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On the distribution of the distance of pairs of random points from a spherical shell

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Abstract. We study the distribution of the distance of a pair of independent random points selected uniformly in concentric spherical shells in $\mathbb{R}^d$. Additionally, we rederive the distribution of the distance between two random points in a ball selected according to the beta distribution. Finally, we also consider the distance between two points following a beta distribution in a concentric spherical shell in some special cases for β and d.

1. Introduction

We investigate geometric models based on certain beta distributions in $\mathbb{R}^d$. Let $\mathbb{1}(\cdot)$ denote the indicator function of a set and $\|x\|$ the Euclidean norm of a vector $x \in \mathbb{R}^d$. We consider the d-dimensional beta distributions $\mu_{d,\beta}$ in the unit ball B^d with the following density function with respect to the Lebesgue measure

$$f_{d,\beta}(x) = c_{d,\beta}(1 - \|x\|^2)^\beta \mathbb{1}(0 \leq \|x\| < 1)$$

for $\beta > -1$ with

$$c_{d,\beta} = \frac{\Gamma(\frac{d}{2} + \beta + 1)}{\pi^{\frac{d}{2}}\Gamma(\beta + 1)}.$$

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Important features of $\mu_{d,\beta}$ are that, on the one hand, $\mu_{d,0}$ is the uniform distribution in B^d, and, on the other hand, as $\beta \to -1^+$, $\mu_{d,\beta}$ converges weakly to the uniform distribution on the unit sphere S^{d-1}.

Several papers have been published recently on random polytope models based on $\mu_{d,\beta}$, see, for example, Grote et al. (2019), Gusakova and Kabluchko (2025), Kabluchko et al. (2019), Kabluchko et al. (2020), Kabluchko and Panzo (2026), Kabluchko and Steigenberger (2025). We refer to a history, recent results, and further references to the above mentioned papers and the book by Kabluchko et al. (2025b).

Let $\mathrm{cl}(\cdot)$ denote the closure of a set in $\mathbb{R}^d$. For $0 \le R < 1$, let $B_R = \mathrm{cl}(B^d \setminus RB^d)$ denote the closed region between the two concentric balls B^d and RB^d. We call B_R the spherical shell of inner radius R and outer radius 1.

For $0 \le R < 1$, consider the restriction $\mu_{d,\beta,R}$ of $\mu_{d,\beta}$ to the spherical shell B_R normalized such that it is a probability distribution. Then $\mu_{d,\beta,R}$ is concentrated in B_R with the following density function with respect to the Lebesgue measure:

$$f_{d,\beta,R}(x) = c_{d,\beta,R}(1 - \|x\|^2)^\beta \mathbb{1}(R \le \|x\| < 1) \tag{1.1}$$

for $\beta > -1$ with a suitable normalizing constant $c_{d,\beta,R}$. We note that $\mu_{d,\beta,0}$ is the beta distribution $\mu_{d,\beta}$ in B^d, and $\mu_{d,0,R}$ is the uniform distribution in B_R for any $0 \le R < 1$.

Let p_1 and p_2 be independent, identically distributed (i.i.d.) random points from B_R chosen according to $\mu_{d,\beta,R}$. We study the density $g^*_{d,\beta,R}(r)$ of the random variable $r = \|p_1 - p_2\|$. In particular, the symbol $g^*_{d,\beta}(r)$ denotes $g^*_{d,\beta,0}(r)$ for brevity. Our argument uses characteristic functions and is based on Lord (1954a). We compute the functions $g^*_{d,\beta}(r)$ and $g^*_{d,0,R}(r)$ explicitly, and we also show that the analogous computation can be carried out for $g^*_{d,\beta,R}$ for any integer β and sufficiently large d, depending on β.

The motivation for examining the distribution of pairs of random points in shells comes, in part, from the theory of random polytopes and their generalizations. It is a common phenomenon in classical random polytope theory that the significant part of an asymptotic formula is generated by random points in small caps of the convex body where the size of caps shrinks with the increase of the number of points. Also, a small part of the boundary of a random polytope is often determined locally by random points that are close. For more information on classical random polytopes see, for example, the surveys Hug (2013), Reitzner (2010) and Schneider (2018).

In recent years, several new geometric models emerged in which the concept of convexity and that of the convex hull are modified, see Bezdek et al. (2026). In classical convexity a convex body is equal to the intersection of its supporting half spaces. In these new models other objects play the role of half spaces. Prominent examples are r-ball convexity, C-ball convexity and H-convexity. In r-ball convexity, the r-ball hull of a set is the intersection of all radius r closed balls containing the set. The concept of r-ball convexity is a special case of C-convexity where instead of radius r balls translates of a fixed convex body C generate the C-ball convex hull. In Kabluchko et al. (2025a), the authors further generalize this by allowing transformations other than just translations. These new models have natural advantages when approximating convex bodies with curvature conditions. In such approximations new phenomena emerged that are not present in the classical convex case, see, for example, Fodor et al. (2014), Marynych and Molchanov (2022); Kabluchko et al. (2025a). In all of the above mentioned models, there are examples where the properties of generalized random polytopes are determined more globally by random points from parts of a shell close to the boundary, with the thickness of the shell shrinking to zero as the number of points tends to infinity. It also a new phenomenon that $n \le d$ i.i.d. uniform random points from a convex body (or beta distributed random points from B^d) may not be in convex position in the sense of the model, meaning that some random points may be contained in the interior of the generalized hull of the others. This phenomenon does not occur with positive probability in the classical convex case which makes it difficult to transfer standard methods to the generalized models. The combination of these two issues makes it important to understand the behaviour of i.i.d. random points from shells.

Investigations of the distribution of the distance of two i.i.d. uniform random points in a convex body K (compact convex set with non-empty interior) go back to the first half of the twentieth century. The density functions for various bodies have been determined by different methods. In the particular case where K is a ball of dimension d, the density function was found by Borel (1925) for $d = 2$, by Deltheil (1926) for $d = 3, 5, 7, 9$ by Crofton's Theorem and by Boursin (1964) for $d = 11, 13$. The general case was solved by Hammersley (1950), and alternative methods were given by Lord (1954a), among others. One of the methods described in Lord (1954a) uses characteristic functions and can be (in theory) applied to radially symmetric distributions that have a density with respect to the Lebesgue measure. We will use this method in this paper. The density function was also determined for some other specific bodies such as cubes, cylinders, etc. For the early history of the topic, see, for example, Kendall and Moran (1963), for more recent references, Mathai (1999, Section 2.6.3).

For general K, Piefke (1978) established a connection between the distribution of random chord lengths and distances of pairs in d-dimensions, extending earlier results for $d = 2$ and 3.

Fairthorne (1964) considered the random model in which two uniform random points are selected from two concentric circular discs, such that one point is from the smaller disc and the other one is from the larger one. He determined the density function of the distance of the two random points. This result was extended to d-dimensions by Ruben (1970).

Although we concentrate only on random distances in this paper, we note that more general models have also been investigated extensively. One such model is when one takes $1 \leq r \leq d$ i.i.d. random points from B^d according to a beta probability distribution. Such random points almost surely span an r-dimensional simplex. This more general model naturally includes both the uniformly distributed case and also the case of random distances of pairs of points (when $r = 1$). For results on mean values and integer moments of the volume of such random r-dimensional simplices see, for example, Miles (1971) and Ruben and Miles (1980). The exact density function of the r-volume was given by Mathai (1983), and it is expressed in several forms, using hypergeometric functions, G-functions, H-functions, series expansions, etc. See also, for example, Pederzoli (1985b,a, 1986). We refer for more detailed information, history and references to Mathai (1999, Section 4.3 and pp. 427–428) and the book by Kabluchko et al. (2025b).

We will use the symbol $\langle \cdot, \cdot \rangle$ for the usual Euclidean scalar product in $\mathbb{R}^d$ whose induced norm is $\| \cdot \|$. The volume of B^d is $\kappa_d = \pi^{\frac{d}{2}} / \Gamma(\frac{d}{2} + 1)$, where $\Gamma(\cdot)$ is Euler's gamma function (see Artin, 1964), and the surface volume of S^{d-1} is $\omega_d = d\kappa_d$, see, for example, Schneider (2014).

We will use Gauss's hypergeometric function ${}_2F_1(a, b, c, z)$, which is defined by the following series for complex numbers with $|z| < 1$,

$$
{}_2F_1(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},
$$

where $a, b, c \in \mathbb{R}$ except when c is a nonpositive integer. The symbol $(x)_k$ is the rising factorial. Furthermore, let

$$
B(z; a, b) = \int_0^z u^{a-1} (1 - u)^{b-1} \, \mathrm{d}u = \frac{z^a}{a} \, {}_2F_1(a, 1 - b, a + 1, z).
$$

be the incomplete beta function. If $z = 1$, then we get the (complete) beta function, which we denote by $B(a, b)$.

The function $g^*_{d,0,0}(r)$ was determined by Hammersley (1950), see also Lord (1954a,b),

$$
g^*_{d,0,0}(r) = \frac{d \, \Gamma(\frac{d}{2} + 1)}{\Gamma(\frac{1}{2}) \Gamma(\frac{d}{2} + \frac{1}{2})} r^{d-1} B\left(1 - \frac{r^2}{4}; \frac{d+1}{2}, \frac{1}{2}\right), \quad 0 \leq r \leq 2. \tag{1.2}
$$

In this paper, we study the model in which two i.i.d. random points p_1 and p_2 are chosen according to the beta distribution $\mu_{d,\beta,R}$ from a spherical shell B_R. We study the density $g^*_{d,\beta,R}(r)$ of the random variable $r = \|p_1 - p_2\|$ using characteristic functions. We calculate explicitly $g^*_{d,0,R}$ as follows.

Theorem 1.1. *Let* $C_d = \frac{d^2\Gamma(\frac{d}{2})}{2\Gamma(\frac{d}{2}+\frac{1}{2})\Gamma(\frac{1}{2})}$. *For* $R \in [0,1)$ *and* $r \in [0,2]$, *let*

$$g_1(d,r,R) = \frac{C_d}{2^{\frac{d}{2}}} \frac{r^{d-1}}{(1-R^d)^2} \int_{\arccos\left(\frac{2-r^2}{2}\right)}^{\pi} \frac{\sin^d \varphi}{(1-\cos\varphi)^{\frac{d}{2}}} \,\mathrm{d}\varphi,$$

$$g_2(d,r,R) = -2C_d \frac{R^d r^{d-1}}{(1-R^d)^2} \int_{\arccos\left(\frac{1+R^2-r^2}{2R}\right)}^{\pi} \frac{\sin^d \varphi}{(1+R^2-2R\cos\varphi)^{\frac{d}{2}}} \,\mathrm{d}\varphi \cdot \mathbb{1}(1-R < r \le 1+R),$$

$$g_3(d,r,R) = -2C_d \frac{R^d r^{d-1}}{(1-R^d)^2} \int_0^{\pi} \frac{\sin^d \varphi}{(1+R^2-2R\cos\varphi)^{\frac{d}{2}}} \,\mathrm{d}\varphi \cdot \mathbb{1}(0 < r \le 1-R),$$

$$g_4(d,r,R) = \frac{C_d}{2^{\frac{d}{2}}} \frac{R^d r^{d-1}}{(1-R^d)^2} \int_{\arccos\left(\frac{2R^2-r^2}{2R^2}\right)}^{\pi} \frac{\sin^d \varphi}{(1-\cos\varphi)^{\frac{d}{2}}} \,\mathrm{d}\varphi \cdot \mathbb{1}(0 < r \le 2R).$$

Then the density of the distance between two independent random points selected uniformly from the spherical shell B_R *is given by*

$$g^*_{d,0,R}(r) = g_1(d,r,R) + \cdots + g_4(d,r,R). \tag{1.3}$$

We note that Theorem 1.1 may also be obtained via the method of Ruben (1970), besides other techniques; however, our argument is short and very direct. Moreover, it can also be carried out in a straightforward, although laborious, way for integer β and sufficiently large d depending on β, see Section 7.

We note that in all dimensions, $g^*_{d,0,0}(r) = g^*_{d,0}(r)$, and if $R \to 1^-$, then $g^*_{d,0,R}(r)$ tends to the density of the distance of two i.i.d. uniform random points from S^{d-1}. We also note that the functions $g_i(d,r,R)$, $i = 1,\dots,4$ can be expressed in terms of incomplete beta integrals and incomplete Gaussian hypergeometric functions by standard substitutions; for details, see Section 8.

The paper is organized as follows. In Section 2, we describe the general method, and in Section 3 we collect some tools from the theory of Bessel functions. We illustrate the method by determining the density function $g^*_{d,\beta}$ in B^d in Section 4. We prove Theorem 1.1 in Section 5, and we provide explicit formulas for $g^*_{d,0,R}$ for $d = 2,3$ cases as examples in Section 6. In Section 7, we show how this calculation can be carried out for positive integer β and sufficiently large dimension d. Finally, in Section 8, we show how one can express the functions in Theorem 1.1 in terms of incomplete beta integrals and incomplete Gaussian hypergeometric functions.

2. The method of characteristic functions

In this section, we recall the main points of the method we use to determine the density $g^*_{d,\beta,R}(r)$. For more details, we refer to Lord (1954a,b) and Mathai (1999).

Let p be a random point in $\mathbb{R}^d$ with a spherically symmetric distribution. Assume that this distribution has a continuous density $f(x)$ with respect to the Lebesgue measure. Then the characteristic function $\phi_p(y)$ of p is

$$\phi_p(y) = \int_{\mathbb{R}^d} e^{i\langle x,y\rangle} f(x) \,\mathrm{d}x,$$

where y is an arbitrary point of $\mathbb{R}^d$. Since the characteristic function of a random variable is determined by its distribution, we indicate either the random variable itself or its density function in the notation, whichever is more convenient. We always use ϕ to denote characteristic functions with indicating the random variable or its density in the subscript. Let p_1 and p_2 i.i.d. random points in $\mathbb{R}^d$ with the same density $f(x)$ with respect to the Lebesgue measure. Then their characteristic functions are the same, and the characteristic function $\phi_{p_1+p_2}(y)$ of p_1+p_2 is $\phi_{p_1+p_2}(y)=\phi_{p_1}(y)\phi_{p_2}(y)$. Since f is assumed to be spherically symmetric, it depends only on $s=\|x\|$. We use the notation $h(s)$ for the radial part of f, that is, the d-dimensional density of p as a function of s. The corresponding one-dimensional density as a function of $\|x\|$ is denoted by $h^*(s)$. The connection between $h(s)$ and $h^*(s)$, by virtue of the spherical symmetry of the distribution of p, is

$$h^*(s)=\omega_d s^{d-1}h(s).$$

Let $\varrho=\|y\|$, and let $\psi_p(\varrho)$ be the radial part of the characteristic function $\phi_p(y)$ in terms of ϱ. Then (for details, see, for example, Lord, 1954a or Mathai, 1999, pp. 289–292)

$$\psi_p(\varrho)=(2\pi)^{\frac{d}{2}}\varrho^{-\frac{d}{2}+1}\int_0^\infty s^{\frac{d}{2}}h(s)J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s, \tag{2.1}$$

where $J_\alpha(z)$ denotes a Bessel function of the first kind defined by the following series

$$J_\alpha(z)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha}. \tag{2.2}$$

Using the inverse Fourier transform (for the details of the argument, see Mathai, 1999 pp. 293, 395), one obtains that

$$h^*(s)=\frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^\infty (s\varrho)^{\frac{d}{2}}J_{\frac{d}{2}-1}(s\varrho)\psi_p(\varrho))\mathrm{d}\varrho. \tag{2.3}$$

Since p has a spherically symmetric distribution, $-p$ has the same density function as p. Thus, if p_1 and p_2 are i.i.d. random points distributed according to f, then the densities of p_1+p_2 and p_1-p_2 are the same.

Therefore, the characteristic function of p_1-p_2 is $\phi_{p_1}(y)\phi_{p_2}(y)$ with radial part $\psi_{p_1}(\varrho)\psi_{p_2}(\varrho)$. If $r=\|p_1-p_2\|$, then using (2.3), we obtain that the one-dimensional density $g^*(r)$ is

$$g^*(r)=\frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^\infty (r\varrho)^{\frac{d}{2}}J_{\frac{d}{2}-1}(r\varrho)\psi_{p_1}(\varrho)\psi_{p_2}(\varrho)\mathrm{d}\varrho. \tag{2.4}$$

In particular, let p_1 and p_2 be i.i.d. random points from B_R distributed according to $\mu_{d,\beta,R}$. We denote the common characteristic function of p_1 and p_2 by $\phi_{d,\beta,R}(y)$ and the radial part of the characteristic function by $\psi_{d,\beta,R}(\varrho)$, where $\varrho=\|y\|$. We recall that $g^*_{d,\beta,R}(r)$ denotes the one-dimensional density of the random variable $r=\|p_1-p_2\|$. We show that both $\psi_{d,\beta,R}(\varrho)$ and $g^*_{d,\beta,R}(r)$ can be evaluated for certain combinations of β and d using known properties of Bessel functions.

Using (2.1), we obtain

$$\phi_{d,\beta,R}(y)=\psi_{d,\beta,R}(\varrho)=(2\pi)^{\frac{d}{2}}\varrho^{-\frac{d}{2}+1}\int_0^\infty s^{\frac{d}{2}}h_{d,\beta,R}(s)J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s. \tag{2.5}$$

where

$$h_{d,\beta,R}(s)=c_{d,\beta,R}(1-s^2)^\beta\mathbb{1}(R\le s<1).$$

Then, by the inversion formula (2.4), the one-dimensional density $g^*_{d,\beta,R}(r)$ is

$$g^*_{d,\beta,R}(r)=\frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^\infty (r\varrho)^{\frac{d}{2}}J_{\frac{d}{2}-1}(r\varrho)\psi^2_{d,\beta,R}(\varrho)\,\mathrm{d}\varrho. \tag{2.6}$$

3. Tools from the theory of Bessel function

In this section, we collect some tools from the theory of Bessel functions that we use in our arguments. For more detailed information and references, the reader may consult the book by Watson (1995). Our main tool is

$$\int_0^\infty \frac{J_\mu(a\varrho)J_\nu(b\varrho)}{\varrho^\lambda}\,\mathrm{d}\varrho,$$

the so-called discontinuous integral of Weber and Schafheitlin. It is assumed that $0 < a, b$ so that the improper integral converges at ∞.

Lemma 3.1. *Assume that $\mu+\nu+1>\lambda>-1$ and $0<b<a$. Then the integral on the left-hand side converges and the following holds*

$$\int_0^\infty \frac{J_\mu(a\varrho)J_\nu(b\varrho)}{\varrho^\lambda}\mathrm{d}\varrho = \frac{b^\nu\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2})}{2^\lambda a^{\nu-\lambda+1}\Gamma(\nu+1)\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2})} \times {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}, \nu+1, \frac{b^2}{a^2}\right). \quad (3.1)$$

Formula (3.1) (see Watson, 1995, (2) on p. 401) was obtained by Sonine (1887) and Schafheitlin (1888) (for a historical discussion, we refer to Watson, 1995, p. 398).

The following formula, which involves the product of three Bessel functions in the integral, can be obtained from the Weber–Schafheitlin integral by substitution, see Watson (1995, 2nd equation in Section 13.4).

Lemma 3.2. *Assume that $\nu>-\frac{1}{2}$, $\mu+\nu+1>\lambda>-1$. Then*

$$\int_0^\infty \frac{J_\mu(a\varrho)J_\nu(b\varrho)J_\nu(c\varrho)}{\varrho^{\lambda+\nu}}\mathrm{d}\varrho = \frac{\left(\frac{1}{2}bc\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\infty\int_0^\pi \frac{J_\mu(a\varrho)J_\nu(\varpi\varrho)}{\varpi^\nu\varrho^\lambda}\sin^{2\nu}\varphi\,\mathrm{d}\varphi\mathrm{d}\varrho, \quad (3.2)$$

where $\varpi=\sqrt{b^2+c^2-2bc\cos\varphi}$, and the integral on the right-hand side is absolutely convergent.

We will use the following formulas to evaluate indefinite integrals that involve Bessel functions. The recursion formula (3.3) is originally from Lommel, see Watson (1995, (4) on p. 133),

$$\int z^{\mu+1}J_\nu(z)\,\mathrm{d}z = -(\mu^2-\nu^2)\int z^{\mu-1}J_\nu(z)\,\mathrm{d}z + (z^{\mu+1}J_{\nu+1}(z)+(\mu-\nu)z^\mu J_\nu(z))+C. \quad (3.3)$$

In the case where $\mu=\nu$, (3.3) reduces to

$$\int z^{\nu+1}J_\nu(z)\,\mathrm{d}z = z^{\nu+1}J_{\nu+1}(z)+C,$$

see also Watson (1995, (1) on p. 132). This yields, with a simple substitution of $r\varrho$, that

$$\int r^{\frac{d}{2}}J_{\frac{d}{2}-1}(r\varrho)\,\mathrm{d}r = r^{\frac{d}{2}}\varrho^{-1}J_{\frac{d}{2}}(r\varrho)+C. \quad (3.4)$$

4. The density $g^*_{d,\beta}(r)$

First, we demonstrate the method by calculating the density in the case where the independent random points p_1 and p_2 are distributed in B^d according to $\mu_{d,\beta}$. Since $f_{d,\beta}(x)$ is rotationally symmetric, the density of x can be written as a function of $s=\|x\|$, that is,

$$h_{d,\beta}(s)=c_{d,\beta}(1-s^2)^\beta\mathbb{1}(0\le s<1).$$

Using formula (2.5) for $\psi_{d,\beta}(\varrho)=\psi_{d,\beta,0}(\varrho)$ together with the (2.2) expansion of the Bessel function $J_{\frac{d}{2}-1}(r\varrho)$, we obtain that

$$\begin{aligned}\psi_{d,\beta}(\varrho)&=(2\pi)^{\frac{d}{2}}\varrho^{-\frac{d}{2}+1}\int_0^\infty s^{\frac{d}{2}}h_{d,\beta}(s)J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s\\&=(2\pi)^{\frac{d}{2}}\varrho^{-\frac{d}{2}+1}\frac{\Gamma(\frac{d}{2}+\beta+1)}{\pi^{\frac{d}{2}}\Gamma(\beta+1)}\int_0^1 s^{\frac{d}{2}}(1-s^2)^\beta\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+\frac{d}{2})}\Big(\frac{s\varrho}{2}\Big)^{2m+\frac{d}{2}-1}\mathrm{d}s\\&=2\frac{\Gamma(\frac{d}{2}+\beta+1)}{\Gamma(\beta+1)}\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+\frac{d}{2})}\Big(\frac{\varrho}{2}\Big)^{2m}\int_0^1(1-s^2)^\beta s^{2m+d-1}\,\mathrm{d}s.\end{aligned}\tag{4.1}$$

Since

$$\int_0^1(1-s^2)^\beta s^{2m+d-1}\,\mathrm{d}s=\frac{1}{2}\frac{\Gamma(m+\frac{d}{2})\Gamma(\beta+1)}{\Gamma(m+\frac{d}{2}+\beta+1)},$$

we obtain from (4.1) that

$$\begin{aligned}\psi_{d,\beta}(\varrho)&=\frac{\Gamma(\frac{d}{2}+\beta+1)}{\Gamma(\beta+1)}\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+\frac{d}{2})}\Big(\frac{\varrho}{2}\Big)^{2m}\frac{\Gamma(m+\frac{d}{2})\Gamma(\beta+1)}{\Gamma(m+\frac{d}{2}+\beta+1)}\\&=\Gamma\Big(\frac{d}{2}+\beta+1\Big)\sum_{m=0}^\infty\frac{(-1)^m}{m!\Gamma(m+\frac{d}{2}+\beta+1)}\Big(\frac{\varrho}{2}\Big)^{2m}\\&=\Gamma\Big(\frac{d}{2}+\beta+1\Big)2^{\frac{d}{2}+\beta}\varrho^{-\frac{d}{2}-\beta}J_{\frac{d}{2}+\beta}(\varrho),\end{aligned}\tag{4.2}$$

where, in the last step, we use the expansion (2.2) again.

Then substituting (4.2) into (2.6), we get

$$\begin{aligned}g^*_{d,\beta}(r)&=\frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^\infty(r\varrho)^{\frac{d}{2}}J_{\frac{d}{2}-1}(r\varrho)\psi^2_{d,\beta}(\varrho)\mathrm{d}\varrho\\&=\frac{2^{\frac{d}{2}+2\beta+1}\Gamma^2(\frac{d}{2}+\beta+1)}{\Gamma(\frac{d}{2})}r^{\frac{d}{2}}\int_0^\infty\varrho^{-\frac{d}{2}-2\beta}J_{\frac{d}{2}-1}(r\varrho)J^2_{\frac{d}{2}+\beta}(\varrho)\,\mathrm{d}\varrho.\end{aligned}\tag{4.3}$$

In order to evaluate (4.3), we first use (3.2) with $\mu=\frac{d}{2}-1,\nu=\frac{d}{2}+\beta,\lambda=\beta,a=r,b=c=1$. Note that as $d\geq 2$ and $\beta>-1$, it holds that $\nu=\frac{d}{2}+\beta>-\frac{1}{2}$ and $\mu+\nu+2=d+\beta+1>\beta+1=\lambda+1>0$, so the conditions of Lemma 3.2 are satisfied. Next, we apply (3.1) with $\mu=\frac{d}{2}+\beta,\nu=\frac{d}{2}-1,\lambda=\beta,a=\varpi,b=r$. As $\mu+\nu+1=d+\beta>\beta=\lambda>-1$, the conditions of Lemma 3.1 are also satisfied. Assuming that $0<r<\varpi$, we get that

$$\begin{aligned}&\int_0^\infty\varrho^{-\frac{d}{2}-2\beta}J_{\frac{d}{2}-1}(r\varrho)J^2_{\frac{d}{2}+\beta}(\varrho)\,\mathrm{d}\varrho\\&\qquad=\frac{r^{\frac{d}{2}-1}}{2^{\frac{d}{2}+2\beta}\Gamma(\frac{1}{2})\Gamma(\beta+1)\Gamma(\frac{d}{2}+\beta+\frac{1}{2})}\int_{A_1}^\pi\frac{\sin^{d+2\beta}\varphi}{\varpi^d}\Big(1-\frac{r^2}{\varpi^2}\Big)^\beta\mathrm{d}\varphi,\end{aligned}$$

where $\varpi=\sqrt{2(1-\cos\varphi)}$ and $A_1=\arccos(\frac{2-r^2}{2})$. Here, in the last step, we also use Euler's transformation

$${}_2F_1(a,b,c;z)=(1-z)^{c-a-b}{}_2F_1(c-a,c-b,c;z).$$

Thus

$$g^*_{d,\beta}(r) = \frac{2^{2\beta+1}\Gamma^2(\frac{d}{2}+\beta+1)}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2})\Gamma(\beta+1)\Gamma(\frac{d}{2}+\beta+\frac{1}{2})} r^{d-1} \int_{A_1}^{\pi} \cos\Big(\frac{\varphi}{2}\Big)^{d+2\beta} \Big(\sin^2\Big(\frac{\varphi}{2}\Big) - \frac{r^2}{4}\Big)^{\beta} \mathrm{d}\varphi$$
$$= \frac{2\Gamma^2(\frac{d}{2}+\beta+1)}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2})\Gamma(\beta+1)\Gamma(\frac{d}{2}+\beta+\frac{1}{2})} r^{d-1}(4-r^2)^{\beta}$$
$$\times \int_0^{1-\frac{r^2}{4}} u^{\frac{d+2\beta-1}{2}}(1-u)^{-\frac{1}{2}}\Big(1-\frac{4}{4-r^2}u\Big)^{\beta} \mathrm{d}u. \tag{4.4}$$

We obtained the last form by first substituting $t = \cos(\varphi/2)$, then $u = t^2$. Note that in the case when $\beta = 0$, (4.4) reduces to Hammersley's formula (1.2). We note that the integral in (4.4) is the incomplete Gaussian hypergeometric function.

5. Proof of Theorem 1.1

Let $R \in [0,1)$, and let the independent random points p_1 and p_2 be chosen from the spherical shell B_R according to the uniform probability distribution $\mu_{d,0,R}$. Thus, p_1 and p_2 have identical (d-dimensional) densities

$$h_{d,0,R}(s) = \frac{1}{\kappa_d(1-R^d)}\mathbb{1}(R \le s \le 1).$$

We recall that $\phi_{d,0,R}(y)$, $y \in \mathbb{R}^d$ is the common characteristic function of p_1 and p_2, and its radial part is denoted by $\psi_{d,0,R}(\varrho)$, where $\varrho = \|y\|$. Then by (2.5),

$$\psi_{d,0,R}(\varrho) = \frac{1}{\kappa_d(1-R^d)}(2\pi)^{\frac{d}{2}}\varrho^{-\frac{d}{2}+1}\int_R^1 s^{\frac{d}{2}} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s.$$

Thus, by (3.4),

$$\psi_{d,0,R}(\varrho) = \frac{2^{\frac{d}{2}-1} d\,\Gamma(\frac{d}{2})}{1-R^d}\varrho^{-\frac{d}{2}}\big(J_{\frac{d}{2}}(\varrho) - R^{\frac{d}{2}}J_{\frac{d}{2}}(R\varrho)\big). \tag{5.1}$$

Let $r = \|p_2 - p_1\|$, as before. Then, using (2.3), we obtain the (one-dimensional) density $g^*_{d,0,R}(r)$ as follows

$$g^*_{d,0,R}(r) = \frac{2^{1-\frac{d}{2}}}{\Gamma(\frac{d}{2})}\int_0^{\infty}(r\varrho)^{\frac{d}{2}} J_{\frac{d}{2}-1}(r\varrho)\psi^2_{d,0,R}(\varrho)\,\mathrm{d}\varrho$$
$$= \frac{2^{\frac{d}{2}-1}d^2\,\Gamma(\frac{d}{2})}{(1-R^d)^2} r^{\frac{d}{2}}\int_0^{\infty}\varrho^{-\frac{d}{2}}\big(J_{\frac{d}{2}}(\varrho) - R^{\frac{d}{2}}J_{\frac{d}{2}}(R\varrho)\big)^2 J_{\frac{d}{2}-1}(r\varrho)\,\mathrm{d}\varrho. \tag{5.2}$$

By expanding the square in (5.2), we get that $g^*_{d,0,R}(r)$ is the sum of the following three terms:

$$g^*_{d,0,R}(r) = \frac{2^{\frac{d}{2}-1}d^2\,\Gamma(\frac{d}{2})}{(1-R^d)^2} r^{\frac{d}{2}}\Bigg($$
$$\int_0^{\infty}\varrho^{-\frac{d}{2}} J^2_{\frac{d}{2}}(\varrho) J_{\frac{d}{2}-1}(r\varrho)\,\mathrm{d}\varrho \tag{5.3}$$
$$-\,2R^{\frac{d}{2}}\int_0^{\infty}\varrho^{-\frac{d}{2}} J_{\frac{d}{2}}(\varrho) J_{\frac{d}{2}}(R\varrho) J_{\frac{d}{2}-1}(r\varrho)\,\mathrm{d}\varrho \tag{5.4}$$
$$+\,R^d\int_0^{\infty}\varrho^{-\frac{d}{2}} J^2_{\frac{d}{2}}(R\varrho) J_{\frac{d}{2}-1}(r\varrho)\,\mathrm{d}\varrho\Bigg). \tag{5.5}$$

We evaluate (5.3)–(5.5), with the help of Lemmas 3.1 and 3.2. Since the integrals in (5.3) and (5.5) are very similar, we work out only (5.5) in detail.

We use Lemma 3.2 with the choice $\mu = \frac{d}{2} - 1$, $\nu = \frac{d}{2}$, $a = r$, $b = c = R$ and $\lambda = 0$. Since $\nu = \frac{d}{2} > -\frac{1}{2}$, and $\mu + \nu + 2 = d + 1 > \lambda + 1 = 1 > 0$, the conditions of Lemma 3.2 are satisfied. Therefore, we obtain that

$$(5.5) = R^d \int_0^\infty \varrho^{-\frac{d}{2}} J_{\frac{d}{2}}^2(R\varrho) J_{\frac{d}{2}-1}(r\varrho)\,\mathrm{d}\varrho$$

$$= \frac{2^{-\frac{d}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})} R^{2d} \int_0^\pi \frac{\sin^d\varphi}{\varpi_4^{\frac{d}{2}}} \int_0^\infty J_{\frac{d}{2}-1}(r\varrho) J_{\frac{d}{2}}(\varpi_4\varrho)\,\mathrm{d}\varrho\mathrm{d}\varphi, \tag{5.6}$$

where $\varpi_4 = \sqrt{2R^2(1-\cos\varphi)}$.

Next, we apply formula (3.1) with $\mu = \frac{d}{2}$, $\nu = \frac{d}{2} - 1$, $a = \varpi_4$, $b = r$, $\lambda = 0$. As $d \geq 2$, it holds that $\nu = \frac{d}{2} - 1 > -\frac{1}{2}$ and $\mu + \nu + 1 = d > \lambda = 0 > -1$, thus the conditions of Lemma 3.1 are satisfied. The condition $0 < r < \varpi_4$ holds precisely when $0 < r \leq 2R$, and then the calculation yields

$$(5.6) = \frac{2^{-\frac{d}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})} R^{2d} \int_{A_4}^\pi \frac{\sin^d\varphi}{\varpi_4^{\frac{d}{2}}} \frac{r^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}{\varpi_4^{\frac{d}{2}}\Gamma(\frac{d}{2})\Gamma(1)} \, {}_2F_1\left(\frac{d}{2}, 0, \frac{d}{2}, \frac{r^2}{\varpi_4^2}\right)\mathrm{d}\varphi$$

$$= \frac{2^{-\frac{d}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})} R^{2d} r^{\frac{d}{2}-1} \int_{A_4}^\pi \left(\frac{\sin\varphi}{\varpi_4}\right)^d \mathrm{d}\varphi, \tag{5.7}$$

with $A_4 = \arccos\left(\frac{2R^2 - r^2}{2R^2}\right)$. It is also clear that if $r \to 2R^-$, then (5.7) tends to 0.

If we apply (3.1) with $\mu = \frac{d}{2} - 1$, $\nu = \frac{d}{2}$, $a = r$, $b = \varpi_4$, $\lambda = 0$, then $0 < \varpi_4 < r$ is satisfied when $r > 2R$, and the calculation yields that $(5.6) = 0$.

By a similar calculation, we obtain from (5.3) that

$$\int_0^\infty \varrho^{-\frac{d}{2}} J_{\frac{d}{2}}^2(\varrho) J_{\frac{d}{2}-1}(r\varrho)\,\mathrm{d}\varrho = \frac{2^{-\frac{d}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})} r^{\frac{d}{2}-1} \int_{A_1}^\pi \left(\frac{\sin\varphi}{\varpi_1}\right)^d \mathrm{d}\varphi, \tag{5.8}$$

with $A_1 = \arccos\left(\frac{2-r^2}{2}\right)$ and $\varpi_1 = \sqrt{2(1-\cos\varphi)}$. This formula is valid for all $r \in [0,2]$ and $R \in [0,1)$.

Now, we turn to the evaluation of the integral (5.4). We use (3.2) with the choice $\mu = \frac{d}{2} - 1$, $\nu = \frac{d}{2}$, $a = r$, $b = R$, $c = 1$, $\lambda = 0$. Since μ, ν and λ are the same as when we used (3.2) in the evaluation of (5.5), the conditions of Lemma 3.2 are satisfied. We obtain that

$$(5.4) = -2R^{\frac{d}{2}} \int_0^\infty \varrho^{-\frac{d}{2}} J_{\frac{d}{2}}(\varrho) J_{\frac{d}{2}}(R\varrho) J_{\frac{d}{2}-1}(r\varrho)\mathrm{d}\varrho$$

$$= -\frac{2^{-\frac{d}{2}+1}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})} R^d \int_0^\infty \int_0^\pi \frac{J_{\frac{d}{2}-1}(r\varrho) J_{\frac{d}{2}}(\varpi_2\varrho)}{\varpi_2^{\frac{d}{2}}} \sin^d\varphi\,\mathrm{d}\varphi\mathrm{d}\varrho$$

$$= -\frac{2^{-\frac{d}{2}+1}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})} R^d \int_0^\pi \frac{\sin^d\varphi}{\varpi_2^{\frac{d}{2}}} \int_0^\infty J_{\frac{d}{2}-1}(r\varrho) J_{\frac{d}{2}}(\varpi_2\varrho)\,\mathrm{d}\varrho\mathrm{d}\,\varphi \tag{5.9}$$

where $\varpi_2 = \sqrt{1 + R^2 - 2R\cos\varphi}$.

Finally, we use (3.1) with $\mu = \frac{d}{2}$, $\nu = \frac{d}{2} - 1$, $a = \varpi_2$, $b = r$ and $\lambda = 0$. Again, as μ, ν and λ are the same as in the evaluation of (5.5) by (3.1), the conditions of Lemma 3.1 are satisfied. The condition $0 < r < \varpi_2$ holds precisely when $0 < r \leq 1 + R$.

If $1-R<r\le 1+R$, then we get that

$$\begin{aligned}(5.9) &= -\frac{2^{-\frac{d}{2}+1}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})}R^d\int_{A_2}^{\pi}\frac{r^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(1)}\,{}_2F_1\left(\frac{d}{2},0,\frac{d}{2},\frac{r^2}{\varpi_2^2}\right)\left(\frac{\sin\varphi}{\varpi_2}\right)^d \mathrm{d}\varphi\\ &= -\frac{2^{-\frac{d}{2}+1}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})}r^{\frac{d}{2}-1}R^d\int_{A_2}^{\pi}\left(\frac{\sin\varphi}{\varpi_2}\right)^d \mathrm{d}\varphi, \end{aligned}\tag{5.10}$$

where $A_2=\arccos\left(\frac{1+R^2-r^2}{2R}\right)$. If $r\to 1+R^-$, then (5.10) $\to 0$.

If $r\le 1-R$, then

$$(5.9) = -\frac{2^{-\frac{d}{2}+1}}{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+\frac{1}{2})}r^{\frac{d}{2}-1}R^d\int_{0}^{\pi}\left(\frac{\sin\varphi}{\varpi_2}\right)^d \mathrm{d}\varphi. \tag{5.11}$$

If $r\to 1-R^+$, then (5.10) tends to (5.11) evaluated at $r=1-R$.

If we use (3.1) with $\mu=\frac{d}{2}-1$, $\nu=\frac{d}{2}$, $a=r$, $b=\varpi_2$ and $\lambda=0$, then the condition $0<\varpi_2<r$ holds when $r>1+R$. In this case we get that (5.9) $=0$.

Now, if we define the functions $g_i(d,r,R)$, $i=1,\dots,4$ as in Theorem 1.1, then (5.7), (5.8), (5.10) and (5.11) show that the density function $g^*_{d,0,R}(r)$ is the sum of the $g_i(d,r,R)$, $i=1,\dots,4$. This finishes the proof of Theorem 1.1.

6. Examples

We provide, as examples, the explicit formulas for the planar ($d=2$) and the 3-dimensional cases.

6.1. *The $d=2$ case.* Direct calculations of the functions $g_i(2,r,R)$, $i=1,\dots,4$ yield the following.

$$\begin{aligned}
g_1(2,r,R) &= \frac{2r}{\pi(1-R^2)^2}\left(\pi-\arccos\left(\frac{2-r^2}{2}\right)-\sqrt{1-\left(\frac{2-r^2}{2}\right)^2}\right),\\
g_2(2,r,R) &= -\frac{8R^2r}{\pi(1-R^2)^2}\cdot\left(\frac{R^2-1}{2R^2}\cdot\frac{\pi}{2}+\frac{(R-1)^2\pi}{4R^2}+\frac{\pi}{2R}-\frac{\sin\left(a(r,R)\right)}{2R}\right.\\
&\quad\left.-\frac{R^2-1}{2R^2}\arctan\left(\frac{R+1}{1-R}\tan\left(\frac{a(r,R)}{2}\right)\right)-\frac{(R-1)^2}{4R^2}a(r,R)-\frac{a(r,R)}{2R}\right),\\
&\quad\text{where } a(r,R)=\arccos\left(\frac{R^2+1-r^2}{2R}\right),\\
g_3(d,r,R) &= -\frac{4R^2r}{(1-R^2)^2},\\
g_4(d,r,R) &= \frac{2R^2r}{\pi(1-R^2)^2}\left(\pi-\arccos\left(\frac{2R^2-r^2}{2R^2}\right)-\sqrt{1-\left(\frac{2R^2-r^2}{2R^2}\right)^2}\right).
\end{aligned}$$

The graph of $g^*_{2,0,R}$ is shown in Figure 6.1 for a few values of R. The $R=1$ case represents the density function of two i.i.d. random points from S^1 chosen according to the normalized arc-length.

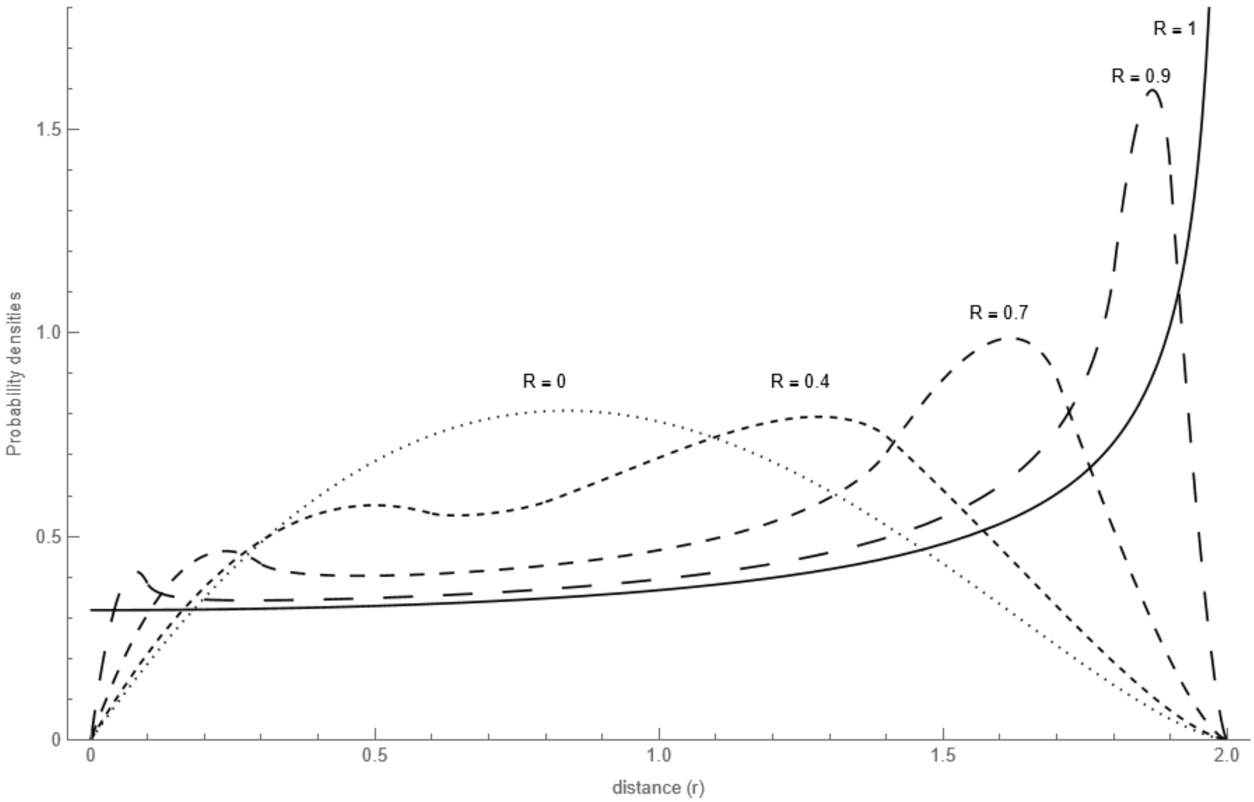

FIGURE 6.1. The graph of $g^*_{2,0,R}(r)$ for various values of R drawn by *Mathematica*.

6.2. *The $d = 3$ case.* If $d = 3$ then we obtain the following formulas.

$$g_1(3, r, R) = \frac{9}{4}\frac{r^2}{(1-R^3)^2}\left(\frac{4}{3} + \frac{1}{12}r^3 - r\right),$$

$$g_2(3, r, R) = -\frac{3}{2}\frac{r^2}{(1-R^3)^2}\left(2 + 2R^3 - \frac{3R^4 + 3 - r^4 - 6R^2 + 6R^2r^2 + 6r^2}{4r}\right),$$

$$g_3(3, r, R) = -6\frac{R^3r^2}{(1-R^3)^2},$$

$$g_4(3, r, R) = \frac{9}{4}\frac{R^3r^2}{(1-R^3)^2}\left(\frac{4}{3} + \frac{1}{12}\frac{r^3}{R^3} - \frac{r}{R}\right).$$

The graph of the function $g^*_{3,0,R}$ is drawn in Figure 6.2 for a few specific values of R. We note that the solid line represents the density of the distance of two i.i.d. random points chosen from the surface S^2 according to the normalized spherical Lebesgue measure; this is marked with $R = 1$ in the figure. In this case, the density function is linear in r.

Not surprisingly, the 3-dimensional density functions are better behaved than in two dimensions; they have a single maximum in the interval $[0, 2]$.

7. Beta distributions in spherical shells

We return to the case of general truncated beta distributions $\mu_{d,\beta,R}$ in spherical shells B_R for $0 \le R < 1$, and we show how the density $g^*_{d,\beta,R}$ can be determined explicitly for integer values of β and sufficiently large d, depending on β. Let $s = \|x\|$ be as before. Then the normalizing constant

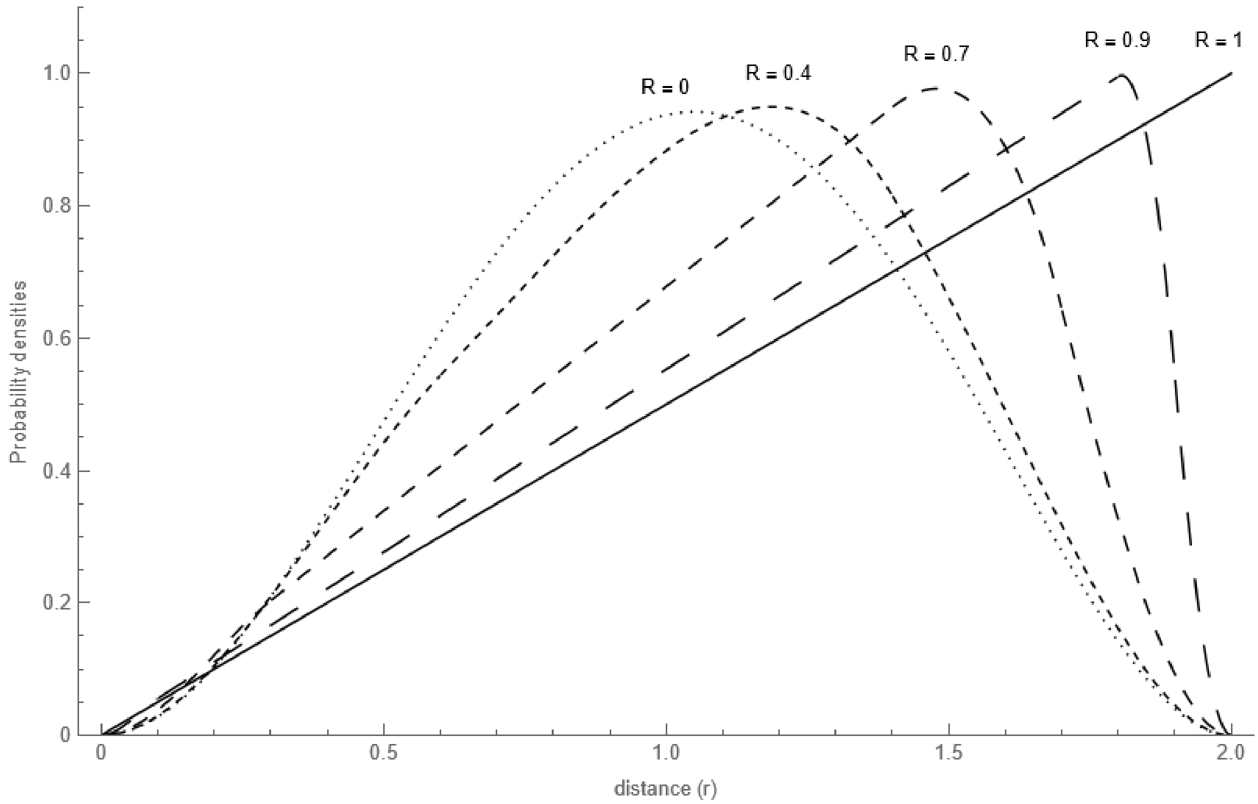

FIGURE 6.2. The function $g^*_{3,0,R}(r)$ for different values of R drawn by *Mathematica*.

$c_{d,\beta,R}$ is

$$c_{d,\beta,R} = \frac{1}{\int_{\mathbb{R}^d} f_{d,\beta,R}(x)\,\mathrm{d}x},$$

where

$$\begin{aligned} c^{-1}_{d,\beta,R} &= \int_{\mathrm{cl}(B^d \setminus RB^d)} (1-\|x\|^2)^\beta \,\mathrm{d}x = \omega_d \int_R^1 (1-s^2)^\beta s^{d-1}\,\mathrm{d}s \\ &= \frac{\omega_d}{2}\left(B\left(1; \frac{d}{2}, \beta+1\right) - B\left(R^2; \frac{d}{2}, \beta+1\right)\right) \\ &= \frac{1}{c_{d,\beta}} - \frac{\omega_d}{2} B\left(R^2; \frac{d}{2}, \beta+1\right). \end{aligned}$$

Now, as $f_{d,\beta,R}$ is rotationally symmetric, the d-dimensional density in terms of s is

$$h_{d,\beta,R}(s) = c_{d,\beta,R}(1-s^2)^\beta \mathbb{1}(R \le s \le 1),$$

and we have that

$$\begin{aligned} \psi_{d,\beta,R}(\varrho) &= (2\pi)^{\frac{d}{2}} \varrho^{-\frac{d}{2}+1} \int_0^\infty s^{\frac{d}{2}} h_{d,\beta,R}(s) J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s \\ &= (2\pi)^{\frac{d}{2}} c_{d,\beta,R} \cdot \varrho^{-\frac{d}{2}+1} \int_R^1 s^{\frac{d}{2}} (1-s^2)^\beta J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s. \end{aligned}$$

If β is a non-negative integer, then the Binomial Theorem yields that

$$\psi_{d,\beta,R}(\varrho) = (2\pi)^{\frac{d}{2}} c_{d,\beta,R} \cdot \varrho^{-\frac{d}{2}+1} \sum_{k=0}^{\beta} (-1)^k \binom{\beta}{k} \int_R^1 s^{\frac{d}{2}+2k} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s. \tag{7.1}$$

Applying (3.3) recursively to each term $\int s^{\frac{d}{2}+2k} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s$ in (7.1), we arrive in k steps to the indefinite integral $\int s^{\frac{d}{2}} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s$, which can be evaluated by (3.4).

In particular, for $k = 1, \ldots, \beta$, the substitution $z = s\varrho$ and repeated application of (3.3) yield, for some constants $c_0, \ldots, c_k$ and $e_1, \ldots, e_k$ depending on d and i, that

$$\begin{aligned}
&\int s^{\frac{d}{2}+2k} J_{\frac{d}{2}-1}(s\varrho)\,\mathrm{d}s \\
&= \varrho^{-\frac{d}{2}-2k-1}\big(c_k z^{\frac{d}{2}+2k} + c_{k-1} z^{\frac{d}{2}+2(k-1)} + \cdots + c_0 z^{\frac{d}{2}}\big) J_{\frac{d}{2}}(z) \\
&\quad + \varrho^{-\frac{d}{2}-2k-1}\big(e_k z^{\frac{d}{2}+2k-1} + e_{k-1} z^{\frac{d}{2}+2(k-1)-1} + \cdots + e_1 z^{\frac{d}{2}+1}\big) J_{\frac{d}{2}-1}(z) + C \\
&= \left(c_k \frac{s^{\frac{d}{2}+2k}}{\varrho} + c_{k-1} \frac{s^{\frac{d}{2}+2(k-1)}}{\varrho^3} + \cdots + c_0 \frac{s^{\frac{d}{2}}}{\varrho^{2k+1}}\right) J_{\frac{d}{2}}(s\varrho) \\
&\quad + \left(e_k \frac{s^{\frac{d}{2}+2k-1}}{\varrho^2} + e_{k-1} \frac{s^{\frac{d}{2}+2(k-1)-1}}{\varrho^4} + \cdots + e_1 \frac{s^{\frac{d}{2}+1}}{\varrho^{2k}}\right) J_{\frac{d}{2}-1}(s\varrho) + C \\
&= \sum_{i=0}^{k} c_i \frac{s^{\frac{d}{2}+2i}}{\varrho^{2(k-i)+1}} J_{\frac{d}{2}}(s\varrho) + \sum_{j=1}^{k} e_j \frac{s^{\frac{d}{2}+2j-1}}{\varrho^{2(k-j+1)}} J_{\frac{d}{2}-1}(s\varrho) + C.
\end{aligned}$$

Thus, after evaluating the definite integrals in (7.1), we get an explicit formula for $\psi_{d,\beta,R}(\varrho)$ in which each term contains a power of ϱ and a Bessel function $J_\nu(\varrho)$ or $J_\nu(R\varrho)$ for $\nu \in \{\frac{d}{2}, \frac{d}{2}-1\}$. Therefore

$$\begin{aligned}
\psi_{d,\beta,R}(\varrho) = (2\pi)^{\frac{d}{2}} c_{d,\beta,R} \cdot \varrho^{-\frac{d}{2}+1} \Bigg(\frac{1}{\varrho} J_{\frac{d}{2}}(\varrho) - \frac{R^{\frac{d}{2}}}{\varrho} J_{\frac{d}{2}}(R\varrho) + \sum_{k=1}^{\beta} \binom{\beta}{k} (-1)^k \\
\times \Bigg(\sum_{i=0}^{k} c_i \frac{1}{\varrho^{2(k-i)+1}} J_{\frac{d}{2}}(\varrho) - \sum_{i=0}^{k} c_i \frac{R^{\frac{d}{2}+2i}}{\varrho^{2(k-i)+1}} J_{\frac{d}{2}}(R\varrho) \\
+ \sum_{j=1}^{k} e_j \frac{1}{\varrho^{2(k-j+1)}} J_{\frac{d}{2}-1}(\varrho) - \sum_{j=1}^{k} e_j \frac{R^{\frac{d}{2}+2j-1}}{\varrho^{2(k-j+1)}} J_{\frac{d}{2}-1}(R\varrho) \Bigg) \Bigg).
\end{aligned}$$

Now, substituting $\psi_{d,\beta,R}(\varrho)$ into

$$g^*_{d,\beta,R}(r) = \frac{2^{1-\frac{d}{2}}}{\Gamma\big(\frac{d}{2}\big)} \int_0^\infty (r\varrho)^{\frac{d}{2}} J_{\frac{d}{2}-1}(r\varrho) \psi^2_{d,\beta,R}(\varrho) \mathrm{d}\varrho, \tag{7.2}$$

then expanding the square $\psi^2_{d,\beta,R}(\varrho)$, one obtains a sum of integrals, each containing the product of a power of ϱ, $J_{\frac{d}{2}-1}(r\varrho)$ and two Bessel functions from among $J_{\frac{d}{2}-1}(\varrho)$, $J_{\frac{d}{2}-1}(R\varrho)$, $J_{\frac{d}{2}}(\varrho)$, $J_{\frac{d}{2}}(R\varrho)$. First, we want to apply Lemma 3.2 to each integral. In each integral, at least two of the Bessel functions have the same order and the exponent of ϱ is between $-\frac{d}{2}-4\beta$ and $-\frac{d}{2}$. So, $\mu, \nu \in \{\frac{d}{2}-1, \frac{d}{2}\}$, and $0 \le \lambda < 4\beta+1$ in Lemma 3.2. Then $\nu > -\frac{1}{2}$ and $\lambda > -1$ are satisfied. The condition $\mu+\nu+1 > \lambda$ puts a lower bound $d \ge 4\beta+1$ on the dimension. If this is satisfied, then Lemma 3.2 can be used. The applicability of Lemma 3.1 also follows as the conditions on μ, ν and λ are weaker than in Lemma 3.2. This process is a straightforward, albeit tedious, computation that yields an explicit formula for $g^*_{d,\beta,R}(r)$ in the form of a sum of functions, each of which comes from an integral in (7.2).

8. Concluding remarks

As mentioned after Theorem 1.1, the functions $g_i(d,r,R)$ can be transformed by standard substitutions in the following way. Substituting first $t=\cos(\varphi/2)$, then $u=t^2$, we obtain

$$\int\left(\frac{\sin\varphi}{\sqrt{1-\cos\varphi}}\right)^d \mathrm{d}\varphi = -2^{\frac{d}{2}}\int u^{\frac{d-1}{2}}(1-u)^{-\frac{1}{2}}\,\mathrm{d}u,$$

$$\int\left(\frac{\sin\varphi}{\sqrt{1+R^2-2R\cos\varphi}}\right)^d \mathrm{d}\varphi = -\frac{2^d}{(R+1)^d}\int u^{\frac{d-1}{2}}(1-u)^{\frac{d-1}{2}}\left(1-\frac{4R}{(R+1)^2}u\right)^{-\frac{d}{2}}\mathrm{d}u.$$

Thus, for $R\in[0,1)$ and $r\in[0,2]$,

$$g_1(d,r,R) = 2^{\frac{d}{2}}C_d\frac{r^{d-1}}{(1-R^d)^2}B\left(1-\frac{r^2}{4};\frac{d+1}{2},\frac{1}{2}\right),$$

$$g_2(d,r,R) = -2^{d+1}C_d\frac{R^d r^{d-1}}{(1-R^d)^2(1+R)^d}\cdot\mathbb{1}(1-R<r\le 1+R) \times\int_0^{\frac{(R+1)^2-r^2}{4R}} u^{\frac{d-1}{2}}(1-u)^{\frac{d-1}{2}}\left(1-\frac{4R}{(R+1)^2}u\right)^{-\frac{d}{2}}\mathrm{d}u,$$

$$g_3(d,r,R) = -2^{d+1}C_d\frac{R^d r^{d-1}}{(1-R^d)^2(1+R)^d}B\left(\frac{d+1}{2},\frac{d+1}{2}\right) \times {}_2F_1\left(\frac{d}{2},\frac{d+1}{2},d+1,\frac{4R}{(R+1)^2}\right)\cdot\mathbb{1}(0<r\le 1-R),$$

$$g_4(d,r,R) = 2^{\frac{d}{2}}C_d\frac{R^d r^{d-1}}{(1-R^d)^2}B\left(1-\frac{r^2}{4R^2};\frac{d+1}{2},\frac{1}{2}\right)\cdot\mathbb{1}(0<r\le 2R),$$

where, for $g_3(d,r,R)$, we also used Euler's integral formula

$$B(b,c-b)\,{}_2F_1(a,b,c,z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,\mathrm{d}x,$$

which holds for $c>b>0$ and $z<1$.

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