ALEA, Lat. Am. J. Probab. Math. Stat. 23, 275–299 (2026)
DOI: 10.30757/ALEA.v23-11

On the Euler-Poincaré Characteristic of the planar Berry's random wave: fluctuations and a perturbation study

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Abstract. We prove a Central Limit Theorem for the Euler-Poincaré characteristic of Berry's random wave model in a growing domain. We also show Gaussian fluctuations for a class of Berry's mixture models that correspond to a perturbation of the initial random field. Finally, some explicit calculations of the variance of the perturbed Berry's model and numerical investigations are provided to illustrate our theoretical results.

1. Introduction and main results

1.1. *Lipschitz-Killing Curvatures and Berry's random wave model.* The study of random fields in $\mathbb{R}^2$ through the geometry of their excursion sets has received a lot of interest in recent literature. The three geometric additive features $\mathcal{L}_k$, $k = 0, 1, 2$ are referred to as either Lipschitz-Killing curvatures (LKCs) in stochastic geometry, curvature measures in differential geometry, intrinsic volumes or Minkowski functionals in integral and convex geometry. These functionals are extensively used in the literature to describe the geometry or the topology of the considered excursion set. Loosely speaking, $\mathcal{L}_0$ is the Euler-Poincaré characteristic of the excursion set (i.e., the difference between

Received by the editors July 3rd, 2024; accepted February 26th, 2026; published under license CC BY 4.0.
2010 *Mathematics Subject Classification.* 60G60, 60F05, 33C10, 60D05.
Key words and phrases. Central Limit Theorem, Euler-Poincaré characteristic, excursion sets, random plane wave, Wiener chaos expansion.
The author E. D.B. has been supported by the French government, through the 3IA Côte d'Azur Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-19-P3IA-0002. The author A.P. T. is a member of INdAM-GNAMPA.

the number of connected components and the number of holes), $\mathcal{L}_1$ is half the perimeter length and $\mathcal{L}_2$ is the area.

The planar Berry's random wave model. Let us denote by T a bounded rectangle contained in $\mathbb{R}^2$, with non empty interior. Obviously,

$$\mathcal{L}_0(T) = 1, \quad \mathcal{L}_1(T) = \frac{1}{2}|\partial T|_1, \quad \mathcal{L}_2(T) = |T|, \tag{1.1}$$

where ∂T stands for the boundary of the set T, $|\cdot|$ the two-dimensional Lebesgue measure and $|\cdot|_1$ the one-dimensional Hausdorff measure.

We will consider here the random excursion set above a real level u, associated to Berry's random wave model, i.e.,

$$A(f, T, u) = \{x \in T : f(x) \geq u\}, \tag{1.2}$$

where f is the isotropic centered Gaussian random field $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ uniquely defined by the covariance function

$$\mathbb{E}[f(x)f(y)] = J_0(\|x - y\|), \tag{1.3}$$

for x, $y \in \mathbb{R}^2$, where J_0 is the order 0 Bessel function. Notice that in our context, since f is the Laplacian eigenfunction with eigenvalue 1 (see Nourdin et al., 2019), it can be expressed through its second derivatives. For the sake of illustration, in Figure 1.1 we generate via the `R` package `RandomFields` (see Schlather et al., 2015), the Gaussian Berry random field f as in (1.3) (first panel) in $T = [-70, 70]^2$, for different choices of the pixelization resolution. In Figure 1.2, we display the excursion sets $A(f, [-70, 70]^2, u)$ in (1.2), for $u = 0, 0.5, 1, 2$, with 140 pixels for side.

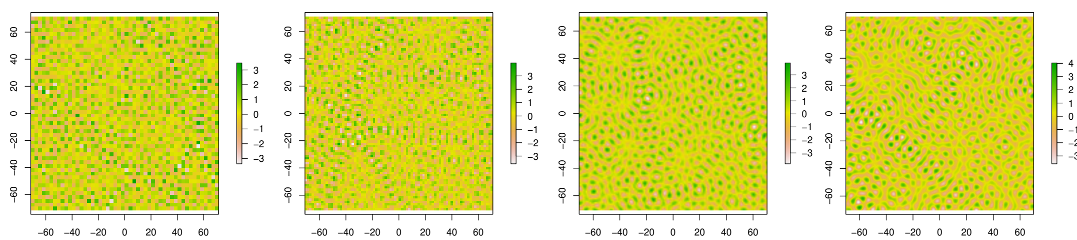

FIGURE 1.1. A random generation in $[-70, 70]^2$ of a Gaussian Berry random field f. From the left to the right: the side of the domain is divided in 50, 80, 140 and 400 pixels respectively.

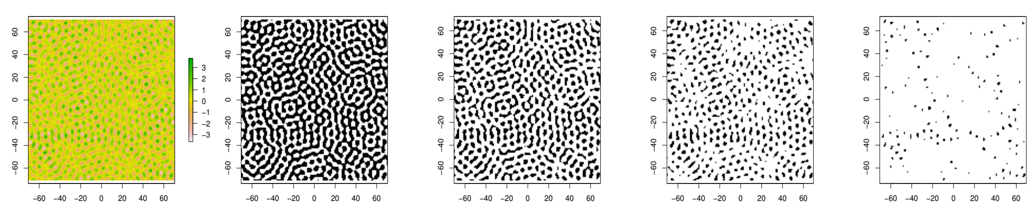

FIGURE 1.2. A random generation in $[-70, 70]^2$, with 140 pixels for side, of a Gaussian Berry random field f (first panel) with associated excursion sets $A(f, [-70, 70]^2, u)$ in (1.2), for $u = 0, 0.5, 1, 2$ (from the left to the right panels).

In this paper we focus on the asymptotic behaviour as the observation domain T tends to $\mathbb{R}^2$, of the first Lipschitz-Killing curvature (the Euler-Poincaré Characteristic, EPC) of the excursion set (1.2) for Berry's random wave model in $\mathbb{R}^2$ defined in (1.3). More precisely, as we are interested in the asymptotics as $T \nearrow \mathbb{R}^2$, instead of considering the EPC of the excursion of f above u, we consider the *modified Euler characteristic*, inspired by Adler and Taylor (2007, Lemma 11.7.1), Estrade and León (2016, Section 1), and Di Bernardino et al. (2017). Roughly speaking, by applying Morse's theorem, both notions coincide on $\mathring{T}$, the interior of the domain T. Indeed, the Euler characteristic of $A(f,T,u)$ is equal to a sum of two terms (see Adler and Taylor, 2007, Chapter 9, for instance). The first one only depends on the restriction of f to $\mathring{T}$, the second one exclusively depends on the behaviour of f on the l-dimensional faces of T, with $0 \leq l < 2$. From now on, we focus on the first term, denoted $\varphi(f,T,u)$, and defined in terms of number of local extrema of f in $\mathring{T}$ above u, as

$$\varphi(f,T,u) = \#\{\text{local extrema of } f \text{ above } u \text{ in } \mathring{T}\} - \#\{\text{local saddle points of } f \text{ above } u \text{ in } \mathring{T}\}. \quad (1.4)$$

The study of Euler characteristic of the excursion sets has received much attention. Let us quote, *e.g.*, Worsley (1994) or Telschow et al. (2023) where the localization of peaks is inferred from the observation of the Euler characteristic of excursion sets in neuroimaging and in the Cosmic Microwave Background radiation; Di Bernardino et al. (2017) or Biermé et al. (2019) where a test of Gaussianity of the underlying random field is produced based on the Euler characteristic of its excursion sets. Considering real data applications, the Euler-Poincaré characteristic has been used for example, to express cancellous bone connectivity (see *e.g.* Odgaard and Gundersen, 1993) and to morphologically model the microstructure of geo-materials (see *e.g.* Roubin, 2023).

On the literature of Central limit theorems (CLT) for LKCs. Recently, the asymptotic Gaussian fluctuations of the LKCs of excursion sets (at any fixed threshold u) have been proven in the case of a Gaussian random field when the observation domain grows to the whole Euclidean space. The interested reader is referred for instance to Bulinski et al. (2012), Pham (2013) for CLTs for the volume. In Kratz and Vadlamani (2018) and Müller (2017) the authors prove a CLT for any Minkowski functional of excursion sets in the general framework of stationary Gaussian fields whose covariance function is decreasing fast enough at infinity (see also Spodarev, 2014 for survey). On the same Gaussian setting for growing domain and fast-decay covariance, in Estrade and León (2016) and Di Bernardino et al. (2017) the asymptotic fluctuations for the Euler-Poincaré characteristic is proved. In this central limit theorems literature, the LKCs exhibit a variance asymptotically proportional to the volume of the growing domain. However, Berry's random wave model does not fall inside these studies. Indeed, the covariance function in (1.3) does not satisfy the assumptions required in these works, since it does not belong to $L^1(\mathbb{R}^2)$. Instead of the class of short-range covariance of the aforementioned literature, we work here with an intermediate-range covariance, i.e, the Bessel function J_0.

For Berry's random wave model on $\mathbb{R}^2$, several results have been shown for specific geometric functionals. The nodal sets (i.e., excursion sets at 0 level) of Berry's random waves in $\mathbb{R}^2$ have been studied in Nourdin et al. (2019) in the high-energy limit setting and their results rigorously confirm the asymptotic behaviour for the variances of the nodal length derived in Berry (2002). The obtained asymptotics, which can be naturally reformulated in terms of the nodal statistics of a single random wave restricted to a compact domain diverging to the whole plane, show that the variance of the nodal length on a domain T on $\mathbb{R}^2$ is asymptotically proportional to area(T)log(area(T)) as T grows up to $\mathbb{R}^2$ (see Theorem 1.1 in Nourdin et al., 2019). Complements of this theory can be found also in Vidotto (2021). In Smutek (2025) the author investigates the fluctuations of the nodal number of two independent real Berry Random Waves, with distinct energies. Furthermore, in Beliaev et al. (2020) critical points for the planar Berry's random wave model have been investigated, while the excursion area can be derived by using the *polyspectra* (stochastic terms in the Wiener chaos decomposition of integral functionals) in Grotto et al. (2024) and the asymptotic behavior of a general class of functionals in Maini and Nourdin (2024). In Estrade and Fournier (2020) anisotropic random waves are considered in any dimension.

In Dalmao et al. (2021) and Dalmao (2023) the authors study the expected length nodal lines of Berry's random waves model in $\mathbb{R}^3$ for a long-range power law covariance model, *i.e.*, the covariance function $r(x) \approx |x|^{\beta-1}$, for $\beta \in (0,1)$, as $x \to \infty$. They explicitly show how the decay of the covariance impacts on the unusual normalizing power of the volume of the growing domain in the behaviour of the asymptotic variance. Indeed, for $0 < \beta < 1/4$, an asymptotic behaviour can be established, see Proposition 3.5 in Dalmao et al. (2021). In this long-ranged dependence, they also exhibit a non-Gaussian limit, which is in hard contrast with the classical short-range covariance situations.

A strongly related problem is the study of the Gaussian Laplace eigenfunctions on the unit sphere $\mathbb{S}^2$. Namely, for $\ell-$th energy level, such centred Gaussian random fields $\{T_\ell(x), x \in \mathbb{S}^2\}$ have the covariance function

$$\mathbb{E}[T_\ell(x)T_\ell(y)] = P_\ell(\cos(\mathrm{dist}(x,y))),$$

where P_ℓ is the $\ell-$th Legendre polynomial and $\mathrm{dist}(x,y)$ is the geodesic distance on the sphere, i.e., $\mathrm{dist}(x,y) = \arccos(\langle x,y\rangle)$, where $\langle x,y\rangle$ is the standard scalar product in $\mathbb{R}^3$. For large ℓ the covariance function locally approximates J_0 by virtue of Hilb's asymptotics (Szegö, 1939, Theorem 8.21.12), and there is hope to relate results for T_ℓ to corresponding questions for the planar field f in (1.3). This is a special case of Berry's famous conjecture (see Berry, 1977) stating that the high energy behaviour (*i.e.*, large ℓ) of Laplace eigenfunctions is universal across "generic" Riemannian manifolds. If proven, this bridge would be extremely useful, because the symmetry of $\mathbb{S}^2$ has been fruitfully exploited for the study of geometric functionals in recent years. The asymptotic behaviour of the nodal length of random spherical harmonics is given in Marinucci et al. (2020) and the CLT for shrinking caps is established in Todino (2020). In Cammarota et al. (2016b) the authors study the limiting distribution of critical points and extrema of random spherical harmonics, in the high energy limit. The correlation between the total number of critical points of random spherical harmonics and the number of critical points with value in any real interval is studied in Cammarota and Todino (2022). In Cammarota et al. (2016a) a precise expression for the asymptotic variance of the Euler-Poincaré characteristic for excursion sets of random spherical harmonics is presented. A quantitative CLT in this high-energy limit has been investigated in Cammarota and Marinucci (2018). Remarkably, for almost all levels the behaviour of the three LKC on $\mathbb{S}^2$ is determined by $H_2(T_\ell)$, the second Wiener chaos component of the initial field T_ℓ. More precisely (see Marinucci, 2023), for $k = 0, 1, 2$,

$$\mathcal{L}_k(A(T_\ell,\mathbb{S}^2,u)) - \mathbb{E}[\mathcal{L}_k(A(T_\ell,\mathbb{S}^2,u))] \overset{\ell\to\infty}{\sim} c_k(u)(\lambda_\ell)^{(2-k)/2}\int_{\mathbb{S}^2} H_2(T_\ell(x))dx, \tag{1.5}$$

where λ_ℓ is the eigenvalue at the energy level ℓ, and the explicit coefficients c_k are such that $c_k(u) = 0$ iff $u = 0$ for $k = 1, 2$, and $c_0(u) = 0$ iff $u \in \{-1, 0, 1\}$. If one knew by Berry's conjecture that Equation (1.5) also holds on the plane, the result of Grotto et al. (2024) would imply that the variance of $\varphi(f,T,u)$ for $u \notin \{-1,0,1\}$ and f in (1.3) is of order $\mathrm{area}(T)^{3/2}$ in contrast to the short-range covariance case (see the aforementioned literature, e.g., Estrade and León, 2016; Kratz and Vadlamani, 2018; Müller, 2017).

A local version of Berry's conjecture has been proven in Canzani and Hanin (2020), via a coupling between a random field on a manifold and a random field on its tangent space. The authors give an estimate for the expectation and variance of the nodal length and the number of critical points for a large class of manifolds. This result has been completed in Dierickx et al. (2023) with a CLT for the nodal length over shrinking balls. In Canzani and Hanin (2020) the authors also provide global estimates for the nodal length and critical points on compact manifolds. As for precise results on the plane confirming Berry's conjecture in the sense of establishing (1.5), one can refer to Nourdin et al. (2019) for the nodal length (see also Wigman, 2024 for an explicit comparison between $\mathbb{R}^2$ and $\mathbb{S}^2$), and, implicitly, in Grotto et al. (2024) for the area.

Our work confirms the prediction of $\mathrm{Var}(\varphi(f,T,u))$ by Berry's conjecture, determines the precise normalising constant as well as a CLT, concluding thereby the study of Berry's random wave LKC on $\mathbb{R}^2$, and providing an instance of a change in behaviour compared to the short-range covariance case. Based on this result, we can find precise asymptotics for a randomly perturbed random wave field. The next section is dedicated to details and implications of our results.

Some conventions. For the rest of the paper, we assume that all random objects are defined on a common probability space $(\Omega, F, \mathbb{P})$, with $\mathbb{E}$ denoting expectation with respect to $\mathbb{P}$. In the whole paper ϕ will be the probability density function and Φ the cumulative distribution function of the standard Gaussian random variable $\mathcal{N}(0,1)$. Given two positive sequences a_n and b_n, we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. We use the symbol $\xrightarrow{\mathcal{L}}$ to denote convergence in distribution.

1.2. *Main results.* Our first goal is to establish the Central Limit Theorem for the modified Euler-Poincaré Characteristic of the excursion above $u \in \mathbb{R}$ in Berry's model in (1.3) in a growing domain $T_N = [-N, N]^2$, with N a positive integer, with $N \to \infty$ (see Theorem 1.1). Secondly, we define a class of Berry's mixture models and we study the obtained variances and the Gaussian fluctuations in this perturbed setting (see Theorem 1.3).

Gaussian fluctuations of the EPC in the planar Berry's model. The desired asymptotic behaviour is obtained exploiting the $L^2(\Omega)$ expansion of modified Euler characteristic into Wiener chaoses, which are orthogonal spaces spanned by Hermite polynomials. First of all, we recall that the Hermite polynomials $H_q(x)$ are defined by $H_0(x) = 1$, and for $q = 2, 3, \ldots$

$$H_q(x) = (-1)^q \frac{1}{\phi(x)} \frac{d^q \phi(x)}{dx^q}, \tag{1.6}$$

with $x \in \mathbb{R}$. We consider the Wiener chaos expansion of $\varphi(f, T_N, u)$ (see, e.g., Estrade and León, 2016, Section 1.2 and Cammarota and Marinucci, 2018).

$$\varphi(f, T_N, u) = \sum_{q=0}^{\infty} \varphi(f, T_N, u)[q], \tag{1.7}$$

where $\varphi(f, T_N, u)[q]$ denotes the projection of $\varphi(f, T_N, u)$ on the q-order chaos component, that is, the space generated by the L^2-completion of linear combinations of the form

$$H_{q_1}(\xi_1) \cdot H_{q_2}(\xi_2) \cdots H_{q_k}(\xi_k), \qquad k \geq 1,$$

with $q_i \in \mathbb{N}$ such that $q_1 + \cdots + q_k = q$, and $(\xi_1, \ldots, \xi_k)$ real standard Gaussian vector.

It results that (after centring) a single term dominates the $L^2(\Omega)$ expansion in (1.7), that is, the projection into the second chaos (see Proposition 2.1). As explained above, this was expected since random spherical harmonics exhibit the same behavior as Berry's random wave (see, e.g., Wigman, 2024).

Theorem 1.1. *Recall that* $|T_N| = (2N)^2$. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be the isotropic standard Berry's Gaussian random field with covariance function in* (1.3). *Denote by* $\lambda_f = \frac{1}{2}$ *the second spectral moment of* f. *Then for any* u *in* $\mathbb{R}$, *with* $u \notin \{0, 1, -1\}$, *it holds that*

$$\frac{\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)]}{\sqrt{\frac{1}{2}\mathcal{L}_1(T_N)\mathcal{L}_2(T_N)}} \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(0, V(u)),$$

where

$$\mathbb{E}[\varphi(f, T_N, u)] = \mathcal{L}_2(T_N)\,(2\pi)^{-\frac{3}{2}}\,\lambda_f\, u\, e^{-u^2/2}, \tag{1.8}$$

and $\mathcal{N}(0, V(u))$ *stands for the centered Gaussian distribution with finite and strictly positive variance,*

$$V(u) = v(u)\,\frac{1}{8\pi^2}\left(\frac{1-\sqrt{2}}{3} + \ln(1+\sqrt{2})\right), \tag{1.9}$$

with

$$v(u) := \phi(u)^2\, u^2 \left(u^2 - 1\right)^2. \tag{1.10}$$

Section 2 is devoted to the proof of Theorem 1.1. Notice that the restriction to $u \notin \{0, 1, -1\}$ in Theorem 1.1 avoids the vanishing of the asymptotic variance in Equation (1.9).

Remark 1.2. Although this statement, and in particular the dependence of the variance on the level u, is quite similar to the spherical case studied in Cammarota and Marinucci (2018) and Cammarota et al. (2016a), our proof strategy is different. In the above references, the authors rely on a Kac-Rice formula type statement to calculate the variance and then compute the second chaos variance that turns out to be the same in the high-frequency asymptotics, allowing them to ignore all the other terms for the CLT. In contrast to this, we compute the order of the variance of each chaos component separately, which requires in particular a careful treatment of the first chaos. This allows us to see "from below" that the variance is dominated by a single component. Moreover, contrary to the spherical case, we cannot rely on a representation of the field and its derivatives as a linear combination of independent random variables. Instead, we show that there is perfect asymptotic correlation between the second chaos component and its part containing only the contribution of the initial field f, which allows us to obtain a CLT as an application of the spectral CLT recently shown by Maini and Nourdin (2024).

Gaussian fluctuations of the EPC in a class of perturbed Berry's models. Our second aim is to establish the Central Limit Theorem for the modified Euler Poincaré Characteristic φ in the case of a *perturbed Berry's model*.

We will study Berry mixture model $\{f_\Lambda(x) : x \in T\}$, with T fixed compact domain in $\mathbb{R}^2$, that have the following general structure:

$$f_\Lambda(x) = g(f(x), \Lambda), \text{ with } \{f(x)\} \perp \Lambda, \, x \in T, \tag{1.11}$$

where Λ is a shape variable with suitable properties and g is a certain link function, such that $g(\cdot, \lambda)$ is strictly increasing for any λ.

Specifically, location mixtures arise for $g(w, \lambda) = \lambda w$ (called *Gaussian scale mixture*), whereas scale mixtures are obtained by setting $g(w, \lambda) = w + \lambda$ (*Gaussian location mixtures*). In the first case $\Lambda > 0$ can be viewed as a random standard deviation parameter embedded in the Gaussian random field f, the second one as a random mean Gaussian model.

Such processes have received considerable attention in the recent spatial statistics literature due to their ability to account for asymmetric lower and upper tails, and for extremal dependence that is stronger than in the purely Gaussian case (see *e.g.*, Krupskii et al., 2018). In spatial extreme-value theory, they lead to extensively studied limit processes of Brown–Resnick type (see *e.g.*, Kabluchko et al., 2009; Wadsworth and Tawn, 2014; Dombry et al., 2016).

This setting allows us to introduce the inverse function $h(\cdot, \lambda)$ of $g(\cdot, \lambda)$ so that from (1.2) we have the following almost sure identity for the excursion set above the level u:

$$A(f_\Lambda, T_N, u) = A(f, T_N, h(u, \Lambda)). \tag{1.12}$$

Using the Central Limit Theorem 1.1 we derive the asymptotic behaviour of the variance of $\varphi(f_\Lambda, T_N, u)$ in the perturbed Berry's model introduced in (1.11) and the Gaussian fluctuations as $T_N \nearrow \mathbb{R}^2$, i.e., $N \to \infty$ (see Theorem 1.3 below). For any function ζ such that $\mathbb{E}[\zeta(A(f_\Lambda, T_N, u))] < \infty$, we can write

$$\mathbb{E}[\zeta(A(f_\Lambda, T_N, u))] = \mathbb{E}[\mathbb{E}[\zeta(A(f, T_N, h(u, \Lambda)))|\Lambda]].$$

Theorem 1.3. *Recall that $|T_N| = (2N)^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the isotropic standard Berry's Gaussian random field with covariance function in* (1.3)*. Let $\{f_\Lambda(x) : x \in T_N\}$ be Berry mixture model in* (1.11)*. Denote by $\lambda_f = \frac{1}{2}$ the second spectral moment of f. Then, for any $u \in \mathbb{R}$, such that $\mathbb{E}\left[\upsilon\left(h(u, \Lambda)\right)\right] \neq 0$, with υ as in* (1.10)*, it holds that*

$$\frac{\varphi(f, T_N, h(u, \Lambda)) - \mathbb{E}[\varphi(f, T_N, h(u, \Lambda))]}{\sqrt{\frac{1}{2}\mathcal{L}_1(T_N)\mathcal{L}_2(T_N)}} \xrightarrow[N \to \infty]{\mathcal{L}} \mathcal{N}(0, V_\Lambda(u)),$$

where

$$\mathbb{E}[\varphi(A(f_\Lambda, T_N, u))] = \mathcal{L}_2(T_N)\,\lambda_f\,\mathbb{E}[\rho_2(h(u,\Lambda))], \tag{1.13}$$

$\rho_2(u) := (2\pi)^{-3/2}e^{-u^2/2}H_1(u) = (2\pi)^{-1/2}\phi(u)\,u$ *and*

$$V_\Lambda(u) = \mathbb{E}\left[\upsilon\left(h(u,\Lambda)\right)\right]\frac{1}{8\pi^2}\left(\frac{1-\sqrt{2}}{3} + \ln(1+\sqrt{2})\right), \tag{1.14}$$

where $h(\cdot,\lambda)$ is the inverse function of $g(\cdot,\lambda)$.

Section 3 is devoted to the proof of Theorem 1.3. Obviously, Theorem 1.1 can be seen as a specific case of the Central Limit Theorem 1.3 for Berry's mixture models where $h(u,\Lambda)$ is an almost surely constant random variable.

The paper is organized as follows. In Section 2 and Section 3 we prove Theorems 1.1 and 1.3, respectively. In Section 4 we present numerical studies for some parametric models for the perturbed Berry's random wave. Finally, technical lemmas exploited in the proof of Theorem 1.1 are collected in Appendix A.

2. Proof of Central Limit Theorem 1.1

The first moment in (1.8) can be obtained via the Rice formulas for the factorial moments of the number of local maxima above u, the number of local minima above u and the number of local saddle points above u (see for instance Adler and Taylor, 2007, Chapter 11 or Azaïs and Wschebor, 2009, Chapter 6).

In Proposition 2.1 below, we compute the asymptotic variance of the EPC, exploiting the Wiener chaos decomposition in (1.7), and conclude that the second chaos is the leading term of the series expansion. Then, in Proposition 2.2, we prove that there is asymptotically full correlation between the Euler-Poincaré characteristic and the second chaotic projection. By combining the variance in Proposition 2.1 and this full correlation result in Proposition 2.2, the Central Limit Theorem 1.1 follows.

Proposition 2.1. *Let $f:\Omega\times\mathbb{R}^2\to\mathbb{R}$ be the isotropic standard Berry's Gaussian random field with covariance function in (1.3). Then for any u in $\mathbb{R}$, as $N\to\infty$, we have*

$$\varphi(f,T_N,u) - \mathbb{E}[\varphi(f,T_N,u)] = \varphi(f,T_N,u)[2] + o(\varphi(f,T_N,u)[2])$$

and then

$$(2\,N)^{-3}\mathrm{Var}(\varphi(f,T_N,u)) = \upsilon(u)\frac{1}{8\pi^2}\left(\frac{1-\sqrt{2}}{3} + \ln(1+\sqrt{2})\right) + O\left(\frac{1}{\sqrt{N}}\right),$$

where $\upsilon(u)$ is defined in (1.10).

The proof of Proposition 2.1 is postponed to Section 2.1.

Proposition 2.2. *Let*

$$\mathcal{S}(f,N) := \int_{[-N,N]^2} H_2(f(x))dx.$$

Then we have

$$|\mathrm{Corr}\left(\varphi(f,T_N,u)[2],\,\mathcal{S}(f,N)\right)| \overset{N\to\infty}{\to} 1.$$

The proof of Proposition 2.2 is postponed to Section 2.2.

Proof of Theorem 1.1: In Proposition 2.1 we establish that, as $N \to \infty$, and $u \neq \{0, -1, 1\}$

$$\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)] = \varphi(f, T_N, u)[2] + o(\varphi(f, T_N, u)[2]).$$

In view of Proposition 2.2 the Euler-Poincaré characteristic is fully correlated in the limit to $\mathcal{S}(f, N)$, i.e.

$$\lim_{N\to\infty} \mathrm{Corr}(\varphi(f, T_N, u), \mathcal{S}(f, N)) = \lim_{N\to\infty} \frac{\mathrm{Cov}(\varphi(f, T_N, u), \mathcal{S}(f, N))}{\sqrt{\mathrm{Var}(\varphi(f, T_N, u))\mathrm{Var}(\mathcal{S}(f, N))}} = 1. \tag{2.1}$$

Moreover,

$$\lim_{N\to\infty} \frac{Var(\varphi(f, T_N, u)[2])}{Var(\varphi(f, T_N, u))} = 1 + o(1).$$

Then, by using the Wasserstein distance d_W and denoting Z a standard Gaussian random variable, we have

$$d_W\left(\frac{\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)]}{\sqrt{Var(\varphi(f, T_N, u))}}, Z\right)$$
$$\leq d_W\left(\frac{\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)]}{\sqrt{Var(\varphi(f, T_N, u))}}, \frac{\varphi(f, T_N, u)[2]}{\sqrt{Var(\varphi(f, T_N, u))}}\right) + d_W\left(\frac{\varphi(f, T_N, u)[2]}{\sqrt{Var(\varphi(f, T_N, u))}}, Z\right).$$

Now we can bound the first term by

$$d_W\left(\frac{\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)]}{\sqrt{Var(\varphi(f, T_N, u)))}}, \qquad \frac{\varphi(f, T_N, u)[2]}{\sqrt{Var(\varphi(f, T_N, u)}}\right)$$
$$\leq \sqrt{\mathbb{E}\left[\frac{\varphi(f, T_N, u) - \mathbb{E}(\varphi(f, T_N, u))) - \varphi(f, T_N, u)[2]}{\sqrt{Var(\varphi(f, T_N, u)))}}\right]^2},$$

which tends to 0 as $N \to \infty$ (see Proposition 2.1). Hence, we have that

$$d_W\left(\frac{\varphi(f, T_N, u)[2]}{\sqrt{Var(\varphi(f, T_N, u))}}, Z\right) = d_W(S(f, N), Z) + o(1).$$

Finally, $d_W(S(f, N), Z) \to 0$, applying the Central Limit Theorem in Maini and Nourdin (2024, Theorem 2). Indeed, the authors show that the spectral condition required for weak convergence is satisfied for the planar Berry random field. The only thing remaining to show is a condition on the Fourier transform $\mathcal{F}$ of the rescaled domain of integration $D := [-1, 1]^2$, namely $|\mathcal{F}[1_D](x)| = O\left(\frac{1}{\|x\|}\right)$ as $\|x\| \to \infty$. We have $|\mathcal{F}[1_D](x)| = \frac{4\sin(x_1)\sin(x_2)}{x_1 x_2}$, which satisfies the necessary condition. □

2.1. *Proof of Proposition 2.1.* To prove Proposition 2.1 we first need to establish the L^2 expansion of $\varphi(f, T_N, u)$ into Wiener chaoses. To this aim we recall a result by Estrade and León (2016, Proposition 1.2) in Lemma 2.3 below, which allows to write the modified EPC as the limit of an integral in L^2. This representation permits the Wiener chaos expansion (after normalising the vector of first and second derivatives to have unit variance). To study the behaviour of different chaos components we need to compute all the covariances between $f, \partial_i f, \partial_{ii} f$ for all $i = 1, 2$. These calculations are made in Lemma 2.4. The asymptotic behaviour of all chaos components in large domain (i.e., $N \to \infty$) can be found in Lemma A.2, A.4, A.5, A.6 and Remark A.3, collected in Appendix A. From these results we establish that the leading terms of the series expansion belong all to the second chaos and they behave as N^3; all the other terms are $o(N^3)$. Then the asymptotic variance in Proposition 2.1 is given by the variance of the dominant terms and we conclude the proof.

The following result is a close adaptation of Estrade and León (2016, Proposition 1.2) to our setup.

Lemma 2.3. *The following convergence holds almost surely and in $L^2(\Omega)$*

$$\varphi(f,T_N,u)=\lim_{\varepsilon\to 0^+}\int_{[-N,N]^2}\begin{vmatrix}\partial_{11}f(x) & \partial_{12}f(x)\\ \partial_{12}f(x) & \partial_{22}f(x)\end{vmatrix}1_{[u,\infty)}\left(-(\partial_{11}f(x)+\partial_{22}f(x))\right)\delta_\varepsilon(\nabla f(x))dx. \quad (2.2)$$

Proof: First, note that the integral appearing in the statement of the lemma can be written through the Wiener chaos decomposition. More precisely, following similar notation as in Estrade and León (2016, Section 1.2), we define

$$\varphi(\varepsilon,T_N):=\int_{T_N}\begin{vmatrix}\partial_{11}f(x) & \partial_{12}f(x)\\ \partial_{12}f(x) & \partial_{22}f(x)\end{vmatrix}1_{[u,\infty)}\left(f(x)\right)\delta_\varepsilon(\nabla f(x))dx, \quad (2.3)$$

which can be rewritten as the r.h.s in (2.2) since f is the Laplacian eigenfunction with eigenvalue 1. We identify any symmetric matrix of size 2×2 with the 3−dimensional vector containing the coefficients on and above the diagonal and write $\tilde{det}$ the associated determinant map. Then we consider the map G_ε defined on $\mathbb{R}^5$ by

$$G_\varepsilon(x,y,z)=\delta_\varepsilon(x)\tilde{det}(y)1_{[u,\infty)}(y),\quad (x,y)\in\mathbb{R}^2\times\mathbb{R}^3,$$

and the map

$$G_u(y,z)=\tilde{det}(y)1_{[u,\infty)}(y),\quad y\in\mathbb{R}^3.$$

We denote by Σ^X the covariance matrix of the 5-dimensional Gaussian vector $X(t)=(\nabla f(t),\nabla^2 f(t))$ and we consider Ξ a 5×5 matrix such that $\Xi^T\Xi=\Sigma^X$. We can thus write, for any $t\in\mathbb{R}^2$, $X(t)=\Xi Y(t)$ with $Y(t)$ a 5-dimensional standard Gaussian vector. The matrix Ξ can be factorized into $\begin{pmatrix}\Xi_1 & 0\\ 0 & \Xi_2\end{pmatrix}$, where $\Xi_1=\sqrt{\lambda_f}I_2$ with λ_f the second spectral moment of f. For $y=(\underline{y},\overline{y})\in\mathbb{R}^5=\mathbb{R}^2\times\mathbb{R}^3$, we define

$$\tilde{G}_\varepsilon(y)=G_\varepsilon(\Xi y)=\delta_\varepsilon(\Xi_1\underline{y})G_u(\Xi_2\overline{y})=\delta_\varepsilon\circ\Xi_1(\underline{y})G_u\circ\Xi_2(\overline{y}).$$

For $n=(n_1,\dots,n_5)\in\mathbb{N}^5$ and $y\in\mathbb{R}^5=\mathbb{R}^2\times\mathbb{R}^3$,

$$\overline{H}_n(y)=\prod_{1\le j\le 5}H_{n_j}(y_j),$$

with H_{n_j} the Hermite polynomials recalled in (1.6). Now we can write

$$G_\varepsilon(y)=\sum_{q=0}^{\infty}\sum_{n\in N^5;|n|=q}\langle\tilde{G}_\varepsilon,\overline{H}_n\rangle\overline{H}_n(y),$$

where the Hermite coefficients $\langle\tilde{G}_\varepsilon,\overline{H}_n\rangle=\mathbb{E}[G_\varepsilon(X)\overline{H}_n(X)]$ can be factorized as $\langle\tilde{G}_\varepsilon,\overline{H}_n\rangle=\langle\delta_\varepsilon\circ\Xi_1,\overline{H}_{\underline{n}}\rangle\langle G_u\circ\Xi_2,\overline{H}_{\overline{n}}\rangle$ given respectively by

$$\langle\delta_\varepsilon\circ\Xi_1,\overline{H}_{\underline{n}}\rangle=\frac{1}{\underline{n}!}\int_{\mathbb{R}^2}\delta_\varepsilon(\sqrt{\lambda_f}y)\overline{H}_{\underline{n}}(y)\phi_2(y)\,dy$$

and

$$\langle G_u\circ\Xi_2,\overline{H}_{\bar{n}}\rangle:=\frac{1}{\bar{n}!}\int_{\mathbb{R}^3}G_u(\Xi_2,\,z)\overline{H}_{\bar{n}}(z)\phi_3(z)\,dz,$$

with ϕ_m, for $m = 2, 4$ the standard Gaussian density on $\mathbb{R}^m$. We take the limit as $\varepsilon \to 0$ to obtain the expansion of the modified EPC. We have that

$$\langle \delta_\varepsilon \circ \Xi_1, \overline{H}_{\underline{n}} \rangle \to_{\varepsilon \to 0} \frac{1}{n!}(2\pi\lambda_f)^{-1}\overline{H}_{\underline{n}}(0).$$

Then, for $n = (\underline{n}, \overline{n}) \in \mathbb{R}^2 \times \mathbb{R}^3$, the Hermite coefficients $\langle G_u, \overline{H}_n \rangle$ are given by

$$\langle G_u, \overline{H}_n \rangle = \frac{1}{\underline{n}!}(2\pi\lambda_f)^{-1}\overline{H}_{\underline{n}}(0)\langle G_u \circ \Xi_2, \bar{n} \rangle.$$

By Estrade and León (2016, Proposition 1.3) the expansion of the modified EPC is finally given by

$$\sum_{q=0}^{\infty} \sum_{n \in \mathbb{N}^5, |n|=q} \langle G_u, \overline{H_n} \rangle h_n,$$

where $|n| = n_1 + \cdots + n_5$ and

$$h_n = \int_{[-N,N]^2} \overline{H}_n(f, \nabla f, \nabla^2 f)(x)\, dx.$$

We refer to Estrade and León (2016, Section 1.2) for more details.

It follows that, the chaotic decomposition in (1.7) can be written as

$$\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)] = \sum_{q=1}^{\infty} \varphi(f, T_N, u)[q] = \sum_{q=1}^{\infty} \sum_{\substack{n \in \mathbb{N}^5, \\ |n|=q}} \langle G_u, \overline{H_n} \rangle h_n. \tag{2.4}$$

□

Now we can analyse the behaviour of different chaos components h_n. To this aim, we will need to consider the polyspectra of the first and second order derivatives of f. The following technical result helps evaluate their covariances.

Lemma 2.4. *We have for x, $y \in \mathbb{R}$ such that $x = (0, 0)$, $y = (0, r)$:*

$$\begin{aligned}
g_1(r) &:= \mathbb{E}[\partial_1 f(x) \partial_1 f(y)] = \frac{J_1(r)}{r}, \\
g_2(r) &:= \mathbb{E}[\partial_2 f(x) \partial_2 f(y)] = \frac{J_0(r) - J_2(r)}{2}, \\
g_3(r) &:= \mathbb{E}[\partial_{11} f(x) \partial_2 f(y)] = \mathbb{E}[\partial_1 f(x) \partial_{12} f(y)] = \frac{J_0(r) - J_2(r)}{2r} - \frac{J_1(r)}{r^2} = -\frac{J_2(r)}{r}, \\
g_4(r) &:= \mathbb{E}[\partial_2 f(x) \partial_{22} f(y)] = \frac{J_3(r) - 3J_1(r)}{4}, \\
g_5(r) &:= \mathbb{E}[\partial_{11} f(x) \partial_{11} f(y)] = \frac{3(J_2(r) - J_0(r))}{2r^2} + \frac{3J_1(r)}{r^3} = \frac{3J_2(r)}{r^2}, \\
g_6(r) &:= \mathbb{E}[\partial_{11} f(x) \partial_{22} f(y)] = \mathbb{E}[\partial_{12} f(x) \partial_{12} f(y)] = \frac{3J_1(r) - J_3(r)}{4r} + \frac{J_0(r) - J_2(r)}{r^2} - \frac{2J_1(r)}{r^3}, \\
g_7(r) &:= \mathbb{E}[\partial_{22} f(x) \partial_{22} f(y)] = \frac{3J_0(r) - 4J_2(r) + J_4(r)}{8} = -\frac{(r^2 - 3)J_2(r)}{r^2},
\end{aligned}$$

while all the other combinations are uncorrelated. The symmetric expressions for $x = (0, r)$, $y = (0, 0)$ are identical up to a possible multiplication by -1.

Proof of Lemma 2.4: We compute, for instance, for g_1 (the proof of the remaining terms is similar):

$$\mathbb{E}[\partial_1 f(x) \partial_1 f(y)] = \partial_{x_1} \partial_{y_1} J_0 \left(\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\| \right) = \partial_{y_1} J_0' (\|x - y\|) \frac{x_1 - y_1}{\|x - y\|}$$

$$\frac{-\|x - y\| - (y_1 - x_1)\frac{x_1 - y_1}{\|x - y\|}}{\|x - y\|^2} J_0'(\|x - y\|) + \frac{(y_1 - x_1)(x_1 - y_1)}{\|x - y\|^2} J_0''(\|x - y\|).$$

This equals $\frac{J_1(r)}{r}$ for both $x = (0,0)$, $y = (0,r)$ and $x = (0,r)$, $y = (0,0)$.
As for the symmetry property, note that all the derivatives have the form

$$h(x_1 - y_1, x_2 - y_2, \|x - y\|),$$

and we can see by induction that the dependence on $(x_2 - y_2)$ is polynomial, either even or odd, depending on the number of the second component derivatives. Therefore, going from $x = (0,0)$, $y = (0,r)$ to $x = (0,r)$, $y = (0,0)$ would not change the final result if the polynomial is even, and would change only the sign if the polynomial is odd. □

In order to obtain the expansion (2.4) of the modified EPC in terms of Hermite polynomials, we also need to normalise the vector of first and second order derivatives. First, let us denote

$$Y_i := \widetilde{\partial_i f(x)} := \frac{\partial_i f(x)}{\sqrt{\mathrm{Var}(\partial_i f(x))}} = \partial_i f(x)\sqrt{2},$$

$$Y_3 := \widetilde{\partial_{11} f(x)} := \frac{\partial_{11} f(x)}{\sqrt{\mathrm{Var}(\partial_{11} f(x))}} = \partial_{11} f(x)\sqrt{\frac{8}{3}},$$

$$Y_4 := \widetilde{\partial_{12} f(x)} := \frac{\partial_{12} f(x)}{\sqrt{\mathrm{Var}(\partial_{12} f(x))}} = \partial_{12} f(x)\sqrt{8},$$

$$Y_5 := \widetilde{\partial_{22} f(x)} := \frac{\partial_{22} f(x)}{\sqrt{\mathrm{Var}(\partial_{22} f(x))}} = \partial_{22} f(x)\sqrt{\frac{8}{3}},$$

where $i = 1, 2$. Next, by considering the limit as $r \to 0$ of $g_3(r)$, $g_4(r)$, and $g_6(r)$ in Lemma 2.4, we see that

$$\mathbb{E}[Y_3 Y_5] = \frac{1}{3},$$

while the other variables are uncorrelated. By setting $Z := \frac{3}{2\sqrt{2}} Y_3 - \frac{1}{2\sqrt{2}} Y_5$, we obtain an i.i.d. vector (Y_1, Y_2, Z, Y_4, Y_5), and with this notation we can write (2.2) as

$$\begin{aligned}
&\begin{vmatrix} \partial_{11} f(x) & \partial_{12} f(x) \\ \partial_{12} f(x) & \partial_{22} f(x) \end{vmatrix} 1_{[u,\infty)} \left(-(\partial_{11} f(x) + \partial_{22} f(x))\right) \delta_\varepsilon(\nabla f(x)) \\
&= \left(\frac{3}{8}\widetilde{\partial_{11} f(x)}\widetilde{\partial_{22} f(x)} - \frac{1}{8}\widetilde{\partial_{12} f(x)}^2\right) 1_{[u,\infty)} \left(-\sqrt{\frac{3}{8}}\left(\widetilde{\partial_{11} f(x)} + \widetilde{\partial_{22} f(x)}\right)\right) \\
&\quad \times \delta_\varepsilon\left(\frac{1}{\sqrt{2}}\widetilde{\partial_1 f(x)}\right) \delta_\varepsilon\left(\frac{1}{\sqrt{2}}\widetilde{\partial_2 f(x)}\right) \\
&= \left(\frac{3}{8}\left(\frac{\sqrt{8}}{3} Z + \frac{1}{3} Y_5\right) Y_5 - \frac{1}{8} Y_4^2\right) 1_{(-\infty,-u]} \left(\frac{1}{\sqrt{3}} Z + \frac{\sqrt{2}}{\sqrt{3}} Y_5\right) \\
&\quad \times \delta_\varepsilon\left(\frac{1}{\sqrt{2}} Y_1\right) \delta_\varepsilon\left(\frac{1}{\sqrt{2}} Y_2\right).
\end{aligned}$$

Now this expression can be expanded in terms of Wiener chaoses. Note that as a function of the isotropic field $(\nabla f, \nabla^2 f)$ it is isotropic, and therefore, when computing the covariance in two different points x and y, we can fix the coordinates at $x = (0,0)$, $y = (0,r)$ once and for all.

In Lemmas A.2, A.4, A.5, A.6 (proved in Appendix A) we compute the covariances that arise from the chaos decomposition. Their results imply that the only summands contributing to the leading order are given by some terms of the second chaos, namely by $H_2(Y_2)$, $H_2(Y_5)$, $H_2(Z)$, $H_1(Y_2)H_1(Y_5)$, $H_1(Y_2)H_1(Z)$, and $H_1(Z)H_1(Y_5)$. Let us call the respective coefficients with which they appear in the chaos decomposition c_2, c_5, c_z, c_{25}, c_{2z}, and c_{z5}. The precise expressions for these coefficients are given in Lemma A.7. Then, let

$$\mathcal{D}_2(f,N) := \int_{[-N,N]^2} \frac{1}{2}c_2 H_2(Y_2(x)) + \frac{1}{2}c_5 H_2(Y_5(x)) + \frac{1}{2}c_z H_2(Z(x)) + c_{z5} H_1(Z(x))H_1(Y_5(x))dx \tag{2.5}$$

denote the leading term of the second chaotic projection of the Wiener chaos expansion of the modified EPC. We have

$$\varphi(f, T_N, u)[2] = \mathcal{D}_2(f,N) + O(N^2\sqrt{N})$$

and

$$\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)] = \varphi(f, T_N, u)[2] + o(\varphi(f, T_N, u)[2]).$$

It follows that the variance $V := \mathrm{Var}(\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)])$ is asymptotically equivalent to

$$\mathbb{E}[\mathcal{D}_2(f,N)^2] =$$
$$\mathbb{E}\left[\left(\int_{[-N,N]^2} \frac{1}{2}c_2 H_2(Y_2(x)) + \frac{1}{2}c_5 H_2(Y_5(x)) + \frac{1}{2}c_z H_2(Z(x)) + c_{z5} H_1(Z(x))H_1(Y_5(x))dx\right)^2\right],$$

where the coefficients c_2, c_5, c_z, c_{z5} are given in Lemma A.7, *i.e.*,

$$(\mathrm{Var}(\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)]))/\mathbb{E}[\mathcal{D}_2(f,N)^2] \to 1, \quad \text{for } N \to \infty.$$

Denoting $r := \|x - y\|$ and plugging in all the normalisations defined previously, we get

$$\begin{aligned} V \sim \int_{[-N,N]^2}\int_{[-N,N]^2} & \frac{1}{2}c_2^2(2g_2(r))^2 + \frac{1}{2}c_2c_5\Big(\frac{4}{\sqrt{3}}g_4(r)\Big)^2 + \frac{1}{2}c_2c_z\Big(\frac{4}{\sqrt{3\cdot 8}}g_4(r)\Big)^2 \\ & + c_2c_{z5}\Big(-\frac{4}{\sqrt{3\cdot 8}}g_4(r)\cdot\frac{4}{\sqrt{3}}g_4(r)\Big) + \frac{1}{2}c_5c_2\Big(\frac{4}{\sqrt{3}}g_4(r)\Big)^2 + \frac{1}{2}c_5^2\Big(\frac{8}{3}g_7(r)\Big)^2 \\ & + \frac{1}{2}c_5c_z\Big(-\frac{\sqrt{8}}{3}g_7(r)\Big)^2 + c_5c_{z5}\Big(-\frac{\sqrt{8}}{3}g_7(r)\cdot\frac{8}{3}g_7(r)\Big) + \frac{1}{2}c_zc_2\Big(\frac{4}{\sqrt{3\cdot 8}}g_4(r)\Big)^2 \\ & + \frac{1}{2}c_zc_5\Big(-\frac{\sqrt{8}}{3}g_7(r)\Big)^2 + \frac{1}{2}c_z^2\Big(\frac{1}{3}g_7(r)\Big)^2 + c_zc_{z5}\Big(-\frac{\sqrt{8}}{3}g_7(r)\cdot\frac{1}{3}g_7(r)\Big) \\ & + c_{z5}c_2\Big(-\frac{4}{\sqrt{3\cdot 8}}g_4(r)\cdot\frac{4}{\sqrt{3}}g_4(r)\Big) + c_{z5}c_5\Big(-\frac{\sqrt{8}}{3}g_7(r)\cdot\frac{8}{3}g_7(r)\Big) \\ & + c_{z5}c_z\Big(-\frac{\sqrt{8}}{3}g_7(r)\cdot\frac{1}{3}g_7(r)\Big) + c_{z5}^2\Big(\frac{8}{3}g_7(r)\frac{1}{3}g_7(r) + \Big(-\frac{\sqrt{8}}{3}g_7(r)\Big)\Big(-\frac{\sqrt{8}}{3}g_7(r)\Big)\Big)dxdy. \end{aligned} \tag{2.6}$$

Since

$$\int_{[-N,N]^2}\int_{[-N,N]^2} g_2^2(r)dxdy \sim \int_{[-N,N]^2}\int_{[-N,N]^2} g_4^2(r)dxdy \sim \int_{[-N,N]^2}\int_{[-N,N]^2} g_7^2(r)dxdy$$

(see Lemma A.6), the sum in (2.6) equals asymptotically

$$\frac{1}{32\pi}\phi(u)^2u^2(u^2-1)^2\int_{[-N,N]^2}\int_{[-N,N]^2} g_2^2(r)dx\,dy.$$

This can be seen by plugging in the values of c_2, c_5, c_z, and c_{z5} and comparing term by term. For instance, the term containing $\Phi(-u)^2$ comes from the terms containing c_5 and c_{z5} and equals asymptotically

$$\int_{[-N,N]^2}\int_{[-N,N]^2} g_2^2(r)dxdy\cdot\left(\frac{64}{9\cdot 2\pi}\frac{1}{16} + \frac{2\cdot 8}{9\pi}\frac{1}{8} - \frac{2\cdot 8\sqrt{8}}{9\pi}\frac{1}{4\sqrt{8}}\right)\Phi(-u) = 0,$$

i.e. it is negligible with respect to the dominating contribution and does not appear in the final result. All the other terms are treated similarly.

We now recall the definition of the υ function in (1.10). It follows that, as $N \to \infty$, the variance of the modified EPC is asymptotically equivalent to

$$\frac{1}{32\pi}\upsilon(u)\int_{[-N,N]^2}\int_{[-N,N]^2} g_2^2(\|x-y\|)dxdy.$$

The last integral is evaluated in Lemma A.6 and it is equal to

$$4(2N)^3\,\frac{1}{\pi}\left(\frac{1-\sqrt{2}}{3}+\ln(1+\sqrt{2})\right).$$

This concludes the proof of Proposition 2.1.

2.2. *Proof of Proposition 2.2.* In Proposition 2.1 we proved that the Euler-Poincaré characteristic behaves as the second chaotic projection, namely

$$\varphi(f,T_N,u)-\mathbb{E}[\varphi(f,T_N,u)]=\varphi(f,T_N,u)[2]+o(\varphi(f,T_N,u)[2]).$$

Moreover, recalling that

$$\mathcal{D}_2(f,N)=\int_{[-N,N]^2}\frac{1}{2}c_2H_2(Y_2(x))+\frac{1}{2}c_5H_2(Y_5(x))+\frac{1}{2}c_zH_2(Z(x))+c_{z5}H_1(Z(x))H_1(Y_5(x))dx,$$

(see Equation (2.5)) we also proved that

$$\varphi(f,T_N,u)[2]=\mathcal{D}_2(f,N)+O(N^2\sqrt{N}).$$

We now show that the entire Euler-Poincaré functional behaves as the second chaos projection of the initial field f (as it happens for all the three Lipschitz-Killing curvatures in the sphere, see for example Cammarota and Marinucci, 2018, Equation (7) and Vidotto, 2022). We note that

$$H_2(f(x))=\frac{1}{3}H_2(Z(x))+\frac{2}{3}H_2(Y_5(x))+\frac{2\sqrt{2}}{3}H_1(Z(x))H_1(Y_5(x)),$$

since f is a Laplace eigenfunction. We can now compute the necessary covariances and variances explicitly, term by term. We recall the notation $r:=\|x-y\|$. Applying the Diagram Formula (see, e.g., Marinucci and Peccati, 2011, Section 4.3.1), we have that

$$\begin{aligned}
\mathbb{E}\left[\mathcal{D}_2(f,N)\mathcal{S}(f,N)\right]=\int_{[-N,N]^2}\int_{[-N,N]^2}&\frac{1}{2}c_2\frac{1}{3}2\left(\frac{4}{\sqrt{3\cdot 8}}g_4(r)\right)^2+2\frac{1}{2}c_2\frac{2}{3}\left(\frac{4}{\sqrt{3}}g_4(r)\right)^2\\
&+2c_2\frac{\sqrt{2}}{3}\left(-\frac{4}{\sqrt{3\cdot 8}}g_4(r)\cdot\frac{4}{\sqrt{3}}g_4(r)\right)+2\frac{1}{2}c_5\frac{1}{3}\left(-\frac{\sqrt{8}}{3}g_7(r)\right)^2\\
&+2\frac{1}{2}c_5\frac{2}{3}\left(\frac{8}{3}g_7(r)\right)^2+2c_5\frac{\sqrt{2}}{3}\left(-\frac{\sqrt{8}}{3}g_7(r)\cdot\frac{8}{3}g_7(r)\right)\\
&+2\frac{1}{2}c_z\frac{1}{3}\left(\frac{1}{3}g_7(r)\right)^2+2\frac{1}{2}c_z\frac{2}{3}\left(-\frac{\sqrt{8}}{3}g_7(r)\right)^2+2c_z\frac{\sqrt{2}}{3}\left(-\frac{\sqrt{8}}{3}g_7(r)\cdot\frac{1}{3}g_7(r)\right)\\
&+2c_{z5}\frac{1}{3}\left(-\frac{\sqrt{8}}{3}g_7(r)\cdot\frac{1}{3}g_7(r)\right)+2c_{z5}\frac{2}{3}\left(-\frac{\sqrt{8}}{3}g_7(r)\cdot\frac{8}{3}g_7(r)\right)\\
&+c_{z5}\frac{\sqrt{2}}{3}2\left(\frac{8}{3}g_7(r)\frac{1}{3}g_7(r)+\left(-\frac{\sqrt{8}}{3}g_7(r)\right)\left(-\frac{\sqrt{8}}{3}g_7(r)\right)\right).
\end{aligned}$$

Due to the equivalence

$$\int_{[-N,N]^2}\int_{[-N,N]^2} g_2^2(r)dxdy \sim \int_{[-N,N]^2}\int_{[-N,N]^2} g_4^2(r)dxdy \sim \int_{[-N,N]^2}\int_{[-N,N]^2} g_7^2(r)dxdy$$

(see Lemma A.6), expression (2.7) is equivalent to

$$\frac{1}{4\sqrt{\pi}}\phi(u)u(u^2-1)\int_{[-N,N]^2}\int_{[-N,N]^2} g_7^2(r)dxdy. \tag{2.7}$$

Moreover, by direct calculation we obtain

$$\mathbb{E}\left[\left(\int_{[-N,N]^2} H_2(f(x))dx\right)^2\right] = 2\int_{[-N,N]^2}\int_{[-N,N]^2} g_7^2(r)dxdy. \tag{2.8}$$

The value of this integral is given in Lemma A.6, postponed to Appendix A. Finally, by using the variance of $\mathcal{D}_2(f,N)$ in Proposition 2.1, the expression in (2.8) and the covariance in (2.7), we get the thesis of Proposition 2.2.

3. Proof of Central Limit Theorem 1.3

First, note that the Berry random field f in (1.3) satisfies regularity assumptions stated in Remark 2 of Di Bernardino et al. (2024), which are sufficient to ensure that the Gaussian Kinematic Formula (GKF, see Theorem 13.2.1 in Adler and Taylor, 2007 or Theorem 4.8.1 in Adler and Taylor, 2011) can be applied. Then, using the tower property of conditional expectation, one can write $\mathbb{E}[\varphi(A(f_\Lambda,T_N,u))] = \mathbb{E}[\mathbb{E}[\varphi(A(f,T_N,h(u,\Lambda)))|\Lambda]]$, to get the expected modified Euler-Poincaré characteristic for perturbed Berry's random field f_Λ in (1.11) (see *e.g.*, Section 4.1 in Di Bernardino et al., 2024). This yields, using Equation (23) in Di Bernardino et al. (2024), to the desired expression $\mathbb{E}[\varphi(A(f_\Lambda,T_N,u))] = \mathcal{L}_2(T_N)\,\lambda_f\,\mathbb{E}[\rho_2(h(u,\Lambda))]$ in (1.13).

We now state and prove the following result which provides the asymptotic variance of the modified EPC of the perturbed model in (1.11).

Proposition 3.1. *By using Equation (1.12), we get*

$$\begin{aligned}(2N)^{-3}\mathrm{Var}(\varphi(f_\Lambda,T_N,u)) &= (2N)^{-3}\mathrm{Var}(\varphi(f,T_N,h(u,\Lambda)) \\ &= \mathbb{E}\left[v\left(h(u,\Lambda)\right)\right]\frac{1}{8\pi^2}\left(\frac{1-\sqrt{2}}{3}+\ln(1+\sqrt{2})\right)+O\left(\frac{1}{\sqrt{N}}\right),\end{aligned}$$

with v as in Equation (1.10).

Proof of Proposition 3.1: In Proposition 2.1 we proved that the second chaotic projection is the leading term of the series expansion in (1.7) and then

$$\varphi(f,T_N,u)-\mathbb{E}[\varphi(f,T_N,u)] = \varphi(f,T_N,u)[2]+\sum_{q=1,q\neq 2}^{\infty}\sum_{\substack{\overline{n}\in\mathbb{N}^5,\\ |n|=q}}\langle G_u,\overline{H_n}\rangle h_n$$

where $\varphi(f,T_N,u)[2] = \mathcal{D}_2(f,N)+O(N^2\sqrt{N})$. Therefore, by independence and monotone convergence theorem, conditioning with respect to Λ and exploiting Proposition 2.1, in particular re-adapting computations in (2.6) to the perturbed model, we obtain, as $N \to \infty$,

$$\begin{aligned}
&\mathbb{E}\big[(\varphi(f_\Lambda, T_N, u) - \mathbb{E}[\varphi(f_\Lambda, T_N, u)])^2\big] \\
&\qquad = \frac{1}{32\pi}\mathbb{E}\left[\upsilon\left(h(u,\Lambda)\right)\right]\mathbb{E}[F_N^2] + \mathbb{E}\left[\sum_{\substack{q=1,\\ q\neq 2}}^{\infty}\sum_{\substack{\overline{n}\in\mathbb{N}^5,\\ |n|=q}}\langle G_{h(u,\Lambda)},\, \overline{H_n}\rangle^2 h_n^2\right] \\
&\qquad = \frac{1}{32\pi}\mathbb{E}\left[\upsilon\left(h(u,\Lambda)\right)\right]\mathbb{E}[F_N^2] + \sum_{\substack{q=1,\\ q\neq 2}}^{\infty}\sum_{\substack{n\in\mathbb{N}^5,\\ |n|=q}}\mathbb{E}\left[\langle G_{h(u,\Lambda)},\, \overline{H_n}\rangle^2\right]\mathbb{E}\left[h_n^2\right],
\end{aligned}$$

with $\upsilon(u)$ defined in (1.10) and

$$\mathbb{E}[F_N^2] = \int_{[-N,N]^2}\int_{[-N,N]^2} g_2^2(\|x-y\|)dxdy,$$

which has been computed in Lemma A.6. We write $\upsilon(u,q) := \mathbb{E}\left[\langle G_{h(u,\Lambda)},\, \overline{H_n}\rangle^2\right]$ and obtain that the variance equals

$$\frac{1}{32\pi}\mathbb{E}\left[\upsilon\left(h(u,\Lambda)\right)\right]\mathbb{E}[F_N^2] + \sum_{\substack{q=1,\\ q\neq 2}}^{\infty}\sum_{\substack{\overline{n}\in\mathbb{N}^5,\\ |\overline{n}|=q}}\upsilon(u,q)\mathbb{E}\left[h_n^2\right].$$

We can note that for fixed u the second summand is dominated by the first summand in N, and therefore the variance is of order of the first summand and this concludes the proof. □

Finally, let us prove the CLT for Berry perturbed field given in Theorem 1.3.

Proof of Theorem 1.3: We show convergence of the distribution function. Let us denote by D_Λ the image of Λ and F_Λ the distribution function of Λ. Let $x \in \mathbb{R}$, then we have by the law of total probability

$$\begin{aligned}
&\mathbb{P}\left(\frac{\varphi(f,T_N,h(u,\Lambda)) - \mathbb{E}[\varphi(f,T_N,h(u,\Lambda))]}{\sqrt{Var(\varphi(f,T_N,h(u,\Lambda)))}} \le x\right) \\
&\quad = \int_{D_\Lambda}\mathbb{P}\left(\frac{\varphi(f,T_N,h(u,\Lambda)) - \mathbb{E}[\varphi(f,T_N,h(u,\Lambda))]}{\sqrt{Var(\varphi(f,T_N,h(u,\Lambda)))}} \le x \,\middle|\, \Lambda = \lambda\right)dF_\Lambda(\lambda) \\
&\quad = \int_{D_\Lambda}\mathbb{P}\left(\frac{\varphi(f,T_N,h(u,\lambda)) - \mathbb{E}[\varphi(f,T_N,h(u,\lambda))]}{\sqrt{Var(\varphi(f,T_N,h(u,\lambda)))}} \le x \,\middle|\, \Lambda = \lambda\right)dF_\Lambda(\lambda) \\
&\quad = \int_{D_\Lambda}\mathbb{P}\left(\frac{\varphi(f,T_N,h(u,\lambda)) - \mathbb{E}[\varphi(f,T_N,h(u,\lambda))]}{\sqrt{Var(\varphi(f,T_N,h(u,\lambda)))}} \le x\right)dF_\Lambda(\lambda),
\end{aligned}$$

since f and Λ are independent. Denoting by Z, as usual, a standard Gaussian random variable, by the dominated convergence theorem combined with the unperturbed CLT given in Theorem 1.1, this expression tends to

$$\int_{D_\Lambda}\mathbb{P}\left(Z \le x\right)dF_\Lambda(\lambda) = \mathbb{P}\left(Z \le x\right),$$

as N tends to infinity, and the statement is proved. □

4. Numerical studies

In this section, we provide numerical studies for some particular perturbed Berry's random wave models to illustrate the performance on finite size samples of the obtained results.

4.1. *Some parametric perturbed Berry's random wave models.* Notice that for some choices of the random shape variable Λ and a specific strictly increasing link function g in (1.11) the variance in Theorem 1.3 can be explicitly computed. By Equation (1.14), this means computing the term $\mathbb{E}\left[\upsilon\left(h(u,\Lambda)\right)\right]$, with function υ as in Equation (1.10) and $h(\cdot,\lambda)$ the inverse function of $g(\cdot,\lambda)$ in the perturbed model in (1.11). One can design several scenarios in which this term can be determined. In particular, if the perturbation is a scaling or a location shift, it suffices to know the value of $\mathbb{E}[\phi\left(h(u,\Lambda)\right)]$. In the following we provide an exemplary illustration for such a computation.

Suppose that the link function is a product, that is, $h(u,\Lambda) = \frac{u}{\Lambda}$. The authors in Di Bernardino et al. (2024) computed explicitly $\mathbb{E}\left[\rho_j\left(\frac{u}{\Lambda}\right)\right]$ for $j = 0,1,2$, where $\rho_j(x) = (2\pi)^{-(1+j)/2}e^{-x^2/2}H_{j-1}(x)$, for different choices of Λ. Using the formula for $j = 2$, they could obtain an expression for the expectation of the EPC (see Equation (1.13)). We shall make use of these calculations to compute the asymptotic variance of the modified EPC.

First, we can combine the recursive formula

$$\rho_{j+1}(u) = -(2\pi)^{-1/2}\rho_j'(u)$$

(consequence of 22.8.8 in Abramowitz and Stegun, 1964) and the relation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$$

(see 22.7.14 in Abramowitz and Stegun, 1964) to obtain the expression

$$\mathbb{E}\left[\rho_j\left(\frac{u}{\Lambda}\right)\right] = -\frac{1}{2\pi}\left(u\frac{d}{du}\mathbb{E}\left[\rho_{j-2}\left(\frac{u}{\Lambda}\right)\right] + (j-2)\mathbb{E}\left[\rho_{j-2}\left(\frac{u}{\Lambda}\right)\right]\right).$$

Note also that, letting $\tilde{u} = \sqrt{2}\,u$, one can write

$$\mathbb{E}\left[\upsilon\left(\frac{u}{\Lambda}\right)\right] = \mathbb{E}\left[\phi\left(\frac{u}{\Lambda}\right)^2\left(\frac{u^2}{\Lambda^2}-1\right)^2\frac{u^2}{\Lambda^2}\right] = \mathbb{E}\left[\frac{e^{-\frac{u^2}{\Lambda^2}}}{2\pi}\left(\frac{u^2}{\Lambda^2}-1\right)^2\frac{u^2}{\Lambda^2}\right] = \mathbb{E}\left[\frac{e^{-\frac{\tilde{u}^2}{2\Lambda^2}}}{2\pi}\left(\frac{\tilde{u}^2}{2\Lambda^2}-1\right)^2\frac{\tilde{u}^2}{2\Lambda^2}\right].$$

Expanding the polynomial part of the expression with Hermite polynomials yields

$$\begin{aligned}
&\mathbb{E}\left[\upsilon\left(\frac{u}{\Lambda}\right)\right]\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{8}\mathbb{E}\left[\phi\left(\frac{\tilde{u}}{\Lambda}\right)H_6\left(\frac{\tilde{u}}{\Lambda}\right)\right] + \frac{11}{8}\mathbb{E}\left[\phi\left(\frac{\tilde{u}}{\Lambda}\right)H_4\left(\frac{\tilde{u}}{\Lambda}\right)\right]\right.\\
&\qquad\left. + \frac{25}{8}\mathbb{E}\left[\phi\left(\frac{\tilde{u}}{\Lambda}\right)H_2\left(\frac{\tilde{u}}{\Lambda}\right)\right] + \frac{7}{8}\mathbb{E}\left[\phi\left(\frac{\tilde{u}}{\Lambda}\right)H_0\left(\frac{\tilde{u}}{\Lambda}\right)\right]\right)\\
&= \frac{1}{8}\left((2\pi)^3\mathbb{E}\left[\rho_7\left(\frac{\tilde{u}}{\Lambda}\right)\right] + 11(2\pi)^2\mathbb{E}\left[\rho_5\left(\frac{\tilde{u}}{\Lambda}\right)\right] + 25(2\pi)\mathbb{E}\left[\rho_3\left(\frac{\tilde{u}}{\Lambda}\right)\right] + 7\mathbb{E}\left[\rho_1\left(\frac{\tilde{u}}{\Lambda}\right)\right]\right).
\end{aligned} \tag{4.1}$$

Heavy-tail perturbation. Let Λ such that Λ^2 is a random variable with a Pareto (Type I) distribution with parameter $\alpha > 0$ ($\Lambda^2 \sim \text{Pa}(\alpha)$), i.e., $\mathbb{P}(\Lambda^2 > x) = x^{-\alpha}$, for $x > 1$ and $\alpha > 0$. Then

$$\begin{aligned}
\mathbb{E}\left[\rho_0\left(\frac{u}{\Lambda}\right)\right] &= u^{-2\alpha}\,2^{\alpha-1}(\pi)^{-1/2}\,\gamma(\alpha+1/2, u^2/2) + \overline{\Phi}(u),\\
\mathbb{E}\left[\rho_1\left(\frac{u}{\Lambda}\right)\right] &= u^{-2\alpha}\,2^{\alpha-1}(\pi)^{-1}\,\alpha\,\gamma(\alpha, u^2/2),\\
\mathbb{E}\left[\rho_2\left(\frac{u}{\Lambda}\right)\right] &= u^{-2\alpha}\,2^{\alpha-1}(\pi)^{-3/2}\,\alpha\,\gamma(\alpha+1/2, u^2/2),
\end{aligned}$$

where $\gamma(a,\cdot)$ stands for the lower incomplete Gamma function, *i.e.*, $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$, for $a > 0, x > 0$ and $\overline{\Phi}$ for the survival standard normal cumulative distribution function. Then we have

$$\begin{aligned}
\mathbb{E}\left[\rho_3\left(\frac{u}{\Lambda}\right)\right] &= -\frac{1}{(2\pi)}2^{\alpha-1}(\pi)^{-1}\,\alpha(u^{-2\alpha}\gamma(\alpha,u^2/2)(1-2\alpha)+2^{1-\alpha}e^{-u^2/2}),\\
\mathbb{E}\left[\rho_5\left(\frac{u}{\Lambda}\right)\right] &= \frac{1}{(2\pi)^2}2^{\alpha-1}(\pi)^{-1}\,\alpha(u^{-2\alpha}\gamma(\alpha,u^2/2)(4\alpha^2-8\alpha+3)+2^{1-\alpha}e^{-u^2/2}(-2\alpha-u^2+4)),\\
\mathbb{E}\left[\rho_7\left(\frac{u}{\Lambda}\right)\right] &= -\frac{1}{(2\pi)^3}2^{\alpha-1}(\pi)^{-1}\,\alpha(u^{-2\alpha}\gamma(\alpha,u^2/2)(-8\alpha^3+36\alpha^2-46\alpha+15)\\
&\qquad +2^{1-\alpha}e^{-u^2/2}(4\alpha^2+2\alpha u^2-18\alpha+u^4-11u^2+23)),
\end{aligned}$$

and then from (4.1)

$$\begin{aligned}
\mathbb{E}\left[\upsilon\left(\frac{u}{\Lambda}\right)\right] &= \frac{1}{8}2^{\alpha-1}(\pi)^{-1}\,\alpha\big((\sqrt{2}u)^{-2\alpha}\gamma(\alpha,u^2)(8\alpha^3+8\alpha^2+8\alpha)-\\
&\qquad 2^{1-\alpha}e^{-u^2}((\sqrt{2}u)^4+2\alpha(\sqrt{2}u)^2+4\alpha^2+4\alpha+4)\big). \qquad (4.2)
\end{aligned}$$

Light-tail perturbation. Let us consider Λ such that $\Lambda^2 \sim \text{Exp}(\theta)$. In Di Bernardino et al. (2024) the authors show that

$$\mathbb{E}\left[\rho_0\left(\frac{u}{\Lambda}\right)\right]=\frac{1}{2}e^{-au},\quad \mathbb{E}\left[\rho_1\left(\frac{u}{\Lambda}\right)\right]=\frac{1}{2\pi}auK_1(au),\quad \mathbb{E}\left[\rho_2\left(\frac{u}{\Lambda}\right)\right]=\frac{1}{4\pi}aue^{-au},$$

with $a=\sqrt{2/\theta}$. Then, we have

$$\begin{aligned}
\mathbb{E}\left[\rho_3\left(\frac{u}{\Lambda}\right)\right] &= -\frac{1}{(2\pi)^2}(-a^2u^2K_0(au)+auK_1(au)),\\
\mathbb{E}\left[\rho_5\left(\frac{u}{\Lambda}\right)\right] &= \frac{1}{(2\pi)^3}(-6a^2u^2K_0(au)+K_1(au)(a^3u^3+3au)),\\
\mathbb{E}\left[\rho_7\left(\frac{u}{\Lambda}\right)\right] &= -\frac{1}{(2\pi)^4}(K_1(au)(13a^3u^3+15au)-(45a^2u^2+a^4u^4)K_0(au)),
\end{aligned}$$

where K_ν is modified Bessel functions of the second kind of order ν. From Equation (4.1), we get

$$\mathbb{E}\left[\upsilon\left(\frac{u}{\Lambda}\right)\right]=\frac{1}{8(2\pi)}\left(a^4\tilde{u}^4K_0(a\tilde{u})-2a^3\tilde{u}^3K_1(a\tilde{u})+4a^2\tilde{u}^2K_0(a\tilde{u})\right).$$

4.2. *Simulated data results.* Since this work provides asymptotic results for growing observation domains, in this section we aim to numerically illustrate the performance on finite size samples of the obtained results. In particular, we will focus on the asymptotic variances and the asymptotic Gaussian fluctuations. Simulations are provided using the `R` package `RandomFields` (see Schlather et al., 2015). To illustrate the asymptotic Gaussian fluctuations stated in Theorem 1.1 for growing domain (i.e., $T=[-N,N]^2$ and $N\to\infty$), in Figure 4.3 we display the p-values of the Shapiro-Wilk Normality Test for the statistics $(\varphi(f,T_N,u)-\mathbb{E}[\varphi(f,T_N,u)])/\sqrt{(2N)^3V(u)}$, with $V(u)$ prescribed by Equation (1.9), as a function of levels u and for several choices of N: $N=70$ (left panel), $N=100$ (center panel), $N=200$ (right panel). The theoretical expected $\mathbb{E}[\varphi(f,T_N,u)]$ is given in Equation (1.8) and the estimation of the EPC φ is evaluated on Berry's Gaussian samples in $T_N=[-N,N]^2$ and $2\,N$ pixels for sides. Notice the global increasing performance of the $p-$values in terms of N and in particular for extreme levels u (from left to right panels in Figure 4.3).

Then, we consider the Berry mixture model f_Λ as in Equation (1.11). In Figure 4.4 we display a realisation of the considered f_Λ random field with $h(u,\Lambda)=\frac{u}{\Lambda}$, where $\Lambda^2\sim\text{Pa}(\alpha)$ with $\alpha=4$ and associated excursions set. Notice that the choice of this Pareto mixture clearly impacts on the standard deviation of the perturbed random field (see Figure 4.4). For this perturbed Berry's model, in Figure 4.5 we illustrate the EPC first and second moments behavior. In particular, we aim to investigate the adequacy between estimated EPC moments and theoretical predicted ones. In Figure 4.5 (fist and second panels) we display the boxplots of the estimated EPC values on

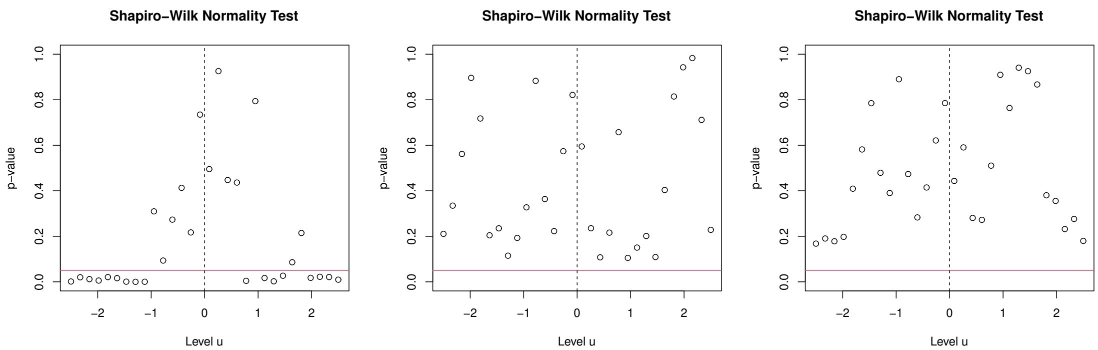

FIGURE 4.3. We display the p-values of the Shapiro-Wilk Normality Test for the statistics $(\varphi(f, T_N, u) - \mathbb{E}[\varphi(f, T_N, u)])/\sqrt{(2N)^3 V(u)}$ as a function of levels u and for several choices of the window $T_N = [-N, N]^2$; $N = 70$ (left panel), $N = 100$ (center panel), $N = 200$ (right panel). Estimation of the EPC φ is provided on 250 Berry's Gaussian samples.

500 mixture Berry samples with $\alpha = 2$ (first panel) and $\alpha = 4$ (second panel) in $[-70, 70]^2$ with 140 pixels for sides. Unsurprisingly, the heavier-tail model for $\alpha = 2$ exhibits a higher variance in the EPC estimation (see first panel in Figure 4.5). For the sake of comparison, in Figure 4.5 (first and second panels), we add the theoretical expectation $u \to \mathbb{E}[\varphi(A(f_\Lambda, [-70, 70]^2, u))]$, for $u \in [-3.5, 3.5]$ (red curve), by using Equation (1.13) and the $\mathbb{E}[\rho_j(\frac{u}{\Lambda}))]$'s expressions obtained in Section 4.1 (*Heavy-tail perturbation*). Finally, the theoretical variance $u \to V_\Lambda(u)$, for $u \in [-3.5, 3.5]$ obtained by Equations (1.14) and (4.2) is displayed in the third panel of Figure 4.5 by using a red curve. The empirical variances obtained on 1000 Monte Carlo simulations are displayed in black dots (see third panel in Figure 4.5). We see that the prescribed asymptotic variance and the empirical counterpart estimated on the $[-70, 70]^2$ domain are in good agreement.

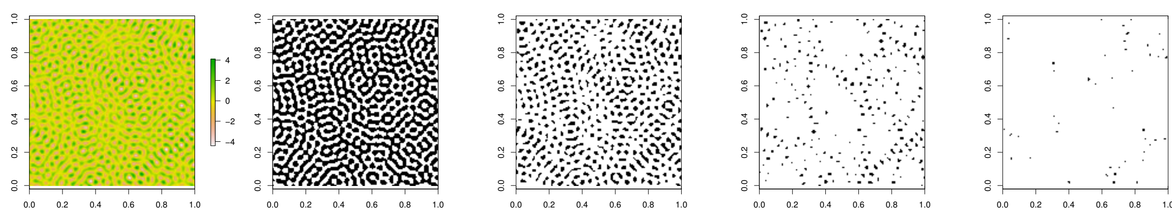

FIGURE 4.4. A random generation in $[-70, 70]^2$ of a mixture Berry random field f_Λ as in Equation (1.11) with $h(u, \Lambda) = \frac{u}{\Lambda}$, where $\Lambda^2 \sim \mathrm{Pa}(\alpha)$ and $\alpha = 4$ (first panel) with associated excursion sets $A(f_\Lambda, [-70, 70]^2, u)$ in Equation (1.2) for $u = 0, 1, 2, 3$ (from the second to the fifth panel). Here we take 140 pixels for side.

Appendix A. Technical lemmas

In this section we collect all the technical results exploited in the proof of Proposition 2.1 which give the asymptotic behaviour of the chaos components of the modified EPC expansion. In particular, we first state Lemma A.1 which is an auxiliary result used in the proof of these technical lemmas below. Then, Lemma A.2, A.4, A.5 and Remark A.3 imply that the asymptotic behaviour of all the terms in the q-chaos for $q \neq 2$ is $o(N^3)$, as $N \to \infty$. Lemma A.6 shows that the second chaos behaves as N^3 and hence it is the leading term of the chaos expansion. Finally in Lemma A.7 we compute the constants of the dominant terms of the second chaotic projection.

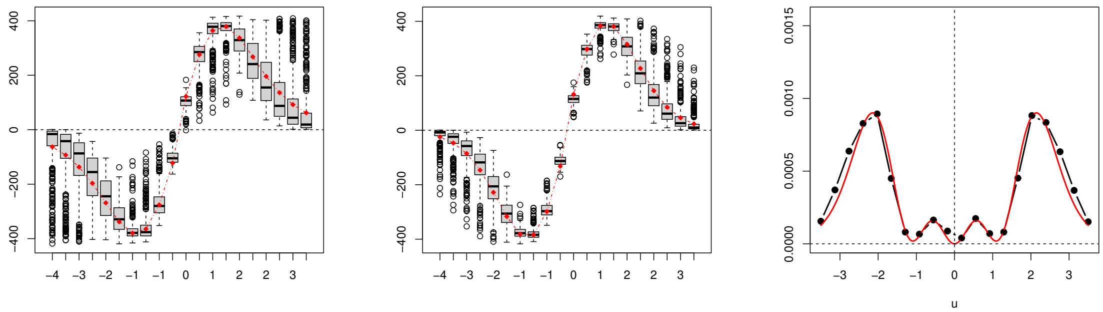

FIGURE 4.5. We consider the perturbed model f_Λ as in Equation (1.11), with $h(u,\Lambda) = \frac{u}{\Lambda}$, where $\Lambda^2 \sim \text{Pa}(\alpha)$ $\alpha = 2$ (first panel), $\alpha = 4$ (second and third panels). Theoretical $u \to \mathbb{E}[\varphi(A(f_\Lambda, [-70,70]^2, u))]$ (first and second panels) and $u \to V_\Lambda(u)$ (third panel), for $u \in [-3.5, 3.5]$ are drawn in red line. Empirical counterparts are represent by using black boxplots on 500 sample simulations (first and second panel) and black dots on 1000 sample simulations (third panel). Here we take 140 pixels for side.

To prove some of the technical lemmas we first need the following simple result.

Lemma A.1. *Let $h : \mathbb{R} \to \mathbb{R}^+$ be a non-negative function. Then, assuming the integrals are well-defined,*

$$\int_{[-N,N]^2} \int_{[-N,N]^2} h(\|x-y\|) dx\, dy \le 8N^2\pi \int_0^{2\sqrt{2}N} h(r) r dr.$$

Proof of Lemma A.1: By a change of variables, we get

$$\int_{[-N,N]^2} \int_{[-N,N]^2} h(\|x-y\|) dx dy = \int_{-2N}^{2N} \int_{-2N}^{2N} (2N - |x_1|)(2N - |x_2|) h\left(\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|\right) dx_1 dx_2,$$

which can then be bounded by

$$4N^2 \int_{-2N}^{2N} \int_{-2N}^{2N} h\left(\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|\right) dx_1 dx_2 \le 4N^2 \int_{B_{2\sqrt{2}N}(0)} h\left(\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|\right) dx_1 dx_2 = 8N^2\pi \int_0^{2\sqrt{2}N} h(r) r dr,$$

with a switch to radial coordinates. □

The following lemma provides some auxiliary statements for the bounds of the chaos components.

Lemma A.2. *We have for $\alpha \in \mathbb{N}^7, |\alpha| = q$, that, as $N \to \infty$,*

$$\int_0^{2\sqrt{2}N} r \prod_{i=1}^{7} |g_i(r)|^{\alpha_i}\, dr = o(N)$$

for $q \ge 3$ and for $q \ge 1$ if at least one of α_1, α_3, α_5, α_6 is nonzero. Moreover,

$$\left| \int_0^{2N\sqrt{2}} r g_i(r) dr \right| = o(N)$$

for $q = 1$ in either case.

Proof of Lemma A.2: First recall the following asymptotic formula for J_m, given, for instance, in 9.2.1 in Abramowitz and Stegun (1964):

$$J_m(r) = \sqrt{\frac{2}{\pi r}} \left(\cos\left(r - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{r}\right) \right), \tag{A.1}$$

as well as the fact that Bessel functions are uniformly bounded by 1.
For $q=1$, the statement in the lemma is obvious for g_1, g_3, g_5 and g_6. More precisely, we have for some constant K and $i=1$, 3, 5, or 6

$$\left|\int_0^{2\sqrt{2}N} r g_i(r)dr\right| \lesssim K + \int_1^{2\sqrt{2}N} \frac{1}{\sqrt{r}}dr \lesssim \sqrt{N} = o(N)$$

thanks to the asymptotics and the bound mentioned above. The proof for $q \geq 3$ follows along the same lines.

For $q=1$ and g_2, g_4, and g_7 we can compute the integral explicitly:

$$\begin{aligned}
&\int_0^{2\sqrt{2}N} r g_2(r)dr = J_0(2\sqrt{2}N) + 2\sqrt{2}NJ_1(2\sqrt{2}N) - 1,\\
&\int_0^{2\sqrt{2}N} r g_4(r)dr = -2\sqrt{2}NJ_2(2\sqrt{2}N),\\
&\int_0^{2\sqrt{2}N} r g_7(r)dr = 2\sqrt{2}NJ_1(2\sqrt{2}N) - \frac{J_1(2\sqrt{2}N)}{2\sqrt{2}N} + J_0(2\sqrt{2}N) - J_2(2\sqrt{2}N) - \frac{1}{2}.
\end{aligned}$$

Again, due to the asymptotics of the Bessel functions, these integrals are sublinear in N. □

Remark A.3. We also have

$$\left|\int_0^{2\sqrt{2}N} r^2 g_2(r)dr\right| \lesssim N\sqrt{N}, \quad \left|\int_0^{2\sqrt{2}N} r^2 g_4(r)dr\right| \lesssim N\sqrt{N}, \quad \left|\int_0^{2\sqrt{2}N} r^2 g_7(r)dr\right| \lesssim N\sqrt{N}$$

as well as

$$\left|\int_0^{2\sqrt{2}N} r^3 g_2(r)dr\right| \lesssim N^2\sqrt{N}, \quad \left|\int_0^{2\sqrt{2}N} r^3 g_4(r)dr\right| \lesssim N^2\sqrt{N}, \quad \left|\int_0^{2\sqrt{2}N} r^3 g_7(r)dr\right| \lesssim N^2\sqrt{N}.$$

by direct calculation, using the identity 6.561 in Gradshteyn and Ryzhik (2007) together with the asymptotics of the Bessel and Lommel functions (Equation (A.1) and an approximation result from Dingle, 1959 respectively).

The following three lemmata (see Lemmas A.4, A.5 and A.6 below) provide bounds and/or precise calculations of all chaos components of the modified Euler-Poincaré characteristic.

Lemma A.4. *The variance of chaos components of order* 3 *or higher, as well as all summands in the chaos decomposition containing at least one of the factors* $\partial_1 f$, $\partial_{11} f$, *and* $\partial_{12} f$ *is of order* $o(N^3)$.

Proof of Lemma A.4: This is a direct consequence of Lemma A.1 and Lemma A.2. □

Lemma A.5. *The variance of the first chaos component involving the factors* $\partial_2 f$ *and/or* $\partial_{22} f$ *is of order* $o(N^3)$, *that is,*

$$\left|\int_{[-N,N]^2}\int_{[-N,N]^2} g_i(\|x-y\|)dxdy\right| = o(N^3)$$

for $i \in \{2, 4, 7\}$.

Proof of Lemma A.5: By a change of variables, we can write

$$\int_{[-N,N]^2}\int_{[-N,N]^2} g_i(\|x-y\|)dxdy = \\ \int_{-2N}^{2N}\int_{-2N}^{2N} (2N-|x_1|)(2N-|x_2|)g_i\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)dx_1dx_2.$$

This integral can be decomposed into the integral over the inscribed ball $B_{2N}(0)$ and the integral over the remaining domain. The integral over $B_{2N}(0)$ can be rewritten with spherical coordinates, and the bound then follows by the second statement of Lemma A.2 and Remark A.3. As for the remaining domain (the four "corners" of the rectangle), note that $g_2\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)$, $g_4\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)$ and $g_7\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)$ are radial functions that decay as $\frac{1}{\sqrt{N}}$ as N tends to infinity. Moreover, their zero sets are (asymptotically) concentric circles with radii that are multiples of 2π. Due to these properties, and also since the area of the remaining domain decays with distance from the origin, we can bound the integral over the remainder by two times the integral between two zero set circles closest to $\partial B_{2N}(0)$ of the absolute value of g_2, g_4 or g_7 times the factor $(2N-|x_1|)(2N-|x_2|)$. This integral is asymptotically bounded by the area of integration $(2\pi(2N)\cdot 2\pi)$ multiplied by a bound over the integrand $(2N)^2\frac{1}{\sqrt{N}}$. This proves the statement also for the remaining domain. □

Lemma A.6. *We have*

$$\begin{aligned}
&\int_{-2N}^{2N}\int_{-2N}^{2N}(2N-|x_1|)(2N-|x_2|)g_2^2\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)dx_1dx_2\\
&\quad\sim\int_{-2N}^{2N}\int_{-2N}^{2N}(2N-|x_1|)(2N-|x_2|)g_4^2\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)dx_1dx_2\\
&\quad\sim\int_{-2N}^{2N}\int_{-2N}^{2N}(2N-|x_1|)(2N-|x_2|)g_7^2\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)dx_1dx_2\\
&\quad=4(2N)^3\,\frac{1}{\pi}\left(\frac{1-\sqrt{2}}{3}+\ln(1+\sqrt{2})\right)+O(N^2\sqrt{N}).
\end{aligned}$$

Proof of Lemma A.6: First note that we can infer from the recurrence relations between Bessel functions that $g_2(r)\sim g_7(r)\sim J_0(r)$ and $g_4(r)\sim -J_1(r)$ as $r\to\infty$. Due to the asymptotics (A.1) we moreover obtain

$$J_0^2(r)\sim\frac{2}{\pi r}\cos^2(r-\pi/4)=\frac{1}{\pi r}(\cos(2r-\pi/2)+1)$$

as well as

$$J_1^2(r)\sim\frac{2}{\pi r}\cos^2(r-\pi/2-\pi/4)=\frac{1}{\pi r}(\cos(2r+\pi/2)+1).$$

Note that, since the integrals appearing in this lemma all tend to infinity with $N\to\infty$, replacing Bessel functions with their approximations up to an additive error of a smaller order is not going to change the asymptotics (nor the relevant constants). Therefore,

$$\begin{aligned}
&\int_{-2N}^{2N}\int_{-2N}^{2N}(2N-|x_1|)(2N-|x_2|)\left(g_2^2\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)-g_4^2\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)\right)dx_1dx_2\\
&\quad\sim\int_{-2N}^{2N}\int_{-2N}^{2N}(2N-|x_1|)(2N-|x_2|)\frac{1}{\pi\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|}2\cos\left(2\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|+\frac{\pi}{2}\right)dx_1dx_2,
\end{aligned}$$

and we can see with the same argument as in the previous lemma that this integral is of order $o(N^3)$.

To show that all the integrals are indeed of order N^3, one can use the fact that the oscillating part (involving $\cos(2r+\pi/2)$ and considered in the computation above) is not contributing to the asymptotics. Indeed from the asymptotics above we have

$$g_2^2(r)=\frac{1+\sin(2r)}{\pi r}+O\left(\frac{1}{r^2}\right)$$

as $r \to \infty$.

This allows us to write explicitly, as $N \to \infty$,

$$\begin{aligned}
&\int_{-2N}^{2N}\int_{-2N}^{2N}(2N-|x_1|)(2N-|x_2|)g_2^2\left(\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|\right)dx_1dx_2\\
&\sim \int_{-2N}^{2N}\int_{-2N}^{2N}(2N-|x_1|)(2N-|x_2|)\frac{1}{\pi\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|}dx_1dx_2\\
&= 4\int_0^{2N}\int_0^{2N}(2N-x_1)(2N-x_2)\frac{1}{\pi\sqrt{x_1^2+x_2^2}}dx_1dx_2\\
&= 4\cdot(2N)^2\int_0^1\int_0^1(2N-2Ny_1)(2N-2Ny_2)\frac{1}{\pi\cdot 2N\sqrt{y_1^2+y_2^2}}dy_1dy_2\\
&= 4(2N)^3\int_0^1\int_0^1(1-y_1)(1-y_2)\frac{1}{\pi\sqrt{y_1^2+y_2^2}}dy_1dy_2\\
&= 4(2N)^3\,\frac{1}{\pi}\left(\frac{1-\sqrt{2}}{3}+\ln(1+\sqrt{2})\right) = 4(2N)^3\,\frac{1}{\pi}\left(\frac{1-\sqrt{2}}{3}+\ln(1+\sqrt{2})\right)\approx 4(2N)^3\,0.2366.
\end{aligned}$$

□

The following Lemma A.7 provides the coefficients of the dominating chaos components (see proof of Proposition 2.1 for the precise definition); the proof exploits Cammarota and Marinucci (2018, Proposition 5) and we omit it for the sake of brevity.

Lemma A.7. *We have*

$$\begin{aligned}
c_2 &= -\frac{1}{\sqrt{\pi}}\frac{1}{4}u\phi(-u),\\
c_5 &= \frac{1}{\sqrt{\pi}}\left(\frac{1}{6}u\phi(-u)+\frac{1}{6}u^3\phi(-u)+\frac{1}{4}\Phi(-u)\right),\\
c_z &= \frac{1}{\sqrt{\pi}}\left(\frac{1}{12}u\phi(-u)+\frac{1}{12}u^3\phi(-u)\right),\\
c_{z5} &= \frac{1}{\sqrt{\pi}}\left(\frac{1}{3\sqrt{8}}u\phi(-u)+\frac{1}{3\sqrt{8}}u^3\phi(-u)+\frac{1}{\sqrt{8}}\Phi(-u)\right),\\
c_{25} &= c_{2z} = 0.
\end{aligned}$$

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