) = [\Lambda^N_{\theta,\alpha,N} T^N_{\theta,\alpha,t} p](\mathbf{x}_N) \tag{4.14}$$

for any symmetric polynomial p in N-variables. Therefore, from (4.13) and (4.14), all moments of the symmetrised measures of $[T^N_{\theta,\alpha+1,t}\Lambda^N_{\theta,\alpha,N}](\mathbf{x}_N, \cdot)$ and $[\Lambda^N_{\theta,\alpha,N} T^N_{\theta,\alpha,t}](\mathbf{x}_N, \cdot)$ are the same.

We observe that $[\Lambda^N_{\theta,\alpha,N} e^{\varepsilon\|\cdot\|}](\mathbf{z}) \le e^{\varepsilon\|\mathbf{z}\|}[\Lambda^N_{\theta,\alpha,N} 1](\mathbf{z}) = e^{\varepsilon\|\mathbf{z}\|}$. From this with the exponential moment estimate Lemma 4.3 and de Jeu (2003, Theorem 1.3), it follows that the symmetrised versions of $[T^N_{\theta,\alpha+1,t}\Lambda^N_{\theta,\alpha,N}](\mathbf{x}_N, \cdot)$ and $[\Lambda^N_{\theta,\alpha,N} T^N_{\theta,\alpha,t} p](\mathbf{x}_N, \cdot)$ are the same probability measure on $[0,\infty)^N$. Thus, these measures coincide on $W^N_{\geq}$, and this completes the proof. □

5. Interpretation of Markov kernels for classical θ in terms of the radial parts of random matrices

In this section, we focus on the classical parameter $\theta = 1/2, 1, 2$. Furthermore, we assume that α is a non-negative integer. For these specific values of θ and α, the kernel $\Lambda^{N+1}_{\theta,\alpha,N}$ has an interpretation in the context of random matrix theory, as described in Theorem 5.3. This interpretation was originally established for $\theta = 1$ in Bufetov and Kawamoto (2025b), and the proof for $\theta = 1/2, 2$ follows by a nearly identical argument. For reader's convenience, we will briefly outline the proof.

5.1. *Conditional eigenvalues of invariant random matrices.* Let $\mathbb{F}$ denote $\mathbb{R}$, $\mathbb{C}$, or the skew field of quaternions $\mathbb{H}$, which corresponds to $\theta = 1/2, 1, 2$, respectively. Let $M_{m,n}(\mathbb{F})$ be the space of $m \times n$ matrices over $\mathbb{F}$, and for brevity write $M_n(\mathbb{F}) = M_{n,n}(\mathbb{F})$. Define the subset $H_n(\mathbb{F}) \subset M_n(\mathbb{F})$ as the space of real symmetric matrices for $\mathbb{F} = \mathbb{R}$, complex Hermitian matrices for $\mathbb{F} = \mathbb{C}$, or quaternion self-dual matrices for $\mathbb{F} = \mathbb{H}$. Furthermore, describe $\mathbb{U}_n(\mathbb{F})$ as the group of orthogonal/unitary/symplectic matrices for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively.

For $m_1 \ge m_2, n_1 \ge n_2$, let $\pi^{m_1,n_1}_{m_2,n_2} : M_{m_1,n_1}(\mathbb{F}) \to M_{m_2,n_2}(\mathbb{F})$ be the natural projection sending an $m_1 \times n_1$ matrix to its upper left $m_2 \times n_2$ corner. We employ the expression $\pi^{m_1}_{m_2,n_2}$ in place of $\pi^{m_1,n_1}_{m_2,n_2}$ if $m_1 = n_1$, and use a similar symbol for $m_2 = n_2$.

We define a map $\mathfrak{eval}_n : H_n(\mathbb{F}) \to W^n$ as

$$\mathfrak{eval}_n(X) = (\lambda_1(X), \ldots, \lambda_n(X)),$$

where $(\lambda_i(X))_{i=1}^n$ are the eigenvalues of X arranged in non-decreasing order. Furthermore, define the radial part $\mathfrak{rad}_n : M_{m,n}(\mathbb{F}) \to W^n_{\geq}$ as $\mathfrak{rad}_n(X) = \mathfrak{eval}_n(X^*X)$. Let a probability measure $P^n_{\mathfrak{eval}}[X]$ on W^n be the distribution of the eigenvalues of a random matrix $X \in H_n(\mathbb{F})$. Similarly, let a probability measure $P^n_{\mathfrak{rad}}[X]$ on $W^n_{\geq}$ denote the distribution of the radial part of a random matrix $X \in M_{m,n}(\mathbb{F})$.

Let $U_{N+1} \in \mathbb{U}_{N+1}(\mathbb{F})$ be a Haar distributed random matrix. For $\mathbf{x} \in W^{N+1}$, let $\mathrm{diag}(x_1, \ldots, x_{N+1})$ denote the square matrix of order $N+1$ with deterministic diagonal elements given by $\mathbf{x}$. Then, the equality

$$\Lambda^{N+1}_{\theta,N}(\mathbf{x}, \cdot) = P^N_{\mathfrak{eval}}[\pi^{N+1}_N(U^*_{N+1}\mathrm{diag}(x_1, \ldots, x_{N+1})U_{N+1})] \tag{5.1}$$

holds for any $\mathbf{x} \in W^{N+1}$. That is, $\Lambda^{N+1}_{\theta,N}(\mathbf{x},\cdot)$ is the pushforward of the distribution of the unitary invariant random matrix $U^*_{N+1}\mathrm{diag}(x_1,\dots,x_{N+1})U_{N+1}$ by $\mathfrak{eval} \circ \pi^{N+1}_N$. The equation (5.1) was proved for $\theta = 1$ in Baryshnikov (2001, Proposition 4.2), and extended to $\theta = 1/2, 2$ in Neretin (2003) (see also Assiotis and Najnudel, 2021, Proposition 1.7). As an immediate consequence of (5.1), we obtain the following lemma:

Lemma 5.1. *If a random matrix $X_{N+1} \in H_{N+1}(\mathbb{F})$ is $\mathbb{U}_{N+1}(\mathbb{F})$-invariant by conjugation in the sense that $U^*_{N+1}X_{N+1}U_{N+1} \overset{\text{law}}{=} X_{N+1}$ for any $U_{N+1} \in \mathbb{U}_{N+1}(\mathbb{F})$, then we have the equality of probability measures*

$$P^{N+1}_{\mathfrak{eval}}[X_{N+1}]\Lambda^{N+1}_{\theta,N} = P^N_{\mathfrak{eval}}[\pi^{N+1}_N(X_{N+1})]. \tag{5.2}$$

5.2. *Interpretation of $\Lambda^{N+1}_{\theta,\alpha,N}$.* In this subsection, α is supposed to be a non-negative integer. We begin by explaining the following interpretation of $\Lambda^N_{\theta,\alpha,N}$ in the context of random matrix theory. This interpretation is established for $\theta = 1$ in Kieburg et al. (2016), and extended to $\theta = 1/2, 2$ in Ahn and Strahov (2022).

Lemma 5.2. *Assume that α is a non-negative integer. Let $V_{N+\alpha+1} \in \mathbb{U}_{N+\alpha+1}(\mathbb{F})$ be a Haar distributed random matrix. Then, for any $\mathbf{z} = (z_1,\dots,z_N) \in W^N_{\geq}$, we have*

$$\Lambda^N_{\theta,\alpha,N}(\mathbf{z},\cdot) = P^N_{\mathfrak{rad}}[\pi^{N+\alpha+1}_{N+\alpha,N}(V_{N+\alpha+1})\mathrm{diag}(\sqrt{z_1},\dots,\sqrt{z_N})]. \tag{5.3}$$

Proof: It is sufficient to show (5.3) for $\mathbf{z} \in \mathring{W}^N_{\geq}$. Applying Ahn and Strahov (2022, Theorem 2.7) with the setting $(m,l,n,\nu) = (N+\alpha+1, N, N, \alpha)$ yields that, for $\mathbf{z} \in \mathring{W}^N_{\geq}$ satisfying $0 < z_1 < \cdots < z_N < 1$, the probability density of

$$P^N_{\mathfrak{rad}}[\pi^{N+\alpha+1}_{N+\alpha,N}(V_{N+\alpha+1})\mathrm{diag}(\sqrt{z_1},\dots,\sqrt{z_N})]$$

is given by (1.5). This result extends naturally to $\mathbf{z} \in \mathring{W}^N_{\geq}$, since we have the scale invariance $\Lambda^N_{\theta,\alpha,N}(\gamma\mathbf{x},\gamma d\mathbf{y}) = \Lambda^N_{\theta,\alpha,N}(\mathbf{x},d\mathbf{y})$ for $\gamma > 0$ from Definition 1.5 or Proposition 1.6. Thus, we obtain (5.3) for $\mathbf{z} \in \mathring{W}^N_{\geq}$, which completes the proof. □

We say a random matrix $X_{m,n} \in M_{m,n}(\mathbb{F})$ is $\mathbb{U}_m(\mathbb{F}) \times \mathbb{U}_n(\mathbb{F})$-invariant if $X_{m,n} \overset{\text{law}}{=} V_m X_{m,n} U_n$ for any fixed matrices $V_m \in \mathbb{U}_m(\mathbb{F}), U_n \in \mathbb{U}_n(\mathbb{F})$. The following theorem was proved for $\theta = 1$ in Bufetov and Kawamoto (2025b).

Theorem 5.3. *Let α be a non-negative integer. For an $\mathbb{U}_{N+\alpha+1}(\mathbb{F}) \times \mathbb{U}_{N+1}(\mathbb{F})$-invariant random matrix $X_{N+\alpha+1,N+1} \in M_{N+\alpha+1,N+1}(\mathbb{F})$, let $X_{N+\alpha,N} := \pi^{N+\alpha+1,N+1}_{N+\alpha,N}(X_{N+\alpha+1,N+1})$ be its truncation. Then, we have the equality of probability measures*

$$P^{N+1}_{\mathfrak{rad}}[X_{N+\alpha+1,N+1}]\Lambda^{N+1}_{\theta,\alpha,N} = P^N_{\mathfrak{rad}}[X_{N+\alpha,N}].$$

Proof: Setting $X_{N+\alpha+1,N} := \pi^{N+\alpha+1,N+1}_{N+\alpha+1,N}(X_{N+\alpha+1,N+1})$, we have

$$\pi^{N+1}_N(X^*_{N+\alpha+1,N+1}X_{N+\alpha+1,N+1}) = X^*_{N+\alpha+1,N}X_{N+\alpha+1,N}.$$

Furthermore, because $X^*_{N+\alpha+1,N+1}X_{N+\alpha+1,N+1} \in H_{N+1}(\mathbb{F})$ is $\mathbb{U}_{N+1}(\mathbb{F})$-invariant by conjugation, the equation (5.2) yields

$$P^{N+1}_{\mathfrak{rad}}[X_{N+\alpha+1,N+1}]\Lambda^{N+1}_{\theta,N} = P^N_{\mathfrak{rad}}[X_{N+\alpha+1,N}]. \tag{5.4}$$

For a random variable $(z_1,\dots,z_N)$ distributed as $P^N_{\mathfrak{rad}}[X_{N+\alpha+1,N}]$, we set

$$D_{N+\alpha+1,N} := \begin{bmatrix} D_N \\ \mathbf{0}_{(\alpha+1)\times N} \end{bmatrix}, \quad D_N := \mathrm{diag}(\sqrt{z_1},\dots,\sqrt{z_N}).$$

Let $U_N \in \mathbb{U}_N(\mathbb{F})$ and $V_{N+\alpha+1} \in \mathbb{U}_{N+\alpha+1}(\mathbb{F})$ be Haar distributed random matrices such that $D_{N+\alpha+1,N}, U_N$, and $V_{N+\alpha+1}$ are independent. Then, for a similar reason as in Defosseux (2010, Lemma 2.4), we obtain

$$X_{N+\alpha+1,N} \overset{\text{law}}{=} V_{N+\alpha+1} D_{N+\alpha+1,N} U_N. \tag{5.5}$$

A direct computation with (5.5) yields

$$X_{N+\alpha,N} = \pi^{N+\alpha+1,N}_{N+\alpha,N}(X_{N+\alpha+1,N}) \overset{\text{law}}{=} \pi^{N+\alpha+1}_{N+\alpha,N}(V_{N+\alpha+1}) D_N U_N.$$

Therefore, using this with the equation (5.3), we have

$$P^N_{\mathfrak{rad}}[X_{N+\alpha,N}] = P^N_{\mathfrak{rad}}[\pi^{N+\alpha+1}_{N+\alpha,N}(V_{N+\alpha+1}) D_N U_N] = P^N_{\mathfrak{rad}}[X_{N+\alpha+1,N}]\Lambda^N_{\theta,\alpha,N}. \tag{5.6}$$

Collecting (5.4) and (5.6) with (1.6), we complete the proof of Theorem 5.3. □

Corollary 5.4. *Let $V_{N+\alpha+1} \in \mathbb{U}_{N+\alpha+1}(\mathbb{F})$ and $U_{N+1} \in \mathbb{U}_{N+1}(\mathbb{F})$ be Haar distributed independent random matrices and $D_{N+\alpha+1,N+1} \in M_{N+\alpha+1,N+1}(\mathbb{F})$ be a deterministic matrix given by*

$$D_{N+\alpha+1,N+1} = \begin{bmatrix} \operatorname{diag}(\sqrt{x_1}, \dots, \sqrt{x_{N+1}}) \\ \mathbf{0}_{\alpha\times(N+1)} \end{bmatrix} \quad \text{for } \mathbf{x} = (x_1, \dots, x_{N+1}) \in W^{N+1}_{\geq}.$$

Then, the probability measure $\Lambda^{N+1}_{\theta,\alpha,N}(\mathbf{x}, \cdot)$ is the same as

$$P^N_{\mathfrak{rad}}\big[\pi^{N+\alpha+1,N+1}_{N+\alpha,N}(V_{N+\alpha+1} D_{N+\alpha+1,N+1} U_{N+1})\big].$$

Proof: Because $V_{N+\alpha+1} D_{N+\alpha+1,N+1} U_{N+1}$ is an $\mathbb{U}_{N+\alpha+1}(\mathbb{F}) \times \mathbb{U}_{N+1}(\mathbb{F})$-invariant random matrix and $P^{N+1}_{\mathfrak{rad}}[V_{N+\alpha+1} D_{N+\alpha+1,N+1} U_{N+1}] = \delta_{\mathbf{x}}$ holds, the statement of this corollary follows from Theorem 5.3. □

Here we show an example obtained as a straightforward consequence of Theorem 5.3; see Forrester (2010, Chapter 3) for Laguerre ensembles below. Let $X \in M_{N+\alpha,N}(\mathbb{F})$ be a random matrix whose probability density is proportional to

$$\exp\big(-\operatorname{Tr} X^* X\big) dX.$$

Then the distribution of the induced Hermitian positive definite matrix $Y := X^*X \in H_N(\mathbb{F})$ is proportional to

$$(\det Y)^{\theta(\alpha+1)-1} \exp(-\operatorname{Tr}(Y))\, dY, \qquad Y \in H_N(\mathbb{F}).$$

The distribution of the squared singular values of X, or eigenvalues of Y, is given by the Laguerre ensemble

$$m^N_{\theta,\alpha}(d\mathbf{x}) := \frac{1}{\mathcal{Z}^N_{\theta,\alpha}} \prod_{1\leq i<j\leq N} |x_j - x_i|^{2\theta} \prod_{i=1}^N x_i^{\theta(\alpha+1)-1} e^{-x_i} d\mathbf{x}, \tag{5.7}$$

where $\mathcal{Z}^N_{\theta,\alpha}$ is the normalising constant. Because X is $\mathbb{U}_{N+\alpha}(\mathbb{F}) \times \mathbb{U}_N(\mathbb{F})$-invariant, we can use (5.6) and Theorem 5.3, and the resulting equations are

$$m^N_{\theta,\alpha+1}\Lambda^N_{\theta,\alpha,N} = m^N_{\theta,\alpha} \tag{5.8}$$
$$m^{N+1}_{\theta,\alpha}\Lambda^{N+1}_{\theta,\alpha,N} = m^N_{\theta,\alpha}, \tag{5.9}$$

for $\theta = 1/2, 1, 2$ and $\alpha \in \{0\} \cup \mathbb{N}$.

Remark 5.5. It is well known that the Laguerre process can be realised as the squared singular value process of a rectangular matrix whose entries are independent $\mathbb{F}$-valued Brownian motions (Bru, 1991; König and O'Connell, 2001). Therefore, the intertwining relations (1.8) and (1.7) can be regarded as dynamical counterparts of (5.8) and (5.9), respectively.

Appendix A. Multivariate Laguerre polynomials and the integral operator $\Lambda^N_{\theta,\alpha,N}$

In this appendix, we investigate the action of the kernel $\Lambda^N_{\theta,\alpha,N}$ on symmetric functions other than Jack polynomials. For any partition λ with $l(\lambda) \leq N$, we define multivariate Laguerre polynomial (or generalised Laguerre polynomial) $L^a_\lambda(\mathbf{x}_N;\theta^{-1})$ by

$$L^a_\lambda(\mathbf{x}_N;\theta^{-1}) := \frac{1}{|\lambda|!}\sum_{\mu\subseteq\lambda}(-1)^{|\mu|}\binom{\lambda}{\mu}_\theta\prod_{i=1}^N\frac{(a+1+\theta(N-i))_{\lambda_i}}{(a+1+\theta(N-i))_{\mu_i}}\frac{P^\theta_\mu(\mathbf{x}_N)}{P^\theta_\mu(1_N)}. \tag{A.1}$$

Our notation $L^a_\lambda(\mathbf{x}_N;\theta^{-1})$ follows the Baker-Forrester style (Baker and Forrester, 1997). Thus, the parameter θ^{-1} is not a typo. We also remark that the notation of the Jack polynomial in Baker and Forrester (1997) is $C^\alpha_\lambda(\mathbf{x}_N)$ (this α is not our α) and the correspondence between our notation and theirs is as follows:

$$\frac{C^{\theta^{-1}}_\lambda(\mathbf{x}_N)}{C^{\theta^{-1}}_\lambda(1_N)} = \frac{P^\theta_\lambda(\mathbf{x}_N)}{P^\theta_\lambda(1_N)}.$$

Multivariate Laguerre polynomials associated with Jack polynomials were introduced as the polynomial eigenfunctions of the following Calogero-Sutherland type operator (Lassalle, 1991):

$$\tilde{H}^{(L)} := D^N_1 + (a+1)E^N_0 - E^N_1.$$

In fact, we have

$$\tilde{H}^{(L)}L^a_\lambda(\mathbf{x}_N;\theta^{-1}) = -|\lambda|L^a_\lambda(\mathbf{x}_N;\theta^{-1}). \tag{A.2}$$

We remark that $\tilde{H}^{(L)} + E^N_1$ appears in the intertwining relation (4.2), where $a := \theta(\alpha+1)-1$. As a generalisation of the equation (4.2), we have the following lemma.

Lemma A.1. *Suppose $\theta > 0$ and $\alpha > -1$. Let λ be a partition with $l(\lambda) \leq N$. Then, for any $k \in \mathbb{N}$ and $\mathbf{x}_N \in W^N_\geq$, we have*

$$[\big(D^N_1 + \theta(\alpha+2)E^N_0\big)^k\Lambda^N_{\theta,\alpha,N}P^\theta_\lambda](\mathbf{x}_N) = [\Lambda^N_{\theta,\alpha,N}\big(D^N_1 + \theta(\alpha+1)E^N_0\big)^kP^\theta_\lambda](\mathbf{x}_N). \tag{A.3}$$

Proof: The proof is by induction on k. The equation (A.3) for $k=1$ is exactly (4.2). Write $C_\alpha := D^N_1 + \theta(\alpha+1)E^N_0$. If we assume that $(C^{k-1}_{\alpha+1}\Lambda^N_{\theta,\alpha,N} - \Lambda^N_{\theta,\alpha,N}C^{k-1}_\alpha)P^\theta_\lambda = 0$ for some k, then we have

$$\begin{aligned}&(C^k_{\alpha+1}\Lambda^N_{\theta,\alpha,N} - \Lambda^N_{\theta,\alpha,N}C^k_\alpha)P^\theta_\lambda\\ &= C_{\alpha+1}(C^{k-1}_{\alpha+1}\Lambda^N_{\theta,\alpha,N} - \Lambda^N_{\theta,\alpha,N}C^{k-1}_\alpha)P^\theta_\lambda + (C_{\alpha+1}\Lambda^N_{\theta,\alpha,N} - \Lambda^N_{\theta,\alpha,N}C_\alpha)C^{k-1}_\alpha P^\theta_\lambda\\ &= 0.\end{aligned}$$

Here, we use the fact that $C^{k-1}_\alpha P^\theta_\lambda$ is given by a linear combination of P^θ_μ with $l(\mu) \leq N$ from (2.4) and (2.6). Thus, we establish the desired result. □

The multivariate Laguerre polynomial $L^a_\lambda(\mathbf{x}_N;\theta^{-1})$ has the following Rodrigues-type formula (Baker and Forrester, 1997, (4.39)) (Baker-Forrester said that "prompted by M. Lassalle"):

$$\frac{(-1)^{|\lambda|}}{|\lambda|!}\exp\big(-D^N_1 - (a+1)E^N_0\big)\frac{P^\theta_\lambda(\mathbf{x}_N)}{P^\theta_\lambda(1_N)} = L^a_\lambda(\mathbf{x}_N;\theta^{-1}). \tag{A.4}$$

From this Rodrigues-type formula, Theorem 1.11, and Lemma A.1, we calculate the action of $\Lambda^N_{\theta,\alpha,N}$ on $L^a_\lambda(\mathbf{x}_N;\theta^{-1})$.

Theorem A.2. *Suppose $\theta > 0$ and $\alpha > -1$. Then, for any λ with $l(\lambda) \leq N$ and $\mathbf{x}_N \in W^N_\geq$, we have*

$$[\Lambda^N_{\theta,\alpha,N}L^a_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) = L^{a+\theta}_\lambda(\mathbf{x}_N;\theta^{-1})c(\lambda,N,\theta\,;\alpha), \tag{A.5}$$

where $a := \theta(\alpha+1)-1$.

Proof: Since the degree of $P^\theta_\lambda(\mathbf{x}_N)$ is $|\lambda|$ and

$$\big(-D^N_1-(a+1)E^N_0\big)^k P^\theta_\lambda(\mathbf{x}_N)=0 \quad \text{for } k>|\lambda|,$$

we have

$$\begin{aligned} L^a_\lambda(\mathbf{x}_N;\theta^{-1}) &= \frac{(-1)^{|\lambda|}}{|\lambda|!}\exp\big(-D^N_1-(a+1)E^N_0\big)\frac{P^\theta_\lambda(\mathbf{x}_N)}{P^\theta_\lambda(1_N)} \\ &= \frac{(-1)^{|\lambda|}}{|\lambda|!}\sum_{k=0}^{|\lambda|}\frac{\big(-D^N_1-\theta(\alpha+1)E^N_0\big)^k}{k!}\frac{P^\theta_\lambda(\mathbf{x}_N)}{P^\theta_\lambda(1_N)} \end{aligned}$$

from (A.4). By applying Lemma A.1 and Theorem 1.11, we obtain

$$\begin{aligned} [\Lambda^N_{\theta,\alpha,N}L^a_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) &= \frac{(-1)^{|\lambda|}}{|\lambda|!}\sum_{k=0}^{|\lambda|}\frac{\Lambda^N_{\theta,\alpha,N}\big(-D^N_1-\theta(\alpha+1)E^N_0\big)^k}{k!}\frac{P^\theta_\lambda(\mathbf{x}_N)}{P^\theta_\lambda(1_N)} \\ &= \frac{(-1)^{|\lambda|}}{|\lambda|!}\sum_{k=0}^{|\lambda|}\frac{\big(-D^N_1-\theta(\alpha+2)E^N_0\big)^k\Lambda^N_{\theta,\alpha,N}}{k!}\frac{P^\theta_\lambda(\mathbf{x}_N)}{P^\theta_\lambda(1_N)} \\ &= \frac{(-1)^{|\lambda|}}{|\lambda|!}\sum_{k=0}^{|\lambda|}\frac{\big(-D^N_1-(a+\theta+1)E^N_0\big)^k}{k!}\frac{P^\theta_\lambda(\mathbf{x}_N)}{P^\theta_\lambda(1_N)}c(\lambda,N,\theta\,;\alpha) \\ &= L^{a+\theta}_\lambda(\mathbf{x}_N;\theta^{-1})c(\lambda,N,\theta\,;\alpha). \end{aligned}$$

□

Remark that, if $N=1$, the equation (A.5) is given Erdélyi et al. (1953, Equation (30) in Section 10.12, Vol.2). In the above proof, we use the intertwining relation (A.3). On the other hand, we give another proof of (A.5) without Lemma A.1. In fact, Theorem 1.11 is equivalent to the following formula:

$$[\Lambda^N_{\theta,\alpha,N}P^\theta_\lambda](\mathbf{x}_N)\prod_{i=1}^N\frac{1}{((N+\alpha+1-i)\theta)_{\lambda_i}} = P^\theta_\lambda(\mathbf{x}_N)\prod_{i=1}^N\frac{1}{((N+\alpha+2-i)\theta)_{\lambda_i}}, \tag{A.6}$$

which is a multivariate analogue of (3.4). Then, from the definition (A.1) and the formula (A.6), we have the conclusion (A.5):

$$\begin{aligned} &[\Lambda^N_{\theta,\alpha,N}L^a_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) \\ &= \frac{1}{|\lambda|!}\sum_{\mu\subseteq\lambda}(-1)^{|\mu|}\binom{\lambda}{\mu}_\theta\prod_{i=1}^N\frac{((N+\alpha+1-i)\theta)_{\lambda_i}}{((N+\alpha+1-i)\theta)_{\mu_i}}\frac{[\Lambda^N_{\theta,\alpha,N}P^\theta_\mu](\mathbf{x}_N)}{P^\theta_\mu(1_N)} \\ &= \frac{1}{|\lambda|!}\sum_{\mu\subseteq\lambda}(-1)^{|\mu|}\binom{\lambda}{\mu}_\theta\prod_{i=1}^N\frac{((N+\alpha+1-i)\theta)_{\lambda_i}}{((N+\alpha+2-i)\theta)_{\mu_i}}\frac{P^\theta_\mu(\mathbf{x}_N)}{P^\theta_\mu(1_N)} \\ &= \frac{1}{|\lambda|!}\sum_{\mu\subseteq\lambda}(-1)^{|\mu|}\binom{\lambda}{\mu}_\theta\prod_{i=1}^N\frac{(a+\theta+1+\theta(N-i))_{\lambda_i}}{(a+\theta+1+\theta(N-i))_{\mu_i}}\frac{P^\theta_\mu(\mathbf{x}_N)}{P^\theta_\mu(1_N)}\prod_{i=1}^N\frac{((N+\alpha+1-i)\theta)_{\lambda_i}}{((N+\alpha+2-i)\theta)_{\lambda_i}} \\ &= L^{a+\theta}_\lambda(\mathbf{x}_N;\theta^{-1})c(\lambda,N,\theta\,;\alpha). \end{aligned}$$

Hence, Theorem A.2 is equivalent to Lemma A.1 under the assumptions of Theorem 1.11 and the Rodrigues-type formula (A.4).

From the formula (A.6), we also have a parameter shift formula of multivariate hypergeometric functions associated with Jack polynomials (see Baker and Forrester, 1997; Kaneko, 1993 and Vilenkin and Klimyk, 1995, Chapter 3):

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; \mathbf{x}_N\right) := \sum_{n=0}^{\infty} \sum_{|\lambda|=n} \frac{(a_1)_\lambda \cdots (a_p)_\lambda}{(b_1)_\lambda \cdots (b_q)_\lambda} \frac{C_\lambda^{\theta^{-1}}(\mathbf{x}_N)}{n!}, \tag{A.7} $$

where $(a)_\lambda = \prod_{i=1}^{l(\lambda)} (a - \theta(i-1))_{\lambda_i}$.

Proposition A.3. *We assume that (A.7) converges. For any $1 \leq j \leq q$, under the condition $b_j = (\beta_j + N)\theta$, we have*

$$ \left[\Lambda_{\theta,\beta_j,N}^N \left\{ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots b_q \end{matrix}; \cdot\right)\right\}\right](\mathbf{x}_N) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{j-1}, b_j + \theta, b_{j+1}, \dots, b_q \end{matrix}; \mathbf{x}_N\right). \tag{A.8} $$

Appendix B. Intertwining of β-Laguerre Ornstein–Uhlenbeck processes

Recall that $\theta = \beta/2$, and let $\theta \geq 1/2$ and $\alpha > -1$ as before. We consider the N-dimensional stochastic differential equation

$$ dX_t^{N,i} = \sqrt{2X_t^{N,i}} dB_t^i - X_t^i dt + \theta\Big(\alpha + 1 + \sum_{1 \leq j \leq N, j \neq i} \frac{2X_t^{N,i}}{X_t^{N,i} - X_t^{N,j}}\Big) dt, \qquad 1 \leq i \leq N, \tag{B.1} $$

where $(B^i)_{i=1}^N$ is the N-dimensional Brownian motion. By slightly modifying the proof of Graczyk and Małecki (2014, Corollary 6.5), we see that the equation (B.1) has a unique strong solution with no collisions and no explosions. We call the unique solution to (B.1) the N-dimensional β-Laguerre Ornstein–Uhlenbeck process for parameter α, and define $\mathfrak{T}_{\theta,\alpha,t}^N$ as the associated Markov semigroup. The infinitesimal generator associated with (B.1) is given by

$$ \mathfrak{A}_{\theta,\alpha}^N := D_1^N + \theta(\alpha+1)E_0^N - E_1^N. $$

The β-Laguerre Ornstein–Uhlenbeck processes satisfy shifted intertwining relations in analogy to Proposition 1.4.

Proposition B.1. *Assume that $\theta \geq 1/2$ and $\alpha > -1$. Then, for any $N \in \mathbb{N}$ and $t \geq 0$, we have*

$$ \mathfrak{T}_{\theta,\alpha,t}^{N+1} \Lambda_{\theta,N}^{N+1} = \Lambda_{\theta,N}^{N+1} \mathfrak{T}_{\theta,\alpha+1,t}^N. \tag{B.2} $$

Proof: This was essentially proved in Assiotis (2019). Actually, $E_1^{N+1} \Lambda_{\theta,N}^{N+1} P_\lambda^\theta = \Lambda_{\theta,N}^{N+1} E_1^N P_\lambda^\theta$ holds from (1.11) and (2.5). Combining this with Assiotis (2019, (21)), we have $\mathfrak{A}_{\theta,\alpha}^{N+1} \Lambda_{\theta,N}^{N+1} P_\lambda^\theta = \Lambda_{\theta,N}^{N+1} \mathfrak{A}_{\theta,\alpha+1}^N P_\lambda^\theta$. Furthermore, an exponential estimate for the β-Laguerre Ornstein–Uhlenbeck processes can be derived from Lemma 4.3. Actually, by a comparison theorem for stochastic differential equations (see Revuz and Yor, 1999, Theorem 3.7 for example), the l_1-norm of the Laguerre Ornstein–Uhlenbeck process is not greater than that of the Laguerre process. Therefore, the argument in Assiotis (2019), with the above modifications, shows that (B.2) holds. □

The following theorem can be proved by the same method in Section 4 via Jack polynomials. On the other hand, by using multivariate Laguerre polynomials instead of Jack polynomials, the proof becomes somewhat simpler, so we present the computation explicitly.

Theorem B.2. *Suppose $\theta \geq 1/2$ and $\alpha > -1$. Then, for any $N \in \mathbb{N}$ and $t \geq 0$, we have*

$$ \mathfrak{T}_{\theta,\alpha+1,t}^N \Lambda_{\theta,\alpha,N}^N = \Lambda_{\theta,\alpha,N}^N \mathfrak{T}_{\theta,\alpha,t}^N. \tag{B.3} $$

Furthermore, we have

$$ \mathfrak{T}_{\theta,\alpha,t}^{N+1} \Lambda_{\theta,\alpha,N}^{N+1} = \Lambda_{\theta,\alpha,N}^{N+1} \mathfrak{T}_{\theta,\alpha,t}^N. \tag{B.4} $$

Proof: The equation (B.4) results from (B.2) and (B.3) immediately. To establish (B.3), by the same argument as in Lemma 4.4 with (A.2), we get

$$\begin{aligned}[\mathfrak{T}^N_{\theta,\alpha,t} L^{\theta(\alpha+1)-1}_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) &= L^{\theta(\alpha+1)-1}_\lambda(\mathbf{x}_N;\theta^{-1}) + \int_0^t [\mathfrak{T}^N_{\theta,\alpha,s} \mathfrak{A}^N_{\theta,\alpha} L^{\theta(\alpha+1)-1}_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) ds \\ &= L^{\theta(\alpha+1)-1}_\lambda(\mathbf{x}_N;\theta^{-1}) - |\lambda| \int_0^t [\mathfrak{T}^N_{\theta,\alpha,s} L^{\theta(\alpha+1)-1}_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) ds,\end{aligned} \tag{B.5}$$

which yields

$$[\mathfrak{T}^N_{\theta,\alpha,t} L^{\theta(\alpha+1)-1}_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) = e^{-|\lambda|t} L^{\theta(\alpha+1)-1}_\lambda(\mathbf{x}_N;\theta^{-1}). \tag{B.6}$$

Combining the equation (B.6) with (A.5), we obtain

$$\begin{aligned}[\Lambda^N_{\theta,\alpha,N} \mathfrak{T}^N_{\theta,\alpha,t} L^{\theta(\alpha+1)-1}_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) &= e^{-|\lambda|t} L^{\theta(\alpha+2)-1}_\lambda(\mathbf{x}_N;\theta^{-1}) c(\lambda,N,\theta;\alpha) \\ &= [\mathfrak{T}^N_{\theta,\alpha+1,t} L^{\theta(\alpha+2)-1}_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N) c(\lambda,N,\theta;\alpha) \\ &= [\mathfrak{T}^N_{\theta,\alpha+1,t} \Lambda^N_{\theta,\alpha,N} L^{\theta(\alpha+1)-1}_\lambda(\cdot;\theta^{-1})](\mathbf{x}_N).\end{aligned} \tag{B.7}$$

Because any symmetric polynomial is a linear combination of multivariate Laguerre polynomials, the equation (B.7) enables us to apply the same argument in the proof of Theorem 1.8, and thus we conclude (B.3). □

Remark B.3. When $\theta = 1$, the equations (B.3) and (B.4) were proved in Bufetov and Kawamoto (2025b) by a different approach.

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