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Loop percolation versus link percolation in the random loop model

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Abstract. In Mülbacher (2021), Peter Mühlbacher showed that in the random loop model without loop weights, a loop phase transition (assuming it exists) cannot occur at the same parameter as the percolation phase transition of the occupied edges. In this work, we give a quantitative version of this result, specifying a minimal gap between the percolation phase transition and a possible loop phase transition. A substantial part of our argument also works for weighted loop models.

1. Introduction and main results

Random loop models arise as graphical representations of various quantum spin systems, such as the quantum Heisenberg (anti-) ferromagnet. These connections were first observed in Aizenman and Nachtergaele (1994); Tóth (1993) and then extended in Ueltschi (2013). We refer the reader to these works for the connections to quantum systems and will exclusively treat the probabilistic versions here.

Given a finite graph, $G = (V, E)$, a *link configuration* is a finite sequence $c = (c_i)_{i\in[n]} = (e_i, s_i)_{i\in[n]}$ with $n \in \mathbb{N}$, $e_i \in E$ and $s_i \in \{-1, 1\}$ for all $i \in [n]$[1]. $s_i = 1$ corresponds to a 'cross' and $s_i = -1$ to a 'double bar' as described in Ueltschi (2013). One element of the sequence is called a *link*.

A link configuration $c = (e_i, s_i)_{i\in[n]}$ gives rise to a loop configuration. The most common way to describe this is through a graphical construction, see e.g. Ueltschi (2013). Here we provide an alternative, equivalent definition. We define the loop configuration $\mathcal{L}(c)$ induced by the link

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[1]We use the notation $[n] := \{1, \dots, n\}$ and $[n]_0 := [n] \cup \{0\}$ for $n \in \mathbb{N}$ throughout.

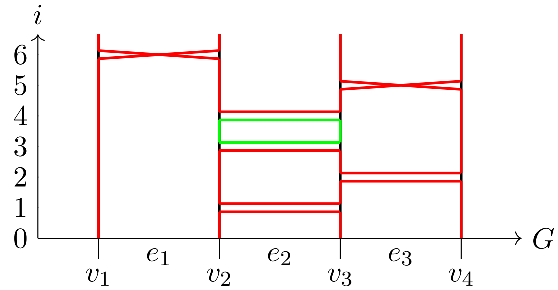


FIGURE 1.1. We consider the link configuration

$$((e_2,-1),(e_3,-1),(e_2,-1),(e_2,-1),(e_3,1),(e_1,1)).$$

Following the rules of the equivalence relation, we obtain two equivalence classes (loops), coloured in green and red.

configuration $c=(c_i)_{i\in[n]}=(e_i,s_i)_{i\in[n]}$ as a partition of the set $X=V\times[n]$. This partition is given by the equivalence classes arising from an equivalence relation $\sim$ which we define as follows: for $(v,j),(w,k)\in X$, we say that $(v,j)\sim(w,k)$ if $v=w$ and $j=k$, or if one of the following conditions is satisfied:

$$\begin{aligned}
&(i) && v=w, k=(j+1) \bmod n, \text{ and } v\notin e_j,\\
&(ii) && \{v,w\}\in E, j=k, e_j=\{v,w\}, \text{ and } s_j=-1,\\
&(iii) && \{v,w\}\in E, j=k, e_{(j-1) \bmod n}=\{v,w\}, \text{ and } s_j=-1,\\
&(iv) && \{v,w\}\in E, k=(j+1) \bmod n, e_j=\{v,w\}, \text{ and } s_j=1.
\end{aligned}$$

We extend $\sim$ first by symmetry, and then by transitivity, to an equivalence relation. $\mathcal{L}(c)$ denotes the set of equivalence classes of the resulting equivalence relation. The connection to the graphical construction appears when interpreting (v,j) as the interval $((j-1)/n, j/n)$ on a real axis attached to v: then (i) tells us that intervals on adjacent levels on the same vertex with no intervening links are connected; (ii) and (iii) say that double bars connect intervals of the same level at both vertices of the edge where they appear; (iv) means that intervals of adjacent levels on neighbouring vertices are connected if a cross is placed on their shared edge. See Figure 1.1.

The random loop measure on the graph $G=(V,E)$ is defined to be the probability measure $\mathbb{P}_{\beta,u,\theta}$ on

$$\mathcal{C}(E)=\{(c_i)_{i\in[n]}=(e_i,s_i)_{i\in[n]}: n\in\mathbb{N}, e_i\in E, \text{ and } s_i\in\{-1,1\} \text{ for all } i\in[n]\}$$

with

$$\mathbb{P}_{\beta,u,\theta}(\{c\}):=(Z_{\beta,u,\theta})^{-1}\,\frac{\beta^{|c|}}{|c|!}u^{\frac{1}{2}\sum_{j=1}^{n}(1+s_j)}(1-u)^{\frac{1}{2}\sum_{j=1}^{n}(1-s_j)}\theta^{|\mathcal{L}(c)|}, \tag{1.1}$$

where $|c|$ is the length of the vector $c\in\mathcal{C}(E)$ and $Z_{\beta,u,\theta}$ is the normalisation constant.

A constructive way of thinking about the random loop measure is to imagine that we first draw the length $|c|$ of the link configuration by evaluating a Poisson random variable with parameter $\beta|E|$. Then for each element c_i of the vector, we assign independently a first component e_i using the uniform distribution on E, and a type s_i, choosing $s_i=1$ with probability u and $s_i=-1$ with probability $1-u$.

Then the quantities

$$N_e(c)=|\{i\in[|c|]: c_i=(e,s) \text{ for some } s\in\{-1,1\}\}|$$

are i.i.d. Poisson random variables with parameter β, and the order in which the elements appear in the sequence is uniformly distributed on the set of all possible orderings. This gives us the connection with the usual way of defining the model. The resulting sequence has law $\mathbb{P}_{\beta,u,1}$. To obtain $\mathbb{P}_{\beta,u,\theta}$, a reweighting with the weight function $\theta^{|\mathcal{L}|}$ is necessary. For $\theta > 1$, this favours configurations with more loops. For integer θ, there is a combinatorial interpretation of colouring each loop with one of θ different colours.

The main question about the random loop model concerns the existence of infinite loops, and is therefore a percolation-type question. For a finite graph $G = (V, E)$ and a link configuration $c \in \mathcal{C}(E)$, we say that $v, w \in V$ are *connected by a loop*, and write $v \overset{c}{\Longleftrightarrow} w$, if $(v, 1)$ and $(w, 1)$ are in the same equivalence class.
For an infinite connected graph $G = (V, E)$, we say that infinite loops exist if we can find an increasing sequence $(E_m)_{m \geqslant 1} \in E^{\mathbb{N}}$ and a vertex $v_0 \in e$ for some $e \in E_1$ such that:
(i) $\bigcup_{m\in\mathbb{N}} E_m = E$, and the subgraph $G_m = (V_m, E_m)$ of G generated by E_m is connected for all m.
(ii) Let $\mathbb{P}_{\beta,u,\theta,m}$ denote the random loop measure on G_m, and let $d(v, w)$ be the graph distance of two vertices v and w. Then

$$\lim_{R\to\infty} \liminf_{m\to\infty} \mathbb{P}_{\beta,u,\theta,m}(v_0 \overset{c}{\Longleftrightarrow} w \text{ for some } w \in V_m \text{ with } d(v_0, w) \geqslant R) > 0. \tag{1.2}$$

While it is usually rather easy to show the absence of infinite loops when β is small (see also below), positive results are much harder to obtain. Two special graphs that are relatively well understood, are the complete graph and regular trees. For $\theta = 1$ and $u = 1$, the random loop model is often referred to as interchange process. On complete graphs, the existence of infinitely long loops for the interchange process follows from Schramm (2005), where it was shown that the re-scaled loop lengths of macroscopic loops converge weakly to the Poisson-Dirichlet distribution PD(1) above the critical threshold $\beta = 1/2$. Björnberg et al. (2019) extended this to $u \in [0, 1)$, yielding convergence to the Poisson-Dirichlet distribution PD(1/2). A more detailed analysis of the expected number of loops of a certain size can be found in Berestycki and Kozma (2015).

On d-regular trees, the existence of infinitely long loops for the interchange process has been shown for $d \geqslant 5$ in Angel (2003) and sharpness of the phase transition has first been proven for $d \geqslant 764$ in Hammond (2015) and in Betz et al. (2021), this was extended to $d \geqslant 3$. Both works, as well as Björnberg and Ueltschi (2018b,a), contribute an asymptotic expansion in d. Here, Betz et al. (2021) allows for $u \in [0, 1]$ and Björnberg and Ueltschi (2018a) allows for $\theta \neq 1$. An interesting result for d-regular graphs, of which d-regular trees are a subclass, was proven in Poudevigne (2022). There it was shown for $\theta \in (0, \infty)$ that macroscopic cycles appear almost surely if one draws the graph uniformly at random among all d-regular graphs. Another study of randomly drawn graphs was carried out in Betz et al. (2018) where Galton-Watson trees with certain conditions on the offspring distribution were considered.

Two specific graphs that also have been studied, are the hypercube and the Hamming graph. For the first, the existence of long loops was shown in Kotecký et al. (2016) and for the latter Adamczak and Kotowski (2021) extends the results of Schramm (2005) also allowing for $\theta \neq 1$ but leaving the question of criticality open.

Graphs with more complex geometries are notoriously difficult to treat. An important recent success is Elboim and Sly (2024), where the existence of loop percolation (in a slightly different sense) is shown in dimensions 5 or higher for the cubic lattice in case $\theta = 1$, $u = 1$. Further rigorous results on the existence of long loops and quantitative bounds on connection probabilities were recently obtained in Betz et al. (2026+), using refined versions of reflection positivity.

In this work, we do not present any results on regimes where (1.2) is valid. Instead, we contribute to the understanding of the region where (1.2) does not hold by comparing the existence of infinite loops and infinite clusters in a percolation model that is called link percolation. For given $c = (e_i, s_i)_{i\in[n]} \in \mathcal{C}(E)$, we define $\eta(c) \in \{0, 1\}^E$ by setting $\eta_e(c) = 1$ if and only if there exists $i \in [n]$ with $e_i = e$. In words, edges are open if at least one link is placed on them. The law of $c \mapsto \eta(c)$ under $\mathbb{P}_{\beta,u,1}$ is just Bernoulli percolation with probability $1 - \mathrm{e}^{-\beta}$ for an open edge, while under the general measure $\mathbb{P}_{\beta,u,\theta}$ a model of dependent percolation emerges. We write $v \overset{c}{\longleftrightarrow} w$ if v and w are in the same $\eta(c)$-percolation cluster. As above, we say that there are infinite clusters in link percolation if we can find an increasing sequence $(E_m)_{m \geqslant 1} \in E^{\mathbb{N}}$ and a vertex $v_0 \in e$ for some $e \in E_1$ such that:

(i) $\bigcup_{m\in\mathbb{N}} E_m = E$, and the subgraph $G_m = (V_m, E_m)$ of G generated by E_m is connected for all m.

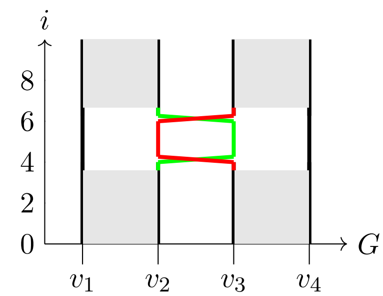
(ii) Let $\mathbb{P}_{\beta,u,\theta,m}$ denote the random loop measure on G_m, and let $d(v, w)$ be the graph distance of vertices v and w. Then,

$$\lim_{R\to\infty} \liminf_{m\to\infty} \mathbb{P}_{\beta,u,\theta,m}(v_0 \overset{c}{\longleftrightarrow} w \text{ for some } w \in V_m \text{ with } d(v_0, w) \geqslant R) > 0. \tag{1.3}$$

It is clear that when (1.3) fails to hold, then (1.2) also does not hold. In Mülbacher (2021), Peter Mühlbacher showed that for graphs of bounded degree, there exists an open interval of parameters where (1.3) holds but (1.2) does not. The purpose of this article is to give a quantitative version of Mühlbacher's result, and at the same time to streamline the proof in several ways. A related result on trees was recently established in Klippel et al. (2025), where a strict inequality of critical parameters for loop and link percolation is proved for a broad class of random trees including Galton–Watson trees.

The main idea for comparing link percolation to loop percolation (i.e., existence of infinite loops) is to find sufficiently many edges that contribute to the former but not to the latter. For $c \in \mathcal{C}(E)$, let $N_e(c)$ be as defined above; that is, $N_e(c)$ denotes the number of times the edge e appears in the sequence $(e_i, s_i)_{i\in[n]}$. We say that an edge e is *blocking* for a link configuration $c = (e_i, s_i)_{i\in[n]}$ if

(1) $N_e(c) = 2$,
(2) $s_i = 1$ if $e_i = e$, i.e., both links on e are crosses, and
(3) no adjacent edges carry any links that go between the two links on e, i.e., if $e_i = e_j = e$ with $i < j$, then $\bigcup_{i<k<j} e_k \cap e = \emptyset$.



In the illustration above, the edge $\{v_2, v_3\}$ is blocking if and only if the fifth element of the link configuration c is not on one of the adjacent edges. If $e = \{v, w\}$ is a blocking edge, a loop that uses one of the links on e to travel from v to w, or vice versa, is not diverted from its current position (because of (3)) before it reaches the other link. Therefore, both links can be deleted from c without affecting whether $x \overset{c}{\Longleftrightarrow} y$ for any vertices $x, y \in V$. So, if we define $B_e(c) = 1$ if e is blocking for c, and $B_e(c) = 0$ otherwise, then a sufficient condition for the absence of infinite loops is that the dependent percolation model

$$c \mapsto (\eta_e(c)(1 - B_e(c)))_{e\in E}$$

has no infinite clusters (again, in the sense of exhausting the graph G with finite approximations). This means that our aim is to find a regime of parameters where the (in general, dependent) percolation model $(\eta_e)_{e\in E}$ has infinite clusters, but $(\eta_e(1 - B_e))_{e\in E}$ has none. Our tool to do this is stochastic domination. We equip the space $\Omega_E = \{0,1\}^E$ with the natural partial order $\leqslant$ given by the entry-wise comparison. A function $f : \Omega_E \to \mathbb{R}$ is increasing if $\omega \leqslant \omega'$ implies $f(\omega) \leqslant f(\omega')$, and $A \subset \Omega_E$ is increasing if its indicator function is an increasing function. For two probability measures μ, ν on Ω_E, μ stochastically dominates ν if for every increasing function f, we have $\nu(f) \leqslant \mu(f)$.

Our main result is for general $\theta > 1$. Write

$$\hat{\theta} = \max\{\theta,\ 1/\theta\}, \tag{1.4}$$

and let $O(c)$ be the set of open edges with respect to $\eta(c)$, i.e., the set of $e \in E$ where $\eta_e(c) = 1$.

Theorem 1.1 (Appearance of blocking events)**.** *Let $G = (V, E)$ be a finite graph with bounded degree K. Define $\mathbb{P}_{\beta,u,\theta}$ as in (1.1). Then for any $E_0 \subset E$, the law of $(B_e)_{e\in E_0}$ under $\mathbb{P}_{\beta,u,\theta}(\cdot|\ O = E_0)$ stochastically dominates a Bernoulli edge percolation measure on E_0 with parameter*

$$\delta(\beta, u, \theta) := \left(\frac{\hat{\theta}(8K-4)(2K-1)}{\beta u^2} + 1 \right)^{-1} \mathrm{e}^{-\beta^+(2K-2)}.$$

The idea of using blocking edges for comparison to percolation is taken from Mülbacher (2021). What is new is the extension to $\theta \neq 1$ and the quantitative bound. For example, we consider the three dimensional lattice ($K = 6$), only crosses ($u = 1$), $\theta = 2$ and $\beta = 0.25$[2]. For these parameters, we get $\delta(\beta, u, \theta) \approx 2.12 \cdot 10^{-5}$.

Theorem 1.1 allows us to take a percolation cluster of the link percolation model $(\eta_e)_{e\in E}$ and to remove each edge of this cluster independently with probability $\delta(\beta, u, \theta)$. If we can find a regime of parameters where the link percolation cluster η percolates, but the 'thinned out' cluster no longer does, we know that for these parameters link percolation occurs, but loop percolation does not. Unfortunately, this argument requires some control on the law of $(\eta_e)_{e\in E}$ itself near the percolation threshold. The only case where we currently have this control is the 'trivial' one where $\theta = 1$ and thus η is just Bernoulli bond percolation with parameter $1 - e^{-\beta}$. Let $p_c(G)$ be the critical probability for the graph G with respect to Bernoulli bond percolation.

Theorem 1.2 (Comparison to percolation)**.** *Let G be an infinite graph with maximal degree $K \in \mathbb{N}$. Assume that $\theta = 1$, and define the existence of loop percolation and link percolation as in (1.2) and (1.3), respectively. Then for all $\beta > 0$, such that*

$$p_c(G) < 1 - \mathrm{e}^{-\beta} < \frac{p_c(G)}{1 - \delta(\beta, u, 1)},$$

loop percolation does not occur, but link percolation does.

Proof: Since $(\eta_e)_{e\in E}$ is Bernoulli bond percolation with parameter $1 - e^{-\beta}$, link percolation occurs for β with $p_c(G) < 1 - \mathrm{e}^{-\beta}$ by the definition of $p_c(G)$. For the proof that loop percolation does not occur when $1 - \mathrm{e}^{-\beta} < \frac{p_c(G)}{1-\delta(\beta,u,1)}$, we use Theorem 1.1.

As discussed in the paragraph before Theorem 1.1, for a link configuration c, $v \overset{c}{\Longleftrightarrow} w$ is only possible if an open path from v to w exists in the edge percolation configuration $(\eta_e(c)(1-B_e(c)))_{e\in E}$. Our task is therefore to show that this percolation model does not exhibit infinite clusters. To do so, we dominate it by a Bernoulli percolation model: let $G = (V, E)$ be a finite graph and $f : \{0, 1\}^E \to \mathbb{R}^+$ be an increasing function. For $E_0 \subset E$ and $\xi \in \{0, 1\}^E$, let $\xi|_{E_0}(e) = \xi(e)\mathbb{1}_{\{e\in E_0\}}$, and define $f_{E_0}(\xi) = f(\xi|_{E_0})$. Note that f_{E_0} is increasing as well. We abbreviate $f(\eta) = f((\eta_e)_{e\in E})$ and analogously $f(\eta(1-B))$. Recall that $O(c)$ is the set of open edges with respect to $\eta(c)$, let c be distributed according to $\mathbb{P}_{\beta,u,\theta}$ (with θ general for now), which also fixes the distributions of η and B. Let X be a Bernoulli edge percolation with probability $1 - \delta(\beta, u, \theta)$ for an edge to be open. We write $\mathbb{E}$ for the expectation with respect to the product measure of $\mathbb{P}_{\beta,u,\theta}$ and the probability

[2] We use that simulations (see Wang et al., 2013) indicate the critical threshold to be ≈ 0.25.

measure that is associated with X, and get

$$\begin{aligned}\mathbb{E}(f(\eta(1-B))) &= \sum_{E_0\subset E}\mathbb{E}(f(\eta(1-B))\,|\,O=E_0)\mathbb{P}(O=E_0)\\ &= \sum_{E_0\subset E}\mathbb{E}(f_{E_0}(1-B)\,|\,O=E_0)\mathbb{P}(O=E_0)\\ &\leqslant \sum_{E_0\subset E}\mathbb{E}(f_{E_0}(X)\,|\,O=E_0)\mathbb{P}(O=E_0)\\ &= \sum_{E_0\subset E}\mathbb{E}(f(\eta X)\,|\,O=E_0)\mathbb{P}(O=E_0)=\mathbb{E}(f(\eta X)).\end{aligned}$$

Now comes the place where we need to assume $\theta=1$: in this case, we know that ηX is a Bernoulli percolation with parameter $(1-\mathrm{e}^{-\beta})(1-\delta(\beta,u,1)$, and therefore we can bound connection probabilities for $\eta(1-B)$ by connection probabilities with respect to this Bernoulli percolation. This shows the claim. □

2. Proof of Theorem 1.1

For this section, we fix $E_0\subset E$ and write $\tilde{\mathbb{P}}(\cdot)=\tilde{\mathbb{P}}_{\beta,u,\theta}(\cdot)=\mathbb{P}_{\beta,u,\theta}(\cdot\,|\,O=E_0)$, which we view as a probability measure on $\mathcal{C}(E_0)$. Note that under $\tilde{\mathbb{P}}$, we have $N_e\geqslant 1$ almost surely for all $e\in E_0$. We also just write δ instead of $\delta(\beta,u,\theta)$, E instead of E_0, and $\mathcal{C}$ instead of $\mathcal{C}(E)$ below. The strategy of the proof is to show a finite energy property for the percolation measure $(B_e)_{e\in E}$, which means that for all $e_0\in E$ and for all $\bar\varepsilon\in\{0,1\}^{E\setminus\{e_0\}}$, we have

$$\tilde{\mathbb{P}}(B_{e_0}=1\,|\,(B_{e'})_{e'\in E\setminus\{e_0\}}=\bar\varepsilon)\ \geqslant\ \delta. \tag{2.1}$$

Then the claim follows by essentially known arguments, which we spell out in Proposition A.1 for the convenience of the reader.

The following equality will be used many times below: given $c\in\{O=E\}$, let $c_+(e,j,s)$ be the link configuration where one link of type s is inserted into the sequence c, at position $j\leqslant|c|+1$ and on edge $e\in E$. Then

$$\tilde{\mathbb{P}}(c_+(e,j,s))=\tilde{\mathbb{P}}(c)\frac{\beta}{|c|+1}\theta^{|\mathcal{L}(c_+(e,j,s))|-|\mathcal{L}(c)|}u^{(1+s)/2}(1-u)^{(1-s)/2}. \tag{2.2}$$

Note that $||\mathcal{L}(c_+(e,j,s))|-|\mathcal{L}(c)||\leqslant 1$, which gives immediate upper and lower bounds.

Let us start by a naive try for proving (2.1) which will fail. The uniform domination would hold if we could show $\tilde{\mathbb{P}}(B_{e_0}(c)=1\,|\,c|_{E\setminus\{e_0\}}=\bar c)\geqslant\delta$ for all $\bar c\in\mathcal{C}(E\setminus\{e_0\})$, because then we can sum over all $\bar c$ leading to a specific $\bar\varepsilon$. But this inequality is not true. The reason is that placing too many links on an edge adjacent to e_0 makes it difficult for e_0 to be blocking: to see this, let $E=\{e_0,e'\}$ so that $e_0\cap e'=\{v\}$. Assume that $\bar c$ is the configuration that has m links on the edge e'. For e_0 to be blocking, we need to place the two links so that none of the links on e' lie between the two links on e_0. In other words, if $c\in\{N_{e_0}=1\}$ and the link on e_0 is at position j, then the additional link that we need to place to get to $\{B_{e_0}=1\}$ has to be either right before or right after that link. Both cases result in the same link configuration. This means that for every such $c\in\{N_{e_0}=1\}$, there is just one $c\in\{B_{e_0}=1\}$ that uses the same position j. As a consequence, (2.2) gives

$$\frac{\tilde{\mathbb{P}}(B_{e_0}=1\,|\,c|_{E\setminus\{e_0\}}=\bar c)}{\tilde{\mathbb{P}}(N_{e_0}=1\,|\,c|_{E\setminus\{e_0\}}=\bar c)}\ \leqslant\ \frac{\hat\theta\beta u^2}{m+1},$$

which shows that uniform domination must fail since m can be made arbitrarily large.

So we need to condition on an event coarse enough to prevent this obstruction from happening, but fine enough to be able to estimate conditional expectations. For this purpose, for $e\in E$, we introduce

$$A_e(k,m) := \{f \in E : \ d(e,f) \in [k,m]\},$$

where $d(e,f)$ is the edge graph distance, i.e., the minimal number of vertices that need to be crossed on a path from the midpoint of e to the midpoint of f. We illustrate to conceptually relevant types of set $A_e(k,m)$ in Figure 2.2. Also, we set $A(k,m) := A_{e_0}(k,m)$. For $\bar{c} \in \mathcal{C}(A(2,\infty))$

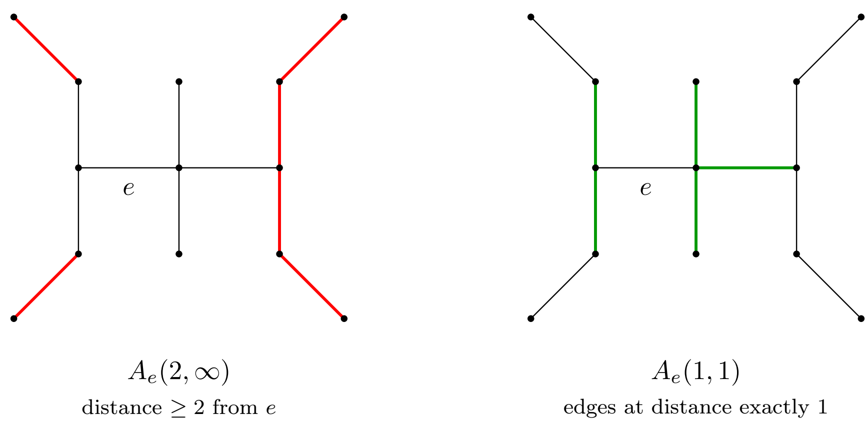


FIGURE 2.2. Illustration of the sets $A_{e_0}(k,n)$ around the distinguished edge e. Red marks $A_e(2,\infty)$, while green marks $A_e(1,1)$.

and $\bar{\varepsilon} \in \{0,1\}^{A(1,2)}$, we define

$$\mathcal{C}_{\bar{c},\bar{\varepsilon}} = \{c \in \mathcal{C} : c|_{A(2,\infty)} = \bar{c},\ B_e = \bar{\varepsilon}_e \text{ for all } e \in A(1,2)\}.$$

In words, $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ consists of sequences that match $\bar{c}$ for edges with distance two or more from e_0, and have the correct blocking structure for all edges in $A(1,\infty)$; we only need to demand the latter for edges in $A(1,2)$, since for all other edges it follows from the knowledge of $\bar{c}|_{A(2,\infty)}$. We illustrate the shared restriction of elements from $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ in Figure 2.3. Note that $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ can be empty in cases where no

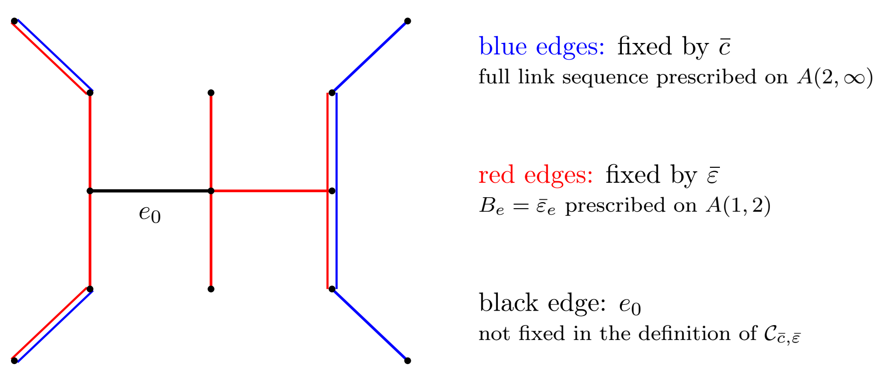


FIGURE 2.3. Illustration of which edges for elements from $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ are affected by $\bar{c}$ and which by $\bar{\varepsilon}$. Blue marks edges fixed by $\bar{c}$, red marks edges fixed by $\bar{\varepsilon}$, and parallel red/blue lines indicate edges affected by both.

placement of links on some $e \in A(1,1)$ is able to produce the required blocking structure on $A(1,2)$.

Elements of $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ can have an arbitrary number of links on the edges of $A(1,1)$, as long as these do not interfere with the desired blocking structure. We wish to restrict the number of these links, and for this purpose we define a partial order on $\mathcal{C}$: we say that $c \leqslant c'$ if c emerges from c' by removing sequence elements of c' but keeping the relative order of the remaining elements resulting in the sequence c. Then $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ and $\mathcal{C}_{\bar{c},\bar{\varepsilon}} \cap \{c \in \mathcal{C} :\ B_{e_0} = 1\}$ contain minimal elements, and we write $\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$ for the union of the two sets of minimal elements. Let us remark already now that for $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$, each edge $\tilde{e} \in A(1,1)$ has at most $\min\{2, K-1\}$ links. The reason is that, on the one hand, if $\bar{\varepsilon}_{\tilde{e}} = 1$, there must be precisely two links on $\tilde{e}$. On the other hand, there are at most $K-1$ edges adjacent to $\tilde{e}$ that are not contained in $A(1,1)$. Each of these edges may be designated as non-blocking, and this may or may not require a link on $\tilde{e}$ to destroy what would otherwise be a blocking structure. Any links beyond these required ones can then be removed. Note that, for any edge $e' \in A(1,1)$ neighbouring $\tilde{e}$ such that two crosses are placed on e' and $\bar{\varepsilon}_{e'} = 0$, there exists a link placed between these two crosses on some other edge neighbouring e': if there would be no such link between the two crosses that are placed on e', we could remove one of the two crosses on e' and still get an element from $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ contradicting minimality. Hence, there is no need to place a link on $\tilde{e}$ to destroy the blocking structure on e' and the number of links in $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$ on edges from $A(1,1)$ can indeed by bounded by $\min\{2, K-1\}$ since there are at most $K-1$ neighbouring edges to an edge from $A(1,1)$ that are not contained in $A(1,1)$.

For $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$, set

$$\mathrm{Ex}(c) = \{\tilde{c} \in \mathcal{C}_{\bar{c},\bar{\varepsilon}} : c \leqslant \tilde{c}\}.$$

As a first step of the proof, we show

Lemma 2.1. *For each $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$, we have*

$$\tilde{\mathbb{P}}(\mathrm{Ex}(c)) \leqslant \mathrm{e}^{\beta^+(2K-2)}\, \tilde{\mathbb{P}}(\{c\}), \tag{2.3}$$

where $\beta^+ := \hat{\theta}\beta$ and $\hat{\theta} = \max\{\theta, \theta^{-1}\}$.

Proof: Let us re-introduce the notation $\tilde{\mathbb{P}}_{\beta,u,\theta} = \tilde{\mathbb{P}}$ for now. We start by a known estimate that relates parameters θ and 1, see e.g. Björnberg and Ueltschi (2018b). For $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$, we have

$$\begin{aligned} \frac{\tilde{\mathbb{P}}_{\beta,u,\theta}(\mathrm{Ex}(c))}{\tilde{\mathbb{P}}_{\beta,u,\theta}(\{c\})} &= \sum_{\tilde{c}\in\mathrm{Ex}(c)} \frac{\beta^{|\tilde{c}|-|c|}|c|!}{|\tilde{c}|!}\theta^{|\mathcal{L}(\tilde{c})|-|\mathcal{L}(c)|}u^{\mathrm{N}_1(\tilde{c})-\mathrm{N}_1(c)}(1-u)^{\mathrm{N}_{-1}(\tilde{c})-\mathrm{N}_{-1}(c)} \\ &\leqslant \sum_{\tilde{c}\in\mathrm{Ex}(c)} \frac{(\beta\hat{\theta})^{|\tilde{c}|-|c|}|c|!}{|\tilde{c}|!}u^{\mathrm{N}_1(\tilde{c})-\mathrm{N}_1(c)}(1-u)^{\mathrm{N}_{-1}(\tilde{c})-\mathrm{N}_{-1}(c)} \\ &= \frac{\tilde{\mathbb{P}}_{\beta^+,u,1}(\mathrm{Ex}(c))}{\tilde{\mathbb{P}}_{\beta^+,u,1}(\{c\})} \end{aligned}$$

From now on, we will work with $\tilde{\mathbb{P}}_{\beta^+,u,1}$ which we abbreviate as $\tilde{\mathbb{P}}_1$. As discussed above, $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$ can not be extended on edges $e \in A(1,1)$ with $\bar{\varepsilon}_e = 1$ without leaving $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$. Also, c can not be extended be adding links on edges from $E\backslash A(1,1)$ without leaving $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ as sequences from $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ are fixed on $A(2,\infty)$. So $\mathrm{Ex}(c)$ arises from c by adding links to the edges $e \in A(1,1)$ where $\bar{\varepsilon}_e = 0$. Assume that there are J of such edges. We construct elements of $\mathrm{Ex}(c)$ by stepwise adding crosses and double bars to each of the J edges, so in total there will be $2J$ steps. We add crosses in odd steps and double bars in even ones. Let m_i be the number of links that we added in the i-th step. In the j-the step, there are thus $|c| + \sum_{k=1}^{j-1} m_k + 1$ positions in the extended sequence in which to place the links. Since they are all of the same type, and on the same edge, the order in which we place them is irrelevant, and we get

$$\begin{aligned} &\frac{\tilde{\mathbb{P}}_1(\mathrm{Ex}(c))}{\tilde{\mathbb{P}}_1(\{c\})} \\ &= \frac{|c|!}{(\beta^+)^{|c|}} \sum_{m_1,\ldots,m_{2J}=0}^{\infty} \frac{(\beta^+)^{|c|+\sum_{k=1}^{2J} m_k}}{(|c| + \sum_{k=1}^{2J} m_k)!} \prod_{j=1}^{2J} u^{m_{2j-1}}(1-u)^{m_{2j}} \frac{\prod_{k=1}^{2J}(|c| + \sum_{j=1}^{k-1} m_j + 1)^{m_k}}{\prod_{k=1}^{2J} m_k!} \end{aligned}$$

Since

$$\frac{\prod_{k=1}^{2J}(|c|+\sum_{j=1}^{k-1}m_j+1)^{m_k}|c|!}{(|c|+\sum_{k=1}^{2J}m_k)!} \leqslant 1,$$

we obtain

$$\begin{aligned}\frac{\tilde{\mathbb{P}}_1(\mathrm{Ex}(c))}{\tilde{\mathbb{P}}_1(\{c\})} &\leqslant \sum_{m_1,\dots,m_{2J}=0}^{\infty}\prod_{k=1}^{2J}\frac{(\beta^+)^{m_k}}{m_k!}\prod_{j=1}^{2J}u^{m_{2j-1}}(1-u)^{m_{2j}}\\ &=\prod_{k=1}^{J}\sum_{m_{2k-1}=1}^{\infty}\frac{(\beta^+u)^{m_{2k-1}}}{(m_{2k-1})!}\sum_{m_{2k}=1}^{\infty}\frac{(\beta^+(1-u))^{m_{2k}}}{(m_{2k})!}=\prod_{k=1}^{J}\mathrm{e}^{\beta^+u}\ \mathrm{e}^{\beta^+(1-u)}\\ &=\mathrm{e}^{\beta^+J}.\end{aligned}$$

The result now follows since $J \leqslant 2K-2$. □

As an immediate consequence, we note that

$$\begin{aligned}\tilde{\mathbb{P}}(\mathcal{C}_{\bar{c},\bar{\varepsilon}}) = \tilde{\mathbb{P}}(\bigcup_{c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}}\mathrm{Ex}(c)) &\leqslant \sum_{c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}}\tilde{\mathbb{P}}(\mathrm{Ex}(c)) \leqslant \mathrm{e}^{\beta^+(2K-1)}\sum_{c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}}\tilde{\mathbb{P}}(\{c\})\\ &= \mathrm{e}^{\beta^+(2K-2)}\,\tilde{\mathbb{P}}(\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}).\end{aligned} \tag{2.4}$$

Next, we show

Lemma 2.2. *For all $\bar{c}\in\mathcal{C}(A(2,\infty))$ and all $\bar{\varepsilon}\in\{0,1\}^{A(1,2)}$, we have*

$$\tilde{\mathbb{P}}(\{B_{e_0}=1\}\cap\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}) \geqslant \delta_0\tilde{\mathbb{P}}(\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}})$$

with $\delta_0 := \left(\frac{\hat{\theta}(8K-4)(2K-1)}{\beta u^2}+1\right)^{-1}$.

We start with a combinatorial consideration that we motivate first by sketching the proof of Lemma 2.2: To obtain the claim given in the Lemma, we show that a fixed fraction of sequences from $\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$ also fulfils $B_{e_0}=1$.
To do so, given a fixed link sequence on $A(1,\infty)$, we study how many free positions on e_0 exist to extend the fixed sequence to an element from $\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$. Extending the fixed sequence only by one link at one of the free positions yields an element from $\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$ with $B_{e_0}=0$ and $N_{e_0}=1$. If we place two crosses such that there is no link from a neighbouring edge surrounded in the link sequence and also such that we maintain the blocking structure on $A(1,1)$, then we get $B_{e_0}=1$. We are going to show that these are the only possibilities to extend the fixed sequence to an element from $\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$.
To get a useful comparison of the probability of adding links in these two singled-out cases, we need to find a lower bound on the number of free positions that grows with $|c|$: For large $|c|$, the probability weight for adding two crosses on e_0 to get $B_{e_0}=1$ is decreasing with rate $\sim 1/|c|$ compared to adding one link. Hence, the combinatorics of adding two crosses need to be growing at least linear in $|c|$ to keep the balance. This is achieved by having many free spots since the combinatorics for adding two links show a quadratic behaviour in the number of free positions while adding one link goes linear with the number of free positions. We prove the lower bound in Lemma 2.4 whose proof works out the just given heuristic argument of quadratic and linear growth in the number of free positions.
For clarity, we outsource one step of the proof to Lemma 2.3. Here, the idea behind the proof of Lemma 2.4 is implicitly encoded: We consider both ends of $e_0=\{v_0,w_0\}$ that is edges incident to v_0 and edges incident to w_0, separately and, to estimate the number of free positions on e_0, determine how much positions are made unavailable by links placed on $A(1,1)$ that are blocking. The latter turn out to be the most problematic to our cause when estimating the number of free positions on

e_0. We start with some notation.
For $e_0 = \{v_0, w_0\}$, let $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block},v_0}}$,

$$E_{\text{block},v_0} := \{e \in E:\ v_0 \in e,\ \bar{\varepsilon}_e = 1\}, \text{ and } \quad Q(c) := \{\tilde{c} \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)}:\ c \leqslant \tilde{c}\}.$$

In words, $Q(c)$ contains the link configurations obtained from c by adding two crosses on each $e \in E_{\text{block},v_0}$ in such a way that all blocking requirements are satisfied. Also, for $\tilde{c} \in Q(c)$, we have $|\tilde{c}| = |c| + 2|E_{\text{block},v_0}|$. For $\tilde{c} = (\tilde{e}_i, \tilde{s}_i)_{i=1}^n \in Q(c)$ and $e \in E_{\text{block},v_0}$, we let $i_e(\tilde{c})$ be the position of the first 'blocking link' on e, and $j_e(\tilde{c})$ the second position, i.e.,

$$i_e(\tilde{c}) = \min\{i \in [|\tilde{c}|]:\ \tilde{e}_i = e\}, \qquad j_e(\tilde{c}) = \max\{i \in [|\tilde{c}|]:\ \tilde{e}_i = e\}.$$

Write ν_c for the uniform distribution on $Q(c)$. Then, we have

Lemma 2.3. *For all* $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block},v_0}}$,

$$\mathbb{E}_{\nu_c}\left(\sum_{e \in E_{\text{block},v_0}} j_e - i_e\right) \leqslant \frac{|E_{\text{block},v_0}|}{2|E_{\text{block},v_0}| + 1}(|c| + 2|E_{\text{block},v_0}| + 1).$$

Proof: Let us write $n = |E_{\text{block},v_0}|$. For given $\tilde{c} \in Q(c)$, we order the numbers $i_e(\tilde{c})$ and $j_e(\tilde{c})$, $e \in E_{\text{block},v_0}$, by size, and write them as $i_1(\tilde{c}) < i_2(\tilde{c}) < \ldots i_n(\tilde{c})$, and the same for the $j_k(\tilde{c})$. Since all $e \in E_{\text{block},v_0}$ share the vertex v_0, blocking intervals cannot overlap, and we obtain

$$0 =: j_0(\tilde{c}) < i_1(\tilde{c}) < j_1(\tilde{c}) < i_2(\tilde{c}) < \cdots < i_n(\tilde{c}) < j_n(\tilde{c}) < i_{n+1}(\tilde{c}) := |c| + 2n + 1 = |\tilde{c}| + 1.$$

Write $I(\tilde{c}) = (i_k(\tilde{c}))_{k \leqslant n}$ and $J(\tilde{c}) = (j_k(\tilde{c}))_{k \leqslant n}$ for the ordered sequence of all initial, respectively final, blocking links, and write $e_k(\tilde{c})$ for the unique element of E_{block,v_0} on which the links with indices $i_k(\tilde{c})$ and $j_k(\tilde{c})$ are located.

Let us fix a sequence $\mathbf{J} = (\mathbf{j}_k)_{k \leqslant n}$ with $0 < \mathbf{j}_1 < \ldots \mathbf{j}_n < |c| + 2n + 1$, and an ordering $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots \boldsymbol{e}_n)$ of E_{block,v_0}. We set

$$Q_{\mathbf{J},\boldsymbol{e}}(c) = \{\tilde{c} \in Q(c):\ J(\tilde{c}) = \mathbf{J}, (e_1(\tilde{c}), \ldots, e_n(\tilde{c})) = \boldsymbol{e}\}.$$

This may be empty, for example if one of the $\mathbf{j}_k$ appears right after the previous one, or right after a link in c that sits on an edge neighbouring $\boldsymbol{e}_k$. When the set is not empty, it always contains the element $\tilde{c}_0 \in Q(c)$ where $i_k(\tilde{c}_0) = j_k(\tilde{c}_0) - 1$. Indeed we can construct all elements of $Q_{\mathbf{J},\boldsymbol{e}}(c)$ by starting from this configuration and swapping the order of the k-th initial blocking link with its preceding link until the preceding link is either a link from c on an edge neighbouring $e_k(\tilde{c}_0)$, or we have reached the position $\mathbf{j}_{k-1}$. From these considerations it is clear that given $J = \mathbf{J}$ and $(e_1, \ldots e_n) = \boldsymbol{e}$, the k-th initial blocking link is distributed uniformly (under ν_c) over an interval that is shorter or equal to $\mathbf{j}_k - \mathbf{j}_{k-1}$. This implies that

$$\mathbb{E}_{\nu_c}(j_k - i_k|\ Q_{\mathbf{J},\boldsymbol{e}}(c)) \leqslant \mathbb{E}_{\nu_c}(i_k - j_{k-1}|\ Q_{\mathbf{J},\boldsymbol{e}}(c))$$

and hence,

$$\mathbb{E}_{\nu_c}(j_k - i_k) \leqslant \mathbb{E}_{\nu_c}(i_k - j_{k-1}).$$

With an analogous proof, we can show for all $k \leqslant n$

$$\mathbb{E}_{\nu_c}(j_k - i_k) \leqslant \mathbb{E}_{\nu_c}(i_{k+1} - j_k).$$

Let $k_0 \leqslant n$ be a maximizer of $k \mapsto \mathbb{E}_{\nu_c}(i_k - j_{k-1})$. Then

$$\begin{aligned}\mathbb{E}_{\nu_c}\left(\sum_{k=1}^{n} j_k - i_k\right) &\leqslant \sum_{k=1}^{k_0-1} \mathbb{E}_{\nu_c}(i_k - j_{k-1}) + \sum_{k=k_0+1}^{n+1} \mathbb{E}_{\nu}(i_k - j_{k-1}) \\ &\leqslant \frac{n}{n+1} \sum_{k=1}^{n+1} \mathbb{E}_{\nu_c}(i_k - j_{k-1}).\end{aligned}$$

Since $\sum_{k=1}^{n}(j_k - i_k) + \sum_{k=1}^{n+1}(i_k - j_{k-1}) = |c| + 2n + 1$, we conclude

$$\mathbb{E}_{\nu_c}\left(\sum_{k=1}^{n} j_k - i_k\right) \leqslant \frac{n}{2n+1}(|c| + 2n + 1)$$

which implies the claim. □

Let $E_{\text{block}} := \{e \in A(1,1) : \bar{\varepsilon}_e = 1\}$. Now, let $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block}}}$. We define

$$Q_1(c) := \{\tilde{c} \in \{N_{e_0} = 1\} \cap \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}} : c \leqslant \tilde{c},\ \tilde{c}|_{\{e_0\}} = (e_0, 1)\}, \tag{2.5}$$

$$Q_2(c) := \{\tilde{c} \in \{B_{e_0} = 1\} \cap \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}} : c \leqslant \tilde{c}\}, \text{ and} \tag{2.6}$$

$$\tilde{Q}(c) := Q_1(c)|_{A(1,\infty)} = Q_2(c)|_{A(1,\infty)}.$$

Moreover, let μ_c denote the uniform measure on $\tilde{Q}(c)$. Given a realisation $\tilde{c} = (\tilde{e}_i, \tilde{s}_i)_{i=1}^{\tilde{n}}$ of μ_c, we want to add links to $\tilde{c}$ to get configurations from $Q_1(c)$ and $Q_2(c)$. We say that we add one cross at $k \in [\tilde{n}]$ or two crosses at $k, l \in [\tilde{n}]$ with $k \leqslant l$ to $\tilde{c}$ on e_0 if we construct the link configuration $c' = (e'_i, s'_i)_{i=1}^{n'}$ with $c'|_{A(1,\infty)} = \tilde{c}$ and, for adding one cross, $e'_k = e_0$ and $c'|_{\{e_0\}} = ((e_0, 1))$ or respectively, for adding two crosses, $e'_k = e'_{l+1} = e_0$ and $c'|_{\{e_0\}} = ((e_0, 1), (e_0, 1))$. Then, there exist disjoint intervals $((a_m, b_m])_{m=1}^{m_0}$ with $a_m, b_m \in \mathbb{N}$ for all $m \in [m_0]$ such that adding two crosses to $\tilde{c}$ yields an element from $Q_2(c)$ if and only if the crosses are added at $k, l \in (a_m, b_m] \cap \mathbb{N}$ for some $m \in [m_0]$. Also, adding one cross yields an element from $Q_1(c)$ if and only if the cross is added at $k \in \mathbb{N} \cap \bigcup_{m\in[m_0]}(a_m, b_m]$. For all $m \in [m_0]$, we set $L_m := b_m - a_m$. Note that we can consider $(a_m, b_m)_{m\in[m_0]}$ and $(L_m)_{m\in[m_0]}$ to be random variables w.r.t. μ_c.

Lemma 2.4. *For all $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block}}}$,*

$$\mathbb{E}_{\mu_c}\left(\sum_{m\in[m_0]} L_m\right) \geqslant \frac{|c| + 1 + |E_{\text{block}}|}{|E_{\text{block}}| + 1}.$$

Proof: Let $\tilde{c} = (\tilde{e}_i, \tilde{s}_i)_{i=1}^{\tilde{n}}$ be a realisation of μ_c. We note that $k \in \bigcup_{m\in[m_0]}(a_m(\tilde{c}), b_m(\tilde{c})] \cap \mathbb{N}$ if and only if there does not exist some $e \in E_{\text{block}}$ with $e = \tilde{e}_i = \tilde{e}_j$ and $i < k < j$. This gives

$$\sum_{m\in[m_0]} L_m(\tilde{c}) \geqslant |c| + 2|E_{\text{block}}| + 1 - \sum_{e\in E_{\text{block}}} (j_e - i_e).$$

Since all link configurations have the same probability under μ_c, we can find some measure $\tilde{\mu}$ on $\tilde{Q}(c)|_{A(1,\infty)\setminus E_{\text{block},v_0}}$ such that we get μ_c by first drawing some $\tilde{c}$ w.r.t. $\tilde{\mu}$ and then drawing a link configuration with $\nu_{\tilde{c}}$ from $Q(\tilde{c})$. This works also when replacing v_0 with w_0 and hence, we get with Lemma 2.3

$$\begin{aligned}\mathbb{E}_{\mu_c}\left(\sum_{m\in[m_0]} L_m\right) &\geqslant |c| + 2|E_{\text{block}}| + 1 \\ &\quad - \left(\frac{|E_{\text{block},v_0}|}{2|E_{\text{block},v_0}| + 1} + \frac{|E_{\text{block},w_0}|}{2|E_{\text{block},w_0}| + 1}\right)(|c| + 2|E_{\text{block}}| + 1).\end{aligned}$$

Since, for $K>0$, on $\{(x,y)\in[0,\infty)^2:\ x+y=K\}$,

$$(x,y)\mapsto \frac{x}{2x+1}+\frac{y}{2y+1}$$

attains its maximum at $(K/2,K/2)$, we conclude the claim. □

Proof of Lemma 2.2: Below, we will show the following equality and inequality for all $\bar{c}\in\mathcal{C}(A(2,\infty))$ and $\bar{\varepsilon}\in\{0,1\}^{A(1,2)}$ such that $\mathcal{C}_{\bar{c},\bar{\varepsilon}}\neq\emptyset$:

$$\tilde{\mathbb{P}}((\{N_{e_0}=1\}\cup\{B_{e_0}=1\})\cap\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}})=\mathbb{P}(\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}), \tag{2.7}$$

$$\tilde{\mathbb{P}}(\{B_{e_0}=1\}\cap\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}})\ \geqslant\ d_0\mathbb{P}(\{N_{e_0}=1\}\cap\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}) \tag{2.8}$$

for $d_0:=\frac{\beta u^2}{\hat{\theta}(8K-4)(2K-1)}$. Once we have this, we obtain the claim with $\delta_0=\frac{d_0}{1+d_0}$.

Equality (2.7) follows from an argument that we have already given in front of Lemma 2.1 but we repeat here in the specific setting for the convenience of the reader: suppose there are at least two links placed on e_0 in $c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$ and $B_{e_0}(c)=0$. This only happens if removing either of these links yields a link configuration not contained in $\mathcal{C}_{\bar{c},\bar{\varepsilon}}$ anymore. This again is only possible if there is some edge $e'\in A(1,1)$ such that c places two crosses on e' without any link on a neighbouring edge that is not e_0, being placed in-between the two crosses and $\bar{\varepsilon}_{e'}=0$. In this case, c is not minimal since one of the two crosses on e' can be removed. Consequently, $c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$ and $B_{e_0}(c)=0$ already implies $N_{e_0}(c)=1$.

To show (2.8), we fix $c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block}}}$, i.e., all the links except the ones on $\{e_0\}\cup E_{\text{block}}$ are fixed. We write

$$q(c)=\frac{\beta^{|c|}}{|c|!}u^{\frac{1}{2}\sum_{j=1}^{n}(1+s_j)}(1-u)^{\frac{1}{2}\sum_{j=1}^{n}(1-s_j)}\theta^{|\mathcal{L}(c)|}$$

for the weight of c (but note that $\tilde{\mathbb{P}}(c)=0$ since the condition of at least one link on $\{e_0\}\cup E_{\text{block}}$ is not met). We remind the reader of the definitions (2.5) and (2.6).

Since adding two links that form a blocking structure never changes the number of loops in the system, we have for all $\tilde{c}\in Q_2(c)$ that

$$\begin{aligned}\tilde{\mathbb{P}}(\tilde{c})&=\frac{\beta^{|c|+2+2|E_{\text{block}}|}}{(|c|+2+2|E_{\text{block}}|)!}u^{2+|E_{\text{block}}|}u^{\frac{1}{2}\sum_{j=1}^{n}(1+s_j)}(1-u)^{\frac{1}{2}\sum_{j=1}^{n}(1-s_j)}\theta^{|\mathcal{L}(c)|}\\&=\frac{(u\beta)^{2+2|E_{\text{block}}|}|c|!}{(|c|+2+2|E_{\text{block}}|)!}q(c).\end{aligned}$$

On the other hand, adding one link to c on the edge e_0 changes the number of loops by exactly one, so that for all $\tilde{c}\in Q_1(c)$, we have

$$\begin{aligned}\tilde{\mathbb{P}}(\tilde{c})+\tilde{\mathbb{P}}(\tilde{c}^{\dagger})\ &\leqslant\ \frac{\beta^{|c|+1+2|E_{\text{block}}|}}{(|c|+1+2|E_{\text{block}}|)!}u^{2|E_{\text{block}}|}u^{\frac{1}{2}\sum_{j=1}^{n}(1+s_j)}(1-u)^{\frac{1}{2}\sum_{j=1}^{n}(1-s_j)}\theta^{|\mathcal{L}(c)|}\hat{\theta}\\&=\frac{\beta u^{2|E_{\text{block}}|}\hat{\theta}|c|!}{(|c|+1+2|E_{\text{block}}|)!}q(c)\end{aligned}$$

where $\tilde{c}^{\dagger}$ is the link configuration where the link on e_0 is replaced by the opposite type compared to $\tilde{c}$. By summation over all $c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block}}}$, that is,

$$\tilde{\mathbb{P}}(\{B_{e_0}=1\}\cap\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}})=\sum_{c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block}}}}\ \sum_{\tilde{c}\in Q_2(c)}\tilde{\mathbb{P}}(\{\tilde{c}\}),\ \text{and}$$

$$\tilde{\mathbb{P}}(\{N_{e_0}=1\}\cap\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}})=\sum_{c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block}}}}\ \sum_{\tilde{c}\in Q_1(c)}(\tilde{\mathbb{P}}(\{\tilde{c}\})+\tilde{\mathbb{P}}(\{\tilde{c}^{\dagger}\})),$$

we can conclude (2.8), once we have established

$$\frac{\beta u^2|Q_2(c)|}{|c|+2+2|E_{\text{block}}|} \geqslant d_0\hat{\theta}|Q_1(c)|. \tag{2.9}$$

We remind the reader of the definition of $(L_m)_{m=1}^{m_0}$ and of the way elements from $Q_1(c)$ and $Q_2(c)$ can be constructed starting with an element from $\tilde{Q}(c)$.
From now on, taking again $c \in \mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}|_{A(1,\infty)\setminus E_{\text{block}}}$, we denote by μ_c the uniform measure on $\tilde{Q}(c)$ as before. Then we get

$$|Q_1(c)| = |\tilde{Q}(c)|\mathbb{E}_{\mu_c}\left(\sum_{m=1}^{m_0}\frac{L_m+1}{2}L_m\right), \text{ and } \quad |Q_2(c)| = |\tilde{Q}(c)|\mathbb{E}_{\mu_c}\left(\sum_{m=1}^{m_0}L_m\right).$$

Noting that, for all $n\in\mathbb{N}$ and $K_0\in\mathbb{R}$, $(x_i)_{i\in[n]_0}\mapsto\sum_{i=0}^n\frac{x_i^2}{2}$ on $\sum_{i=0}^n x_i = K_0$ attains its minimum at $(x_i)_{i\in[n]_0} = \frac{K_0}{n+1}(1)_{i\in[n]_0}$ and that, for $\tilde{c}\in Q_1(c)|_{A(1,\infty)}$, $m_0 \leqslant 2K-2$, we get

$$\mathbb{E}_{\mu_c}\left(\sum_{m=1}^{m_0}\frac{L_m+1}{2}L_m\right) \geqslant \frac{1}{2}\mathbb{E}_{\mu_c}\left(m_0\left(\frac{\sum_{m=1}^{m_0}L_m}{m_0}\right)^2\right) \geqslant \frac{1}{4K-2}\mathbb{E}_{\mu_c}\left(\sum_{m=1}^{m_0}L_m\right)^2$$

where we have used Jensen's inequality. Hence, (2.9) follows by

$$\mathbb{E}_{\mu_c}\left(\sum_{m=1}^{m_0}L_m\right) \geqslant (4K-2)\frac{|c|+2+2|E_{\text{block}}|}{\beta u^2}\hat{\theta}d_0$$

as a consequence of Lemma 2.4. □

Proof of Theorem 1.1: We need to show (2.1). We use Lemmata 2.1 and 2.2 to get for all $\bar{c}\in\mathcal{C}(A(2,\infty))$ and $\bar{\varepsilon}\in\{0,1\}^{A(1,2)}$

$$\tilde{\mathbb{P}}(\{B_{e_0}=1\}\cap\mathcal{C}_{\bar{c},\bar{\varepsilon}}) \geqslant \tilde{\mathbb{P}}(\{B_{e_0}=1\}\cap\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}) \geqslant \delta_0\tilde{\mathbb{P}}(\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}) \geqslant \delta_0\,\mathrm{e}^{-\beta^+(2K-2)}\,\tilde{\mathbb{P}}(\mathcal{C}_{\bar{c},\bar{\varepsilon}})$$

where we have summed inequality (2.3) over all $c\in\mathcal{C}^{\min}_{\bar{c},\bar{\varepsilon}}$. Let $\tilde{\varepsilon}\in\{0,1\}^{E\setminus\{e_0\}}$ be arbitrary and set $\bar{\varepsilon}:=\tilde{\varepsilon}|_{A(1,2)}$. By summing the estimate over all the $\bar{c}$ such that, for all $e'\in A(3,\infty)$, we have $B_{e'}(\bar{c})=\tilde{\varepsilon}_{e'}$, we conclude

$$\tilde{\mathbb{P}}(B_{e_0}=1,\ (B_{e'})_{e'\in E\setminus\{e\}}=\tilde{\varepsilon}) \geqslant \delta_0\,\mathrm{e}^{-\beta^+(2K-2)}\,\tilde{\mathbb{P}}((B_{e'})_{e'\in E\setminus\{e\}}=\tilde{\varepsilon})$$

which yields the claim. □

Appendix A. Stochastic domination

This result provides stochastic domination in the case of local domination making it possible to find a coupling between the locally dominating and dominated measure. We will consider measures on $\{0,1\}^I$ for some set I at most countable. For simplification, we identify I with $[N]$ for some $N\in\mathbb{N}\cup\{\infty\}$ with $[\infty]:=\mathbb{N}$. A similar statement with a slightly different assumption to the one of the following proposition can be found in Liggett et al. (1997).

Proposition A.1. *Let X and Y be $\{0,1\}^I$-valued a random variables on such that for every finite set $J\subseteq I$, all $(\varepsilon_j)_{j\in J}\subseteq\{0,1\}$ and every $i\in I\setminus J$, we have*

$$\mathbb{P}\left(X_i=1\,\middle|\,\forall j\in J:\ X_j=\varepsilon_j\right) \geqslant \mathbb{P}\left(Y_i=1\,\middle|\,\forall j\in J:\ Y_j=\varepsilon_j\right).$$

Then Y is stochastically dominated by X.

Proof: We define functions $(m_k)_{k\in\mathbb{N}}$ by

$$m_k((\varepsilon_j,\tilde{\varepsilon}_j)_{j\in[k]}) := \begin{cases} 0 \text{ if } (\varepsilon_k,\tilde{\varepsilon}_k)=(0,1), \\ \mathbb{P}\left(X_k=1\middle|\forall j\ \leqslant\ k-1:\ X_j=\varepsilon_j\right)-\mathbb{P}\left(Y_k=1\middle|\forall j\ \leqslant\ k-1:\ Y_j=\varepsilon_j\right) \\ \quad \text{if } (\varepsilon_k,\tilde{\varepsilon}_k)=(1,0), \\ \mathbb{P}\left(Y_k=1\middle|\forall j\ \leqslant\ k-1:\ Y_j=\varepsilon_j\right) \text{ if } (\varepsilon_k,\tilde{\varepsilon}_k)=(1,1), \\ 1-\mathbb{P}\left(Y_k=1\middle|\forall j\ \leqslant\ k-1:\ Y_j=\varepsilon_j\right) \text{ if } (\varepsilon_k,\tilde{\varepsilon}_k)=(0,0). \end{cases}$$

Using Kolmogorovs extension theorem, we find a random variable Z on $\{0,1\}^I\times\{0,1\}^I$ such that for all $k\in\mathbb{N}$ and $(\varepsilon_j,\tilde{\varepsilon}_j)_{j\in[k]}$

$$\mathbb{P}(Z=(\varepsilon_j,\tilde{\varepsilon}_j)_{j\in[k]})=\prod_{l=1}^{k} m_l((\varepsilon_j,\tilde{\varepsilon}_j)_{j\in[l]}).$$

A straightforward calculation shows $Z(\cdot,\{0,1\}^I)\overset{d}{=}X(\cdot)$ and $Z(\{0,1\}^I,\cdot)\overset{d}{=}Y(\cdot)$. By the definition of Z, we have $Z\in\{(a_i,b_i)_{i\in I}\in\{0,1\}^I:\ a_i\ \geqslant\ b_i\}$ $\mathbb{P}$-a.s.. This implies that Y is stochastically dominated by X. □

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