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On free energy of non-convex multi-species spin glasses

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Abstract. In Chen and Mourrat (2025), it was shown that if the limit of the free energy in a non-convex vector spin glass model exists, it must be a critical value of a certain functional. In this work, we extend this result to multi-species spin glass models with non-convex interactions, where spins from different species may lie in distinct vector spaces. Since the species proportions may be irrational and the existence of the limit of the free energy is not generally known, non-convex multi-species models cannot be approximated by vector spin models in a straightforward manner, necessitating more careful treatment.

1. Introduction

In mean-field vector spin glass models with general non-convex interactions, the main result in Chen and Mourrat (2025) shows that the limit of free energy, if exists, must be a critical value of some functional. Moreover, up to a small perturbation, the limit distribution of the spin overlap is determined by the critical point. In this work, we extend these results to the setting of multi-species spin glasses, where spins are grouped into different species and interactions are between species. We consider the general setting where spins from different species can take values in different vector spaces. Hence, the model considered here can be described as a multi-species vector spin glass model with possibly non-convex interactions. Main results corresponding to those in Chen and Mourrat (2025) are stated as Theorems in Section 1.2 and some less important results from Chen and Mourrat (2025) are adapted and collected in Section 5.3.

We clarify the necessity for this separate work. If species proportions are rational, then the multi-species model can be reduced to a vector spin glass model, to which results in Chen and Mourrat (2025) are directly applicable. If not all proportions are rational, it is tempting to simply take a sequence of models with rational proportions to approximate the non-rational one. This is indeed the argument used by Panchenko (2015) to treat the multi-species Sherrington–Kirkpatrick model (see the second paragraph in Panchenko, 2015, Section 5). However, this argument relies on that the limit of free energy exists for each approximation model, which is not known if the interaction is non-convex. Therefore, results for non-convex multi-species models are not direct corollaries from those Chen and Mourrat (2025) for vector spin glasses. This justifies our pursuits here.

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Moreover, a direct motivation is from Chen and Mourrat (2024) which studies the simultaneous replica symmetry breaking (RSB) in vector and multi-species spin glasses. The relation satisfied by critical points is the key structure for the simultaneous RSB. This work supplies the relation in the multi-species setting and thus together with Chen and Mourrat (2025) forms the groundwork for Chen and Mourrat (2024).

Aside from results in the general non-convex setting, we include results in the convex case in Section 6. In particular, we prove the Parisi formula (Proposition 6.1) for convex multi-species vector spin glass models; and we adapt the main results for non-convex models to the convex setting and summarize them in Corollary 6.6.

1.1. *Setting.* Before describing the multi-species model, we start with some basic notation used throughout this work.

1.1.1. *Matrices, inner products, and paths.* Throughout, we write $\mathbb{R}_+ = [0, +\infty)$.

For $m, n \in \mathbb{N}$, we denote by $\mathbb{R}^{m\times n}$ the space of all $m \times n$ real matrices. For any $a = (a_{ij})_{1\leqslant i\leqslant m,\,1\leqslant j\leqslant n} \in \mathbb{R}^{m\times n}$, we denote its j-th column vector as $a_{\bullet j} = (a_{ij})_{1\leqslant i\leqslant m}$ and its i-th row vector as $a_{i\bullet} = (a_{ij})_{1\leqslant j\leqslant n}$. If not specified, a vector is understood to be a column vector. For a matrix or vector a, we denote by $a^\intercal$ its transpose. For $a, b \in \mathbb{R}^{m\times n}$, we write $a \cdot b = \sum_{ij} a_{ij}b_{ij}$, $|a| = \sqrt{a\cdot a}$, and similarly for vectors. More generally, for any finite set I and $a, b \in \mathbb{R}^I$, we write

$$a \cdot b = \sum_{i\in I} a_i b_i; \qquad |a| = \sqrt{a\cdot a}. \tag{1.1}$$

For each $D \in \mathbb{N}$, let S^D be the linear space of $D \times D$ real symmetric matrices. For $a, b \in S^D$, viewing them as elements in $\mathbb{R}^{D\times D}$, we write $a \cdot b = \sum_{ij} a_{ij}b_{ij}$ and $|a| = \sqrt{a\cdot a}$. Let S^D_+ (resp. S^D_{++}) be the subset consisting of positive semi-definite (resp. definite) matrices. For $a, b \in S^D$, we write $a \geqslant b$ provided $a - b \in S^D_+$, which gives a natural partial order on S^D_+. In particular, when $D = 1$, we have $S^D_+ = \mathbb{R}_+$.

Let $\mathcal{Q}(D)$ be the collection of right-continuous increasing paths $q : [0, 1) \to S^D_+$. Here, q is said to be increasing provided that $q(r') \geqslant q(r)$ in S^D_+ for every $0 \leqslant r \leqslant r' < 1$. For $p \in [1,\infty)$, we write $\mathcal{Q}_p(D) = \mathcal{Q}(D) \cap L^p([0, 1); S^D)$. For $q \in \mathcal{Q}_\infty(D)$, we set $q(1) = \lim_{s\nearrow 1} q(s)$ which exists by monotonicity.

1.1.2. *Species and proportions.* Fix a finite set $\mathscr{S}$, the elements of which are interpreted as symbols for different species. For each N, let $I_{N,s}$, for $s \in \mathscr{S}$, be a partition of $\{1, \ldots , N\}$. We interpret each $I_{N,s}$ as the set of indices for the s-species.

For each N, we set

$$\lambda_{N,s} = |I_{N,s}|/N, \quad \forall s \in \mathscr{S}; \qquad \lambda_N = (\lambda_{N,s})_{s\in\mathscr{S}}\,. \tag{1.2}$$

We interpret $\lambda_{N,s}$ as the proportion of the s-species at size N. Introducing the notation for the discrete simplex

$$\blacktriangle_N = \Big\{(\lambda_s)_{s\in\mathscr{S}} \;\Big|\; \lambda_s \in [0, 1] \cap (\mathbb{Z}/N),\ \forall s \in \mathscr{S};\ \sum_{s\in\mathscr{S}} \lambda_s = 1\Big\} \tag{1.3}$$

we have $\lambda_N \in \blacktriangle_N$. We often assume that $(\lambda_N)_{N\in\mathbb{N}}$ converges to some λ_∞ belonging to the continuous simplex

$$\blacktriangle_\infty = \Big\{(\lambda_s)_{s\in\mathscr{S}} \;\Big|\; \lambda_s \in [0, 1],\ \forall s \in \mathscr{S};\ \sum_{s\in\mathscr{S}} \lambda_s = 1\Big\}. \tag{1.4}$$

1.1.3. *Spins, overlaps, and interaction.* We fix some $\kappa_s \in \mathbb{N}$ for each $s \in \mathscr{S}$ we write $\kappa = (\kappa_s)_{s\in\mathscr{S}}$. We assume that spins in the s-species are vectors in $\mathbb{R}^{\kappa_s}$. Denote by

$$\Sigma = \sqcup_{s\in\mathscr{S}}[-1,+1]^{\kappa_s} \tag{1.5}$$

the state space for a single spin (here $\sqcup$ stands for the disjoint union). For $s \in \mathscr{S}$, let μ_s be a finite nonnegative measure supported on $[-1,+1]^{\kappa_s}$. We extend μ_s trivially to Σ. For each $N \in \mathbb{N}$, a spin configuration consisting of N spins is denoted by $\sigma = (\sigma_{\bullet 1}, \ldots, \sigma_{\bullet N}) \in \Sigma^N$ where each $\sigma_{\bullet n} = (\sigma_{kn})_{1\leqslant k\leqslant \kappa_s}$ is a (column) vector in $\mathbb{R}^{\kappa_s}$ if $n \in I_{N,s}$. We sample a configuration $\sigma = (\sigma_{\bullet 1}, \ldots \sigma_{\bullet N})$ by independently drawing each $\sigma_{\bullet n}$ according to μ_s if $n \in I_{N,s}$. In other words, denoting P_{N,λ_N} as the distribution of σ, we have

$$\mathrm{d}P_{N,\lambda_N}(\sigma) = \otimes_{s\in\mathscr{S}} \otimes_{n\in I_{N,s}} \mathrm{d}\mu_s(\sigma_{\bullet n}). \tag{1.6}$$

For two spin configurations σ, σ' of size N and $s \in \mathscr{S}$, we consider the $\mathbb{R}^{\kappa_s\times\kappa_s}$-valued overlap of the s-species:

$$R_{N,\lambda_N,s}(\sigma,\sigma') = \frac{1}{N}\sigma_{\bullet I_{N,s}}(\sigma'_{\bullet I_{N,s}})^\intercal \in [-1,+1]^{\kappa_s\times\kappa_s} \tag{1.7}$$

where

$$\sigma_{\bullet I_{N,s}} = (\sigma_{kn})_{1\leqslant k\leqslant\kappa_s,\, n\in I_{N,s}} \tag{1.8}$$

is a $\kappa_s \times |I_{N,s}|$-matrix and similarly for $\sigma'_{\bullet I_{N,s}}$. Putting them together, we consider the $\prod_{s\in\mathscr{S}} \mathbb{R}^{\kappa_s\times\kappa_s}$-valued overlap:

$$R_{N,\lambda_N}(\sigma,\sigma') = \left(R_{N,\lambda_N,s}(\sigma,\sigma')\right)_{s\in\mathscr{S}}.$$

Notice that the overlap depends on the partition $(I_{N,s})_{s\in\mathscr{S}}$. Here, we choose to only display the dependence on λ_N because the distribution of the overlap under Gibbs measures to be introduced only depends on the proportion.

Let $\xi : \prod_{s\in\mathscr{S}} \mathbb{R}^{\kappa_s\times\kappa_s} \to \mathbb{R}$ be a smooth function and assume the existence of a centered Gaussian process $(H_N(\sigma))_{\sigma\in\Sigma^N}$ with covariance

$$\mathbb{E}\left[H_N(\sigma)H_N(\sigma')\right] = N\xi\left(R_{N,\lambda_N}(\sigma,\sigma')\right). \tag{1.9}$$

The form of ξ and the construction of $H_N(\sigma)$ are similar to those in the vector spin case. We refer to Chen and Mourrat (2025, Section 1.5) for the detail.

Example (Multi-species Sherrington–Kirkpatrick model, Sherrington and Kirkpatrick, 1975). The model considered in Panchenko (2015) corresponds to the case where $\kappa_s = 1$, $\mu_s = \delta_{-1} + \delta_{+1}$ for every $s \in \mathscr{S}$, and ξ is given by $\xi(a) = \sum_{s,s'\in\mathscr{S}} \Delta^2_{ss'} a_s a_{s'}$ where $\Delta^2 = (\Delta^2_{ss'})_{s,s'\in\mathscr{S}}$ is a matrix in $S_+^{|\mathscr{S}|}$.

1.1.4. *Cascade.* To describe the external fields in the model, we need first to recall the Ruelle probability cascade (RPC). Let $\mathfrak{R}$ denote the RPC with overlap uniformly distributed over $[0,1]$ (see Panchenko, 2013b, Theorem 2.17). Precisely, $\mathfrak{R}$ is a random probability measure on the unit sphere of a separable Hilbert space, with the inner product denoted by $\alpha \wedge \alpha'$. Let α and α' be independent samples from $\mathfrak{R}$. Then, the law of $\alpha \wedge \alpha'$ under $\mathbb{E}\mathfrak{R}^{\otimes 2}$ is the uniform distribution over $[0,1]$, where $\mathbb{E}$ integrates the randomness of $\mathfrak{R}$. This overlap distribution uniquely determines $\mathfrak{R}$ (see Panchenko, 2013b, Theorem 2.13). Almost surely, the support of $\mathfrak{R}$ is ultrametric in the induced topology. For rigorous definitions and basic properties, we refer to Panchenko (2013b, Chapter 2) (also see Dominguez and Mourrat, 2024c, Chapter 5). We also refer to Chen and Mourrat (2025, Section 4) for the construction and properties of $\mathfrak{R}$ useful in this work. Throughout, we denote by $\langle\cdot\rangle_{\mathfrak{R}} = \mathfrak{R}^{\otimes\mathbb{N}}$ the tensorized version of $\mathfrak{R}$.

1.1.5. *External fields.* Recall the space $\mathcal{Q}(D)$ of right-continuous increasing paths from Section 1.1.1. We use the following shorthand notation:

$$\mathcal{Q}^{\mathscr{S}}_{\square}(\kappa) = \prod_{s\in\mathscr{S}} \mathcal{Q}_{\square}(\kappa_s) \tag{1.10}$$

where $\square$ is a placeholder for subscripts. We explicitly construct the external field parametrized by any $q = (q_s)_{s\in\mathscr{S}} \in \mathcal{Q}^{\mathscr{S}}_{\infty}(\kappa)$. For almost every realization of $\mathfrak{R}$, every $s \in \mathscr{S}$, and every $n \in I_{N,s}$, let $(w^{q_s}_n(\alpha))_{\alpha\in\operatorname{supp}\mathfrak{R}}$ be the $\mathbb{R}^{\kappa_s}$-valued centered Gaussian process with covariance

$$\mathbb{E}\left[w^{q_s}_n(\alpha) w^{q_s}_n(\alpha')^{\intercal}\right] = q_s(\alpha \wedge \alpha'). \tag{1.11}$$

The existence of such a process and its properties are given in Chen and Mourrat (2025, Section 4). Conditioned on $\mathfrak{R}$, we assume that all these processes, indexed by s and n, are independent. For each s, we write $w^{q_s}_{I_{N_s}} = (w^{q_s}_n)_{n\in I_{N,s}}$. Recall the notation in (1.8). For each $N \in \mathbb{N}$ and $q \in \mathcal{Q}^{\mathscr{S}}_{\infty}(\kappa)$, we define

$$W^q_N(\sigma,\alpha) = \sum_{s\in\mathscr{S}} w^{q_s}_{I_{N,s}}(\alpha)\cdot \sigma_{\bullet I_{N,s}} \tag{1.12}$$

which, conditioned on $\mathfrak{R}$, is a centered Gaussian process with covariance

$$\mathbb{E}\left[W^q_N(\sigma,\alpha)W^q_N(\sigma',\alpha')\right] \overset{(1.7),(1.11)}{=} Nq(\alpha\wedge\alpha')\cdot R_{N,\lambda_N}(\sigma,\sigma') \tag{1.13}$$

where the dot product follows the rule as in (1.1).

1.1.6. *Hamiltonian and free energy.* Now, for $N \in \mathbb{N}$, $\lambda_N \in \blacktriangle_N$, $t \in \mathbb{R}_+$, and $q \in \mathcal{Q}^{\mathscr{S}}_{\infty}(\kappa)$, we consider the Hamiltonian

$$H^{t,q}_N(\sigma,\alpha) = \sqrt{2t}H_N(\sigma) - tN\xi\left(R_{N,\lambda_N}(\sigma,\sigma)\right) + \sqrt{2}W^q_N(\sigma,\alpha) - Nq(1)\cdot R_{N,\lambda_N}(\sigma,\sigma) \tag{1.14}$$

where $q(1) = (q_s(1))_{s\in\mathscr{S}} \in \prod_{s\in\mathscr{S}} S^{\kappa_s}_+$ and we understand $q(1)\cdot R_{N,\lambda_N}(\sigma,\sigma) = \sum_{s\in\mathscr{S}} q_s(1)\cdot R_{N,\lambda_N,s}(\sigma,\sigma)$ still in the sense of entry-wise product. Here, $R_{N,\lambda_N}(\sigma,\sigma)$ is the overlap between the same sample and is thus often called the self-overlap. The two terms in (1.14) involving the self-overlap are respectively the variances of $\sqrt{t}H_N(\sigma)$ and $W^q_N(\sigma,\alpha)$. These two terms are often called the self-overlap correction, which resembles the drift term in an exponential martingale.

We define the associated free energy and Gibbs measure

$$\overline{F}_{N,\lambda_N}(t,q) = -\frac{1}{N}\mathbb{E}\log\iint \exp\left(H^{t,q}_N(\sigma,\alpha)\right)\mathrm{d}P_{N,\lambda_N}(\sigma)\mathrm{d}\mathfrak{R}(\alpha), \tag{1.15}$$

$$\langle\cdot\rangle_{N,\lambda_N} \propto \exp\left(H^{t,q}_N(\sigma,\alpha)\right)\mathrm{d}P_{N,\lambda_N}(\sigma)\mathrm{d}\mathfrak{R}(\alpha). \tag{1.16}$$

Here, $\mathbb{E}$ first averages over all the Gaussian randomness in $H_N(\sigma)$ and $W^q_N(\sigma,\alpha)$ and then the randomness in $\mathfrak{R}$. This particular order of integration is needed to ensure that there is no measurability issues (see Chen and Mourrat, 2025, Lemma 4.5). Notice the additional minus sign on the right-hand side of (1.15). We have omitted the dependence on t and q from the notation of $\langle\cdot\rangle_{N,\lambda_N}$, which should be clear from the context. The dependence of $\overline{F}_{N,\lambda_N}(t,q)$ on the partition $(I_{N,s})_{s\in\mathscr{S}}$ is only through λ_N, which is the reason for us to include it in the notation. We omit the dependence on λ_N in the notation of $H^{t,q}_N$, H_N, and W^q_N, which we prefer to keep implicit.

We can view $\overline{F}_{N,\lambda_N}$ as a function of $(t,q) \in \mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_{\infty}(\kappa)$. By continuity in Proposition 4.1, we can extend $\overline{F}_{N,\lambda_N}$ to the domain $\mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_{1}(\kappa)$.

1.1.7. *Initial condition, functional, and critical points.* For each $\lambda \in \mathbb{R}^{\mathscr{S}}_+$, there is a function $\psi_\lambda : \mathcal{Q}^{\mathscr{S}}_1(\kappa) \to \mathbb{R}$ (Lemma 4.10) such that $\overline{F}_{N,\lambda_N}(0,\cdot) = \psi_{\lambda_N}$ (see Lemma 4.11). For any $\lambda_\infty \in \blacktriangle_\infty$, $(t,q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$, we consider the functional

$$\mathcal{J}_{\lambda_\infty,t,q}(q',p) = \psi_{\lambda_\infty}(q') + \langle p, q-q'\rangle_{L^2} + t\int_0^1 \xi(p) \tag{1.17}$$

defined for $q' \in \mathcal{Q}^{\mathscr{S}}_2(\kappa)$, $p \in L^2([0,1],\prod_{s\in\mathscr{S}} S^{\kappa_s})$. Here, $\langle\cdot,\cdot\rangle_{L^2}$ is the inner product in $L^2([0,1],\prod_{s\in\mathscr{S}} S^{\kappa_s})$ and the last integral is $\int_0^1 \xi(p(s))\mathrm{d}s$. We call (1.17) the Hamilton–Jacobi functional due to its close connection to the Hamilton–Jacobi equation (1.27). For instance, in Theorem 1.6 to be stated, the variational formula representation for the limit of free energy is exactly the Hopf–Lax formula. We refer to Chen and Mourrat (2025, Section 1) for more detail on the Hamilton–Jacobi equation approach to spin glass.

We say that a pair $(q',p) \in \mathcal{Q}^{\mathscr{S}}_2(\kappa) \times L^2([0,1],\prod_{s\in\mathscr{S}} S^{\kappa_s})$ is a **critical point** of the functional $\mathcal{J}_{\lambda_\infty,t,q}$ if

$$q = q' - t\nabla\xi(p) \qquad \text{and} \qquad p = \partial_q\psi_{\lambda_\infty}(q'). \tag{1.18}$$

Here, the derivative $\partial_q\psi_{\lambda_\infty}$ is understood in the sense of Fréchet, which is rigorously defined in (4.1). The differentiability of ψ_{λ_∞} is ensured by Lemma 4.10.

Heuristically, at any critical point (q',p), the derivatives of $\mathcal{J}_{\lambda_\infty,t,q}$ in q' and p are both zero. Critical points and the value of the functional at these points are important to our main results to be stated.

1.1.8. *Regular paths.* We introduce subsets of $\mathcal{Q}^{\mathscr{S}}(\kappa)$ consisting of regular paths that play important roles in the differentiability of limits of free energy on $\mathcal{Q}^{\mathscr{S}}_2(\kappa)$. For any $D \in \mathbb{N}$ and any matrix $a \in S^D_+$, we define

$$\mathsf{Ellipt}(a) = \frac{\max_{u\in\mathbb{R}^D:\,|u|=1} u^\intercal a u}{\min_{u\in\mathbb{R}^D:\,|u|=1} u^\intercal a u}. \tag{1.19}$$

In other words, $\mathsf{Ellipt}(a)$ is the ratio of the largest eigenvalue over the smallest eigenvalue of a. We have $a \in S^D_{++}$ if and only if $\mathsf{Ellipt}(a) < +\infty$. For $D \in \mathbb{N}$ and $c > 0$, we define

$$\begin{aligned}\mathcal{Q}_{\uparrow,c}(D) = \Big\{q \in \mathcal{Q}(D) \;\Big|\; q(0) = 0 \text{ and } \forall r \leqslant r' \in [0,1),\ q(r') - q(r) \geqslant c(r'-r)\mathbf{Id}_D \\ \text{and } \mathsf{Ellipt}(q(r') - q(r)) \leqslant c^{-1}\Big\};\end{aligned} \tag{1.20}$$

$$\mathcal{Q}_{\uparrow}(D) = \cup_{c>0}\mathcal{Q}_{\uparrow,c}(D); \qquad \mathcal{Q}_{\infty,\uparrow}(D) = \mathcal{Q}_\infty(D)\cap\mathcal{Q}_\uparrow(D). \tag{1.21}$$

Here, $\mathbf{Id}_D$ is the $D\times D$ identity matrix. Let $\mathcal{Q}_{\uparrow,c}(\kappa)$, $\mathcal{Q}_\uparrow(\kappa)$, and $\mathcal{Q}_{\infty,\uparrow}(\kappa)$ be given as in (1.10).

1.2. *Main results.* In the results to be stated below, we say that $\left(\overline{F}_{N,\lambda_N}\right)_{N\in\mathbb{N}}$ converges to some f if $(\overline{F}_{N,\lambda_N}(t,q))_{N\in\mathbb{N}}$ converges to $f(t,q)$ at every $(t,q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_1(\kappa)$. Namely, the convergence is pointwise.

The result below adapts Chen and Mourrat (2025, Theorem 1.1) partially. The missing part to recover the full version (Theorem 1.6) is formulated into Claim 1.5. We discuss this issue after the statements of four main theorems.

Theorem 1.1. *If ξ is a convex function on $\prod_{s\in\mathscr{S}} S^{\kappa_s}_+$ and $(\lambda_N)_{N\in\mathbb{N}}$ converges to some λ_∞, then for every $(t,q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$, we have*

$$\lim_{N\to\infty} \overline{F}_{N,\lambda_N}(t,q) = \sup_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \mathcal{J}_{\lambda_\infty,t,q}\left(q + t\nabla\xi(p), p\right). \tag{1.22}$$

This theorem is in fact simply the Parisi formula, which can be rewritten in a more familiar form in Proposition 6.1.

The next two results adapt Chen and Mourrat (2025, Theorems 1.2 and 1.3) respectively to the setting here. In the following, the Gateaux differentiability is defined in Section 4.1.

Theorem 1.2 (Critical point representation)**.** *Assume that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges to some* λ_∞ *and that* $(\overline{F}_{N,\lambda_N})_{N\in\mathbb{N}}$ *converges to some* f*. Then, for every* $(t,q)\in\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$*, there exists* $(q',p)\in\mathcal{Q}_2^{\mathscr{S}}(\kappa)\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$ *that is a critical point of* $\mathscr{J}_{\lambda_\infty,t,q}$ *and is such that*

$$\lim_{N\to\infty}\overline{F}_{N,\lambda_N}(t,q)=\mathscr{J}_{\lambda_\infty,t,q}(q',p). \tag{1.23}$$

Theorem 1.3 (Critical point identification)**.** *Assume that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges to some* λ_∞ *and that* $(\overline{F}_{N,\lambda_N})_{N\in\mathbb{N}}$ *converges to some* f*. For every* $(t,q)\in\mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$*, if* $f(t,\cdot)$ *is Gateaux differentiable at* q*, then letting* $p=\partial_q f(t,q)$ *and* $q'=q+t\nabla\xi(p)$*, we have that* $(q',p)\in\mathcal{Q}_2^{\mathscr{S}}(\kappa)\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$ *is a critical point of* $\mathscr{J}_{\lambda_\infty,t,q}$ *and is such that* (1.23) *holds.*

Theorem 1.2 states that, if the free energy converges, then the limit must be a critical value of Hamilton–Jacobi functional. For $t>0$ sufficiently small and $q=0$, namely, in the replica symmetry regime, Theorem 1.2 was shown in Dey and Wu (2021) for the multi-species Sherrington–Kirkpatrick model. Theorem 1.3 shows that, at every differentiable point in $\mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$ of the limit, the critical point is uniquely determined. By Proposition 4.8, such differentiability points are dense in $\mathcal{Q}_2^{\mathscr{S}}(\kappa)$.

Next, we state an adapted version of Chen and Mourrat (2025, Theorem 1.4). By adding a small perturbation of quadratic interaction, we can show that the overlap $R_{N,\lambda_N}(\sigma,\sigma')$ under $\mathbb{E}\langle\cdot\rangle_{N,\lambda_N}$ converges in law to p from the critical point (q',p). Here, σ' is an independent copy of σ sampled from $\langle\cdot\rangle_{N,\lambda}$ introduced in (1.16). The perturbation is given by an additional Gaussian Hamiltonian $(\widehat{H}_N(\sigma))_{\sigma\in\Sigma^N}$ characterized by the covariance

$$\mathbb{E}\left[\widehat{H}_N(\sigma)\widehat{H}_N(\sigma')\right]=N\left|R_{N,\lambda_N}(\sigma,\sigma')\right|^2. \tag{1.24}$$

Letting $(g_{s,n,n',k,k'})$ be a family of i.i.d. standard Gaussian variables independent of all other randomness, we can explicitly construct

$$\widehat{H}_N(\sigma):=\frac{1}{\sqrt{N}}\sum_{s\in\mathscr{S}}\sum_{n,n'\in I_{N,s}}\sum_{k,k'=1}^{\kappa_s}g_{s,n,n',k,k'}\sigma_{kn}\sigma_{k'n'}.$$

For every $\widehat{t}\geqslant 0$, to $H_N^{t,q}(\sigma,\alpha)$ in (1.14), we add the quantity

$$\sqrt{2\widehat{t}}\,\widehat{H}_N(\sigma)-\widehat{t}N\left|R_{N,\lambda_N}(\sigma,\sigma)\right|^2$$

and let $\widehat{F}_{N,\lambda_N}(t,\widehat{t},q)$ be the corresponding free energy, which is precisely expressed in (5.53). The corresponding functional (see (1.17)) becomes

$$\widehat{\mathscr{J}}_{\lambda_\infty,t,\widehat{t},q}(q',p):=\psi_{\lambda_\infty}(q')+\langle p,q-q'\rangle_{L^2}+t\int_0^1\xi(p)+\widehat{t}\int_0^1|p|^2. \tag{1.25}$$

Theorem 1.4 (Identification of overlap distribution)**.** *Assume that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges to some* λ_∞ *and that* $(\widehat{F}_{N,\lambda_N})_{N\in\mathbb{N}}$ *converges to some* $\widehat{f}$ *along some subsequence. Suppose also that, for some* $(t,q)\in\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$ *and* $\widehat{t}>0$*, we have that* $\widehat{f}(t,\widehat{t},\cdot)$ *is Gateaux differentiable at* q*, and that* $\widehat{f}(t,\cdot,q)$ *is differentiable at* $\widehat{t}$*. Then letting* $p=\partial_q\widehat{f}(t,\widehat{t},q)$ *and* $q'=q+t\nabla\xi(p)+2\widehat{t}p$*, we have that* $(q',p)\in\mathcal{Q}_2^{\mathscr{S}}(\kappa)\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$ *is a critical point of* $\widehat{\mathscr{J}}_{\lambda_\infty,t,\widehat{t},q}$*. Moreover, as* N *tends to infinity along the said subsequence, the overlap* $R_{N,\lambda_N}(\sigma,\sigma')$ *under* $\mathbb{E}\langle\cdot\rangle_{N,\lambda_N}$ *converges in law to* $p(U)$*, where* U *is the uniform random variable over* $[0,1]$*.*

In the above theorem, we do not require that the free energy converges as N tends to infinity. Any subsequential convergence is enough. We can always extract such a convergent subsequence through a precompactness result such as Proposition 4.2. If $\widehat{F}_{N,\lambda_N}$ does converge along the full sequence, then an application of Theorem 1.2 yields a representation of the free energy itself as

$$\lim_{N\to\infty} \widehat{F}_{N,\lambda_N}(t,\widehat{t},q) = \widehat{\mathscr{J}}_{\lambda_\infty,t,\widehat{t},q}(q',p),$$

for the same critical point (q',p) of $\widehat{\mathscr{J}}_{\lambda_\infty,t,\widehat{t},q}$. Here and in the above theorem, that (q',p) is a critical point of $\widehat{\mathscr{J}}_{\lambda_\infty,t,\widehat{t},q}$ means that

$$q = q' - t\nabla\xi(p) - 2\widehat{t}p \quad \text{and} \quad p = \partial_q\psi_{\lambda_\infty}(q').$$

As in Theorem 1.3, for each $t \geqslant 0$, almost every $(\widehat{t},q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$ is a point that satisfies the assumptions of Theorem 1.4. In particular, the set of such points is dense in $\mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$.

Adapted versions of Chen and Mourrat (2025, Propositions 1.5 and 1.6) in the introduction of Chen and Mourrat (2025) are adapted into Propositions 5.17 and 5.18.

Lastly, we come back to the issue with adapting Chen and Mourrat (2025, Theorem 1.1). The missing piece is summarized as follows.

Claim 1.5 (Free energy upper bound)**.** *Assume that $(\lambda_N)_{N\in\mathbb{N}}$ converges to some λ_∞. For every $(t,q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$, we have*

$$\liminf_{N\to\infty} \overline{F}_{N,\lambda_N}(t,q) \geqslant f(t,q) \tag{1.26}$$

where $f : \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa) \to \mathbb{R}$ is the solution to the equation

$$\begin{cases} \partial_t f - \int_0^1 \xi(\partial_q f) = 0, & \text{on } \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa), \\ f(0,\cdot) = \psi_{\lambda_\infty}, & \text{on } \mathcal{Q}^{\mathscr{S}}_2(\kappa). \end{cases} \tag{1.27}$$

Moreover, if ξ is a convex function on $\prod_{s\in\mathscr{S}} S^{\kappa_s}_+$, then f admits the Hopf–Lax representation at every $(t,q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$:

$$f(t,q) = \sup_{q'\in q+\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \inf_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \mathscr{J}_{\lambda_\infty,t,q}\left(q',p\right). \tag{1.28}$$

Parts (1.26) and (1.28) of this claim respectively adapt Mourrat (2023, Theorem 3.4) and Chen and Xia (2025a, Corollary 4.14) (for vector spins) to the multi-species setting. The modification to the proofs should be straightforward but tedious. Hence, we only formulate this claim here without proving it. Given this claim, we can recover the full version of Chen and Mourrat (2025, Theorem 1.1).

Theorem 1.6. *Assume that Claim 1.5 is valid. If ξ is a convex function on $\prod_{s\in\mathscr{S}} S^{\kappa_s}_+$ and $(\lambda_N)_{N\in\mathbb{N}}$ converges to some λ_∞, then for every $(t,q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$, we have*

$$\lim_{N\to\infty} \overline{F}_{N,\lambda_N}(t,q) = \sup_{q'\in q+\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \inf_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \mathscr{J}_{\lambda_\infty,t,q}\left(q',p\right) = f(t,q) \tag{1.29}$$

where f is the solution to (1.27).

Convexity is needed here to identify the limit of free energy to be the solution of (1.27). Without convexity and assuming the convergence of free energy, we can only show that the limit satisfies (1.27) at every differentiability point (see (5.48) in Proposition 5.12). This is not enough to identify the limit because we do not have the uniqueness of solutions in the sense of satisfying the equation on a dense set.

1.3. *Related works.* This work concerns the limit of free energy. In physics, the replica method has been developed and proven to be powerful (Mézard et al., 1987; Parisi, 1979, 1980a,b, 1983). As a special case of the multi-species setting, the bipartite spin glass model has been considered in Fyodorov et al. (1987a,b); Hartnett et al. (2018); Korenblit and Shender (1985), where formulas for the limit free energy in terms of functionals similar to $\mathscr{J}_{\lambda_\infty,t,q}$ in (1.17) has been obtained using the replica method.

Another perspective is based on Hamilton–Jacobi equations, which is explored in Mourrat (2021b, 2023); see also Dominguez and Mourrat (2024c). The functional $\mathscr{J}_{\lambda_\infty,t,q}$ is closely related to the characteristic lines of the Hamilton–Jacobi equation (1.27). Our main result can be restated as that the limit of free energy, if exists, attains its value along a certain characteristic line. The conjecture of Mourrat (2021b, 2023) (see also Dominguez and Mourrat, 2024c, Question 6.11) is that, in the general non-convex case, the free energy $\overline{F}_{N,\lambda_N}$ converges to the unique solution of (1.27)

In the convex case, much is known. The limit of free energy in the Sherrington–Kirkpatrick models has been mathematically identified as the Parisi formula in Guerra (2003); Talagrand (2006b) (also see Talagrand, 2011a,b), which is then extended to more general scalar models (Panchenko, 2013a,b, 2014) using an argument based on the ultrametricity of the asymptotic Gibbs measure (see also Aizenman and Contucci, 1998; Ghirlanda and Guerra, 1998; Guerra, 1996). Then, this is extended further to multi-species and vector spin glasses in Barra et al. (2015); Panchenko (2015, 2018b,a) under the assumption that ξ is convex on the whole space. The setting closest to the multi-species vector model considered here is the multi-species Potts model studied in Jagannath et al. (2018). Here, Theorem 1.1 (see also Proposition 6.1) encompasses these results with a weaker assumption that ξ is convex on positive semi-definite matrices. For spherical spins, the limit of free energy has been identified for scalar models (Chen, 2013; Talagrand, 2006a) and multi-type models (Bates and Sohn, 2022a,b; Ko, 2020; Panchenko and Talagrand, 2007).

In the context of spin glasses, the connection to the Hamilton–Jacobi equation was first explored in Agliari et al. (2012); Barra et al. (2010) and further in various settings in Barra et al. (2013); Agliari et al. (2012); Barra et al. (2011, 2014). The Parisi formula in the scalar case was rewritten as the Hopf–Lax formula for the solution of an equation of the form in (1.27) in Mourrat (2022) (later extended in Mourrat and Panchenko, 2020). Here, the equation is infinite-dimensional and its well-posedness together with variational representations was proved in Chen and Xia (2025a). For non-convex models, the best result so far is the lower bound as in (1.26) established in Mourrat (2021b, 2023). In particular, when ξ is not convex, the Hopf–Lax formula as in (1.29) is false (see Mourrat, 2021b, Section 6). Also, since ψ_{λ_∞} does not seem to be convex in general, another possibility, the Hopf formula

$$\sup_{p\in\mathcal{Q}^{\mathscr{L}}_\infty(\kappa)}\ \inf_{q'\in\mathcal{Q}^{\mathscr{L}}_\infty(\kappa)} \mathscr{J}_{\lambda_\infty,t,q}\left(q',p\right) \tag{1.30}$$

cannot be the limit of free energy.

For some non-convex models, other partial bounds for the limit free energy were obtained in Alberici et al. (2020, 2021). Some works also focus on the high-temperature regime, including Barra et al. (2011); Dey and Wu (2021); Genovese (2023). We also mention that additional symmetry in the model can lead to simplification (Bates and Sohn, 2024; Chen, 2024; Issa, 2024).

Formulas of the form in (1.30) appear in some problems from high-dimensional statistical inference (Barbier et al., 2016; Barbier and Macris, 2019; Barbier et al., 2017; Chen et al., 2022; Chen, 2022; Chen and Xia, 2022, 2023; Kadmon and Ganguli, 2019; Lelarge and Miolane, 2019; Lesieur et al., 2017; Luneau et al., 2020; Mayya and Reeves, 2019; Miolane, 2017; Mourrat, 2020, 2021a; Reeves, 2020; Reeves et al., 2019) (see also Dominguez and Mourrat, 2024c, Chapter 4 for a Hamilton–Jacobi approach). In certain problems on sparse graphs (Dominguez and Mourrat, 2024a,b; Kireeva and Mourrat, 2024), an issue similar to the non-convexity of ψ_{λ_∞} occurs and thus invalidates the Hopf formula. More broadly, connections between mean-field models and Hamilton–Jacobi equations have been noticed in Brankov and Zagrebnov (1983); Newman (1986). We refer to a recent survey (Bauerschmidt et al., 2024) for related topics.

In certain spherical models with multiple types and non-convex ξ, the limit of free energy, if exists, was explicitly identified in Subag (2025, 2023a,b,c). Also see the related work Baik and Lee (2020) on the bipartite spherical Sherrington–Kirkpatrick model. A more geometric analysis of the energy landscape can be found in Ben Arous et al. (2022); Huang and Sellke (2026); Kivimae (2023); McKenna (2024). For scalar models, also see Auffinger and Ben Arous (2013); Auffinger et al. (2013); Fyodorov (2004); Subag (2017).

1.4. *Outline of the paper.* In Section 2, we recall the Gaussian interpolation technique and use it to show two results: the continuity of $\overline{F}_{N,\lambda_N}$ in λ_N and the estimate on the discrepancy between $\overline{F}_{N,\lambda_N}$ and $\overline{F}_{N+M,\lambda_{N+M}}$.

In Section 3, we relate the multi-species spin glass model with rational species proportions to some vector spin glass model. In particular, we show that, if $(\lambda_{N,s})_{N\in\mathbb{N}}$ converges to a rational number for every $s \in \mathscr{S}$, then the free energy of the multi-species model is asymptotically equal to that of a vector spin model. In this case, results from Chen and Mourrat (2025) for vector spin glasses are directly applicable. It remains to consider the case where $(\lambda_{N,s})_{N\in\mathbb{N}}$ converges to an irrational number for some $s \in \mathscr{S}$, which we refer to as the irrational case.

In preparation for treating the irrational case, in Section 4, we collect analytic properties of the free energy $\overline{F}_{N,\lambda_N}$, its limit (if exists), and the initial condition $\overline{F}_{N,\lambda_N}(0,\cdot)$. These are results adapted from those in Chen and Mourrat (2025, Sections 3 and 5).

To handle the general case including the irrational case, we need to redo some of the cavity computation in Section 5. After importing cavity computation results for the rational case from Chen and Mourrat (2025, Section 6), we need an additional approximation procedure to obtain the results in the general case. This allows us to prove corresponding results from Chen and Mourrat (2025, Section 7) adapted to the current setting. In particular, we prove Theorems 1.2, 1.3, and 1.4 in Section 5.3.

In Section 6, we apply those results to the convex setting, namely, the one where ξ is convex. We recover corresponding results from Chen and Mourrat (2025, Section 8) and prove Theorems 1.1 and 1.6,

2. Gaussian interpolation and two estimates

In this section, we record two estimates on the continuity of the free energy in terms of λ_N in Lemma 2.3 and the discrepancy of free energy with different configuration sizes in Lemma 2.4. To prove them, we first recall the Gaussian interpolation technique stated as in Lemma 2.1 and Corollary 2.2, which will also be needed later.

Lemma 2.1 (Gaussian interpolation technique)**.** *Let $\mathfrak{P}$ be a random probability measure on some Polish space $\mathcal{X}$. Suppose that, conditioned on $\mathfrak{P}$, there are two independent centered Gaussian process $(\mathbf{G}_i(\mathbf{x}))_{\mathbf{x}\in\operatorname{supp}\mathfrak{P}}$ and two bounded deterministic function $\mathbf{D}_i : \mathcal{X} \to \mathbb{R}$ for $i \in \{0,1\}$. Also, assume that there are deterministic functions $\mathbf{V}_i : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ for $i \in \{0,1\}$ such that, conditioned on $\mathfrak{P}$,*

$$\mathbf{V}_i(\mathbf{x},\mathbf{x}') = \mathbb{E}\mathbf{G}_i(\mathbf{x})\mathbf{G}_i(\mathbf{x}'), \quad \forall \mathbf{x},\mathbf{x}' \in \operatorname{supp}\mathfrak{P}.$$

For $r \in [0,1]$, define $\mathbf{G}_r = \sqrt{1-r}\mathbf{G}_0 + \sqrt{r}\mathbf{G}_1$, $\mathbf{D}_r = (1-r)\mathbf{D}_0 + r\mathbf{D}_1$, and

$$\varphi(r) = -\mathbb{E}\log\int_{\mathcal{X}} \exp\left(\sqrt{2}\mathbf{G}_r(\mathbf{x}) + \mathbf{D}_r(\mathbf{x})\right) \mathrm{d}\mathfrak{P}(\mathbf{x})$$

where $\mathbb{E}$ *first averages over all Gaussian randomness (conditioned on* $\mathfrak{P}$*) and then that of* $\mathfrak{P}$*. Also, take the Gibbs measure* $\langle\cdot\rangle_r \propto \exp\left(\sqrt{2}\mathbf{G}_r(\mathbf{x}) + \mathbf{D}_r(\mathbf{x})\right) \mathrm{d}\mathfrak{P}$*. Then, we have*

$$\varphi(1) - \varphi(0) = \int_0^1 \mathbb{E}\left\langle \mathbf{V}_1(\mathbf{x},\mathbf{x}') - \mathbf{V}_0(\mathbf{x},\mathbf{x}') - \mathbf{V}_1(\mathbf{x},\mathbf{x}) + \mathbf{V}_0(\mathbf{x},\mathbf{x}) - \mathbf{D}_1(\mathbf{x}) + \mathbf{D}_0(\mathbf{x})\right\rangle_r \mathrm{d}r$$

where $\mathbf{x}'$ *is an independent copy of* $\mathbf{x}$ *under* $\langle\cdot\rangle_r$*.*

We have an immediate corollary.

Corollary 2.2 (Interpolation with self-overlap correction)**.** *Under the same setting of Lemma 2.1, we further assume that* $\mathbf{D}_i(\mathbf{x}) = -\mathbf{V}_i(\mathbf{x},\mathbf{x})$ *for every* $\mathbf{x} \in \mathcal{X}$ *and* $i \in \{0,1\}$*. Then, we have*

$$\varphi(1) - \varphi(0) = \int_0^1 \mathbb{E}\left\langle \mathbf{V}_1(\mathbf{x},\mathbf{x}') - \mathbf{V}_0(\mathbf{x},\mathbf{x}')\right\rangle_r \mathrm{d}r$$

Proof of Lemma 2.1: We can compute

$$\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r) = -\mathbb{E}\left\langle (2-2r)^{-1/2}\mathbf{G}_1(\mathbf{x}) - (2r)^{-1/2}\mathbf{G}_0(\mathbf{x}) + \mathbf{D}_1(\mathbf{x}) - \mathbf{D}_0(\mathbf{x})\right\rangle_r.$$

Apply the Gaussian integration by parts (e.g. see Panchenko, 2013b, Lemma 1.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r) = \mathbb{E}\left\langle \mathbf{V}_1(\mathbf{x},\mathbf{x}') - \mathbf{V}_0(\mathbf{x},\mathbf{x}') - \mathbf{V}_1(\mathbf{x},\mathbf{x}) + \mathbf{V}_0(\mathbf{x},\mathbf{x}) - \mathbf{D}_1(\mathbf{x}) + \mathbf{D}_0(\mathbf{x})\right\rangle_r$$

which gives the desired result. □

Now, with the Gaussian interpolation technique, we can start to prove the first estimate.

Lemma 2.3. *Let* $C_\xi = \max_{x\in\prod_{s\in\mathscr{S}}[-1,+1]^{\kappa_s\times\kappa_s}} |\nabla\xi(x)|$*,* $C_\mu = \sqrt{\sum_{s\in\mathscr{S}} |\log\mu_s(\mathbb{R}^{\kappa_s})|^2}$*, and* $|\kappa|_\infty = \max_{s\in\mathscr{S}}\kappa_s$*. Then, for every* $N\in\mathbb{N}$*,* $t\in\mathbb{R}_+$*,* $q\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$*, and* λ_N*,* $\lambda'_N \in \blacktriangle_N$*, we have*

$$\left|\overline{F}_{N,\lambda_N}(t,q) - \overline{F}_{N,\lambda'_N}(t,q)\right| \leqslant \left(|\kappa|_\infty C_\xi t + |\kappa|_\infty |q|_{L^1} + C_\mu\right)\left|\lambda_N - \lambda'_N\right|. \tag{2.1}$$

Here and henceforth, the L^p-norm of q for $p\in[0,\infty]$ is given as

$$|q|_{L^p} = \left(\int_0^1 |q(r)|^p \mathrm{d}r\right)^{\frac{1}{p}}, \qquad |q|_{L^\infty} = \operatorname*{ess\,sup}_{r\in[0,1]} |q(r)|$$

where $|q(r)|$ is the norm in $\prod_{s\in\mathscr{S}} S^{\kappa_s}$ as given in (1.1).

Proof: Throughout this proof, we fix t and q and thus omit them from the notation of different versions of free energy. We write $R_N = R_{N,\lambda_N}$ and $P_N = P_{N,\lambda_N}$ for brevity.

Let $I_N = (I_{N,s})_{s\in\mathbb{N}}$ be the partition associated with λ_N. Let $J = (J_s)_{s\in\mathbb{N}}$ be a collection of subsets $J_s \subseteq I_{N,s}$. For each configuration $\sigma\in\Sigma^N$, we write $\sigma^J = \left(\sigma_{\bullet n}\mathbf{1}_{n\in\cup_{s\in\mathscr{S}}J_s}\right)_{1\leqslant n\leqslant N}$. We define $\overline{F}^J_{N,I_N}$ in the same way as (1.15) but with every instance of σ in $\exp(\cdots)$ replaced by σ^J. Notice that in this definition, we keep $P_N(\sigma)$ in $\overline{F}^J_{N,I_N}$ intact, which is source of the dependence on I_N. Define $\overline{F}^J_N$ by further replacing $\mathrm{d}P_n(\sigma)$ in $\overline{F}^J_{N,I_N}$ by

$$\mathrm{d}P^J_N\left((\sigma_{\bullet n})_{n\in\cup_s J_s}\right) = \otimes_{s\in\mathscr{S}}\otimes_{n\in J_s}\mathrm{d}\mu_s(\sigma_{\bullet n}). \tag{2.2}$$

It is straightforward to see that if $J' = (J'_s)$ is a partition of a subset of $\{1,\ldots,N\}$ and $|J'_s| = |J_s|$ for every s, we have

$$\overline{F}^J_N = \overline{F}^{J'}_N. \tag{2.3}$$

Writing $|I_N \setminus J| = \left(\sum_{s\in\mathscr{S}} |I_{N,s} \setminus J_s|^2\right)^{\frac{1}{2}}$, we claim

$$\left|\overline{F}_{N,\lambda_N} - \overline{F}^J_{N,I_N}\right| \leqslant (tC_\xi + |q|_{L^1})\, |\kappa|_\infty N^{-1} |I_N \setminus J|, \tag{2.4}$$

$$\left|\overline{F}^J_{N,I_N} - \overline{F}^J_N\right| \leqslant C_\mu N^{-1} |I_N \setminus J|. \tag{2.5}$$

Before proving them, we first use them to deduce the desired estimate.

Let $(I'_{N,s})_{s\in\mathbb{N}}$ be a partition with proportions given by λ'_N. We can find a bijection ι from $\{1,\dots,N\}$ to itself such that, setting $J_s = I_{N,s} \cap \iota^{-1}(I'_{N,s})$ for each s, we have $J_s = I_{N,s}$ or $\iota(J_s) = I'_{N,s}$ for every $s \in \mathscr{S}$. This property implies

$$|I_{N,s} \setminus J_s| + \left|I'_{N,s} \setminus \iota(J_s)\right| = \left||I_{N,s}| - |I'_{N,s}|\right| = N\left|\lambda_{N,s} - \lambda'_{N,s}\right|, \quad \forall s \in \mathscr{S}. \tag{2.6}$$

Write $J = (J_s)_{s\in\mathscr{S}}$ and $\iota(J) = (\iota(J_s))_{s\in\mathscr{S}}$. We can apply (2.4) and (2.5) to the pair J and I_N and the pair $\iota(J)$ and I'_N. These along with the triangle inequality give

$$\begin{aligned}\left|\overline{F}_{N,\lambda_N} - \overline{F}_{N,\lambda'_N}\right| &\leqslant \left|\overline{F}^J_N - \overline{F}^{\iota(J)}_N\right| + CN^{-1}\left|I_N \setminus J\right| + CN^{-1}\left|I'_N \setminus \iota(J_s)\right| \\ &\overset{(2.3),\,(2.6)}{=} 0 + C|\lambda_{N,s} - \lambda'_{N,s}|\end{aligned}$$

for $C = (tC_\xi + |q|_{L^1})\, |\kappa|_\infty + C_\mu$ as announced.

It remains to verify (2.4) and (2.5). We first show the latter. Recall that the reference measure in $\overline{F}^J_{N,I_N}$ is $\mathrm{d}P_N(\sigma)$ as in (1.6) and that in $\overline{F}^J_N$ is $\mathrm{d}P^J_N(\cdots)$ as in (2.2). Also notice that the term $\exp(\cdots)$ in $\overline{F}^J_{N,I_N}$ does not depend on spins $\sigma_{\bullet n}$ for $n \notin \cup_{s\in\mathscr{S}} J_s$. Hence, we can compute

$$\overline{F}^J_{N,I_N} = \overline{F}^J_N - N^{-1} \sum_{s\in\mathscr{S}} |I_s \setminus J_s| \log \mu_s(\mathbb{R}^{\kappa_s})$$

which gives (2.5).

To show (2.4), we need an interpolation argument. Let us rewrite the terms inside $\exp(\cdots)$ in $\overline{F}_{N,\lambda_N}(t,q)$ (see (1.15)) as $\sqrt{2}\mathbf{G}_1(\sigma,\alpha) + \mathbf{D}_1(\sigma)$, where $\mathbf{G}(\sigma,\alpha)$ collects the Gaussian terms (conditioned on $\mathfrak{R}$) and $\mathbf{D}(\sigma)$ collects the deterministic terms. Set $\mathbf{G}_0(\sigma,\alpha) = \mathbf{G}_1(\sigma^J,\alpha)$ and $\mathbf{D}_0(\sigma) = \mathbf{D}_1(\sigma^J)$. Notice that

$$\mathbb{E}\mathbf{G}_1(\sigma,\alpha)\mathbf{G}_1(\sigma',\alpha') \overset{(1.9),\,(1.11)}{=} Nt\xi(R_N(\sigma,\sigma')) + Nq(\alpha\wedge\alpha')\cdot R_N(\sigma,\sigma')$$

and also the covariance of $\mathbf{G}_0$ is the same with σ above replaced by σ^J. We can apply Corollary 2.2 in the setting (with $\varphi(1) = N\overline{F}_{N,\lambda_N}$ and $\varphi(0) = N\overline{F}^J_{N,I_N}$). Thus, we get

$$N\overline{F}_{N,\lambda_N} - N\overline{F}^J_{N,I_N} = \int_0^1 \mathbb{E}\left\langle t(\xi(R_N) - \xi(R^J_N)) + q(\alpha\wedge\alpha')\cdot(R_N - R^J_N)\right\rangle_r \mathrm{d}r$$

where we used the shorthand $R_N = R_N(\sigma,\sigma')$ and $R^J_N = R_N(\sigma^J,\sigma'^J)$. We denote their s-coordinate as $R_{N,s}$ and $R^J_{N,s}$. We recall an important property of invariance of cascade: for every $r \in [0,1]$,

$$\mathbb{E}\left\langle q(\alpha\wedge\alpha')\right\rangle_r = \mathbb{E}\left\langle q(\alpha\wedge\alpha')\right\rangle_{\mathfrak{R}} = \int_0^1 q(u)\mathrm{d}r. \tag{2.7}$$

This property is restated as in Lemma 5.3 later. By the definition of σ^J, almost surely, we have $|R_{N,s} - R^J_{N,s}| \leqslant \kappa_s N^{-1}|I_{N,s} \setminus J_s|$. Hence, we have $|R_N - R^J_N| \leqslant |\kappa|_\infty N^{-1}|I_N \setminus J|$. Using this and (2.7), we get

$$\left|N\overline{F}_{N,\lambda_N} - N\overline{F}^J_{N,I_N}\right| \leqslant tC_\xi|\kappa|_\infty|I_N \setminus J| + |q|_{L^1}|\kappa|_\infty|I_N \setminus J|$$

which implies (2.4). ☐

Now, we turn to the second estimate. We again take $|\kappa|_\infty = \max_{s\in\mathscr{S}} \kappa_s$.

Lemma 2.4. *There are constants* C_ξ *depending only on* ξ *(over* $\prod_{s\in\mathscr{S}}[-1,+1]^{\kappa_s\times\kappa_s}$*) and* C_μ *depending only on* μ *such that the following holds. For every* $N, M \in \mathbb{N}$ *with* $N \geqslant M$*,* $t \in \mathbb{R}_+$*,* $q \in \mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$*,* $\lambda_N \in \blacktriangle_N$*, and* $\lambda_{N+M} \in \blacktriangle_{N+M}$*, we have*

$$\begin{aligned}&\left|N\overline{F}_{N,\lambda_N}(t,q) - (N+M)\overline{F}_{N+M,\lambda_{N+M}}(t,q)\right| \\ &\leqslant 4\left(|\kappa|_\infty C_\xi t + |\kappa|_\infty |q|_{L^1} + C_\mu\right)\left((N+M)\left|\lambda_N - \lambda_{N+M}\right| + M|\mathscr{S}|\right).\end{aligned}$$

Proof: Fixing any t and q, we omit them from the notation of free energy. Let $I_N = (I_{N,s})_{s\in\mathscr{S}}$ and $I_{N+M} = (I_{N+M,s})_{s\in\mathscr{S}}$ be partitions associated with λ_N and λ_{N+M}. Let us fix any partition $I'_{N+M} = (I'_{N+M,s})_{s\in\mathscr{S}}$ of $\{1,\dots,M\}$ satisfying

$$I_{N,s} \subseteq I'_{N+M,s}, \quad \forall s \in \mathscr{S}. \tag{2.8}$$

Let λ'_{N+M} be the associated proportions. Before proceeding, we derive some simple estimates. By (2.8), we get

$$|I'_{N+M,s}| - |I_{N,s}| \leqslant M, \quad \forall s \in \mathscr{S}. \tag{2.9}$$

Using this, we also have

$$|\lambda'_{N+M,s} - \lambda_{N,s}| \overset{(1.2)}{\leqslant} \frac{|I'_{N+M,s}| - |I_{N,s}|}{N+M} \overset{(2.8)}{\leqslant} \frac{M}{N+M}. \tag{2.10}$$

We first compare $\overline{F}_{N,\lambda_N}$ with $\overline{F}_{N+M,\lambda'_{N+M}}$. Denote by $\sigma \in \Sigma^{N+M}$ the configuration appearing in $\overline{F}_{N+M,\lambda'_{N+M}}$. For every such $\sigma = (\sigma_{\bullet n})_{1\leqslant n\leqslant N+M}$, write $\overline{\sigma} = (\sigma_{\bullet n})_{1\leqslant n\leqslant N}$. By (2.8) and the fact that I_N is a partition of $\{1,\dots,N\}$, we have $\overline{\sigma} = (\sigma_{\bullet n})_{n\in\cup_{s\in\mathscr{S}} I_{N,s}}$ and $\overline{\sigma}$ is the configuration appearing in $\overline{F}_{N,\lambda_N}$.

Let $\sqrt{2}\mathbf{G}_1(\sigma,\alpha)$ and $\mathbf{D}_1(\sigma)$ (resp. $\sqrt{2}\mathbf{G}_0(\sigma,\alpha)$ and $\mathbf{D}_0(\sigma)$) collect respectively Gaussian terms and deterministic terms inside $\exp(\cdots)$ in $\overline{F}_{N,\lambda_N}$ (resp. $\overline{F}_{N+M,\lambda'_{N+M}}$). Notice that $\mathbf{G}_1$ and $\mathbf{D}_1$ depends on σ only through $\overline{\sigma}$. With this setup, we apply Corollary 2.2 (with $\varphi(1) = N\overline{F}_{N,\lambda_N}$ and $\varphi(0) = (N+M)\overline{F}_{N+M,\lambda'_{N+M}}$). We can use (1.9) and (1.13) to compute the covariances of $\mathbf{G}_0$ and $\mathbf{G}_1$. Then, Corollary 2.2 implies

$$\begin{aligned}&\left|N\overline{F}_{N,\lambda_N} - (N+M)\overline{F}_{N+M,\lambda'_{N+M}}\right| \\ &= \int_0^1 \mathbb{E}\left\langle Nt\xi(\overline{R}) - (N+M)t\xi(R) + q(\alpha\wedge\alpha')\cdot(N\overline{R} - (N+M)R)\right\rangle_r \mathrm{d}r\end{aligned} \tag{2.11}$$

where we used the short hand $\overline{R} = R_{N,\lambda_N}(\overline{\sigma},\overline{\sigma}')$ and $R = R_{N+M,\lambda'_{N+M}}(\sigma,\sigma')$. Notice that $\overline{R}$ and R depend on the partition I_N and I'_{N+M} respectively (see (1.7)). By the definition of $\overline{\sigma}$, we have the following a.s. entry-wise bound

$$\left|N\overline{R}_s - (N+M)R_s\right| \leqslant \kappa_s\left(|I'_{N+M,s}| - |I_{N,s}|\right) \overset{(2.9)}{\leqslant} \kappa_s M, \quad \forall s \in \mathscr{S}.$$

Using this and $|\overline{R}| \leqslant |\kappa|_\infty|\mathscr{S}|$, we can get

$$\left|\overline{R} - R\right| \leqslant \frac{\left|N\overline{R} - (N+M)R\right|}{N+M} + \frac{M|\overline{R}|}{N+M} \leqslant \frac{2|\kappa|_\infty M|\mathscr{S}|}{N}.$$

Using this and (2.7), we can bound the absolute value of the integrand in (2.11) by

$$\begin{aligned}&\mathbb{E}\left\langle Nt\left|\xi(\overline{R}) - \xi(R)\right| + Mt\left|\xi(R)\right| + \left|q(\alpha\wedge\alpha')\right|\left(N\left|\overline{R} - R\right| + M\left|R\right|\right)\right\rangle_r \\ &\leqslant 3\left(tC_\xi + |q|_{L^1}\right)|\kappa|_\infty M|\mathscr{S}|\end{aligned}$$

where C_ξ only depends on ξ over $\prod_{s\in\mathscr{S}}[-1,+1]^{\kappa_s\times\kappa_s}$. Inserting this back to (2.11), we get

$$\left|N\overline{F}_{N,\lambda_N} - (N+M)\overline{F}_{N+M,\lambda'_{N+M}}\right| \leqslant 3\left(tC_\xi + |q|_{L^1}\right)|\kappa|_\infty M|\mathscr{S}|.$$

On the other hand, Lemma 2.3 yields

$$\left|\overline{F}_{N+M,\lambda'_{N+M}} - \overline{F}_{N+M,\lambda_{N+M}}\right| \leqslant (tC_\xi|\kappa|_\infty + |q|_{L^1}|\kappa|_\infty + C_\mu)\left|\lambda'_{N+M} - \lambda_{N+M}\right|.$$

Notice that

$$\left|\lambda'_{N+M} - \lambda_{N+M}\right| \overset{(2.10)}{\leqslant} |\lambda_N - \lambda_{N+M}| + \frac{M|\mathscr{S}|}{N+M}$$

Combining the above two displays, we get the desired result. □

3. Relation to vector spin glasses

In this section, we show that if $\lambda_{N,s}$ is rational for every $s \in \mathscr{S}$, we can equate $\overline{F}_{N,\lambda_N}$ to the free energy of a vector spin glass. We start by introducing the setting of vector spin glass models in Section 3.1 and then prove Lemma 3.1 and Corollary 3.2.

3.1. *Vector spin glasses.* Let $D \in \mathbb{N}$ be the dimension for vector-valued spins and let P_1^{vec} be a finite nonnegative measure supported on a compact set in $\mathbb{R}^D$. We view P_1^{vec} as the distribution for a single spin. For each $N \in \mathbb{N}$, a spin configuration with size N is denoted by $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{dn})_{1\leqslant d\leqslant D,\,1\leqslant n\leqslant N}$. Column vectors $\boldsymbol{\sigma}_{\bullet n}$ are individual spins in $\boldsymbol{\sigma}$. We sample $\boldsymbol{\sigma}$ by independently drawing each $\boldsymbol{\sigma}_{\bullet n}$ according to P_1. More precisely, denoting the distribution of $\boldsymbol{\sigma}$ as P_N^{vec}, we have

$$\mathrm{d}P_N^{\mathrm{vec}}(\boldsymbol{\sigma}) = \otimes_{n=1}^N \mathrm{d}P_1^{\mathrm{vec}}(\boldsymbol{\sigma}_{\bullet n}). \tag{3.1}$$

Given a smooth function $\boldsymbol{\xi} : \mathbb{R}^{D\times D} \to \mathbb{R}$, for each $N \in \mathbb{N}$, we assume the existence of a centered Gaussian process $(H_N^{\mathrm{vec}}(\boldsymbol{\sigma}))_{\boldsymbol{\sigma}\in\mathbb{R}^{D\times N}}$ with covariance

$$\mathbb{E}H_N^{\mathrm{vec}}(\boldsymbol{\sigma})H_N^{\mathrm{vec}}(\boldsymbol{\sigma}') = N\boldsymbol{\xi}\left(\frac{\boldsymbol{\sigma}\boldsymbol{\sigma}'^{\intercal}}{N}\right). \tag{3.2}$$

We interpret $\frac{\boldsymbol{\sigma}\boldsymbol{\sigma}'^{\intercal}}{N}$ as the $\mathbb{R}^{D\times D}$-valued overlap between configurations $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$. For $\mathbf{q} \in \mathcal{Q}_\infty(D)$ (see Section 1.1.1), conditioned on $\mathfrak{R}$, let $(\mathbf{w}^{\mathbf{q}}(\alpha))_{\alpha\in\operatorname{supp}\mathfrak{R}}$ be the $\mathbb{R}^D$-valued centered Gaussian process with covariance

$$\mathbb{E}\mathbf{w}^{\mathbf{q}}(\alpha)\mathbf{w}^{\mathbf{q}}(\alpha') = \mathbf{q}(\alpha\wedge\alpha'). \tag{3.3}$$

Conditioned on $\mathfrak{R}$, for each $i \in \{1,\ldots,N\}$, let $\mathbf{w}_i^{\mathbf{q}}$ be independent copies of $\mathbf{w}^{\mathbf{q}}$. Then, we set

$$W_N^{\mathbf{q}}(\alpha) = \left(\mathbf{w}_1^{\mathbf{q}}(\alpha),\ldots,\mathbf{w}_N^{\mathbf{q}}(\alpha)\right), \quad \forall \alpha \in \operatorname{supp}\mathfrak{R}. \tag{3.4}$$

We view $W_N^{\mathbf{q}}(\alpha)$ as an $\mathbb{R}^{D\times N}$-valued process with column vectors $\mathbf{w}_i^{\mathbf{q}}$. For each $N \in \mathbb{N}$, $t \in \mathbb{R}_+$, and $\mathbf{q} \in \mathcal{Q}_\infty(D)$, we consider the Hamiltonian and free energy:

$$H_N^{\mathrm{vec},t,\mathbf{q}}(\boldsymbol{\sigma},\alpha) = \sqrt{2t}H_N^{\mathrm{vec}}(\boldsymbol{\sigma}) - N\boldsymbol{\xi}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^{\intercal}/N\right) + \sqrt{2}W_N^{\mathbf{q}}(\alpha)\cdot\boldsymbol{\sigma} - \mathbf{q}(1)\cdot\boldsymbol{\sigma}\boldsymbol{\sigma}^{\intercal}, \tag{3.5}$$

$$\overline{F}_N^{\mathrm{vec}}(t,\mathbf{q}) = -\frac{1}{N}\mathbb{E}\log\iint \exp\left(H_N^{\mathrm{vec},t,\mathbf{q}}(\boldsymbol{\sigma},\alpha)\right)\mathrm{d}P_N^{\mathrm{vec}}(\boldsymbol{\sigma})\mathrm{d}\mathfrak{R}(\alpha), \tag{3.6}$$

where the expectation is first taken over Gaussian randomness in H_N^{vec} and $W_N^{\mathbf{q}}$ and then over the randomness in $\mathfrak{R}$.

3.2. *Reduction.* For $M \in \mathbb{N}$, we call a collection $(\mathsf{M}_s)_{s\in\mathscr{S}}$ of subsets a **weak partition** of $\{1,\dots,M\}$ if $\cup_{s\in\mathscr{S}}\mathsf{M}_s = \{1,\dots,M\}$ and $\mathsf{M}_s \cap \mathsf{M}_{s'} = \emptyset$ whenever $s \neq s'$. This differs from the standard notion in that we allow M_s to be empty. We work with the multi-species spin glass with system size MN for $N \in \mathbb{N}$ and with species proportion satisfying

$$\lambda_{MN,s} = |\mathsf{M}_s|/M, \quad \forall s \in \mathscr{S} \tag{3.7}$$

for some weak partition $(\mathsf{M}_s)_{s\in\mathscr{S}}$ of $\{1,\dots,M\}$. Under this assumption, among MN spins of the multi-species configuration σ, there are exactly $|\mathsf{M}_s|N$ spins belonging to the s-species for each $s \in \mathscr{S}$.

We want to map this model to a vector spin model with spins in $\mathbb{R}^\Delta$ and size N, where

$$\Delta = \Delta(M,\lambda_{MN}) = \sum_{s\in\mathscr{S}} \lambda_{MN,s} M\kappa_s = \sum_{s\in\mathscr{S}} |\mathsf{M}_s|\kappa_s. \tag{3.8}$$

Notice that Δ does not depend on N due to the form of $\lambda_{MN,s}$ in (3.7). We reorder $\{1,2,\dots,\Delta\}$ by fixing an arbitrary bijection:

$$\{1,2,\dots,\Delta\} \quad \longleftrightarrow \quad \cup_{s\in\mathscr{S}}\{(m,k):\ m\in\mathsf{M}_s,\ k\in\{1,\dots,\kappa_s\}\}. \tag{3.9}$$

Under this re-ordering, we view $a \in \mathbb{R}^{\Delta\times\Delta}$ and $b \in \mathbb{R}^{\Delta\times N}$ as $a = (a_{(m,k)(m',k')})_{m,m';k,k'}$ and $b = (b_{(m,k)n})_{m,k,n}$. For $a \in \mathbb{R}^{\Delta\times\Delta}$, we use the notation, for $m \in \mathsf{M}_s$ and $m' \in \mathsf{M}_{s'}$

$$a_{(m,\bullet)(m',\bullet)} = \left(a_{(m,k)(m',k')}\right)_{1\leqslant k\leqslant\kappa_s,\,1\leqslant k'\leqslant\kappa_{s'}} \in \mathbb{R}^{\kappa_s\times\kappa_{s'}}. \tag{3.10}$$

We can fix a bijection $\sigma \mapsto \boldsymbol{\sigma}$ from Σ^{MN} (see (1.5)) to $[-1,+1]^{\Delta\times N}$ with the property:

$$\forall i \in I_{N,s},\ \exists m \in \mathsf{M}_s,\ \exists n \in \{1,\dots,N\}: \quad \sigma_{\bullet i} \mapsto \boldsymbol{\sigma}_{(m,\bullet)n} = (\boldsymbol{\sigma}_{(m,k)n})_{1\leqslant k\leqslant\kappa_s}. \tag{3.11}$$

In words, every single spin in the s-species is mapped to a sub-column vector crossing κ_s rows with indices (m,k), for $k \in \{1,\dots,\kappa_s\}$, for some $m \in \mathsf{M}_s$. Under such a bijection, we have the following correspondence of overlaps (recall the overlap (1.7) in the multi-species setting):

$$R_{MN,\lambda_{MN},s}(\sigma,\sigma') = (MN)^{-1}\sum_{m\in\mathsf{M}_s}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'^{\intercal}\right)_{(m,\bullet)(m,\bullet)}, \quad \forall s\in\mathscr{S}, \tag{3.12}$$

where we have used the notation in (3.10) and each entry in $(\boldsymbol{\sigma}\boldsymbol{\sigma}'^{\intercal})_{(m,\bullet)(m,\bullet)}$ is given by

$$\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'^{\intercal}\right)_{(m,k)(m,k')} = \sum_{n=1}^N \boldsymbol{\sigma}_{(m,k)n}\boldsymbol{\sigma}'_{(m,k')n}, \quad \forall k,k' \in \{1,\dots,\kappa_s\}. \tag{3.13}$$

This is the usual matrix multiplication in view of the re-ordering (3.9).

We describe the corresponding single vector spin distribution. Let P_1^{vec} be the finite nonnegative measure supported on $[-1,1]^\Delta$ given by

$$\mathrm{d}P_1^{\mathrm{vec}}(\boldsymbol{\sigma}) = \otimes_{s\in\mathscr{S}}\otimes_{m\in\mathsf{M}_s}\mathrm{d}\mu_s\left(\boldsymbol{\sigma}_{(m,\bullet)1}\right) \tag{3.14}$$

Suggested by (3.12), for ξ in (1.9), we take

$$\boldsymbol{\xi}(a) = M\xi\left(\left(M^{-1}\sum_{m\in\mathsf{M}_s} a_{(m,\bullet)(m,\bullet)}\right)_{s\in\mathscr{S}}\right), \quad \forall a \in \mathbb{R}^{\Delta\times\Delta}, \tag{3.15}$$

Lastly, we describe the corresponding parameter for the external field. To each $q \in \mathcal{Q}_\infty^{\mathscr{S}}(\kappa)$ (see (1.10)), we associate $\mathbf{q} \in \mathcal{Q}_\infty(\Delta)$ given by

$$\mathbf{q}_{(m,\bullet)(m',\bullet)} = 0, \quad \forall m \neq m'; \qquad \mathbf{q}_{(m,\bullet)(m,\bullet)} = q_s, \quad \forall m \in \mathsf{M}_s. \tag{3.16}$$

Lemma 3.1 (Relation between multi-species and vector spin glass)**.** *Let $M \in \mathbb{N}$ and let $(\mathsf{M}_s)_{s\in\mathscr{S}}$ be a weak partition of $\{1,\dots,M\}$. Correspondingly, let P^{vec}_1 and $\boldsymbol{\xi}$ be given as in (3.14) and (3.15) and let $\left(\overline{F}^{\mathrm{vec}}_N\right)_{N\in\mathbb{N}}$ be the associated free energy given in (3.6). In particular, the vector spin is Δ-dimensional for Δ in (3.8). Then, for every $N\in\mathbb{N}$, λ_{MN} satisfying (3.7), and $(t,q)\in\mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$, we have*

$$\overline{F}_{MN,\,\lambda_{MN}}(t,q) = M^{-1}\overline{F}^{\mathrm{vec}}_N(t,\mathbf{q}) \tag{3.17}$$

with $\mathbf{q}$ given in (3.16).

Proof: We identify the two types of configurations σ and $\boldsymbol{\sigma}$ via a bijection described as in (3.11). Under this identification, we can see that $P_{MN,\lambda_{MN}}$ in (1.6) is the same as P^{vec}_N in (3.1) with P^{vec}_1 in (3.14). Next, using (3.12) and (3.15), we have $N\boldsymbol{\xi}(\boldsymbol{\sigma}\boldsymbol{\sigma}'^{\intercal}/N) = MN\xi(R_{MN,\lambda_{MN}}(\sigma,\sigma'))$. From this and the covariances (1.9) and (3.2), we can deduce that $(H_{MN}(\sigma))_\sigma \overset{\mathrm{d}}{=} (H^{\mathrm{vec}}_N(\boldsymbol{\sigma}))_{\boldsymbol{\sigma}}$. It remains to verify that the external fields have the same distribution. We can compute

$$\begin{aligned}\mathbb{E}W^q_{MN}(\alpha,\sigma)W^q_{MN}(\alpha',\sigma') &\overset{(1.13)}{=} MN\sum_{s\in\mathscr{S}} q_s(\alpha\wedge\alpha')\cdot R_{MN,\lambda_{MN},s}(\sigma,\sigma')\\ &\overset{(1.7),(3.12)}{=} \sum_{s\in\mathscr{S}} q_s(\alpha\wedge\alpha')\cdot\sum_{m\in\mathsf{M}_s}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'^{\intercal}\right)_{(d\bullet)(d\bullet)} \overset{(3.16)}{=} \mathbf{q}(\alpha\wedge\alpha')\cdot\boldsymbol{\sigma}\boldsymbol{\sigma}'^{\intercal}.\end{aligned}$$

This implies that, under the identification (3.11) and conditioned on $\mathfrak{R}$, we have

$$\left(W^q_{MN}(\alpha,\sigma)\right)_\sigma \overset{\mathrm{d}}{=} \left(W^{\mathbf{q}}_N(\alpha)\cdot\boldsymbol{\sigma}\right)_{\boldsymbol{\sigma}} \tag{3.18}$$

where the latter appears in $\overline{F}^{\mathrm{vec}}_N(t,\mathbf{q})$ (see (3.6)). The equivalence of distributions of various Gaussian terms also ensures that the self-overlap correction terms are the same in $\overline{F}_{MN,\,\lambda_{MN}}(t,q)$ and $\overline{F}^{\mathrm{vec}}_N(t,\mathbf{q})$. Hence, under the identification between σ and $\boldsymbol{\sigma}$ as in (3.11), we have thus verified

$$\left(H^{t,q}_{MN}(\sigma,\alpha)\right)_{\sigma\in\Sigma^{MN}} \overset{\mathrm{d}}{=} \left(H^{\mathrm{vec},t,\mathbf{q}}_{MN}(\boldsymbol{\sigma},\alpha)\right)_{\boldsymbol{\sigma}\in[-1,+1]^{\Delta\times N}} \tag{3.19}$$

where the two sides are given in (1.14) and (3.5). In particular, this yields (3.17). □

For any $r\in\mathbb{R}$, write $\lceil r\rceil = \min\{n\in\mathbb{N} : n\geqslant r\}$.

Corollary 3.2 (Equivalence in the rational case)**.** *Assume that there are $M\in\mathbb{N}$ and a weak partition $(\mathsf{M}_s)_{s\in\mathscr{S}}$ of $\{1,\dots,M\}$ such that*

$$\lim_{N\to\infty}\lambda_{N,s} = |\mathsf{M}_s|/M, \quad \forall s\in\mathscr{S}.$$

Let $\overline{F}^{\mathrm{vec}}_N$ be the free energy with P^{vec}_1 and $\boldsymbol{\xi}$ specified in (3.14) and (3.15). For every $t\in\mathbb{R}_+$ and $q\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$, let $\mathbf{q}$ be given as in (3.16) and we have

$$\lim_{N\to\infty}\left|\overline{F}_{N,\lambda_N}(t,q) - M^{-1}\overline{F}^{\mathrm{vec}}_{\lceil N/M\rceil}(t,\mathbf{q})\right| = 0.$$

Proof: Fixing t and q, we omit them and $\mathbf{q}$ from the notation. For two sequences $(r_N)_{N\in\mathbb{N}}$ and $(r'_N)_{N\in\mathbb{N}}$ of real numbers, we write $r_N\approx r'_N$ provided $\lim_{N\to\infty}|r_N - r'_N| = 0$. We need the following estimates for every $N\in\mathbb{N}$,

$$0\leqslant \lceil N/M\rceil M - N\leqslant M, \qquad \left|\overline{F}_{N,\lambda_N}\right|\leqslant C \tag{3.20}$$

for some constant $C>0$. The latter estimate follows from Jensen's inequality applied to the expression of $\overline{F}_{N,\lambda_N}$ (see (1.15)). Set $\lambda_\infty = (|\mathsf{M}_s|/M)_{s\in\mathscr{S}}$. Now, we can get

$$\overline{F}_{N,\lambda_N} \overset{\mathrm{L.}2.3}{\approx} \overline{F}_{N,\lambda_\infty} \overset{(3.20),\,\mathrm{L.}2.4}{\approx} \overline{F}_{\lceil N/M\rceil M,\,\lambda_\infty} \overset{\mathrm{L.}3.1}{=} M^{-1}\overline{F}^{\mathrm{vec}}_{\lceil N/M\rceil}$$

which gives the desired result. □

4. Analytic properties of the free energy

We study analytic properties of the free energy, its limits (if it exists), and the initial condition at $t = 0$. In Section 4.1, we show that the free energy is differentiable and monotone, and that any subsequence of $(\overline{F}_{N,\lambda_N})_{N\in\mathbb{N}}$ is precompact in the local uniform topology. In Section 4.2, we show that the free energy is locally semi-concave uniformly in N and then deduce properties of any subsequential limit. In Section 4.3, we identify $\overline{F}_{N,\lambda_N}(0,\cdot)$ and establish its regularity properties. Results in this section adapt those in Chen and Mourrat (2025, Sections 3 and 5).

4.1. *Differentiability, monotonicity, and precompactness.* Recall the notation of $\mathcal{Q}_\square(D)$ in Section 1.1.1 and $\mathcal{Q}_\square(\kappa)$ in (1.10) for spaces of paths.

Let G be either $\mathcal{Q}_2^{\mathscr{S}}(\kappa)$, $\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$, or $\mathcal{Q}_2(D)$ for some $D\in\mathbb{N}$. Let L^2 be the ambient Hilbert space for G. A function $g : G\to\mathbb{R}$ is said to be **Fréchet differentiable** at some $q\in G$ if there is a unique $y\in L^2$ such that

$$\lim_{r\to 0}\sup_{\substack{q'\in G\setminus\{q\}\\ |q'-q|_{L^2}\leqslant r}}\frac{|g(q')-g(q)-\langle y,q'-q\rangle_{L^2}|}{|q'-q|_{L^2}}=0. \tag{4.1}$$

In this case, we call y the **Fréchet derivative** of g at q.

For every $q\in G$, we define

$$\mathrm{Adm}(G,q)=\left\{e\in L^2\;\middle|\;\exists r>0:\ \forall r'\in[0,r],\ q+r'e\in G\right\}$$

to be set of directions along which a small line segment starting from q belongs to G. We say that $g : G\to\mathbb{R}$ is **Gateaux differentiable** at $q\in G$ if

- $g'(q,e)=\lim_{r\searrow 0}\frac{g(q+re)-g(q)}{r}$ exists for every $e\in\mathrm{Adm}(G,q)$;
- there is a unique $y\in L^2$ such that $g'(q,e)=\langle y,e\rangle_{L^2}$ for every $e\in\mathrm{Adm}(G,q)$.

In this case, we call y the **Gateaux derivative** of g at q.

If g is differentiable at q in either of the two senses, we denote its derivative at q by $\partial_q g(q)$, which is an element in L^2.

For any $D\in\mathbb{N}$ and $u\in\mathbb{R}_+$, we define

$$\mathcal{Q}_{\infty,\leqslant u}(D)=\{q\in\mathcal{Q}_\infty(D):\ |q(r)|\leqslant u,\quad\forall r\in[0,1)\}.$$

Then, for any $\lambda=(\lambda_s)_{s\in\mathscr{S}}\in\mathbb{R}_+^{\mathscr{S}}$, we define

$$\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda}(\kappa)=\prod_{s\in\mathscr{S}}\mathcal{Q}_{\infty,\leqslant\lambda_s}(\kappa_s). \tag{4.2}$$

Recall $\overline{F}_{N,\lambda_N}$ in (1.15) and $\langle\cdot\rangle_{N,\lambda_N}$ in (1.16).

Proposition 4.1 (Differentiability of $\overline{F}_{N,\lambda_N}$)**.** *Let $N\in\mathbb{N}$, $\lambda_N\in\blacktriangle_N$, and let $\overline{F}_{N,\lambda_N}$ be given as in (1.15). We have for every $t,t'\in\mathbb{R}_+$ and $q,q'\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$ that*

$$\left|\overline{F}_{N,\lambda_N}(t,q)-\overline{F}_{N,\lambda_N}(t',q')\right|\leqslant\left|q-q'\right|_{L^1}+|t-t'|\sup_{|a|\leqslant 1}|\xi(a)|.$$

In particular, the free energy in (1.15) can be extended by continuity to $\mathbb{R}_+\times\mathcal{Q}_1^{\mathscr{S}}(\kappa)$. Moreover, the restriction of the function $\overline{F}_{N,\lambda_N}$ to $\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$ is Fréchet (and Gateaux) differentiable everywhere, jointly in its two variables. We denote its Fréchet (and Gateaux) derivative in q by $\partial_q\overline{F}_{N,\lambda_N}(t,q)=\partial_q\overline{F}_{N,\lambda_N}(t,q,\cdot)\in L^2([0,1];\prod_{s\in\mathscr{S}}S^{\kappa_s})$. For every $t\geqslant 0$, we have, for every $q\in\mathcal{Q}_2^{\mathscr{S}}(\kappa)$,

$$\partial_q\overline{F}_{N,\lambda_N}(t,q)\in\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda_N}(\kappa),\qquad\left|\partial_t\overline{F}_{N,\lambda_N}(t,q)\right|\leqslant\sup_{|a|\leqslant 1}|\xi(a)|, \tag{4.3}$$

and, for every $q \in \mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$ *and* $\pi \in L^2([0,1]; \prod_{s\in\mathscr{S}} S^{\kappa_s})$,

$$\begin{aligned} \left\langle \pi, \partial_q \overline{F}_{N,\lambda_N}(t,q)\right\rangle_{L^2} &= \mathbb{E}\left\langle \pi\left(\alpha\wedge\alpha'\right)\cdot R_{N,\lambda_N}(\sigma,\sigma')\right\rangle_{N,\lambda_N} \\ \partial_t \overline{F}_{N,\lambda_N}(t,q) &= \mathbb{E}\left\langle \xi\left(R_{N,\lambda_N}(\sigma,\sigma')\right)\right\rangle_{N,\lambda_N}. \end{aligned} \tag{4.4}$$

Finally, for every $r \in [1,+\infty]$, $t \geqslant 0$ *and* $q, q' \in \mathcal{Q}^{\mathscr{S}}_2(\kappa)$ *with* $q'-q \in L^r$, *we have*

$$\left|\partial_q \overline{F}_{N,\lambda_N}(t,q') - \partial_q \overline{F}_{N,\lambda_N}(t,q)\right|_{L^r} \leqslant 16N\left|q'-q\right|_{L^r}. \tag{4.5}$$

In particular, the mapping $q \mapsto \partial_q \overline{F}_{N,\lambda_N}(t,q)$ *can be extended to* $\mathcal{Q}^{\mathscr{S}}_1(\kappa)$ *by continuity, and the properties in* (4.3) *and* (4.5) *remain valid with* $q, q' \in \mathcal{Q}^{\mathscr{S}}_1(\kappa)$.

The proof follows the same lines as those for Chen and Mourrat (2025, Proposition 5.1), which are based on the Gaussian interpolation arguments. We only need to explain the first inclusion in (4.3). This follows from the first line in (4.4), the fact that $|R_{N,\lambda_N,s}(\sigma,\sigma')| \leqslant \lambda_{N,s}$ due to (1.7), and the invariance of cascade in Lemma 5.3.

Proposition 4.2 (Precompactness of $\overline{F}_{N,\lambda_N}$)**.** *For every* $r \in (1,+\infty]$ *and any sequence* $(\lambda_N)_{N\in\mathbb{N}}$ *with* $\lambda_N \in \blacktriangle_N$, *any subsequence of* $\left(\overline{F}_{N,\lambda_N}\right)_{N\in\mathbb{N}}$ *has a further subsequence which converges uniformly on every bounded metric ball in* $\mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_r(\kappa)$.

This proposition can be proved in the same way as Chen and Mourrat (2025, Proposition 3.3) using the following lemma.

Lemma 4.3 (Compact embedding of paths)**.** *Let* $r \in (1,+\infty]$ *and let* $(q_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{Q}^{\mathscr{S}}_r(\kappa)$ *such that*

$$\sup_{N\in\mathbb{N}} |q_n|_{L^r} < +\infty.$$

Then, there exists a subsequence $(q_{n_k})_{k\in\mathbb{N}}$ *and some* $q \in \mathcal{Q}^{\mathscr{S}}_r(\kappa)$ *such that, for every* $r' \in [1,r)$, *this subsequence converges almost everywhere on* $[0,1]$ *and in* $L^{r'}$ *to* q.

This lemma is a straightforward adaption of Chen and Mourrat (2025, Lemma 3.4) (a special case with $|\mathscr{S}| = 1$).

Next, we describe a monotonicity property of the free energy in the second variable. Given a nonempty closed convex cone $\mathcal{C}$ in a Hilbert space L^2, we define the dual cone of $\mathcal{C}^*$ by

$$\mathcal{C}^* = \left\{p \in L^2 \;\middle|\; \forall q \in \mathcal{C},\ \langle p, q\rangle_{L^2} \geqslant 0\right\}. \tag{4.6}$$

For each $s \in \mathscr{S}$, the dual cone of $\mathcal{Q}_2(\kappa_s)$ has a simple characterization given by Chen and Mourrat (2025, Lemma 3.5) (see also Chen and Xia, 2025a, Lemma 3.4(2)):

$$\mathcal{Q}_2(\kappa_s)^* = \left\{p \in L^2 \;\middle|\; \forall t \in [0,1),\ \int_t^1 p(r)\mathrm{d}r \in S^{\kappa_s}_+\right\}.$$

Since any $p \in \prod_{s\in\mathscr{S}} \mathcal{Q}_2(\kappa_s)^*$ satisfies exactly $\langle p, q\rangle_{L^2} = \sum_{s\in\mathscr{S}} \langle p_s, q_s\rangle_{L^2} \geqslant 0$ for every $q \in \mathcal{Q}^{\mathscr{S}}_2(\kappa)$, we can identify

$$\left(\mathcal{Q}^{\mathscr{S}}_2(\kappa)\right)^* = \prod_{s\in\mathscr{S}} \mathcal{Q}_2(\kappa_s)^*.$$

For a subset G of $L^2([0,1]; \prod_{s\in\mathscr{S}} S^{\kappa_s})$ and a function $g : G \to \mathbb{R}$, we say that g is $\left(\mathcal{Q}^{\mathscr{S}}_2(\kappa)\right)^*$**-increasing** if for every $q, q' \in G$, we have

$$q - q' \in \left(\mathcal{Q}^{\mathscr{S}}_2(\kappa)\right)^* \quad\Longrightarrow\quad g(q) \geqslant g(q'). \tag{4.7}$$

Proposition 4.4 (Monotonicity of free energy)**.** *For every* $N \in \mathbb{N}$, $\lambda_N \in \blacktriangle_N$, *and* $t \in \mathbb{R}_+$, *the function* $\overline{F}_{N,\lambda_N}(t,\cdot)$ *is* $\left(\mathcal{Q}^{\mathscr{S}}_2(\kappa)\right)^*$*-increasing.*

Proof: Fix any N, λ_N, and t. We apply Lemma 3.1 with M, N therein substituted with $N, 1$ respectively. Hence, M_s in (3.7) corresponds to $I_{N,s}$ associated with λ_N. Then, Lemma 3.1 gives $\overline{F}_{N,\lambda_N}(t,q) = N^{-1}\overline{F}_1^{\mathrm{vec}}(t,\mathbf{q})$ for every $q \in \mathcal{Q}_\infty^{\mathscr{S}}(\kappa)$ and $\mathbf{q} \in \mathcal{Q}_\infty(\Delta)$ given in (3.16). We are allowed by Proposition 4.1 to take $q \in \mathcal{Q}_2^{\mathscr{S}}(\kappa)$ instead of $\mathcal{Q}_\infty^{\mathscr{S}}(\kappa)$. Given any q', we define $\mathbf{q}'$ in the same was as in (3.16).

Now, let us assume $q - q' \in \left(\mathcal{Q}_2^{\mathscr{S}}(\kappa)\right)^*$ and we want to show $\mathbf{q} - \mathbf{q}' \in \mathcal{Q}_2(\Delta)^*$ defined as in (4.6). Let $\mathbf{p}$ be any element in $\mathcal{Q}_2(\Delta)$. Using the notation in (3.10) and the form of $\mathbf{q}$ and $\mathbf{q}'$ given in (3.16), we can compute

$$\left\langle \mathbf{p}, \mathbf{q} - \mathbf{q}' \right\rangle_{L^2} = \sum_{s\in\mathscr{S}} \sum_{m\in\mathsf{M}_s} \left\langle \mathbf{p}_{(m,\bullet)(m,\bullet)},\ \mathbf{q}_{(m,\bullet)(m,\bullet)} - \mathbf{q}'_{(m,\bullet)(m,\bullet)} \right\rangle_{L^2}.$$

Since $\mathbf{p}$ is an increasing path in S_+^Δ and each $\mathbf{p}_{(m,\bullet)(m,\bullet)}$ is a projection of $\mathbf{p}$ into a minor matrix, we have that $\mathbf{p}_{(m,\bullet)(m,\bullet)}$ is an increasing path in $S_+^{\kappa_s}$ for $m \in \mathsf{M}_s$. From the second part in (3.16), we can see $\mathbf{q}_{(m,\bullet)(m,\bullet)} - \mathbf{q}'_{(m,\bullet)(m,\bullet)} \in \mathcal{Q}_2(\kappa_s)^*$ (from $q - q' \in \left(\mathcal{Q}_2^{\mathscr{S}}(\kappa)\right)^*$). Therefore, by the definition of dual cones in (4.6), the right-hand side of the above display is nonnegative. Since $\mathbf{p}$ is arbitrary, we conclude that $\mathbf{q} - \mathbf{q}' \in \mathcal{Q}_2(\Delta)^*$.

By the result in the setting of vector spin glasses in Chen and Mourrat (2025, Proposition 3.6), we have that $\overline{F}_1^{\mathrm{vec}}(t,\cdot)$ is $\mathcal{Q}_2(\Delta)^*$-nondecreasing defined in the same way as in (4.7). Therefore, $\overline{F}_1^{\mathrm{vec}}(t,\mathbf{q}) \geqslant \overline{F}_1^{\mathrm{vec}}(t,\mathbf{q}')$ and thus $\overline{F}_{N,\lambda_N}(t,q) \geqslant \overline{F}_{N,\lambda_N}(t,q')$, which gives the desired result. □

4.2. *Semi-concavity and consequences.* Recall the definition of $\mathcal{Q}_{\uparrow,c}^{\mathscr{S}}(\kappa)$ from (1.20) and (1.10). For any increasing path q, we denote by $\dot{q}$ its distributional derivative.

Proposition 4.5 (Semi-concavity of the free energy)**.** *There exists a constant $C < +\infty$ (depending only on $(\mu_s)_{s\in\mathscr{S}}$, $(\kappa_s)_{s\in\mathscr{S}}$, and ξ) such that, for every $N \in \mathbb{N}$, $\lambda_N \in \blacktriangle_N$, $c > 0$, $t, t' \geqslant c$, $q, q' \in \mathcal{Q}_{\uparrow,c}^{\mathscr{S}}(\kappa)$ with $\dot{q} - \dot{q}' \in L^2$, and $r \in [0,1]$,*

$$\begin{aligned}(1-r)\overline{F}_{N,\lambda_N}(t,q) + r\overline{F}_{N,\lambda_N}(t,' q') - \overline{F}_{N,\lambda_N}\left((1-r)(t,q) + r(t',q')\right)\\ \leqslant Cr(1-r)c^{-2}\left((t-t')^2 + \left|\dot{q} - \dot{q}'\right|_{L^2}^2\right).\end{aligned} \tag{4.8}$$

Proof: The argument is the same as that for Chen and Mourrat (2025, Propositions 3.7 and 3.8). Here, we sketch similar parts and highlight the differences.

By a density argument and Proposition 4.1, it suffices to consider smooth q and q'. We then approximate them by piece-wise constant paths. For each $K \in \mathbb{N}$ and $k \in \{0, \dots, K\}$, we set

$$q_k = q\Big(\frac{k}{K+1}\Big), \qquad \forall k \in \{0,\dots,K\}; \qquad q^K = \sum_{k=0}^K q_k \mathbf{1}_{\left[\frac{k}{K+1}, \frac{k+1}{K+1}\right)} \in \mathcal{Q}^{\mathscr{S}}(\kappa).$$

Recall Ellipt from (1.19). Due to $q \in \mathcal{Q}_{\uparrow,c}^{\mathscr{S}}(\kappa)$, we have

$$\frac{c}{K+1}\mathbf{Id}_{\kappa_s} \leqslant q_{k,s} - q_{k-1,s}, \quad \mathsf{Ellipt}(q_{k,s} - q_{k-1,s}) \leqslant c^{-1}, \quad \forall k \in \{0,\dots,K\},\ s \in \mathscr{S}. \tag{4.9}$$

Here, $\mathbf{Id}_{\kappa_s}$ is the $\kappa_s \times \kappa_s$ identity matrix. Similarly, we construct $(q'_k)_{0\leqslant k\leqslant K}$ and q'^K, which satisfy an analogous version of (4.9). Also set $q_{-1} = q'_{-1} = 0$. We claim that for a constant C as announced in the statement, we have

$$\begin{aligned}(1-r)\overline{F}_{N,\lambda_N}\left(t,q^K\right) + r\overline{F}_{N,\lambda_N}(t,' q'^K) - \overline{F}_{N,\lambda_N}\left((1-r)\left(t,q^K\right) + r\left(t',q'^K\right)\right)\\ \leqslant Cr(1-r)c^{-2}\Big((t-t')^2 + (K+1)\sum_{k=0}^K \left|(q_k - q_{k-1}) - (q'_k - q'_{k-1})\right|^2\Big).\end{aligned} \tag{4.10}$$

Then, we can use approximation arguments by sending $K \to \infty$ as in the proof of Chen and Mourrat (2025, Proposition 3.8) (below (3.34)) to get (4.8).

In the remainder, we explain the proof of (4.10). For preparation, we need some properties on the matrix square root. For every $h \in S^D_{++}$ (see Section 1.1.1) and $a \in S^D$ for any $D \in \mathbb{N}$, we define the first and second order derivatives of matrix square root:

$$\mathcal{D}_{\sqrt{h}}(a) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(\sqrt{h + \varepsilon a} - \sqrt{h} \right), \qquad \mathcal{D}^2_{\sqrt{h}}(a) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(\mathcal{D}_{\sqrt{h+\varepsilon a}}(a) - \mathcal{D}_{\sqrt{h}}(a) \right).$$

Denoting by $h_{\min}$ the smallest eigenvalue of h. Then, Chen and Mourrat (2025, (3.23) and (3.24)) give

$$\left| \mathcal{D}_{\sqrt{h}}(a) \right| \leqslant |a| h_{\min}^{-\frac{1}{2}}/2, \qquad \left| \mathcal{D}^2_{\sqrt{h}}(a) \right| \leqslant |a|^2 h_{\min}^{-\frac{3}{2}}/4. \tag{4.11}$$

To prove (4.10), we need to bound the Hessian of $\overline{F}_{N,\lambda_N}$ after a change of variables. As in the proof of Chen and Mourrat (2025, Proposition 3.7), we focus on the semi-concavity in the second variable for the brevity of presentation. Henceforth, we omit the first variable from the notation by fixing some t and writing $\overline{F}_{N,\lambda_N}(\cdot) = \overline{F}_{N,\lambda_N}(t, \cdot)$. For simplicity, we write $\mathbf{S} = \prod_{s \in \mathscr{S}} S^{\kappa_s}$ and $\mathbf{S}_+ = \prod_{s \in \mathscr{S}} S^{\kappa_s}_+$. We view $\mathbf{S}^{K+1}$ as a subset of the linear space $\left(\prod_{s \in \mathscr{S}} \mathbb{R}^{\kappa_s \times \kappa_s} \right)^{K+1}$ with inner product given by the entry-wise dot product as in (1.1).

When considering $\overline{F}_{N,\lambda_N}$ over paths of the form q^K, we can think of $\overline{F}_{N,\lambda_N}$ as a function of $(q_k)_{0 \leqslant k \leqslant K} \in \mathbf{S}^{K+1}_+$. For each $x = (x_k)_{0 \leqslant k \leqslant K} \in \mathbf{S}^{K+1}_+$, we write $\sqrt{x} = \left(\sqrt{x_k} \right)_{0 \leqslant k \leqslant K}$ and $\sqrt{x_k} = \left(\sqrt{x_{k,s}} \right)_{s \in \mathscr{S}}$ with each $x_k = (x_{k,s})_{s \in \mathscr{S}} \in \mathbf{S}_+$. Then, we define a function $G_N : \mathbf{S}^{K+1} \to \mathbb{R}$ through the relation

$$\overline{F}_{N,\lambda_N}\left(q^K\right) = \overline{F}_{N,\lambda_N}\left((q_k)_{0 \leqslant k \leqslant K}\right) = G_N\left(\left(\sqrt{q_k - q_{k-1}} \right)_{0 \leqslant k \leqslant K} \right). \tag{4.12}$$

We consider the function $\mathbf{S}^{K+1} \ni y \mapsto G_N(y)$. For each $k \in \{0, \ldots, K\}$ and $s \in \mathscr{S}$, we define $\partial_{y_{k,s}} G_N(y) \in S^{\kappa_s}$ via the relation: for every $a \in S^{\kappa_s}$,

$$a \cdot \partial_{y_{k,s}} G_N(y) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} G_N(y_1, \ldots, y_{k-1}, y_k + \varepsilon \overline{a}, y_{k+1}, \ldots, y_K) \Big|_{\varepsilon = 0}$$

where $\overline{a} = (\overline{a}_{s'})_{s' \in \mathscr{S}}$ satisfies $\overline{a}_{s'} = 0$ when $s' \neq s$, and $\overline{a}_s = a$. The Hessian $\nabla^2 G_N(y)$ viewed as a linear map from $\mathbf{S}^{K+1}$ to $\mathbf{S}^{K+1}$ is defined through: for every $\mathbf{a} \in \mathbf{S}^{K+1}$,

$$\mathbf{a} \cdot \nabla^2 G_N(y) \mathbf{a} = \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} G_N(y + \mathbf{a}) \Big|_{\varepsilon = 0}.$$

By the same computation for Chen and Mourrat (2025, (3.28) and (3.29)), we can find a constant C as in the statement of the proposition such that

$$\left| \partial_{y_{k,s}} G_N(y) \right| \leqslant C |y_{k,s}|, \qquad \mathbf{a} \cdot \nabla^2 G_N(y) \mathbf{a} \leqslant C |\mathbf{a}|^2 \tag{4.13}$$

for every k and every $\mathbf{a}$. The computation is basic and involves the Gaussian interpolation technique as in Corollary 2.2 and Jensen's inequality.

Next, we introduce the change of variable: for every $x \in S^{K+1}_+$, we set $\widetilde{G}_N(x) = G_N(\sqrt{x})$. We can make sense of the Hessian of $\widetilde{G}_N$ in the same way as above. As in the second step of the proof of Chen and Mourrat (2025, Proposition 3.7), we can compute, for every $\mathbf{a} \in \mathbf{S}^{K+1}$

$$\mathbf{a} \cdot \widetilde{\nabla}^2 G_N(x) \mathbf{a} = \mathbf{D} \cdot \nabla^2 G_N(\sqrt{x}) \cdot \mathbf{D} + \sum_{k,\, s} \mathbf{D}^2_{k,s} \cdot \partial_{y_{k,s}} G_N(\sqrt{x})$$

where

$$\mathbf{D} = \left(\mathcal{D}_{\sqrt{x_{k,s}}}(a_{k,s}) \right)_{0 \leqslant k \leqslant K,\, s \in \mathscr{S}} \in \mathbf{S}^K, \qquad \mathbf{D}^2_{k,s} = \mathcal{D}^2_{\sqrt{x_{k,s}}}(a_{k,s}).$$

Then, we get

$$\begin{aligned}\mathbf{a}\cdot\widetilde{\nabla}^2 G_N(x)\mathbf{a} &\overset{(4.13)}{\leqslant} C\,|\mathbf{D}|^2 + C\sum_{k=0}^{K}\left|\mathbf{D}_k^2\right|\,|\sqrt{x_k}| \\ &\overset{(4.11)}{\leqslant} \frac{C}{4}\sum_{k,\,s}|a_{k,s}|^2\left((x_{k,s})_{\min}\right)^{-1} + \frac{C}{4}\sum_{k,\,s}|a_{k,s}|^2\left((x_{k,s})_{\min}\right)^{-3/2}\left|\sqrt{x_{k,s}}\right|.\end{aligned}$$

It is easy to see that

$$\left|\sqrt{x_{k,s}}\right|^2 \leqslant \kappa_s \mathsf{Ellipt}(x_{k,s})(x_{k,s})_{\min}.$$

Combining the above two displays, we have that for every $x\in\mathbf{S}^{K+1}$ satisfying

$$\frac{c}{K+1}\mathbf{Id}_{\kappa_s}\leqslant x_{k,s},\quad \mathsf{Ellipt}(x_{k,s})\leqslant c^{-1},\quad \forall k\in\{0,\ldots,K\},\ s\in\mathscr{S}, \tag{4.14}$$

we have

$$\mathbf{a}\cdot\widetilde{\nabla}^2 G_N(x)\mathbf{a}\leqslant C(K+1)c^{-2}|a|^2$$

where we have absorbed κ_s into C. Hence, for every x satisfying (4.14) and x' satisfying an analogous version of (4.14), we have

$$(1-r)\widetilde{G}_N(x)+r\widetilde{G}_N(x')\leqslant \widetilde{G}_N\left((1-r)x+rx'\right)+Cr(1-r)c^{-1}(K+1)\left|x-x'\right|^2.$$

In view of (4.12), we have $\overline{F}_{N,\lambda_N}\left(q^K\right)=\widetilde{G}_N\left((q_k-q_{k-1})_{0\leqslant k\leqslant K}\right)$. Recall that in the relation (4.10) to prove, both q and q' satisfy (4.9). Therefore, we are allowed to use the above display to deduce (4.10). ☐

As a consequence of the semi-concavity, we have the following result. Recall the definition of $\mathcal{Q}_{\uparrow}^{\mathscr{S}}(\kappa)$ from (1.21) and (1.10).

Proposition 4.6 (Convergence of derivatives)**.** *Suppose that $\overline{F}_{N,\lambda_N}$ converges pointwise to some limit f along a subsequence $(N_n)_{n\in\mathbb{N}}$.*

(1) For every $t\in\mathbb{R}_+$, if $f(t,\cdot)$ is Gateaux differentiable at $q\in\mathcal{Q}_{\uparrow}^{\mathscr{S}}(\kappa)$, then $\partial_q\overline{F}_{N_n,\lambda_{N_n}}(t,q,\cdot)$ converges in L^r to $\partial_q f(t,q,\cdot)$ for every $r\in[1,+\infty)$.

(2) For every $q\in\mathcal{Q}_{1}^{\mathscr{S}}(\kappa)$, if $f(\cdot,q)$ is differentiable at $t>0$, then $\partial_t\overline{F}_{N_n,\lambda_{N_n}}(t,q)$ converges to $\partial_t f(t,q)$.

The proof is the same as that for Chen and Mourrat (2025, Proposition 5.4) based on the semi-concavity proved in Proposition 4.5.

Remark 4.7 (Convergence of derivatives for perturbed free energy). Let $(N_n)_{N\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$. For each $n\in\mathbb{N}$, let M_n satisfy $N_n\in M_n\mathbb{N}$, let $\lambda_{N_n}\in\blacktriangle_{M_n}$ (see (1.3)), let x_n be any perturbation parameter from (5.9) (the space of which depends on M_n). Then, we consider the perturbed free energy $\overline{F}^{M_n,x_n}_{N_n,\lambda_{N_n}}$ in (5.11) or $\widetilde{F}^{M_n,x_n}_{N_n,\lambda_{N_n}}$ in (5.13). For them, we can also prove Propositions 4.5 and Proposition 4.6 (along any further subsequence of $(N_n)_{n\in\mathbb{N}}$). The proofs are the same.

Next, we state the result on the regularity of limits of the free energy. In the statement below, we use the notion of "Gaussian null sets" which is a natural generalization of "Lebesgue null sets" to infinite dimensions. We refer to Chen and Mourrat (2025, Definition 4.2) for the exact definition. The only property important to us here is that the complement of a Gaussian null set is dense, which allows us to say that any limit of free energy is differentiable on a dense set.

Proposition 4.8 (Regularity of the limit)**.** *Suppose that* $(\overline{F}_{N,\lambda_N})_{N\in\mathbb{N}}$ *converges pointwise to some* f *along some subsequence. Then, for every* $r\in(1,\infty]$*, the function* $\overline{F}_{N,\lambda_N}$ *converges locally uniformly on* $\mathbb{R}_+\times\mathcal{Q}_r^{\mathscr{S}}(\kappa)$ *along this subsequence. The limit* f *satisfies the same Lipschitz, monotone, and local semi-concave properties of* $\overline{F}_{N,\lambda_N}$ *in Propositions 4.4, 4.1, and 4.5. Moreover,*

- *for each* $t\geqslant 0$*, there is a Gaussian null set* $\mathcal{N}_t$ *of* $L^2([0,1],\prod_{s\in\mathscr{S}}S^{\kappa_s})$ *such that* $f(t,\cdot):\mathcal{Q}_2^{\mathscr{S}}(\kappa)\to\mathbb{R}$ *is Gateaux differentiable at every point in* $\mathcal{Q}_2^{\mathscr{S}}(\kappa)\setminus\mathcal{N}_t$ *and* $\mathcal{Q}_{\infty,\uparrow}^{\mathscr{S}}\setminus\mathcal{N}_t$ *is dense in* $\mathcal{Q}_2^{\mathscr{S}}(\kappa)$*;*
- *there is a Gaussian null set* $\mathcal{N}$ *of* $\mathbb{R}\times L^2([0,1],\prod_{s\in\mathscr{S}}S^{\kappa_s})$ *such that* $f:\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)\to\mathbb{R}$ *is Gateaux differentiable on* $(\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa))\setminus\mathcal{N}$ *and* $(\mathbb{R}_+\times\mathcal{Q}_{\infty,\uparrow}^{\mathscr{S}}(\kappa))\setminus\mathcal{N}$ *is dense in* $\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$*.*

This corresponds to Chen and Mourrat (2025, Proposition 5.3) and can be deduced by the same argument. Hence, we omit the proof here.

4.3. *Initial condition.* We want to describe the initial condition $\overline{F}_{N,\lambda_N}(0,\cdot)$. For any finite positive measure ν on $[-1,1]^D$ for some $D\in\mathbb{N}$ and for $\mathbf{q}\in\mathcal{Q}_\infty(D)$, define

$$\psi^{\mathrm{vec}}_\nu(\mathbf{q}) = -\mathbb{E}\log\iint\exp\left(\sqrt{2}\mathbf{w}^{\mathbf{q}}(\alpha)\cdot\boldsymbol{\sigma}-\mathbf{q}(1)\cdot\boldsymbol{\sigma}\boldsymbol{\sigma}^\intercal\right)\mathrm{d}\nu(\boldsymbol{\sigma})\mathrm{d}\mathfrak{R}(\alpha) \tag{4.15}$$

where $\mathbf{w}^{\mathbf{q}}$ is the Gaussian process with covariance given in (3.3). This is related to the initial condition in the vector spin glass. Indeed, comparing with the expression of the free energy in (3.6), we have

$$\overline{F}^{\mathrm{vec}}_N(0,\mathbf{q}) = \psi^{\mathrm{vec}}_{P_1^{\mathrm{vec}}}(\mathbf{q}).$$

This identity holds clearly for $N=1$ and the general case follows from a standard property of the cascade measure (see Chen and Mourrat, 2025, Proposition 3.2).

In the multi-species setting, given $\lambda=(\lambda_s)_{s\in\mathscr{S}}\in\mathbb{R}_+^{\mathscr{S}}$, we define, for $q\in\mathcal{Q}_\infty^{\mathscr{S}}(\kappa)$,

$$\psi_\lambda(q) = \sum_{s\in\mathscr{S}}\lambda_s\psi^{\mathrm{vec}}_{\mu_s}(q_s) \tag{4.16}$$

where $(\mu_s)_{s\in\mathscr{S}}$ are the fixed distributions of spins of different species (see Section 1.1.3).

We recall the result Chen and Mourrat (2025, Corollary 5.2) on the regularity of the initial condition (4.15) in the vector spin model. For $\mathbf{q}\in\mathcal{Q}_\infty(D)$ and a positive measure ν on $\mathbb{R}^D$, we introduce the following Gibbs measure:

$$\langle\cdot\rangle^{\mathrm{vec}}_{\nu,\mathbf{q}} = \exp\left(\mathbf{w}^{\mathbf{q}}(\alpha)\cdot\sigma-\mathbf{q}(1)\cdot\sigma\sigma^\intercal\right)\mathrm{d}\nu(\sigma)\mathrm{d}\mathfrak{R}(\alpha)$$

where $\mathbf{w}^{\mathbf{q}}(\alpha)$ is given as in (3.3).

Lemma 4.9 (Regularity of vector-spin initial condition)**.** *For any* $D\in\mathbb{N}$ *and any positive measure* ν *supported on* $[-1,+1]^D$*, the function* ψ^{vec}_ν *given in (4.15) can be extended to* $\mathcal{Q}_1(D)$ *and satisfies*

$$\left|\psi^{\mathrm{vec}}_\nu(\mathbf{q})-\psi^{\mathrm{vec}}_\nu(\mathbf{q}')\right|\leqslant|\mathbf{q}-\mathbf{q}'|_{L^1},\quad\forall\mathbf{q},\mathbf{q}'\in\mathcal{Q}_1(D).$$

The restriction $\psi^{\mathrm{vec}}_\nu:\mathcal{Q}_2(D)\to\mathbb{R}$ *is Fréchet (and Gateaux) differentiable everywhere; we denote its Fréchet (and Gateaux) derivative by* $\partial_{\mathbf{q}}\psi^{\mathrm{vec}}_\nu(\mathbf{q})=\partial_{\mathbf{q}}\psi^{\mathrm{vec}}_\nu(\mathbf{q},\cdot)\in L^2([0,1];S^D)$*. We have, for every* $\mathbf{q}\in\mathcal{Q}_2(D)$*,*

$$\partial_{\mathbf{q}}\psi^{\mathrm{vec}}_\nu(\mathbf{q})\in\mathcal{Q}_{\infty,\leqslant 1}(D), \tag{4.17}$$

and, for every $\mathbf{q}\in\mathcal{Q}_\infty(D)$ *and* $\pi\in L^2([0,1];S^D)$*,*

$$\langle\pi,\partial_{\mathbf{q}}\psi^{\mathrm{vec}}_\nu(\mathbf{q})\rangle_{L^2}=\mathbb{E}\left\langle\pi\left(\alpha\wedge\alpha'\right)\cdot\sigma\sigma'^\intercal\right\rangle^{\mathrm{vec}}_{\nu,\mathbf{q}}.$$

Moreover, for every $r\in[1,+\infty]$ *and* $\mathbf{q},\mathbf{q}'\in\mathcal{Q}_2(D)$ *with* $\mathbf{q}-\mathbf{q}'\in L^r$*, we have*

$$\left|\partial_{\mathbf{q}}\psi^{\mathrm{vec}}_\nu(\mathbf{q})-\partial_{\mathbf{q}}\psi^{\mathrm{vec}}_\nu(\mathbf{q}')\right|_{L^r}\leqslant 16\left|\mathbf{q}-\mathbf{q}'\right|_{L^r}. \tag{4.18}$$

In particular, the mapping $\mathbf{q} \mapsto \partial_{\mathbf{q}} \psi^{\mathrm{vec}}_{\nu}(\mathbf{q})$ *can be extended to* $\mathcal{Q}_1(D)$ *by continuity, and the properties in* (4.17) *and* (4.18) *remain valid with* $\mathbf{q}, \mathbf{q}' \in \mathcal{Q}_1(D)$*.*

Lemma 4.10 (Multi-species initial condition)**.** *For any* $\lambda \in \mathbb{R}^{\mathscr{S}}_{+}$*,* ψ_λ *given in* (4.16) *can be extended to* $Q^{\mathscr{S}}_1(\kappa)$ *and satisfies*

$$\left|\psi_\lambda(q) - \psi_\lambda(q')\right| \leqslant \sum_{s\in\mathscr{S}} \lambda_s |q_s - q'_s|_{L^1}, \quad \forall q, q' \in Q^{\mathscr{S}}_1(\kappa).$$

The restriction $\psi_\lambda : \mathcal{Q}^{\mathscr{S}}_2(\kappa) \to \mathbb{R}$ *is Féchet (and Gateaux) differentiable everywhere. At every* $q \in \mathcal{Q}^{\mathscr{S}}_2(\kappa)$*, its derivative is given by*

$$\partial_q \psi_\lambda(q) = \left(\lambda_s \partial_{q_s} \psi^{\mathrm{vec}}_{\mu_s}(q_s)\right)_{s\in\mathscr{S}} \in \mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda}(\kappa).$$

Proof: The extendability and Lipschitzness follows from (4.16) and Lemma 4.9. The expression for the derivative follows from (4.16) and the definition of derivatives. The range $\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda}(\kappa)$ is clear from (4.17). □

Lemma 4.11 (Initial condition)**.** *For every* $N \in \mathbb{N}$*,* $q \in \mathcal{Q}^{\mathscr{S}}_1(\kappa)$*, and* $\lambda_N \in \blacktriangle_N$*, we have*

$$\overline{F}_{N,\lambda_N}(0,q) = \psi_{\lambda_N}(q).$$

Proof: We only need to prove identity at $q \in \mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$ and the general case follows by extension. For each $s \in \mathscr{S}$ and $n \in I_{N,s}$, we define

$$X_{s,n}(\alpha) = \log \int \exp\left(\sqrt{2} w^{q_s}_n(\alpha)\cdot\sigma_{\bullet n} - q_s(1)\cdot\sigma_{\bullet n}\sigma^{\intercal}_{\bullet n}\right) \mathrm{d}\mu_s(\sigma_{\bullet n})$$

where $w^{q_s}_n$ is introduced in (1.11). Comparing this with ψ^{vec}_ν in (4.19), we have

$$\psi^{\mathrm{vec}}_{\mu_s}(q_s) = -\mathbb{E}\log\int \exp\left(X_{s,n}(\alpha)\right) \mathrm{d}\mathfrak{R}(\alpha). \tag{4.19}$$

On the other hand, using the expression in (1.15), we can rewrite

$$-N\overline{F}_{N,\lambda_N}(0,q) = \mathbb{E}\log\int \exp\left(\sum_{s\in\mathscr{S}}\sum_{n\in I_{N,s}} X_{s,n}(\alpha)\right)\mathrm{d}\mathfrak{R}(\alpha).$$

By a basic property of cascades stated for example in Dominguez and Mourrat (2024c, Corollary 5.26), the right-hand side in the above display is equal to

$$\sum_{s\in\mathscr{S}}\sum_{n\in I_{N,s}} \mathbb{E}\log\int \exp\left(X_{s,n}(\alpha)\right)\mathrm{d}\mathfrak{R}(\alpha) \overset{(4.19)}{=} -\sum_{s\in\mathscr{S}} |I_{N,s}|\,\psi^{\mathrm{vec}}_{\mu_s}(q_s)$$

which together with the definition of λ_N in (1.2) implies the announced result. □

5. Cavity computation and proofs in the general case

In Section 3, we have shown that, if the limit of species proportions $(\lambda_{N,s})_{s\in\mathscr{S}}$ are all rational, the multi-species model is equivalent to the vector spin glass model (see Corollary 3.2). Hence, we can directly apply cavity computation results in Chen and Mourrat (2025, Sections 6 and 7) stated for vector spin glasses. However, in the case where some entries in the limit of λ_N are irrational, such argument no longer works. To handle this, we need an additional approximation argument.

In Section 5.1, we give definitions of various objects appearing in the cavity computation and then state the cavity computation results, Lemmas 5.5 and 5.7, assuming that the species proportions are rational. These two lemmas are straightforward adaptions of results in Chen and Mourrat (2025). In Section 5.2, we consider the (general) irrational case and use approximation to extend the two results to Lemmas 5.9 and 5.10. In Section 5.3, we apply these two lemmas to prove the results corresponding to those in Chen and Mourrat (2025, Section 7). In particular, we prove Theorems 1.2, 1.3, and 1.4 here.

5.1. *Cavity computation.* We start with introducing definitions necessary for the cavit computation.

5.1.1. *Parisi functional.* Recall the definition of ψ_λ from (4.16). For ξ in (1.9), we define $\theta : \prod_{s\in\mathscr{S}} \mathbb{R}^{\kappa_s\times\kappa_s} \to \mathbb{R}$ by

$$\theta(a) = a\cdot\nabla\xi(a) - \xi(a). \tag{5.1}$$

Here, $\nabla\xi : \prod_{s\in\mathscr{S}} \mathbb{R}^{\kappa_s\times\kappa_s} \to \prod_{s\in\mathscr{S}} \mathbb{R}^{\kappa_s\times\kappa_s}$ is the gradient of ξ defined with respect to the entry-wise inner product structure on the linear space $\prod_{s\in\mathscr{S}} \mathbb{R}^{\kappa_s\times\kappa_s}$.

For $t\in\mathbb{R}_+$, $q\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$, and $\lambda\in\mathbb{R}^{\mathscr{S}}_+$, we set

$$\mathscr{P}_{\lambda,t,q}(p) = \psi_\lambda(q+t\nabla\xi(p)) - t\int_0^1 \theta(p(r))\mathrm{d}r. \tag{5.2}$$

Comparing this with (1.17), we have

$$\mathscr{P}_{\lambda_\infty,t,q}(p) = \mathscr{J}_{\lambda_\infty,t,q}(q+t\nabla\xi(p),p). \tag{5.3}$$

5.1.2. *Hamiltonian from the cavity computation.* Then, we describe the Gibbs measure appearing in the cavity computation and the corresponding free energy. Fix any $M\in\mathbb{N}$ interpreted as the number of cavity spins. The Hamiltonian arising from the cavity computation is a centered Gaussian process $\big(\widetilde{H}_N(\sigma)\big)_{\sigma\in\Sigma^N}$ with covariance

$$\mathbb{E}\left[\widetilde{H}_N(\sigma)\widetilde{H}_N(\sigma')\right] = (N+M)\xi\Big(\frac{N}{N+M}R_{N,\lambda_N}(\sigma,\sigma')\Big) \tag{5.4}$$

which should be compared with $H_N(\sigma)$ given in (1.9). Notice that we have kept M implicit from the notation. We assume that $\widetilde{H}_N(\sigma)$ is independent from other randomness. Let $\widetilde{W}^q_N(\sigma,\alpha)$ be an independent copy of $W^q_N(\sigma,\alpha)$ given in (1.12). Analogous to $H^{t,q}_N(\sigma,\alpha)$ in (1.14), we define

$$\begin{aligned}\widetilde{H}^{t,q}_N(\sigma,\alpha) = \sqrt{2t}\widetilde{H}_N(\sigma) - t(N+M)\xi\left(\frac{N}{N+M}R_{N,\lambda_N}(\sigma,\sigma)\right)\\ +\widetilde{W}^q_N(\sigma,\alpha) - q(1)\cdot R_{N,\lambda_N}(\sigma,\sigma).\end{aligned} \tag{5.5}$$

We prefer to omit M in the notation.

5.1.3. *Perturbations.* To ensure Ghirlanda–Guerra identities, we need to introduce perturbation terms to the Hamiltonian. Since we want to apply Lemma 3.1 and results from Chen and Mourrat (2025) for vector spin glasses, the perturbation to be introduced below is not the most suitable choice. Indeed, in the multi-species setting, we only need Ghirlanda–Guerra identities for the overlap array $\big(R_{MN,\lambda_{MN}}(\sigma^l,\sigma^{l'})\big)_{l,l'\in\mathbb{N}}$ (see (1.7)) which contains less entries than $\big(N^{-1}\boldsymbol{\sigma}^l\boldsymbol{\sigma}^{l'}\big)_{l,l'\in\mathbb{N}}$ in the matching vector spin glass model (see (3.12)). However, to use results from Chen and Mourrat (2025), we need to employ perturbation in the style of vector spin glasses. In Remark 5.8, we describe the most suitable perturbation in the multi-species setting and corresponding results without proofs.

Henceforth, we fix any $M\in\mathbb{N}$ and consider the multi-species model with size MN for $N\in\mathbb{N}$. We assume that λ_{MN} satisfies (3.7) for some fixed weak partition $(\mathsf{M}_s)_{s\in\mathscr{S}}$ of $\{1,\ldots,M\}$. As in (3.8), let $\Delta=\Delta(M,\lambda_{MN})$ be the dimension for the matching vector spin model (which is independent of N). As in (3.11), fix any such bijection (for each N) and let $\boldsymbol{\sigma}$ be the image of σ after this mapping.

Let (r_n) be an enumeration of $[0,1]\cap\mathbb{Q}$ and $(a_n)_{n\in\mathbb{N}}$ be an enumeration of elements in $S^\Delta_+\cap\mathbb{Q}^{\Delta\times\Delta}$. Conditioned on $\mathfrak{R}$, for every $N\in\mathbb{N}$ and every $h=(h_i)_{1\leqslant i\leqslant 4}\in\mathbb{N}^4$, let $\left(H^h_{MN}(\sigma,\alpha)\right)_{\sigma\in\Sigma^N,\,\alpha\in\operatorname{supp}\mathfrak{R}}$ be an independent centered Gaussian process with covariance

$$\mathbb{E}\left[H^h_{MN}(\sigma,\alpha)H^h_{MN}(\sigma',\alpha')\right]=N\left(a_{h_1}\cdot\left(N^{-1}\boldsymbol{\sigma}\boldsymbol{\sigma}'^\intercal\right)\right)^{\odot h_2}+\lambda_{h_3}\alpha\wedge\alpha'\Big)^{h_4}. \tag{5.6}$$

Here, $\odot$ denotes the Schur product, namely, $a\odot b=(a_{ij}b_{ij})_{i,j}$ for two matrices a and b of the same dimension. The existence of such a process is explained in Chen and Mourrat (2025, Section 6.1.1). For each $h\in\mathbb{N}^4$, we write $|h|_1=\sum_{i=1}^4 h_i$ and let $c_h>0$ be a constant such that

$$c_h\sqrt{N^{-1}\mathbb{E}\left[H^h_{MN}(\sigma,\alpha)H^h_{MN}(\sigma,\alpha)\right]}\leqslant 2^{-|h|_1}, \tag{5.7}$$

uniformly over $\sigma\in\Sigma^N$, $\alpha\in\operatorname{supp}\mathfrak{R}$, and $N\in\mathbb{N}$. Fix an orthonormal basis bas of S^Δ. We define the space of perturbation parameters:

$$\mathsf{pert}(M,\lambda_{MN})=[0,3]^{\mathbb{N}^4\times\mathsf{bas}} \tag{5.8}$$

where the dependence on λ_{MN} and M is through Δ as in (3.8). For every

$$x=\left((x_h)_{h\in\mathbb{N}^4},\ (x_e)_{e\in\mathsf{bas}}\right)\in\mathsf{pert}(M,\lambda_{MN}), \tag{5.9}$$

we set

$$H^x_{MN}(\sigma,\alpha)=\sum_{h\in\mathbb{N}^4}x_hc_hH^h_{MN}(\sigma,\alpha)+\frac{1}{|\mathsf{bas}|}\sum_{e\in\mathsf{bas}}x_e e\cdot\boldsymbol{\sigma}\boldsymbol{\sigma}^\intercal. \tag{5.10}$$

Compared with the standard perturbation as in Panchenko (2015, 2018b,a), the additional second sum ensures the concentration of the self-overlap $N^{-1}\boldsymbol{\sigma}\boldsymbol{\sigma}'^\intercal$ (which implies the concentration of $R_{MN,\lambda_{MN}}(\sigma,\sigma)$). Recall the original Hamiltonian $H^{t,q}_{MN}(\sigma,\alpha)$ from (1.14) (with MN substituted for N therein). For each $N\in\mathbb{N}$, we define the free energy with perturbation

$$\begin{aligned}&\overline{F}^{M,x}_{MN,\lambda_{MN}}(t,q)\\&=-\frac{1}{MN}\mathbb{E}\log\iint\exp\left(H^{t,q}_{MN}(\sigma,\alpha)+N^{-\frac{1}{16}}H^x_{MN}(\sigma,\alpha)\right)\mathrm{d}P_{MN,\lambda_{MN}}(\sigma)\mathrm{d}\mathfrak{R}(\alpha).\end{aligned} \tag{5.11}$$

The choice of $\frac{1}{16}$ is inconsequential and can be replaced by any number in $(\frac{1}{32},\frac{1}{8})$. This factor is needed to ensure that the perturbation is weak enough not to change the limit of free energy and strong enough to ensure the validity of the Ghirlanda–Guerra identities. For each $N\in\mathbb{N}$, we denote the associated Gibbs measure by

$$\langle\cdot\rangle^{M,x}_{MN,\lambda_{MN},x}\propto\exp\left(H^{t,q}_{MN}(\sigma,\alpha)+N^{-\frac{1}{16}}H^x_{MN}(\sigma,\alpha)\right)\mathrm{d}P_{MN,\lambda_{MN}}(\sigma)\mathrm{d}\mathfrak{R}(\alpha) \tag{5.12}$$

where the value for (t,q) will be clear from the context.

Let $\widetilde{H}^{t,q}_{MN}(\sigma,\alpha)$ be given as in (5.5) with MN substituted for N therein. For each $N\in\mathbb{N}$, we denote the associated perturbed free energy and Gibbs measure by

$$\begin{aligned}&\widetilde{F}^{M,x}_{MN,\lambda_{MN}}(t,q)\\&=-\frac{1}{MN}\mathbb{E}\log\iint\exp\left(\widetilde{H}^{t,q}_{MN}(\sigma,\alpha)+N^{-\frac{1}{16}}H^x_{MN}(\sigma,\alpha)\right)\mathrm{d}P_{MN,\lambda_{MN}}(\sigma)\mathrm{d}\mathfrak{R}(\alpha);\end{aligned} \tag{5.13}$$

$$\langle\cdot\rangle^{\circ,M,x}_{MN,\lambda_{MN}}\propto\exp\left(\widetilde{H}^{t,q}_{MN}(\sigma,\alpha)+N^{-\frac{1}{16}}H^x_{MN}(\sigma,\alpha)\right)\mathrm{d}P_{MN,\lambda_{MN}}(\sigma)\mathrm{d}\mathfrak{R}(\alpha). \tag{5.14}$$

The symbol $\circ$ in the superscript signifies “cavity”.

In the notation for the free energy and the Gibbs measure in (5.11), (5.12), (5.13), and (5.14), we have emphasized the dependence on M (the cavity dimension) in the superscript.

We denote by (σ,α) the canonical random variable under $\langle\cdot\rangle^{M,x}_{MN,\lambda_{MN}}$ and $\langle\cdot\rangle^{\circ,M,x}_{MN,\lambda_{MN}}$. We write $\left(\sigma^l,\alpha^l\right)_{l\in\mathbb{N}}$ to denote independent copies of (σ,α).

Lemma 5.1. *Let $M \in \mathbb{N}$ and let λ_{MN} satisfy (3.7). There is a constant $C > 0$ depending only κ, μ, and ξ such that for every N, $t \in \mathbb{R}_+$, and $q \in \mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$, we have*

$$\sup_{x \in \mathsf{pert}(M,\lambda_{MN})} \left| \overline{F}_{MN,\lambda_{MN}}(t,q) - \overline{F}^{M,x}_{MN,\lambda_{MN}}(t,q) \right| \leqslant CN^{-\frac{1}{16}}, \tag{5.15}$$

$$\sup_{x \in \mathsf{pert}(M,\lambda_{MN})} \left| \overline{F}^{M,x}_{MN,\lambda_{MN}}(t,q) - \widetilde{F}^{M,x}_{MN,\lambda_{MN}}(t,q) \right| \leqslant C|t|N^{-1}. \tag{5.16}$$

Proof: Comparing $\overline{F}_{MN,\lambda_{MN}}$ given in (1.15) and $\overline{F}^{M,x}_{MN,\lambda_{MN}}$ in (5.11), we see that the additional term is $N^{-\frac{1}{16}} H^x_{MN}(\sigma,\alpha)$. The definition of Δ in (3.8) implies that $|\Delta| \leqslant M|\kappa|_\infty$ for $|\kappa|_\infty = \max_{s\in\mathscr{S}} \kappa_s$. Since every entry in $\boldsymbol{\sigma} \in \mathbb{R}^{\Delta\times N}$ lies in $[-1,+1]$, we have $|\boldsymbol{\sigma}\boldsymbol{\sigma}'| \leqslant MN|\kappa|_\infty$. Using this, the choice of c_h in (5.7), and the presence of $|\mathsf{bas}|$ in (5.10), we can apply the interpolation argument in Lemma 2.1 to get (5.15).

Comparing $\overline{F}^{M,x}_{MN,\lambda_{MN}}$ in (5.11) and $\widetilde{F}^{M,x}_{MN,\lambda_{MN}}$ in (5.13), the difference lies in the terms associated with $\sqrt{2t}H_{MN}(\sigma)$ and $\sqrt{2t}\widetilde{H}_{MN}(\sigma)$. The variance of the former is given in (1.9) and latter in (5.4). There is a constant C_ξ depending on ξ such that

$$\left| (MN+M)\xi\Big(\frac{MN}{MN+M} R_{MN,\lambda_{MN}}(\sigma,\sigma')\Big) - MN\xi\Big(R_{MN,\lambda_{MN}}(\sigma,\sigma')\Big) \right| \leqslant C_\xi M.$$

Using this and the interpolation argument in Lemma 2.1, we can get (5.16). □

Remark 5.2. We clarify that the law of $(\sigma^l,\alpha^l)_{l\in\mathbb{N}}$ under $\langle\cdot\rangle^{M,x}_{MN,\lambda_{MN}}$ or $\langle\cdot\rangle^{\circ,M,x}_{MN,\lambda_{MN}}$ depends on the partition $(I_{MN,s})_{s\in\mathscr{S}}$ of $\{1,\ldots,N\}$. But, the law of overlaps

$$\left(R_{MN,\lambda_{MN}}(\sigma^l,\sigma^{l'}),\, \alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$$

(see (1.7)) under these Gibbs measures only depend on the proportions λ_N. Since we are more interested in the law of overlaps, we prefer to display in the notation only the dependence on λ_{MN}. □

Since we have introduced several Gibbs measures, it is a good place state the following important property (e.g. see Chen and Mourrat, 2025, Proposition 4.8). Recall the notation $\langle\cdot\rangle_{\mathfrak{R}}$ from Section 1.1.4.

Lemma 5.3 (Invariance of cascades)**.** *Let $\langle\cdot\rangle$ be the one of the following Gibbs measures: $\langle\cdot\rangle_{N,\lambda_N}$ in (1.16), $\langle\cdot\rangle^{M,x}_{MN,\lambda_{MN}}$ in (5.12), $\langle\cdot\rangle^{\circ,M,x}_{MN,\lambda_{MN}}$ in (5.14), or any interpolation of these appearing in Lemma 2.1. Then, the law of $\left(\alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ under $\mathbb{E}\langle\cdot\rangle$ is equal to that under $\mathbb{E}\langle\cdot\rangle_{\mathfrak{R}}$. In particular, $\alpha\wedge\alpha'$ is uniformly distributed on $[0,1]$ under $\mathbb{E}\langle\cdot\rangle$.*

Later, we need the next result for the convergence of overlap arrays.

Lemma 5.4 (Criterion for convergence of overlap array)**.** *Let $(\alpha^l)_{l\in\mathbb{N}}$ be i.i.d. samples from $\langle\cdot\rangle_{\mathfrak{R}}$. Let E be a fixed compact Euclidean set and let $(\Omega_n,\nu_n)_{n\in\mathbb{N}}$ be a sequence of probability spaces such that, for each n, the following holds:*

- *$(\alpha^l)_{l\in\mathbb{N}}$ are random variables on Ω_n and the law of $\left(\alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ under ν_n is equal to that under $\mathbb{E}\langle\cdot\rangle_{\mathfrak{R}}$;*
- *there is a random array $R_n = \left(R_n^{l,l'}\right)_{l,l'}$ of E-valued random variables on Ω_n and the law of R_n under ν_n is invariant under permutation of labels.*

Let $a \in E$ and $p : [0,1] \to E$ be bounded and measurable. Then, we have

$$\lim_{n\to\infty} \nu_n\left(\left|R_n^{1,2} - p\left(\alpha^1\wedge\alpha^2\right)\right|\right) = 0, \qquad \lim_{n\to\infty} \nu_n\left(\left|R_n^{1,1} - a\right|\right) = 0, \tag{5.17}$$

if and only if $\left(R_n^{l,l'},\, \alpha^l \wedge \alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ *under* ν_n *converges in law to*

$$\left(\left(p\big(\alpha^l \wedge \alpha^{l'}\big),\ \alpha^l \wedge \alpha^{l'}\right)\mathbf{1}_{l\neq l'} + (a,1)\mathbf{1}_{l=l'}\right)_{l,l'\in\mathbb{N}}$$

under $\mathbb{E}\left\langle\cdot\right\rangle_{\mathfrak{R}}$*, as* n *tends to infinity.*

Proof: We first show "$\Longrightarrow$". Since the convergence in law is defined to be convergence over finitely many entries, by considering test functions of the form of a product of functions of one entry, it suffices to prove convergence for each entry. The convergence of any diagonal entry ($l = l'$) is obvious. For non-diagonal entry, by symmetry, we only need to consider the one with index $(1,2)$. Let $g : E\times\mathbb{R}\to\mathbb{R}$ be any Lipschitz function. Then, writing $R = R_n^{1,2}$ and $Q = \alpha^1\wedge\alpha^2$, we have

$$\begin{aligned}\left|\nu_n\left(g(R,Q)\right) - \mathbb{E}\left\langle g(p(Q),Q)\right\rangle_{\mathfrak{R}}\right| &= \left|\nu_n\left(g(R,Q)\right) - \nu_n\left(g(p(Q),Q)\right)\right| \\ &\leqslant \|g\|_{\mathrm{Lip}}\nu_n\left(|R-p(Q)|\right)\end{aligned}$$

which converges to zero by (5.17). This completes the proof of "$\Longrightarrow$". To see "$\Longleftarrow$", we first notice that it is sufficient to prove (5.17) with $|\cdot|$ replaced by $|\cdot|^2$ (random variables are bounded because E is compact). Then, we can expand the square and use the convergence in law to get the desired result. □

5.1.4. *Two results from the cavity computation.* Recall the definition of the discrete simplex $\blacktriangle_N$ in (1.3). Denote by $\overline{\mathrm{conv}}$ the operator taking the closed convex hull of some set in a finite-dimensional linear space. Define

$$\mathcal{K} = \prod_{s\in\mathscr{S}} \overline{\mathrm{conv}}\left\{\tau\tau^{\intercal} :\ \tau\in\operatorname{supp}\mu_s\right\}. \tag{5.18}$$

Recall the definition of $\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda}(\kappa)$ from (4.2) and the space of perturbation parameters in (5.8).

Lemma 5.5. *Let* $M\in\mathbb{N}$ *and* $\lambda^\star\in\blacktriangle_M$*. Set* $\lambda_{MN}=\lambda^\star$ *for every* N*. For every* $(t,q)\in\mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_{\infty}(\kappa)$*, there are sequences* $\left(N_k^{\pm}\right)_{k\in\mathbb{N}}$ *of strictly increasing integers,* $\left(x_k^{\pm}\right)_{k\in\mathbb{N}}$ *of parameters in* $\mathsf{pert}(M,\lambda^\star)$*,* $p_\pm\in\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda^\star}(\kappa)$*, and* $a_\pm\in\mathcal{K}$ *satisfying* $a_\pm\geqslant p_\pm$ *such that*

(1) $\left(R_{MN_k^\pm,\lambda^\star}(\sigma^l,\sigma^{l'}),\, \alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ *under* $\mathbb{E}\left\langle\cdot\right\rangle^{\circ,M,x_k^\pm}_{MN_k^\pm,\lambda^\star}$ *converges in law to*

$$\left(\left(p_\pm\big(\alpha^l\wedge\alpha^{l'}\big),\ \alpha^l\wedge\alpha^{l'}\right)\mathbf{1}_{l\neq l'} + (a_\pm,1)\mathbf{1}_{l=l'}\right)_{l,l'\in\mathbb{N}}$$

under $\mathbb{E}\left\langle\cdot\right\rangle_{\mathfrak{R}}$*, as* k *tends to infinity;*

(2) and

$$\mathscr{P}_{\lambda^\star,t,q}(p_-)\leqslant\liminf_{N\to\infty}\overline{F}_{MN,\lambda^\star}(t,q)\leqslant\limsup_{N\to\infty}\overline{F}_{MN,\lambda^\star}(t,q)\leqslant\mathscr{P}_{\lambda^\star,t,q}(p_+).$$

The two Gibbs measures appearing in Part (1) are given in (5.14) and Section 1.1.4, respectively. In the statement, $a_\pm\geqslant p_\pm$ and more generally $a\geqslant p$ means

$$a_s - p_s(r)\in S_+^{\kappa_s},\quad \forall r\in[0,1),\ s\in\mathscr{S}. \tag{5.19}$$

Proof: We fix any weak partition $(\mathsf{M}_s)_{s\in\mathscr{S}}$ of $\{1,\dots,M\}$ satisfying $\lambda^\star = (|\mathsf{M}_s|/M)_{s\in\mathscr{S}}$. Hence, the assumption $\lambda_{MN}=\lambda^\star$ ensures (3.7). Fix any (t,q). Let $\mathbf{q}$ be given as in (3.16) and $\overline{F}^{\mathrm{vec}}_N$ be given by Lemma 3.1. We directly define $\overline{F}^{\mathrm{vec},x}_N(t,\mathbf{q}) = M\overline{F}^{M,x}_{MN,\lambda_{MN}}(t,q)$ and $\widetilde{F}^{\mathrm{vec},x}_N(t,\mathbf{q}) = M\widetilde{F}^{M,x}_{MN,\lambda_{MN}}(t,q)$. Allowed by the bijection between σ and $\boldsymbol{\sigma}$ in (3.11), we can view $\left\langle\cdot\right\rangle^{M,x}_{MN,\lambda_{MN}}$ (see (5.12)) and $\left\langle\cdot\right\rangle^{\circ,M,x}_{MN,\lambda_{MN}}$ (see (5.14)) as the Gibbs measure associated with $\overline{F}^{\mathrm{vec},x}_N(t,\mathbf{q})$ and $\widetilde{F}^{\mathrm{vec},x}_N(t,\mathbf{q})$, respectively.

Recall that in the proof of Lemma 3.19, we used various identities to derive the equivalence (3.19) between the multi-species Hamiltonian and the one in the vector spin glass model. As a consequence, we can match the corresponding free energies. Similar arguments together with the definition of $H^x_{MN}(\sigma,\alpha)$ in (5.10) can be used to match $\overline{F}^{\mathrm{vec},x}_N(t,\mathbf{q})$ and $\widetilde{F}^{\mathrm{vec},x}_N(t,\mathbf{q})$ exactly (Chen and Mourrat, 2025, (6.6) and (6.8)). Similarly, the Gibbs measures $\langle\cdot\rangle^{M,x}_{MN,\lambda_{MN}}$ and $\langle\cdot\rangle^{\circ,M,x}_{MN,\lambda_{MN}}$ match exactly (Chen and Mourrat, 2025, (6.7) and (6.9)) (denoted by $\langle\cdot\rangle_{N,x}$ and $\langle\cdot\rangle^{\circ}_{N,x}$ therein).

Hence, applying Chen and Mourrat (2025, Corollary 6.11) (with M therein set to be 1), we get that there are $(N^{\pm}_k)_{k\in\mathbb{N}}$, $(x^{\pm}_k)_{k\in\mathbb{N}}$, $\mathbf{p}_\pm \in \mathcal{Q}_{\infty,\leqslant 1}(\Delta)$, and $\mathbf{a}_\pm \in \overline{\mathrm{conv}}\,\{\boldsymbol{\sigma}\boldsymbol{\sigma}^\intercal : \boldsymbol{\sigma}\in \operatorname{supp} P^{\mathrm{vec}}_1\}$ (see (3.14)) satisfying $\mathbf{a}_\pm \geqslant \mathbf{p}_\pm$ such that

(i) $\left(N^{-1}\boldsymbol{\sigma}^l\left(\boldsymbol{\sigma}^{l'}\right)^\intercal,\, \alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ under $\mathbb{E}\,\langle\cdot\rangle^{\circ,M,x^\pm_k}_{MN^\pm_k,\lambda^\star}$ converges in law to

$$\left(\left(\mathbf{p}_\pm\left(\alpha^l\wedge\alpha^{l'}\right),\ \alpha^l\wedge\alpha^{l'}\right)\mathbf{1}_{l\neq l'} + (\mathbf{a}_\pm,1)\mathbf{1}_{l=l'}\right)_{l,l'\in\mathbb{N}}$$

under $\mathbb{E}\,\langle\cdot\rangle_{\mathfrak{R}}$, as k tends to infinity;

(ii) we have

$$\mathscr{P}^{\mathrm{vec}}_{t,q}(\mathbf{p}_-) \leqslant \liminf_{N\to\infty} \overline{F}^{\mathrm{vec}}_N(t,\mathbf{q}) \leqslant \limsup_{N\to\infty}\overline{F}^{\mathrm{vec}}_N(t,\mathbf{q}) \leqslant \mathscr{P}^{\mathrm{vec}}_{t,q}(\mathbf{p}_+).$$

Here, the functional (see Chen and Mourrat, 2025, (6.15)) is given by, for every $\mathbf{p}\in\mathcal{Q}_\infty(\Delta)$,

$$\mathscr{P}^{\mathrm{vec}}_{t,q}(\mathbf{p}) = \psi^{\mathrm{vec}}_{P^{\mathrm{vec}}_1}(\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})) - t\int_0^1 \boldsymbol{\theta}(\mathbf{p}(r))\mathrm{d}r \tag{5.20}$$

where $\psi^{\mathrm{vec}}_{P^{\mathrm{vec}}_1}$ is given as in (4.15) and $\boldsymbol{\theta}$ is given by $\boldsymbol{\theta} = \mathbf{b}\cdot\nabla\boldsymbol{\xi}(\mathbf{b}) - \boldsymbol{\xi}(\mathbf{b})$ for every $\mathbf{b}\in\mathbb{R}^{\Delta\times\Delta}$ ($\boldsymbol{\xi}$ as in (3.15)).

In the following, we use the above statement to complete the proof.

Recall the reparametrization of $\{1,\ldots,\Delta\}$ given in (3.9) and recall notation in (3.10). We set

$$p_\pm = \left(M^{-1}\sum_{m\in\mathsf{M}_s}(\mathbf{p}_\pm)_{(m,\bullet)(m,\bullet)}\right)_{s\in\mathscr{S}}, \qquad a_\pm = \left(M^{-1}\sum_{m\in\mathsf{M}_s}(\mathbf{a}_\pm)_{(m,\bullet)(m,\bullet)}\right)_{s\in\mathscr{S}}. \tag{5.21}$$

Using this and the relation between $R_{MN,\lambda_{MN}}(\sigma,\sigma')$ and $N^{-1}\boldsymbol{\sigma}\boldsymbol{\sigma}'^\intercal$ in (3.12), we can deduce Part (1) from Part (i) in the above.

Next, we verify properties of $p_\pm$ and $a_\pm$. First of all, it is clear that $p_\pm\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$. By (5.21) and $\mathbf{a}_\pm\geqslant\mathbf{p}_\pm$, we can deduce $a_\pm\geqslant p_\pm$. From the definition of P^{vec}_1 in (3.14), the fact that $\mathbf{a}_\pm\in\overline{\mathrm{conv}}\{\cdots\}$ in the above, and (5.21), we have $a_\pm\in\mathcal{K}$ defined in (5.18). Lastly, from (5.21), we have $|p_{\pm,s}(r)|\leqslant M^{-1}|\mathsf{M}_s||\mathbf{p}_\pm(r)|\leqslant\lambda^\star_s$ for every $r\in[0,1)$ and $s\in\mathscr{S}$. Hence, we can conclude $p_\pm\in\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda^\star}(\kappa)$.

Lastly, we verify Part (2). We want to match the functional appearing in Part (2) with that in Part (ii). We start with the second term in the functionals. By computing $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\boldsymbol{\xi}(\mathbf{b}+\varepsilon\mathbf{b})\big|_{\varepsilon=0}$, we can get from the definition of $\boldsymbol{\xi}$ in (3.15) that, for any $\mathbf{b},\mathbf{b}'\in\mathbb{R}^{\Delta\times\Delta}$,

$$\mathbf{b}'\cdot\nabla\boldsymbol{\xi}(\mathbf{b}) = \left(\sum_{m\in\mathsf{M}_s}\mathbf{b}'_{(m,\bullet)(m,\bullet)}\right)_{s\in\mathscr{S}}\cdot\nabla\xi\left(\left(M^{-1}\sum_{m\in\mathsf{M}_s}\mathbf{b}_{(m,\bullet)(m,\bullet)}\right)_{s\in\mathscr{S}}\right). \tag{5.22}$$

Comparing the above with (5.21), we get $\mathbf{p}_\pm\cdot\nabla\boldsymbol{\xi}(\mathbf{p}_\pm) = Mp_\pm\cdot\nabla\xi(p_\pm)$. By (5.21) and the relation between $\boldsymbol{\xi}$ and ξ in (3.15), we get $\boldsymbol{\xi}(\mathbf{p}_\pm) = M\xi(p_\pm)$. Recall the definition of θ in (5.1). Then, we can conclude $\boldsymbol{\theta}(\mathbf{p}_\pm) = M\theta(p_\pm)$. It remains to identify the first term on the right on (5.20).

In the notation in (3.9) and (3.10), from (5.21) and (5.22) we can see that $\nabla\boldsymbol{\xi}(\mathbf{p}_\pm)$ and $\nabla\xi(p_\pm)$ satisfy (3.16) with $\mathbf{q}$ and q therein replaced by them respectively. Hence, we have

$$\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p}_\pm) \text{ and } q+t\nabla\xi(p_\pm) \text{ satisfy (3.16)} \tag{5.23}$$

with $\mathbf{q}$ and q therein substituted with this pair. This along with Lemma 3.1 (at $N=1$ and $t=0$) implies

$$\overline{F}_{M,\lambda_M}(0,q+t\nabla\xi(p_\pm)) = M^{-1}\overline{F}_1^{\mathrm{vec}}(0,\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p}_\pm)). \tag{5.24}$$

In view of (3.6) and (4.15), we have $\overline{F}_1^{\mathrm{vec}}(0,\cdot)=\psi_{P_1^{\mathrm{vec}}}^{\mathrm{vec}}$. Therefore, we get

$$\psi_{P_1^{\mathrm{vec}}}^{\mathrm{vec}}(\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})) = \overline{F}_1^{\mathrm{vec}}(0,\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})) \overset{(5.24),\ \mathrm{L.}4.11}{=} M\psi_{\lambda^\star}(q+t\nabla\xi(p)).$$

Inserting this and $\boldsymbol{\theta}(\mathbf{p}_\pm)=M\theta(p_\pm)$ into (5.20) and comparing it with (5.2), we thus obtain

$$\mathscr{P}^{\mathrm{vec}}_{t,q}(\mathbf{p}_\pm) = M\mathscr{P}_{\lambda^\star,t,q}(p_\pm). \tag{5.25}$$

This along with Part (ii) and Lemma 3.1 yields Part (2). The proof is now complete. □

To state the second result from the cavity computation, we need to introduce the notation for the overlap of cavity spins. From the definition of the overlap in (1.7) and the free energy in (1.15), it is clear that we can reorder the elements in the partition $(I_{N,s})_{s\in\mathscr{S}}$ as long as the proportions $\lambda_{N,s}$ are preserved. Hence, for every $N\in\mathbb{N}$, we can assume

$$I_{M,s}\subseteq I_{MN,s},\quad \forall s\in\mathscr{S} \tag{5.26}$$

so that the indices for cavity spins are fixed. Then, for every $N\in\mathbb{N}$ and every $\sigma\in\Sigma^{MN}$, we define

$$\sigma_{\circ,s} = (\sigma_{kn})_{1\leqslant k\leqslant\kappa_s,\, n\in I_{M,s}},\qquad \forall s\in\mathscr{S}. \tag{5.27}$$

to play the role of cavity spins. Then, for every $N\in\mathbb{N}$ and $\sigma,\sigma'\in\Sigma^{MN}$, we consider the overlap of cavity spins

$$\begin{aligned} R^{\circ,M}_{MN,\lambda_{MN},s}(\sigma,\sigma') &= \frac{1}{M}\sigma_{\circ,s}\left(\sigma'_{\circ,s}\right)^\intercal,\quad \forall s\in\mathscr{S};\\ R^{\circ,M}_{MN,\lambda_{MN}}(\sigma,\sigma') &= \left(R^{\circ,M}_{MN,\lambda_{MN},s}(\sigma,\sigma')\right)_{s\in\mathscr{S}}. \end{aligned} \tag{5.28}$$

We state a simple observation to be used later.

Lemma 5.6. *Let $M,N\in\mathbb{N}$, $(t,q)\in\mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$, $\lambda^\star\in\blacktriangle_M$, and $x\in\mathsf{pert}(M,\lambda^\star)$. Then at (t,q) we have, for every bounded measurable $\pi:[0,1]\to\prod_{s\in\mathscr{S}}S^{\kappa_s}$,*

$$\mathbb{E}\left\langle\pi(\alpha\wedge\alpha')\cdot R_{MN,\lambda_{MN}}(\sigma,\sigma')\right\rangle^{M,x}_{MN,\lambda^\star} = \mathbb{E}\left\langle\pi(\alpha\wedge\alpha')\cdot R^{\circ,M}_{MN,\lambda_{MN}}(\sigma,\sigma')\right\rangle^{M,x}_{MN,\lambda^\star}. \tag{5.29}$$

Proof: We consider the bijection $\sigma\mapsto\boldsymbol{\sigma}$ given in (3.11) and assume (5.26). In addition, we can choose the bijection $\sigma\mapsto\boldsymbol{\sigma}$ in (3.11) to ensure that, for every $N\in\mathbb{N}$, the cavity spins in $(\sigma_{\circ,s})_{s\in\mathscr{S}}$ (see (5.27)) with $\sigma\in\Sigma^{MN}$ are mapped to $\boldsymbol{\tau}=(\boldsymbol{\sigma}_{(m,k)N})_{m,k}\in\mathbb{R}^\Delta$, the last column vector of $\boldsymbol{\sigma}\in\mathbb{R}^{\Delta\times N}$ (here Δ is given in (3.8)). Hence, analogous to (3.12), we have

$$R^{\circ,M}_{MN,\lambda_{MN}}(\sigma,\sigma') = \left(M^{-1}\sum_{m\in\mathsf{M}_s}\left(\boldsymbol{\tau}\boldsymbol{\tau}'^\intercal\right)_{(m,\bullet)(m,\bullet)}\right)_{s\in\mathscr{S}} \tag{5.30}$$

in the notation introduced in (3.10). In the following, we write $\langle\cdot\rangle=\langle\cdot\rangle^{M,x}_{MN,\lambda^\star}$. Since the Hamiltonian in (5.12) only depends on σ only through the overlaps $(\boldsymbol{\sigma}^l\boldsymbol{\sigma}^{l'\intercal})_{l,l'\in\mathbb{N}}$, the law of $\boldsymbol{\sigma}$ under $\mathbb{E}\langle\cdot\rangle$ is invariant if we permute the indices of the column vectors. Using this and the expression in (3.13), we have, for every $s\in\mathscr{S}$,

$$\begin{aligned} \mathbb{E}\left\langle\pi_s(\alpha\wedge\alpha')\cdot\sum_{m\in\mathsf{M}_s}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'^\intercal\right)_{(m,\bullet)(m,\bullet)}\right\rangle &= \sum_{n=1}^N\mathbb{E}\left\langle\pi_s(\alpha\wedge\alpha')\cdot\sum_{m\in\mathsf{M}_s}\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'^\intercal\right)_{(m,\bullet)n(m,\bullet)n}\right\rangle\\ = \sum_{n=1}^N\mathbb{E}\left\langle\pi_s(\alpha\wedge\alpha')\cdot\sum_{m\in\mathsf{M}_s}\left(\boldsymbol{\tau}\boldsymbol{\tau}'^\intercal\right)_{(m,\bullet)(m,\bullet)}\right\rangle &= N\mathbb{E}\left\langle\pi_s(\alpha\wedge\alpha')\cdot\sum_{m\in\mathsf{M}_s}\left(\boldsymbol{\tau}\boldsymbol{\tau}'^\intercal\right)_{(m,\bullet)(m,\bullet)}\right\rangle \end{aligned}$$

where π_s is the s-component of $\pi = (\pi_s)_{s\in\mathscr{S}}$. This together with (3.12) and (5.30) yields (5.29). $\square$

Lastly, for every $\lambda_M \in \blacktriangle_M$ and $q \in \mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$, we define the Gibbs measure

$$\langle\cdot\rangle_{M,\lambda_M,\mathfrak{R},q} \propto \exp\left(\sqrt{2}W^q_M(\sigma,\alpha) - Mq(1)\cdot R_{M,\lambda_M}(\sigma,\sigma)\right)\mathrm{d}P_M(\sigma)\mathrm{d}\mathfrak{R}(\alpha) \tag{5.31}$$

where $W^q_M(\sigma,\alpha)$ and P_M are given as in (1.12) and (1.6), respectively. In view of (1.15), we can see that this Gibbs measure is the one associated with $\overline{F}_{M,\lambda_M}(0,q)$.

Lemma 5.7. *Let $M\in\mathbb{N}$ and $\lambda^\star \in \blacktriangle_M$. Set $\lambda_{MN} = \lambda^\star$ for every N. For any $(t,q)\in\mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$ and any sequence* seq *of increasing integers, there are a subsequence $(N_k)_{k\in\mathbb{N}}$ of* seq*, $(x_k)_{k\in\mathbb{N}}$ of parameters in* $\mathsf{pert}(M,\lambda^\star)$*, $p\in\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda^\star}$, and $a\in\mathcal{K}$ (see (5.18)) satisfying $a\geqslant p$ (see (5.19)) such that*

(1) $\left(R_{MN_k,\lambda^\star}(\sigma^l,\sigma^{l'}),\ \alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ under $\mathbb{E}\langle\cdot\rangle^{\circ,M,x_k}_{MN_k,\lambda^\star}$ converges in law to

$$\left(\left(p(\alpha^l\wedge\alpha^{l'}),\ \alpha^l\wedge\alpha^{l'}\right)\mathbf{1}_{l\neq l'} + (a,1)\mathbf{1}_{l=l'}\right)_{l,l'\in\mathbb{N}}$$

under $\mathbb{E}\langle\cdot\rangle_{\mathfrak{R}}$, as k tends to infinity;

(2) for every bounded continuous $g:\left(\prod_{s\in\mathscr{S}}\mathbb{R}^{\kappa_s\times\kappa_s}\right)\times\mathbb{R}\to\mathbb{R}$, we have

$$\begin{aligned}\lim_{k\to\infty}\mathbb{E}\left\langle g\left(R^{\circ,M}_{M(N_k+1),\lambda^\star}(\sigma,\sigma'),\ \alpha\wedge\alpha'\right)\right\rangle^{M,x_k}_{M(N_k+1),\lambda^\star}\\ = \mathbb{E}\left\langle g\left(R_{M,\lambda^\star}(\sigma,\sigma'),\ \alpha\wedge\alpha'\right)\right\rangle_{M,\lambda^\star,\mathfrak{R},q+t\nabla\xi(p)}.\end{aligned}$$

The two Gibbs measures in Part (1) are given in (5.14) and Section 1.1.4, respectively. The two Gibbs measure in Part (2) are given in (5.12) and (5.31), respectively.

Proof: We match $\overline{F}_{MN,\lambda_{MN}}$, its perturbed versions, and the associated Gibbs measures with those of $\overline{F}^{\mathrm{vec}}_N$ as described in the first two paragraphs in the proof of Lemma 5.5.

Then, applying Chen and Mourrat (2025, Corollary 6.12) (with M therein set to be 1), we get that there are $(N_k)_{k\in\mathbb{N}}$, $(x_k)_{k\in\mathbb{N}}$, $\mathbf{p}\in\mathcal{Q}_{\infty,\leqslant 1}(\Delta)$, and $\mathbf{a}\in\overline{\mathrm{conv}}\left\{\boldsymbol{\sigma}\boldsymbol{\sigma}^\intercal : \boldsymbol{\sigma}\in\mathrm{supp}\,P^{\mathrm{vec}}_1\right\}$ (see (3.14)) satisfying $\mathbf{a}\geqslant\mathbf{p}$ such that

(i) $\left(N^{-1}\boldsymbol{\sigma}^l(\boldsymbol{\sigma}^{l'})^\intercal,\ \alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ under $\mathbb{E}\langle\cdot\rangle^{\circ,M,x_k}_{MN_k,\lambda^\star}$ converges in law to

$$\left(\left(\mathbf{p}(\alpha^l\wedge\alpha^{l'}),\alpha^l\wedge\alpha^{l'}\right)\mathbf{1}_{l\neq l'} + (\mathbf{a},1)\mathbf{1}_{l=l'}\right)_{l,l'\in\mathbb{N}}$$

under $\mathbb{E}\langle\cdot\rangle_{\mathfrak{R}}$ as k tends to infinity;

(ii) for every bounded continuous $\mathbf{g}:\mathbb{R}^{\Delta\times\Delta}\times\mathbb{R}\to\mathbb{R}$, we have

$$\lim_{k\to\infty}\mathbb{E}\left\langle\mathbf{g}\left(\boldsymbol{\tau}\boldsymbol{\tau}'^\intercal,\ \alpha\wedge\alpha'\right)\right\rangle^{M,x_k}_{M(N+1),\lambda^\star} = \mathbb{E}\left\langle\mathbf{g}\left(\boldsymbol{\tau}\boldsymbol{\tau}'^\intercal,\ \alpha\wedge\alpha'\right)\right\rangle^{\mathrm{vec}}_{\mathfrak{R},\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})}.$$

Here, $\boldsymbol{\tau} = (\boldsymbol{\sigma}_{(m,k)N+1})_{m,k}\in\mathbb{R}^\Delta$ is the last column vector of $\boldsymbol{\sigma}\in\mathbb{R}^{\Delta\times(N+1)}$ and the Gibbs measure on the right-hand side is defined as in Chen and Mourrat (2025, (6.17)):

$$\langle\cdot\rangle^{\mathrm{vec}}_{\mathfrak{R},\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})} \propto \exp\left(\sqrt{2}W^{\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})}_1(\alpha)\cdot\boldsymbol{\tau} - \left(\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})\right)(1)\cdot\boldsymbol{\tau}\boldsymbol{\tau}^\intercal\right)\mathrm{d}P^{\mathrm{vec}}_1(\boldsymbol{\tau})\mathrm{d}\mathfrak{R}(\alpha)$$

where $W^{\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})}_1$ is given as in (3.4).

We set p and a analogously as in (5.21). Using the same argument below (5.21), we can get Part (1) from Part (i).

For the second part, as argued above (5.30), we can choose the bijection $\sigma\mapsto\boldsymbol{\sigma}$ in (3.11) to satisfy

$$R^{\circ,M}_{M(N+1),\lambda^\star}(\sigma,\sigma') = \left(M^{-1}\sum_{m\in\mathsf{M}_s}\left(\boldsymbol{\tau}\boldsymbol{\tau}'^\intercal\right)_{(m,\bullet)(m,\bullet)}\right)_{s\in\mathscr{S}}. \tag{5.32}$$

Similar to (5.23), we have that $\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})$ and $q+t\nabla\xi(p)$ satisfy (3.16), which allows us to use the identity (3.18) to see that, under the identification of $\boldsymbol{\sigma}$ and σ in (3.11), $(\sigma^l,\alpha^l)_{l\in\mathbb{N}}$ under $\mathbb{E}\left\langle\cdot\right\rangle^{\mathrm{vec}}_{\mathfrak{R},\mathbf{q}+t\nabla\boldsymbol{\xi}(\mathbf{p})}$ has the same law under $\mathbb{E}\left\langle\cdot\right\rangle_{M,\lambda^\star,\mathfrak{R},q+t\nabla\xi(p)}$. Therefore, Part (2) follows from Part (ii) and (5.32). Properties of p and a can be verified similarly as in the proof of Lemma 5.5. ☐

Remark 5.8 (Perturbation specific to multi-species models). In the above, the perturbation was introduced in the style of vector spins (presence of $\boldsymbol{\sigma}$ in (5.6) and (5.10)) because we want to directly apply Lemma 3.1 and results from Chen and Mourrat (2025) for vector spins. In fact, results in Lemmas 5.5 and 5.7 hold for the following perturbation specific to the multi-species models as used in Panchenko (2015); Bates and Sohn (2022b). Instead of only considering systems with size MN due to reliance on the assumption (3.7), we are able to define the perturbation for each $N\in\mathbb{N}$ (but later we only apply to systems with sizes MN). Let (r_n) be an enumeration of $[0,1]\cap\mathbb{Q}$ and $(a_n)_{n\in\mathbb{N}}$ be an enumeration of elements in $\prod_{s\in\mathscr{S}}S^{\kappa_s}_+$ with rational entries. Conditioned on $\mathfrak{R}$, for every $h=(h_i)_{1\leqslant i\leqslant 4}\in\mathbb{N}^4$, let $\left(H^h_N(\sigma,\alpha)\right)_{\sigma\in\Sigma^N,\,\alpha\in\operatorname{supp}\mathfrak{R}}$ be an independent centered Gaussian process with covariance

$$\mathbb{E}\left[H^h_N(\sigma,\alpha)H^h_N(\sigma',\alpha')\right]=N\left(a_{h_1}\cdot\left(R_{N,\lambda_N}(\sigma,\sigma')\right)^{\odot h_2}+\lambda_{h_3}\alpha\wedge\alpha'\right)^{h_4}.$$

The existence of such a process is explained in Chen and Mourrat (2025, Section 6.1.1). Fix c_h similarly as in (5.7) but this time for every N instead of MN. Let bas be an orthonormal basis of $\prod_{s\in\mathscr{S}}S^{\kappa_s}$. Then, we set

$$H^x_N(\sigma,\alpha)=\sum_{h\in\mathbb{N}^4}x_hc_hH^h_N(\sigma,\alpha)+\sum_{e\in\mathsf{bas}}x_e e\cdot R_{N,\lambda_N}(\sigma,\sigma).$$

Let M be the dimension of cavity as fixed in Lemmas 5.5 and 5.7. Let $\widetilde{H}^{t,q}_N(\sigma,\alpha)$ be given as in (5.5) relative to this M. Then, for each $N\in\mathbb{N}$, we define

$$\left\langle\cdot\right\rangle_{N,\lambda_N,x}\propto\exp\left(H^{t,q}_N(\sigma,\alpha)+N^{-\frac{1}{16}}H^x_N(\sigma,\alpha)\right)\mathrm{d}P_{N,\lambda_N}(\sigma)\mathrm{d}\mathfrak{R}(\alpha),$$

$$\left\langle\cdot\right\rangle^\circ_{N,\lambda_N,x}\propto\exp\left(\widetilde{H}^{t,q}_N(\sigma,\alpha)+N^{-\frac{1}{16}}H^x_N(\sigma,\alpha)\right)\mathrm{d}P_{N,\lambda_N}(\sigma)\mathrm{d}\mathfrak{R}(\alpha).$$

Then, Lemma 5.5 and 5.7 hold when the Gibbs measures therein are replaced by these two (at size MN). To see this, one needs to redo cavity computations in Chen and Mourrat (2025, Section 6) using this perturbation. The key part is to prove that Ghirlanda–Guerra identities hold for the overlap $\left(R_{N,\lambda_N}(\sigma^l,\sigma^{L'})\right)_{l,l'\in\mathbb{N}}$ under this perturbation (see Chen and Mourrat, 2025, Proposition 6.8). These modifications are straightforward but tedious. ☐

5.2. *Approximation.* Recall the definition of the discrete simplex $\blacktriangle_N$ from (1.3). Recall the continuous simplex $\blacktriangle_\infty$ from (1.4). In this section, we fix any $\lambda_\infty\in\blacktriangle_\infty$. We want to use approximation arguments to study the limit of $\overline{F}_{N,\lambda_N}$ and the limit law of overlaps, for $(\lambda_N)_{N\in\mathbb{N}}$ converging to λ_∞.

To use previous results, it is convenient to work with another sequence of proportions other than $(\lambda_N)_{N\in\mathbb{N}}$. We fix two sequences $(M_n)_{n\in\mathbb{N}}$ and $(\lambda^\star_n)_{n\in\mathbb{N}}$ satisfying

$$\lambda^\star_n\in\blacktriangle_{M_n},\quad\forall n\in\mathbb{N};\qquad\lim_{n\to\infty}\lambda^\star_n=\lambda_\infty.\tag{5.33}$$

To simplify our notation for approximations, for $a,b\in\mathbb{R}$ and $\varepsilon>0$, we write $a\lesssim_\varepsilon b$ provided $a\leqslant b+\varepsilon$; also, we write $a\approx_\varepsilon b$ provided $|a-b|\leqslant\varepsilon$.

Recall the functional $\mathscr{P}_{\lambda,t,q}$ in (5.2) and Gibbs measures introduced in Section 1.1.4, (5.12), (5.14), and (5.31). Also recall the space of perturbation parameters in (5.8). We have two results and we now state the first.

Lemma 5.9. *Assume that $(\lambda_N)_{N\in\mathbb{N}}$ converges to some λ_∞. Let $(t,q)\in\mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$. Let $(M_n)_{n\in\mathbb{N}}$ and $(\lambda^\star_n)_{n\in\mathbb{N}}$ satisfy (5.33). Then, there are*

- *$(N_n^\pm)_{n\in\mathbb{N}}$ of strictly increasing positive integers satisfying $N_n^\pm \in M_n\mathbb{N}$ for each n,*
- *$(x_n^\pm)_{n\in\mathbb{N}}$ of perturbation parameters satisfying $x_n^\pm \in \mathsf{pert}(M_n, \lambda_n^\star)$ for each n,*
- *$p_\pm$ in $\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda_\infty}(\kappa)$ and $a_\pm$ in $\prod_{s\in\mathscr{S}} S_+^{\kappa_s}$ satisfying $a_\pm \geqslant p_\pm$ (see (5.19)),*

such that the following holds at (t,q):

(1) $\left(R_{N_n^\pm,\lambda_n^\star}(\sigma^l,\sigma^{l'}),\ \alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ under $\mathbb{E}\left\langle\cdot\right\rangle^{\circ,M_n,x_n^\pm}_{N_n^\pm,\lambda_n^\star}$ converges in law to

$$\left(\left(p_\pm(\alpha^l\wedge\alpha^{l'}),\ \alpha^l\wedge\alpha^{l'}\right)\mathbf{1}_{l\neq l'} + (a_\pm,1)\mathbf{1}_{l=l'}\right)_{l,l'\in\mathbb{N}}$$

under $\mathbb{E}\left\langle\cdot\right\rangle_{\mathfrak{R}}$, as n tends to infinity;

(2) we have

$$\limsup_{n\to\infty}\overline{F}_{N_n^\pm,\lambda_n^\star}(t,q) \leqslant \mathscr{P}_{\lambda_\infty,t,q}(p_+), \qquad \liminf_{n\to\infty}\overline{F}_{N_n^\pm,\lambda_n^\star}(t,q) \geqslant \mathscr{P}_{\lambda_\infty,t,q}(p_-).$$

Moreover, if $(\overline{F}_{N,\lambda_N})_{N\in\mathbb{N}}$ converges pointwise to some f, then both $\left(\overline{F}_{N_n^\pm,\lambda_n^\star}\right)_{n\in\mathbb{N}}$ and $\left(\widetilde{F}^{M_n,x_n^\pm}_{N_n^\pm,\lambda_n^\star}\right)_{n\in\mathbb{N}}$ converge pointwise to f.

Proof: We shall only consider the case with superscript + and the other case can be treated similarly. Henceforth, we omit + from the notation. Fix any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of strictly positive real numbers satisfying $\lim_{n\to\infty}\varepsilon_n = 0$.

For each $n\in\mathbb{N}$, let $(N_k^n)_{k\in\mathbb{N}}$, $(x_k^n)_{k\in\mathbb{N}}$, p_n, and a_n be given by Lemma 5.5 for the + case with M_n and $\lambda_n^\star$ substituted for M and $\lambda^\star$ therein. By Part (1) of Lemma 5.5 (and also Lemma 5.4), there is $\mathbf{k}_1\in\mathbb{N}$ such that, for every $k\geqslant\mathbf{k}_1$, we have

$$\mathbb{E}\left\langle\left|R_{M_nN_k^n,\lambda_n^\star}(\sigma,\sigma') - p_n(\alpha\wedge\alpha')\right|\right\rangle^{\circ,M_n,x_k^n}_{M_nN_k^n,\lambda_n^\star} \approx_{\varepsilon_n} 0, \tag{5.34}$$

$$\mathbb{E}\left\langle\left|R_{M_nN_k^n,\lambda_n^\star}(\sigma,\sigma) - a_n\right|\right\rangle^{\circ,M_n,x_k^n}_{M_nN_k^n,\lambda_n^\star} \approx_{\varepsilon_n} 0. \tag{5.35}$$

By Part (2) of Lemma 5.5, there is $\mathbf{k}_2\in\mathbb{N}$ such that, for every $k\geqslant\mathbf{k}_2$, we have

$$\overline{F}_{M_nN_k^n,\lambda_n^\star}(t,q) \lesssim_{\varepsilon_n} \limsup_{N\to\infty}\overline{F}_{M_nN,\lambda_n^\star}(t,q) \leqslant \mathscr{P}_{\lambda_n^\star,t,q}(p_n). \tag{5.36}$$

Fix any k_n satisfying $k\geqslant\max_{i\in\{1,2\}}\mathbf{k}_i$ and set $N_n = M_nN_{k_n}^n$ and $x_n = x_{k_n}^n$. Since there is no upper bound on k_n, we can choose larger k_n to ensure that $(N_n)_{n\in\mathbb{N}}$ is strictly increasing as announced and

$$(N_n/M_n)^{-1} \leqslant \varepsilon_n \tag{5.37}$$

which will needed later.

By passing to a subsequence of $(N_n)_{n\in\mathbb{N}}$ and using the compactness result in Lemma 4.3, we may assume that there are $p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$ such that $(p_n)_{n\in\mathbb{N}}$ converges to p pointwise and in L^1. Also, we can assume that $(a_n)_{n\in\mathbb{N}}$ converges to some a. Recall from Lemma 5.5 that we have $p_n\in\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda_n^\star}(\kappa)$. The pointwise convergence and (5.33) implies $p\in\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda_\infty}(\kappa)$. Clearly, we also have $a\geqslant p$. Hence, we have verified properties of p and a.

We are ready to verify the two main parts of Lemma 5.9. From the convergence of $(a_n)_{n\in\mathbb{N}}$ and (5.35), we immediately get

$$\lim_{n\to\infty}\mathbb{E}\left\langle\left|R_{N_n,\lambda_n}(\sigma,\sigma) - a\right|\right\rangle^{\circ,M_n,x_n}_{N_n,\lambda_n^\star} = 0. \tag{5.38}$$

Using the invariance of cascades in Lemma 5.3 and the convergence of $(p_n)_{n\in\mathbb{N}}$, we have

$$\lim_{n\to\infty}\mathbb{E}\left\langle\left|p_n(\alpha\wedge\alpha') - p(\alpha\wedge\alpha')\right|\right\rangle^{\circ,M_n,x_n}_{N_n,\lambda_n^\star} = 0$$

which together with (5.34) implies

$$\lim_{n\to\infty} \mathbb{E}\left\langle \left| R_{N_n,\lambda_n}(\sigma,\sigma') - p(\alpha\wedge\alpha')\right|\right\rangle^{\circ,M_n,x_n}_{N_n,\lambda^\star_n} = 0. \tag{5.39}$$

Combining (5.38) and (5.39) with Lemma 5.4, we can conclude Part (1). From its definition in (5.2), we can see that $\mathscr{P}_{\lambda^\star_n,t,q}(p_n)$ is continuous jointly in $\lambda^\star_n$ and p_n (due to Lemmas 4.9 and 4.10). This along with (5.36) yields the first side in Part (2). As we explained previously, the other side can be treated by the same method.

Now, we turn to the last statement. Here, we display the superscript ±. For each (t',q'), let $C_{t',q'}$ be the constant appearing on the right-hand side in (2.1) of Lemma 2.3, which only depends on $(\kappa_s)_{s\in\mathscr{S}}$, ξ, $(\mu_s)_{s\in\mathscr{S}}$, and (t',q'). We set $\varepsilon^\pm_n = C_{t',q'}|\lambda^\star_n - \lambda_{N^\pm_n}|$. Since both $(\lambda^n_\star)_{n\to\infty}$ and $(\lambda_N)_{N\to\infty}$ converge to λ_∞, we have $\lim_{n\to\infty}\varepsilon^\pm_n = 0$. Lemma 2.3 implies that, for every (t',q'),

$$\overline{F}_{N^\pm_n,\lambda^\star_n}(t',q') \approx_{\varepsilon^\pm_n} \overline{F}_{N^\pm_n,\lambda_{N^\pm_n}}(t',q') \tag{5.40}$$

which gives the convergence of $\overline{F}_{N^\pm_n,\lambda^\star_n}$ to f. By Lemma 5.1, we have

$$\left|\overline{F}_{N^\pm_n,\lambda^\star_n}(t',q') - \widetilde{F}^{M_n,x^\pm_n}_{N^\pm_n,\lambda^\star_n}(t',q')\right| \leqslant C(1+|t'|)(N^\pm_n/M_n)^{-\frac{1}{16}}.$$

This along with (5.37) and (5.40) implies that $\widetilde{F}^{M_n,x^\pm_n}_{N^\pm_n,\lambda^\star_n}(t',q')$ converges pointwise to f as $n\to\infty$. □

Recall the cavity overlap from (5.28). We state the second result.

Lemma 5.10. *Assume that $(\lambda_N)_{N\in\mathbb{N}}$ converges to some λ_∞. Let $(t,q)\in\mathbb{R}_+\times\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$. Let $(M_n)_{n\in\mathbb{N}}$ and $(\lambda^\star_n)_{n\in\mathbb{N}}$ satisfy (5.33). Let seq be any strictly increasing sequence in $\mathbb{N}$. Then, there are*

- *a subsequence $(N_n)_{n\in\mathbb{N}}$ of seq satisfying $N_n\in M_n\mathbb{N}$ for every n,*
- *$(x_n)_{n\in\mathbb{N}}$ of perturbation parameters satisfying $x_n\in\mathsf{pert}(M_n,\lambda^\star_n)$ for each n,*
- *p in $\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda_\infty}(\kappa)$ and a in $\prod_{s\in\mathscr{S}}S^{\kappa_s}_+$ satisfying $a\geqslant p$,*

such that the following holds at (t,q):

(1) $\left(R_{N_n,\lambda^\star_n}(\sigma^l,\sigma^{l'}),\ \alpha^l\wedge\alpha^{l'}\right)_{l,l'\in\mathbb{N}}$ under $\mathbb{E}\langle\cdot\rangle^{\circ,M_n,x_n}_{N_n,\lambda^\star_n}$ converges in law to

$$\left(\left(p_\pm(\alpha^l\wedge\alpha^{l'}),\ \alpha^l\wedge\alpha^{l'}\right)\mathbf{1}_{l\neq l'} + (a_\pm,1)\mathbf{1}_{l=l'}\right)_{l,l'\in\mathbb{N}}$$

under $\mathbb{E}\langle\cdot\rangle_{\mathfrak{R}}$, as n tends to infinity;

(2) for any bounded continuous $\pi:[0,1]\to\left(\prod_{s\in\mathscr{S}}S^{\kappa_s}\right)$, we have

$$\lim_{n\to\infty}\mathbb{E}\left\langle\pi\left(\alpha\wedge\alpha'\right)\cdot R^{\circ,M_n}_{N_n+M_n,\lambda^\star_n}\right\rangle^{M_n,x_n}_{N_n+M_n,\lambda^\star_n} = \langle\pi,\partial_q\psi_{\lambda_\infty}(q+t\nabla\xi(p))\rangle_{L^2}.$$

Moreover, if $(\overline{F}_{N,\lambda_N})_{N\in\mathsf{seq}}$ converges pointwise to some f, then both $\left(\overline{F}^{M_n,x_n}_{N_n+M_n,\lambda^\star_n}\right)_{n\in\mathbb{N}}$ and $\left(\widetilde{F}^{M_n,x_n}_{N_n,\lambda^\star_n}\right)_{n\in\mathbb{N}}$ converge pointwise to f.

Proof: Fix any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of strictly positive real numbers satisfying $\lim_{n\to\infty}\varepsilon_n=0$.

For each n, we choose the corresponding parameters. Define $\mathsf{seq}(n) = (\lfloor N/M_n\rfloor)_{N\in\mathsf{seq}}$. Apply Lemma 5.7 to M_n, $\lambda^\star_n$, and $\mathsf{seq}(n)$, we get the corresponding $(N^n_k)_{k\in\mathbb{N}}$, $(x^n_k)_{k\in\mathbb{N}}$, $p_n\in\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda^\star_n}(\kappa)$, and a_n. By Part (1) of Lemma 5.7 (together with Lemma 5.4), there is $\mathbf{k}_1\in\mathbb{N}$ such that, for every $k\geqslant\mathbf{k}_1$, we have

$$\mathbb{E}\left\langle\left|R_{M_nN^n_k,\lambda^\star_n}(\sigma,\sigma') - p_n(\alpha\wedge\alpha')\right|\right\rangle^{\circ,M_n,x^n_k}_{M_nN^n_k,\lambda^\star_n} \approx_{\varepsilon_n} 0, \tag{5.41}$$

$$\mathbb{E}\left\langle\left|R_{M_nN^n_k,\lambda^\star_n}(\sigma,\sigma) - a_n\right|\right\rangle^{\circ,M_n,x^n_k}_{M_nN^n_k,\lambda^\star_n} \approx_{\varepsilon_n} 0. \tag{5.42}$$

By Part (2) of Lemma 5.7, there is $\mathbf{k}_2 \in \mathbb{N}$ such that, for every $k \geqslant \mathbf{k}_2$, we have

$$\begin{aligned} &\mathbb{E}\left\langle \pi\left(\alpha \wedge \alpha'\right) \cdot R^{\circ, M_n}_{M_n(N^n_k+1), \lambda^\star_n}(\sigma, \sigma') \right\rangle^{M_n, x^n_k}_{M_n(N^n_k+1), \lambda^\star_n} \\ &\approx_{\varepsilon_n} \mathbb{E}\left\langle \pi\left(\alpha \wedge \alpha'\right) \cdot R_{M_n, \lambda^\star_n}(\sigma, \sigma') \right\rangle_{M_n, \lambda^\star_n, \mathfrak{R}, q + t\nabla\xi(p_n)} . \end{aligned} \tag{5.43}$$

These are preparations for Parts (1) and (2) of the lemma to prove. But before proceeding, we first prove the last statement in the lemma. For each $k \in \mathbb{N}$, let $\widetilde{N}^n_k$ from seq satisfy $\lfloor \widetilde{N}^n_k / M_n \rfloor = N^n_k$. Then, we can estimate, at every $(t', q') \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{L}}_\infty(\kappa)$,

$$\begin{aligned} \left| \overline{F}_{\widetilde{N}^n_k, \lambda_{\widetilde{N}^n_k}} - \overline{F}^{M_n, x^n_k}_{M_n(N^n_k+1), \lambda^\star_n} \right| \leqslant \left| \overline{F}_{\widetilde{N}^n_k, \lambda_{\widetilde{N}^n_k}} - \overline{F}_{\widetilde{N}^n_k, \lambda^\star_n} \right| + \left| \overline{F}_{\widetilde{N}^n_k, \lambda^\star_n} - \overline{F}_{M_n(N^n_k+1), \lambda^\star_n} \right| \\ + \left| \overline{F}_{M_n(N^n_k+1), \lambda^\star_n} - \overline{F}^{M_n, x^n_k}_{M_n(N^n_k+1), \lambda^\star_n} \right| . \end{aligned}$$

Applying Lemmas 2.3, 2.4, and 5.1 to the three terms respectively, we can see that the left-hand side is bounded by a constant $C_{t', q'}$ times

$$\left| \lambda_{\widetilde{N}^n_k} - \lambda^\star_n \right| + \frac{M_n}{\widetilde{N}^n_k} + (N^n_k)^{-1/16} .$$

A similar bound holds for the difference between $\overline{F}_{\widetilde{N}^n_k, \lambda_{\widetilde{N}^n_k}}$ and $\widetilde{F}^{M_n, x^n_k}_{N_n, \lambda^\star_n}$. Notice that the last two terms in this bound vanish as k tends to infinity (recall that n is temporarily fixed) while the first term tends to $|\lambda_\infty - \lambda^\star_n|$. Recall that $(\varepsilon_n)_{n\in\mathbb{N}}$ is an arbitrary vanishing sequence that we choose. Here, for convenience of notation, we can assume that we have chosen it to satisfy $\varepsilon_n \geqslant 2|\lambda_\infty - \lambda^\star_n|$. Then, we can find $\mathbf{k}_3 \in \mathbb{N}$ (independent of (t', q')) such that the error term in the above display is bounded by ε_n for all $k \geqslant \mathbf{k}_3$. Hence, at every (t', q') and for every $k \geqslant \mathbf{k}_3$, we have

$$\overline{F}_{\widetilde{N}^n_k, \lambda_{\widetilde{N}^n_k}} \approx_{C_{t', q'}\varepsilon_n} \overline{F}^{M_n, x^n_k}_{M_n(N^n_k+1), \lambda^\star_n}, \qquad \overline{F}_{\widetilde{N}^n_k, \lambda_{\widetilde{N}^n_k}} \approx_{C_{t', q'}\varepsilon_n} \widetilde{F}^{M_n, x^n_k}_{N_n, \lambda^\star_n} . \tag{5.44}$$

Fix some k_n satisfying $k \geqslant \max_{i\in\{1,2,3\}} \mathbf{k}_i$ and set

$$N_n = M_n N^n_{k_n}, \qquad x_n = x^n_{k_n} . \tag{5.45}$$

This is done for each n. We can choose larger k_n to ensure that both $(N_n)_{n\in\mathbb{N}}$ and $(\widetilde{N}^n_{k_n})_{n\in\mathbb{N}}$ are strictly increasing. To verify the last statement, recall that our choice of $\widetilde{N}^n_k$ ensures that $\overline{F}_{\widetilde{N}^n_k, \lambda_{\widetilde{N}^n_k}}$ in (5.44) belongs to the sequence $(\overline{F}_{N, \lambda_N})_{N \in \mathsf{seq}}$. Now inserting (5.45) into (5.44), we can see that the last statement holds.

Now we turn to Parts (1) and (2). By passing to a subsequence of $(N_n)_{n\in\mathbb{N}}$ and using the compactness result in Lemma 4.3, we can find some $p \in \mathcal{Q}^{\mathscr{L}}_\infty(\kappa)$ such that $(p_n)_{n\in\mathbb{N}}$ converges to p pointwise and in L^1. By the same argument as in the proof of Lemma 5.9, we can verify that $p \in \mathcal{Q}^{\mathscr{L}}_{\infty, \leqslant \lambda_\infty}(\kappa)$ and $a \geqslant p$.

Notice that (5.41) and (5.42) have the same form as in (5.34) and (5.35). Using the same argument in Lemma 5.9, we can verify Part (1) of Lemma 5.10.

For Part (2), recall that the Gibbs measure on the right of (5.43) is given in (5.31). Comparing it with (1.15) and (1.16), we can see that it is exactly the Gibbs measure associated with $\overline{F}_{M_n, \lambda^\star_n}(0, q + t\nabla\xi(p_n))$. Using this and the computation in (4.4), we can see that the right-hand side in (5.43) is equal to

$$\left\langle \pi, \partial_q \overline{F}_{M_n, \lambda^\star_n}(0, q + t\nabla\xi(p_n)) \right\rangle_{L^2} \overset{\text{L.4.11}}{=} \left\langle \pi, \partial_q \psi_{\lambda^\star_n}(q + t\nabla\xi(p_n)) \right\rangle_{L^2} .$$

By Lemmas 4.9 and 4.10, $\partial_q \psi_{\lambda^\star_n}(q + t\nabla\xi(p_n))$ is continuous in $\lambda^\star_n$ and p_n. Hence, the above display along with (5.43) gives Part (2) after n is sent to infinity. □

5.3. *Results concerning critical points.* We use Lemma 5.9 to prove the following proposition, which is the counterpart to Chen and Mourrat (2025, Proposition 7.1). Recall the definition of $\mathcal{Q}_{\infty,\uparrow}(D)$ from (1.21) and that of $\mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$ as in (1.10).

Proposition 5.11 (Critical point identification, I)**.** *Suppose that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges to some* $\lambda_\infty \in \blacktriangle_\infty$ *and that* $\left(\overline{F}_{N,\lambda_N}\right)_{N\in\mathbb{N}}$ *converges pointwise to some* f*. Let* $t\in\mathbb{R}_+$ *and* $q\in\mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$*. If* $f(t,\cdot)$ *is Gateaux differentiable at* q*, then* $f(t,q)=\mathscr{P}_{\lambda_\infty,t,q}\left(\partial_q f(t,q)\right)$*.*

Proof: Let the sequences and parameters be given by Lemma 5.9. Since $\overline{F}_{N_n^\pm,\lambda_n^\star}$ also converges to f due to Lemma 5.9, we get from Part (2) of this lemma that

$$\mathscr{P}_{\lambda_\infty,t,q}(p_-)\leqslant f(t,q)\leqslant \mathscr{P}_{\lambda_\infty,t,q}(p_+). \tag{5.46}$$

It remains to identify $p_\pm$. For brevity, we write $\widetilde{F}_n^\pm=\widetilde{F}^{M_n,x_n^\pm}_{N_n^\pm,\lambda_n^\star}$ and $\langle\cdot\rangle^\circ_{\pm,n}=\langle\cdot\rangle^{\circ,M_n,x_n^\pm}_{N_n^\pm,\lambda_n^\star}$. Lemma 5.9 gives that $\overline{F}_n^\pm$ converges to f. As a consequence of Proposition 4.6 and Remark 4.7, we have the convergence of $\partial_q\widetilde{F}_n^\pm(t,q)$ to $\partial_q f(t,q)$. By a similar computation for (4.4), we have that, for every continuous $\pi\in L^\infty([0,1];\prod_{s\in\mathscr{S}}S^{\kappa_s})$,

$$\left\langle \pi,\, \partial_q\widetilde{F}_n^\pm(t,q)\right\rangle_{L^2}=\mathbb{E}\left\langle \pi(\alpha\wedge\alpha')\cdot R_{N_n^\pm,\lambda_n^\star}(\sigma,\sigma')\right\rangle^\circ_{\pm,n}$$

We send n to infinity and apply Lemma 5.9 (1) to the right-hand side to get

$$\langle\pi,\,\partial_q f(t,q)\rangle_{L^2}=\mathbb{E}\left\langle\pi(\alpha\wedge\alpha')\cdot p_\pm(\alpha\wedge\alpha')\right\rangle_{\mathfrak{R}}\overset{\mathrm{L.}5.3}{=}\langle\pi,p_\pm\rangle_{L^2}$$

which implies that $p_\pm=\partial_q f(t,q)$. Inserting this to (5.46), we get the desired result. □

The following corresponds to Chen and Mourrat (2025, Proposition 7.2) and we apply Lemma 5.10.

Proposition 5.12 (Critical point identification, II)**.** *Suppose that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges to some* $\lambda_\infty\in\blacktriangle_\infty$ *and that* $\left(\overline{F}_{N,\lambda_N}\right)_{N\in\mathbb{N}}$ *converges pointwise to some* f *along a subsequence* $(N_k)_{k\in\mathbb{N}}$*. Let* $t\geqslant 0$*. If* $f(t,\cdot)$ *is Gateaux differentiable at some* $q\in\mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$*, then*

$$\partial_q f(t,q)=\partial_q\psi_{\lambda_\infty}(q+t\nabla\xi(\partial_q f(t,q))). \tag{5.47}$$

If, in addition, $t>0$*, and* $f(\cdot,q)$ *is differentiable at* t*, then*

$$\partial_t f(t,q)-\int_0^1\xi\left(\partial_q f(t,q)\right)=0. \tag{5.48}$$

Proof: Denote by seq the sequence $(N_k)_{k\in\mathbb{N}}$. Let $(N_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, p, and a be given by Lemma 5.10 corresponding to seq and (t,q) in the statement of this proposition.

We first show (5.48). We write $\widetilde{F}_n=\widetilde{F}^{M_n,x_n}_{N_n,\lambda_n^\star}$ and $\langle\cdot\rangle^\circ_n=\langle\cdot\rangle^{\circ,M_n,x_n}_{N_n,\lambda_n^\star}$ (appearing in Lemma 5.10 (1)). The last statement of Lemma 5.10 ensures that $\widetilde{F}_n$ converges to f pointwise. Since $\langle\cdot\rangle^\circ_n$ is the Gibbs measure associated with $\widetilde{F}_n$ (see (5.13) and (5.14)), a similar computation for (4.4) gives that, for every bounded continuous $\pi:[0,1]\to\prod_{s\in\mathscr{S}}S^{\kappa_s}$,

$$\left\langle\pi,\partial_q\widetilde{F}_n(t,q)\right\rangle_{L^2}=\mathbb{E}\left\langle\pi(\alpha\wedge\alpha')\cdot R\right\rangle^\circ_n,\qquad \partial_t\widetilde{F}_n(t,q)=\mathbb{E}\left\langle\xi(R)\right\rangle^\circ_n \tag{5.49}$$

where we used the short hand $R=R_{N_n,\lambda_n^\star}(\sigma,\sigma')$. By Proposition 4.6 and Remark 4.7, $\partial_q\widetilde{F}_n(t,q)$ converges to $\partial_q f(t,q)$. Using this and Lemma 5.10 (1), we can send n to infinity in the first relation of the above display to get

$$\langle\pi,\partial_q f(t,q)\rangle_{L^2}=\mathbb{E}\left\langle\pi(\alpha\wedge\alpha')\cdot p(\alpha\wedge\alpha')\right\rangle_{\mathfrak{R}}\overset{\mathrm{L.}5.3}{=}\langle\pi,p\rangle_{L^2}.$$

Varying π, we get

$$\partial_q f(t,q)=p. \tag{5.50}$$

Under the additional assumption of differentiability at t, we can use Proposition 4.6 and Remark 4.7 to get the convergence of $\partial_t \widetilde{F}_n(t,q)$ to $\partial_t f(t,q)$. Sending n to infinity in the second relation in (5.49), we get

$$\partial_t f(t,q) = \mathbb{E}\left\langle \xi(p(\alpha\wedge\alpha'))\right\rangle_{\mathfrak{R}} \overset{\text{L.5.3}}{=} \int_0^1 \xi(p(r))\mathrm{d}r$$

which along with (5.50) implies (5.48).

Now, we write $\overline{F}_n = \overline{F}^{M_n,x_n}_{N_n+M_n,\lambda_n^\star}$, which by Lemma 5.10 converges pointwise to f. Let $\langle\cdot\rangle_n = \langle\cdot\rangle^{M_n,x_n}_{N_n+M_n,\lambda_n^\star}$ be the Gibbs measure associated with $\overline{F}_n$ which appears in Lemma 5.10 (2). Similar to (4.4), we can compute

$$\begin{aligned}\left\langle \pi, \partial_q \overline{F}_n(t,q)\right\rangle_{L^2} &= \mathbb{E}\left\langle \pi(\alpha\wedge\alpha')\cdot R_{N_n+M_n,\lambda_n^\star}(\sigma,\sigma')\right\rangle_n \\ &\overset{\text{L.5.6}}{=} \mathbb{E}\left\langle \pi(\alpha\wedge\alpha')\cdot R^{\circ,M_n}_{N_n+M_n,\lambda_n^\star}(\sigma,\sigma')\right\rangle_n\end{aligned}$$

for any bounded continuous $\pi : [0,1]\to\prod_{s\in\mathscr{S}} S^{\kappa_s}$, where Lemma 5.6 is applied with N_n+M_n and M_n substituted for N and M therein. Again Proposition 4.6 and Remark 4.7 together give the convergence of $\partial_q\overline{F}_n(t,q)$ to $\partial_t f(t,q)$. Sending n to infinity and applying Lemma 5.10 (2), we get

$$\left\langle \pi, \partial_q f(t,q)\right\rangle_{L^2} = \left\langle \pi, \partial_q \psi_{\lambda_\infty}(q+t\nabla\xi(p))\right\rangle_{L^2}.$$

Varying π and inserting (5.50), we obtain (5.47). ☐

Proof of Theorem 1.3: Let p and q' be given in the statement. The relation (5.47) in Proposition 5.12 implies that (q',p) is a critical point defined in (1.18). The convergence in (1.23) follows from Proposition 5.11 and (5.3). ☐

The following corresponds to Chen and Mourrat (2025, Proposition 7.3).

Proposition 5.13 (Critical point representation)**.** *Suppose that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges to some* $\lambda_\infty\in\blacktriangle_\infty$ *and that* $\left(\overline{F}_N\right)_{N\in\mathbb{N}}$ *converges pointwise to some* f. *For every* $(t,q)\in\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$*, there is* $p\in\mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda_\infty}(\kappa)$ *such that*

$$f(t,q) = \mathscr{P}_{\lambda_\infty,t,q}(p), \qquad p = \partial_q\psi_{\lambda_\infty}(q+t\nabla\xi(p)). \tag{5.51}$$

The proof is based on Propositions 5.11 and 5.12 together with approximation arguments. We prefer to omit the detail since the proof is exactly the same as that for Chen and Mourrat (2025, Proposition 7.3). We only mention that Propositions 5.3, 5.4, 7.1, and 7.2 in Chen and Mourrat (2025) used in that proof correspond to Propositions 4.8, 4.6, 5.11, and 5.12 here; and Corollary 5.2 there corresponds to Lemma 5.10 here.

Proof of Theorem 1.2: Let p be given in Proposition 5.13 and set $q' = q+t\nabla\xi(p)$. The second relation in (5.51) ensures that (q',p) is a critical point defined in (1.18). The first relation in (5.51) and (5.3) yield (1.23). ☐

Recall the Gibbs measure $\langle\cdot\rangle_{N,\lambda_N}$ from (1.16) associated with the original free energy. We consider the array of conditional overlaps:

$$R^{l,l'}_{N,\lambda_N,\sigma|\alpha} = \mathbb{E}\left\langle R_{N,\lambda_N}(\sigma^l,\sigma^{l'})\,\middle|\,\alpha^l\wedge\alpha^{l'}\right\rangle_{N,\lambda_N}, \quad \forall l,l'\in\mathbb{N} \tag{5.52}$$

where the conditional expectation is taken with respect to $\mathbb{E}\langle\cdot\rangle_{N,\lambda_N}$ (not $\langle\cdot\rangle_{N,\lambda_N}$). Also recall $\langle\cdot\rangle_{\mathfrak{R}}$ from Section 1.1.4. Also, $\sigma|\alpha$ in the subscript of $R^{l,l'}_{N,\lambda_N,\sigma|\alpha}$ is purely symbolic to indicate the conditioning.

The next result is the version of Chen and Mourrat (2025, Proposition 7.4) in the multi-species setting. Recall the Gibbs measure $\langle\cdot\rangle_{N,\lambda_N}$ as in (1.16).

Proposition 5.14 (Convergence of conditional overlap)**.** *Suppose that* $(\overline{F}_{N,\lambda_N})_{N\in\mathbb{N}}$ *converges pointwise to some* f *along a subsequence* $(N_k)_{k\in\mathbb{N}}$ *and let* $t \in \mathbb{R}_+$ *(here, convergence of* $(\lambda_N)_{N\in\mathbb{N}}$ *is not assumed). If* $f(t,\cdot)$ *is Gateaux differentiable at some* $q \in \mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$*, then* $\big(R^{l,l'}_{N,\lambda_N,\sigma|\alpha}\big)_{l,l'\in\mathbb{N}:l\neq l'}$ *under* $\mathbb{E}\left\langle\cdot\right\rangle_{N_k,\lambda_{N_k}}$ *(at* (t,q)*) converges in law to* $\big(p(\alpha^l\wedge\alpha^{l'})\big)_{l,l'\in\mathbb{N}:l\neq l'}$ *under* $\mathbb{E}\left\langle\cdot\right\rangle_{\mathfrak{R}}$*, as* k *tends to infinity, where* $p = \partial_q f(t,q)$*.*

This result does not reply on Propositions 5.11 or 5.12. The same proof for Chen and Mourrat (2025, Proposition 7.4) works here. There, Propositions 4.8, 5.4 and display (5.7) correspond to Lemma 5.3, Proposition 4.6, and (4.4) here.

Before preceding, we extract a useful representation for $R^{l,l'}_{N,\lambda_N,\sigma|\alpha}$ (defined in (5.52)) from the proof of Chen and Mourrat (2025, Proposition 7.4). The following rephrases Chen and Mourrat (2025, (7.10)).

Lemma 5.15 (Representation of conditional overlap)**.** *For every* $N \in \mathbb{N}$*,* $\lambda_N \in \blacktriangle_N$*, and* $(t,q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$*, we write* $p_N = \partial_q \overline{F}_{N,\lambda_N}(t,q)$ *as an element in* $\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$*. Then, we have*

$$R^{l,l'}_{N,\lambda_N,\sigma|\alpha} = p_N(\alpha^l\wedge\alpha^{l'})$$

for every $l,l' \in \mathbb{N}$ *with* $l \neq l'$*, a.s. under* $\mathbb{E}\left\langle\cdot\right\rangle_{N,\lambda_N}$*.*

Next, we describe the convergence of the overlap when there is a small perturbation.

For $N \in \mathbb{N}$ and $\lambda_N \in \blacktriangle_N$, let $\widehat{H}_N(\sigma)$ be the Hamiltonian with quadratic interaction that was introduced in (1.24). Recall the definition of the Hamiltonian $H^{t,q}_N(\sigma)$ in (1.14). For every $(t,\widehat{t},q) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$, we consider

$$\begin{aligned}&\widehat{F}_{N,\lambda_N}\left(t,\widehat{t},q\right)\\&= -\frac{1}{N}\log\iint \exp\left(H^{t,q}_N(\sigma,\alpha) + \sqrt{2\widehat{t}}\widehat{H}_N(\sigma) - \widehat{t}N\left|R_{N,\lambda_N}(\sigma,\sigma)\right|^2\right)\mathrm{d}P_{N,\lambda_N}(\sigma)\mathrm{d}\mathfrak{R}(\alpha).\end{aligned} \tag{5.53}$$

We denote the associated Gibbs measure by $\left\langle\cdot\right\rangle_{N,\lambda_N,\widehat{t}}$, where we omit the dependence on (t,q). Let $\lambda_\infty \in \blacktriangle_\infty$ and recall the functional $\widehat{\mathscr{J}}_{\lambda_\infty,t,\widehat{t},q}(q',p)$ defined for $q' \in \mathcal{Q}^{\mathscr{S}}_2(\kappa)$ and $p \in L^2([0,1],\prod_{s\in\mathscr{S}} S^{\kappa_s})$, which was introduced previously in (1.25). The following result is an adaption of Chen and Mourrat (2025, Proposition 7.5). The most interesting is the third part.

Proposition 5.16 (Convergence of overlap under perturbation)**.** *Suppose that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges to some* $\lambda_\infty \in \blacktriangle_\infty$ *and that* $\left(\widehat{F}_N\right)_{N\in\mathbb{N}}$ *converges pointwise to some* f *along a subsequence* $(N_k)_{k\in\mathbb{N}}$*. Then, for each* $t \geqslant 0$*, the function* $f(t,\cdot,\cdot) : \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa) \to \mathbb{R}$ *is Gateaux differentiable (jointly in the two variables) on a subset of* $\mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$ *that is dense in* $\mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$*. Moreover, for every* $\widehat{t} \geqslant 0$ *and every* $q \in \mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$ *of Gateaux differentiability of* $f(t,\widehat{t},\cdot)$*, the following holds for* $p = \partial_q f(t,\widehat{t},q)$ *and* $q' = q + t\nabla\xi(p) + 2\widehat{t}p$*:*

(1) $p = \partial_q\psi(q')$*;*

(2) if $(N_k)_{k\in\mathbb{N}}$ *is the full sequence* $(N)_{N\in\mathbb{N}}$*, then*

$$\lim_{N\to\infty}\widehat{F}_{N,\lambda_N}\left(t,\widehat{t},q\right) = \widehat{\mathscr{J}}_{\lambda_\infty,t,\widehat{t},q}(q',p);$$

(3) if $\widehat{t} > 0$ *and* $f(t,\cdot,q)$ *is differentiable at* $\widehat{t}$*, then* $\left(R_{N_k,\lambda_{N_k}}(\sigma^l,\sigma^{l'})\right)_{l,l'\in\mathbb{N}:l\neq l'}$ *under* $\mathbb{E}\left\langle\cdot\right\rangle_{N_k,\lambda_{N_k},\widehat{t}}$ *converges in law to* $\left(p(\alpha^l\wedge\alpha^{l'})\right)_{l,l'\in\mathbb{N}:l\neq l'}$ *under* $\mathbb{E}\left\langle\cdot\right\rangle_{\mathfrak{R}}$*, as* k *tends to infinity.*

The proof is a straightforward adaption of that for Chen and Mourrat (2025, Proposition 7.5). In that proof, Propositions 5.3, 7.1, 7.2, and 7.4 correspond to Propositions 4.8, 5.11, 5.12, and 5.14 here.

Proof of Theorem 1.4: The theorem follows from Proposition 5.16. ☐

The next result adapts Chen and Mourrat (2025, Proposition 1.5) to the multi-species setting.

Proposition 5.17 (Uniqueness of critical point at high temperature)**.** *There exists* $t_{\mathrm{c}} > 0$ *such that for every* $t \in [0, t_{\mathrm{c}})$ *and* $q \in \mathcal{Q}^{\mathscr{S}}_2(\kappa)$*, the function* $\mathscr{J}_{t,q}$ *has a unique critical point in* $\mathcal{Q}^{\mathscr{S}}_2(\kappa) \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$*.*

The proof is the same as that for Chen and Mourrat (2025, Proposition 1.5). The estimates in (5.23) and (5.21) used therein correspond to those in Lemma 4.9, which should be used together with Lemma 4.10 here.

The next result adapts Chen and Mourrat (2025, Proposition 1.6).

Proposition 5.18 (Relevant critical points must be stable)**.** *Assume that* $(\lambda_N)_{N\in\mathbb{N}}$ *converges to* λ_∞*. For each* $n \in \mathbb{N}$*, let* $(t_n, q_n) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$ *and let* $(q'_n, p_n) \in \mathcal{Q}^{\mathscr{S}}_2(\kappa) \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$ *be a critical point of* $\mathscr{J}_{\lambda_\infty,t_n,q_n}$ *such that*

$$\lim_{N\to\infty} \overline{F}_{N,\lambda_N}(t_n, q_n) = \mathscr{J}_{\lambda_\infty,t_n,q_n}(q'_n, p_n).$$

Suppose that (t_n, q_n) *converges towards* $(t, q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$*. Then,* $(q'_n, p_n)_{n\in\mathbb{N}}$ *is precompact in* $\mathcal{Q}^{\mathscr{S}}_2(\kappa) \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$*. Moreover, any subsequential limit* $(q', p) \in \mathcal{Q}^{\mathscr{S}}_2(\kappa) \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$ *is a critical point* $\mathscr{J}_{\lambda_\infty,t,q}$ *and is such that*

$$\lim_{N\to\infty} \overline{F}_{N,\lambda_N}(t, q) = \mathscr{J}_{\lambda_\infty,t,q}(q', p).$$

Again, the proof is the same. Proposition 3.1, Lemma 3.4, and Corollary 5.2 used there correspond to Proposition 4.1 (Lipschitzness), Lemma 4.3, and Lemma 4.15 (together with Lemma 4.10).

6. Results for convex models

We apply results in Section 5.3 to the case where ξ is convex. The results proved here correspond to those in Chen and Mourrat (2025, Section 8). In particular, we prove Theorems 1.1 and 1.6.

Recall $\mathcal{K}$ from (5.18) and $\mathscr{P}_{\lambda,t,q}$ from (5.2). The following is the version of Chen and Mourrat (2025, Proposition 8.1) for the multi-species setting.

Proposition 6.1 (Parisi formula for enriched model)**.** *Let* $(\lambda_N)_{N\in\mathbb{N}}$ *converge to some* $\lambda_\infty \in \blacktriangle_\infty$*. If* ξ *is convex on* $\prod_{s\in\mathscr{S}} S^{\kappa_s\times\kappa_s}_+$*, then we have, for every* $(t, q) \in \mathbb{R}_+ \times \mathcal{Q}^{\mathscr{S}}_2(\kappa)$*,*

$$\lim_{N\to\infty} \overline{F}_{N,\lambda_N}(t, q) = \sup_{p\in\mathcal{Q}^{\mathscr{S}}(\kappa):\, \exists a\in\mathcal{K},\, a\geqslant p} \mathscr{P}_{\lambda_\infty,t,q}(p) = \sup_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \mathscr{P}_{\lambda_\infty,t,q}(p). \tag{6.1}$$

The relation $a \geqslant p$ under the supremum is understood as in (5.19).

Proof: We first prove this for rational λ_∞. Suppose that there is $M \in \mathbb{N}$ such that $\lambda_\infty \in \blacktriangle_M$. Then, we can use Corollary 3.2 to match $\overline{F}_{N,\lambda_N}$ with $\overline{F}^{\mathrm{vec}}_N$ as described therein. Due to its definition in (3.15), we can deduce that $\boldsymbol{\xi}$ is convex on S^Δ_+ from the convexity of ξ. By Chen and Mourrat (2025, Proposition 8.1), we have the Parisi formula: for every $(t, \mathbf{q}) \in \mathbb{R}_+ \times \mathcal{Q}_2(\Delta)$,

$$\lim_{N\to\infty} \overline{F}^{\mathrm{vec}}_N(t, \mathbf{q}) = \sup_{\mathbf{p}\in\mathcal{Q}(\Delta):\, \exists \mathbf{a}\in\mathcal{K}^{\mathrm{vec}},\, \mathbf{a}\geqslant\mathbf{p}} \mathscr{P}^{\mathrm{vec}}_{t,\mathbf{q}}(\mathbf{p}) = \sup_{\mathbf{p}\in\mathcal{Q}_\infty(\Delta)} \mathscr{P}^{\mathrm{vec}}_{t,\mathbf{q}}(\mathbf{p}) \tag{6.2}$$

where $\mathscr{P}^{\mathrm{vec}}_{t,\mathbf{q}}(\mathbf{p})$ is given as in (5.20) and $\mathcal{K}^{\mathrm{vec}} = \overline{\mathrm{conv}}\,\{\boldsymbol{\sigma}\boldsymbol{\sigma}^\intercal : \boldsymbol{\sigma} \in \operatorname{supp} P^{\mathrm{vec}}_1\}$. Now, fix any (t, q) and let $\mathbf{q}$ be given as in (3.16). For every $\mathbf{q}$ and $\mathbf{a}$ appearing in (6.2), let p and a be given as in (5.21) (without $\pm$). Hence, the setup is the same as in the proof of Lemma 5.5 (with $\lambda^\star$ therein substituted with λ_∞. By the argument in the paragraph below (5.21), we have $a \geqslant p$ and $a \in \mathcal{K}$. Also, we have $\mathscr{P}^{\mathrm{vec}}_{t,q}(\mathbf{p}) = M\mathscr{P}_{\lambda_\infty,t,q}(p)$ as verified in (5.25). This correspondence between (a, p) and $(\mathbf{a}, \mathbf{p})$ is bijective. Hence, by (6.2) and Corollary 3.2, we get (6.1) when all entries of λ_∞ are rational.

The general case follows from the rational case and a continuity argument. Fix a sequence $(\lambda^n_\infty)_{n\in\mathbb{N}}$ in $\blacktriangle_\infty$ with rational entries such that this sequence converges to λ_∞. For each n, fix any $(\lambda^n_N)_{N\in\mathbb{N}}$ converging to λ^n_∞. For each n, Proposition 5.13 gives $p_n \in \mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda^n_\infty}(\kappa)$ such that

$$\lim_{N\to\infty}\overline{F}_{N,\lambda^n_N}(t,q) = \mathscr{P}_{\lambda^n_\infty,t,q}(p_n). \tag{6.3}$$

Notice that the sequence $(p_n)_{n\in\mathbb{N}}$ is bounded uniformly in $Q^{\mathscr{S}}_\infty(\kappa)$. This along with the Lipschitzness of ψ^{vec}_ν in Lemma 4.9 implies that $\psi^{\mathrm{vec}}_{\mu_s}((q+t\nabla\xi(p_n))_s)$ is bounded uniformly in s and n. Hence, the definition of ψ_λ in (4.16) and that of $\mathscr{P}_{\lambda,t,q}$ in (5.2) imply

$$\left|\mathscr{P}_{\lambda^n_\infty,t,q}(p_n) - \mathscr{P}_{\lambda_\infty,t,q}(p_n)\right| \leqslant C\left|\lambda^n_\infty - \lambda_\infty\right|, \quad \forall n\in\mathbb{N}, \tag{6.4}$$

for some constant C. Also, Lemma 2.3 yields

$$\limsup_{N\to\infty}\left|\overline{F}_{N,\lambda^n_N}(t,q) - \overline{F}_{N,\lambda^n_N}(t,q)\right| \leqslant C'\left|\lambda^n_\infty - \lambda_\infty\right|, \quad \forall n\in\mathbb{N}, \tag{6.5}$$

for some constant C'. For $r, r' \geqslant \mathbb{R}$ and $\varepsilon > 0$, let us write $r \lesssim_\varepsilon r'$ for $r \leqslant r' + \varepsilon$. Then, for every ε, we can find $p_\varepsilon \in \mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$ and $n\in\mathbb{N}$ sufficiently large such that

$$\begin{aligned}\sup_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \mathscr{P}_{\lambda_\infty,t,q}(p) \lesssim_\varepsilon \mathscr{P}_{\lambda_\infty,t,q}(p_\varepsilon) &\overset{(4.16),(5.2)}{\lesssim_\varepsilon} \mathscr{P}_{\lambda^n_\infty,t,q}(p_\varepsilon) \\ &\overset{(6.1)}{\leqslant} \lim_{N\to\infty}\overline{F}_{N,\lambda^n_N}(t,q) \overset{(6.5)}{\lesssim_\varepsilon} \liminf_{N\to\infty}\overline{F}_{N,\lambda_N}(t,q)\end{aligned}$$

On the other hand, for every ε, there is $n\in\mathbb{N}$ such that

$$\begin{aligned}\limsup_{N\to\infty}\overline{F}_{N,\lambda_N}(t,q) &\overset{(6.5)}{\lesssim_\varepsilon} \lim_{N\to\infty}\overline{F}_{N,\lambda^n_N}(t,q) \overset{(6.3)}{=} \mathscr{P}_{\lambda^n_\infty,t,q}(p_n) \\ &\overset{(6.4)}{\lesssim_\varepsilon} \mathscr{P}_{\lambda_\infty,t,q}(p_n) \leqslant \sup_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \mathscr{P}_{\lambda_\infty,t,q}(p).\end{aligned}$$

The above two displays yield one identity in (6.1) in the general case. The other identity in (6.1) can be deduced similarly. □

Proof of Theorem 1.1: We can obtain (1.22) by using Proposition 6.1 and the relation in (5.3) between functionals. □

Proof of Theorem 1.6: Recall the definition of θ in (5.1). Under the assumption that ξ is convex on $\prod_{s\in\mathscr{S}} S^{\kappa_s}_+$, we can follow the same argument for (8.5) in the proof of Chen and Mourrat (2025, Proposition 8.1) to get

$$\int_0^1 \theta(p(r))\mathrm{d}r = \sup_{p'\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\left\{\left\langle\nabla\xi(p), p'\right\rangle_{L^2} - \int_0^1 \xi\left(p'(r)\right)\mathrm{d}r\right\}, \quad \forall p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa). \tag{6.6}$$

Recall the definition of $\mathscr{P}_{\lambda_\infty,t,q}$ in (5.2). Inserting (6.6) to the right-hand side of (6.1), we get

$$\begin{aligned}&\limsup_{N\to\infty}\overline{F}_{N,\lambda_N}(t,q) \\ &= \sup_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\inf_{p'\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\left\{\psi_{\lambda_\infty}(q+t\nabla\xi(p)) - \left\langle t\nabla\xi(p), p'\right\rangle_{L^2} + t\int_0^1 \xi\left(p'(r)\right)\mathrm{d}r\right\} \\ &\overset{(1.17)}{\leqslant} \sup_{q'\in q+\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\inf_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\mathcal{J}_{\lambda_\infty,t,q}\left(q',p\right).\end{aligned}$$

On the other hand, (1.26) and (1.28) in Claim 1.5 together yield

$$\liminf_{N\to\infty}\overline{F}_{N,\lambda_N}(t,q) \geqslant \sup_{q'\in q+\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\ \inf_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)} \mathscr{J}_{\lambda_\infty,t,q}\left(q',p\right). \tag{6.7}$$

The two above displays then give (1.29). ☐

Recall that in the statement of Theorem 1.6, we assumed Claim 1.5. The same claim also allows us to derive a different version of (1.29), which adapts Chen and Mourrat (2025, Corollary 8.2) to the current setting. For every $a\in\prod_{s\in\mathscr{S}}\mathbb{R}^{\kappa_s\times\kappa_s}$, we define

$$\xi^*(a) = \sup_{b\in\prod_{s\in\mathscr{S}}S_+^{\kappa_s}}\{a\cdot b-\xi(b)\}. \tag{6.8}$$

Corollary 6.2 (Alternative form of Hopf–Lax formula)**.** *Assume that Claim 1.5 is valid. If ξ is a convex function on $\prod_{s\in\mathscr{S}}S_+^{\kappa_s}$ and $(\lambda_N)_{N\in\mathbb{N}}$ converges to some λ_∞, then for every $(t,q)\in\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$, we have*

$$\lim_{N\to\infty}\overline{F}_{N,\lambda_N}(t,q) = \sup_{q'\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\left\{\psi_{\lambda_\infty}(q+q')-t\int_0^1\xi^*\left(t^{-1}q'\right)\right\}. \tag{6.9}$$

Proof: We fix any (t,q) and denote the two sides in (6.9) by LHS and RHS. By the convexity of ξ and the same argument as that for Chen and Mourrat (2025, (8.4)), it is easy to see $\theta(a)=\xi^*(\nabla(a))$ for every $a\in\prod_{s\in\mathscr{S}}S_+^{\kappa_s}$. Inserting this to the right-hand side in (6.1) and recalling the definition of $\mathscr{P}_{\lambda_\infty,t,q}$ in (5.2), we can get LHS $\leqslant$ RHS. On the other hand, by (6.8), we get

$$\sup_{p\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\left\{t^{-1}\left\langle q',p\right\rangle_{L^2}-\int_0^1\xi(p(r))\mathrm{d}r\right\}\leqslant\int_0^1\xi^*\left(t^{-1}q'(r)\right)\mathrm{d}r.$$

Inserting this to the right-hand side of (6.7) (which requires Claim 1.5) and using the definition of $\mathscr{J}_{\lambda_\infty,t,q}$ in (1.17) (together with changing q' to $q+q'$), we get LHS $\geqslant$ RHS. ☐

We can extract from Proposition 6.1 a more familiar form of the Parisi formula. For every $s\in\mathscr{S}$, $\mathbf{q}\in\mathcal{Q}_\infty(\kappa_s)$, and $a\in S^{\kappa_s}$, we define

$$X_s(\mathbf{q},a)=\mathbb{E}\log\iint\exp\left(\mathbf{w}^{\mathbf{q}}(\alpha)\cdot\boldsymbol{\tau}-\frac{1}{2}\mathbf{q}(1)\cdot\boldsymbol{\tau}\boldsymbol{\tau}^\intercal+a\cdot\boldsymbol{\tau}\boldsymbol{\tau}^\intercal\right)\mathrm{d}\mu_s(\boldsymbol{\tau})\mathrm{d}\mathfrak{R}(\alpha)$$

where $\mathbf{w}^{\mathbf{q}}$ is the $\mathbb{R}^{\kappa_s}$-valued process given as in (3.3). For every $\pi\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$ and $x\in\prod_{s\in\mathscr{S}}S^{\kappa_s}$, we define

$$\mathscr{P}_{\lambda_\infty}(\pi,x)=\sum_{s\in\mathscr{S}}\lambda_{\infty,s}X_s((\nabla\xi(\pi))_s,x_s)+\frac{1}{2}\int_0^1\theta(\pi(r))\mathrm{d}r$$

where $(\nabla\xi(\pi))_s$ is the $S_+^{\kappa_s}$-valued path in the s-coordinate of $\nabla\xi(\pi)\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)$. Recall the Hamiltonian $H_N(\sigma)$ from (1.9).

Corollary 6.3 (Parisi formula for free energy with correction)**.** *Let $(\lambda_N)_{N\to\infty}$ converge to some $\lambda_\infty\in\blacktriangle_\infty$. If ξ is convex on $\prod_{s\in\mathscr{S}}S_+^{\kappa_s}$, then,*

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\iint\exp\left(H_N(\sigma)-\frac{N}{2}\xi\left(R_{N,\lambda_N}(\sigma,\sigma)\right)\right)\mathrm{d}P_{N,\lambda_N}(\sigma)\mathrm{d}\mathfrak{R}(\alpha)=\inf_{\pi\in\mathcal{Q}^{\mathscr{S}}_\infty(\kappa)}\mathscr{P}_{\lambda_\infty}(\pi,0).$$

Proof: Due to (3.3), we have $\mathbf{w}^{\mathbf{q}}\overset{\mathrm{d}}{=}\sqrt{2}\mathbf{w}^{\mathbf{q}/2}$. By (4.15), we can see $X_s(\mathbf{q},0)=-\psi^{\mathrm{vec}}_{\mu_s}(\mathbf{q}/2)$, which by (4.16) gives $\sum_{s\in\mathscr{S}}\lambda_{\infty,s}X_s((\nabla\xi(\pi))_s,x_s)=-\psi_{\lambda_\infty}(\frac{1}{2}(\nabla\xi(\pi))$. Hence, we have $\mathscr{P}_{\lambda_\infty}(\pi,0)=-\mathscr{P}_{\lambda_\infty,\frac{1}{2},0}(\pi)$ given as in (5.2). On the other hand, notice that expression after the limit on the left-hand side of the above display is equal to $-\overline{F}_{N,\lambda_N}(\frac{1}{2},0)$ (see (1.15)). Therefore, this result follows from Proposition 6.1 with $(\frac{1}{2},0)$ substituted for (t,q) therein. ☐

This corollary corresponds to Chen and Mourrat (2025, Corollary 8.3) and the next result to Chen and Mourrat (2025, Proposition 8.4). We can get the Parisi formula without the correction term $-\frac{N}{2}\xi\left(R_{N,\lambda_N}(\sigma,\sigma)\right)$.

Proposition 6.4 (Parisi formula)**.** *Let* $(\lambda_N)_{N\to\infty}$ *converge to some* $\lambda_\infty \in \blacktriangle_\infty$*. If* ξ *is convex on* $\prod_{s\in\mathscr{S}} S_+^{\kappa_s}$*, then,*

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\iint \exp\left(H_N(\sigma)\right)\mathrm{d}P_{N,\lambda_N}(\sigma)\mathrm{d}\mathfrak{R}(\alpha) = \sup_{y\in S_+}\inf_{\pi\in\mathcal{Q}_\infty^{\mathscr{S}}(\kappa)}\left\{\mathscr{P}_{\lambda_\infty}(\pi,y)-\frac{1}{2}\xi^*(2y)\right\}$$
$$= \sup_{z\in S_+}\inf_{\substack{y\in S_+\\ \pi\in\mathcal{Q}_\infty^{\mathscr{S}}(\kappa)}}\left\{\mathscr{P}_{\lambda_\infty}(\pi,y)-y\cdot z+\frac{1}{2}\xi(z)\right\}$$

where we used the shorthand $S_+ = \prod_{s\in\mathscr{S}} S_+^{\kappa_s}$*.*

Proof: The argument involves some Hamilton–Jacobi equation as in Mourrat and Panchenko (2020, Section 5) (also see Chen, 2023, Section 5). One can follow the same steps as in the proof of Chen and Mourrat (2025, Proposition 8.4). The difference is that there the PDE is considered on $\mathbb{R}_+\times S_+^D$ for some $D\in\mathbb{N}$ but here we need to adapt it to $\mathbb{R}_+\times\left(\prod_{s\in\mathscr{S}} S_+^{\kappa_s}\right)$. The only new ingredient is to show that the Hopf–Lax formula and the Hopf formula still hold on this domain. In the following, we explain how to prove this.

By Chen and Xia (2025b, Propositions 6.3 and 6.4), it is sufficient to verify that the convex cone $\prod_{s\in\mathscr{S}} S_+^{\kappa_s}$ satisfies the *Fenchel–Moreau property* as described in Chen and Xia (2025b, Definition 6.1). Using a straightforward modification of the argument for S_+^D in Chen and Xia (2024, Proposition 5.1), we can verify that $\prod_{s\in\mathscr{S}} S_+^{\kappa_s}$ is also a *perfect cone* described in Chen and Xia (2024, Definition 2.1). By Chen and Xia (2024, Corollary 2.3), every perfect cone satisfies the Fenchel–Moreau property. Hence, the Hopf–Lax formula and the Hopf formula are valid in our setting. ☐

The next result adapts Chen and Mourrat (2025, Proposition 8.6).

Proposition 6.5 (Differentiability of Parisi formula)**.** *Let* $(\lambda_N)_{N\to\infty}$ *converge to some* $\lambda_\infty \in \blacktriangle_\infty$*, let* ξ *be convex on* $\prod_{s\in\mathscr{S}} S_+^{\kappa_s}$*, and let* f *be the pointwise limit of* $(\overline{F}_{N,\lambda_N})_{N\in\mathbb{N}}$ *given by Proposition 6.1.*

- *For each* $t\in\mathbb{R}_+$*, the function* $f(t,\cdot)$ *is Gateaux differentiable everywhere on* $\mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$*.*
- *The function* f *is Gateaux differentiable everywhere on* $(0,\infty)\times\mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$*.*

A straightforward modification of the proof of Chen and Mourrat (2025, Proposition 8.6) works here. Let us mention how to substitute results here for those in that proof. Lemma 6.4 corresponds to Lemma 5.1 here; Propositions 5.3, 5.4, 7.1, and 8.1 correspond to Propositions 4.8, 4.6, 5.11, and 6.1; Corollary 5.2 and 6.11 correspond to Lemma 4.9 (to be used together with Lemma 4.10) and Lemma 5.9. Only Proposition 2.7 does not have a restatement here (which states that a Lipschitz function is differentiable "almost everywhere" in infinite dimensions), but it easily adapts to the setting here.

As in Chen and Mourrat (2025, Corollary 8.7), we can summarize the results in the convex case as follows.

Corollary 6.6. *Let* $(\lambda_N)_{N\to\infty}$ *converge to some* $\lambda_\infty\in\blacktriangle_\infty$ *and let* ξ *be convex on* $\prod_{s\in\mathscr{S}} S_+^{\kappa_s}$*. Then, the sequence* $\left(\overline{F}_{N,\lambda_N}\right)_{N\in\mathbb{N}}$ *converges pointwise to some limit* f *on* $\mathbb{R}_+\times\mathcal{Q}_2^{\mathscr{S}}(\kappa)$*. At every* $(t,q)\in(0,\infty)\times\mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$*, the function* f *is Gateaux differentiable (jointly in its two variables) and satisfies*

$$\partial_t f(t,q) - \int_0^1 \xi\left(\partial_q f(t,q)\right) = 0. \tag{6.10}$$

For every $t \in \mathbb{R}_+$, $f(t,\cdot)$ *is Gateaux differentiable at every* $q \in \mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$ *and the following holds for* $p = \partial_q f(t,q)$ *and* $p_N = \partial_q \overline{F}_{N,\lambda_N}(t,q)$*:*

(1) $p_\infty \in \mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda_\infty}(\kappa)$, $p_N \in \mathcal{Q}^{\mathscr{S}}_{\infty,\leqslant\lambda_N}(\kappa)$ *for every* $N \in \mathbb{N}$*, and* $(p_N)_{N\in\mathbb{N}}$ *converges to* p *in* L^r *for every* $r \in [1,\infty)$ *as* N *tends to infinity;*

(2) $f(t,q) = \mathscr{P}_{\lambda_\infty,t,q}(p)$ *and* $p = \partial_q \psi_{\lambda_\infty}(q + t\nabla\xi(p))$*;*

(3) $p_N(\alpha\wedge\alpha') = \mathbb{E}\left\langle R_{N,\lambda_N}(\sigma,\sigma') \,|\, \alpha\wedge\alpha'\right\rangle_{N,\lambda_N}$ *almost surely under* $\mathbb{E}\langle\cdot\rangle_{N,\lambda_N}$ *for every* N*, and the overlap array* $\left(p_N(\alpha^\ell\wedge\alpha^{\ell'})\right)_{\ell,\ell'\in\mathbb{N}:\,\ell\neq\ell'}$ *under* $\mathbb{E}\langle\cdot\rangle_{N,\lambda_N}$ *converges in law to* $\left(p(\alpha^\ell\wedge\alpha^{\ell'})\right)_{\ell,\ell'\in\mathbb{N}:\,\ell\neq\ell'}$ *under* $\mathbb{E}\langle\cdot\rangle_{\mathfrak{R}}$ *as* N *tends to infinity.*

Proof: The existence of f is given by Proposition 6.1. The differentiability of f follows from Proposition 6.5. Proposition 5.12 yields (6.10). In Part (1), the range for p_N is due to (4.3) in Proposition 4.1; the convergence of $(p_N)_{N\in\mathbb{N}}$ follows from Proposition 4.6; the range for p is a consequence of these two results (because we can extract a subsequence converging a.e. on $[0,1]$). Part (2) follows from Propositions 5.11 and 5.12. Part (3) is due to Lemma 5.15 and Proposition 5.14. ☐

We can strengthen Part (3) to the convergence of the unconditioned overlap under the additional assumption that ξ is strictly convex. The next result adapts Chen and Mourrat (2025, Proposition 8.8) and the original proof can be modified easily (Propositions 4.8 and 5.4 used therein correspond to Lemma 5.3 and Proposition 4.6).

Proposition 6.7. *Let* $(\lambda_N)_{N\to\infty}$ *converge to some* $\lambda_\infty \in \blacktriangle_\infty$*, let* ξ *be strictly convex over* $\prod_{s\in\mathscr{S}}\mathbb{R}^{\kappa_s\times\kappa_s}$*, let* $t\in(0,\infty)$ *and* $q \in \mathcal{Q}^{\mathscr{S}}_{\infty,\uparrow}(\kappa)$*, and let* p *be as in Corollary 6.6. Then, the off-diagonal overlap array* $\left(R_{N,\lambda_N}(\sigma^l,\sigma^{l'})\right)_{l,l'\in\mathbb{N}:\,l\neq l'}$ *under* $\mathbb{E}\langle\cdot\rangle_{N,\lambda_N}$ *converges in law to* $\left(p(\alpha^l\wedge\alpha^{l'})\right)_{l,l'\in\mathbb{N}:\,l\neq l'}$ *under* $\mathbb{E}\langle\cdot\rangle_{\mathfrak{R}}$ *as* N *tends to infinity.*

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